Publ. RIMS, Kyoto Univ.
17 (1981), 657–663

On Betti Numbers of Complement of Hyperplanes

By

Hiroaki TERAO*

§1. Introduction

Let X be a non-void finite family of hyperplanes in $\boldsymbol{C}^{n+1}$ or $\boldsymbol{P}^{n+1}(\boldsymbol{C})$. Denote $\boldsymbol{P}^{n+1}(\boldsymbol{C})$ simply by $\boldsymbol{P}^{n+1}$. By $|X|$ denote we the union of all hyperplanes belonging to X. In this article we give some formulas (Theorem A, B) for computing the Betti numbers of $\boldsymbol{C}^{n+1}\backslash|X|$ or $\boldsymbol{P}^{n+1}\backslash|X|$.

Define a set

$$L(X)=\{\bigcap_{H\in A} H;\ A\subset X\}\cup\{\text{the ambient space }(\boldsymbol{C}^{n+1}\text{ or }\boldsymbol{P}^{n+1})\}\backslash\{\emptyset\}$$

and introduce a partial order $\succ$ into $L(X)$ by

$$s\succ t \Longleftrightarrow s\subset t \qquad (s,\ t\in L(X)).$$

If $L(X)$ has a unique maximal element, then X is said to be *central*. In other words, X is central if and only if $\bigcap_{H\in X} H\neq\emptyset$.

Recall the following

(1.1) **Definition.** The *Möbius function* μ: $L(X)\to\boldsymbol{Z}$ is inductively defined by

$$\mu(0)=1\,,$$
$$\mu(s)=-\sum_{\substack{t\prec s\\ t\neq s}}\mu(t)\,,$$

where 0 stands for the ambient space (the minimal element in $L(X)$).

By $r(s)$ we denote the length of the longest chain in $L(X)$ below s ($s\in L(X)$).

In this article we call a non-void finite family of hyperplanes in $\boldsymbol{C}^{n+1}$ (or $\boldsymbol{P}^{n+1}$) an *affine* (resp. *projective*) *n-arrangement*. Then we have

(1.2) **Theorem A.** *Let* X *be an affine* (*or projective*) *n-arrangement.*

Communicated by S. Nakano, September 16, 1980.
* Department of Mathematics, International Christian University, Mitaka 181, Japan.
Partially supported by Kakenhi No. 574047.

Then the Poincaré polynomial of $\boldsymbol{C}^{n+1}\backslash|X|$ *(resp.* $\boldsymbol{P}^{n+1}\backslash|X|$*) equals*

$$\sum_{s\in L(X)}\mu(s)(-t)^{r(s)}$$

$$(resp.\ \sum_{s\in L(X)}\mu(s)\{(-t)^{r(s)}-(-t)^{n+2}\}/(1+t))\,.$$

Remark. When X is central affine, this result was proved by Orlik-Solomon [1] (5.2). Moreover they explicitly determined the graded $\boldsymbol{C}$-algebra structure of $H^*(\boldsymbol{C}^{n+1}\backslash|X|, \boldsymbol{C})$. There it was also announced, without proof, that their method would go well in case that X is a non-central affine arrangement.

We will define the freeness of any affine (or projective) arrangement and the generalized exponents of a free affine (resp. projective) arrangement in Section 4 (resp. Section 3). All of the definitions are given via the case of a free central affine arrangement studied in [3] [4] [5].

The following theorem gives another formula for the Betti numbers of $\boldsymbol{C}^{n+1}\backslash|X|$ (or $\boldsymbol{P}^{n+1}\backslash|X|$) by using the generalized exponents:

(1.3) **Theorem B.** *Let X be a free affine (or projective) n-arrangement and $(d_0,\dots, d_n)$ be its generalized exponents. Then the Poincaré polynomial of* $\boldsymbol{C}^{n+1}\backslash|X|$ *(resp.* $\boldsymbol{P}^{n+1}\backslash|X|$*) equals*

$$\prod_{i=0}^{n}(1+d_it)\,.$$

Remark. This result was obtained in [5] when X is central affine. Our proof is nothing other than the reduction to the case.

The following Sections 2, 3 and 4 are devoted to the proofs of Theorems A and B. Section 2 is for the central affine case, Section 3 for the projective case, and Section 4 for the (non-central) affine case.

§2. The Central Affine Case

In this section we briefly review some known results on a central affine n-arrangement X.

By an appropriate coordinate change we can assume that $\bigcap_{H\in X}H$ contains the origin $\boldsymbol{0}$ of $\boldsymbol{C}^{n+1}$. Let $Q\in\boldsymbol{C}[z_0,\dots, z_n]$ be a defining equation of X, that is, $\boldsymbol{V}(Q)=|X|$. By $\mathcal{O}$ denote we $\mathcal{O}_{\boldsymbol{C}^{n+1},\boldsymbol{0}}$. Then

$D(X):=\{\theta$; a germ at the origin of holomorphic vector field such that
$\theta\cdot Q\in Q\cdot\mathcal{O}\}$

is an $\mathcal{O}$-module. We call X to be *free* if $D(X)$ is a free $\mathcal{O}$-module.

Assume that X is free. Let $\{\theta_0,\dots,\theta_n\}$ be a free basis for $D(X)$ such that each θ_i is homogeneous of degree d_i (see [3] 2.10). Then we call the integers $(d_0,\dots,d_n)$ the *generalized exponents* of X. They depend only on X (see [3] 2.12).

Throughout this article $b_i(S)$ stands for the i-th Betti number of a topological space S for any integer i ($b_i(S)=0$ if $i<0$).

The following "trinity" was proved in [1] (5.2) and [5] (Main Theorem):

Theorem A, B (*central affine version*).

$$(-1)^i \sum_{\substack{s\in L(X)\\ r(s)=i}} \mu(s)=b_i(\boldsymbol{C}^{n+1}\backslash|X|)=\pi_i(d_0,\dots,d_n)$$

for any integer i, where $\pi_i\in\boldsymbol{Z}[t_0,\dots,t_n]$ is the elementary symmetric polynomial of degree i ($\pi_i=0$ if $i<0$ or $i>n+1$).

§3. The Projective Case

Let $X(\neq\emptyset)$ be a projective n-arrangement. Let $Q\in\boldsymbol{C}[z_0,\dots,z_{n+1}]$ be a homogeneous polynomial defining $|X|\subset\boldsymbol{P}^{n+1}$. Then there exists a central affine $(n+1)$-arrangement $\tilde{X}$ such that

$$V(Q)=|\tilde{X}|\subset\boldsymbol{C}^{n+2}.$$

(3.1) **Proposition.**

$$b_i(\boldsymbol{P}^{n+1}\backslash|X|)+b_{i-1}(\boldsymbol{P}^{n+1}\backslash|X|)=b_i(\boldsymbol{C}^{n+2}\backslash|\tilde{X}|)$$

for any integer i.

Proof. Consider the natural projection

$$\pi\colon \boldsymbol{C}^{n+2}\backslash|\tilde{X}|\longrightarrow\boldsymbol{P}^{n+1}\backslash|X|,$$

then this is a $\boldsymbol{C}^*$-bundle. So we have the Gysin exact sequence

$$\begin{aligned}\cdots&\longrightarrow H^q(\boldsymbol{P}^{n+1}\backslash|X|)\xrightarrow{\pi^*}H^q(\boldsymbol{C}^{n+2}\backslash|\tilde{X}|)\longrightarrow H^{q-1}(\boldsymbol{P}^{n+1}\backslash|X|)\\ &\longrightarrow H^{q+1}(\boldsymbol{P}^{n+1}\backslash|X|)\xrightarrow{\pi^*}\cdots.\end{aligned}$$

What we have to prove is the injectivity of each π^* above.

Let φ be a rational q-form on $\boldsymbol{P}^{n+1}$ whose pole is only along $|X|$. Assume that $\pi^{\cdot}\varphi=d\eta$ for some homogeneous rational $(q-1)$-form η on $\boldsymbol{C}^{n+2}$ with pole only along $|\tilde{X}|$, where $\pi^{\cdot}\varphi$ means the pull-back of φ by π.

Then there exists a rational $(q-1)$-form ψ on $\boldsymbol{P}^{n+1}$ with pole only along

$|X|$ such that

$$\pi^{*}\psi = -\langle \theta, \frac{1}{\deg Q}\frac{dQ}{Q} \wedge \eta \rangle .$$

Here $\langle \theta, \ \rangle$ stands for the contraction with the Euler vector field

$$\theta = \sum_{i=0}^{n+1} z_i(\partial/\partial z_i).$$

Then we can show

$$d\psi = \varphi$$

by a direct but lengthy computation (or by applying (2.6), (2.7) and (2.9) in [2]).

These facts imply that each π^* is injective and thus (3.1). Q.E.D.

For any integer i, we have

$$\text{(3.2)} \qquad b_i(\boldsymbol{P}^{n+1}\backslash|X|) = \sum_{j=0}^{i} (-1)^{i-j} b_j(\boldsymbol{C}^{n+2}\backslash|\tilde{X}|)$$

in the light of (3.1).

Define an injective mapping

$$\rho\colon L(X) \longrightarrow L(\tilde{X})$$

by $\rho(s)=$(the closure of $\pi^{-1}(s)$ in $\boldsymbol{C}^{n+2}$) ($s \in L(X)$), where π is the natural projection: $\boldsymbol{C}^{n+2}\backslash\{\boldsymbol{0}\} \to \boldsymbol{P}^{n+1}$. Then it is easy to see that

$$r(\rho(s)) = r(s), \quad \text{and} \quad \mu(\rho(s)) = \mu(s), \quad (s \in L(X)).$$

Notice that im $\rho \supset \{t \in L(\tilde{X});\ r(t) < n+2\}$. Thus we have

$$\text{(3.3)} \qquad \begin{aligned} & b_i(\boldsymbol{P}^{n+1}\backslash|X|) \\ &= \sum_{j=0}^{i} (-1)^{i-j} b_j(\boldsymbol{C}^{n+2}\backslash|\tilde{X}|) \quad \text{(by (3.2))} \\ &= (-1)^i \sum_{j=0}^{i} \sum_{\substack{t \in L(\tilde{X}) \\ r(t)=j}} \mu(t) \\ &= (-1)^i \sum_{j=0}^{i} \sum_{\substack{s \in L(X) \\ r(s)=j}} \mu(s) \end{aligned}$$

for $i < n+2$. It is obvious that

$$b_i(\boldsymbol{P}^{n+1}\backslash|X|) = 0 \quad \text{if} \quad i \geqq n+2.$$

Thus a brief computation leads us to Theorem A (projective version).

(3.4) **Definition.** We call X to be *free* if $\tilde{X}$ is free.

Assume that X is free. Let $(d_0, d_1, \dots, d_n)$ be the generalized exponents of $\tilde{X}$, then we can assume that $d_0 = 1$ (due to the existence of the Euler vector field) because $\tilde{X} \neq \emptyset$. The *generalized exponents of* X are defined by $(d_1, \dots, d_n)$.

For any integer i, we have

$$\begin{aligned} & b_i(\boldsymbol{P}^{n+1} \backslash |X|) \\ &= \sum_{j=0}^{i} (-1)^{i-j} b_j(\boldsymbol{C}^{n+2} \backslash |\tilde{X}|) \quad \text{(by (3.2))} \\ &= \sum_{j=0}^{i} (-1)^{i-j} \pi_j(1, d_1, \dots, d_n) \\ &= \pi_i(d_1, \dots, d_n), \end{aligned}$$

where π_j's $(j \leqq i)$ are the elementary symmetric polynomials of degree j (with n or $(n+1)$-variables). This proves Theorem B (projective version).

§ 4. The (Non-Central) Affine Case

Let H_∞ be a hyperplane in $\boldsymbol{P}^{n+1}$, then we can identify $\boldsymbol{C}^{n+1}$ with a Zariski open $\boldsymbol{P}^{n+1} \backslash H_\infty$ of $\boldsymbol{P}^{n+1}$. Let X be a (perhaps non-central) affine n-arrangement. Define a projective n-arrangement

$$X_\infty = X \cup \{H_\infty\}.$$

We can regard $L(X)$ as a subset of $L(X_\infty)$ by a correspondence

$$s \longmapsto \text{the closure of } s \text{ in } \boldsymbol{P}^{n+1}$$

$(s \in L(X))$. Put $L = L(X)$ and $L_\infty = L(X_\infty)$. By $L(i)$ (or $L_\infty(i)$) we denote a set

$$\{t \in L;\ r(t) = i\} \qquad (\text{resp. } \{t \in L_\infty;\ r(t) = i\})$$

for any integer i.

Define

$$M(s) := \{t \in L;\ r(t) = i-1,\ t \prec s\}$$

for any $s \in L_\infty(i) \backslash L$. By μ we denote the Möbius function on L_∞. Then we have

(4.1) **Lemma.** *Let* $i < n+2$, *then*

1) $L(i-1) = \bigcup_{s \in L_\infty(i) \backslash L} M(s)$ *(disjoint)*,

2) $\mu(s) = -\sum_{t \in M(s)} \mu(t)$ *for any* $s \in L_\infty(i) \backslash L$.

Proof. 1): For any $t \in L(i-1)$, we have

$$r(t \cap H_\infty)=i,\ t \in M(t \cap H_\infty), \text{ and } t \cap H_\infty \in L_\infty(i)\backslash L.$$

This implies that

$$L(i-1)=\bigcup_{s \in L_\infty(i)\backslash L} M(s).$$

Next assume that $t \in M(s)$ $(s \in L_\infty(i)\backslash L)$, then $t \cap H_\infty \prec s$ and $r(s)=i = r(t \cap H_\infty)$. Thus $s=t \cap H_\infty$, which implies that

$$M(s) \cap M(s')=\emptyset \quad (s \neq s',\ s, s' \in L_\infty(i)\backslash L).$$

2): We prove by an induction on i. When $i \leqq 0$, $L_\infty(i)\backslash L=\emptyset$. So 2) holds true trivially. Let $s \in L_\infty(i)\backslash L$ and $k<i$. Then we have

$$\sum_{\substack{t \in L_\infty(k)\backslash L \\ t \prec s}} \mu(t) = - \sum_{\substack{t \in L_\infty(k)\backslash L \\ t \prec s}} \sum_{u \in M(t)} \mu(u) \tag{4.2}$$

because of the assumption of our induction. Notice that

$$\{u \in L(k-1);\ u \prec s\} = \bigcup_{\substack{t \in L_\infty(k)\backslash L \\ t \prec s}} M(t) \quad \text{(disjoint)} \tag{4.3}$$

due to 1). Thus we have

$$\sum_{\substack{t \in L_\infty(k)\backslash L \\ t \prec s}} \mu(t) = - \sum_{\substack{u \in L(k-1) \\ u \prec s}} \mu(u)$$

by (4.2) and (4.3). Therefore we obtain

$$\begin{aligned}
\sum_{\substack{t \in L_\infty \backslash L \\ t \prec s \\ t \neq s}} \mu(t) &= \sum_{k=0}^{i-1} \sum_{\substack{t \in L_\infty(k)\backslash L \\ t \prec s}} \mu(t) \\
&= - \sum_{k=0}^{i-1} \sum_{\substack{u \in L(k-1) \\ u \prec s}} \mu(u) \\
&= - \sum_{\substack{u \in L \\ r(u) < i-1 \\ u \prec s}} \mu(u).
\end{aligned}$$

Finally we have

$$\begin{aligned}
\mu(s) &= - \sum_{\substack{t \in L_\infty \\ t \prec s \\ t \neq s}} \mu(t) \\
&= - \sum_{t \in M(s)} \mu(t) - \sum_{\substack{u \in L \\ r(u) < i-1 \\ u \prec s}} \mu(u) - \sum_{\substack{t \in L_\infty \backslash L \\ t \prec s \\ t \neq s}} \mu(t) \\
&= - \sum_{t \in M(s)} \mu(t).
\end{aligned}$$

Q.E.D.

(4.4) Proposition.

$$\sum_{s \in L(i)} \mu(s) - \sum_{s \in L(i-1)} \mu(s) = \sum_{s \in L_\infty(i)} \mu(s)$$

for any integer $i<n+2$.

Proof.

$$\sum_{s\in L_\infty(i)}\mu(s)-\sum_{s\in L(i)}\mu(s)=\sum_{s\in L_\infty(i)\setminus L}\mu(s)$$
$$=-\sum_{s\in L_\infty(i)\setminus L}\sum_{t\in M(s)}\mu(t)\quad \text{(by (4.1), 2))}$$
$$=-\sum_{t\in L(i-1)}\mu(t)\quad \text{(by (4.1), 1))}.$$

Q.E.D.

We shall prove Theorem A as follows:

$$b_i(\boldsymbol{C}^{n+1}\setminus|X|)=b_i(\boldsymbol{P}^{n+1}\setminus|X_\infty|)$$
$$=(-1)^i\sum_{j=0}^{i}\sum_{s\in L_\infty(j)}\mu(s)\quad \text{(by (3.3))}$$
$$=(-1)^i\sum_{s\in L(i)}\mu(s)\quad \text{(by (4.4))}$$

for $i<n+2$. If $i\geqq n+2$, then

$$b_i(\boldsymbol{C}^{n+1}\setminus|X|)=0=(-1)^i\sum_{s\in L(i)}\mu(s)$$

because $L(i)=\emptyset$.

(4.5) **Definition.** An affine n-arrangement X is said to be *free* if X_∞ is a free projective n-arrangement. Let X be free. Then the *generalized exponents of* X are defined to be the generalized exponents of X_∞.

Then this definition is consistent with the definition in Section 2.

Theorem B is immediately derived from Theorem B (projective version) and the very definition (4.5) of the generalized exponents of an affine n-arrangement.

References

[1] Orlik, P. and Solomon, L., Combinatorics and topology of complements of hyperplanes, *Invent. Math.*, **56** (1980), 167–189.

[2] Terao, H., Forms with logarithmic pole and the filtration by the order of the pole, *Proc. Intern. Sympo. on Algebraic Geometry, Kyoto*, 1977, Kinokuniya, Tokyo, 1978, 673–685.

[3] ———, Arrangements of hyperplanes and their freeness I, *J. Fac. Sci. Univ. Tokyo Sect. IA Math.*, **27** (1980), 293–312.

[4] ———, Arrangements of hyperplanes and their freeness II–the Coxeter equality–, *ibid.*, 313–320.

[5] ———, Generalized exponents of a free arrangement of hyperplanes and Shepherd-Todd-Brieskorn formula, *Invent. Math.*, **63** (1981), 159–179.