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Calabi–Yau pointed Hopf algebras of finite Cartan type

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Abstract. We study the Calabi–Yau property of pointed Hopf algebra $U(\mathcal{D}, \lambda)$ of finite Cartan type. It turns out that this class of pointed Hopf algebras constructed by N. Andruskiewitsch and H.-J. Schneider contains many Calabi–Yau Hopf algebras. To give concrete examples of new Calabi–Yau Hopf algebras, we classify the Calabi–Yau pointed Hopf algebras $U(\mathcal{D}, \lambda)$ of dimension less than 5.

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Contents

Introduction

In [3], N. Andruskiewitsch and H.-J. Schneider classified pointed Hopf algebras with finite Gelfand–Kirillov dimension, which are domains with a finitely generated abelian group of group-like elements, and a positive braiding. In the same paper, the authors constructed a class of Hopf algebra $U(\mathcal{D}, \lambda)$, generalizing the quantized enveloping algebra $U_q(\mathfrak{g})$ of a finite dimensional semisimple Lie algebra $\mathfrak{g}$. These pointed Hopf algebras turn out to be Artin–Schelter (AS-) Gorenstein Hopf algebras. AS-Gorenstein Hopf algebras have been recently intensively studied (e.g. [6], [7], [15], [16], [25], [26]). One of the important properties of an AS-Gorenstein Hopf algebra is the existence of a homological integral which generalizes Sweedler's classical integral of a finite dimensional Hopf algebra, cf. [15]. Brown and Zhang proved that the rigid dualizing complex of an AS-Gorenstein Hopf algebra is determined by its homological integral and antipode [7]. He, Van Oystaeyen and Zhang used homological integrals to investigate the Calabi–Yau property of cocommutative Hopf algebras [12]. They successfully classified the low-dimensional cocommutative Calabi–Yau Hopf algebras over an algebraically closed field of characteristic zero.

The main aim of this paper is to find out when a pointed Hopf algebra $U(\mathcal{D}, \lambda)$ of finite Cartan type is Calabi–Yau, and to classify those low-dimensional Calabi–Yau pointed Hopf algebras. It turns out that the class $U(\mathcal{D}, \lambda)$ of pointed Hopf algebras contains many Calabi–Yau Hopf algebras. Most of them are of types different from the quantum groups $U_q(\mathfrak{g})$, which were proved to be Calabi–Yau by Chemla [8]. This give us more interesting examples of Calabi–Yau Hopf algebras. The paper is organized as follows.

In Section 1, we recall the pointed Hopf algebras $U(\mathcal{D}, \lambda)$ of finite Cartan type, the definition of a Calabi–Yau algebra, the notion of a homological integral and a (rigid) dualizing complex over a Noetherian algebra. In Section 2, we study the Calabi–Yau pointed Hopf algebras of finite Cartan type. We give a necessary and sufficient condition for a Hopf algebra $U(\mathcal{D}, \lambda)$ to be Calabi–Yau, and calculate the rigid dualizing complex of $U(\mathcal{D}, \lambda)$ (Theorem 2.3). In [8], Chemla calculated the rigid dualizing complexes of quantum groups $U_q(\mathfrak{g})$. As a consequence of the characterization theorem, the quantum groups $U_q(\mathfrak{g})$ are Calabi–Yau Hopf algebras.

A pointed Hopf algebra of the form $U(\mathcal{D}, \lambda)$ is a (cocycle) deformation of the smash product $\mathcal{B}(V) \# \Bbbk\Gamma$, where V is the space of skew-primitive elements of $U(\mathcal{D}, \lambda)$, $\mathcal{B}(V)$ is the Nichols algebra of V and $\Bbbk\Gamma$ is the group algebra of the group formed by group-like elements of $U(\mathcal{D}, \lambda)$. Our second aim in this paper is to study the Calabi–Yau property of the Nichols algebra $\mathcal{B}(V)$. The algebra $\mathcal{B}(V)$ is an $\mathbb{N}^{p+1}$-filtered algebra. By analyzing the rigid dualizing complex of the associated graded algebra $\mathbb{Gr}\ \mathcal{B}(V)$, we obtain the rigid dualizing complex of $\mathcal{B}(V)$. We then are able to give a necessary and sufficient condition for the algebra $\mathcal{B}(V)$ to be Calabi–Yau, which forms the main result of Section 3 (see Theorem 3.9).

In Section 4, we discuss the relation between the Calabi–Yau property of a pointed Hopf algebra $U(\mathcal{D}, \lambda)$ and the Calabi–Yau property of the associated Nichols algebra $\mathcal{B}(V)$. It turns out that for a pointed Hopf algebra $U(\mathcal{D}, \lambda)$ and its associated Nichols algebra $\mathcal{B}(V)$, if one of them is CY, the other one is not.

In the final Section 5, we classify the Calabi–Yau pointed Hopf algebras $U(\mathcal{D}, \lambda)$ of dimension less than 5. It turns out that $U_q(\mathfrak{sl}_2)$ is the only known non-cocommutative example in the classification. The other non-cocommutative Hopf algebras are new examples of Calabi–Yau Hopf algebras.

1. Preliminaries

Throughout this paper, we fix an algebraically closed field $\Bbbk$. All vector spaces, algebras are over $\Bbbk$. The unadorned tensor $\otimes$ means $\otimes_{\Bbbk}$. Given an algebra A, we write A^{op} for the opposite algebra of A and A^{e} for the enveloping algebra $A \otimes A^{\mathrm{op}}$ of A.

Let A be an algebra. For a left A-module M and an algebra automorphism $\phi\colon A \to A$, ${}_{\phi}M$ stands for the left A-module twisted by the automorphism ϕ. Similarly, for a right A-module N, we have N_{ϕ}. Observe that $A_{\phi} \cong {}_{\phi^{-1}}A$ as A-A-bimodules. $A_{\phi} \cong A$ as A-A-bimodules if and only if ϕ is an inner automorphism.

Let A be a Hopf algebra, and $\xi\colon A \to \Bbbk$ an algebra homomorphism. We write $[\xi]$ to be the *winding* homomorphism of ξ defined by

$$[\xi](a) = \sum \xi(a_1)a_2,$$

for any $a \in A$.

A Noetherian algebra in this paper means a *left and right* Noetherian algebra.

1.1. Pointed Hopf algebra $U(\mathcal{D}, \lambda)$. In this section we recall the definitions and basic properties of Nichols algebras and pointed Hopf algebras of finite Cartan type. More details can be found in [3]. We fix the following terminology:

- A free abelian group Γ of finite rank s.
- A Cartan matrix $(a_{ij}) \in \mathbb{Z}^{\theta\times\theta}$ of finite type, where $\theta \in \mathbb{N}$. Denote by $(d_1, \dots, d_\theta)$ a diagonal matrix of positive integers such that $d_i a_{ij} = d_j a_{ji}$, which is minimal with this property.
- A set $\mathcal{X}$ of connected components of the Dynkin diagram corresponding to the Cartan matrix (a_{ij}). If $1 \leqslant i, j \leqslant \theta$, then $i \sim j$ means that they belong to the same connected component.
- A family $(q_I)_{I\in\mathcal{X}}$ of elements in $\Bbbk$ which are *not* roots of unity.
- Elements $g_1, \dots, g_\theta \in \Gamma$ and characters $\chi_1, \dots, \chi_\theta \in \widehat{\Gamma}$ such that

$$\chi_j(g_i)\chi_i(g_j) = q_I^{d_i a_{ij}}, \quad \chi_i(g_i) = q_I^{d_i} \quad \text{for all } 1 \leqslant i, j \leqslant \theta,\ I \in \mathcal{X}. \tag{1}$$

Let $\mathcal{D}$ be the collection $\mathcal{D}(\Gamma, (a_{ij})_{1\leqslant i,j\leqslant\theta}, (q_I)_{I\in\mathcal{X}}, (g_i)_{1\leqslant i\leqslant\theta}, (\chi_i)_{1\leqslant i\leqslant\theta})$. A *linking datum* $\lambda = (\lambda_{ij})$ for $\mathcal{D}$ is a collection of elements $(\lambda_{ij})_{1\leqslant i<j\leqslant\theta, i\nsim j} \in \{0, 1\}$ such that $\lambda_{ij} = 0$ if $g_i g_j = 1$ or $\chi_i\chi_j \neq \varepsilon$. We write the datum $\lambda = 0$ if $\lambda_{ij} = 0$ for all $1 \leqslant i < j \leqslant \theta$. The datum $(\mathcal{D}, \lambda) = (\Gamma, (a_{ij}), q_I, (g_i), (\chi_i), (\lambda_{ij}))$ is called a *generic datum of finite Cartan type* for group Γ.

Definition 1.1 ([3], Section 4). Let $(\mathcal{D}, \lambda)$ be a generic datum of finite Cartan type. Let $U(\mathcal{D}, \lambda)$ be the algebra generated by $x_1, \dots, x_\theta$ and $y_1^{\pm 1}, \dots, y_s^{\pm 1}$ subject to the relations

$$y_m^{\pm 1} y_h^{\pm 1} = y_h^{\pm 1} y_m^{\pm 1}, \quad y_m^{\pm 1} y_m^{\mp 1} = 1, \quad 1 \leqslant m, h \leqslant s,$$
$$\text{group action: } y_h x_j = \chi_j(y_h) x_j y_h, \quad 1 \leqslant j \leqslant \theta,\ 1 \leqslant h \leqslant s,$$
$$\text{Serre relations: } (\mathrm{ad}_c\, x_i)^{1-a_{ij}}(x_j) = 0, \quad 1 \leqslant i \neq j \leqslant \theta,\ i \sim j,$$
$$\text{linking relations: } x_i x_j - \chi_j(g_i) x_j x_i = \lambda_{ij}(1 - g_i g_j), \quad 1 \leqslant i < j \leqslant \theta,\ i \nsim j,$$

where ad_c is the braided adjoint representation defined in [3], Section 1.

For a generic datum of finite Cartan type $(\mathcal{D}, \lambda)$, denote by $q_{ji} = \chi_i(g_j)$. Then equation (1) reads as follows:

$$q_{ii} = q_I^{d_i} \quad \text{and} \quad q_{ij} q_{ji} = q_I^{d_i a_{ij}} \quad \text{for all } 1 \leqslant i, j \leqslant \theta,\ I \in \mathcal{X}. \tag{2}$$

Let V be a Yetter–Drinfeld module over the group algebra $\Bbbk\Gamma$ with basis $x_i \in V_{g_i}^{\chi_i}$, $1 \leqslant i \leqslant \theta$. In other words, V is a braided vector space with basis $x_1, \dots, x_\theta$, whose braiding is given by

$$c(x_i \otimes x_j) = q_{ij} x_j \otimes x_i.$$

It can be easily derived from the proof of [3], Theorem 4.3, that the Nichols algebra $\mathcal{B}(V)$ is isomorphic to the algebra

$$\Bbbk\langle x_1, \dots, x_\theta \mid (\mathrm{ad}_c\, x_i)^{1-a_{ij}}(x_j) = 0,\ 1 \leqslant i, j \leqslant \theta,\ i \neq j \rangle.$$

We refer to [1], Section 2, for the definition of a Nichols algebra.

Let Φ be the root system corresponding to the Cartan matrix (a_{ij}) with $\{\alpha_1, \dots, \alpha_\theta\}$ a set of fix simple roots, and $\mathcal{W}$ the Weyl group. We fix a reduced decomposition of the longest element $w_0 = s_{i_1} \dots s_{i_p}$ of $\mathcal{W}$ in terms of the simple reflections. Then the positive roots are precisely

$$\beta_1 = \alpha_{i_1},\ \beta_2 = s_{i_1}(\alpha_{i_2}),\ \dots,\ \beta_p = s_{i_1} \dots s_{i_{p-1}}(\alpha_{i_p}).$$

If $\beta_i = \sum_{i=1}^{\theta} m_i \alpha_i$, then we write

$$g_{\beta_i} = g_1^{m_1} \dots g_\theta^{m_\theta} \quad \text{and} \quad \chi_{\beta_i} = \chi_1^{m_1} \dots \chi_\theta^{m_\theta}.$$

Similarly, we write $q_{\beta_j \beta_i} = \chi_{\beta_i}(g_{\beta_j})$.

Root vectors for a quantum group $U_q(\mathfrak{g})$ were defined by Lusztig [17]. Up to a non-zero scalar, each root vector can be expressed as an iterated braided commutator. As in [2], Section 4.1, this definition can be generalized to a pointed Hopf algebras $U(\mathcal{D}, \lambda)$. For each positive root β_i, $1 \leqslant i \leqslant p$, the root vector x_{β_i} is defined by the same iterated braided commutator of the elements $x_1, \dots, x_\theta$, but with respect to the general braiding.

Remark 1.2. If $\beta_j = \alpha_l$, then we have $x_{\beta_j} = x_l$, that is, $x_1, \dots, x_\theta$ are the simple root vectors.

Lemma 1.3 ([3], Theorem 4.3). *Let $(\mathcal{D},\lambda) = (\Gamma,(a_{ij}),q_I,(g_i),(\chi_i),(\lambda_{ij}))$ be a generic datum of finite Cartan type for Γ. The algebra $U(\mathcal{D},\lambda)$ as defined in Definition 1.1 is a pointed Hopf algebra with comultiplication structure determined by*

$$\Delta(y_h) = y_h \otimes y_h, \quad \Delta(x_i) = x_i \otimes 1 + g_i \otimes x_i, \quad 1 \leqslant h \leqslant s,\ 1 \leqslant i \leqslant \theta.$$

Furthermore, $U(\mathcal{D},\lambda)$ has a PBW-basis given by monomials in the root vectors

$$\{x_{\beta_1}^{a_1} \dots x_{\beta_p}^{a_p} y\},$$

for $a_i \geqslant 0$, $1 \leqslant i \leqslant p$, and $y \in \Gamma$. The coradical filtration of $U(\mathcal{D},\lambda)$ is given by

$$U(\mathcal{D},\lambda)_N = \operatorname{span}\{x_{\beta_{i_1}} \dots x_{\beta_{i_j}} y \mid j \leqslant N,\ y \in \Gamma\}.$$

There is an isomorphism of graded Hopf algebras $\operatorname{Gr} U(\mathcal{D},\lambda) \cong \mathcal{B}(V) \# \Bbbk\Gamma \cong U(\mathcal{D},0)$. The algebra $U(\mathcal{D},\lambda)$ has finite Gelfand–Kirillov dimension and is a domain.

In [3], degrees of the PBW basis elements are defined by

$$\deg(x_{\beta_1}^{a_1} \dots x_{\beta_p}^{a_p} y) = (a_1,\dots,a_p,\textstyle\sum_{i=1}^p a_i \mathrm{ht}(\beta_i)) \in \mathbb{N}^{p+1}, \tag{3}$$

where $\mathrm{ht}(\beta)$ is the height of the root β. That is, if $\beta = \sum_{i=1}^\theta m_i\alpha_i$, then $\mathrm{ht}(\beta) = \sum_{i=1}^\theta m_i$. Order the elements in $(\mathbb{Z}^{\geqslant 0})^{p+1}$ as follows:

$$\begin{aligned}&(a_1,\dots,a_p,a_{p+1}) < (b_1,\dots,b_p,b_{p+1}) \text{ if and only if there is some}\\ &1 \leqslant k \leqslant p+1 \text{ such that } a_i = b_i \text{ for } i \geqslant k \text{ and } a_{k-1} < b_{k-1}.\end{aligned} \tag{4}$$

For $m \in \mathbb{N}^{p+1}$, let $F_m U(\mathcal{D},\lambda)$ be the space spanned by the monomials $x_{\beta_1}^{a_1} \dots x_{\beta_p}^{a_p} y$ such that $\deg(x_{\beta_1}^{a_1} \dots x_{\beta_p}^{a_p} y) \leqslant m$. Then we obtain a filtration on the algebra $U(\mathcal{D},\lambda)$.

Lemma 1.4. *If the root vectors x_{β_i}, x_{β_j} belong to the same connected component and $j > i$, then*

$$[x_{\beta_i}, x_{\beta_j}]_c = \sum_{\boldsymbol{a} \in \mathbb{N}^p} \rho_{\boldsymbol{a}} x_{\beta_1}^{a_1} \dots x_{\beta_p}^{a_p}, \tag{5}$$

where $\rho_{\boldsymbol{a}} \in \Bbbk$ and $\rho_{\boldsymbol{a}} \neq 0$ only when $\boldsymbol{a} = (a_1,\dots,a_p)$ is such that $a_s = 0$ for $s \leqslant i$ or $s \geqslant j$. In particular, in $U(\mathcal{D},0)$, the equation (5) holds for all root vectors x_{β_i}, x_{β_j} with $i < j$.

Proof. This follows from [3], Proposition 2.2, and the classical relations that hold for a quantum group $U_q(\mathfrak{g})$ (see [9], Theorem 9.3, for example). It was actually proved in step VI of the proof of Theorem 4.3 in [3]. □

Lemma 1.5. *The filtration defined by PBW basis is an algebra filtration. The associated graded algebra* $\mathbb{G}\mathrm{r}\, U(\mathcal{D}, \lambda)$ *is generated by* x_{β_i}, $1 \leqslant i \leqslant p$, *and* y_h, $1 \leqslant h \leqslant s$, *subject to the relations*

$$\begin{aligned} y_h^{\pm 1} y_m^{\pm 1} &= y_m^{\pm 1} y_h^{\pm 1}, \quad y_h^{\pm 1} y_h^{\mp 1} = 1, \quad 1 \leqslant h, m \leqslant s, \\ y_h x_{\beta_i} &= \chi_{\beta_i}(y_h) x_{\beta_i} y_h, \quad 1 \leqslant i \leqslant p, \ 1 \leqslant h \leqslant s, \\ x_{\beta_i} x_{\beta_j} &= \chi_{\beta_j}(g_{\beta_i}) x_{\beta_j} x_{\beta_i}, \quad 1 \leqslant i < j \leqslant p. \end{aligned}$$

Proof. This follows from Lemma 1.4 and the linking relations. □

Note that the associated graded algebra $\mathbb{G}\mathrm{r}\, U(\mathcal{D}, \lambda)$ is an $\mathbb{N}^{p+1}$-graded algebra.

1.2. Calabi–Yau algebras. Following [11], we call an algebra *Calabi–Yau of dimension d* if

(i) A is homologically smooth, that is, A has a bounded resolution of finitely generated projective A-A-bimodules;

(ii) There are A-A-bimodule isomorphisms

$$\mathrm{Ext}^i_{A^{\mathrm{e}}}(A, A^{\mathrm{e}}) \cong \begin{cases} 0, & i \neq d, \\ A, & i = d. \end{cases}$$

In the sequel Calabi–Yau will be abbreviated to CY for short.

In [12], the CY property of Hopf algebras was discussed by using the homological integrals of Artin–Schelter Gorenstein (AS-Gorenstein for short) algebras [12], Theorem 2.3.

Let us recall the definition of an AS-Gorenstein algebra (cf. [7]).

(i) Let A be a left Noetherian augmented algebra with a fixed augmentation map $\varepsilon\colon A \to \mathbb{k}$. The algebra A is said to be *left AS-Gorenstein* if

 (a) $\mathrm{injdim}\, {}_A A = d < \infty$,

 (b) $\dim \mathrm{Ext}^i_A({}_A\mathbb{k}, {}_A A) = \begin{cases} 0, & i \neq d, \\ 1, & i = d, \end{cases}$

 where injdim stands for injective dimension. A *right AS-Gorenstein algebras* can be defined similarly.

(ii) An algebra A is said to be *AS-Gorenstein* if it is both left and right AS-Gorenstein (relative to the same augmentation map ε).

(iii) An AS-Gorenstein algebra A is said to be *regular* if, in addition, the global dimension of A is finite.

Let A be a Noetherian algebra. If the injective dimension of ${}_A A$ and A_A are both finite, then these two integers are equal by [29], Lemma A. We call this common value the injective dimension of A. The left global dimension and the right global dimension of a Noetherian algebra are equal [24], Example 4.1.1. When the global dimension is finite, then it is equal to the injective dimension.

In order to study infinite dimensional Noetherian Hopf algebras, Lu, Wu and Zhang introduced the concept of a homological integral for an AS-Gorenstein Hopf algebra in [15], which is a generalization of an integral of a finite dimensional Hopf algebra. In [7], homological integrals were defined for general AS-Gorenstein algebras.

Let A be a left AS-Gorenstein algebra with $\operatorname{injdim}_A A = d$. Then $\operatorname{Ext}^d_A({}_A\Bbbk, {}_AA)$ is a 1-dimensional right A-module. Any nonzero element in $\operatorname{Ext}^d_A({}_A\Bbbk, {}_AA)$ is called a *left homological integral* of A. We write $\int^l_A$ for $\operatorname{Ext}^d_A({}_A\Bbbk, {}_AA)$. Similarly, if A is right AS-Gorenstein, any nonzero element in $\operatorname{Ext}^d_A(\Bbbk_A, A_A)$ is called a *right homological integral* of A. Write $\int^r_A$ for $\operatorname{Ext}^d_A(\Bbbk_A, A_A)$.

$\int^l_A$ and $\int^r_A$ are called *left and right homological integral modules* of A, respectively.

CY algebras are closely related to algebras having rigid dualizing complexes. The non-commutative version of a dualizing complex was first introduced by Yekutieli.

Definition 1.6 ([27], cf. [21], Definition 6.1). Assume that A is a (graded) Noetherian algebra. Then an object $\mathcal{R}$ of $D^{\mathrm{b}}(A^{\mathrm{e}})$ ($D^{\mathrm{b}}(\mathrm{GrMod}(A^{\mathrm{e}}))$) is called a *dualizing complex* (in the graded sense) if it satisfies the following conditions:

(i) $\mathcal{R}$ is of finite injective dimension over A and A^{op}.

(ii) The cohomology of $\mathcal{R}$ is given by bimodules which are finitely generated on both sides.

(iii) The natural morphisms $A \to \operatorname{RHom}_A(\mathcal{R}, \mathcal{R})$ and $A \to \operatorname{RHom}_{A^{\mathrm{op}}}(\mathcal{R}, \mathcal{R})$ are isomorphisms in $D(A^{\mathrm{e}})$ ($D(\mathrm{GrMod}(A^{\mathrm{e}}))$).

Roughly speaking, a dualizing complex is a complex $\mathcal{R} \in D^{\mathrm{b}}(A^{\mathrm{e}})$ such that the functor

$$\operatorname{RHom}_A(-, \mathcal{R})\colon D^{\mathrm{b}}_{\mathrm{fg}}(A) \to D^{\mathrm{b}}_{\mathrm{fg}}(A^{\mathrm{op}}) \tag{6}$$

is a duality, with adjoint $\operatorname{RHom}_{A^{\mathrm{op}}}(-, \mathcal{R})$ (cf. [27], Propositions 3.4 and 3.5). Here $D^{\mathrm{b}}_{\mathrm{fg}}(A)$ is the full triangulated subcategory of $D(A)$ consisting of bounded complexes with finitely generated cohomology modules.

In the above definition the algebra A is a Noetherian algebra. In this case, a dualizing complex in the graded sense is also a dualizing complex in the usual sense.

Dualizing complexes are not unique up to isomorphism. To overcome this weakness, Van den Bergh introduced the concept of a rigid dualizing complex in [21], Definition 8.1.

Definition 1.7. Let A be a (graded) Noetherian algebra. A dualizing complex $\mathcal{R}$ over A is called *rigid* (in the graded sense) if

$$\operatorname{RHom}_{A^{\mathrm{e}}}(A, {}_A\mathcal{R} \otimes \mathcal{R}_A) \cong \mathcal{R}$$

in $D(A^{\mathrm{e}})$ $(D(\mathrm{GrMod}(A^{\mathrm{e}})))$.

Note again that if A^{e} is Noetherian then the graded version of this definition implies the ungraded version.

Lemma 1.8 (Cf. [7], Proposition 4.3, and [21], Proposition 8.4). *Let A be a Noetherian algebra. Then the following two conditions are equivalent:*

(a) *A has a rigid dualizing complex $\mathcal{R} = A_{\psi}[s]$, where ψ is an algebra automorphism and $s \in \mathbb{Z}$.*

(b) *A has finite injective dimension d and there is an algebra automorphism ϕ such that*

$$\mathrm{Ext}^i_{A^{\mathrm{e}}}(A, A^{\mathrm{e}}) \cong \begin{cases} 0, & i \neq d, \\ A_{\phi}, & i = d, \end{cases}$$

as A-A-bimodules.

In this case, $\phi = \psi^{-1}$ and $s = d$.

The following corollary follows immediately from Lemma 1.8 and the definition of a CY algebra. It characterizes the Noetherian CY algebras.

Corollary 1.9. *Let A be a Noetherian algebra which is homologically smooth. Then A is a CY algebra of dimension d if and only if A has a rigid dualizing complex $A[d]$.*

2. Calabi–Yau pointed Hopf algebras of finite Cartan type

In this section we calculate the rigid dualizing complex of a pointed Hopf algebra $U(\mathcal{D}, \lambda)$ and study its Calabi–Yau property.

Before we give the main theorem of this section, let us recall the Koszul complex of quadratic algebras (cf. [20]). Let V be a finite dimensional vector space and $T(V)$ the tensor algebra of V. Suppose that A is a quadratic algebra, that is, $A = T(V)/\langle R\rangle$, where $R \subseteq V \otimes V$. The quadratic dual algebra of A, denoted by $A^!$, is the quadratic algebra $T(V^*)/\langle R^{\perp}\rangle$. Let $\{x_i\}_{i=1,\dots,n}$ be a basis of V and $\{x_i^*\}_{i=1,\dots,n}$ be the dual basis of V^*. Introduce the canonical element $e = \sum_{i=1}^n x_i \otimes x_i^* \in A \otimes A^!$. The right multiplication by e defines a complex

$$\cdots \to A \otimes A_j^{!*} \xrightarrow{d_j} A \otimes A_{j-1}^{!*} \to \cdots \to A \otimes A_1^{!*} \to A \to \Bbbk \to 0. \tag{7}$$

This complex is called the Koszul complex of A. The algebra A is Koszul if and only if the complex (7) is a resolution of ${}_A\Bbbk$.

Let $\mathcal{K}$ be the bimodule complex defined by

$$\mathcal{K}\colon \cdots \to A \otimes A_j^{!*} \otimes A \xrightarrow{D_j} A \otimes A_{j-1}^{!*} \otimes A \to \cdots \to A \otimes A \to 0. \tag{8}$$

The differentials $D_j\colon A \otimes A^{!*}_j \otimes A \to A \otimes A^{!*}_{j-1} \otimes A$, $1 \leqslant j \leqslant n$, are defined by $D_j = d^l_j + (-1)^j d^r_j$, where $d^l_j(1 \otimes a \otimes 1) = \sum_{i=1}^n x_i \otimes a \cdot x^*_i \otimes 1$ and $d^r_j(1 \otimes a \otimes 1) = \sum_{i=1}^n 1 \otimes x^*_i \cdot a \otimes x_i$, for any $1 \otimes a \otimes 1 \in A \otimes A^{!*}_j \otimes A$. The complex $\mathcal{K}$ is called the Koszul bimodule complex of A. If A is Koszul, then $\mathcal{K} \to A \to 0$ is exact.

In the rest of this section we fix a generic datum of finite Cartan type:

$$(\mathcal{D}, \lambda) = (\Gamma, (a_{ij})_{1\leqslant i,j\leqslant\theta}, (q_I)_{I\in\mathcal{X}}, (g_i)_{1\leqslant i\leqslant\theta}, (\chi_i)_{1\leqslant i\leqslant\theta}, (\lambda_{ij})_{1\leqslant i<j\leqslant\theta, i\nsim j}),$$

where Γ is a free abelian group of rank s. Let $x_{\beta_1}, \dots, x_{\beta_p}$ be the root vectors. Recall from Remark 1.2 that there are $1 \leqslant j_k \leqslant p$, $1 \leqslant k \leqslant \theta$, such that $x_{\beta_{j_k}} = x_k$.

The algebra $A = U(\mathcal{D}, \lambda)$ has a natural $\mathbb{N}$-filtration. It is given by $F_m A = \{x^{a_1}_{\beta_1} \dots x^{a_p}_{\beta_p} y \mid \sum_{i=1}^p a_i \mathrm{ht}(\beta_i)) \leqslant m\}$. In the following, we use $\mathrm{Gr}\, A$ to denote the associated $\mathbb{N}$-graded algebra.

Lemma 2.1. *The Hopf algebra $A = U(\mathcal{D}, \lambda)$ is Noetherian with finite global dimension bounded by $p + s$.*

Proof. The group algebra $\Bbbk\Gamma$ is isomorphic to a Laurent polynomial algebra with s variables. So $\Bbbk\Gamma$ is Noetherian of global dimension s. By Lemma 1.4, the algebra $\mathrm{Gr}\, A \cong U(\mathcal{D}, 0)$ is an iterated Ore extension of $\Bbbk\Gamma$. Indeed, if $x_{\beta_1}, \dots, x_{\beta_p}$ are the root vectors of A, then

$$\mathrm{Gr}\, A \cong \Bbbk\Gamma[x_{\beta_1}; \tau_1, \delta_1][x_{\beta_2}; \tau_2, \delta_2] \dots [x_{\beta_p}; \tau_p, \delta_p],$$

where τ_j, $1 \leqslant j \leqslant p$, is an algebra automorphism such that $\tau_j(x_{\beta_i})$ is just a scalar multiple of x_{β_i} for $i < j$, and δ_j is a τ_j-derivation such that $\delta_j(x_{\beta_i})$, $i < j$, is a linear combination of monomials in $x_{\beta_{i+1}}, \dots, x_{\beta_{j-1}}$. By [18], Theorems 1.2.9 and 7.5.3, we have that $\mathrm{Gr}\, A$ is Noetherian of global dimension less than $p + s$. Now it follows from [18], Theorem 1.6.9 and Corollary 7.6.18, that the algebra A is Noetherian of global dimension less than $p + s$. □

Theorem 2.2. *Let $(\mathcal{D}, \lambda)$ be a generic datum of finite Cartan and A the Hopf algebra $U(\mathcal{D}, \lambda)$. Then A is Noetherian AS-regular of global dimension $p + s$, where s is the rank of Γ and p is the number of the positive roots of the Cartan matrix. The left homological integral module $\int^l_A$ of A is isomorphic to $\Bbbk_\xi$, where $\xi\colon A \to \Bbbk$ is an algebra homomorphism defined by $\xi(g) = (\prod_{i=1}^p \chi_{\beta_i})(g)$ for all $g \in \Gamma$ and $\xi(x_i) = 0$ for all $1 \leqslant i \leqslant \theta$.*

Proof. We first show that

$$\mathrm{Ext}^i_A({}_A\Bbbk, {}_AA) \cong \begin{cases} 0, & i \neq p + s, \\ \Bbbk_\xi, & i = p + s. \end{cases}$$

With Lemma 1.5 and Lemma 2.1, the method in [8], Proposition 3.2.1, for computing the group $\mathrm{Ext}^*_{U_q(\mathfrak{g})}({}_{U_q(\mathfrak{g})}\Bbbk, {}_{U_q(\mathfrak{g})}U_q(\mathfrak{g}))$ also works in the case of $A = U(\mathcal{D}, \lambda)$. The difference is that the right A-module structure on $\mathrm{Ext}^{p+s}_A({}_A\Bbbk, {}_AA)$ is not trivial in the case of $U(\mathcal{D}, \lambda)$. Let $C = \mathbb{G}\mathrm{r}\, U(\mathcal{D}, \lambda)$. We also have $\mathrm{Ext}^i_A({}_A\Bbbk, {}_AA) = 0$ for $i \neq p + s$ and $\mathrm{Ext}^{p+s}_C({}_C\Bbbk, {}_CC) \cong \mathrm{Ext}^{p+s}_A({}_A\Bbbk, {}_AA)$ as right Γ-modules.

We now give the structure of $\mathrm{Ext}^*_C({}_C\Bbbk, {}_CC)$. Let B be the algebra

$$\Bbbk\langle x_{\beta_1}, \dots, x_{\beta_p} \mid x_{\beta_i} x_{\beta_j} = \chi_{\beta_j}(g_{\beta_i}) x_{\beta_j} x_{\beta_i},\ 1 \leqslant i < j \leqslant p\rangle.$$

Then $C = B \# \Bbbk\Gamma$. We have the isomorphisms

$$\begin{aligned}\mathrm{RHom}_C(\Bbbk, C) &\cong \mathrm{RHom}_C(\Bbbk\Gamma \otimes_{\Bbbk\Gamma} \Bbbk, C)\\ &\cong \mathrm{RHom}_{\Bbbk\Gamma}(\Bbbk, \mathrm{RHom}_C(\Bbbk\Gamma, C))\\ &\cong \mathrm{RHom}_{\Bbbk\Gamma}(\Bbbk, \Bbbk\Gamma) \otimes^L_{\Bbbk\Gamma} \mathrm{RHom}_C(\Bbbk\Gamma, C).\end{aligned}$$

Let

$$0 \to B \otimes B^{!*}_p \to \cdots \to B \otimes B^{!*}_j \to \cdots \to B \otimes B^{!*}_1 \to B \to \Bbbk \to 0 \qquad (9)$$

be the Koszul complex of B (cf. complex (7)). It is a projective resolution of $\Bbbk$. Each $B^{!*}_j$ is a left $\Bbbk\Gamma$-module defined by

$$\begin{aligned}[g(\beta)](x^*_{\beta_{i_1}} \wedge \cdots \wedge x^*_{\beta_{i_j}}) &= \beta(g^{-1}(x^*_{\beta_{i_1}} \wedge \cdots \wedge x^*_{\beta_{i_j}}))\\ &= \beta(g^{-1}(x^*_{\beta_{i_1}}) \wedge \cdots \wedge g^{-1}(x^*_{\beta_{i_j}}))\\ &= \prod_{t=1}^{j} \chi_{\beta_{i_t}}(g) \beta(x^*_{\beta_{i_1}} \wedge \cdots \wedge x^*_{\beta_{i_j}}).\end{aligned}$$

Thus, each $B \otimes B^{!*}_j$ is a $B \# \Bbbk\Gamma$-module defined by

$$(c \# g) \cdot (b \otimes \beta) = (c \# g)(b) \otimes g(\beta)$$

for any $b \otimes \beta \in B \otimes B^{!*}_j$ and $c \# g \in B \# \Bbbk\Gamma$. It is not difficult to see that the complex (9) is an exact sequence of $B \# \Bbbk\Gamma$ modules. Tensoring it with $\Bbbk\Gamma$, we obtain the following exact sequence of $B \# \Bbbk\Gamma$-modules

$$\begin{aligned}0 \to B \otimes B^{!*}_p \otimes \Bbbk\Gamma \to \dots \to B \otimes B^{!*}_j \otimes \Bbbk\Gamma \to \cdots&\\ \cdots \to B \otimes B^{!*}_1 \otimes \Bbbk\Gamma \to B \otimes \Bbbk\Gamma \to \Bbbk\Gamma \to 0,&\end{aligned}$$

where the Γ-action is diagonal. Each $B \otimes B^{!*}_j \otimes \Bbbk\Gamma$ is a free $B \# \Bbbk\Gamma$-module. Therefore, we obtain a projective resolution of $\Bbbk\Gamma$ over $B \# \Bbbk\Gamma$.

The complex

$$\begin{aligned}0 \to \mathrm{Hom}_C(B \otimes \Bbbk\Gamma, C) \to \mathrm{Hom}_C(B \otimes B^{!*}_1 \otimes \Bbbk\Gamma, C) \to \cdots&\\ \cdots \to \mathrm{Hom}_C(B \otimes B^{!*}_p \otimes \Bbbk\Gamma, C) \to 0&\end{aligned}$$

is isomorphic to the complex

$$0 \to C \to B_1^! \otimes C \to \cdots \to B_{p-1}^! \otimes C \xrightarrow{\delta_p} B_p^! \otimes C \to 0.$$

This complex is exact except at $B_p^! \otimes C$, whose cohomology is isomorphic to $B_p^! \otimes \Bbbk\Gamma$. So $\mathrm{RHom}_C(\Bbbk\Gamma, C) \cong B_p^! \otimes \Bbbk\Gamma[p]$. We have

$$(x_{\beta_1}^* \wedge \cdots \wedge x_{\beta_p}^*) \otimes g = (\textstyle\prod_{i=1}^p \chi_{\beta_i})(g) g((x_{\beta_1}^* \wedge \cdots \wedge x_{\beta_p}^*) \otimes 1)$$

for all $g \in \Gamma$. The group Γ is a free abelian group of rank s, so $\mathrm{RHom}_{\Bbbk\Gamma}(\Bbbk, \Bbbk\Gamma) \cong \Bbbk[s]$. Therefore, we obtain

$$\mathrm{RHom}_{\Bbbk\Gamma}(\Bbbk, \Bbbk\Gamma) \otimes_{\Bbbk\Gamma}^L \mathrm{RHom}_C(\Bbbk\Gamma, C) \cong \Bbbk_{\xi'}[p+s],$$

where ξ' is defined by $\xi'(g) = (\prod_{i=1}^p \chi_{\beta_i})(g)$ for all $g \in \Gamma$ and $\xi'(x_{\beta_j}) = 0$ for all $1 \leqslant j \leqslant p$. That is,

$$\mathrm{Ext}_C^i({}_C\Bbbk, {}_CC) \cong \begin{cases} 0, & i \neq p+s, \\ \Bbbk_{\xi'}, & i = p+s. \end{cases}$$

$\mathrm{Ext}_A^{p+s}({}_A\Bbbk, {}_AA)$ is a 1-dimensional right A-module. Let m be a basis of the module $\mathrm{Ext}_A^{p+s}({}_A\Bbbk, {}_AA)$. It follows from the right version of [19], Lemma 2.13 (1), that $m \cdot x_i = 0$ for all $1 \leqslant i \leqslant \theta$. Since $\mathrm{Ext}_C^{p+s}({}_C\Bbbk, {}_CC) \cong \mathrm{Ext}_A^{p+s}({}_A\Bbbk, {}_AA)$ as right Γ-modules, we have showed that

$$\mathrm{Ext}_A^i({}_A\Bbbk, {}_AA) \cong \begin{cases} 0, & i \neq p+s, \\ \Bbbk_{\xi}, & i = p+s. \end{cases}$$

Similarly, we have

$$\dim \mathrm{Ext}_A^i(\Bbbk_A, A_A) = \begin{cases} 0, & i \neq p+s, \\ 1, & i = p+s. \end{cases}$$

By Lemma 2.1, the algebra A is AS-regular of global dimension $p+s$. □

Now we can give a necessary and sufficient condition for a pointed Hopf algebra $U(\mathcal{D}, \lambda)$ to be CY.

Theorem 2.3. *Let $(\mathcal{D}, \lambda)$ be a generic datum of finite Cartan type and A the Hopf algebra $U(\mathcal{D}, \lambda)$. Let s be the rank of Γ and p the number of the positive roots of the Cartan matrix.*

(a) *The rigid dualizing complex of the Hopf algebra $A = U(\mathcal{D}, \lambda)$ is ${}_\psi A[p+s]$, where ψ is defined by $\psi(x_k) = \prod_{i=1, i\neq j_k}^p \chi_{\beta_i}(g_k) x_k$ for all $1 \leqslant k \leqslant \theta$ and $\psi(g) = (\prod_{i=1}^p \chi_{\beta_i})(g)$ for all $g \in \Gamma$, where each j_k, $1 \leqslant k \leqslant \theta$, is the integer such that $\beta_{j_k} = \alpha_k$.*

(b) *The algebra A is CY if and only if $\prod_{i=1}^p \chi_{\beta_i} = \varepsilon$ and $\mathcal{S}_A^2$ is an inner automorphism.*

Proof. (a) By [7], Proposition 4.5, and Theorem 2.2, the rigid dualizing complex of A is isomorphic to ${}_{[\xi]\mathcal{S}_A^2}A[p+s]$, where ξ is the algebra homomorphism defined in Theorem 2.2. It is not difficult to see that

$$([\xi]\mathcal{S}_A^2)(g) = (\textstyle\prod_{i=1}^p \chi_{\beta_i})(g)$$

for all $g \in \Gamma$. For $1 \leqslant k \leqslant \theta$, we have $\Delta(x_k) = x_k \otimes 1 + g_k \otimes x_k$ and $\mathcal{S}_A^2(x_k) = \chi_k(g_k^{-1})x_k$. If j_k is the integer such that $\beta_{j_k} = \alpha_k$, then $\chi_{\beta_{j_k}}(g_k) = \chi_k(g_k)$. So

$$\begin{aligned}
([\xi]\mathcal{S}_A^2)(x_k) &= \chi_k(g_k^{-1})[\xi](x_k) \\
&= \chi_k(g_k^{-1}) \prod_{i=1}^{p} \chi_{\beta_i}(g_k)(x_k) \\
&= \prod_{i=1, i\neq j_k}^{p} \chi_{\beta_i}(g_k)(x_k).
\end{aligned}$$

(b) follows from Theorem 2.2 and [12], Theorem 2.3. □

Remark 2.4. From Theorem 2.3, we can see that for a pointed Hopf algebra $U(\mathcal{D}, \lambda)$, it is CY if and only if its associated graded algebra $U(\mathcal{D}, 0)$ is CY.

Corollary 2.5. *Assume that* $A = U(\mathcal{D}, \lambda)$. *For every* A-A-*bimodule* M, *there are isomorphisms*

$$\mathrm{HH}^i(A, M) \cong \mathrm{HH}_{p+s-i}(A, {}_{\psi^{-1}}M), \quad 0 \leqslant i \leqslant p+s,$$

where ψ *is the algebra automorphism defined in Theorem 2.3.*

Proof. This follows from [7], Corollary 5.2, and Theorem 2.2. □

3. Calabi–Yau Nichols algebras of finite Cartan type

As we noted in Remark 2.4, the CY property of $U(\mathcal{D}, \lambda)$ is determined by the CY property of $U(\mathcal{D}, 0)$, which is equal to the smash product $\mathcal{B}(V)\,\#\,\Bbbk\Gamma$ of the Nichols algebra $\mathcal{B}(V)$ with the group algebra $\Bbbk\Gamma$. It is natural to ask whether or not the CY property of $U(\mathcal{D}, 0)$ depends on the CY property of $\mathcal{B}(V)$. In this section, we work out a criterion for the Nichols algebra $\mathcal{B}(V)$ to be Calabi–Yau, and answer the question in Section 4. We fix a generic datum of finite Cartan type:

$$(\mathcal{D}, 0) = (\Gamma, (a_{ij})_{1\leqslant i,j\leqslant\theta}, (q_I)_{I\in\mathcal{X}}, (g_i)_{1\leqslant i\leqslant\theta}, (\chi_i)_{1\leqslant i\leqslant\theta}, 0),$$

where Γ is a free abelian group of rank s. Let V be the generic braided vector space with basis $\{x_1, \dots, x_\theta\}$ whose braiding is given by

$$c(x_i \otimes x_j) = q_{ij} x_j \otimes x_i$$

for all $1 \leqslant i, j \leqslant \theta$, where $q_{ij} = \chi_j(g_i)$. Recall that the Nichols algebra $\mathcal{B}(V)$ is generated by x_i, $1 \leqslant i \leqslant \theta$, subject to the relations

$$\mathrm{ad}_c(x_i)^{1-a_{ij}} x_j = 0, \quad 1 \leqslant i, j \leqslant \theta,\ i \neq j,$$

where ad_c is the braided adjoint representation.

By [3], Theorem 4.3, the Nichols algebra $\mathcal{B}(V)$ is a subalgebra of $U(\mathcal{D}, 0)$, and the monomials in root vectors

$$\{x_{\beta_1}^{a_1} \dots x_{\beta_p}^{a_p} \mid a_i \geqslant 0,\ 1 \leqslant i \leqslant p\}$$

form a PBW basis of the Nichols algebra $\mathcal{B}(V)$. The degree (cf. (3)) of each PBW basis element is defined by

$$\deg(x_{\beta_1}^{a_1} \dots x_{\beta_p}^{a_p}) = (a_1, \dots, a_p, \textstyle\sum a_i \mathrm{ht}(\beta_i)) \in (\mathbb{Z}^{\geqslant 0})^{p+1},$$

where $\mathrm{ht}(\beta_i)$ is the height of β_i.

The following result is a direct consequence of Lemma 1.4.

Lemma 3.1. *In the Nichols algebra $\mathcal{B}(V)$, for $j > i$, we have*

$$[x_{\beta_i}, x_{\beta_j}]_c = \sum_{\boldsymbol{a} \in \mathbb{N}^p} \rho_{\boldsymbol{a}} x_{\beta_1}^{a_1} \dots x_{\beta_p}^{a_p},$$

where $\rho_{\boldsymbol{a}} \in \Bbbk$ and $\rho_{\boldsymbol{a}} \neq 0$ only when $\boldsymbol{a} = (a_1, \dots, a_p)$ satisfies $a_k = 0$ for $k \leqslant i$ and $k \geqslant j$.

Order the PBW basis elements by degree as in (4). By Lemma 3.1, we obtain the following corollary.

Corollary 3.2. *The Nichols algebra $\mathcal{B}(V)$ is an $\mathbb{N}^{p+1}$-filtered algebra whose associated graded algebra $\mathbb{G}\mathrm{r}\ \mathcal{B}(V)$ is isomorphic to the algebra*

$$\Bbbk\langle x_{\beta_1}, \dots, x_{\beta_p} \mid x_{\beta_i} x_{\beta_j} = \chi_{\beta_j}(g_{\beta_i}) x_{\beta_j} x_{\beta_i},\ 1 \leqslant i < j \leqslant p\rangle,$$

where $x_{\beta_1}, \dots, x_{\beta_p}$ are the root vectors of $\mathcal{B}(V)$.

For elements $\{x_{\beta_1}^{a_1} \dots x_{\beta_p}^{a_p}\}$, where $a_1, \dots, a_p \geqslant 0$, define

$$d_0(x_{\beta_1}^{a_1} \dots x_{\beta_p}^{a_p}) = \sum_{i=1}^{p} a_i \mathrm{ht}(\beta_i).$$

Then $R = \mathcal{B}(V)$ is a graded algebra with grading given by d_0. Let $R^{(0)} = R$. Define $d_1(x_{\beta_1}^{a_1} \dots x_{\beta_p}^{a_p}) = a_p$. We obtain an $\mathbb{N}$-filtration on $R^{(0)}$. Let $R^{(1)} = \mathrm{Gr}\, R^{(0)}$ be the associated graded algebra. In a similar way, we define $d_2(x_{\beta_1}^{a_1} \dots x_{\beta_p}^{a_p}) = a_{p-1}$ and let

$R^{(2)} = \mathrm{Gr}\, R^{(1)}$ be the associated graded algebra. Inductively, we obtain a sequence of $\mathbb{N}$-filtered algebras $R^{(0)}, \dots, R^{(p)}$ such that $R^{(i)} = \mathrm{Gr}\, R^{(i-1)}$ for $1 \leqslant i \leqslant p$ and $R^{(p)} = \mathbb{G}\mathrm{r}\, R$.

The algebra R^{e} has a PBW basis

$$\{x_{\beta_1}^{a_1} \dots x_{\beta_p}^{a_p} \otimes x_{\beta_p}^{b_p} \star \dots \star x_{\beta_1}^{b_1} \mid a_1, \dots, a_p, b_1, \dots, b_p \geqslant 0\},$$

where "$\star$" denotes the multiplication in R^{op}. Similarly, define a degree on each element as

$$\begin{aligned} &\deg(x_{\beta_1}^{a_1} \dots x_{\beta_p}^{a_p} \otimes x_{\beta_p}^{b_p} \star \dots \star x_{\beta_1}^{b_1}) \\ &\qquad = (a_1 + b_1, \dots, a_p + b_p, \textstyle\sum (a_i + b_i)\mathrm{ht}\beta_i) \in (\mathbb{Z}^{\geqslant 0})^{(p+1)}. \end{aligned}$$

Then R^{e} is an $\mathbb{N}^{p+1}$-filtered algebra whose associated graded algebra $\mathbb{G}\mathrm{r}(R^{\mathrm{e}})$ is isomorphic to $(\mathbb{G}\mathrm{r}\, R)^{\mathrm{e}}$.

In a similar way, we obtain a sequence of $\mathbb{N}$-filtered algebras $(R^{\mathrm{e}})^{(0)}, \dots, (R^{\mathrm{e}})^{(p)}$ such that $(R^{\mathrm{e}})^{(i)} = \mathrm{Gr}((R^{\mathrm{e}})^{(i-1)})$ for $1 \leqslant i \leqslant p$ and $(R^{\mathrm{e}})^{(p)} = \mathbb{G}\mathrm{r}\, R^{\mathrm{e}}$. In fact, $(R^{\mathrm{e}})^{(i)} = (R^{(i)})^{\mathrm{e}}$ for $0 \leqslant i \leqslant p$.

Lemma 3.3. *Let $R = \mathcal{B}(V)$ be the Nichols algebra of V. Then the algebra R^{e} is Noetherian.*

Proof. The sequence $(R^{\mathrm{e}})^{(0)}, \dots, (R^{\mathrm{e}})^{(p)}$ is a sequence of algebras, each of which is the associated graded algebra of the previous one with respect to an $\mathbb{N}$-filtration. The algebra $(R^{\mathrm{e}})^{(p)}$ is isomorphic to $(\mathbb{G}\mathrm{r}\, R)^{\mathrm{e}}$, which is Noetherian. By [18], Theorem 1.6.9, the algebra R^{e} is Noetherian. □

Lemma 3.4. *The algebra $R = \mathcal{B}(V)$ is homologically smooth.*

Proof. Since R^{e} is Noetherian by Lemma 3.3 and R is a finitely generated R^{e}-module, it is sufficient to prove that the projective dimension $\mathrm{projdim}_{R^{\mathrm{e}}} R$ is finite. The filtration on each $(R^{(i)})^{\mathrm{e}}$, $0 \leqslant i \leqslant p-1$, is bounded below. In addition, from the proof of the foregoing Lemma 3.3, each $(R^{(i)})^{\mathrm{e}}$ is Noetherian for $0 \leqslant i \leqslant p$. Therefore, $(R^{(i)})^{\mathrm{e}}$ is a Zariskian algebra for each $0 \leqslant i \leqslant p-1$. It is clear that each $R^{(i)}$, $1 \leqslant i \leqslant p-1$, viewed as an $(R^{(i)})^{\mathrm{e}}$-module has a good filtration. By [14], Corollary 5.8, we have

$$\begin{aligned} \mathrm{projdim}_{R^{\mathrm{e}}} R &= \mathrm{projdim}_{(R^{(0)})^{\mathrm{e}}} R^{(0)} \\ &\leqslant \mathrm{projdim}_{(R^{(1)})^{\mathrm{e}}} R^{(1)} \\ &\;\;\vdots \\ &\leqslant \mathrm{projdim}_{(R^{(p)})^{\mathrm{e}}} R^{(p)} = \mathrm{projdim}_{(\mathbb{G}\mathrm{r}\, R)^{\mathrm{e}}} \mathbb{G}\mathrm{r}\, R. \end{aligned}$$

The algebra $\mathbb{G}\mathrm{r}\, R$ is a quantum polynomial algebra of p variables. From the Koszul bimodule complex of $\mathbb{G}\mathrm{r}\, R$ (cf. (8)), we obtain $\mathrm{projdim}_{(\mathbb{G}\mathrm{r}\, R)^{\mathrm{e}}} \mathbb{G}\mathrm{r}\, R = p$. Therefore, $\mathrm{projdim}_{R^{\mathrm{e}}} R \leqslant p$ and R is homologically smooth. □

Proposition 3.5. *Let $R = \mathcal{B}(V)$ be the Nichols algebra of V.*

(a) *R is AS-regular of global dimension p.*

(b) *The rigid dualizing complex of R in the graded sense is isomorphic to ${}_\varphi R(l)[p]$ for some integer l and some $\mathbb{N}$-graded algebra automorphism φ of degree 0.*

(c) *The rigid dualizing complex in the ungraded sense is just ${}_\varphi R[p]$.*

Proof. Let $x_{\beta_1}, \dots, x_{\beta_p}$ be the root vectors. By Lemma 3.1, we can use a similar argument to the proof of Lemma 2.1 to show that the algebra R is an iterated graded Ore extension of $\Bbbk[x_{\beta_1}]$. Indeed,

$$R \cong \Bbbk[x_{\beta_1}][x_{\beta_2}; \tau_2, \delta_2] \dots [x_{\beta_p}; \tau_p, \delta_p],$$

where τ_j, $2 \leqslant j \leqslant p$, is an algebra automorphism such that $\tau_j(x_{\beta_i})$ is just a scalar multiple of x_{β_i} for $i < j$, and δ_j is a τ_j-derivation such that $\delta_j(x_{\beta_i})$, $i < j$, is a linear combination of monomials in $x_{\beta_{i+1}}, \dots, x_{\beta_{j-1}}$. It is well known that AS-regularity is preserved under graded Ore extension (see [30], Proposition 3.2, for instance). The algebra $\Bbbk[x_{\beta_1}]$ is an AS-regular algebra of dimension 1, so R is an AS-regular algebra of dimension p. Therefore, the rigid dualizing complex of R in the graded case is isomorphic to ${}_\varphi R(l)[p]$ for some graded algebra automorphism φ and some $l \in \mathbb{Z}$ [27]. By Lemma 3.3, R^{e} is Noetherian. Thus the rigid dualizing complex ${}_\varphi R(l)[p]$ in the graded case implies the rigid dualizing complex ${}_\varphi R[p]$ in the ungraded case. □

We claim that the automorphism φ in Proposition 3.5 is just a scalar multiplication. We need some preparations to prove this claim.

If R is a Γ-module algebra, then the algebra R^{e} is also a Γ-module algebra with the natural action $g(r \otimes s) := g(r) \otimes g(s)$ for all $g \in \Gamma$ and $r, s \in R$.

Lemma 3.6. *Let R be a Γ-module algebra such that $\Bbbk^\times$ is the group of units of R. Assume that U is an $R^{\mathrm{e}} \# \Bbbk\Gamma$-module and $U \cong R_\phi$ for an algebra automorphism ϕ, as $R^{\mathrm{e}} \# \Bbbk\Gamma$-modules. Then*

(a) *the algebra automorphism ϕ preserves the Γ-action;*

(b) *the $R^{\mathrm{e}} \# \Bbbk\Gamma$-module structures on U (up to isomorphism) are parameterized by $\operatorname{Hom}(\Gamma, \Bbbk)$, the set of group homomorphisms from Γ to $\Bbbk^\times$.*

Proof. (a) Fix an isomorphism $U \cong R_\phi$. Let $u \in U$ be the element mapped to $1 \in R$. Then $U = Ru$ and we have $g(ru) = g(r)g(u)$ for all $r \in R$ and $g \in \Gamma$. To determine the Γ-action on U, we only need to determine $g(u)$ for $g \in \Gamma$. Since $g(u) \in U$, there is some $r_g \in R$ such that $g(u) = r_g u$. On the other hand, we have

$$U = g(U) = g(Ru).$$

So there is some $s \in R$, such that $u = g(s) r_g u$. Since the element u forms an R-basis of U, the element r_g has a left inverse. Similarly, there is some $s' \in R$ such that $u = r_g ug(s')$. Since $U \cong R_\phi$ as R-R-bimodules, we have

$$\phi(r)u = ur \tag{10}$$

for any $r \in R$. So $u = r_g ug(s') = r_g \phi(g(s'))u$. Thus r_g has a right inverse as well. Consequently, r_g is a unit in R and $r_g \in \Bbbk^\times$. We have $g(h(u)) = (gh)(u)$ for $g, h \in \Gamma$, that is, $r_{gh} = r_g r_h$. Therefore, the Γ-action on U defines a group homomorphism from Γ to $\Bbbk^\times$, denoted by $\chi\colon \Gamma \to \Bbbk^\times$. Since U is an $R^{\mathrm{e}} \# \Bbbk\Gamma$-module, we have $g(rus) = g(r)g(u)g(s)$, for any $r, s \in R$ and $g \in \Gamma$. To show that ϕ preserves the Γ-action, we compute $g(\phi(r)(u))$. On one hand, we have

$$g(\phi(r)u) = g(ur) = g(u)g(r) = \chi(g)ug(r) \overset{(10)}{=} \chi(g)\phi(g(r))u.$$

On the other hand, we have

$$g(\phi(r)u) = g(\phi(r))g(u) = \chi(g)g(\phi(r))u.$$

So $g(\phi(r)) = \phi(g(r))$. That is, the automorphism ϕ preserves the Γ-action.

(b) In Part (a) we have shown that a Γ-action on U is determined by a group homomorphism from Γ to $k^\times$ such that U is an $R^{\mathrm{e}} \# \Bbbk\Gamma$-module.

Suppose there are two Γ-actions on U such that they are isomorphic. We write these two actions as $g^{\cdot 1}(u) = r_g u$ and $g^{\cdot 2}(u) = s_g u$. Denote by U_1 and U_2 the Γ-modules with these two actions respectively. Let $f : U_1 \to U_2$ be an $R^{\mathrm{e}} \# \Bbbk\Gamma$-module isomorphism. Then $f(u) = ru$ for some unit $r \in R$. Since the set of units of R is $\Bbbk^\times$, we have $r \in \Bbbk^\times$. On one hand, we have

$$f(g^{\cdot 1}(u)) = f(r_g u) = r_g ru.$$

On the other hand, we also have

$$f(g^{\cdot 1}(u)) = g^{\cdot 2}(f(u)) = g^{\cdot 2}(ru) = rg^{\cdot 2}(u) = s_g ru.$$

Therefore, $r_g = s_g$, and (b) follows. □

If U is an $R^{\mathrm{e}} \# \Bbbk\Gamma$-module, then we can define an $(R \# \Bbbk\Gamma)^{\mathrm{e}}$-module $U \# \Bbbk\Gamma$. It is isomorphic to $U \otimes \Gamma$ as vector space with bimodule structure given by

$$(r \# h)(u \otimes g) := rh(u) \otimes hg, \quad (u \otimes g)(r \# h) := ug(r) \otimes gh$$

for any $r \# h \in R \# H$ and $u \otimes g \in U \otimes \Gamma$.

Lemma 3.7. *Let R be a Γ-module algebra with $\Bbbk^\times$ being the group of units, and let U be an $R^{\mathrm{e}} \# \Bbbk\Gamma$-module. Assume that $U \cong R_\phi$ as $R^{\mathrm{e}} \# \Bbbk\Gamma$-modules, where ϕ is an algebra automorphism. If the Γ-action on U is defined by a group homomorphism $\chi\colon \Gamma \to \Bbbk^\times$, then $U \# \Bbbk\Gamma \cong (R \# \Bbbk\Gamma)_\psi$ as $(R \# \Bbbk\Gamma)^{\mathrm{e}}$-modules, where ψ is the algebra automorphism defined by $\psi(r \# g) = \chi(g^{-1})\phi(r) \# g$ for any $r \# g \in R \# \Bbbk\Gamma$.*

Proof. The homomorphism ψ defined in the lemma is clearly bijective. First we check that it is an algebra homomorphism. For any $r \# g,\ s \# h \in R \# \Bbbk\Gamma$, we have

$$\begin{aligned}\psi((r \# g)(s \# h)) &= \psi(rg(s) \# gh)\\ &= \chi(h^{-1}g^{-1})\phi(rg(s)) \# gh\\ &= \chi(h^{-1}g^{-1})\phi(r)\phi(g(s)) \# gh\\ &= \chi(h^{-1}g^{-1})\phi(r)g(\phi(s)) \# gh\\ &= (\phi(r)\chi(g^{-1}) \# g)(\phi(s)\chi(h^{-1}) \# h)\\ &= \psi(r \# g)\psi(s \# h).\end{aligned}$$

The forth equation holds since ϕ preserves the Γ-action by Lemma 3.6.

Next we show that $U \# \Bbbk\Gamma \cong (R \# \Bbbk\Gamma)_\psi$ as $(R \# \Bbbk\Gamma)^{\mathrm{e}}$-modules. Fix an isomorphism $U \cong R_\phi$ and let $u \in U$ be the element mapped to $1 \in R$. We define a homomorphism $\Phi\colon U \# \Bbbk\Gamma \to (R \# \Bbbk\Gamma)_\psi$ by $\Phi(ru \otimes g) = \chi(g^{-1})r \# g$. It is easy to see that Φ is an isomorphism of left $R \# \Bbbk\Gamma$-modules. Now we show that it is a right $R \# \Bbbk\Gamma$-module homomorphism. Indeed, we have

$$\begin{aligned}\Phi(u(r \# g)) &= \Phi(ur \otimes g)\\ &= \Phi(\phi(r)u \otimes g)\\ &= \chi(g^{-1})\phi(r) \# g\\ &= \Phi(u)\psi(r \# g)\\ &= \Phi(u) \cdot (r \# g).\end{aligned}$$ □

Now we can prove the following lemma.

Lemma 3.8. *Keep the notations as in Proposition 3.5. The actions of φ on generators $x_1, \dots, x_\theta$ are just scalar multiplications.*

Proof. By Proposition 3.5 and Lemma 1.8, we have R-R-bimodule isomorphisms

$$\operatorname{Ext}^i_{R^{\mathrm{e}}}(R, R^{\mathrm{e}}) \cong \begin{cases}0, & i \neq p,\\ R_\varphi, & i = p.\end{cases}$$

The group Γ is a free abelian group of rank s, so the algebra $\Bbbk\Gamma$ is a CY algebra of dimension s. Following from [10], Section 2, R_φ is an $R^{\mathrm{e}} \# \Bbbk\Gamma$-module and there are $(R \# \Bbbk\Gamma)^{\mathrm{e}}$-bimodule isomorphisms

$$\operatorname{Ext}^i_{(R\#\Bbbk\Gamma)^{\mathrm{e}}}(R \# \Bbbk\Gamma, (R \# \Bbbk\Gamma)^{\mathrm{e}}) \cong \begin{cases}0, & i \neq p+s,\\ (R_\varphi) \# \Bbbk\Gamma, & i = p+s.\end{cases}$$

For the sake of completeness, we sketch the proof here. By Lemma 3.4, R is homologically smooth. That is, R has a bimodule projective resolution

$$0 \to P_q \to \cdots \to P_1 \to P_0 \to R \to 0, \tag{11}$$

with each P_i being finitely generated as an R-R-bimodule.

$\mathrm{Ext}^*_{R^e}(R, R^e)$ are the cohomologies of the complex $\mathrm{Hom}_{R^e}(P_\bullet, R^e)$. The algebra R^e is an $R^e \# \Bbbk\Gamma$-module defined by

$$((c \otimes d) \# g) \cdot (a \otimes b) = g(a)d \otimes cg(b)$$

for any $a \otimes b \in R^e$ and $(c \otimes d) \# g \in R^e \# \Bbbk\Gamma$. Then each $\mathrm{Hom}_{R^e}(P_i, R^e)$ is an $R^e \# \Bbbk\Gamma$-module as well,

$$[((c \otimes d) \# g) \cdot f](x) = ((c \otimes d) \# g) \cdot f(x), \tag{12}$$

where $(c \otimes d) \# g \in R^e \# \Bbbk\Gamma$, $f \in \mathrm{Hom}_{R^e}(P_i, R^e)$ and $x \in P_i$. Now $\mathrm{Hom}_{R^e}(P_\bullet, R^e)$ is a complex of left $R^e \# \Bbbk\Gamma$-modules. Thus we obtain that $\mathrm{Ext}^p_{R^e}(R, R^e) \cong R_\varphi$ is an $R^e \# \Bbbk\Gamma$-module.

Let $A = R \# \Bbbk\Gamma$. Observe that A^e is an $R^e \# \Bbbk\Gamma$-A^e-bimodule. The left $\Bbbk\Gamma$-module action is defined by

$$g \cdot (a \# h \otimes b \# k) = g(a)gh \otimes b \# kg^{-1}$$

for any $a \# h \otimes b \# k \in A^e$ and $g \in \Gamma$. The left R^e-action and right A^e-action are given by multiplication. Let W be the vector space $\Bbbk\Gamma \otimes \Bbbk\Gamma$. Then $R^e \otimes W$ is also an $R^e \# \Bbbk\Gamma$-A^e-bimodule defined by

$$((c \otimes d) \# g) \cdot (a \otimes b \otimes h \otimes k) = cg(a) \otimes g(b)d \otimes gh \otimes kg^{-1}$$

and

$$(a \otimes b \otimes h \otimes k) \cdot (c \# h' \otimes d \# k') = ah(c) \otimes ((k^{-1}k'^{-1})d)b \otimes hh' \otimes k'k.$$

It is not difficult to see that the morphism $f : A^e \to R^e \otimes W$ defined by

$$f(a \# h \otimes b \# k) = a \otimes k^{-1}(b) \otimes h \otimes k$$

is an isomorphism of $R^e \# \Bbbk\Gamma$-A^e-bimodules.

Let P be a finitely generated projective R^e-module. The $\Bbbk\Gamma$-A^e-bimodule structure of $R^e \otimes W$ induces a $\Bbbk\Gamma$-A^e-bimodule structure on $\mathrm{Hom}_{R^e}(P, R^e \otimes W)$. We define a $\Bbbk\Gamma$-A^e-bimodule structure on $\mathrm{Hom}_{R^e}(P, R^e) \otimes W$ by

$$g \cdot (f \otimes h \otimes k) = g \cdot f \otimes gh \otimes kg^{-1}$$

and

$$(f \otimes h \otimes k) \cdot (c \# h' \otimes d \# k') = (h(c) \otimes (k^{-1}k'^{-1})d) \cdot f \otimes hh' \otimes k'k,$$

where the $R^e \# \Bbbk\Gamma$-module structure on $\mathrm{Hom}_{R^e}(P, R^e)$ is defined in (12). Now the canonical isomorphism from $\mathrm{Hom}_{R^e}(P, R^e) \otimes W$ to $\mathrm{Hom}_{R^e}(P, R^e \otimes W)$ is a $\Bbbk\Gamma$-A^e-bimodule isomorphism.

Since R admits a resolution like (11) with each P_i finitely generated, we have

$$\mathrm{Ext}^i_{R^e}(R, R^e \otimes W) \cong \mathrm{Ext}^i_{R^e}(R, R^e) \otimes W$$

as $\Bbbk\Gamma$-A^{e}-bimodules for all $i \geqslant 0$. On the other hand, we have Stefan's spectral sequence [23]:

$$\mathrm{Ext}^m_{\Bbbk\Gamma}(\Bbbk, \mathrm{Ext}^n_{R^{\mathrm{e}}}(R, A^{\mathrm{e}})) \Rightarrow \mathrm{Ext}^{m+n}_{A^{\mathrm{e}}}(A, A^{\mathrm{e}}).$$

Thus for $m, n \geqslant 0$ we have

$$\begin{aligned}\mathrm{Ext}^m_{\Bbbk\Gamma}(\Bbbk, \mathrm{Ext}^n_{R^{\mathrm{e}}}(R, A^{\mathrm{e}})) &\cong \mathrm{Ext}^m_{\Bbbk\Gamma}(\Bbbk, \mathrm{Ext}^n_{R^{\mathrm{e}}}(R, R^{\mathrm{e}} \otimes W)) \\ &\cong \mathrm{Ext}^m_{\Bbbk\Gamma}(\Bbbk, \mathrm{Ext}^n_{R^{\mathrm{e}}}(R, R^{\mathrm{e}}) \otimes W).\end{aligned}$$

Hence, $\mathrm{Ext}^m_{\Bbbk\Gamma}(\Bbbk, \mathrm{Ext}^n_{R^{\mathrm{e}}}(R, A^{\mathrm{e}})) = 0$ except that $m = s$ and $n = p$. Therefore,

$$\mathrm{Ext}^i_{(R\#\Bbbk\Gamma)^{\mathrm{e}}}(R \# \Bbbk\Gamma, (R \# \Bbbk\Gamma)^{\mathrm{e}}) = 0$$

for $i \neq p + s$ and

$$\mathrm{Ext}^{p+s}_{A^{\mathrm{e}}}(A, A^{\mathrm{e}}) \cong \mathrm{Ext}^s_{\Bbbk\Gamma}(\Bbbk, \mathrm{Ext}^p_{R^{\mathrm{e}}}(R, A^{\mathrm{e}})).$$

Let M be a left $\Bbbk\Gamma$-module. One can consider it as a $\Bbbk\Gamma$-$\Bbbk\Gamma$-bimodule M_ε with the trivial right $\Bbbk\Gamma$-module action. The algebra $\Bbbk\Gamma$ is a CY algebra of dimension s. From Van den Bergh's duality theorem ([22], Theorem 1) we obtain the following isomorphisms:

$$\mathrm{Ext}^s_{\Bbbk\Gamma}(\Bbbk, M) \cong \mathrm{HH}^s(\Bbbk\Gamma, M_\varepsilon) \cong \mathrm{HH}_0(\Bbbk\Gamma, M_\varepsilon) \cong \mathrm{Tor}^{\Bbbk\Gamma}_0(\Bbbk, M).$$

Now we have the following isomorphisms of right A^{e}-modules:

$$\begin{aligned}\mathrm{Ext}^{p+s}_{A^{\mathrm{e}}}(A, A^{\mathrm{e}}) &\cong \mathrm{Ext}^s_{\Bbbk\Gamma}(\Bbbk, \mathrm{Ext}^p_{R^{\mathrm{e}}}(R, A^{\mathrm{e}})) \\ &\cong \mathrm{Ext}^s_{\Bbbk\Gamma}(\Bbbk, \mathrm{Ext}^p_{R^{\mathrm{e}}}(R, R^{\mathrm{e}}) \otimes W) \\ &\cong \mathrm{Ext}^s_{\Bbbk\Gamma}(\Bbbk, R_\varphi \otimes W) \\ &\cong \mathrm{Tor}^{\Bbbk\Gamma}_0(\Bbbk, R_\varphi \otimes W) \\ &\cong \Bbbk \otimes_{\Bbbk\Gamma} R_\varphi \otimes W.\end{aligned}$$

If we look at the $\Bbbk\Gamma$-A^{e}-bimodule structure on $R_\varphi \otimes W$ carefully, we obtain

$$\Bbbk \otimes_{\Bbbk\Gamma} R_\varphi \otimes W \cong R_\varphi \# \Bbbk\Gamma$$

as right A^{e}-modules.

Since the connected graded algebra R is a domain by Lemma 1.3, the group of units of R is $\Bbbk^\times$. Following Lemma 3.6 and 3.7, we have $(R_\varphi) \# \Bbbk\Gamma \cong (R \# \Bbbk\Gamma)_{\bar{\psi}}$, where $\bar{\psi}$ is the algebra automorphism defined by $\bar{\psi}(r \# g) = \varphi(r)\chi(g^{-1})$ for some algebra homomorphism $\chi\colon \Gamma \to \Bbbk$.

On the other hand, since $A = R \# \Bbbk\Gamma \cong U(\mathcal{D}, 0)$, we have A-A-bimodule isomorphisms

$$\mathrm{Ext}^i_{A^{\mathrm{e}}}(A, A^{\mathrm{e}}) \cong \begin{cases} 0, & i \neq p + s, \\ A_\psi, & i = p + s, \end{cases}$$

where ψ is the algebra automorphism defined in Theorem 2.3.

Therefore, we obtain an A-A-bimodule isomorphism $A_{\bar{\psi}} \cong A_{\psi}$. That is, $\bar{\psi}$ and ψ differ only by an inner automorphism. By Lemma 1.3, the graded algebra A is a domain; and the invertible elements of A are in $\Bbbk\Gamma$. The actions of ψ and the group actions on generators $x_1, \dots, x_\theta$ are just scalar multiplications. Thus the actions of $\bar{\psi}$ on $x_1, \dots, x_\theta$ are also scalar multiplications. Since $\bar{\psi}(x_i) = \varphi(x_i)$ for all $1 \leqslant i \leqslant \theta$, we obtain the desired result. □

Now we are ready to prove the main theorem of this section.

Theorem 3.9. *Let V be a generic braided vector space of finite Cartan type, and $R = \mathcal{B}(V)$ the Nichols algebra of V. Let p be the number of the positive roots of the Cartan matrix. For each $1 \leqslant k \leqslant \theta$, let j_k be the integer such that $\beta_{j_k} = \alpha_k$.*

(a) *The rigid dualizing complex is isomorphic to ${}_{\varphi}R[p]$, where φ is the algebra automorphism defined by*

$$\varphi(x_k) = (\prod_{i=1}^{j_k-1} \chi_k^{-1}(g_{\beta_i}))(\prod_{i=j_k+1}^{p} \chi_{\beta_i}(g_k))x_k$$

for all $1 \leqslant k \leqslant \theta$.

(b) *The algebra R is a CY algebra if and only if*

$$\prod_{i=1}^{j_k-1} \chi_k(g_{\beta_i}) = \prod_{i=j_k+1}^{p} \chi_{\beta_i}(g_k)$$

for all $1 \leqslant k \leqslant \theta$.

Proof. (a) Note that $\mathbb{G}\mathrm{r}\ R$ is isomorphic to the following quantum polynomial algebra:

$$\Bbbk\langle x_{\beta_1}, \dots, x_{\beta_p} \mid x_{\beta_i} x_{\beta_j} = \chi_{\beta_j}(g_{\beta_i}) x_{\beta_j} x_{\beta_i},\ 1 \leqslant i < j \leqslant p \rangle.$$

By [21], Proposition 8.2 and Theorem 9.2, $\mathbb{G}\mathrm{r}\ R$ has a rigid dualizing complex ${}_{\bar{\xi}}\mathbb{G}\mathrm{r}\ R[p] (\cong \mathbb{G}\mathrm{r}\ R_{\bar{\xi}^{-1}}[p])$, where $\bar{\xi}$ is defined by

$$\bar{\xi}(x_{\beta_k}) = \chi_{\beta_k}^{-1}(g_{\beta_1}) \dots \chi_k^{-1}(g_{\beta_{k-1}}) \chi_{\beta_{k+1}}(g_{\beta_k}) \dots \chi_{\beta_p}(g_{\beta_k}) x_{\beta_k}$$

for all $1 \leqslant k \leqslant p$.

On the other hand, it follows from Proposition 3.5 and Lemma 3.8 that R has a rigid dualizing complex ${}_{\varphi}R$, where φ is an algebra automorphism such that for each $1 \leqslant k \leqslant \theta$, $\varphi(x_k)$ is a scalar multiple of x_k. Assume that $\varphi(x_k) = l_k x_k$, with $l_k \in \Bbbk$.

Let $R^{(0)}, \dots, R^{(p)}$ be the sequence of algebras defined after Corollary 3.2. By Lemma 3.1, and applying a similar argument to the one in the proof of Proposition 3.5, we obtain that each $R^{(i)}$, $0 \leqslant i \leqslant p$, is an iterated Ore extension of the polynomial algebra $\Bbbk[x]$. Thus each of them is AS-regular. It follows from [28], Proposition 1.1, that each $R^{(i)}$, $1 \leqslant i \leqslant p$, has a rigid dualizing complex ${}_{\varphi^{(i)}}(R^{(i)})[p]$, where $\varphi^{(i)} = \operatorname{Gr}\varphi^{(i-1)}$ and $\varphi^{(0)} = \varphi$. Since for each $1 \leqslant k \leqslant \theta$, $\varphi(x_k) = l_k x_k$, we have $\varphi^{(p)}(x_k) = l_k x_k$. Because $R^{(p)} = \mathbb{G}\mathrm{r}\, R$, there is a bimodule isomorphism ${}_{\varphi^{(p)}}(R^{(p)}) \cong {}_{\bar{\xi}}(\mathbb{G}\mathrm{r}\, R)$. We obtain $\varphi^{(p)} = \bar{\xi}$, as R is connected. Therefore, for each $1 \leqslant k \leqslant \theta$,

$$l_k x_k = \bar{\xi}(x_k) = (\prod_{i=1}^{j_k-1} \chi_k^{-1}(g_{\beta_i}))(\prod_{i=j_k+1}^{p} \chi_{\beta_i}(g_k))x_k,$$

where j_k is the integer such that $\beta_{j_k} = \alpha_k$.

Now we can conclude that $\varphi(x_k) = (\prod_{i=1}^{j_k-1} \chi_k^{-1}(g_{\beta_i}))(\prod_{i=j_k+1}^{p} \chi_{\beta_i}(g_k))x_k$, for each $1 \leqslant k \leqslant \theta$.

(b) The algebra R is homologically smooth by Lemma 3.4. It follows from Corollary 1.9 that R is CY if and only if $R \cong {}_{\varphi}R$ as bimodules. That is, R is CY if and only if $\varphi = \mathrm{id}$. Hence (b) follows from (a). □

Example 3.10. Let $(\mathcal{D}, \lambda) = (\Gamma, (a_{ij}), (q_I), (g_i), (\chi_i), 0)$ be a generic datum such that the Cartan matrix is of type A_2. This defines a braided vector space V. Let $\{x_1, x_2\}$ be a basis of V. The braiding of V is given by

$$c(x_i \otimes x_j) = \chi_j(g_i)x_j \otimes x_i, \quad i, j = 1, 2.$$

The Nichols algebra $R = \mathcal{B}(V)$ of V is generated by x_1 and x_2 subject to the relations

$$x_1^2x_2 - q_{12}x_1x_2x_1 - q_{11}q_{12}x_1x_2x_1 + q_{11}q_{12}^2x_2x_1^2 = 0,$$
$$x_2^2x_1 - q_{21}x_2x_1x_2 - q_{22}q_{21}x_2x_1x_2 + q_{22}q_{21}^2x_1x_2^2 = 0,$$

where $q_{ij} = \chi_j(g_i)$. The element $s_1s_2s_1$ is the longest element in the Weyl group $\mathcal{W}$. Let α_1 and α_2 be the two simple roots. Then the positive roots are

$$\beta_1 = \alpha_1, \quad \beta_2 = \alpha_1 + \alpha_2, \quad \beta_3 = \alpha_2.$$

By Theorem 3.9, the algebra R is CY if and only if

$$\chi_{\beta_2}(g_1)\chi_{\beta_3}(g_1) = (\chi_1\chi_2^2)(g_1) = 1$$

and

$$\chi_2(g_{\beta_1})\chi_2(g_{\beta_2}) = \chi_2(g_1^2g_2) = 1.$$

That is, $q_{11}q_{12}^2 = q_{22}q_{12}^2 = 1$. By equation (2), we have $q_{11}^{-1} = q_{22}^{-1} = q_{12}q_{21}$.

Now we conclude that the algebra R is CY if and only if there is some $q \in \Bbbk^\times$, which is not a root of unity and satisfies the following relations

$$q_{11} = q_{22} = q^2 \quad \text{and} \quad q_{12} = q_{21} = q^{-1}.$$

In other words, the braiding is of DJ-type. Then the algebra R is an AS-regular algebra of type A (see [4] for terminology). This coincides with Proposition 5.4 in [5].

Example 3.11. Let R be a Nichols algebra of type B_2. That is, R is generated by x_1 and x_2 subject to the relations

$$x_1^3x_2 - q_{12}x_1^2x_2x_1 - q_{11}q_{12}x_1^2x_2x_1 + q_{11}q_{12}^2x_1x_2x_1^2 \\ -q_{11}^2q_{12}(x_1^2x_2x_1 - q_{12}x_1x_2x_1^2 - q_{11}q_{12}x_1x_2x_1^2 + q_{11}q_{12}^2x_2x_1^3) = 0$$

and

$$x_2^2x_1 - q_{21}x_2x_1x_2 - q_{22}q_{21}x_2x_1x_2 + q_{22}q_{21}^2x_1x_2^2 = 0,$$

where $q_{ij} \in \Bbbk$ for $1 \leqslant i, j \leqslant 2$ and $q_{12}q_{21} = q_{11}^{-2} = q_{22}^{-1}$. Applying a similar argument, we obtain that R is CY if and only if there is some $q \in \Bbbk^\times$, which is not a root of unity and satisfies

$$q_{11} = q, \quad q_{12} = q^{-1}, \quad q_{21} = q^{-1} \quad \text{and} \quad q_{22} = q^2.$$

4. Relation between the Calabi–Yau property of pointed Hopf algebras and Nichols algebras

We keep the notations as in the previous section. Let $(\mathcal{D}, \lambda)$ be a generic datum of finite Cartan type. In this section, we discuss the relation between the CY property of the algebra $U(\mathcal{D}, \lambda)$ and that of the corresponding Nichols algebra $\mathcal{B}(V)$. It turns out that if one of them is CY, then the other one is not.

Lemma 4.1. *For each* $1 \leqslant k \leqslant \theta$*, we have*

$$\prod_{i=1, i\neq j_k}^{p} \chi_{\beta_i}(g_k) = \Big(\prod_{i=1}^{j_k-1} \chi_k^{-1}(g_{\beta_i})\Big)\Big(\prod_{i=j_k+1}^{p} \chi_{\beta_i}(g_k)\Big).$$

Proof. Let $\omega_0 = s_{i_1} \dots s_{i_p}$ be the fixed reduced decomposition of the longest element ω_0 in the Weyl group. It is clear that ω_0^{-1} is also of maximal length. By Lemma 3.11 in [13], for each $1 \leqslant k \leqslant \theta$, there exists $1 \leqslant t \leqslant p$ such that

$$s_k s_{i_1} \dots s_{i_{t-1}} = s_{i_1} \dots s_{i_t}.$$

That is, $\omega_0 = s_k s_{i_1} \dots s_{i_{t-1}} s_{i_{t+1}} \dots s_{i_p}$. Set

$$\beta_1' = \alpha_k,\ \beta_2' = s_k(\alpha_{i_1}),\ \dots,\ \beta_p' = s_k s_{i_1} \dots s_{i_{t-1}} s_{i_{t+1}} \dots s_{i_{p-1}}(\alpha_{i_p}).$$

Applying a similar argument to the one in the proof of Theorem 3.9, we conclude that the rigid dualizing complex of the algebra $R = \mathcal{B}(V)$ is isomorphic to ${}_{\varphi'}R[p]$. The algebra automorphism φ' is defined by

$$\varphi'(x_l) = \Big(\prod_{i=1}^{j_l'-1} \chi_l^{-1}(g_{\beta_i'})\Big)\Big(\prod_{i=j_l'+1}^{p} \chi_{\beta_i'}(g_l)\Big)x_l$$

for each $1 \leqslant l \leqslant \theta$, where j_l', $1 \leqslant l \leqslant \theta$, are the integers such that $\beta'_{j_l'} = \alpha_l$. In particular, we have

$$\varphi'(x_k) = (\prod_{i=2}^{p} \chi_{\beta_i'}(g_k))x_k.$$

The rigid dualizing complex is unique up to isomorphism, so ${}_{\varphi'}R \cong {}_{\varphi}R$ as R-R-bimodules, where φ is the algebra automorphism defined in Theorem 3.9. Since the graded algebra R is connected, we have $\varphi' = \varphi$. In particular, $\varphi'(x_k) = \varphi(x_k)$, that is,

$$\prod_{i=2}^{p} \chi_{\beta_i'}(g_k) = (\prod_{i=1}^{j_k-1} \chi_k^{-1}(g_{\beta_i}))(\prod_{i=j_k+1}^{p} \chi_{\beta_i}(g_k)).$$

Both $\beta_1, \dots, \beta_p$ and $\beta_1', \dots, \beta_p'$ are enumerations of positive roots. We have $\alpha_k = \beta_1' = \beta_{j_k}$. Therefore,

$$\prod_{i=2}^{p} \chi_{\beta_i'}(g_k) = \prod_{i=1, i\neq j_k}^{p} \chi_{\beta_i}(g_k).$$

It follows that

$$(\prod_{i=1}^{j_k-1} \chi_k^{-1}(g_{\beta_i}))(\prod_{i=j_k+1}^{p} \chi_{\beta_i}(g_k)) = \prod_{i=1, i\neq j_k}^{p} \chi_{\beta_i}(g_k). \qquad \square$$

Proposition 4.2. *If $A = U(\mathcal{D}, \lambda)$ is a CY algebra, then the rigid dualizing complex of the Nichols algebra $R = \mathcal{B}(V)$ is isomorphic to ${}_{\varphi}R[p]$, where φ is defined by $\varphi(x_k) = \chi_k^{-1}(g_k)x_k$, for all $1 \leqslant k \leqslant \theta$.*

Proof. By Theorem 3.9, the rigid dualizing complex of R is isomorphic to ${}_{\varphi}R[p]$, where φ is defined by

$$\varphi(x_k) = (\prod_{i=1}^{j_k-1} \chi_k^{-1}(g_{\beta_i}))(\prod_{i=j_k+1}^{p} \chi_{\beta_i}(g_k))x_k$$

for all $1 \leqslant k \leqslant \theta$. If A is a CY algebra, then $\prod_{i=1}^{p} \chi_{\beta_i} = \varepsilon$ by Theorem 2.3. Therefore, for $1 \leqslant k \leqslant \theta$,

$$(\prod_{i=1}^{j_k-1} \chi_k^{-1}(g_{\beta_i}))(\prod_{i=j_k+1}^{p} \chi_{\beta_i}(g_k)) = \prod_{i=1, i\neq j_k}^{p} \chi_{\beta_i}(g_k) = \chi_k^{-1}(g_k),$$

where the first equation follows from Lemma 4.1. Now $\varphi(x_k) = \chi_k^{-1}(g_k)x_k$ for all $1 \leqslant k \leqslant \theta$. $\square$

Note that $\chi_k(g_k) \neq 1$ for all $1 \leqslant k \leqslant \theta$. So the algebra $R = \mathcal{B}(V)$ is not CY if $A = U(\mathcal{D}, \lambda)$ is a CY algebra.

Proposition 4.3. *If the Nichols algebra $R = \mathcal{B}(V)$ is a CY algebra, then the rigid dualizing complex of $A = U(\mathcal{D}, \lambda)$ is isomorphic to ${}_{\psi}A[p+s]$, where ψ is defined by $\psi(x_k) = x_k$ for all $1 \leqslant k \leqslant \theta$ and $\psi(g) = \prod_{i=1}^{p} \chi_{\beta_i}(g)$ for all $g \in \Gamma$.*

Proof. If the Nichols algebra R is CY, then by Theorem 3.9 and Lemma 4.1 we have

$$\prod_{i=1, i\neq j_k}^{p} \chi_{\beta_i}(g_k) = \Big(\prod_{i=1}^{j_k-1} \chi_k^{-1}(g_{\beta_i})\Big)\Big(\prod_{i=j_k+1}^{p} \chi_{\beta_i}(g_k)\Big) = 1$$

for each $1 \leqslant k \leqslant \theta$. Now the statement follows from Theorem 2.3. □

With the assumption of Proposition 4.3, for all $1 \leqslant k \leqslant \theta$, we have

$$\psi(g_k) = \prod_{i=1}^{p} \chi_{\beta_i}(g_k) = \chi_k(g_k) g_k \neq g_k.$$

Since the invertible elements of A are in $\Bbbk\Gamma$ and Γ is an abelian group, ψ cannot be an inner automorphism. So the algebra A is not CY.

Example 4.4. Let R be the algebra in Example 3.10. Assume that $\Gamma = \langle y_1, y_2 \rangle \cong \mathbb{Z}^2$ and $g_i = y_i$, $i = 1, 2$. The characters χ_1 and χ_2 are given by the following table.

	y_1	y_2
χ_1	q^2	q^{-1}
χ_2	q^{-1}	q^2

Here q is not a root of unity.

The algebra R is a CY algebra. But the algebra $A = R \# \Bbbk\Gamma$ is not. The rigid dualizing complex of A is isomorphic to ${}_{\psi}A[5]$, where ψ is defined by $\psi(x_i) = x_i$ and $\psi(y_i) = q^2 y_i$ for $i = 1, 2$.

Example 4.5. Let A be an algebra with generators $y_1^{\pm 1}$, $y_2^{\pm 1}$, x_1 and x_2 subject to the relations

$$\begin{gathered} y_h^{\pm 1} y_h^{\mp 1} = 1, \quad 1 \leqslant h, m \leqslant 2, \\ y_1 x_1 = q x_1 y_1, \quad y_1 x_2 = q^{-1} x_2 y_1, \\ y_2 x_1 = q^{\frac{k}{l}} x_1 y_2, \quad y_2 x_2 = q^{-\frac{k}{l}} x_2 y_2, \\ x_1 x_2 - q^{-k} x_2 x_1 = 1 - y_1^k y_2^l, \end{gathered}$$

where $k, l \in \mathbb{Z}^+$ and $q \in \Bbbk$ is not a root of unity.

By Theorem 2.3, the algebra A is a CY algebra of dimension 4. Let R be the corresponding Nichols algebra of A. The rigid dualizing complex of R is isomorphic to ${}_{\varphi}R[2]$, where φ is defined by $\varphi(x_1) = q^{-k} x_1$ and $\varphi(x_2) = q^k x_2$.

5. Classification of Calabi–Yau pointed Hopf algebra $U(\mathcal{D},\lambda)$ of lower dimensions

In this section we assume that $\Bbbk = \mathbb{C}$. We shall classify CY pointed Hopf algebras $U(\mathcal{D},\lambda)$ of dimension less than 5, where $(\mathcal{D},\lambda)$ is a generic datum of finite Cartan type. In a generic datum $(\Gamma, (a_{ij}), (q_I), (g_i), (\chi_i), (\lambda_{ij}))$ of finite Cartan type, (q_I) are determined by (χ_i) and (g_i). In the following, we will omit (q_I) for simplicity.

Let $(\mathcal{D},\lambda) = (\Gamma, (a_{ij}), (g_i), (\chi_i), (\lambda_{ij}))$ be a generic datum of finite Cartan type. Then $\chi_i(g_i)$ are not roots of unity for $1 \leqslant i \leqslant \theta$. Hence, in the classification, we exclude the case where the group is trivial. If the group Γ in a datum $(\mathcal{D},\lambda) = (\Gamma, (a_{ij}), (g_i), (\chi_i), (\lambda_{ij}))$ is trivial, then the algebra $U(\mathcal{D},0)$ (in this case, $U(\mathcal{D},0)$ has no non-trivial lifting) is the universal enveloping algebra $U(\mathfrak{g})$, where the Lie algebra $\mathfrak{g}$ is generated by x_i, $1 \leqslant i \leqslant \theta$, subject to the relations

$$(\operatorname{ad} x_i)^{1-a_{ij}} x_j = 0, \quad 1 \leqslant i, j \leqslant \theta,\ i \neq j.$$

We have $\operatorname{tr}(\operatorname{ad} x) = 0$ for all $x \in \mathfrak{g}$. Therefore, $U(\mathfrak{g})$ is CY by [12], Lemma 4.1. We list those of dimension less than 5 in the following table.

Case	CY dimension	Cartan matrix	Lie algebra	
			basis	relations
1	1	A_1	x	
2	2	$A_1 \times A_1$	x, y	abelian Lie algebra
3	3	$A_1 \times A_1 \times A_1$	x, y, z	abelian Lie algebra
4	3	A_2	x, y, z	$[x,y] = z, [x,z] = [y,z] = 0$
5	4	$A_1 \times \cdots \times A_1$	x, y, z, w	abelian Lie algebra
6	4	$A_1 \times A_2$	x, y, z, w	$[x,y] = z, [x,z] = [y,z] = 0,$ $[x,w] = [y,w] = [z,w] = 0$
7	4	B_2	x, y, z, w	$[x,y] = z, [x,z] = w,$ $[x,w] = [y,z] = [y,w] = [z,w] = 0$

Remark 5.1. The Lie algebra in case 4 is the Heisenberg algebra. In [12], the authors classified those 3-dimensional Lie algebras whose universal enveloping algebras are CY algebras. Beside the algebras in case 3 and case 4, the other two Lie algebras are

- The 3-dimensional simple Lie algebra $\mathfrak{sl}_2$;
- The Lie algebra $\mathfrak{g}$, where $\mathfrak{g}$ has a basis $\{x, y, z\}$ such that $[x,y] = y$, $[x,z] = -z$ and $[y,z] = 0$.

Now let

$$(\mathcal{D},\lambda) = (\Gamma, (a_{ij})_{1\leqslant i,j\leqslant\theta}, (g_i)_{1\leqslant i\leqslant\theta}, (\chi_i)_{1\leqslant i\leqslant\theta}, (\lambda_{ij})_{1\leqslant i<j\leqslant\theta, i\nsim j})$$

and

$$(\mathcal{D}',\lambda')=(\Gamma',(a'_{ij})_{1\leqslant i,j\leqslant\theta'},(g'_i)_{1\leqslant i\leqslant\theta'},,(\chi'_i)_{1\leqslant i\leqslant\theta'},(\lambda'_{ij})_{1\leqslant i<j\leqslant\theta',i\nsim j})$$

be two generic data of finite Cartan type for groups Γ and Γ', where Γ and Γ' are both free abelian groups of finite rank.

The data $(\mathcal{D},\lambda)$ and $(\mathcal{D}',\lambda')$ are said to be *isomorphic* if $\theta=\theta'$ and if there exist a group isomorphism $\varphi\colon\Gamma\to\Gamma'$, a permutation $\sigma\in\mathbb{S}_\theta$, and elements $0\neq\alpha_i\in\Bbbk$ for all $1\leqslant i\leqslant\theta$ subject to the relations

$$\begin{aligned}\varphi(g_i)&=g'_{\sigma(i)}\quad\text{for all } 1\leqslant i\leqslant\theta,\\ \chi_i&=\chi'_{\sigma(i)}\varphi\quad\text{for all } 1\leqslant i\leqslant\theta,\\ \lambda_{ij}&=\begin{cases}\alpha_i\alpha_j\lambda'_{\sigma(i)\sigma(j)} & \text{if } \sigma(i)<\sigma(j),\\ -\alpha_i\alpha_j\chi_j(g_i)\lambda'_{\sigma(j)\sigma(i)} & \text{if } \sigma(i)>\sigma(j),\end{cases}\end{aligned}$$

for all $1\leqslant i<j\leqslant\theta$ and $i\nsim j$. In this case the triple $(\varphi,\sigma,(\alpha_i))$ is called an *isomorphism* from $(\mathcal{D},\lambda)$ to $(\mathcal{D}',\lambda')$.

If $(\mathcal{D},\lambda)$ and $(\mathcal{D}',\lambda)$ are isomorphic, then we can deduce that $a_{ij}=a'_{\sigma(i)\sigma(j)}$ for all $1\leqslant i,j\leqslant\theta$ [3].

The following corollary can be immediately obtained from the definition of isomorphic data.

Corollary 5.2. *Suppose that $(\mathcal{D},0)$ is a generic datum of finite Cartan type formed by $(\Gamma,(a_{ij}),(g_i),(\chi_i),0)$. Assume that $\varphi\colon\Gamma\to\Gamma'$ is a group isomorphism and σ is a permutation in $\mathbb{S}_\theta$. Then $(\mathcal{D},0)$ is isomorphic to $(\mathcal{D}',0)$, where $\mathcal{D}'$ is formed by $(\Gamma',(a_{\sigma^{-1}(i)\sigma^{-1}(j)}),(\varphi(g_{\sigma^{-1}(i)})),(\chi_{\sigma^{-1}(i)}\varphi^{-1}))$.*

Let $(\mathcal{D},\lambda)$ be a generic datum of finite Cartan type. By [3], the pointed Hopf algebra $U(\mathcal{D},\lambda)$ is uniquely determined by the datum $(\mathcal{D},\lambda)$. Let $\mathrm{Isom}((\mathcal{D},\lambda),(\mathcal{D}',\lambda'))$ be the set of all isomorphisms from $(\mathcal{D},\lambda)$ to $(\mathcal{D}',\lambda')$. For A, B two Hopf algebras, we denote by $\mathrm{Isom}(A,B)$ the set of all Hopf algebra isomorphisms from A to B.

Lemma 5.3 ([3], Theorem 4.5). *Let $(\mathcal{D},\lambda)$ and $(\mathcal{D}',\lambda')$ be two generic data of finite Cartan type. Then the Hopf algebras $U(\mathcal{D},\lambda)$ and $U(\mathcal{D}',\lambda')$ are isomorphic if and only if $(\mathcal{D},\lambda)$ is isomorphic to $(\mathcal{D}',\lambda')$. More precisely, let $x_1,\dots,x_\theta$ (resp. $x'_1,\dots,x'_\theta$) be the simple root vectors in $U(\mathcal{D},\lambda)$ (resp. $U(\mathcal{D}',\lambda')$), and let $g_1,\dots,g_\theta$ (resp. $g'_1,\dots,g'_\theta$) be the group-like elements in $\mathcal{D}$ (resp. $\mathcal{D}'$). Then the map*

$$\mathrm{Isom}(U(\mathcal{D},\lambda),U(\mathcal{D}',\lambda'))\to\mathrm{Isom}((\mathcal{D},\lambda),(\mathcal{D}',\lambda')),$$

given by $\phi\mapsto(\varphi,\sigma,(\alpha_i))$ where $\varphi(g)=\phi(g)$, $\varphi(g_i)=g'_{\sigma(i)}$, $\phi(x_i)=\alpha_i x'_{\sigma(i)}$ for all $g\in\Gamma$, $1\leqslant i\leqslant\theta$, is bijective.

The following lemma is well known.

Lemma 5.4. *If Γ is a free abelian group of rank s, then the algebra $\Bbbk\Gamma$ is a CY algebra of dimension s.*

If Γ is a free abelian group of finite rank, we denote by $|\Gamma|$ the rank of Γ.

Proposition 5.5. *Let A be the algebra $U(\mathcal{D}, \lambda)$, where $(\mathcal{D}, \lambda)$ is a generic datum of finite Cartan type for a group Γ. Then*

(a) *A is CY of dimension 1 if and only if $A = \Bbbk\mathbb{Z}$,*

(b) *A is CY of dimension 2 if and only if $A = \Bbbk\Gamma$, where Γ is a free abelian group of rank 2.*

Proof. (a) is clear.

(b) It is sufficient to show that if A is CY of dimension 2, then A is the group algebra of a free abelian group of rank 2. By Theorem 2.2, if the global dimension of A is 2. Then the following possibilities arise:

(i) $|\Gamma| = 2$, $A = \Bbbk\Gamma$ is the group algebra of a free abelian group of rank 2;

(ii) $|\Gamma| = 1$ and the Cartan matrix of A is of type A_1.

Let A be a pointed Hopf algebra of type (ii) and let the datum

$$(\mathcal{D}, \lambda) = (\Gamma, (g_i), (\chi_i), (a_{ij}), (\lambda_{ij}))$$

be as follows:

- $\Gamma = \langle y_1 \rangle \cong \mathbb{Z}$;
- $g_1 = y_1^k$ for some $k \in \mathbb{Z}$;
- $\chi_1 \in \widehat{\Gamma}$ is defined by $\chi_1(y_1) = q$, where q is not a root of unity;
- the Cartan matrix is of type A_1;
- $\lambda = 0$.

Observe that in this case, the linking parameter must be 0. In addition, there is only one root vector, that is, the simple root vector x_1. Since $q \neq 1$, we have $\chi_1 \neq \varepsilon$. So the algebra A is not CY by Theorem 2.3.

Therefore, if A is CY, then A is of type (i). Hence, the classification is complete. □

Proposition 5.6. *Let A be the algebra $U(\mathcal{D}, \lambda)$, where $(\mathcal{D}, \lambda)$ is a generic datum of finite Cartan type for a group Γ. If A is CY of dimension 3, then the group Γ and the Cartan matrix (a_{ij}) are given by one of the following two cases.*

Case	$\lvert\Gamma\rvert$	*Cartan matrix*
1	3	*trivial*
2	1	$A_1 \times A_1$

The non-isomorphic classes of CY algebras in each case are given as follows.

Case 1: *The group algebra of a free abelian group of rank* 3.

Case 2: (I) *The datum* $(\mathcal{D}, \lambda) = (\Gamma, (g_1, g_2), (\chi_1, \chi_2), (a_{ij})_{1 \leqslant i,j \leqslant 2}, \lambda_{12})$ *is given as follows:*

- $\Gamma = \langle y_1 \rangle \cong \mathbb{Z}$;
- $g_1 = g_2 = y_1^k$ *for some* $k \in \mathbb{Z}^+$;
- $\chi_1(y_1) = q$, *where* $q \in \Bbbk$ *is not a root of unity and* $0 < |q| < 1$, *and* $\chi_2 = \chi_1^{-1}$;
- $(a_{ij})_{1 \leqslant i,j \leqslant 2}$ *is the Cartan matrix of type* $A_1 \times A_1$;
- $\lambda_{12} = 0$.

(II) *The datum* $(\mathcal{D}, \lambda) = (\Gamma, (g_1, g_2), (\chi_1, \chi_2), (a_{ij})_{1 \leqslant i,j \leqslant 2}, \lambda_{12})$ *is given as follows:*

- $\Gamma = \langle y_1 \rangle \cong \mathbb{Z}$;
- $g_1 = g_2 = y_1^k$ *for some* $k \in \mathbb{Z}^+$;
- $\chi_1(y_1) = q$, *where* $q \in \Bbbk$ *is not a root of unity and* $0 < |q| < 1$, *and* $\chi_2 = \chi_1^{-1}$;
- $(a_{ij})_{1 \leqslant i,j \leqslant 2}$ *is the Cartan matrix of type* $A_1 \times A_1$;
- $\lambda_{12} = 1$.

Proof. By Remark 2.4, it is sufficient to discuss the graded case and consider the non-trivial liftings. We first show that the algebras listed in the proposition are all CY. Case 1 follows from Lemma 5.4. Now we discuss case 2. The root system of the Cartan matrix of type $A_1 \times A_1$ has two simple roots, say α_1 and α_2. They are also the positive roots. First we have $\chi_1\chi_2 = \varepsilon$. Since $\mathcal{S}_A^2(x_i) = \chi_i(g_i^{-1})x_i$, $i = 1, 2$, $g_1 = g_2 = y_1^k$, we have $\mathcal{S}_A^2(x_i) = {y_1}^{-k} x_i {y_1}^k$ for $i = 1, 2$. It is easy to see that $\mathcal{S}_A^2(y_1) = y_1$. It follows that $\mathcal{S}_A^2$ is an inner automorphism. Thus the algebras in case 2 are CY by Theorem 2.3.

Now we show that the classification is complete.

If A is of global dimension 3, then the following possibilities for the group Γ and the Cartan matrix (a_{ij}) arise:

(i) $|\Gamma| = 3$, A is the group algebra of a free abelian group of rank 3.

(ii) $|\Gamma| = 2$ and the Cartan matrix of A is of type A_1.

(iii) $|\Gamma| = 1$ and the Cartan matrix of A is of type $A_1 \times A_1$.

Similar to the case of global dimension 2, A cannot be CY if A is of type (ii).

Now let A be a CY graded algebra of type (iii). Then we have $\chi_2(g_1)\chi_1(g_2) = 1$ (cf. equation (1)). In addition, we have $\chi_1\chi_2 = \varepsilon$ by Theorem 2.3. It follows that $1 = \chi_2(g_1)\chi_1(g_2) = \chi_1^{-1}(g_1)\chi_1(g_2)$. Let $\Gamma = \langle y_1 \rangle$ and assume that $g_1 = y_1^k$, $g_2 = y_1^l$ for some $k, l \in \mathbb{Z}$. Then $\chi_1(y_1^{l-k}) = 1$. Since $\chi_1(y_1)$ is not a root of unity, we have $k = l$, that is, $g_1 = g_2 = y_1^k$. Therefore, $A \cong U(\mathcal{D}, 0)$, where the datum $\mathcal{D}$ is given by

- $\Gamma = \langle y_1 \rangle \cong \mathbb{Z}$;
- $g_1 = g_2 = y_1^k$, for some $k \in \mathbb{Z}$;
- $\chi_1(y_1) = q$, where $q \in \Bbbk$ is not a root of unity, and $\chi_2 = \chi_1^{-1}$;
- $(a_{ij})_{1 \leqslant i,j \leqslant 2}$ is the Cartan matrix of type $A_1 \times A_1$.

Let $\mathcal{D}'$ be another datum given by

- $\Gamma' = \langle y_1' \rangle \cong \mathbb{Z}$;
- $g_1' = g_2' = y_1'^{k'}$ for some $k' \in \mathbb{Z}$;
- $\chi_1'(y_1') = q'$, where $q' \in \Bbbk$ is not a root of unity, and $\chi_2' = \chi_1'^{-1}$;
- $(a_{ij}')_{1 \leqslant i,j \leqslant 2}$ is the Cartan matrix of type $A_1 \times A_1$.

Assume that $(\mathcal{D}', 0)$ is isomorphic to $(\mathcal{D}, 0)$ via an isomorphism $(\varphi, \sigma, (\alpha_i))$. Then φ is a group automorphism such that $\varphi(y_1) = y_1'$ or $\varphi(y_1) = y_1'^{-1}$. Since $\sigma \in \mathbb{S}_2$, we have $\sigma = \mathrm{id}$ or $\sigma = (12)$. From an easy computation, there are four possibilities for k' and q':

- $k' = k$ and $q' = q$;
- $k' = -k$ and $q' = q$;
- $k' = k$ and $q' = q^{-1}$;
- $k' = -k$ and $q' = q^{-1}$.

This shows that $A = U(\mathcal{D}, 0)$ is isomorphic to an algebra in (I) of case 2. In addition, every pair $(k, q) \in \mathbb{Z}^+ \times \Bbbk$ such that $0 < |q| < 1$ determines a non-isomorphic algebra in (I) of case 2. Each algebra in (I) of case 2 has only one non-trivial lifting which is isomorphic to an algebra in (II).

Thus we have completed the classification. □

We list all CY Hopf algebras $U(\mathcal{D}, \lambda)$ of dimension 3 in terms of generators and relations in the following table. Note that in each case q is not a root of unity.

CY algebras of dimension 3

Case	Generators	Relations
Case 1	$y_h, y_h^{-1}, 1 \leqslant h \leqslant 3$	$y_h^{\pm 1} y_m^{\pm 1} = y_m^{\pm 1} y_h^{\pm 1}, y_h^{\pm 1} y_h^{\mp 1} = 1,$ $1 \leqslant h, m \leqslant 3$
Case 2 (I)	$y_1^{\pm 1}, x_1, x_2$	$y_1 y_1^{-1} = y_1^{-1} y_1 = 1, y_1 x_1 = q x_1 y_1,$ $y_1 x_2 = q^{-1} x_2 y_1, 0 < \lvert q \rvert < 1,$ $x_1 x_2 - q^{-k} x_2 x_1 = 0, k \in \mathbb{Z}^+$
Case 2 (II)	$y_1^{\pm 1}, x_1, x_2$	$y_1 y_1^{-1} = y_1^{-1} y_1 = 1, y_1 x_1 = q x_1 y_1,$ $y_1 x_2 = q^{-1} x_2 y_1, 0 < \lvert q \rvert < 1,$ $x_1 x_2 - q^{-k} x_2 x_1 = (1 - y_1^{2k}), k \in \mathbb{Z}^+$

Proposition 5.7. *Let A be the algebra $U(\mathcal{D}, \lambda)$, where $(\mathcal{D}, \lambda)$ is a generic datum of finite Cartan type for a group Γ. If A is CY of dimension 4, then the group Γ and the Cartan matrix (a_{ij}) are given by one of the following two cases.*

Case	$\lvert\Gamma\rvert$	*Cartan matrix*
1	4	*trivial*
2	2	$A_1 \times A_1$

In each case the non-isomorphic classes of CY algebras are given as follows.

Case 1: *The group algebra of a free abelian group of rank* 4.

Case 2: (I) *The datum* $(\mathcal{D}, \lambda) = (\Gamma, (g_1, g_2), (\chi_1, \chi_2), (a_{ij})_{1\leqslant i,j\leqslant 2}, \lambda_{12})$ *is given by*

- $\Gamma = \langle y_1, y_2\rangle \cong \mathbb{Z}^2$;
- $g_1 = g_2 = y_1^k$ *for some* $k \in \mathbb{Z}^+$;
- $\chi_1(y_1) = q_1$, $\chi_1(y_2) = q_2$, *where* $q_1, q_2 \in \Bbbk$ *with* $0 < |q_1| < 1$ *and* q_1 *is not a root of unity,* $\chi_2 = \chi_1^{-1}$;
- $(a_{ij})_{1\leqslant i,j\leqslant 2}$ *is the Cartan matrix of type* $A_1 \times A_1$;
- $\lambda_{12} = 0$.

(II) *The datum* $(\mathcal{D}, \lambda) = (\Gamma, (g_1, g_2), (\chi_1, \chi_2), (a_{ij})_{1\leqslant i,j\leqslant 2}, \lambda_{12})$ *is given by*

- $\Gamma = \langle y_1, y_2\rangle \cong \mathbb{Z}^2$;
- $g_1 = g_2 = y_1^k$ *for some* $k \in \mathbb{Z}^+$;
- $\chi_1(y_1) = q_1$, $\chi_1(y_2) = q_2$, *where* $q_1, q_2 \in \Bbbk$ *with* $0 < |q_1| < 1$ *and* q_1 *is not a root of unity,* $\chi_2 = \chi_1^{-1}$;
- $(a_{ij})_{1\leqslant i,j\leqslant 2}$ *is the Cartan matrix of type* $A_1 \times A_1$;
- $\lambda_{12} = 1$.

Let A and B be two algebras in case (I) (*or* (II)) *defined by triples* (k, q_1, q_2) *and* (k', q_1', q_2') *respectively. They are isomorphic if and only if* $k = k'$, $q_1 = q_1'$ *and there is some integer b such that* $q_2' = q_1^b q_2$ *or* $q_2' = q_1^b q_2^{-1}$.

(III) *The datum* $(\mathcal{D}, \lambda) = (\Gamma, (g_1, g_2), (\chi_1, \chi_2), (a_{ij})_{1\leqslant i,j\leqslant 2}, \lambda_{12})$ *is given by*

- $\Gamma = \langle y_1, y_2\rangle \cong \mathbb{Z}^2$;
- $g_1 = y_1^k$, $g_2 = y_2^l$ *for some* $k, l \in \mathbb{Z}^+$;
- $\chi_1(y_1) = q$, $\chi_1(y_2) = q^{\frac{k}{l}}$, *where* $q \in \Bbbk$ *is not a root of unity and* $0 < |q| < 1$, $\chi_2 = \chi_1^{-1}$;
- $(a_{ij})_{1\leqslant i,j\leqslant 2}$ *is the Cartan matrix of type* $A_1 \times A_1$;
- $\lambda_{12} = 0$.

(IV) *The datum* $(\mathcal{D}, \lambda) = (\Gamma, (g_1, g_2), (\chi_1, \chi_2), (a_{ij})_{1\leqslant i,j\leqslant 2}, \lambda_{12})$ *is given by*

- $\Gamma = \langle y_1, y_2 \rangle \cong \mathbb{Z}^2$;
- $g_1 = y_1^k$, $g_2 = y_2^l$ *for some* $k, l \in \mathbb{Z}^+$;
- $\chi_1(y_1) = q$, $\chi_1(y_2) = q^{\frac{k}{l}}$, *where* $q \in \Bbbk$ *is not a root of unity and* $0 < |q| < 1$, $\chi_2 = \chi_1^{-1}$;
- $(a_{ij})_{1\leqslant i,j\leqslant 2}$ *is the Cartan matrix of type* $A_1 \times A_1$;
- $\lambda_{12} = 1$.

(V) *The datum* $(\mathcal{D}, \lambda) = (\Gamma, (g_1, g_2), (\chi_1, \chi_2), (a_{ij})_{1\leqslant i,j\leqslant 2}, \lambda_{12})$ *is given by*

- $\Gamma = \langle y_1, y_2 \rangle \cong \mathbb{Z}^2$;
- $g_1 = y_1^k$, $g_2 = y_1^{l_1} y_2^{l_2}$ *for some* $k, l_1, l_2 \in \mathbb{Z}^+$, $k \neq l_1$, $0 < l_1 < l_2$;
- $\chi_1(y_1) = q$, $\chi_1(y_2) = q^{\frac{k-l_1}{l_2}}$, *where* $q \in \Bbbk$ *is not a root of unity and* $0 < |q| < 1$, $\chi_2 = \chi_1^{-1}$;
- $(a_{ij})_{1\leqslant i,j\leqslant 2}$ *is the Cartan matrix of type* $A_1 \times A_1$;
- $\lambda_{12} = 0$.

(VI) *The datum* $(\mathcal{D}, \lambda) = (\Gamma, (g_1, g_2), (\chi_1, \chi_2), (a_{ij})_{1\leqslant i,j\leqslant 2}, \lambda_{12})$ *is given by*

- $\Gamma = \langle y_1, y_2 \rangle \cong \mathbb{Z}^2$;
- $g_1 = y_1^k$, $g_2 = y_1^{l_1} y_2^{l_2}$ *for some* $k, l_1, l_2 \in \mathbb{Z}^+$, $k \neq l_1$ *and* $0 < l_1 < l_2$;
- $\chi_1(y_1) = q$, $\chi_1(y_2) = q^{\frac{k-l_1}{l_2}}$, *where* $q \in \Bbbk$ *is not a root of unity and* $0 < |q| < 1$, $\chi_2 = \chi_1^{-1}$;
- $(a_{ij})_{1\leqslant i,j\leqslant 2}$ *is the Cartan matrix of type* $A_1 \times A_1$;
- $\lambda_{12} = 1$.

Proof. We first show that the algebras listed in the proposition are all CY. That the algebra in case 1 is a CY algebra follows from Lemma 5.4. In case 2, we have $\chi_1\chi_2 = \varepsilon$ and $\mathcal{S}_A^2$ is an inner automorphism in each subcase. Indeed, $\mathcal{S}_A^2(x_i) = g_1^{-1} x_i g_1$ and $\mathcal{S}_A^2(y_i) = g_1^{-1} y_i g_1 = y_i$, $i = 1, 2$. Thus the algebras in case 2 are CY by Theorem 2.3.

Now we show that the classification is complete and the algebras on the list are non-isomorphic to each other.

If A is of global dimension 4, then the group Γ and the Cartan matrix (a_{ij}) must be one of the following types:

(i) $|\Gamma| = 4$ and A is the group algebra of a free abelian group of rank 4.

(ii) $|\Gamma| = 3$ and the Cartan matrix of A is of type A_1.

(iii) $|\Gamma| = 2$ and the Cartan matrix of A is of type $A_1 \times A_1$.

(iv) $|\Gamma| = 1$ and the Cartan matrix of A is of type $A_1 \times A_1 \times A_1$.

(v) $|\Gamma| = 1$ and the Cartan matrix of A is of type A_2.

Let A be a CY algebra of dimension 4. Similar to the case of global dimension 2, A cannot be of type (ii). We claim that A cannot be of type (iv) and (v) either.

Assume that A is of type (iv), put $\Gamma = \langle y_1 \rangle$, $g_i = y_1^{m_i}$ for some $0 \neq m_i \in \mathbb{Z}$ and $\chi_i(y_1) = q_i$ for some $q_i \in \Bbbk$, $1 \leqslant i \leqslant 3$. Then $q_{ij} = q_j^{m_i}$ for $1 \leqslant i, j \leqslant 3$. Because each q_{ii} is not a root of unity, each q_i is not a root of unity either. Since $q_{ij} q_{ji} = 1$, we have

$$q_1^{m_2} q_2^{m_1} = 1, \quad q_1^{m_3} q_3^{m_1} = 1, \quad q_2^{m_3} q_3^{m_2} = 1.$$

Then $q_1^{2m_2m_3} = 1$. But q_1 is not a root of unity. So A cannot be of type (iv).

In the case of type (v), there are three positive roots in the root system. They are α_1, α_2 and $\alpha_1 + \alpha_2$, where α_1 and α_2 are the simple roots. If A is CY, then $\chi_1^2 \chi_2^2 = \varepsilon$ by Theorem 2.3. So we have $q_{11}^2 q_{21}^2 = 1$ and $q_{12}^2 q_{22}^2 = 1$. However, $q_{21} q_{12} = q_{11}^{-1}$ (equation (2)). Thus $q_{22}^2 = 1$. But q_{22} is not a root of unity. So A cannot be of type (v) either.

Now to show that the classification is complete, we only need to show that if A is a CY pointed Hopf algebra of type (iii), then A is isomorphic to an algebra in case 2. Each algebra in (I), (III) and (V) of case 2 has only one non-trivial lifting, which is isomorphic to an algebra in (II), (IV) and (VI) respectively. By Remark 2.4, it suffices to show that if A is a graded CY pointed Hopf algebra of type (iii), then A is isomorphic to an algebra in (I), (III) and (V) of case 2.

Let $\Gamma = \langle y_1, y_2 \rangle$ be a free abelian group of rank 2. We write $\chi_1(y_1) = q_1$, $\chi_1(y_2) = q_2$ and $g_1 = y_1^{k_1} y_2^{k_2}$, $g_2 = y_1^{l_1} y_2^{l_2}$, where $\chi_1(g_1) = q_1^{k_1} q_2^{k_2}$ is not a root of unity, and $k_1, k_2, l_1, l_2 \in \mathbb{Z}$. Following Theorem 2.3, we have $\chi_1 \chi_2 = \varepsilon$. So $q_{21} = q_1^{l_1} q_2^{l_2}$ and $q_{12} = q_1^{-k_1} q_2^{-k_2}$. We also have $q_{12} q_{21} = 1$ (equation (2)). Thus $q_1^{l_1 - k_1} q_2^{l_2 - k_2} = 1$. Therefore, $A \cong U(\mathcal{D}, 0)$, where the datum $\mathcal{D}$ is formed by

- $\Gamma = \langle y_1, y_2 \rangle \cong \mathbb{Z}^2$;
- (a_{ij}) is the Cartan matrix of type $A_1 \times A_1$;
- $g_1 = y_1^{k_1} y_2^{k_2}$, $g_2 = y_1^{l_1} y_2^{l_2}$, $k_1, k_2, l_1, l_2 \in \mathbb{Z}$;
- $\chi_1(y_1) = q_1$, $\chi_1(y_2) = q_2$, where $\chi_1(g_1) = q_1^{k_1} q_2^{k_2}$ is not a root of unity and $q_1^{l_1 - k_1} q_2^{l_2 - k_2} = 1$, and $\chi_2 = \chi_1^{-1}$.

In the above datum $\mathcal{D}$, we may assume that $k_1 > 0$ and $k_2 = 0$. Then q_1 is not a root of unity. We show that there is a group isomorphism $\varphi \colon \Gamma \to \Gamma'$, where $\Gamma' = \langle y_1', y_2' \rangle$ is also a free abelian group of rank 2 such that $\varphi(y_1^{k_1} y_2^{k_2}) = y_1'^k$ and $k > 0$.

The integers k_1 and k_2 cannot be both equal to 0. If $k_2 = 0$ and $k_1 > 0$, then it is done. If $k_2 = 0$ and $k_1 < 0$, then $\varphi(y_1) = y_1'^{-1}$ and $\varphi(y_2) = y_2'^{-1}$ defines a desired isomorphism.

Similarly, we can obtain a desired isomorphism when $k_1 = 0$ and $k_2 \neq 0$.

If $k_1, k_2 \neq 0$, then there are some $k, \bar{k}_1, \bar{k}_2 \in \mathbb{Z}$ such that $k_1 = \bar{k}_1 k$, $k_2 = \bar{k}_2 k$,

$k > 0$ and $(\bar{k}_1, \bar{k}_2) = 1$, that is, $\bar{k}_1$ and $\bar{k}_2$ have no common divisors. We can find integers a, b such that $a\bar{k}_1 + b\bar{k}_2 = 1$. Let $\varphi\colon \Gamma \to \Gamma'$ be the group isomorphism defined by $\varphi(y_1) = {y'_1}^a {y'_2}^{-\bar{k}_2}$ and $\varphi(y_2) = {y'_1}^b {y'_2}^{\bar{k}_1}$. Then $\varphi(y_1^{k_1} y_2^{k_2}) = {y'_1}^k$ and $k > 0$. In conclusion, we have proved the claim.

If $l_2 = 0$, then we have $q_1^{l_1 - k_1} = 1$. Since q_1 is not a root of unity, we have $l_1 = k_1$. Applying a similar argument to the one in case 2 of Proposition 5.6, we find that A is isomorphic to an algebra in (I) of case 2.

Next we consider the case when $l_2 \neq 0$. In case $l_1 = 0$, like what we did for k_1 and k_2, we may assume that $l_2 > 0$. If $0 < |q_1| < 1$, then A is isomorphic to an algebra in (III) of case 2. Otherwise, the datum $(\mathcal{D}, 0)$ is isomorphic to the datum given by

- $\Gamma = \langle y_1, y_2 \rangle \cong \mathbb{Z}^2$;
- $g'_1 = y_1^{l_2}, g'_2 = y_2^{k_1}, k_1, l_2 \in \mathbb{Z}^+$;
- $\chi'_1(y_1) = q_1^{-\frac{k_1}{l_2}}, \chi'_1(y_2) = q_1^{-1}, \chi'_2 = {\chi'_1}^{-1}$.
- (a_{ij}) is the Cartan matrix of type $A_1 \times A_1$;
- $\lambda_{12} = 0$

via the isomorphism $(\varphi, (12), \alpha_1 = \alpha_2 = 1)$, where φ is the algebra automorphism defined by $\varphi(y_1) = y_2$ and $\varphi(y_2) = y_1$. So A is isomorphic to an algebra in (III) of case 2 as well.

If $l_1 \neq 0$ and $l_2 > 0$, then there is an integer c such that $0 \leqslant l_1 + cl_2 < l_2$. Let $\Gamma' = \langle y'_1, y'_2 \rangle$ be a free abelian group of rank 2, and $\varphi\colon \Gamma \to \Gamma'$ the group isomorphism defined by $\varphi(y_1) = y'_1$ and $\varphi(y_2) = {y'_1}^c y'_2$. Then $\varphi(y_1^{k_1}) = {y'_1}^{k_1}$ and $\varphi(y_1^{l_1} y_2^{l_2}) = {y'_1}^{l_1 + cl_2} {y'_2}^{l_2}$.

If $l_1 \neq 0$ and $l_2 < 0$, then there are integers $\bar{l}_1, \bar{l}_2$, such that $l_1 = \bar{l}_1 l$, $l_2 = \bar{l}_2 l$, $l > 0$ and $(\bar{l}_1, \bar{l}_2) = 1$. So $\bar{l}_2 < 0$. We can find integers a, b such that $a\bar{l}_1 + b\bar{l}_2 = 1$. Since for any integer d, $(a + d\bar{l}_2)\bar{l}_1 + (b - d\bar{l}_1)\bar{l}_2 = a\bar{l}_1 + b\bar{l}_2 = 1$, we may assume that $0 \leqslant a < -\bar{l}_2$. Let $\Gamma' = \langle y'_1, y'_2 \rangle$ be a free abelian group of rank 2, and $\varphi\colon \Gamma \to \Gamma'$ be the group isomorphism defined by $\varphi(y_1) = {y'_1}^a {y'_2}^{-\bar{l}_2}$ and $\varphi(y_2) = {y'_1}^b {y'_2}^{\bar{l}_1}$. Then $\varphi(y_1^{k_1}) = {y'_1}^{ak_1} {y'_2}^{-\bar{l}_2 k_1}$ and $\varphi(y_1^{l_1} y_2^{l_2}) = {y'_1}^l$.

In summary, by Corollary 5.2, we may assume that $l_2 > 0$ and $0 \leqslant l_1 < l_2$. If $l_1 = 0$, then we go back to the case we just discussed. If $l_1 \neq 0$ and $0 < |q_1| < 1$, then A is isomorphic to an algebra in (V). Otherwise, $(\mathcal{D}, 0)$ is isomorphic to the datum given by

- $\Gamma = \langle y_1, y_2 \rangle \cong \mathbb{Z}^2$;
- $g'_1 = y_1^l, g'_2 = y_1^{ak_1} y_2^{k_1 \bar{l}_2}$, $\bar{l}_1, \bar{l}_2 \in \mathbb{Z}^+$ are the integers such that $l\bar{l}_1 = l_1$, $l\bar{l}_2 = l_2$ and $(\bar{l}_1, \bar{l}_2) = 1$, $a, b \in \mathbb{Z}$ are the integers such that $a\bar{l}_1 + b\bar{l}_2 = 1$ and $0 < a < \bar{l}_2$;

- $\chi_1'(y_1) = q_1^{-\frac{k_1\bar{l}_2}{l_2}}$, $\chi_1'(y_2) = q_1^{\frac{ak_1-l}{l_2}}$, $\chi_2' = \chi_1'^{-1}$;
- (a_{ij}) is the Cartan matrix of type $A_1 \times A_1$;
- $\lambda_{12} = 0$

via the isomorphism $(\varphi, (12), \alpha_1 = \alpha_2 = 1)$, where φ is the isomorphism defined by $\varphi(y_1) = y_1^a y_2^{\bar{l}_2}$ and $\varphi(y_2) = y_1^b y_2^{-\bar{l}_1}$. It follows that A is isomorphic to an algebra in (V) as well.

It is clear that the algebras from different cases and subcases are non-isomorphic to each other. It is sufficient to show that the algebras in the same subcases in case 2 are non-isomorphic. Each algebra in (II), (IV) and (VI) is a lifting of an algebra in (I), (III) and (V) respectively. So it is sufficient to discuss the cases (I), (III) and (V).

First we discuss the case (I). Let $\mathcal{D}$ and $\mathcal{D}'$ be two data given by

- $\Gamma = \langle y_1, y_2 \rangle \cong \mathbb{Z}^2$;
- $g_1 = g_2 = y_1^k$ for some $k \in \mathbb{Z}^+$;
- $\chi_1(y_1) = q_1$, $\chi_1(y_2) = q_2$, where $q_1, q_2 \in \Bbbk$ satisfy that $0 < |q_1| < 1$ and q_1 is not a root of unity, and $\chi_2 = \chi_1^{-1}$;
- $(a_{ij})_{1\leqslant i,j\leqslant 2}$ is the Cartan matrix of type $A_1 \times A_1$

and

- $\Gamma = \langle y_1', y_2' \rangle \cong \mathbb{Z}^2$;
- $g_1 = g_2 = y_1^{k'}$ for some $k' \in \mathbb{Z}^+$;
- $\chi_1(y_1) = q_1'$, $\chi_1(y_2) = q_2'$, where $q_1', q_2' \in \Bbbk$ satisfy that $0 < |q_1'| < 1$ and q_1' is not a root of unity, and $\chi_2 = \chi_1^{-1}$;
- $(a_{ij}')_{1\leqslant i,j\leqslant 2}$ is the Cartan matrix of type $A_1 \times A_1$,

respectively. Assume that $(\varphi, \sigma, \alpha)$ is an isomorphism from $(\mathcal{D}, 0)$ to $(\mathcal{D}', 0)$. Say $\varphi(y_1) = y_1'^a y_2'^c$ and $\varphi(y_2) = y_1'^b y_2'^d$. Since $g_1 = g_2$ and $g_1' = g_2'$, we have $\varphi(y_1^k) = y_1'^{k'}$. Moreover, $k, k' > 0$. So $a = 1$, $c = 0$ and $d = \pm 1$. Consequently, we have $k = k'$, $q_1 = q_1'$. If $\sigma = \mathrm{id}$, then $q_2' = q_1^{-b} q_2$. Otherwise, $q_2' = q_1^b q_2^{-1}$. We have identified the isomorphic algebras in (I).

Similarly, it is direct to show that each triple $(k, l, q) \in \mathbb{Z}^+ \times \mathbb{Z}^+ \times \Bbbk$ such that $0 < |q| < 1$ determines a non-isomorphic algebra in (III).

Now we show that the algebras in (V) are non-isomorphic. Let $\mathcal{D}$ and $\mathcal{D}'$ be the data given by

- $\Gamma = \langle y_1, y_2 \rangle \cong \mathbb{Z}^2$;
- $g_1 = y_1^k$, $g_2 = y_1^{l_1} y_2^{l_2}$ such that $k, l_1, l_2 \in \mathbb{Z}^+$ and $0 < l_1 < l_2$;
- $\chi_1(y_1) = q$, where $q \in \Bbbk$ is not a root of unity, $0 < |q| < 1$, and $\chi_1(y_2) = q^{\frac{k-l_1}{l_2}}$ and $\chi_2 = \chi_1^{-1}$;

- $(a_{ij})_{1\leqslant i,j\leqslant 2}$, the Cartan matrix of type $A_1 \times A_1$

and

- $\Gamma' = \langle y_1', y_2' \rangle$ is also a free abelian group of rank 2;
- $g_1' = {y_1'}^{k'}$, $g_2' = {y_1'}^{l_1'} {y_2'}^{l_2'}$ such that $k', l_1', l_2' \in \mathbb{Z}^+$ and $0 < l_1' < l_2'$;
- $\chi_1'(y_1') = q'$, where $q' \in \Bbbk$ is not a root of unity, $0 < |q'| < 1$, and $\chi_1'(y_2') = {q'}^{\frac{k'-l_1'}{l_2'}}$ and $\chi_2' = {\chi_1'}^{-1}$;
- $(a_{ij})_{1\leqslant i,j\leqslant 2}$, the Cartan matrix of type $A_1 \times A_1$,

respectively. We claim that $(\mathcal{D}, 0)$ and $(\mathcal{D}', 0)$ are isomorphic if and only if $q = q'$, $k = k'$, $l_1 = l_1'$ and $l_2 = l_2'$.

Assume that $(\mathcal{D}, 0)$ is isomorphic to $(\mathcal{D}', 0)$ via an isomorphism $(\varphi, \sigma, \alpha_1 = \alpha_2 = 1)$. Suppose that $\varphi(y_1) = {y_1'}^a {y_2'}^c$ and $\varphi(y_2) = {y_1'}^b {y_2'}^d$, with $a, b, c, d \in \mathbb{Z}$.

Either $\sigma = \mathrm{id}$ or $\sigma = (12)$. If $\sigma = \mathrm{id}$, then $\varphi(g_i) = g_i'$, $i = 1, 2$. So

$$ {y_1'}^{ak} {y_2'}^{ck} = {y_1'}^{k'} \quad \text{and} \quad {y_1'}^{al_1+bl_2} {y_2'}^{cl_1+dl_2} = {y_1'}^{l_1'} {y_2'}^{l_2'}. $$

Since φ is an isomorphism, we have $ad - bc = \pm 1$. Because $k, k', l_2, l_2' > 0$, $0 < l_1 < l_2$ and $0 < l_1' < l_2'$, it follows that $b = c = 0$ and $a = d = 1$. Therefore, $k = k'$, $l_1 = l_1'$, $l_2 = l_2'$, and $q = q'$. Namely, $(\mathcal{D}, 0) = (\mathcal{D}', 0)$

If $\sigma = (12)$, then $\varphi(g_i) = g_{3-i}'$, $i = 1, 2$. This implies that

$$ {y_1'}^{ak} {y_2'}^{ck} = {y_1'}^{l_1'} {y_2'}^{l_2'} \quad \text{and} \quad {y_1'}^{al_1+bl_2} {y_2'}^{cl_1+dl_2} = {y_1'}^{k'}. $$

We can find integers $\bar{l}_1$ and $\bar{l}_2$ such that $l_1 = \bar{l}_1 l$, $l_2 = \bar{l}_2 l$, $l > 0$ and $(\bar{l}_1, \bar{l}_2) = 1$.

Since $ad - bc = \pm 1$, we have $(c, d) = 1$. From $ck = l_2' > 0$ and $cl_1 + dl_2 = 0$, it follows that $c = \bar{l}_2$ and $d = -\bar{l}_1$. If $ad - bc = 1$, we have

$$ k' = al_1 + bl_2 = l(a\bar{l}_1 + b\bar{l}_2) = -l(ad - bc) = -l < 0, $$

a contradiction!

If $ad - bc = -1$, we have

$$ q' = \chi_1'(y_1') = \chi_2 \varphi^{-1}(y_1) = \chi_2(y_1^{\bar{l}_1} y_2^{\bar{l}_2}) = q^{-\bar{l}_2 \frac{k}{l_2}}. $$

But $\bar{l}_2, k, l_2 > 0$ and $0 < |q|, |q'| < 1$. We get a contraction as well. In summary, we have proved the claim. □

Now we list all pointed CY Hopf algebras $U(\mathcal{D}, \lambda)$ of dimension 4 in terms of generators and relations in the following table. Note that q_1 and q are not roots of unity.

CY algebras of dimension 4

Case	Generators	Relations
Case 1	$y_h, y_h^{-1}, 1 \leqslant h \leqslant 4$	$y_h^{\pm 1} y_m^{\pm 1} = y_m^{\pm 1} y_h^{\pm 1}, y_h^{\pm 1} y_h^{\mp 1} = 1,$ $1 \leqslant h, m \leqslant 4$
Case 2 (I)	$y_1^{\pm 1}, y_2^{\pm 1}, x_1, x_2$	$y_h^{\pm 1} y_m^{\pm 1} = y_m^{\pm 1} y_h^{\pm 1}, y_h^{\pm 1} y_h^{\mp 1} = 1,$ $1 \leqslant h, m \leqslant 2, y_1 x_1 = q_1 x_1 y_1,$ $y_1 x_2 = q_1^{-1} x_2 y_1, y_2 x_1 = q_2 x_1 y_2,$ $y_2 x_2 = q_2^{-1} x_2 y_2, 0 < \lvert q_1 \rvert < 1,$ $x_1 x_2 - q_1^{-k} x_2 x_1 = 0, k \in \mathbb{Z}^+$
Case 2 (II)	$y_1^{\pm 1}, y_2^{\pm 1}, x_1, x_2$	$y_h^{\pm 1} y_m^{\pm 1} = y_m^{\pm 1} y_h^{\pm 1}, y_h^{\pm 1} y_h^{\mp 1} = 1,$ $1 \leqslant h, m \leqslant 2, y_1 x_1 = q_1 x_1 y_1,$ $y_1 x_2 = q_1^{-1} x_2 y_1, y_2 x_1 = q_2 x_1 y_2,$ $y_2 x_2 = q_2^{-1} x_2 y_2, 0 < \lvert q_1 \rvert < 1,$ $x_1 x_2 - q_1^{-k} x_2 x_1 = 1 - y_1^{2k}, k \in \mathbb{Z}^+$
Case 2 (III)	$y_1^{\pm 1}, y_2^{\pm 1}, x_1, x_2$	$y_h^{\pm 1} y_m^{\pm 1} = y_m^{\pm 1} y_h^{\pm 1}, y_h^{\pm 1} y_h^{\mp 1} = 1,$ $1 \leqslant h, m \leqslant 2, y_1 x_1 = q x_1 y_1,$ $y_1 x_2 = q^{-1} x_2 y_1, y_2 x_1 = q^{\frac{k}{l}} x_1 y_2,$ $y_2 x_2 = q^{-\frac{k}{l}} x_2 y_2,$ $x_1 x_2 - q^{-k} x_2 x_1 = 0, k, l \in \mathbb{Z}^+,$ $0 < \lvert q \rvert < 1$
Case 2 (IV)	$y_1^{\pm 1}, y_2^{\pm 1}, x_1, x_2$	$y_h^{\pm 1} y_m^{\pm 1} = y_m^{\pm 1} y_h^{\pm 1}, y_h^{\pm 1} y_h^{\mp 1} = 1,$ $1 \leqslant h, m \leqslant 2, y_1 x_1 = q x_1 y_1,$ $y_1 x_2 = q^{-1} x_2 y_1, y_2 x_1 = q^{\frac{k}{l}} x_1 y_2,$ $y_2 x_2 = q^{-\frac{k}{l}} x_2 y_2,$ $x_1 x_2 - q^{-k} x_2 x_1 = 1 - y_1^k y_2^l,$ $k, l \in \mathbb{Z}^+, 0 < \lvert q \rvert < 1$
Case 2 (V)	$y_1^{\pm 1}, y_2^{\pm 1}, x_1, x_2$	$y_h^{\pm 1} y_m^{\pm 1} = y_m^{\pm 1} y_h^{\pm 1}, y_h^{\pm 1} y_h^{\mp 1} = 1,$ $1 \leqslant h, m \leqslant 2, y_1 x_1 = q x_1 y_1,$ $y_1 x_2 = q^{-1} x_2 y_1, y_2 x_1 = q^{\frac{k-l_1}{l_2}} x_1 y_2,$ $y_2 x_2 = q^{-\frac{k-l_1}{l_2}} x_2 y_2,$ $x_1 x_2 - q^{-k} x_2 x_1 = 0, k, l_1, l_2 \in \mathbb{Z}^+,$ $0 < l_1 < l_2, 0 < \lvert q \rvert < 1$
Case 2 (VI)	$y_1^{\pm 1}, y_2^{\pm 1}, x_1, x_2$	$y_h^{\pm 1} y_m^{\pm 1} = y_m^{\pm 1} y_h^{\pm 1}, y_h^{\pm 1} y_h^{\mp 1} = 1,$ $1 \leqslant h, m \leqslant 2, y_1 x_1 = q x_1 y_1,$ $y_1 x_2 = q^{-1} x_2 y_1, y_2 x_1 = q^{\frac{k-l_1}{l_2}} x_1 y_2,$ $y_2 x_2 = q^{-\frac{k-l_1}{l_2}} x_2 y_2,$ $x_1 x_2 - q^{-k} x_2 x_1 = 1 - y_1^{k+l_1} y_2^{l_2},$ $k, l_1, l_2 \in \mathbb{Z}^+, 0 < l_1 < l_2, 0 < \lvert q \rvert < 1$

Let $\mathfrak{g}$ be a semisimple Lie algebra and $U_q(\mathfrak{g})$ its quantized enveloping algebra. By [7], Proposition 6.4, the global dimension of the algebra $U_q(\mathfrak{g})$ is the dimension of $\mathfrak{g}$. Thus, if $U_q(\mathfrak{g})$ is of global dimension less than 5, then $U_q(\mathfrak{g})$ is isomorphic to $U_q(\mathfrak{sl}_2)$, which is of global dimension 3. That is, among the algebras of the form $U_q(\mathfrak{g})$, only $U_q(\mathfrak{sl}_2)$ appears in the lists of Propositions 5.5, 5.6 and 5.7. The algebra $U_q(\mathfrak{sl}_2)$ is isomorphic to $U(\mathcal{D}, \lambda)$ with the datum given by

- $\Gamma = \langle g \rangle$, a free abelian group of rank 1;
- the Cartan matrix is of type $A_1 \times A_1$;
- $g_1 = g_2 = g$;
- $\chi_1(g) = q^{-2}$, $\chi_2(g) = q^2$, where q is not a root of unity;
- $\lambda_{12} = 1$.

It belongs to (II) of case 2 of Proposition 5.6.

The family of pointed Hopf algebras $U(\mathcal{D}, \lambda)$ provide more examples of CY Hopf algebras of higher dimensions. From the classification of CY pointed Hopf algebras $U(\mathcal{D}, \lambda)$ of dimensions less than 5, we see that the Cartan matrices are either trivial or of type $A_1 \times \cdots \times A_1$. The following example provides a CY pointed Hopf algebra of type $A_2 \times A_1$ of dimension 7.

Example 5.8. Let A be $U(\mathcal{D}, \lambda)$ with the datum $(\mathcal{D}, \lambda)$ given by

- $\Gamma = \langle y_1, y_2, y_3 \rangle$, a free abelian group of rank 3;
- the Cartan matrix is

$$\begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix};$$

- $g_i = y_i$, $1 \leqslant i \leqslant 3$;
- χ_i, $1 \leqslant i \leqslant 3$, are given by the following table, where q is not a root of unity.

	y_1	y_2	y_3
χ_1	q	q^{-2}	q^4
χ_2	q	q	q^{-2}
χ_3	q^{-4}	q^2	q^{-4}

- $\lambda = 0$.

In other words, A is the algebra with generators x_i, $y_j^{\pm 1}$, $1 \leqslant i, j \leqslant 3$, subject to the relations

$$y_i^{\pm 1} y_j^{\pm 1} = y_j^{\pm 1} y_i^{\pm 1}, \quad y_j^{\pm 1} y_j^{\mp 1} = 1, \quad 1 \leqslant i, j \leqslant 3,$$
$$y_j(x_i) = \chi_i(y_j) x_i y_j, \quad 1 \leqslant i, j \leqslant 3,$$
$$x_1^2 x_2 - q x_1 x_2 x_1 - q^2 x_1 x_2 x_1 + q^3 x_2 x_1^2 = 0,$$
$$x_2^2 x_1 - q^{-2} x_2 x_1 x_2 - q^{-1} x_2 x_1 x_2 + q^{-3} x_1 x_2^2 = 0,$$
$$x_1 x_3 = x_3 x_1.$$

The non-trivial liftings of A are also CY.

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