J. Noncommut. Geom. 7 (2013), 981–1006
DOI 10.4171/JNCG/141

**Journal of Noncommutative Geometry**
© European Mathematical Society

# Homotopical algebra for C*-algebras

Otgonbayar Uuye

**Abstract.** The category of fibrant objects is a convenient framework to do homotopy theory, introduced and developed by Ken Brown. In this paper, we apply it to the category of C*-algebras. In particular, we get a unified treatment of (ordinary) homotopy theory for C*-algebras, KK-theory and E-theory, since all of these can be expressed as the homotopy theory of a category of fibrant objects.

*Mathematics Subject Classification* (2010). 46L85; 55U35.

*Keywords.* Category of fibrant objects, abstract homotopy theory, C*-algebras, KK-theory, K-theory

## 0. Introduction

Basic homotopy theory for C*-algebras can be developed in a way analogous to the homotopy theory for topological spaces, using the Gelfand–Naimark duality between pointed compact Hausdorff spaces and abelian C*-algebras. This is carried out, for example, by Rosenberg in [Ros82] and Schochet in [Sch84]. Thus, for instance, we have a version of the Puppe exact sequence, with essentially the same proof (cf. [Sch84], Proposition 2.6).

There is one big difference: the homotopy theory for C*-algebras does not admit a Quillen model category structure, as first pointed out by Andersen–Grodal (see Appendix). This is unfortunate, since model categories provide a standard and powerful framework to study various aspects of homotopy theories. However, it turns out that not everything is lost: the category of C*-algebras behave as if it was the subcategory of the fibrant objects in a model category, and this is enough for many purposes because many proofs in model category theory start by reducing to the case of (co)fibrant objects.

The notion of a "category of fibrant objects" is abstracted and developed by Brown in [Bro74]. In this paper, we apply Brown's theory to the category of C*-algebras. In Section 1, we review some basic facts about abstract homotopy theory in the setting of category of fibrant objects.

In Section 2, we first apply the abstract theory of Section 1 to the ordinary homotopy theory for C*-algebras (this essentially recovers [Sch84]). We also show that the Meyer–Nest's UCT category (cf. [MN06]), Kasparov's KK-theory (cf. [Kas80], [Kas88]), and Connes–Higson's E-theory (cf. [Hig90], [CH90]) can be described as the homotopy category of a category of fibrant objects. As a corollary, we get a unified treatment of the triangulated structures on these categories.

In addition to ordinary homotopy theory, we also have shape theories for (separable) C*-algebras (cf. [EK86], [Bla85]). In [Dad94], Dadarlat constructed the *strong* shape category and showed that it is equivalent to the asymptotic homotopy category of separable C*-algebras of Connes–Higson (cf. [CH90]).

Unfortunately and unlike the commutative case (cf. [Cat81], [CH81]), we do not (yet) have a category of fibrant objects whose homotopy category describes the strong shape category. However, as we show in Section 2.5, the suspension-stable version considered by Thom (cf. [Tho03]) does arise as the stable homotopy category of a category of fibrant objects. We also show that Thom's connective K-theory category fits well in this framework (cf. loc.cit.).

Needless to say, Brown's theory of category of fibrant objects is not the only way to approach the homotopy theory for C*-algebras. The main "reason" for the failure for the existence of a model structure on the category of C*-algebras is that the category is too small, so an alternative approach would be to enlarge the category of C*-algebras. Joachim–Johnson produced a model category structure for KK-theory by enlarging the category of C*-algebras to a suitable category of topological algebras (cf. [JJ06]). Østvær developed a homotopy theory by enlarging the category of C*-algebras to the category of C*-spaces (cf. [Øst10]). Cuntz described an alternative construction of bivariant $K$-theories in [Cun98].

We also note that Voigt computed the K-theory of free orthogonal quantum groups in [Voi11] using Meyer–Nest's triangulated category approach to the Baum–Connes conjecture (cf. [MN06]). This seems to be the first concrete results in the theory of operator algebras which can be proved only using abstract homotopy-theoretic methods.

Applications of the framework developed in this paper will appear elsewhere.

**Acknowledgments** This research was supported by the Danish National Research Foundation (DNRF) through the Centre for Symmetry and Deformation at the University of Copenhagen. I thank the referee for many useful suggestions and Ilan Barnea for pointing out a mistake in an earlier version.

## 1. Abstract homotopy theory

For the convenience of the reader we recall some basic notions and results from abstract homotopy theory. See [Qui67], [Bro74], [Hel68], [KP97], [GJ99] for details.

**1.1. Categories of fibrant objects.** The following is our main definition.

**Definition 1.1** (Brown [Bro74]). Let $\mathbf{C}$ be category with terminal object $*$ and let $\mathbf{F} \subseteq \mathbf{C}$ and $\mathbf{W} \subseteq \mathbf{C}$ be distinguished subcategories. We say that $\mathbf{C}$ is a *category of fibrant objects* if the following conditions (F0)–(FW2) hold.

(F0) The class $\mathbf{F}$ is closed under composition.

(F1) Isomorphisms of $\mathbf{C}$ are in $\mathbf{F}$.

(F2) The pullback in $\mathbf{C}$ of a morphism in $\mathbf{F}$ exists and is in $\mathbf{F}$.

(F3) For any object $B$ of $\mathbf{C}$, the morphism $B \to *$ is in $\mathbf{F}$.

Morphisms of $\mathbf{F}$ are called *fibrations* and denoted by $\twoheadrightarrow$.

(W1) Isomorphisms of $\mathbf{C}$ are in $\mathbf{W}$.

(W2) If two of $f$, $g$ and $gf$ are in $\mathbf{W}$, then so is the third.

Morphisms of $\mathbf{W}$ are called *weak equivalences* and denoted by $\xrightarrow{\sim}$.

(FW1) The pullback in $\mathbf{C}$ of a morphism in $\mathbf{W} \cap \mathbf{F}$ is in $\mathbf{W} \cap \mathbf{F}$.

Morphisms of $\mathbf{W} \cap \mathbf{F}$ are called *trivial fibrations* and denoted by $\overset{\sim}{\twoheadrightarrow}$.

(FW2) For any object $B$ of $\mathbf{C}$, the diagonal map $B \to B \times B$ admits a factorization

$$B \underset{s}{\overset{\sim}{\twoheadrightarrow}} B^I \underset{d}{\twoheadrightarrow} B \times B,$$

where $s \in \mathbf{W}$ is a weak equivalence, $d = (d_0, d_1) \in \mathbf{F}$ is a fibration.

The object $B^I$ or more precisely the quadruple $(B^I, s, d_0, d_1)$ is called a *path-object* of $B$.

If there is no risk for confusion, we simply say that $\mathbf{C}$ is a category of fibrant objects. If the terminal object is also an *initial* object, we say that $\mathbf{C}$ is a *pointed* category of fibrant objects.

**Remark 1.2.** The condition (F0) is superfluous since $\mathbf{F}$ is assumed to be a subcategory. But it is convenient to have a notation for this property.

The conditions (F1) and (W1) imply that $\mathbf{F}$ and $\mathbf{W}$ contain all objects of $\mathbf{C}$.

The conditions (F2) and (F3) imply that $\mathbf{C}$ is has finite products.

**Remark 1.3.** Dually there is a notion of a *category of cofibrant objects*.

The following is the motivating example.

**Example 1.4.** For any model category $\mathbf{M}$, the full subcategory $\mathbf{M}_f$ consisting of the fibrant objects in $\mathbf{M}$ is naturally a category of fibrant objects, by restricting the weak equivalences and the fibrations to $\mathbf{M}_f$.

In particular, if **Top** denotes the category of compactly generated weakly Hausdorff topological spaces and continuous maps, then

(1) **Top**, homotopy equivalences, Hurewicz fibrations,

(2) **Top**, weak homotopy equivalences, Serre fibrations,

are examples of categories of fibrant objects. In this paper, we only consider the latter one.

A more algebraic example is the following: let $R$ be a ring and let $\mathbf{Ch}(R)$ denote the category of chain complexes of left $R$-modules and chain maps. Then

(3) $\mathbf{Ch}(R)$, quasi-isomorphisms, degreewise epimorphisms,

is a category of fibrant objects. In these three examples, all objects are fibrant, i.e., $\mathbf{M}_f = \mathbf{M}$.

**Definition 1.5.** A functor between categories of fibrant objects is said to be *exact* if it preserves all the relevant structure: it sends the terminal object to the terminal object, fibrations to fibrations, weak equivalences to weak equivalences and pullbacks (of fibrations) to pullbacks.

**Example 1.6.** Let $\mathbf{C}$ be a category of fibrant objects and let $\mathbf{A} \subseteq \mathbf{C}$ be a full *reflective* subcategory, i.e., the inclusion $i : \mathbf{A} \to \mathbf{C}$ is a right-adjoint. Suppose that for any $B \in \mathbf{A}$, a path-object $B^I$ can be chosen to lie in $\mathbf{A}$. Then $\mathbf{A}$ is a category of fibrant objects by restricting weak equivalences and fibrations, since limits in $\mathbf{A}$ can be computed in $\mathbf{C}$; and the inclusion $i : \mathbf{A} \to \mathbf{C}$ is exact.

Occasionally, we find it convenient to isolate the notions of weak equivalences and fibrations.

**Definition 1.7.** Let $\mathbf{C}$ be a category. A *subcategory of weak equivalences* is a subcategory $\mathbf{W} \subseteq \mathbf{C}$ satisfying (W1) and (W2). If $\mathbf{C}$ has a terminal object, a *subcategory of fibrations* is a subcategory $\mathbf{F} \subseteq \mathbf{C}$ satisfying (F0)–(F3).

## 1.2. Fibre and homotopy fibre

**Lemma 1.8** (Factorization Lemma). *Let $f : A \to B$ be a morphism in a category of fibrant objects. Consider the diagram*

$$\begin{tikzcd}
Nf \arrow[r, "p"] \arrow[d, equal] & B \\
Nf \arrow[r, "d_0^*(f)"] \arrow[d, shift left] & B^I \arrow[u, "d_1"'] \arrow[d, shift left, "d_0"] \\
A \arrow[r, "f"] \arrow[u, bend left, "i"] & B, \arrow[u, bend left, "s"]
\end{tikzcd}$$

*where $(B^I, s, d_0, d_1)$ is a path-object for $B$ and $Nf$ is the pullback $A \times_B B^I$ and $p$ is the composition $d_1 \circ d_0^*(f)$ and $i$ is the map determined by the section $s$.*

*Then $p$ is a fibration and $i$ is a right inverse to a trivial fibration* (*in particular, a weak equivalence*) *and $f = p \circ i$.*

*Proof.* See [Bro74], Factorization Lemma. □

**Definition 1.9.** We call $Nf$ a *mapping path-object* of $f$.

**Corollary 1.10.** *Let* **C** *be a category of fibrant objects and let* **D** *be a category with weak equivalences. Let* $F : \mathbf{C} \to \mathbf{D}$ *be a functor that sends trivial fibrations to weak equivalences. Then* $F$ *sends weak equivalences to weak equivalences.*

Now we consider pointed categories.

**Definition 1.11.** Let $p$ be a fibration in a pointed category of fibrant objects. The *fibre* $F$ of $f$ is the pullback

$$\begin{tikzcd} F \arrow[r, "i"] \arrow[d] & E \arrow[d, "p"] \\ * \arrow[r] & B. \end{tikzcd}$$

We express this situation by the diagram

$$F \overset{i}{\rightarrowtail} E \overset{p}{\twoheadrightarrow} B \ .$$

**Definition 1.12.** Let $f : A \to B$ be a morphism in a pointed category of fibrant objects. The *homotopy fibre* $Ff$ of $f$ is the fibre of $Nf \overset{p}{\twoheadrightarrow} B$, where $p$ is as in the Factorization Lemma (Lemma 1.8).

**Lemma 1.13.** *Let* $p$ *be a fibration in a pointed category of fibrant objects with fibre* $F$*. Then the natural map*

$$F \to Fp$$

*is a weak equivalence.*

*Proof.* Apply [Bro74], Lemma 4.3, to

$$\begin{tikzcd} F \arrow[r, tail] \arrow[d] & E \arrow[r, two heads, "p"] \arrow[d, "\wr"] & B \arrow[d, equal] \\ Fp \arrow[r, tail] & Np \arrow[r, two heads] & B. \end{tikzcd}$$

□

## 1.3. Homotopy category

**Notation 1.14.** If **C** is a category, we write $\mathrm{Ob}\,\mathbf{C}$ for the objects of **C** and write $\mathrm{Mor}_{\mathbf{C}}(A, B)$ for the space of morphisms from $A$ to $B$, with $A, B \in \mathbf{C}$.

**Definition 1.15.** The *homotopy category* of a category $\mathbf{C}$ of fibrant objects with weak equivalences $\mathbf{W}$ is the localization

$$\mathbf{Ho}(\mathbf{C}) := \mathbf{C}[\mathbf{W}^{-1}].$$

In other words, there is given a functor $\gamma\colon \mathbf{C} \to \mathbf{Ho}(\mathbf{C})$, called the *localization functor*, with the property that for any functor $k\colon \mathbf{C} \to \mathbf{D}$ such that $k(t)$ is invertible in $\mathbf{D}$ for all $t \in \mathbf{W}$ there exists a unique functor $\mathbf{Ho}(\mathbf{C}) \to \mathbf{D}$ making the diagram

$$\begin{tikzcd} & \mathbf{Ho}(\mathbf{C}) \arrow[dr, dashed] & \\ \mathbf{C} \arrow[ur, "\gamma"] \arrow[rr, "k"] & & \mathbf{D} \end{tikzcd}$$

commutative.

Often we write $[A, B]_{\mathbf{C}}$ for $\mathrm{Mor}_{\mathbf{Ho}(\mathbf{C})}(A, B)$. Note that there is no guarantee that $[A, B]_{\mathbf{C}}$ is a small set (see Corollary 1.19).

**Definition 1.16.** Let $\mathbf{C}$ be a category of fibrant objects. Two morphisms

$$f_0, f_1\colon A \rightrightarrows B$$

are said to be *right-homotopic* if for some path-object $(B^I, s, d_0, d_1)$ of $B$, there is a morphism $h\colon A \to B^I$ such that $f_0 = d_0 h$ and $f_1 = d_1 h$.

The two are said to be *homotopic* if there is a weak equivalence $t\colon A' \to A$ such that $f_0 t, f_1 t\colon A' \rightrightarrows B$ are right-homotopic.

Right-homotopy and homotopy are equivalence relations, and moreover, homotopy is compatible with the composition in $\mathbf{C}$ (cf. [Bro74], Section 2).

**Definition 1.17.** Let $\mathbf{C}$ be a category of fibrant objects. We denote the category of *homotopy classes* in $\mathbf{C}$ by $\pi\mathbf{C}$ and let $\pi\colon \mathbf{C} \to \pi\mathbf{C}$ denote the quotient functor.

The following is the fundamental result of Brown.

**Theorem 1.18** (Brown [Bro74], Theorem 2.1). *Let $\mathbf{C}$ be a category of fibrant objects. Then $\pi\mathbf{W} \subseteq \pi\mathbf{C}$ admits a* calculus of right fractions.

*It follows that, for $A, B \in \mathbf{C}$,*

$$[A, B]_{\mathbf{C}} \cong \operatorname*{colim}_{A' \xrightarrow{\sim} A} \mathrm{Mor}_{\pi\mathbf{C}}(A', B)$$

*and hence if $\gamma\colon \mathbf{C} \to \mathbf{Ho}(\mathbf{C})$ is the localization functor, then*

(1) *any morphism in $[A, B]_{\mathbf{C}}$ can be written as a right-fraction*

$$A \xleftarrow{\gamma(t)^{-1}} A' \xrightarrow{\gamma(f)} B$$

*where $t \in \mathbf{W}$ is a weak equivalence, and*

(2) *if* $f_0$, $f_1$ *are morphisms in* $\mathrm{Mor}_{\mathbf{C}}(A, B)$, *then* $\gamma(f_0) = \gamma(f_1)$ *if and only if* $f_0$ *and* $f_1$ *are homotopic, i.e.,* $\pi(f_0) = \pi(f_1)$.

**Corollary 1.19.** *Let* $\mathbf{C}$ *be a category of fibrant objects and let* $A$ *be an object in* $\mathbf{C}$. *Suppose that the category* $\mathbf{W}_A$ *of weak equivalences over* $A$ *is "coinitially small" i.e there exists a* set $S_A$ *of objects in* $\mathbf{C}$ *such that for any* $A' \xrightarrow{\sim} A$, *there is a* $A'' \xrightarrow{\sim} A'$ *such that* $A'' \in S_A$, *then* $[A, B]_{\mathbf{C}}$ *is a small set for every* $B \in \mathbf{C}$. □

*Proof.* See [GZ67], Proposition 2.4. □

Now we consider pointed categories.

**Definition 1.20.** Let $B$ be an object of a pointed category of fibrant objects. A *loop object* of $B$ is the fibre $\Omega B$ of $(d_0, d_1)\colon B^I \to B \times B$, where $(B^I, s, d_0, d_1)$ is a path-object of $B$.

**Lemma 1.21.** *Let* $\mathbf{C}$ *be a pointed category of fibrant objects. Then* $\Omega$ *defines a functor*

$$\Omega\colon \mathbf{Ho}(\mathbf{C}) \to \mathbf{Ho}(\mathbf{C}),$$

*called the* loop object functor.

(1) *For any* $B \in \mathbf{C}$, *the object* $\Omega B$ *is naturally a group object in* $\mathbf{Ho}(\mathbf{C})$ *and* $\Omega^2 B$ *is naturally an abelian group object in* $\mathbf{Ho}(\mathbf{C})$.

(2) *For any fibration* $p\colon E \twoheadrightarrow B$ *with fibre* $F$, *there is a natural right-action* $F \times \Omega B \to F$ *in* $\mathbf{Ho}(\mathbf{C})$. *In particular, we have a natural map* $\Omega B \to F$ *in* $\mathbf{Ho}(\mathbf{C})$.

*Proof.* See [Bro74], Section 4. □

**Theorem 1.22.** *Let* $\mathbf{C}$ *be a pointed category of fibrant objects and let* $p\colon E \twoheadrightarrow B$ *be a fibration with fibre* $F$. *Then for any* $D \in \mathbf{C}$, *there is an exact sequence*

$$\begin{aligned} \cdots \to [D, \Omega^2 B]_{\mathbf{C}} \to [D, \Omega F]_{\mathbf{C}} \to [D, \Omega E]_{\mathbf{C}} \\ \to [D, \Omega B]_{\mathbf{C}} \to [D, F]_{\mathbf{C}} \to [D, E]_{\mathbf{C}} \to [D, B]_{\mathbf{C}}. \end{aligned}$$

*Proof.* See [Bro74], Section 4, and [Qui67], Section I.3. □

Note that while $\mathbf{Ho}(\mathbf{C})$ depends only on the weak equivalences, the loop object functor $\Omega$ depends also on the fibrations.

**Definition 1.23.** Let $\mathbf{C}$ be a pointed category of fibrant objects. We define the *stable homotopy category* of $\mathbf{C}$ as the category

$$\mathbf{SHo}(\mathbf{C}) := \mathbf{Ho}(\mathbf{C})[\Omega^{-1}],$$

obtained from $\mathbf{Ho}(\mathbf{C})$ by inverting the endofunctor $\Omega$.

Objects of $\mathbf{SHo}(\mathbf{C})$ are $(A, n)$ with $A \in \mathbf{Ho}(\mathbf{C})$ and $n \in \mathbb{Z}$ and the morphisms are given by

$$\mathrm{Mor}_{\mathbf{SHo}(\mathbf{C})}((A,n),(B,m)) := \operatorname*{colim}_{k\to\infty}[\Omega^{n+k}A, \Omega^{m+k}B]_{\mathbf{C}}.$$

For $n \in \mathbb{Z}$, we have natural functors, also denoted by $\Omega^n$,

$$\Omega^n \colon \mathbf{Ho}(\mathbf{C}) \to \mathbf{SHo}(\mathbf{C}), \quad A \mapsto (A, n),$$

which send morphisms in $\mathrm{Mor}_{\mathbf{Ho}(\mathbf{C})}(A, B)$ to the corresponding element in $\mathrm{Mor}_{\mathbf{SHo}(\mathbf{C})}((A,n),(B,n))$.

**Theorem 1.24.** *Let* $\mathbf{C}$ *be a pointed category of fibrant objects. Then the stable homotopy category* $\mathbf{SHo}(\mathbf{C})$ *is a* triangulated category *with the* shift

$$\Sigma = \Omega^{-1} \colon \mathbf{SHo}(\mathbf{C}) \to \mathbf{SHo}(\mathbf{C})$$

*given by* $(A,n) \mapsto (A, n-1)$ *and the* distinguished triangles *given by triangles isomorphic to triangles of the form*

$$(\Omega B, n) \to (F, n) \to (E, n) \to (B, n),$$

*where* $n \in \mathbb{Z}$ *and* $E \to B$ *is a fibration,* $F \to E$ *is the fibre inclusion and* $\Omega B \to F$ *is the morphism obtained from Lemma* 1.21.

*Proof.* See [Hel68] or [Hov99], [May01]. □

**Remark 1.25.** We note that for any $f \in [A, B]_{\mathbf{C}}$ and $n \in \mathbb{Z}$, we have a natural distinguished triangle

$$(\Omega B, n) \to (Ff, n) \to (A, n) \xrightarrow{\Omega^n f} (B, n).$$

**Definition 1.26.** We say that a pointed category of fibrant objects $\mathbf{C}$ is *stable* if the loop functor $\Omega \colon \mathbf{Ho}(\mathbf{C}) \to \mathbf{Ho}(\mathbf{C})$ is invertible.

**Remark 1.27.** If $\mathbf{C}$ is a *stable* pointed category of fibrant objects, then

$$\Omega^0 \colon \mathbf{Ho}(\mathbf{C}) \to \mathbf{SHo}(\mathbf{C})$$

is an equivalence of categories. In particular, $\mathbf{Ho}(\mathbf{C})$ is naturally a triangulated category with shift $\Sigma = \Omega^{-1} \colon \mathbf{Ho}(\mathbf{C}) \to \mathbf{Ho}(\mathbf{C})$.

**Example 1.28.** Let $\mathbf{M}$ be a pointed Quillen model category and let $\mathbf{M}_f$ be the full subcategory of fibrant objects in $\mathbf{M}$, considered a category of fibrant objects as in Example 1.4. Then the inclusion $\mathbf{M}_f \to \mathbf{M}$ induces an equivalence $\mathbf{Ho}(\mathbf{M}_f) \cong \mathbf{Ho}(\mathbf{M})$, with compatible loop objects and fibration sequences. Compare [Bro74] and [Qui67].

### 1.4. Homology theories and localizations

**Definition 1.29.** A *homology theory* on a pointed category of fibrant objects $\mathbf{C}$ is a homology theory on $\mathbf{SHo}(\mathbf{C})$, i.e., an exact functor $\mathcal{H}\colon \mathbf{SHo}(\mathbf{C}) \to \mathbf{Ab}$.

**Definition 1.30.** Let $\mathbf{C}$ be a pointed category of fibrant objects and let $\mathcal{H}$ be a homology theory on $\mathbf{C}$.

A morphism $t\colon A \to B$ is said to be an *$\mathcal{H}$-equivalence* if the induced maps

$$(\Omega^n t)_*\colon \mathcal{H}(A,n) \to \mathcal{H}(B,n)$$

are isomorphisms for all $n \in \mathbb{Z}$.

An object $F \in \mathbf{C}$ is said to be *$\mathcal{H}$-acyclic* if $\mathcal{H}(F,n) = 0$ for all $n \in \mathbb{Z}$.

Note that since homology theories are homotopy invariant by definition, weak equivalences in $\mathbf{C}$ are $\mathcal{H}$-equivalences.

**Lemma 1.31.** *Let $\mathbf{C}$ be a pointed category of fibrant objects and let $\mathcal{H}$ be a homology theory on $\mathbf{C}$. Then a morphism $t$ in $\mathbf{C}$ is an $\mathcal{H}$-equivalence if and only if its homotopy fibre $Ft$ is $\mathcal{H}$-acyclic.*

*Proof.* Clear from the long-exact sequence associated to the distinguished triangle of Remark 1.25. □

**Corollary 1.32.** *Let $\mathbf{C}$ be a pointed category of fibrant objects and let $\mathcal{H}$ be a homology theory on $\mathbf{C}$. Then a fibration $p \in \mathbf{C}$ with fibre $F$ is an $\mathcal{H}$-equivalence if and only if $F$ is $\mathcal{H}$-acyclic.*

*Proof.* By Lemma 1.13, the natural map $F \to Fp$ is a weak equivalence, hence an $\mathcal{H}$-equivalence. The proof is complete by Lemma 1.31. □

**Theorem 1.33.** *Let $\mathbf{C}$ be a pointed category of fibrant objects and let $\mathcal{H}$ be a homology theory on $\mathbf{C}$. Then $\mathcal{H}$-equivalences and fibrations define a pointed category of fibrant objects on $\mathbf{C}$, denoted by $R_{\mathcal{H}}\mathbf{C}$, with the same path and loop objects as in $\mathbf{C}$.*

*Proof.* It is clear that $\mathcal{H}$-equivalences form a subcategory of weak equivalences. Hence we need to show the compatibility conditions (FW1) and (FW2) are satisfied.

(FW1) Let $p\colon E \twoheadrightarrow B$ be a fibration which is also an $\mathcal{H}$-equivalence. We need to show that for any $f\colon A \to B$, the pullback $f^*(p)$ is again an $\mathcal{H}$-equivalence. But this is immediate from Corollary 1.32 applied to the diagram:

$$\begin{tikzcd} F \arrow[r, tail] \arrow[d, equal] & E \times_B A \arrow[r, two heads, "f^*(p)"] \arrow[d] & A \arrow[d, "f"] \\ F \arrow[r, tail] & E \arrow[r, two heads, "p"] & B, \end{tikzcd} \tag{1.1}$$

where $F$ is the fibre of $p$.

(FW2) Since weak equivalences are $\mathcal{H}$-equivalences, path-objects in $\mathbf{C}$ also give path-objects in the new category of fibrant objects $R_{\mathcal{H}}\mathbf{C}$. $\square$

**Definition 1.34.** Let $\mathbf{C}$ be a pointed category of fibrant objects and let $S \subseteq \mathbf{C}$ be a class of morphisms. We say that a morphism $t \in \mathrm{Mor}_{\mathbf{C}}(A, B)$ is a $S^{-1}$*-weak equivalence* if for any homology theory $\mathcal{H} : \mathbf{SHo}(\mathbf{C}) \to \mathbf{Ab}$ such that every $s \in S$ is an $\mathcal{H}$-equivalence, $t$ is an $\mathcal{H}$-equivalence.

**Theorem 1.35.** *Let* $\mathbf{C}$ *be a pointed category of fibrant objects and let* $S \subseteq \mathbf{C}$ *be a class of morphisms. Then* $S^{-1}$*-weak equivalences and fibrations define a pointed category of fibrant objects, denoted by* $\mathbf{R}_S\mathbf{C}$*. The stable homotopy category* $\mathbf{SHo}(\mathbf{R}_S\mathbf{C})$ *is naturally equivalent to the Verdier localization* $\mathbf{SHo}(\mathbf{C})[(\Omega^0 S)^{-1}]$ *as a triangulated category.*

*Proof.* Considering all homology theories $\mathcal{H} : \mathbf{SHo}(\mathbf{C}) \to \mathbf{Ab}$ in which every $t \in S$ is an $\mathcal{H}$-equivalence in Theorem 1.33, we see that $\mathbf{R}_S\mathbf{C}$ is indeed a category of fibrant objects.

Now consider the natural triangulated functor $Q : \mathbf{SHo}(\mathbf{C}) \to \mathbf{SHo}(\mathbf{R}_S\mathbf{C})$ induced by $\mathbf{C} \to \mathbf{R}_S\mathbf{C}$. Since any $s \in S$ is a $S^{-1}$-weak equivalence, we see that $Q(\Omega^0 s)$ is invertible in $\mathbf{SHo}(\mathbf{R}_S\mathbf{C})$.

We show that $Q$ is the universal triangulated functor that invert $\Omega^0 S \subseteq \mathbf{SHo}(\mathbf{C})$. Indeed, let $R : \mathbf{SHo}(\mathbf{C}) \to (\mathbf{P}, \Omega^{-1})$ be a triangulated functor such that morphisms in $R(\Omega^0 S) \subseteq \mathrm{Mor}_{\mathbf{P}}(R(A, 0), R(B, 0))$ are all invertible.

Let $t \in \mathrm{Mor}_{\mathbf{C}}(A, B)$ be a $S^{-1}$-weak equivalence. Then for any $D \in \mathbf{P}$,

$$\mathcal{H} : \mathbf{SHo}(\mathbf{C}) \to \mathbf{Ab}, \quad (A, n) \mapsto \mathrm{Mor}_{\mathbf{P}}(D, R(A, n)),$$

is a homology theory by [Tho03], Theorem 2.3.8, and every $s \in S$ is an $\mathcal{H}$-equivalence, hence we see that $t$ too is an $\mathcal{H}$-equivalence. By Yoneda's Lemma, $R(\Omega^0 t)$ is invertible in $\mathbf{P}$. Thus $R$ induces a functor $R_* : \mathbf{Ho}(\mathbf{R}_S\mathbf{C}) \to \mathbf{P}$ which is easily seen to intertwine the $\Omega$'s, hence induces a functor $\widehat{R} : \mathbf{SHo}(\mathbf{R}_S\mathbf{C}) \to \mathbf{P}$. Since $R$ is a triangulated homology theory, $\widehat{R}$ is a triangulated functor and $R = \widehat{R} \circ Q$. The uniqueness of $\widehat{R}$ is clear. $\square$

In other words, $\mathbf{SHo}(\mathbf{R}_S\mathbf{C})$ is the universal triangulated homology theory for which all morphisms of $S$ are equivalences (cf. [Tho03], Definition 2.3.3).

## 2. Applications to the category of C*-algebras

Let $\mathbf{C}^*$ denote the category of *$C^*$-algebras* and *$*$-homomorphisms*. It is complete and cocomplete and pointed – the zero object is the zero algebra 0 – symmetric monoidal category with respect to the *maximal* tensor product, which we denote by $\otimes$ (instead of the more standard notation $\otimes_{\max}$, since we will not consider any other tensor product). We refer to [Mey08] for the details.

The category $\mathbf{C}^*$ is naturally enriched over **Top**, the Cartesian closed category of compactly generated weakly Hausdorff topological spaces. Indeed, since C*-algebras are normed, they are compactly generated and weakly Hausdorff as spaces, hence there is a forgetful functor $\mathbf{C}^* \to \mathbf{Top}$. For C*-algebras $A$ and $B$, we give $\mathrm{Mor}_{\mathbf{C}^*}(A, B)$ the subspace topology from $\mathrm{Mor}_{\mathbf{Top}}(A, B)$ via the forgetful functor. It is easy to see that $\mathrm{Mor}_{\mathbf{C}^*}(A, B)$ is a closed subspace of $\mathrm{Mor}_{\mathbf{Top}}(A, B)$, hence itself a compactly generated weakly Hausdorff space.

Let $\mathbf{A}^* \subset \mathbf{C}^*$ denote the full subcategory of *abelian* C*-algebras. By the Gelfand–Naimark duality, $\mathbf{A}^*$ is equivalent to the opposite category of the category $\mathbf{CH}_*$ of pointed, compact Hausdorff topological spaces and pointed continuous maps. If $X$ is a compact Hausdorff space, we write $C(X)$ for the (unital) C*-algebra of continuous functions on $X$. If in addition $X$ has a base point, we write $C_0(X)$ for the C*-algebra of continuous functions on $X$ vanishing at the base point.

**Remark 2.1.** The category $\mathbf{C}^*$ of C*-algebras is also enriched over the category of Hausdorff spaces, using the compact-open topology on morphism spaces. However, in order to facilitate the connection to algebraic topology, we use the compactly generated compact-open topology. Note that if $A$ is separable, then the compact-open topology on $\mathrm{Mor}_{\mathbf{C}^*}(A, B)$ is metrizable, hence compactly generated.

**Lemma 2.2.** *Let $B$ be a C*-algebra and let $X$ be a compact Hausdorff space. Then the set of maps* $\mathrm{Mor}_{\mathbf{Top}}(X, B)$ *is naturally a C*-algebra isomorphic to* $C(X) \otimes B$.

*Proof.* By [Str], Proposition 2.13, the topology on $\mathrm{Mor}_{\mathbf{Top}}(X, B)$ coincides with the topology given by the norm $\|f\| := \sup_{x\in X} \|f(x)\|_B$. The rest is standard (cf. [WO93], Corollary T.6.17). □

**Notation 2.3.** Let $B$ be a C*-algebra and let $X$ be a compact Hausdorff space. We write $B^X$ for the C*-algebra $\mathrm{Mor}_{\mathbf{Top}}(X, B) \cong C(X) \otimes B$. For $x \in X$, the *evaluation* map $f \mapsto f(x)$ is denoted by $\mathrm{ev}_x\colon B^X \to B$.

The following is the main property of the enrichment that we use. See also [JJ06], Proposition 3.4, and [Mey08], Proposition 24.

**Lemma 2.4.** *Let $A$ and $B$ be C*-algebras and let $X$ be a compact Hausdorff space. Then there is an identification*

$$\mathrm{Mor}_{\mathbf{Top}}(X, \mathrm{Mor}_{\mathbf{C}^*}(A, B)) \cong \mathrm{Mor}_{\mathbf{C}^*}(A, B^X) \tag{2.1}$$

*natural in $A$, $B$ and $X$.*

*Proof.* Since $A$ and $B$ are compactly generated weakly Hausdorff spaces, we have a natural identification

$$\mathrm{Mor}_{\mathbf{Top}}(X, \mathrm{Mor}_{\mathbf{Top}}(A, B)) \cong \mathrm{Mor}_{\mathbf{Top}}(A, \mathrm{Mor}_{\mathbf{Top}}(X, B))$$

by [Str], Proposition 2.12. Hence by Lemma 2.2,

$$\mathrm{Mor}_{\mathbf{Top}}(X, \mathrm{Mor}_{\mathbf{Top}}(A, B)) \cong \mathrm{Mor}_{\mathbf{Top}}(A, B^X).$$

Now it is easy to check that this restricts to the identification in (2.1). □

Often we will make this identification implicitly.

**Remark 2.5.** Lemma 2.2 and Lemma 2.4 have pointed analogues.

Let $\mathbf{Top}_*$ denote the category of pointed spaces and pointed maps. Since C*-algebras have a natural base point 0 and $*$-homomorphisms are pointed maps, there is in fact a forgetful functor $\mathbf{C}^* \to \mathbf{Top}_*$ and $\mathbf{C}^*$ is enriched over $\mathbf{Top}_*$.

Let $A$ and $B$ are C*-algebras and let $X$ be a pointed compact Hausdorff space. Let $B^X$ denote the C*-algebra $C_0(X) \otimes B \cong \mathrm{Mor}_{\mathbf{Top}_*}(X, B)$. Then it follows from Lemma 2.4 that there is a natural identification

$$\mathrm{Mor}_{\mathbf{Top}_*}(X, \mathrm{Mor}_{\mathbf{C}^*}(A, B)) \cong \mathrm{Mor}_{\mathbf{C}^*}(A, B^X).$$

**Corollary 2.6.** *For any $D \in \mathbf{C}^*$, the functor $\mathrm{Mor}_{\mathbf{C}^*}(D, -)\colon \mathbf{C}^* \to \mathbf{Top}$ preserves pullbacks.*

*Proof.* Let $D$ be fixed and let $F := \mathrm{Mor}_{\mathbf{C}^*}(D, -)$.

Consider a pullback diagram

$$\begin{tikzcd} A \times_B E \arrow[r] \arrow[d] & E \arrow[d] \\ A \arrow[r] & B \end{tikzcd}$$

in $\mathbf{C}^*$. We need to prove that the natural map

$$\Phi\colon F(A \times_B E) \to F(A) \times_{F(B)} F(E)$$

is a homeomorphism. It is clear that $\Phi$ is a continuous bijection. Hence it suffices to prove that for any $X$ compact Hausdorff, a map $X \to F(A \times_B E)$ is continuous if the compositions $X \to F(A)$ and $X \to F(E)$ are continuous. However, this follows from Lemma 2.4 and its proof. □

## 2.1. Ordinary homotopy theory

**Notation 2.7.** We denote the interval $[0,1] := \{x \in \mathbf{R} \mid 0 \leq x \leq 1\}$ by $I$.

**Definition 2.8.** Let $A$ and $B$ be C*-algebras. Two $*$-homomorphisms $f_0, f_1\colon A \to B$ are said to be *homotopic* if there exists a $*$-homomorphism $F\colon A \to B^I$, called a homotopy, such that $f_0 = \mathrm{ev}_0 \circ F$ and $f_1 = \mathrm{ev}_1 \circ F$, where $\mathrm{ev}_t\colon B^I \to B$ is the evaluation map at $t \in [0,1]$. We denote the set of homotopy classes of $*$-homomorphisms $A \to B$ by

$$[A, B] := \{\text{homotopy classes of maps } A \to B\}.$$

**Remark 2.9.** By Lemma 2.4, two $*$-homomorphisms $f_0, f_1\colon A \to B$ are homotopic if and only if $\pi_0(f_0) = \pi_0(f_1)$ in $\pi_0(\mathrm{Mor}_{\mathbf{C}^*}(A,B))$, where $\pi_0$ is the path-connected components functor.

The (ordinary) *homotopy category* of C*-algebras is the category of C*-algebras and homotopy classes of $*$-homomorphisms. In view of Remark 2.9, we denote this category $\pi_0\mathbf{C}^*$:

$$\mathrm{Mor}_{\pi_0\mathbf{C}^*}(A,B) := \pi_0 \,\mathrm{Mor}_{\mathbf{C}^*}(A,B) = [A,B].$$

We have a natural functor $\pi_0\colon \mathbf{C}^* \to \pi_0\mathbf{C}^*$.

We now give $\mathbf{C}^*$ the structure of a category of fibrant objects, whose homotopy category is $\pi_0\mathbf{C}^*$, following [Sch84]. We consider **Top** as a category of fibrant objects using weak homotopy equivalences and Serre fibrations (see Example 1.4) and we "pullback" this structure to $\mathbf{C}^*$ using Corollary 2.6.

**Definition 2.10.** A $*$-homomorphism $t \in \mathbf{C}^*$ is called a *homotopy equivalence* if $\pi_0(t)$ is invertible in $\pi_0\mathbf{C}^*$.

**Lemma 2.11.** *Let $F \in \mathbf{C}^*$. If $f_0, f_1 \in \mathrm{Mor}_{\mathbf{C}^*}(A,B)$ are homotopic, then the maps $f_0 \otimes \mathrm{id}_F,\ f_1 \otimes \mathrm{id}_F \in \mathrm{Mor}_{\mathbf{C}^*}(A \otimes F, B \otimes F)$ are homotopic. In particular, the functor $A \mapsto A \otimes F$ preserves homotopy equivalences.*

*Proof.* Clear. □

**Lemma 2.12.** *If $f_0, f_1 \in \mathrm{Mor}_{\mathbf{C}^*}(A,B)$ are homotopic then for any $D \in \mathbf{C}^*$, the induced maps $(f_0)_*, (f_1)_*\colon \mathrm{Mor}_{\mathbf{C}^*}(D,A) \to \mathrm{Mor}_{\mathbf{C}^*}(D,B)$ are homotopic in* **Top**.

*Proof.* This follows from Lemma 2.4. □

**Proposition 2.13.** *Let $t \in \mathrm{Mor}_{\mathbf{C}^*}(A,B)$. Then $t$ is a homotopy equivalence if and only if the induced map*

$$t_*\colon \mathrm{Mor}_{\mathbf{C}^*}(D,A) \to \mathrm{Mor}_{\mathbf{C}^*}(D,B)$$

*is a weak homotopy equivalence in* **Top** *for all $D \in \mathbf{C}^*$.*

*Proof.* If $t \in \mathrm{Mor}_{\mathbf{C}^*}(A, B)$ is a homotopy equivalence, then for any $D \in \mathbf{C}^*$, the induced map $t_*\colon \mathrm{Mor}_{\mathbf{C}^*}(D, A) \to \mathrm{Mor}_{\mathbf{C}^*}(D, B)$ is a homotopy equivalence by Lemma 2.12, hence a weak homotopy equivalence. Conversely, suppose that the induced map $t_*\colon \mathrm{Mor}_{\mathbf{C}^*}(D, A) \to \mathrm{Mor}_{\mathbf{C}^*}(D, B)$ is a weak homotopy equivalence in **Top** for all $D \in \mathbf{C}^*$. Then in particular, $\pi_0(t)_* = \pi_0(t_*)\colon \pi_0 \mathrm{Mor}_{\mathbf{C}^*}(D, A) \to \pi_0 \mathrm{Mor}_{\mathbf{C}^*}(D, B)$ is a bijection for all $D \in \mathbf{C}^*$. By Yoneda's Lemma, $\pi_0(t)$ is invertible. □

**Definition 2.14.** A $*$-homomorphism $p\colon E \to B$ is called a *Schochet fibration* if the induced map

$$p_*\colon \mathrm{Mor}_{\mathbf{C}^*}(D, E) \to \mathrm{Mor}_{\mathbf{C}^*}(D, B)$$

has the path lifting property (i.e., the right lifting property with respect to the inclusion $\{0\} \hookrightarrow [0, 1]$) in **Top** for all $D \in \mathbf{C}^*$.

**Definition 2.15.** Let $f\colon A \to B$ be a $*$-homomorphism. Let $Nf$ denote the pullback

$$\begin{tikzcd} NF \arrow[r] \arrow[d] & B^I \arrow[d, "\mathrm{ev}_0"] \\ A \arrow[r, "f"] & B. \end{tikzcd}$$

**Lemma 2.16.** *A $*$-homomorphism $p\colon E \to B$ is a Schochet fibration if and only if the natural map $E^I \to Np$ splits.*

*Proof.* See [Sch84], Proposition 1.10. □

**Lemma 2.17.** *For any $F \in \mathbf{C}^*$, the functor $A \mapsto A \otimes F$ preserves pullbacks and Schochet fibrations.*

*Proof.* The functor $A \mapsto A \otimes F$ preserves pullbacks by [Ped99], Remark 3.10, and it preserves Schochet fibrations by Lemma 2.16. See [Sch84], Proposition 1.11. □

**Proposition 2.18.** *Let $p \in \mathrm{Mor}_{\mathbf{C}^*}(E, B)$. Then $p$ is a Schochet fibration if and only if the induced map*

$$p_*\colon \mathrm{Mor}_{\mathbf{C}^*}(D, E) \to \mathrm{Mor}_{\mathbf{C}^*}(D, B)$$

*is a Serre fibration (i.e., has the right lifting property with respect to the natural inclusion $\{0\} \times I^n \hookrightarrow [0, 1] \times I^n$ for all $n \geq 0$) in* **Top** *for all $D \in \mathbf{C}^*$.*

*Proof.* Clearly, Serre fibrations have the path lifting property. Hence it is enough to show that if $p$ is a Schochet fibration then $p_*$ is a Serre fibration. For any compact Hausdorff space $X$, by Lemma 2.4, the map $p_*\colon \mathrm{Mor}_{\mathbf{C}^*}(D, E) \to \mathrm{Mor}_{\mathbf{C}^*}(D, B)$ has the right lifting property with respect to $\{0\} \times X \hookrightarrow [0, 1] \times X$ if and only if the map $(\mathrm{id}_{C(X)} \otimes p)_*\colon \mathrm{Mor}_{\mathbf{C}^*}(D, C(X) \otimes E) \to \mathrm{Mor}_{\mathbf{C}^*}(D, C(X) \otimes B)$ has the path lifting property. Now the proof is complete by Lemma 2.17. □

The following theorem is contained in [Sch84].

**Theorem 2.19.** *The category of $C^*$-algebras $\mathbf{C}^*$ is a pointed category of fibrant objects with weak equivalences the homotopy equivalences and fibrations the Schochet fibrations, whose homotopy category is the ordinary homotopy category, i.e.,* $\mathbf{Ho}(\mathbf{C}^*) = \pi_0\mathbf{C}^*$.

*Proof.* Properties (F0), (F1), (F3) follow from Proposition 2.18. Properties (W1) and (W2) are clear (or use Proposition 2.13). For properties (F2) and (FW1), use Corollary 2.6 in addition.

For (FW2): Let $[a,b]$ be a compact interval, $a < b$, let

$$B^{[a,b]} := \mathrm{Mor}_{\mathbf{Top}}([a,b], B) \cong C[a,b] \otimes B,$$

and let $\mathrm{ev}_t \colon B^{[a,b]} \to B$ denote the evaluation at $t \in [a,b]$ (cf. Notation 2.3). Then the map $(\mathrm{ev}_a, \mathrm{ev}_b) \colon B^{[a,b]} \to B \times B$ is a Schochet fibration, since the rectangle $[0,1] \times [a,b]$ retracts to the union of its three sides $\sqsubset$. The constant-path map $s \colon B \to B^{[a,b]}$ is a homotopy equivalence with homotopy inverse $\mathrm{ev}_a$. Thus $(B^{[a,b]}, s, \mathrm{ev}_a, \mathrm{ev}_b)$ is a path-object for $B$. For fixed $a$ and $b$, this is functorial.

It follows from the construction of the path-object in $\mathbf{C}^*$ that two $*$-homomorphisms are right-homotopic if and only if they are homotopic in the sense of Definition 2.8 and this happens if and only if they are homotopic in the sense of Definition 1.16. Hence

$$\mathbf{Ho}(\mathbf{C}^*) = \pi\mathbf{C}^* = \pi_0\mathbf{C}^*. \qquad \square$$

Note that $\mathbf{C}^*$ has a functorial path-object, given by $C[0,1] \otimes B = B^I$, hence also a functorial loop object $\Omega B := C_0(0,1) \otimes B$.

**Remark 2.20.** Schochet called these maps *cofibrations* in [Sch84], because, under the Gelfand–Naimark duality, the condition in Definition 2.14 for a $*$-homomorphism of abelian algebras corresponds to the homotopy extension property for the corresponding map of (pointed compact Hausdorff) spaces.

In a similar way, it is customary that $\mathrm{Mor}_{\mathbf{Top}_*}(S^1, B) \cong C_0(S^1) \otimes B \cong C_0(0,1) \otimes B$ is called the *suspension* of $B$, since $C_0(S^1) \otimes C_0(X) \cong C_0(S^1 \wedge X)$ for $B = C_0(X)$, where $X$ is a pointed compact Hausdorff space. See also Remark A.3.

However, for the sake of consistency, in this paper we will keep our notations and terminologies compatible with that of Section 1.

The stable homotopy category $\mathbf{SHo}(\mathbf{C}^*)$ is the *suspension-stable homotopy category of $C^*$-algebras* studied by Rosenberg [Ros82] and Schochet [Sch84].

**Remark 2.21.** Let $\mathbf{sC}^*$ denote the category of *separable* C*-algebras. Then considering only $D$ separable in Definitions 2.10 and 2.14, we get a structure of a category of fibrant objects on $\mathbf{sC}^*$.

**Lemma 2.22.** *All Schochet fibrations are surjective.*

*Proof.* Let $p\colon E \twoheadrightarrow B$ be a Schochet fibration. Consider the universal algebra generated by a positive contraction:

$$C := \mathbf{C}^*(x \mid 0 \le x \le 1) = C_0(0,1].$$

Then for any $b \in B$, $0 \le b \le 1$, there is a path

$$[0,1] \ni r \mapsto (x \mapsto rb) \in \mathrm{Mor}_{\mathbf{C}^*}(C, B),$$

which lifts to $0 \in \mathrm{Mor}_{\mathbf{C}^*}(C, E)$ at $r = 0$. Lifting the path to $\mathrm{Mor}_{\mathbf{C}^*}(C, E)$, we get $e \in E$, $0 \le e \le 1$, such that $p(e) = b$. It follows that $p$ is surjective. □

**Remark 2.23.** The following are well known and/or easy to see.

(1) The localization $\mathbf{C}^* \to \mathbf{Ho}(\mathbf{C}^*)$ preserves arbitrary coproducts and arbitrary products:

$$[\textstyle\coprod_{i\in\Lambda} A_i, B]_{\mathbf{C}^*} \cong \prod_{i\in\Lambda}[A_i, B]_{\mathbf{C}^*}, \quad [A, \prod_{i\in\Lambda} B_i]_{\mathbf{C}^*} \cong \prod_{i\in\Lambda}[A, B_i]_{\mathbf{C}^*}.$$

(2) The loop functor $\Omega\colon \mathbf{Ho}(\mathbf{C}^*) \to \mathbf{Ho}(\mathbf{C}^*)$ preserves finite products,

$$\Omega(B_1 \times B_2) \cong \Omega B_1 \times \Omega B_2,$$

but not finite coproducts (for example, the natural map $\Omega\mathbb{C} \coprod \Omega\mathbb{C} \to \Omega(\mathbb{C} \coprod \mathbb{C})$ is not a homotopy equivalence).

(3) The loop functor $\Omega\colon \mathbf{Ho}(\mathbf{C}^*) \to \mathbf{Ho}(\mathbf{C}^*)$ does *not* preserve infinite products, and in particular does not admit a left-adjoint; see Appendix.

(4) The "stable homotopy functor" $\Omega^0\colon \mathbf{Ho}(\mathbf{C}^*) \to \mathbf{SHo}(\mathbf{C})^*$ preserves finite products but not finite coproducts.

**2.2. C*-stable homotopy theory.** Let $\mathcal{K}$ denote the C*-algebra of compact operators on a separable infinite-dimensional Hilbert space.

**Proposition 2.24.** *Defining the weak equivalences to be*

$$\{t \in \mathbf{C}^* \mid t \otimes \mathrm{id}_{\mathcal{K}} \textit{ is a homotopy equivalence}\}$$

*and the fibrations to be*

$$\{p \in \mathbf{C}^* \mid p \otimes \mathrm{id}_{\mathcal{K}} \textit{ is a Schochet fibration}\}$$

*defines a category of fibrant objects on* $\mathbf{C}^*$*, denoted by* $\mathbf{M}$.

*Proof.* This is clear since $- \otimes \mathrm{id}_{\mathcal{K}}$ preserves pullbacks. □

Let $e_{11}\colon \mathbb{C} \to \mathcal{K}$ denote a rank one projection. Then for any $B \in \mathbf{M}$, the morphism $\mathrm{id}_B \otimes e_{11}$ is a weak equivalence in $\mathbf{M}$. It follows that $\mathbf{Ho}(\mathbf{M})$ is the "monoidal" localization $\mathbf{Ho}(\mathbf{C}^*)[\otimes e_{11}^{-1}]$:

$$[A, B]_{\mathbf{M}} \cong [A, B \otimes \mathcal{K}]_{\mathbf{C}^*} \cong [A \otimes \mathcal{K}, B \otimes \mathcal{K}]_{\mathbf{C}^*}.$$

In the notation of [Hig90], the categories $\mathbf{Ho}(\mathbf{M})$ and $\mathbf{SHo}(\mathbf{M})$ are the not necessarily separable versions of **TH** and **TS** respectively. When restricted to the abelian algebras, $\mathbf{SHo}(\mathbf{M})$ gives the *kk* groups of Dadarlat–McClure [DM00].

**2.3. Topological K-theory.** Taking $\mathcal{H}$ to be topological K-theory in Theorem 1.33, we get a category $\mathbf{K} = R_K\mathbf{C}^*$ of fibrant objects whose weak equivalences are K-equivalences and fibrations are Schochet fibrations. Compare [JJ06] and [MN06]. It follows from Theorem 2.25 that $\mathbf{Ho}(\mathbf{K})$ has small hom sets.

Let $\mathcal{K}$ be the algebra of compact operators on a separable Hilbert space and let $e_{11}\colon \mathbb{C} \to \mathcal{K}$ denote a rank one projection. Then

$$\mathrm{id}_A \otimes e_{11}\colon A \to A \otimes \mathcal{K}$$

is a K-equivalence. We also have a natural isomorphism $\Omega^2 A \to A \otimes \mathcal{K}$ in $\mathbf{Ho}(\mathbf{K})$, since Bott periodicity can be implemented by a boundary map associated to a Toeplitz type extension. It follows that

$$\Omega\colon [A, B]_{\mathbf{K}} \to [\Omega A, \Omega B]_{\mathbf{K}}$$

is invertible. Hence $\mathbf{K}$ is stable and the natural functor $\mathbf{Ho}(\mathbf{K}) \to \mathbf{SHo}(\mathbf{K})$ is an equivalence of categories. In particular, $\mathbf{Ho}(\mathbf{K})$ is a triangulated category in a natural way, and $\mathbf{SHo}(\mathbf{C})^* \to \mathbf{Ho}(\mathbf{K})$ is a triangulated functor.

The following is a version of the Universal Coefficient Theorem of Rosenberg and Schochet (cf. [RS87]). It can be deduced from results in [MN06], but we give a self-contained proof.

**Theorem 2.25.** *For $B \in \mathbf{K}$, we have*

$$[\mathbb{C}, B]_{\mathbf{K}} \cong K_0(B). \tag{2.2}$$

*More generally, for $A,\ B \in \mathbf{K}$, there is a natural short exact sequence*

$$\mathrm{Ext}(K_{*-1}(A), K_*(B)) \rightarrowtail [A, B]_{\mathbf{K}} \twoheadrightarrow \mathrm{Hom}(K_*(A), K_*(B)),$$

*where*

$$\mathrm{Hom}(K_*(A), K_*(B)) := \bigoplus_{i=0,1} \mathrm{Hom}_{\mathbb{Z}}(K_i(A), K_i(B))$$

*and*

$$\mathrm{Ext}(K_{*-1}(A), K_*(B)) := \bigoplus_{i=0,1} \mathrm{Ext}^1_{\mathbb{Z}}(K_{i-1}(A), K_i(B)).$$

*Proof.* We have a natural (additive) map

$$[A, B]_{\mathbf{K}} \to \mathrm{Hom}_{\mathbb{Z}}(K_*(A), K_*(B)). \tag{2.3}$$

We claim that this is an isomorphism if $K_*(A)$ is free – for $A = \mathbb{C}$ we get (2.2).

Indeed, suppose that $K_*(A)$ is free. First recall that we have natural isomorphisms

$$K_0(D) = [q\mathbb{C}, D \otimes \mathcal{K}]_{\mathbf{C}^*}, \quad K_1(D) = [\Omega\mathbb{C}, D \otimes \mathcal{K}]_{\mathbf{C}^*},$$

where $q\mathbb{C}$ is the kernel of the folding map $(\mathbb{C} \coprod \mathbb{C} \to \mathbb{C})$. We have a K-equivalence $q\mathbb{C} \xrightarrow{\sim} \mathbb{C}$.

Then it is clear that any map $K_*(A) \to K_*(B)$ can be implemented by an element of the form

$$\begin{tikzcd}
(\coprod_I q\mathbb{C}) \coprod (\coprod_J \Omega\mathbb{C}) \arrow[r] \arrow[d, "\wr"] & B \otimes \mathcal{K} \\
A \otimes \mathcal{K} & \\
A \arrow[u, "\wr"] & B \arrow[uu, "\wr"]
\end{tikzcd}$$

in $\mathbf{Ho}(\mathbf{K})$. Hence (2.3) is surjective. To see injectivity of (2.3), let

$$A \xleftarrow{\sim} A' \to B \tag{2.4}$$

be a morphism in $[A, B]_{\mathbf{K}}$ that maps to $0 \in \mathrm{Hom}(K_*(A), K_*(B))$. Then we can complete (2.4) to a homotopy-commutative diagram

$$\begin{tikzcd}
(\coprod_I q\mathbb{C}) \coprod (\coprod_J \Omega\mathbb{C}) \arrow[r, "\sim"] \arrow[d, "\wr"] & A' \otimes \mathcal{K} \arrow[r] & B \otimes \mathcal{K} \\
A \otimes \mathcal{K} & & \\
A \arrow[u, "\wr"] & A' \arrow[l, "\sim"'] \arrow[r] \arrow[uu, "\wr"] & B \arrow[uu, "\wr"]
\end{tikzcd}$$

in $\mathbf{Ho}(\mathbf{C}^*)$. Then the top horizontal map is null-homotopic, i.e., zero in $\mathbf{Ho}(\mathbf{C}^*)$, hence zero in $\mathbf{Ho}\,\mathbf{K}$. In other words, (2.3) is injective if $K_*(A)$ is free.

The general case follows using a geometric resolution of $K_*(A)$. See for instance [Uuy11]. □

**2.4. KK-theory.** In the next two subsections, we will concentrate on the category $\mathbf{sC}^*$ of *separable* C*-algebras. We refer to [Bla98], Chapter VIII, for details about Kasparov's KK-theory.

Recall that, in the Cuntz picture of KK-theory (cf. [Cun87]), we have

$$\mathrm{KK}(A, B) := [qA, B \otimes \mathcal{K}]_{\mathbf{C}^*} = [qA, B]_{\mathbf{M}},$$

where $qA$ is the kernel of the folding map $\mathrm{id}_A \coprod \mathrm{id}_A \colon A \coprod A \to A$.

**Lemma 2.26.** *Let $E \to B$ be a Schochet fibration with fibre $F$. Then for any $D \in \mathbf{C}^*$, we have a natural 6-term exact sequence:*

$$\begin{tikzcd} \mathrm{KK}(D, F) \arrow[r] & \mathrm{KK}(D, E) \arrow[r] & \mathrm{KK}(D, B) \arrow[d] \\ \mathrm{KK}(D, \Omega B) \arrow[u] & \mathrm{KK}(D, \Omega E) \arrow[l] & \mathrm{KK}(D, \Omega F). \arrow[l] \end{tikzcd}$$

*Proof.* This follows from the fibre exact sequence (cf. Theorem 1.22) in $\mathbf{Ho}(\mathbf{M})$ (or $\mathbf{Ho}(\mathbf{C}^*)$) and Bott periodicity. □

**Definition 2.27.** A $*$-homomorphisms $t \colon A \to B$ in $\mathbf{sC}^*$ is called a KK-equivalence if

$$t_* \colon \mathrm{KK}(D, A) \to \mathrm{KK}(D, B)$$

is an isomorphism for all $D \in \mathbf{sC}^*$.

The following example is the cornerstone of the Cuntz picture of KK-theory.

**Example 2.28.** For any $A \in \mathbf{sC}^*$, the composition

$$qA \rightarrowtail A \coprod A \xrightarrow{\mathrm{id}_A \coprod 0} A$$

is a KK-equivalence.

In particular, we have an identification

$$\mathrm{KK}(A, B) \cong [qA, qB]_{\mathbf{M}} = [qA \otimes \mathcal{K}, qB \otimes \mathcal{K}]_{\mathbf{C}^*}.$$

Under this identification, composition of KK-elements corresponds to composition of homotopy classes (cf. [Cun87]). In particular, a $*$-homomorphism is a KK-equivalence if and only if it determines an invertible element in KK, as expected.

**Theorem 2.29.** *The category of* separable *$C^*$-algebras forms a category of fibrant objects with weak equivalences the* KK*-equivalences and fibrations the Schochet fibrations, denoted by* **KK**, *whose homotopy category* $\mathbf{Ho}(\mathbf{KK})$ *is equivalent to the* KK*-category of Kasparov. It follows that Kasparov's* KK*-category is a stable triangulated category.*

*Proof.* The category of fibrant objects structure follows from Theorem 1.33, since $\mathrm{KK}(D, -)$ gives a homology theory on $\mathbf{C}^*$ in the sense of Definition 1.29 for all $D$ by Lemma 2.26.

Now the functor $\mathbf{Ho}(\mathbf{KK}) \to \mathrm{KK}$ given by $A \mapsto qA \otimes \mathcal{K}$ is easily seen to be an equivalence of categories. Stability follows from Bott periodicity. □

**Remark 2.30.** Note that in Theorem 2.29, we can take the semi-split surjections, i.e., surjections with a completely positive contractive splitting, to be the fibrations. Indeed, the only nontrivial part is (FW1): if $p : E \to B$ is a semi-split surjection which is also a KK-equivalence and $f : A \to B$ is arbitrary, then the pullback $f^*(p)$ is also a KK-equivalence. However, this is clear since if $p$ is a semi-split surjection with kernel $F$, then $F \to Fp$ is a KK-equivalence (see [Bla98], Theorem 19.5.5), hence $p$ is a KK-equivalence if and only $F$ is KK-contractible if and only if $f^*(p)$ is a KK-equivalence (see diagram (1.1)).

Note also that Schochet fibrations and semi-split surjections give rise to the same class of distinguished triangles in $\mathbf{Ho}(\mathbf{KK}) \cong \mathbf{SHo}(\mathbf{KK})$.

**2.5. Universal homology theories.** We consider $\mathbf{sC}^*$ as a category of fibrant objects with weak equivalences the homotopy equivalences and fibrations the Schochet fibrations. In this subsection, we identify various localizations of $\mathbf{sC}^*$.

**Definition 2.31.** A *fibre homology theory* on $\mathbf{sC}^*$ is a homology theory the pointed category of fibrant objects $\mathbf{sC}^*$ in the sense of Definition 1.29, i.e., a homological functor on the triangulated category $\mathbf{SHo}(\mathbf{sC}^*)$ to $\mathbf{Ab}$.

**Definition 2.32.** We say that a fibre homology theory $\mathcal{H}$ on $\mathbf{sC}^*$ is *excisive* with respect to a surjection $p$, if the inclusion $\ker(p) \to Fp$ is an $\mathcal{H}$-equivalence. A *homology theory* on $\mathbf{sC}^*$ is a fibre homology theory excisive with respect to all surjections.

**Definition 2.33.** We say that a morphism $t \in \mathbf{sC}^*$ is a *weak equivalence* if it is an $\mathcal{H}$-equivalence for all homology theories $\mathcal{H}$ on $\mathbf{sC}^*$.

**Remark 2.34.** Note that homotopy equivalences are weak equivalences.

**Theorem 2.35.** *The category $\mathbf{sC}^*$ forms a pointed category of fibrant objects with weak equivalences as in Definition 2.33 and fibrations the Schochet fibrations, whose stable homotopy category is triangulated equivalent to the stable homotopy category of* [Tho03], *Theorem* 3.3.5.

By [Dad94], the stable homotopy category mentioned above is equivalent to the suspension-stable version of the strong shape category.

*Proof.* It follows from Theorem 1.35 that the stable homotopy category is a *universal* triangulated homology theory in the sense of [Tho03], Definition 2.3.3. Then [Tho03], Theorem 3.3.6, finishes the proof. □

For a Hilbert space $H$, let $e_H: \mathbb{C} \to \mathcal{K}(H)$ denote a rank one projection.

**Definition 2.36.** A (fibre) homology theory $\mathcal{H}$ on $\mathbf{sC}^*$ is said to be

(1) *matrix-invariant* if $\mathrm{id}_B \otimes e_H$ is an $\mathcal{H}$-equivalence for all $B \in \mathbf{sC}^*$ and $H$ finite dimensional, and

(2) *$C^*$-invariant* if $\mathrm{id}_B \otimes e_H$ is an $\mathcal{H}$-equivalence for all $B \in \mathbf{sC}^*$ and $H$ separable.

**Definition 2.37.** A morphism $t \in \mathbf{sC}^*$ is said to be

(1) a **bu**-*equivalence* if it induces isomorphism on all matrix-invariant homology theories, and

(2) an *E-equivalence* if it induces isomorphism on all C*-invariant homology theories.

**Theorem 2.38.** (1) *The category* $\mathbf{sC}^*$ *forms a pointed category of fibrant objects with weak equivalences the* **bu**-*equivalences and fibrations the Schochet fibrations, whose stable homotopy category is triangulated equivalent to the category* **bu** *of* [Tho03], *Theorem* 4.2.1.

(2) *The category* $\mathbf{sC}^*$ *forms a stable pointed category of fibrant objects with weak equivalences the E-equivalences and fibrations the Schochet fibrations, whose homotopy category is a triangulated category, equivalent to the E-theory of Higson.*

*Proof.* This follows from Theorem 1.35 and the universal properties of **bu** and $E$ (cf. [Tho03]). □

**Remark 2.39.** (1) Let $p: E \to B$ be a surjection with kernel $F$. Then $p$ is a weak equivalence in the sense of Definition 2.33 if and only if $F$ is $\mathcal{H}$-acyclic for all homology theories $\mathcal{H}$ on $\mathbf{sC}^*$. Indeed, we have a map of extensions where the vertical maps are all weak equivalences:

$$\begin{tikzcd}
0 \arrow[r] & F \arrow[r, "i"] \arrow[d, "\wr"] & E \arrow[r, "p"] \arrow[d, "\wr"] & B \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & Fp \arrow[r, tail] & Np \arrow[r, two heads] & B \arrow[r] & 0.
\end{tikzcd} \tag{2.5}$$

Hence the claim follows from the naturality of the long exact sequence associated to homology theories. It follows that in Theorem 2.35 and Theorem 2.38, we can take the fibrations to be all surjections. However, the distinguished triangles in the stable homotopy category would be the same (see the diagram (2.5)).

(2) We can also describe the **KK**-category of Kasparov as the universal split-exact triangulated homology theory in a similar way.

## Appendix: No Quillen model structure (following Andersen–Grodal)

As noted in the introduction, the homotopy theory of C*-algebras does not come from a Quillen model structure. This was perhaps first pointed out as part of a 1997 preprint by Andersen–Grodal [AG97], where they also established a *Baues fibration category structure* [Bau89] on $C^*$-algebras (a notion very similar to a category of fibrant objects; see [Bau89], Remark I.1a.6). Since their work however remains unpublished, we, by permission of the authors, reproduce their non-existence argument in this appendix.

Recall that if $\mathbf{M}$ is a Quillen model category, then the full subcategory $\mathbf{M}_f$ of fibrant objects in $\mathbf{M}$ is a category of fibrant objects (cf. Example 1.4).

**Theorem A.1.** *Let* $\mathbf{C}^*$ *denote the pointed category of fibrant objects of Theorem 2.19. Then* $\mathbf{C}^*$ *is not the full subcategory of fibrant objects of a Quillen model category.*

The essential part of the proof is to see that the loop functor does not admit a left adjoint, as already remarked on in Remark 2.23 (3).

**Lemma A.2.** *Let* $\mathbf{M}_f$ *be the full subcategory fibrant objects of a Quillen model category* $\mathbf{M}$*, considered as a category of fibrant objects as in Example 1.4. Then the loop functor*

$$\Omega\colon \mathbf{Ho}(\mathbf{M}_f) \to \mathbf{Ho}(\mathbf{M}_f) \tag{A.1}$$

*admits a left-adjoint.*

*Proof.* Follows from Theorem I.1.1 and Theorem I.2.2 of [Qui67] and the definitions. □

The following Lemma is clear.

**Lemma A.3.** *Let* $\mathbf{A}^* \subseteq \mathbf{C}^*$ *denote the full subcategory consisting of abelian* $C^*$*-algebras. Then* $\mathbf{A}^*$ *is a reflective (monoidal) subcategory of* $\mathbf{C}^*$ *– the left-adjoint of the inclusion* $i\colon \mathbf{A}^* \to \mathbf{C}^*$ *is the abelianization* $(-)^{\mathrm{ab}}\colon \mathbf{C}^* \to \mathbf{A}^*$:

$$\mathrm{Mor}_{\mathbf{A}^*}(D^{\mathrm{ab}}, B) \cong \mathrm{Mor}_{\mathbf{C}^*}(D, iB) \tag{A.2}$$

*for* $D \in \mathbf{C}^*$, $B \in \mathbf{A}^*$. □

In particular, $\mathbf{A}^*$ is a pointed category of fibrant objects (cf. Example 1.6).

**Corollary A.4.** *The homotopy category* $\mathbf{Ho}(\mathbf{A}^*)$ *is a full reflective subcategory of* $\mathbf{Ho}(\mathbf{C}^*)$ *and the loop functor*

$$\Omega\colon \mathbf{Ho}(\mathbf{A}^*) \to \mathbf{Ho}(\mathbf{A}^*)$$

*is the restriction of* $\Omega\colon \mathbf{Ho}(\mathbf{C}^*) \to \mathbf{Ho}(\mathbf{C}^*)$ *to* $\mathbf{Ho}(\mathbf{A}^*)$.

*Proof.* The adjunction (A.2) descends to the homotopy categories and gives and adjunction:

$$[D^{\mathrm{ab}}, B]_{\mathbf{A}^*} \cong [D, iB]_{\mathbf{C}^*},$$

for $D \in \mathbf{C}^*$, $B \in \mathbf{A}^*$. See also [Bro74], Adjoint Functor Lemma. The rest of the statements are clear. □

Consequently, we see that $\mathbf{SHo}(\mathbf{A})^*$ is a full triangulated subcategory of $\mathbf{SHo}(\mathbf{C})^*$.

**Lemma A.5.** *The loop functor* $\Omega\colon \mathbf{Ho}(\mathbf{A}^*) \to \mathbf{Ho}(\mathbf{A}^*)$ *does not admit a left-adjoint.*

*Proof.* By Gelfand–Naimark duality, the category $\mathbf{CM}_*$ of pointed compact Hausdorff spaces is contravariantly equivalent to $\mathbf{A}^*$, hence form a category of *cofibrant* objects. We need to show that the functor

$$\Sigma = S^1 \wedge -\colon \mathbf{Ho}(\mathbf{CM}_*) \to \mathbf{Ho}(\mathbf{CM}_*) \tag{A.3}$$

does not admit a *right*-adjoint. We show that, in fact, the functor

$$\mathbf{Ho}(\mathbf{CM}_*) \to \mathbf{Set}_*, \quad X \mapsto [\Sigma X, S^1]_{\mathbf{CM}_*},$$

is *not* representable, where $\mathbf{Set}_*$ denote the category of pointed sets. Indeed, suppose that for some $Y \in \mathbf{Ho}(\mathbf{CM}_*)$ we have a natural identification

$$[\Sigma X, S^1]_{\mathbf{CM}_*} \cong [X, Y]_{\mathbf{CM}_*}.$$

Let $\mathbf{Top}_*$ denote the category of pointed compactly generated weakly Hausdorff topological spaces. Then $\mathbf{CM}_*$ is a full (reflective) subcategory of $\mathbf{Top}_*$ and $\mathbf{Ho}(\mathbf{CM}_*)$ is a full subcategory of $\mathbf{Ho}(\mathbf{Top}_*)$. Moreover, the functor $\Sigma$ of (A.3) is the restriction of

$$\Sigma = S^1 \wedge -\colon \mathbf{Ho}(\mathbf{Top}_*) \to \mathbf{Ho}(\mathbf{Top}_*),$$

which does have a right-adjoint

$$\Omega = \mathrm{Mor}_{\mathbf{Top}_*}(S^1, -)\colon \mathbf{Ho}(\mathbf{Top}_*) \to \mathbf{Ho}(\mathbf{Top}_*).$$

Hence for $X \in \mathbf{CM}_*$, we have

$$[X, Y]_{\mathbf{Top}_*} \cong [X, Y]_{\mathbf{CM}_*} \cong [\Sigma X, S^1]_{\mathbf{CM}_*} \cong [X, \Omega S^1]_{\mathbf{Top}_*}.$$

Moreover, by Yoneda's Lemma, the natural identification above must be induced by a map $f\colon Y \to \Omega S^1$ of $\mathbf{Top}_*$. This is a contradiction, for since $Y$ is compact, $f$ cannot be surjective on $\pi_0$. □

**Corollary A.6.** *The loop functor* $\Omega\colon \mathbf{Ho}(\mathbf{C}^*) \to \mathbf{Ho}(\mathbf{C}^*)$ *does not admit a left-adjoint.*

*Proof.* Suppose that $\Sigma\colon \mathbf{Ho}(\mathbf{C}^*) \to \mathbf{Ho}(\mathbf{C}^*)$ is a left-adjoint of $\Omega$. It follows that the composition

$$(-)^{\mathrm{ab}} \circ \Sigma \circ i : \mathbf{Ho}(\mathbf{A}^*) \to \mathbf{Ho}(\mathbf{C}^*) \to \mathbf{Ho}(\mathbf{C}^*) \to \mathbf{Ho}(\mathbf{A}^*)$$

is a left-adjoint of $\Omega\colon \mathbf{Ho}(\mathbf{A}^*) \to \mathbf{Ho}(\mathbf{A}^*)$, contradicting Lemma A.5. □

Now Theorem A.1 follows from Lemma A.2 and Corollary A.6.

## References


[AG97] K. Andersen and J. Grodal, A Baues fibration category structure on Banach and $C^*$-algebras. Manuscript 1997. http://www.math.ku.dk/ jg/papers/fibcat.pdf

[Bau89] H. J. Baues, *Algebraic homotopy*. Cambridge Stud. Adv. Math. 15, Cambridge University Press, Cambridge 1989. Zbl 0688.55001 MR 985099

[Bla85] B. Blackadar, Shape theory for $C^*$-algebras. *Math. Scand.* **56** (1985), 249–275. Zbl 0615.46066 MR 813640

[Bla98] B. Blackadar, *K-theory for operator algebras*. 2nd ed., Math. Sci. Res. Inst. Publ. 5, Cambridge University Press, Cambridge 1998. Zbl 0913.46054 MR 1656031

[Bro74] K. S. Brown, Abstract homotopy theory and generalized sheaf cohomology. *Trans. Amer. Math. Soc.* **186** (1973), 419–458. Zbl 0245.55007 MR 0341469

[CH81] A. Calder and H. M. Hastings, Realizing strong shape equivalences. *J. Pure Appl. Algebra* **20** (1981), 129–156. Zbl 0457.55004 MR 601680

[Cat81] F. Cathey, Strong shape theory. In *Shape theory and geometric topology* (Dubrovnik, 1981), Lecture Notes in Math. 870, Springer-Verlag, Berlin 1981, 215–238. Zbl 0473.55011 MR 643532

[CH90] A. Connes and N. Higson, Déformations, morphismes asymptotiques et $K$-théorie bivariante. *C. R. Acad. Sci. Paris Sér. I Math.* **311** (1990), 101–106. Zbl 0717.46062 MR 1065438

[Cun87] J. Cuntz, A new look at $KK$-theory. *K-Theory* **1** (1987), 31–51. Zbl 0636.55001 MR 899916

[Cun98] J. Cuntz, A general construction of bivariant $K$-theories on the category of $C^*$-algebras. In *Operator algebras and operator theory* (Shanghai, 1997), Contemp. Math. 228, Amer. Math. Soc., Providence, RI, 1998, 31–43. Zbl 0930.46055 MR 1667652

[Dad94] M. Dadarlat, Shape theory and asymptotic morphisms for $C^*$-algebras. *Duke Math. J.* **73** (1994), 687–711. Zbl 0847.46028 MR 1262931

[DM00] M. Dadarlat and J. McClure, When are two commutative $C^*$-algebras stably homotopy equivalent? *Math. Z.* **235** (2000), 499–523. Zbl 0969.46050 MR 1800209

[EK86] E. G. Effros and J. Kaminker, Homotopy continuity and shape theory for $C^*$-algebras. In *Geometric methods in operator algebras* (Kyoto, 1983), Pitman Res. Notes Math. Ser. 123, Longman Sci. Tech., Harlow 1986, 152–180. Zbl 0645.46051 MR 866493

[GZ67] P. Gabriel and M. Zisman, *Calculus of fractions and homotopy theory*. Ergeb. Math. Grenzgeb. 35, Springer-Verlag, Berlin 1967. Zbl 0186.56802 MR 0210125

[GJ99] P. G. Goerss and J. F. Jardine, *Simplicial homotopy theory*. Progr. Math. 174, Birkhäuser, Basel 1999. Zbl 0949.55001 MR 1711612

[Hel68] A. Heller, Stable homotopy categories. *Bull. Amer. Math. Soc.* **74** (1968), 28–63. Zbl 0177.25605 MR 0224090

[Hig90] N. Higson, Categories of fractions and excision in $KK$-theory. *J. Pure Appl. Algebra* **65** (1990), 119–138. Zbl 0718.46053 MR 1068250

[Hov99] M. Hovey, *Model categories*. Math. Surveys Monographs 63, Amer. Math. Soc., Providence, RI, 1999. Zbl 0909.55001 MR 1650134

[JJ06] M. Joachim and M. W. Johnson, Realizing Kasparov's $KK$-theory groups as the homotopy classes of maps of a Quillen model category. In *An alpine anthology of homotopy theory*, Contemp. Math. 399, Amer. Math. Soc., Providence, RI, 2006, 163–197. Zbl 1103.46048 MR 2222510

[KP97] K. H. Kamps and T. Porter, *Abstract homotopy and simple homotopy theory*. World Scientific, Singapore 1997. Zbl 0890.55014 MR 1464944

[Kas80] G. G. Kasparov, The operator $K$-functor and extensions of $C^*$-algebras. *Izv. Akad. Nauk SSSR Ser. Mat.* **44** (1980), 571–636; English transl. *Math. USSR-Izv.* **16** (1981), 513–572. Zbl 0464.46054 MR 582160

[Kas88] G. G. Kasparov, Equivariant $KK$-theory and the Novikov conjecture. *Invent. Math.* **91** (1988), 147–201. Zbl 0647.46053 MR 918241

[May01] J. P. May, The additivity of traces in triangulated categories. *Adv. Math.* **163** (2001), 34–73. Zbl 1007.18012 MR 1867203

[Mey08] R. Meyer, Categorical aspects of bivariant $K$-theory. In *$K$-theory and noncommutative geometry*, EMS Ser. Congr. Rep., Eur. Math. Soc., Zürich 2008, 1–39. Zbl 1157.19004 MR 2513331

[MN06] R. Meyer and R. Nest, The Baum–Connes conjecture via localisation of categories. *Topology* **45** (2006), 209–259. Zbl 1092.19004 MR 2193334

[Øst10] P. A. Østvær, *Homotopy theory of $C^*$-algebras*. Front. Math., Birkhäuser, Basel 2010. Zbl 1213.46072 MR 2723902

[Ped99] Gert K. Pedersen, Pullback and pushout constructions in $C^*$-algebra theory. *J. Funct. Anal.* **167** (1999), 243–344. Zbl 0944.46063 MR 1716199

[Qui67] D. G. Quillen, *Homotopical algebra*. Lecture Notes in Math. 43, Springer-Verlag, Berlin 1967. Zbl 0168.20903 MR 0223432

[Ros82] J. Rosenberg, The role of $K$-theory in non-commutative algebraic topology. In *Operator algebras and $K$-theory* (San Francisco, Calif., 1981), Contemp. Math. 10, Amer. Math. Soc., Providence, R.I., 1982, 155–182. Zbl 0486.46052 MR 658514

[RS87] J. Rosenberg and C. Schochet, The Künneth theorem and the universal coefficient theorem for Kasparov's generalized $K$-functor. *Duke Math. J.* **55** (1987), 431–474. Zbl 0644.46051 MR 894590

[Sch84] C. Schochet, Topological methods for $C^*$-algebras III: Axiomatic homology. *Pacific J. Math.* **114** (1984), 399–445. Zbl 0491.46061 MR 757510

[Str] Neil P. Strickland, The category of CGWH spaces. Preprint 2009. http://neil-strickland.staff.shef.ac.uk/courses/homotopy/cgwh.pdf

[Tho03] Andreas Thom, Connective $E$-theory and bivariant homology for $C^*$-algebras. Ph.D. thesis, University of Münster, Münster 2003. Zbl 1030.46100

[Uuy11] O. Uuye, The Baum–Connes conjecture for $KK$-theory. *J. K-Theory* **8** (2011), 3–29. Zbl 1243.19004 MR 2826278

[Voi11] C. Voigt, The Baum–Connes conjecture for free orthogonal quantum groups. *Adv. Math.* **227** (2011), 1873–1913. Zbl 05918252 MR 2803790

[WO93] N. E. Wegge-Olsen, *$K$-theory and $C^*$-algebras*. Oxford University Press, Oxford 1993. Zbl 0780.46038 MR 1222415

Received January 3, 2011; revised October 3, 2011

O. Uuye, School of Mathematics, Cardiff University, Senghennydd Road, Cardiff, Wales, UK, CF24 4AG

E-mail: UuyeO@cardiff.ac.uk