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Scalar curvature for the noncommutative two torus

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Abstract. We give a local expression for the *scalar curvature* of the noncommutative two torus $A_\theta = C(\mathbb{T}_\theta^2)$ equipped with an arbitrary translation invariant complex structure and Weyl factor. This is achieved by evaluating the value of the (analytic continuation of the) *spectral zeta functional* $\zeta_a(s) := \mathrm{Trace}(a\triangle^{-s})$ at $s = 0$ as a linear functional in $a \in C^\infty(\mathbb{T}_\theta^2)$. A new, purely noncommutative, feature here is the appearance of the *modular automorphism group* from the theory of type III factors and quantum statistical mechanics in the final formula for the curvature. This formula coincides with the formula that was recently obtained independently by Connes and Moscovici in their paper [15].

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Contents

1. Introduction

In this paper we give a local expression for the *scalar curvature* of the noncommutative two torus $A_\theta = C(\mathbb{T}_\theta^2)$ equipped with an arbitrary translation invariant complex structure and Weyl factor. More precisely, for any complex number τ in the upper half plane, representing the conformal class of a metric on $\mathbb{T}_\theta^2$, and a Weyl factor given by a positive invertible element $k \in C^\infty(\mathbb{T}_\theta^2)$, we give an explicit formula for an element $R = R(\tau, k) \in C^\infty(\mathbb{T}_\theta^2)$ that is the scalar curvature of the underlying noncommutative Riemannian manifold $\mathbb{T}_\theta^2$. This is achieved by evaluating the value of the (analytic continuation of the) *spectral zeta functional* $\zeta_a(s) := \mathrm{Trace}(a\triangle^{-s})$ at

$s = 0$ as a linear functional in $a \in C^\infty(\mathbb{T}_\theta^2)$. A new, purely noncommutative, feature here is the appearance of the *modular automorphism group* from the theory of type III factors and quantum statistical mechanics in the final formula for curvature. This formula exactly reproduces the formula that was recently obtained independently by Connes and Moscovici in their recent paper [15]. It also reduces, for $\tau = \sqrt{-1}$, to a formula that was earlier obtained by Alain Connes for the scalar curvature of the noncommutative two torus.

Our main result (Theorem 5.2 below) extends and refines the recent work on *Gauss–Bonnet theorem* for the noncommutative two torus that was initiated in the pioneering work of Connes and Tretkoff in [16] (cf. also [6], [5] for a preliminary version) and its later generalization in [17]. In fact after applying the standard trace of the noncommutative torus to the scalar curvature R one obtains, for all values of τ and k, the value 0. This is the Gauss–Bonnet theorem for the noncommutative two torus and, in the commutative case, is equivalent to the classical Gauss–Bonnet theorem for a surface of genus 1.

The backbone of the present paper is *noncommutative differential geometry program* [7], [8], [10], [12]. According to parts of this theory that is relevant here the metric information on a noncommutative space is fully encoded as a *spectral triple* on the noncommutative algebra of coordinates on that space. Various technical results corroborates, in fact fully justifies, this vision. First of all, *Connes' reconstruction theorem* [11] guarantees that in the commutative case, the notion of spectral triple is strong enough to fully recover the Riemannian (spin) manifold from its natural spectral triple data defined using the Dirac operator acting on spinors. Secondly, as it is shown in [9], [10], [12], ideas of spectral geometry, in particular formulation of several invariants of a Riemannian manifold like volume and scalar curvature in terms of asymptotics of the trace of the heat kernel of Laplacians and Dirac operators, have very natural extensions in the noncommutative setting and recover the classical results in the commutative case. Other relevant results are the Connes–Moscovici local index formula [13] and Chamseddine–Connes spectral action principle [3]. In passing to the noncommutative case, sooner or later one must face the prospect of type III algebras and the lack of trace on them. It was exactly for this reason that *twisted spectral triples* were introduced by Connes and Moscovici in [14]. The spectral triple at the foundation of the present paper was defined in [16] and is in fact, via the right action corresponding to the Tomita anti-linear unitary map, a twisted spectral triple.

One of the main technical tools employed in this paper is Connes' pseudodifferential operators and their symbol calculus on the noncommutative torus [7] and the use of the asymptotic expansion of the heat kernel in computing zeta values. This, however, by itself is not enough and, similar to [6], [16], [17], one needs an extra and intricate argument to express $\zeta_a(0)$ in terms of the modular operator defined by the Weyl factor. As a first step, the calculation of the asymptotic expansion of the heat operator for arbitrary values of the conformal class is quite involved and must be performed by a computer. We found it impossible to carry this step without the use of symbolic calculations. Finally we should mention that, as is explained in [16], [17], there is a close relationship between the subject of this paper and scale invariance in spectral action [3], [4] on the one hand, and non-unimodular (or twisted) spectral triples [14] on the other hand.

This paper is organized as follows. In Section 2 we recall a twisted spectral triple on the noncommutative torus from [16] and the conformal structures of this noncommutative space. An important idea here is to determine the conformal class of a metric by defining a complex structure on the noncommutative torus, and perturbing this metric by changing the tracial volume form to a KMS state by means of a Weyl factor given by an invertible positive smooth element [16]. In Section 3 we give a spectral definition for the scalar curvature of the noncommutative torus equipped with a general metric. We also recall the pseudodifferential calculus [7] for the special case of the canonical dynamical system defining the noncommutative torus and explain how this will provide a method for computing a local expression for the scalar curvature of this noncommutative Riemannian manifold. In Section 4 we illustrate the process of finding this local expression by means of pseudodifferential calculus on the noncommutative torus and heat kernel techniques. Another crucial technique here, as in [6], [16], [17], is to use the *modular automorphism* to permute elements of the noncommutative torus with the Weyl factor. In fact this prepares the ground for using functional calculus to write the final formula for the scalar curvature in a concise form. Considering the lengthy computations and formulas in this section, the final concise formula shows some magical cancelations and simplifications after the necessary rearrangements and permutations by means of the modular automorphism. In Section 5 we simplify our formula for the scalar curvature of the noncommutative torus in terms of the logarithm of the Weyl factor. Here again the modular automorphism is used crucially to find some identities that relate the derivative of the Weyl factor and the derivative of its logarithm with respect to the noncommutative coordinates of the noncommutative torus.

The definition of the scalar curvature for spectral triples in terms of the second term of the heat expansion was given in [12], Definition 1.147 of Section 11.1. The refinement used here as well as in [15] is to introduce the chiral scalar curvature from which the scalar curvature using the Laplacian on functions is easily deduced, (see also [2] for a variant).

We would like to express our indebtedness to Alain Connes for motivating and enlightening discussions and for much help during the various stages of the work on this paper. At several crucial stages he generously shared his insight and ideas with us and communicated their relevant joint results in [15] with us. This gave us a good chance of finding potential errors in the computations. In fact the idea of using the *full Laplacian*, on functions and 1-forms, as opposed to just functions, was suggested to us by him. While in the commutative case one can recover the curvature from zeta functionals from the Laplacian on functions, this is no more the case in the noncommutative case. We would also like to heartily thank Henri Moscovici for a push in the right direction at an early stage. After the appearance of our Gauss–Bonnet paper [17], Henri and Alain kindly pointed out to us that the calculations in that paper might be quite relevant for computing the scalar curvature of the noncommutative two torus. We would like to thank the Institute for Research in Fundamental Sciences (IPM) in Tehran where part of this paper was written during our stay there for their hospitality and support. Finally the first author would like to thank IHES for kind support and excellent environment during his visit in Summer 2011 where part of this work was carried out.

2. Preliminaries

Let Σ be a closed, oriented, 2-dimensional smooth manifold equipped with a Riemannian metric g. The scalar curvature of (Σ, g) can be expressed by a local formula in terms of the symbol of the Laplacian $\triangle_g = d^*d$, where d is the de Rham differential operator acting on smooth functions on Σ. In fact using the Cauchy integral formula, for any $t > 0$ one can write

$$e^{-t\triangle_g} = \frac{1}{2\pi i}\int_C e^{-t\lambda}(\triangle_g - \lambda)^{-1}\, d\lambda,$$

where C is a curve in the complex plane that goes around the non-negative real axis in the clockwise direction without touching it. The operator $e^{-t\triangle_g}$ has a smooth kernel $K(t, x, y)$ and there is an asymptotic expansion of the form

$$K(t,x,x) \sim t^{-1}\sum_{n=0}^{\infty} e_{2n}(x, \triangle_g)t^n \quad (t \to 0).$$

The term $e_2(x, \triangle_g)$ turns out to be a constant multiple of the scalar curvature of (Σ, g).

As a first step towards computing the *scalar curvature* of the noncommutative two torus, we recall the notion of the perturbed spectral triple attached to $(\mathbb{T}^2_\theta, \tau, k)$, where $\tau \in \mathbb{C} \setminus \mathbb{R}$ represents the conformal class of a metric on the noncommutative two tours $\mathbb{T}^2_\theta$, and $k \in C^\infty(\mathbb{T}^2_\theta)$ is the Weyl factor by the aid of which one can vary inside the conformal class of the metric [16], [17].

2.1. The irrational rotation algebra. Let θ be an irrational number. Recall that the irrational rotation C*-algebra A_θ is, by definition, the universal unital C*-algebra generated by two unitaries U, V satisfying

$$VU = e^{2\pi i\theta}UV.$$

One usually thinks of A_θ as the algebra of continuous functions on the noncommutative two torus $\mathbb{T}^2_\theta$. There is a continuous action of $\mathbb{T}^2$, $\mathbb{T} = \mathbb{R}/2\pi\mathbb{Z}$, on A_θ by C*-algebra automorphisms $\{\alpha_s\}$, $s \in \mathbb{R}^2$, defined by

$$\alpha_s(U^mV^n) = e^{is.(m,n)}U^mV^n.$$

The space of smooth elements for this action, that is, those elements $a \in A_\theta$ for which the map $s \mapsto \alpha_s(a)$ is C^∞ will be denoted by A_θ^∞. It is a dense subalgebra of A_θ which can be alternatively described as the algebra of elements in A_θ whose (noncommutative) Fourier expansion has rapidly decreasing coefficients:

$$A_\theta^\infty = \{\textstyle\sum_{m,n\in\mathbb{Z}} a_{m,n} U^m V^n \mid \sup_{m,n\in\mathbb{Z}}(|m|^k |n|^q |a_{m,n}|) < \infty \text{ for all } k,q \in \mathbb{Z}\}.$$

There is a unique normalized trace $\mathfrak{t}$ on A_θ whose restriction on smooth elements is given by

$$\mathfrak{t}(\textstyle\sum_{m,n\in\mathbb{Z}} a_{m,n} U^m V^n) = a_{0,0}.$$

The infinitesimal generators of the above action of $\mathbb{T}^2$ on A_θ are the derivations $\delta_1, \delta_2 \colon A_\theta^\infty \to A_\theta^\infty$ defined by

$$\delta_1(U) = U, \quad \delta_1(V) = 0, \quad \delta_2(U) = 0, \quad \delta_2(V) = V.$$

In fact, δ_1, δ_2 are analogues of the differential operators $\frac{1}{i}\partial/\partial x$, $\frac{1}{i}\partial/\partial y$ acting on the smooth functions on the ordinary two torus. We have $\delta_j(a^*) = -\delta_j(a)^*$ for $j = 1,2$ and all $a \in A_\theta^\infty$. Moreover, since $\mathfrak{t} \circ \delta_j = 0$, for $j = 1,2$, we have the analogue of the integration by parts formula:

$$\mathfrak{t}(a\delta_j(b)) = -\mathfrak{t}(\delta_j(a)b) \quad \text{for all a,b } \in A_\theta^\infty.$$

We define an inner product on A_θ by

$$\langle a, b\rangle = \mathfrak{t}(b^* a), \quad a, b \in A_\theta,$$

and complete A_θ with respect to this inner product to obtain a Hilbert space denoted by $\mathcal{H}_0$. The derivations δ_1, δ_2, as unbounded operators on $\mathcal{H}_0$, are formally selfadjoint and have unique extensions to selfadjoint operators.

2.2. Conformal structures on $\mathbb{T}_\theta^2$. To any complex number $\tau = \tau_1 + i\tau_2$, $\tau_1, \tau_2 \in \mathbb{R}$, with non-zero imaginary part, we can associate a complex structure on the noncommutative two torus by defining

$$\partial = \delta_1 + \bar{\tau}\delta_2, \quad \partial^* = \delta_1 + \tau\delta_2.$$

To the conformal structure defined by τ, corresponds a positive Hochschild two cocycle on A_θ^∞ given by (cf. [10])

$$\psi(a,b,c) = -\mathfrak{t}(a\partial b \partial^* c).$$

We note that ∂ is an unbounded operator on $\mathcal{H}_0$ and ∂^* is its formal adjoint. The analogue of the space of $(1,0)$-forms on the ordinary two torus is defined to be the Hilbert space completion of the space of finite sums $\sum a\partial b$, $a, b \in A_\theta^\infty$, with respect to the inner product defined above, and it is denoted by $\mathcal{H}^{(1,0)}$.

Now we can vary inside the conformal class of the metric [16] by choosing a smooth selfadjoint element $h = h^* \in A_\theta^\infty$ and define a linear functional φ on A_θ by

$$\varphi(a) = \mathfrak{t}\,(ae^{-h}), \quad a \in A_\theta.$$

In fact, φ is a positive linear functional which is not a trace, however, it is a twisted trace, and satisfies the KMS condition at $\beta = 1$ for the 1-parameter group $\{\sigma_t\}$, $t \in \mathbb{R}$, of inner automorphisms $\sigma_t = \Delta^{-it}$ where the modular operator for φ is given by (cf. [16])

$$\Delta(x) = e^{-h} x e^{h};$$

moreover, the 1-parameter group of automorphisms σ_t is generated by the derivation $-\log\Delta$ where

$$\log\Delta(x) = [-h, x], \quad x \in A_\theta^\infty.$$

We define an inner product $\langle\ ,\ \rangle_\varphi$ on A_θ by

$$\langle a, b\rangle_\varphi = \varphi(b^*a), \quad a, b \in A_\theta.$$

The Hilbert space obtained from completing A_θ with respect to this inner product will be denoted by $\mathcal{H}_\varphi$.

2.3. Spectral triple on A_θ. In this section, we recall the Connes–Tretkoff ordinary and twisted spectral triple over A_θ and A_θ^{op} respectively.

Let us view the operator ∂ as an unbounded operator from $\mathcal{H}_\varphi$ to $\mathcal{H}^{(1,0)}$ and denote it by ∂_φ. Similar to [16], we construct an *even spectral triple* by considering the left action of A_θ on the Hilbert space

$$\mathcal{H} = \mathcal{H}_\varphi \oplus \mathcal{H}^{(1,0)},$$

and the operator

$$D = \begin{pmatrix} 0 & \partial_\varphi^* \\ \partial_\varphi & 0 \end{pmatrix} : \mathcal{H} \to \mathcal{H}.$$

Then the *Laplacian* has the form

$$\triangle := D^2 = \begin{pmatrix} \partial_\varphi^*\partial_\varphi & 0 \\ 0 & \partial_\varphi\partial_\varphi^* \end{pmatrix}.$$

We also note that the *grading* is given by

$$\gamma = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} : \mathcal{H} \to \mathcal{H}.$$

It is shown in [17], [16] that the operator

$$\partial_\varphi^*\partial_\varphi : \mathcal{H}_\varphi \to \mathcal{H}_\varphi,$$

is anti-unitarily equivalent to

$$k\partial^*\partial k\colon \mathcal{H}_0 \to \mathcal{H}_0,$$

where $k := e^{h/2}$ acts on $\mathcal{H}_0$ by left multiplication. In a similar manner, we have the following equivalence for the other half of the Laplacian.

Lemma 2.1. *The operator* $\partial_\varphi \partial_\varphi^*\colon \mathcal{H}^{(1,0)} \to \mathcal{H}^{(1,0)}$ *is anti-unitarily equivalent to*

$$\partial^* k^2 \partial\colon \mathcal{H}^{(1,0)} \to \mathcal{H}^{(1,0)},$$

where k^2 *acts by left multiplication.*

Proof. One can easily see that the formal adjoint of $\partial_\varphi\colon \mathcal{H}_\varphi \to \mathcal{H}^{(1,0)}$ is given by $R_{k^2}\partial^*$, where R_{k^2} denotes the right multiplication by k^2. Let J be the involution on $\mathcal{H}^{(1,0)}$ given by $J(a) = a^*$. Then we have

$$J\partial_\varphi\partial_\varphi^* J = J\partial R_{k^2}\partial^* J = J\partial J J R_{k^2} J J \partial^* J = \partial^* k^2 \partial. \qquad \square$$

In [16], a twisted spectral triple is also constructed over A_θ^{op}. In fact, considering the Tomita anti-linear unitary map J_φ in $\mathcal{H}_\varphi$, and the corresponding unitary right action of A_θ in $\mathcal{H}_\varphi$ given by $a \mapsto J_\varphi a^* J_\varphi$, it is shown that $(A_\theta^{\mathrm{op}}, \mathcal{H}, D)$ is a twisted spectral triple in the sense that the twisted commutators

$$Da^{\mathrm{op}} - (k^{-1}ak)^{\mathrm{op}} D$$

are bounded operators for all $a \in A_\theta^\infty$.

3. Scalar curvature

In their book [12], Connes and Marcolli give a definition for the *scalar curvature* of *spectral triples of metric dimension* 4. This uses residues of the zeta function at its poles and cannot be applied to spectral triples of metric dimension 2, as is the case for the noncommutative two torus. For spectral triples of metric dimension 2, it is the value of the zeta functional at $s = 0$ that gives the scalar curvature. The general definition of *scalar curvature* for spectral triples of metric dimension 2, reduces to the following definition in the case of the noncommutative two torus (cf. also [15] for further explanations, motivations, and extensions, and [2] for a related proposal). The scalar curvature of the spectral triple attached to $(\mathbb{T}_\theta^2, \tau, k)$ in Subsection 2.3 is the unique element $R \in A_\theta^\infty$ satisfying the equation

$$\mathrm{Trace}(a\triangle^{-s})_{|s=0} + \mathrm{Trace}(aP) = \mathfrak{t}(aR) \quad \text{for all } a \in A_\theta^\infty,$$

where P is the projection onto the kernel of $\triangle$. The first term on the left hand side of this formula denotes the value at the origin, $\zeta_a(0)$, of the zeta function

$$\zeta_a(s) := \mathrm{Trace}(a\triangle^{-s}), \quad \mathrm{Re}(s) \gg 0.$$

This function has a holomorphic continuation to $\mathbb{C} \setminus \{1\}$, in particular its value at the origin is defined (cf. the proof of Proposition 3.5).

In a similar manner, for the graded case, where the additional data of grading γ is involved, the *chiral scalar curvature* R^γ is the unique element $R^\gamma \in A_\theta^\infty$ which satisfies the equation

$$\mathrm{Trace}(\gamma a \triangle^{-s})_{|s=0} + \mathrm{Trace}(\gamma a P) = \mathfrak{t}(a R^\gamma) \quad \text{for all } a \in A_\theta^\infty.$$

Proposition 3.5 will provide the means for finding a local expression for R and R^γ. First we recall the pseudodifferential calculus that we shall use for finding these elements.

3.1. Connes' pseudodifferential operators on $\mathbb{T}_\theta^2$. For a non-negative integer n, the space of differential operators on A_θ^∞ of order at most n is defined to be the vector space of operators of the form

$$\sum_{j_1+j_2 \le n} a_{j_1,j_2} \delta_1^{j_1} \delta_2^{j_2}, \quad j_1, j_2 \ge 0,\ a_{j_1,j_2} \in A_\theta^\infty.$$

The notion of a differential operator on A_θ^∞ can be generalized to the notion of a pseudodifferential operator using operator valued symbols [7]. In fact this is achieved by considering the pseudodifferential calculus associated to C*-dynamical systems [7], [1], for the canonical dynamical system $(A_\theta^\infty, \{\alpha_s\})$. In the sequel, we shall use the notation $\partial_1 = \frac{\partial}{\partial \xi_1}$, $\partial_2 = \frac{\partial}{\partial \xi_2}$.

Definition 3.1. For an integer n, a smooth map $\rho\colon \mathbb{R}^2 \to A_\theta^\infty$ is said to be a symbol of order n if for all non-negative integers i_1, i_2, j_1, j_2,

$$\|\delta_1^{i_1} \delta_2^{i_2} \partial_1^{j_1} \partial_2^{j_2} \rho(\xi)\| \le c(1+|\xi|)^{n-j_1-j_2},$$

where c is a constant, and if there exists a smooth map $k\colon \mathbb{R}^2 \to A_\theta^\infty$ such that

$$\lim_{\lambda\to\infty} \lambda^{-n} \rho(\lambda\xi_1, \lambda\xi_2) = k(\xi_1, \xi_2).$$

The space of symbols of order n is denoted by S_n.

To a symbol ρ of order n one can associate an operator on A_θ^∞, denoted by P_ρ, given by

$$P_\rho(a) = (2\pi)^{-2} \iint e^{-is\cdot\xi} \rho(\xi) \alpha_s(a)\, ds\, d\xi.$$

The operator P_ρ is said to be a pseudodifferential operator of order n. For example, the differential operator $\sum_{j_1+j_2\le n} a_{j_1,j_2} \delta_1^{j_1} \delta_2^{j_2}$ is associated with the symbol $\sum_{j_1+j_2\le n} a_{j_1,j_2} \xi_1^{j_1} \xi_2^{j_2}$ via the above formula.

Definition 3.2. Two symbols $\rho, \rho' \in S_k$ are said to be equivalent if and only if $\rho - \rho'$ is in S_n for all integers n. The equivalence of the symbols will be denoted by $\rho \sim \rho'$.

The following lemma shows that the space of pseudodifferential operators is an algebra and one can find the symbol of the product of pseudodifferential operators up to the above equivalence. Also, the adjoint of a pseudodifferential operator, with respect to the inner product defined on $\mathcal{H}_0$ in Section 2, is a pseudodifferential operator with the symbol given in the following proposition up to the equivalence (cf. [7]).

Proposition 3.3. *Let P and Q be pseudodifferential operators with symbols ρ and ρ' respectively. Then the adjoint P^* and the product PQ are pseudodifferential operators with the symbols*

$$\sigma(P^*) \sim \sum_{\ell_1,\ell_2 \geq 0} \frac{1}{\ell_1!\ell_2!} \partial_1^{\ell_1} \partial_2^{\ell_2} \delta_1^{\ell_1} \delta_2^{\ell_2} (\rho(\xi))^*,$$

$$\sigma(PQ) \sim \sum_{\ell_1,\ell_2 \geq 0} \frac{1}{\ell_1!\ell_2!} \partial_1^{\ell_1} \partial_2^{\ell_2} (\rho(\xi)) \delta_1^{\ell_1} \delta_2^{\ell_2} (\rho'(\xi)).$$

Definition 3.4. Let ρ be a symbol of order n. It is said to be elliptic if $\rho(\xi)$ is invertible for $\xi \neq 0$ and if there exists a constant c such that

$$\|\rho(\xi)^{-1}\| \leq c(1 + |\xi|)^{-n}$$

for sufficiently large $|\xi|$.

For example, the flat Laplacian $\delta_1^2 + 2\tau_1\delta_1\delta_2 + |\tau|^2\delta_2^2$ is an elliptic pseudodifferential operator (cf. [16], [17]).

3.2. Local expression for scalar curvature. Here we explain how one can find a local expression for the scalar curvature of the noncommutative two torus. This will justify the lengthy computations in the following sections.

Using the Cauchy integral formula, one has

$$e^{-t\triangle} = \frac{1}{2\pi i} \int_C e^{-t\lambda} (\triangle - \lambda)^{-1} \, d\lambda \tag{1}$$

where C is a curve in the complex plane that goes around the non-negative real axis in the *clockwise* direction without touching it. Similar to the formula in [18], one can approximate the inverse of the operator $(\triangle - \lambda)$ by a pseudodifferential operator B_λ whose symbol has an expansion of the form

$$b_0(\xi,\lambda) + b_1(\xi,\lambda) + b_2(\xi,\lambda) + \cdots$$

where $b_j(\xi,\lambda)$ is a symbol of order $-2-j$, and

$$\sigma(B_\lambda(\triangle - \lambda)) \sim 1.$$

Proposition 3.5. *The scalar curvature of the ungraded spectral triple attached to* $(\mathbb{T}_\theta^2, \tau, k)$ *is equal to*

$$\frac{1}{2\pi i} \int_{\mathbb{R}^2} \int_C e^{-\lambda} b_2(\xi, \lambda)\, d\lambda\, d\xi,$$

where b_2 *is defined as above.*

Proof. Using the Mellin transform we have

$$\triangle^{-s} = \frac{1}{\Gamma(s)} \int_0^\infty (e^{-t\triangle} - P) t^{s-1}\, dt,$$

where P denotes the orthogonal projection on Ker$(\triangle)$. Therefore for any $a \in A_\theta^\infty$, we have

$$a\triangle^{-s} = \frac{1}{\Gamma(s)} \int_0^\infty a(e^{-t\triangle} - P) t^{s-1}\, dt,$$

which gives

$$\mathrm{Trace}(a\triangle^{-s}) = \frac{1}{\Gamma(s)} \int_0^\infty (\mathrm{Trace}(ae^{-t\triangle}) - \mathrm{Trace}(aP)) t^{s-1}\, dt.$$

Appealing to the Cauchy integral formula (1) and using similar arguments to those in [18], one can derive an asymptotic expansion:

$$\mathrm{Trace}(ae^{-t\triangle}) \sim t^{-1} \sum_{n=0}^\infty B_{2n}(a, \triangle) t^n \quad (t \to 0).$$

Using this asymptotic expansion and the fact that Γ has a simple pole at 0 with residue 1, one can see that the zeta function

$$\zeta_a(s) = \mathrm{Trace}(a\triangle^{-s}), \qquad \mathrm{Re}(s) \gg 0,$$

has a meromorphic extension to the whole plane with no pole at 0 and

$$\zeta_a(0) = B_2(a, \triangle) - \mathrm{Trace}(aP).$$

Also one can see that

$$B_2(a, \triangle) = \frac{1}{2\pi i} \iint_C e^{-\lambda}\, \mathfrak{t}(ab_2(\xi, \lambda))\, d\lambda\, d\xi = \frac{1}{2\pi i} \mathfrak{t}\Big(a \iint_C e^{-\lambda} b_2(\xi, \lambda)\, d\lambda\, d\xi\Big),$$

where, as above, b_2 is the symbol of order -4 in $\sigma(B_\lambda)$. Hence the scalar curvature is equal to

$$\frac{1}{2\pi i} \int_{\mathbb{R}^2} \int_C e^{-\lambda} b_2(\xi, \lambda)\, d\lambda\, d\xi. \qquad \square$$

Note that for the purpose of computing the scalar curvature, using a homogeneity argument, one can set $\lambda = -1$ and multiply the final answer by -1 (cf. [16], [17]). In the sequel, we will write b_2 for $b_2(\xi, -1)$.

4. Computation of the scalar curvature

Following the recipe given in Section 3.2 we compute the two components of the scalar curvature for the noncommutative two torus corresponding to the Laplacian of the perturbed spectral triple attached to $(\mathbb{T}_\theta^2, \tau, k)$.

4.1. The computations for $k\partial^*\partial k$. In [17], it is shown that the symbol of the operator $k\partial^*\partial k$, '*the Laplacian on functions*', is equal to $a_2(\xi) + a_1(\xi) + a_0(\xi)$, where

$$\begin{aligned}
a_2(\xi) &= \xi_1^2 k^2 + |\tau|^2 \xi_2^2 k^2 + 2\tau_1 \xi_1 \xi_2 k^2, \\
a_1(\xi) &= 2\xi_1 k \delta_1(k) + 2|\tau|^2 \xi_2 k \delta_2(k) + 2\tau_1 \xi_1 k \delta_2(k) + 2\tau_1 \xi_2 k \delta_1(k), \\
a_0(\xi) &= k\delta_1^2(k) + |\tau|^2 k \delta_2^2(k) + 2\tau_1 k \delta_1 \delta_2(k).
\end{aligned}$$

It is also shown that one can find the terms b_j inductively. In fact the equation

$$\begin{aligned}
&(b_0 + b_1 + b_2 + \cdots)\, \sigma(k\partial^*\partial k + 1) \\
&\qquad = (b_0 + b_1 + b_2 + \cdots)((a_2 + 1) + a_1 + a_0) \sim 1,
\end{aligned}$$

has a solution where each b_j can be chosen to be a symbol of order $-2-j$. In fact, treating 1 as a symbol of order 2, we let $a_2' = a_2 + 1, a_1' = a_1, a_0' = a_0$. Then the above equation yields

$$\sum_{\substack{j,\ell_1,\ell_2 \geq 0 \\ k=0,1,2}} \frac{1}{\ell_1!\ell_2!} \partial_1^{\ell_1} \partial_2^{\ell_2}(b_j) \delta_1^{\ell_1} \delta_2^{\ell_2}(a_k') \sim 1.$$

By decomposing the latter into terms of order $-n$, $n = 0, 1, 2, \ldots$, we find

$$\begin{aligned}
b_0 &= a_2'^{-1} = (a_2 + 1)^{-1} = (\xi_1^2 k^2 + |\tau|^2 \xi_2^2 k^2 + 2\tau_1 \xi_1 \xi_2 k^2 + 1)^{-1}, \\
b_1 &= -(b_0 a_1 b_0 + \partial_1(b_0)\delta_1(a_2) b_0 + \partial_2(b_0)\delta_2(a_2) b_0), \\
b_2 &= -(b_0 a_0 b_0 + b_1 a_1 b_0 + \partial_1(b_0)\delta_1(a_1) b_0 + \partial_2(b_0)\delta_2(a_1) b_0 \\
&\quad + \partial_1(b_1)\delta_1(a_2) b_0 + \partial_2(b_1)\delta_2(a_2) b_0 + (1/2)\partial_{11}(b_0)\delta_1^2(a_2) b_0 \\
&\quad + (1/2)\partial_{22}(b_0)\delta_2^2(a_2) b_0 + \partial_{12}(b_0)\delta_1\delta_2(a_2) b_0).
\end{aligned}$$

After a direct computation, we find a lengthy formula for b_2. In order to integrate b_2 over the ξ-plane we pass to the coordinates

$$\xi_1 = r\cos\theta - r\frac{\tau_1}{\tau_2}\sin\theta, \quad \xi_2 = \frac{r}{\tau_2}\sin\theta, \tag{2}$$

where θ ranges from 0 to 2π and r ranges from 0 to ∞. After the integration with respect to θ, up to a factor of $\frac{r}{\tau_2}$ which is the Jacobian of the change of variables, one gets

$$
\begin{aligned}
&4|\tau|^2\pi r^6 b_0^2 k^2\delta_2(k)b_0^2k^3\delta_2(k)b_0k + 4\tau_1\pi r^6 b_0^2k^2\delta_2(k)b_0^2k^3\delta_1(k)b_0k\\
&+ 4|\tau|^2\pi r^6 b_0^2k^2\delta_2(k)b_0^2k^4\delta_2(k)b_0 + 4\tau_1\pi r^6 b_0^2k^2\delta_2(k)b_0^2k^4\delta_1(k)b_0\\
&+ 4\tau_1\pi r^6 b_0^2k^2\delta_1(k)b_0^2k^3\delta_2(k)b_0k + 4\pi r^6 b_0^2k^2\delta_1(k)b_0^2k^3\delta_1(k)b_0k\\
&+ 4\tau_1\pi r^6 b_0^2k^2\delta_1(k)b_0^2k^4\delta_2(k)b_0 + 4\pi r^6 b_0^2k^2\delta_1(k)b_0^2k^4\delta_1(k)b_0\\
&+ 4|\tau|^2\pi r^6 b_0^2k^3\delta_2(k)b_0^2k^2\delta_2(k)b_0k + 4\tau_1\pi r^6 b_0^2k^3\delta_2(k)b_0^2k^2\delta_1(k)b_0k\\
&+ 4|\tau|^2\pi r^6 b_0^2k^3\delta_2(k)b_0^2k^3\delta_2(k)b_0 + 4\tau_1\pi r^6 b_0^2k^3\delta_2(k)b_0^2k^3\delta_1(k)b_0\\
&+ 4\tau_1\pi r^6 b_0^2k^3\delta_1(k)b_0^2k^2\delta_2(k)b_0k + 4\pi r^6 b_0^2k^3\delta_1(k)b_0^2k^2\delta_1(k)b_0k\\
&+ 4\tau_1\pi r^6 b_0^2k^3\delta_1(k)b_0^2k^3\delta_2(k)b_0 + 4\pi r^6 b_0^2k^3\delta_1(k)b_0^2k^3\delta_1(k)b_0\\
&+ 8|\tau|^2\pi r^6 b_0^3k^4\delta_2(k)b_0k\delta_2(k)b_0k + 8\tau_1\pi r^6 b_0^3k^4\delta_2(k)b_0k\delta_1(k)b_0k\\
&+ 8|\tau|^2\pi r^6 b_0^3k^4\delta_2(k)b_0k^2\delta_2(k)b_0 + 8\tau_1\pi r^6 b_0^3k^4\delta_2(k)b_0k^2\delta_1(k)b_0\\
&+ 8\tau_1\pi r^6 b_0^3k^4\delta_1(k)b_0k\delta_2(k)b_0k + 8\pi r^6 b_0^3k^4\delta_1(k)b_0k\delta_1(k)b_0k\\
&+ 8\tau_1\pi r^6 b_0^3k^4\delta_1(k)b_0k^2\delta_2(k)b_0 + 8\pi r^6 b_0^3k^4\delta_1(k)b_0k^2\delta_1(k)b_0\\
&+ 8|\tau|^2\pi r^6 b_0^3k^5\delta_2(k)b_0\delta_2(k)b_0k + 8\tau_1\pi r^6 b_0^3k^5\delta_2(k)b_0\delta_1(k)b_0k\\
&+ 8|\tau|^2\pi r^6 b_0^3k^5\delta_2(k)b_0k\delta_2(k)b_0 + 8\tau_1\pi r^6 b_0^3k^5\delta_2(k)b_0k\delta_1(k)b_0\\
&+ 8\tau_1\pi r^6 b_0^3k^5\delta_1(k)b_0\delta_2(k)b_0k + 8\pi r^6 b_0^3k^5\delta_1(k)b_0\delta_1(k)b_0k\\
&+ 8\tau_1\pi r^6 b_0^3k^5\delta_1(k)b_0k\delta_2(k)b_0 + 8\pi r^6 b_0^3k^5\delta_1(k)b_0k\delta_1(k)b_0\\
&- 4|\tau|^2\pi r^4 b_0k\delta_2(k)b_0^2k^2\delta_2(k)b_0k - 4\tau_1\pi r^4 b_0k\delta_2(k)b_0^2k^2\delta_1(k)b_0k\\
&- 4|\tau|^2\pi r^4 b_0k\delta_2(k)b_0^2k^3\delta_2(k)b_0 - 4\tau_1\pi r^4 b_0k\delta_2(k)b_0^2k^3\delta_1(k)b_0\\
&- 4\tau_1\pi r^4 b_0k\delta_1(k)b_0^2k^2\delta_2(k)b_0k - 4\pi r^4 b_0k\delta_1(k)b_0^2k^2\delta_1(k)b_0k\\
&- 4\tau_1\pi r^4 b_0k\delta_1(k)b_0^2k^3\delta_2(k)b_0 - 4\pi r^4 b_0k\delta_1(k)b_0^2k^3\delta_1(k)b_0\\
&- 8|\tau|^2\pi r^4 b_0^2k^2\delta_2(k)b_0k\delta_2(k)b_0k - 8\tau_1\pi r^4 b_0^2k^2\delta_2(k)b_0k\delta_1(k)b_0k\\
&- 12|\tau|^2\pi r^4 b_0^2k^2\delta_2(k)b_0k^2\delta_2(k)b_0 - 12\tau_1\pi r^4 b_0^2k^2\delta_2(k)b_0k^2\delta_1(k)b_0\\
&- 8\tau_1\pi r^4 b_0^2k^2\delta_1(k)b_0k\delta_2(k)b_0k - 8\pi r^4 b_0^2k^2\delta_1(k)b_0k\delta_1(k)b_0k\\
&- 12\tau_1\pi r^4 b_0^2k^2\delta_1(k)b_0k^2\delta_2(k)b_0 - 12\pi r^4 b_0^2k^2\delta_1(k)b_0k^2\delta_1(k)b_0\\
&- 12|\tau|^2\pi r^4 b_0^2k^3\delta_2(k)b_0\delta_2(k)b_0k - 12\tau_1\pi r^4 b_0^2k^3\delta_2(k)b_0\delta_1(k)b_0k\\
&- 16|\tau|^2\pi r^4 b_0^2k^3\delta_2(k)b_0k\delta_2(k)b_0 - 16\tau_1\pi r^4 b_0^2k^3\delta_2(k)b_0k\delta_1(k)b_0\\
&- 12\tau_1\pi r^4 b_0^2k^3\delta_1(k)b_0\delta_2(k)b_0k - 12\pi r^4 b_0^2k^3\delta_1(k)b_0\delta_1(k)b_0k\\
&- 16\tau_1\pi r^4 b_0^2k^3\delta_1(k)b_0k\delta_2(k)b_0 - 16\pi r^4 b_0^2k^3\delta_1(k)b_0k\delta_1(k)b_0\\
&- 4\pi r^4 b_0^3k^4\delta_1^2(k)b_0k - 4|\tau|^2\pi r^4 b_0^3k^4\delta_2^2(k)b_0k - 8\tau_1\pi r^4 b_0^3k^4\delta_1\delta_2(k)b_0k\\
&- 8|\tau|^2\pi r^4 b_0^3k^4\delta_2(k)^2b_0 - 8\tau_1\pi r^4 b_0^3k^4\delta_2(k)\delta_1(k)b_0\\
&- 8\tau_1\pi r^4 b_0^3k^4\delta_1(k)\delta_2(k)b_0 - 8\pi r^4 b_0^3k^4\delta_1(k)^2b_0 - 4\pi r^4 b_0^3k^5\delta_1^2(k)b_0
\end{aligned}
$$

$$
\begin{aligned}
&-4|\tau|^2\pi r^4 b_0^3 k^5 \delta_2^2(k) b_0 - 8\tau_1\pi r^4 b_0^3 k^5 \delta_1\delta_2(k) b_0 \\
&+4|\tau|^2\pi r^2 b_0 k \delta_2(k) b_0 \delta_2(k) b_0 k + 4\tau_1\pi r^2 b_0 k \delta_2(k) b_0 \delta_1(k) b_0 k \\
&+8|\tau|^2\pi r^2 b_0 k \delta_2(k) b_0 k \delta_2(k) b_0 + 8\tau_1\pi r^2 b_0 k \delta_2(k) b_0 k \delta_1(k) b_0 \\
&+4\tau_1\pi r^2 b_0 k \delta_1(k) b_0 \delta_2(k) b_0 k + 4\pi r^2 b_0 k \delta_1(k) b_0 \delta_1(k) b_0 k \\
&+8\tau_1\pi r^2 b_0 k \delta_1(k) b_0 k \delta_2(k) b_0 + 8\pi r^2 b_0 k \delta_1(k) b_0 k \delta_1(k) b_0 \\
&+2\pi r^2 b_0^2 k^2 \delta_1^2(k) b_0 k + 2|\tau|^2\pi r^2 b_0^2 k^2 \delta_2^2(k) b_0 k + 4\tau_1\pi r^2 b_0^2 k^2 \delta_1\delta_2(k) b_0 k \\
&+8|\tau|^2\pi r^2 b_0^2 k^2 \delta_2(k)^2 b_0 + 8\tau_1\pi r^2 b_0^2 k^2 \delta_2(k)\delta_1(k) b_0 \\
&+8\tau_1\pi r^2 b_0^2 k^2 \delta_1(k)\delta_2(k) b_0 + 8\pi r^2 b_0^2 k^2 \delta_1(k)^2 b_0 + 6\pi r^2 b_0^2 k^3 \delta_1^2(k) b_0 \\
&+6|\tau|^2\pi r^2 b_0^2 k^3 \delta_2^2(k) b_0 + 12\tau_1\pi r^2 b_0^2 k^3 \delta_1\delta_2(k) b_0 - 2\pi b_0 k \delta_1^2(k) b_0 \\
&-2|\tau|^2\pi b_0 k \delta_2^2(k) b_0 - 4\tau_1\pi b_0 k \delta_1\delta_2(k) b_0,
\end{aligned}
$$

where

$$b_0 = (r^2k^2+1)^{-1}.$$

Since we are in the noncommutative case, where $b_0 = (r^2k^2+1)^{-1}$ and $\delta_j(k)$, $j = 1, 2$, do not necessarily commute, for the computation of $\int_0^\infty \bullet\, r dr$ of these terms, we need to use the modular automorphism Δ to permute k with elements of A_θ^∞. In the next two subsections we explain how this calculation is performed for the above types of terms.

4.1.1. Terms with two factors of the form b_0^i, $i \geq 1$. These are the following terms:

$$
\begin{aligned}
&-4\pi r^4 b_0^3 k^4 \delta_1^2(k) b_0 k - 4|\tau|^2\pi r^4 b_0^3 k^4 \delta_2^2(k) b_0 k - 8\tau_1\pi r^4 b_0^3 k^4 \delta_1\delta_2(k) b_0 k \\
&-8|\tau|^2\pi r^4 b_0^3 k^4 \delta_2(k)^2 b_0 - 8\tau_1\pi r^4 b_0^3 k^4 \delta_2(k)\delta_1(k) b_0 \\
&-8\tau_1\pi r^4 b_0^3 k^4 \delta_1(k)\delta_2(k) b_0 - 8\pi r^4 b_0^3 k^4 \delta_1(k)^2 b_0 - 4\pi r^4 b_0^3 k^5 \delta_1^2(k) b_0 \\
&-4|\tau|^2\pi r^4 b_0^3 k^5 \delta_2^2(k) b_0 - 8\tau_1\pi r^4 b_0^3 k^5 \delta_1\delta_2(k) b_0 + 2\pi r^2 b_0^2 k^2 \delta_1^2(k) b_0 k \\
&+2|\tau|^2\pi r^2 b_0^2 k^2 \delta_2^2(k) b_0 k + 4\tau_1\pi r^2 b_0^2 k^2 \delta_1\delta_2(k) b_0 k + 8|\tau|^2\pi r^2 b_0^2 k^2 \delta_2(k)^2 b_0 \\
&+8\tau_1\pi r^2 b_0^2 k^2 \delta_2(k)\delta_1(k) b_0 + 8\tau_1\pi r^2 b_0^2 k^2 \delta_1(k)\delta_2(k) b_0 + 8\pi r^2 b_0^2 k^2 \delta_1(k)^2 b_0 \\
&+6\pi r^2 b_0^2 k^3 \delta_1^2(k) b_0 + 6|\tau|^2\pi r^2 b_0^2 k^3 \delta_2^2(k) b_0 + 12\tau_1\pi r^2 b_0^2 k^3 \delta_1\delta_2(k) b_0 \\
&-2\pi b_0 k \delta_1^2(k) b_0 - 2|\tau|^2\pi b_0 k \delta_2^2(k) b_0 - 4\tau_1\pi b_0 k \delta_1\delta_2(k) b_0.
\end{aligned}
$$

The computation of $\int_0^\infty \bullet\, r dr$ of these terms is achieved by the following lemma of Connes and Tretkoff proved in [16].

Lemma 4.1. *For any $\rho \in A_\theta^\infty$ and every non-negative integer m, one has*

$$\int_0^\infty \frac{k^{2m+2}u^m}{(k^2u+1)^{m+1}} \rho \frac{1}{(k^2u+1)}\, du = \mathcal{D}_m(\rho),$$

where $\mathcal{D}_m = \mathcal{L}_m(\Delta)$, Δ is the modular automorphism introduced in Section 2 and $\mathcal{L}_m$ is the modified logarithm,

$$\mathcal{L}_m(u) = \int_0^\infty \frac{x^m}{(x+1)^{m+1}} \frac{1}{(xu+1)}\, dx$$
$$= (-1)^m (u-1)^{-(m+1)} \Big(\log u - \sum_{j=1}^{m} (-1)^{j+1} \frac{(u-1)^j}{j} \Big).$$

Using this lemma, one can see that $\int_0^\infty \bullet\, r dr$ of the above terms, up to an overall factor of π, is equal to

$$\begin{aligned}
&-2\mathcal{D}_2\Delta^{1/2}(k^{-1}\delta_1^2(k)) - 2|\tau|^2\mathcal{D}_2\Delta^{1/2}(k^{-1}\delta_2^2(k)) - 4\tau_1\mathcal{D}_2\Delta^{1/2}(k^{-1}\delta_1\delta_2(k))\\
&-4|\tau|^2\mathcal{D}_2(k^{-2}\delta_2(k)^2) - 4\tau_1\mathcal{D}_2(k^{-2}\delta_2(k)\delta_1(k)) - 4\tau_1\mathcal{D}_2(k^{-2}\delta_1(k)\delta_2(k))\\
&-4\mathcal{D}_2(k^{-2}\delta_1(k)^2) - 2\mathcal{D}_2(k^{-1}\delta_1^2(k)) - 2|\tau|^2\mathcal{D}_2(k^{-1}\delta_2^2(k))\\
&-4\tau_1\mathcal{D}_2(k^{-1}\delta_1\delta_2(k)) + \mathcal{D}_1\Delta^{1/2}(k^{-1}\delta_1^2(k)) + |\tau|^2\mathcal{D}_1\Delta^{1/2}(k^{-1}\delta_2^2(k))\\
&+2\tau_1\mathcal{D}_1\Delta^{1/2}(k^{-1}\delta_1\delta_2(k)) + 4|\tau|^2\mathcal{D}_1(k^{-2}\delta_2(k)^2) + 4\tau_1\mathcal{D}_1(k^{-2}\delta_2(k)\delta_1(k))\\
&+4\tau_1\mathcal{D}_1(k^{-2}\delta_1(k)\delta_2(k)) + 4\mathcal{D}_1(k^{-2}\delta_1(k)^2) + 3\mathcal{D}_1(k^{-1}\delta_1^2(k))\\
&+3|\tau|^2\mathcal{D}_1(k^{-1}\delta_2^2(k)) + 6\tau_1\mathcal{D}_1(k^{-1}\delta_1\delta_2(k)) - \mathcal{D}_0(k^{-1}\delta_1^2(k))\\
&-|\tau|^2\mathcal{D}_0(k^{-1}\delta_2^2(k)) - 2\tau_1\mathcal{D}_0(k^{-1}\delta_1\delta_2(k)).
\end{aligned}$$

Hence, up to an overall factor of π, $\int_0^\infty \bullet\, r dr$ of the terms with two positive powers of b_0 is equal to

$$\begin{aligned}
&f_1(\Delta)(k^{-1}\delta_1^2(k)) + f_2(\Delta)(k^{-2}\delta_1(k)^2)\\
&\quad + |\tau|^2 f_1(\Delta)(k^{-1}\delta_2^2(k)) + |\tau|^2 f_2(\Delta)(k^{-2}\delta_2(k)^2)\\
&\quad + \tau_1 f_1(\Delta)(k^{-1}\delta_1\delta_2(k)) + \tau_1 f_2(\Delta)(k^{-2}\delta_1(k)\delta_2(k))\\
&\quad + \tau_1 f_1(\Delta)(k^{-1}\delta_2\delta_1(k)) + \tau_1 f_2(\Delta)(k^{-2}\delta_2(k)\delta_1(k)),
\end{aligned} \tag{3}$$

where

$$\begin{aligned}
f_1(u) &:= -2\mathcal{L}_2(u)u^{1/2} - 2\mathcal{L}_2(u) + \mathcal{L}_1(u)u^{1/2} + 3\mathcal{L}_1(u) - \mathcal{L}_0(u)\\
&= -\frac{u^{1/2}(2-2u+(1+u)\log u)}{(-1+u^{1/2})^3(1+u^{1/2})^2}
\end{aligned} \tag{4}$$

and

$$f_2(u) := -4\mathcal{L}_2(u) + 4\mathcal{L}_1(u) = 2\frac{-1+u^2-2u\log u}{(-1+u)^3}. \tag{5}$$

4.1.2. Terms with three factors of the form b_0^i, $i \geq 1$. These terms are the following:

$$
\begin{aligned}
&4|\tau|^2\pi r^6 b_0^2k^2\delta_2(k)b_0^2k^3\delta_2(k)b_0k + 4\tau_1\pi r^6 b_0^2k^2\delta_2(k)b_0^2k^3\delta_1(k)b_0k\\
&+ 4|\tau|^2\pi r^6 b_0^2k^2\delta_2(k)b_0^2k^4\delta_2(k)b_0 + 4\tau_1\pi r^6 b_0^2k^2\delta_2(k)b_0^2k^4\delta_1(k)b_0\\
&+ 4\tau_1\pi r^6 b_0^2k^2\delta_1(k)b_0^2k^3\delta_2(k)b_0k + 4\pi r^6 b_0^2k^2\delta_1(k)b_0^2k^3\delta_1(k)b_0k\\
&+ 4\tau_1\pi r^6 b_0^2k^2\delta_1(k)b_0^2k^4\delta_2(k)b_0 + 4\pi r^6 b_0^2k^2\delta_1(k)b_0^2k^4\delta_1(k)b_0\\
&+ 4|\tau|^2\pi r^6 b_0^2k^3\delta_2(k)b_0^2k^2\delta_2(k)b_0k + 4\tau_1\pi r^6 b_0^2k^3\delta_2(k)b_0^2k^2\delta_1(k)b_0k\\
&+ 4|\tau|^2\pi r^6 b_0^2k^3\delta_2(k)b_0^2k^3\delta_2(k)b_0 + 4\tau_1\pi r^6 b_0^2k^3\delta_2(k)b_0^2k^3\delta_1(k)b_0\\
&+ 4\tau_1\pi r^6 b_0^2k^3\delta_1(k)b_0^2k^2\delta_2(k)b_0k + 4\pi r^6 b_0^2k^3\delta_1(k)b_0^2k^2\delta_1(k)b_0k\\
&+ 4\tau_1\pi r^6 b_0^2k^3\delta_1(k)b_0^2k^3\delta_2(k)b_0 + 4\pi r^6 b_0^2k^3\delta_1(k)b_0^2k^3\delta_1(k)b_0\\
&+ 8|\tau|^2\pi r^6 b_0^3k^4\delta_2(k)b_0k\delta_2(k)b_0k + 8\tau_1\pi r^6 b_0^3k^4\delta_2(k)b_0k\delta_1(k)b_0k\\
&+ 8|\tau|^2\pi r^6 b_0^3k^4\delta_2(k)b_0k^2\delta_2(k)b_0 + 8\tau_1\pi r^6 b_0^3k^4\delta_2(k)b_0k^2\delta_1(k)b_0\\
&+ 8\tau_1\pi r^6 b_0^3k^4\delta_1(k)b_0k\delta_2(k)b_0k + 8\pi r^6 b_0^3k^4\delta_1(k)b_0k\delta_1(k)b_0k\\
&+ 8\tau_1\pi r^6 b_0^3k^4\delta_1(k)b_0k^2\delta_2(k)b_0 + 8\pi r^6 b_0^3k^4\delta_1(k)b_0k^2\delta_1(k)b_0\\
&+ 8|\tau|^2\pi r^6 b_0^3k^5\delta_2(k)b_0\delta_2(k)b_0k + 8\tau_1\pi r^6 b_0^3k^5\delta_2(k)b_0\delta_1(k)b_0k\\
&+ 8|\tau|^2\pi r^6 b_0^3k^5\delta_2(k)b_0k\delta_2(k)b_0 + 8\tau_1\pi r^6 b_0^3k^5\delta_2(k)b_0k\delta_1(k)b_0\\
&+ 8\tau_1\pi r^6 b_0^3k^5\delta_1(k)b_0\delta_2(k)b_0k + 8\pi r^6 b_0^3k^5\delta_1(k)b_0\delta_1(k)b_0k\\
&+ 8\tau_1\pi r^6 b_0^3k^5\delta_1(k)b_0k\delta_2(k)b_0 + 8\pi r^6 b_0^3k^5\delta_1(k)b_0k\delta_1(k)b_0\\
&- 4|\tau|^2\pi r^4 b_0k\delta_2(k)b_0^2k^2\delta_2(k)b_0k - 4\tau_1\pi r^4 b_0k\delta_2(k)b_0^2k^2\delta_1(k)b_0k\\
&- 4|\tau|^2\pi r^4 b_0k\delta_2(k)b_0^2k^3\delta_2(k)b_0 - 4\tau_1\pi r^4 b_0k\delta_2(k)b_0^2k^3\delta_1(k)b_0\\
&- 4\tau_1\pi r^4 b_0k\delta_1(k)b_0^2k^2\delta_2(k)b_0k - 4\pi r^4 b_0k\delta_1(k)b_0^2k^2\delta_1(k)b_0k\\
&- 4\tau_1\pi r^4 b_0k\delta_1(k)b_0^2k^3\delta_2(k)b_0 - 4\pi r^4 b_0k\delta_1(k)b_0^2k^3\delta_1(k)b_0\\
&- 8|\tau|^2\pi r^4 b_0^2k^2\delta_2(k)b_0k\delta_2(k)b_0k - 8\tau_1\pi r^4 b_0^2k^2\delta_2(k)b_0k\delta_1(k)b_0k\\
&- 12|\tau|^2\pi r^4 b_0^2k^2\delta_2(k)b_0k^2\delta_2(k)b_0 - 12\tau_1\pi r^4 b_0^2k^2\delta_2(k)b_0k^2\delta_1(k)b_0\\
&- 8\tau_1\pi r^4 b_0^2k^2\delta_1(k)b_0k\delta_2(k)b_0k - 8\pi r^4 b_0^2k^2\delta_1(k)b_0k\delta_1(k)b_0k\\
&- 12\tau_1\pi r^4 b_0^2k^2\delta_1(k)b_0k^2\delta_2(k)b_0 - 12\pi r^4 b_0^2k^2\delta_1(k)b_0k^2\delta_1(k)b_0\\
&- 12|\tau|^2\pi r^4 b_0^2k^3\delta_2(k)b_0\delta_2(k)b_0k - 12\tau_1\pi r^4 b_0^2k^3\delta_2(k)b_0\delta_1(k)b_0k\\
&- 16|\tau|^2\pi r^4 b_0^2k^3\delta_2(k)b_0k\delta_2(k)b_0 - 16\tau_1\pi r^4 b_0^2k^3\delta_2(k)b_0k\delta_1(k)b_0\\
&- 12\tau_1\pi r^4 b_0^2k^3\delta_1(k)b_0\delta_2(k)b_0k - 12\pi r^4 b_0^2k^3\delta_1(k)b_0\delta_1(k)b_0k\\
&- 16\tau_1\pi r^4 b_0^2k^3\delta_1(k)b_0k\delta_2(k)b_0 - 16\pi r^4 b_0^2k^3\delta_1(k)b_0k\delta_1(k)b_0\\
&+ 4|\tau|^2\pi r^2 b_0k\delta_2(k)b_0\delta_2(k)b_0k + 4\tau_1\pi r^2 b_0k\delta_2(k)b_0\delta_1(k)b_0k\\
&+ 8|\tau|^2\pi r^2 b_0k\delta_2(k)b_0k\delta_2(k)b_0 + 8\tau_1\pi r^2 b_0k\delta_2(k)b_0k\delta_1(k)b_0
\end{aligned}
$$

$$+ 4\tau_1 \pi r^2 b_0 k \delta_1(k) b_0 \delta_2(k) b_0 k + 4\pi r^2 b_0 k \delta_1(k) b_0 \delta_1(k) b_0 k$$
$$+ 8\tau_1 \pi r^2 b_0 k \delta_1(k) b_0 k \delta_2(k) b_0 + 8\pi r^2 b_0 k \delta_1(k) b_0 k \delta_1(k) b_0.$$

For computing $\int_0^\infty \bullet r dr$ of these terms we will use the following lemma in which two variable modified logarithm functions appear. This lemma can be proved along the same lines as in [16].

Lemma 4.2. *For any $\rho, \rho' \in A_\theta^\infty$ and positive integers m, m', we have*

$$\int_0^\infty \frac{1}{(k^2u+1)^m} \rho \frac{k^{2(m+m')} u^{m+m'-1}}{(k^2u+1)^{m'}} \rho' \frac{1}{k^2u+1}\, du = \mathcal{D}_{m,m'}(\Delta_{(1)}, \Delta_{(2)})(\rho\rho'),$$

where the function $\mathcal{D}_{m,m'}$ is defined by

$$\mathcal{D}_{m,m'}(u,v) = \int_0^\infty \frac{1}{(xu^{-1}+1)^m} \frac{x^{m+m'-1}}{(x+1)^{m'}} \frac{1}{xv+1}\, dx$$

and $\Delta_{(1)}$ and $\Delta_{(2)}$ respectively denote the action of Δ on the first and second factor of the product.

Proof. Using the change of variable $s = \log u + h$, we have

$$\int_0^\infty \frac{1}{(k^2u+1)^m} \rho \frac{k^{2(m+m')} u^{m+m'-1}}{(k^2u+1)^{m'}} \rho' \frac{1}{k^2u+1}\, du$$
$$= \int_{-\infty}^\infty \frac{1}{(e^s+1)^m} \rho \frac{e^{s(m+m'-1)} k^2}{(e^s+1)^{m'}} \rho' \frac{k^{-2}}{e^s+1}\, d(e^s)$$
$$= \int_{-\infty}^\infty \frac{1}{(e^s+1)^m} \rho \frac{e^{s(m+m'-1/2)}}{(e^s+1)^{m'}} \Delta^{-1/2}(\rho') \frac{e^{s/2}}{e^s+1}\, ds$$
$$= \int_{-\infty}^\infty \frac{1}{(e^s+1)^m} \rho \frac{e^{s(m+m'-1/2)}}{(e^s+1)^{m'}} \Delta^{-1/2}(\rho') \int_{-\infty}^\infty \frac{e^{its}}{e^{\pi t}+e^{-\pi t}}\, dt\, ds$$
$$= \int_{-\infty}^\infty \frac{1}{(e^s+1)^m} \rho \frac{e^{s(m+m'-1/2)}}{(e^s+1)^{m'}} \int_{-\infty}^\infty \frac{e^{its}}{e^{\pi t}+e^{-\pi t}} \Delta^{it-1/2}(\rho')\, dt\, ds$$
$$= \int_{-\infty}^\infty \frac{1}{(e^s+1)^m} \rho \frac{e^{s(m+m'-1/2)}}{(e^s+1)^{m'}} \frac{e^{(s+\log\Delta)/2}}{e^{s+\log\Delta}+1} \Delta^{-1/2}(\rho')\, ds$$
$$= \int_{-\infty}^\infty \frac{e^{ms/2}}{(e^s+1)^m} \Delta^{m/2}(\rho) \frac{e^{s(m/2+m'-1/2)}}{(e^s+1)^{m'}} \frac{e^{(s+\log\Delta)/2}}{e^{s+\log\Delta}+1} \Delta^{-1/2}(\rho')\, ds$$
$$= \int_{-\infty}^\infty \prod_{j=1}^m \int_{-\infty}^\infty \frac{e^{it_j s}}{e^{\pi t_j}+e^{-\pi t_j}}\, dt_j\, \Delta^{m/2}(\rho) \frac{e^{s(m/2+m'-1/2)}}{(e^s+1)^{m'}}$$

$$\cdot \frac{e^{(s+\log\Delta)/2}}{e^{s+\log\Delta}+1}\Delta^{-1/2}(\rho')\,ds$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty} \frac{\Delta^{-is\sum_{j=1}^m t_j+m/2}(\rho)e^{is\sum_{j=1}^m t_j}}{\prod_{j=1}^m(e^{\pi t_j}+e^{-\pi t_j})}\,dt_1\ldots dt_m$$

$$\cdot \frac{e^{s(m/2+m'-1/2)}}{(e^s+1)^{m'}}\frac{e^{(s+\log\Delta)/2}}{e^{s+\log\Delta}+1}\Delta^{-1/2}(\rho')\,ds$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty} \frac{\Delta_{(1)}^{-is\sum_{j=1}^m t_j}e^{is\sum_{j=1}^m t_j}}{\prod_{j=1}^m(e^{\pi t_j}+e^{-\pi t_j})}\,dt_1\ldots dt_m$$

$$\cdot \frac{e^{s(m/2+m'-1/2)}}{(e^s+1)^{m'}}\frac{e^{(s+\log\Delta_{(2)})/2}}{e^{s+\log\Delta_{(2)}}+1}\,ds\,(\Delta^{m/2}(\rho)\Delta^{-1/2}(\rho'))$$

$$= \int_{-\infty}^{\infty}\frac{e^{m(s-\log\Delta_{(1)})/2}}{(e^{s-\log\Delta_{(1)}}+1)^m}\frac{e^{s(m/2+m'-1/2)}}{(e^s+1)^{m'}}\frac{e^{(s+\log\Delta_{(2)})/2}}{e^{s+\log\Delta_{(2)}}+1}\,ds\,(\Delta^{m/2}(\rho)\Delta^{-1/2}(\rho'))$$

$$= \int_{-\infty}^{\infty}\frac{e^{s/2}\Delta_{(1)}^{-m/2}}{(e^{s-\log\Delta_{(1)}}+1)^m}\frac{e^{s(m+m'-1)}}{(e^s+1)^{m'}}\frac{e^{(s+\log\Delta_{(2)})/2}}{e^{s+\log\Delta_{(2)}}+1}\,ds\,(\Delta^{m/2}(\rho)\Delta^{-1/2}(\rho'))$$

$$= \int_{0}^{\infty}\frac{1}{(x\Delta_{(1)}^{-1}+1)^m}\frac{x^{m+m'-1}}{(x+1)^{m'}}\frac{1}{x\Delta_{(2)}+1}\,dx\,(\rho\rho'). \qquad \square$$

In this paper, we only need these cases for our computations:

$$\begin{aligned}\mathcal{D}_{1,1}(u,v) = {}& ((-1+v)\log[\tfrac{1}{u}]\\ &-(-1+\tfrac{1}{u})\log[v])/((-1+\tfrac{1}{u})(-1+v)(-(\tfrac{1}{u})+v));\end{aligned}$$

$$\begin{aligned}\mathcal{D}_{2,2}(u,v) = {}& (u((-1+v)((-1+\tfrac{1}{u})(\tfrac{1}{u}-v)(1+1/u^2-(1+\tfrac{1}{u})v)\\ &+((-1+3/u-2v)(-1+v)\log[\tfrac{1}{u}])/u)\\ &-((-1+\tfrac{1}{u})^3\log[v])/u))/((-1+\tfrac{1}{u})^3(\tfrac{1}{u}-v)^2(-1+v)^2);\end{aligned}$$

$$\begin{aligned}\mathcal{D}_{1,2}(u,v) = {}& ((-1+v)^2\log[\tfrac{1}{u}]+(-1+\tfrac{1}{u})((\tfrac{1}{u}-v)(-1+v)\\ &-(-1+\tfrac{1}{u})\log[v]))/((-1+\tfrac{1}{u})^2(\tfrac{1}{u}-v)(-1+v)^2);\end{aligned}$$

$$\begin{aligned}\mathcal{D}_{2,1}(u,v) = {}& (u((-1+v)((-1+\tfrac{1}{u})(\tfrac{1}{u}-v)+((1-2/u+v)\log[\tfrac{1}{u}])/u)\\ &+((-1+\tfrac{1}{u})^2\log[v])/u))/((-1+\tfrac{1}{u})^2(\tfrac{1}{u}-v)^2(-1+v));\end{aligned}$$

$$\begin{aligned}\mathcal{D}_{3,1}(u,v) = {}& (u^2((-1+v)((-1+\tfrac{1}{u})(\tfrac{1}{u}-v)(5/u^2+v-(3(1+v))/u)\\ &-(2(1+3/u^2+v+v^2-(3(1+v))/u)\log[\tfrac{1}{u}])/u^2)\\ &+(2(-1+\tfrac{1}{u})^3\log[v])/u^2))/(2(-1+\tfrac{1}{u})^3(\tfrac{1}{u}-v)^3(-1+v)).\end{aligned}$$

Using this lemma, $\int_0^\infty \bullet r\,dr$ of the terms listed in the beginning of this subsection, up to an overall factor of π, is equal to

$$
\begin{aligned}
&2\mathcal{D}_{2,2}(\Delta_{(1)},\Delta_{(2)})(\Delta^{-1}(\delta_1(k)k^{-1})\Delta^{1/2}(k^{-1}\delta_1(k)))\\
&+2\mathcal{D}_{2,2}(\Delta_{(1)},\Delta_{(2)})(\Delta^{-1}(\delta_1(k)k^{-1})(k^{-1}\delta_1(k)))\\
&+2\mathcal{D}_{2,2}(\Delta_{(1)},\Delta_{(2)})(\Delta^{-3/2}(\delta_1(k)k^{-1})\Delta^{1/2}(k^{-1}\delta_1(k)))\\
&+2\mathcal{D}_{2,2}(\Delta_{(1)},\Delta_{(2)})(\Delta^{-3/2}(\delta_1(k)k^{-1})(k^{-1}\delta_1(k)))\\
&+4\mathcal{D}_{3,1}(\Delta_{(1)},\Delta_{(2)})(\Delta^{-2}(\delta_1(k)k^{-1})\Delta^{1/2}(k^{-1}\delta_1(k)))\\
&+4\mathcal{D}_{3,1}(\Delta_{(1)},\Delta_{(2)})(\Delta^{-2}(\delta_1(k)k^{-1})(k^{-1}\delta_1(k)))\\
&+4\mathcal{D}_{3,1}(\Delta_{(1)},\Delta_{(2)})(\Delta^{-5/2}(\delta_1(k)k^{-1})\Delta^{1/2}(k^{-1}\delta_1(k)))\\
&+4\mathcal{D}_{3,1}(\Delta_{(1)},\Delta_{(2)})(\Delta^{-5/2}(\delta_1(k)k^{-1})(k^{-1}\delta_1(k)))\\
&-2\mathcal{D}_{1,2}(\Delta_{(1)},\Delta_{(2)})(\Delta^{-1/2}(\delta_1(k)k^{-1})\Delta^{1/2}(k^{-1}\delta_1(k)))\\
&-2\mathcal{D}_{1,2}(\Delta_{(1)},\Delta_{(2)})(\Delta^{-1/2}(\delta_1(k)k^{-1})(k^{-1}\delta_1(k)))\\
&-4\mathcal{D}_{2,1}(\Delta_{(1)},\Delta_{(2)})(\Delta^{-1}(\delta_1(k)k^{-1})\Delta^{1/2}(k^{-1}\delta_1(k)))\\
&-6\mathcal{D}_{2,1}(\Delta_{(1)},\Delta_{(2)})(\Delta^{-1}(\delta_1(k)k^{-1})(k^{-1}\delta_1(k)))\\
&-6\mathcal{D}_{2,1}(\Delta_{(1)},\Delta_{(2)})(\Delta^{-3/2}(\delta_1(k)k^{-1})\Delta^{1/2}(k^{-1}\delta_1(k)))\\
&-8\mathcal{D}_{2,1}(\Delta_{(1)},\Delta_{(2)})(\Delta^{-3/2}(\delta_1(k)k^{-1})(k^{-1}\delta_1(k)))\\
&+2\mathcal{D}_{1,1}(\Delta_{(1)},\Delta_{(2)})(\Delta^{-1/2}(\delta_1(k)k^{-1})\Delta^{1/2}(k^{-1}\delta_1(k)))\\
&+4\mathcal{D}_{1,1}(\Delta_{(1)},\Delta_{(2)})(\Delta^{-1/2}(\delta_1(k)k^{-1})(k^{-1}\delta_1(k)))\\
&+|\tau|^2\Big(2\mathcal{D}_{2,2}(\Delta_{(1)},\Delta_{(2)})(\Delta^{-1}(\delta_2(k)k^{-1})\Delta^{1/2}(k^{-1}\delta_2(k)))\\
&+2\mathcal{D}_{2,2}(\Delta_{(1)},\Delta_{(2)})(\Delta^{-1}(\delta_2(k)k^{-1})(k^{-1}\delta_2(k)))\\
&+2\mathcal{D}_{2,2}(\Delta_{(1)},\Delta_{(2)})(\Delta^{-3/2}(\delta_2(k)k^{-1})\Delta^{1/2}(k^{-1}\delta_2(k)))\\
&+2\mathcal{D}_{2,2}(\Delta_{(1)},\Delta_{(2)})(\Delta^{-3/2}(\delta_2(k)k^{-1})(k^{-1}\delta_2(k)))\\
&+4\mathcal{D}_{3,1}(\Delta_{(1)},\Delta_{(2)})(\Delta^{-2}(\delta_2(k)k^{-1})\Delta^{1/2}(k^{-1}\delta_2(k)))\\
&+4\mathcal{D}_{3,1}(\Delta_{(1)},\Delta_{(2)})(\Delta^{-2}(\delta_2(k)k^{-1})(k^{-1}\delta_2(k)))\\
&+4\mathcal{D}_{3,1}(\Delta_{(1)},\Delta_{(2)})(\Delta^{-5/2}(\delta_2(k)k^{-1})\Delta^{1/2}(k^{-1}\delta_2(k)))\\
&+4\mathcal{D}_{3,1}(\Delta_{(1)},\Delta_{(2)})(\Delta^{-5/2}(\delta_2(k)k^{-1})(k^{-1}\delta_2(k)))\\
&-2\mathcal{D}_{1,2}(\Delta_{(1)},\Delta_{(2)})(\Delta^{-1/2}(\delta_2(k)k^{-1})\Delta^{1/2}(k^{-1}\delta_2(k)))\\
&-2\mathcal{D}_{1,2}(\Delta_{(1)},\Delta_{(2)})(\Delta^{-1/2}(\delta_2(k)k^{-1})(k^{-1}\delta_2(k)))\\
&-4\mathcal{D}_{2,1}(\Delta_{(1)},\Delta_{(2)})(\Delta^{-1}(\delta_2(k)k^{-1})\Delta^{1/2}(k^{-1}\delta_2(k)))\\
&-6\mathcal{D}_{2,1}(\Delta_{(1)},\Delta_{(2)})(\Delta^{-1}(\delta_2(k)k^{-1})(k^{-1}\delta_2(k)))\\
&-6\mathcal{D}_{2,1}(\Delta_{(1)},\Delta_{(2)})(\Delta^{-3/2}(\delta_2(k)k^{-1})\Delta^{1/2}(k^{-1}\delta_2(k)))\\
&-8\mathcal{D}_{2,1}(\Delta_{(1)},\Delta_{(2)})(\Delta^{-3/2}(\delta_2(k)k^{-1})(k^{-1}\delta_2(k)))
\end{aligned}
$$

$$
\begin{aligned}
&+2\mathcal{D}_{1,1}(\Delta_{(1)},\Delta_{(2)})(\Delta^{-1/2}(\delta_2(k)k^{-1})\Delta^{1/2}(k^{-1}\delta_2(k)))\\
&+4\mathcal{D}_{1,1}(\Delta_{(1)},\Delta_{(2)})(\Delta^{-1/2}(\delta_2(k)k^{-1})(k^{-1}\delta_2(k)))\Big)\\
&+\tau_1\Big(2\mathcal{D}_{2,2}(\Delta_{(1)},\Delta_{(2)})(\Delta^{-1}(\delta_1(k)k^{-1})\Delta^{1/2}(k^{-1}\delta_2(k)))\\
&+2\mathcal{D}_{2,2}(\Delta_{(1)},\Delta_{(2)})(\Delta^{-1}(\delta_1(k)k^{-1})(k^{-1}\delta_2(k)))\\
&+2\mathcal{D}_{2,2}(\Delta_{(1)},\Delta_{(2)})(\Delta^{-3/2}(\delta_1(k)k^{-1})\Delta^{1/2}(k^{-1}\delta_2(k)))\\
&+2\mathcal{D}_{2,2}(\Delta_{(1)},\Delta_{(2)})(\Delta^{-3/2}(\delta_1(k)k^{-1})(k^{-1}\delta_2(k)))\\
&+4\mathcal{D}_{3,1}(\Delta_{(1)},\Delta_{(2)})(\Delta^{-2}(\delta_1(k)k^{-1})\Delta^{1/2}(k^{-1}\delta_2(k)))\\
&+4\mathcal{D}_{3,1}(\Delta_{(1)},\Delta_{(2)})(\Delta^{-2}(\delta_1(k)k^{-1})(k^{-1}\delta_2(k)))\\
&+4\mathcal{D}_{3,1}(\Delta_{(1)},\Delta_{(2)})(\Delta^{-5/2}(\delta_1(k)k^{-1})\Delta^{1/2}(k^{-1}\delta_2(k)))\\
&+4\mathcal{D}_{3,1}(\Delta_{(1)},\Delta_{(2)})(\Delta^{-5/2}(\delta_1(k)k^{-1})(k^{-1}\delta_2(k)))\\
&-2\mathcal{D}_{1,2}(\Delta_{(1)},\Delta_{(2)})(\Delta^{-1/2}(\delta_1(k)k^{-1})\Delta^{1/2}(k^{-1}\delta_2(k)))\\
&-2\mathcal{D}_{1,2}(\Delta_{(1)},\Delta_{(2)})(\Delta^{-1/2}(\delta_1(k)k^{-1})(k^{-1}\delta_2(k)))\\
&-4\mathcal{D}_{2,1}(\Delta_{(1)},\Delta_{(2)})(\Delta^{-1}(\delta_1(k)k^{-1})\Delta^{1/2}(k^{-1}\delta_2(k)))\\
&-6\mathcal{D}_{2,1}(\Delta_{(1)},\Delta_{(2)})(\Delta^{-1}(\delta_1(k)k^{-1})(k^{-1}\delta_2(k)))\\
&-6\mathcal{D}_{2,1}(\Delta_{(1)},\Delta_{(2)})(\Delta^{-3/2}(\delta_1(k)k^{-1})\Delta^{1/2}(k^{-1}\delta_2(k)))\\
&-8\mathcal{D}_{2,1}(\Delta_{(1)},\Delta_{(2)})(\Delta^{-3/2}(\delta_1(k)k^{-1})(k^{-1}\delta_2(k)))\\
&+2\mathcal{D}_{1,1}(\Delta_{(1)},\Delta_{(2)})(\Delta^{-1/2}(\delta_1(k)k^{-1})\Delta^{1/2}(k^{-1}\delta_2(k)))\\
&+4\mathcal{D}_{1,1}(\Delta_{(1)},\Delta_{(2)})(\Delta^{-1/2}(\delta_1(k)k^{-1})(k^{-1}\delta_2(k)))\\
&+2\mathcal{D}_{2,2}(\Delta_{(1)},\Delta_{(2)})(\Delta^{-1}(\delta_2(k)k^{-1})\Delta^{1/2}(k^{-1}\delta_1(k)))\\
&+2\mathcal{D}_{2,2}(\Delta_{(1)},\Delta_{(2)})(\Delta^{-1}(\delta_2(k)k^{-1})(k^{-1}\delta_1(k)))\\
&+2\mathcal{D}_{2,2}(\Delta_{(1)},\Delta_{(2)})(\Delta^{-3/2}(\delta_2(k)k^{-1})\Delta^{1/2}(k^{-1}\delta_1(k)))\\
&+2\mathcal{D}_{2,2}(\Delta_{(1)},\Delta_{(2)})(\Delta^{-3/2}(\delta_2(k)k^{-1})(k^{-1}\delta_1(k)))\\
&+4\mathcal{D}_{3,1}(\Delta_{(1)},\Delta_{(2)})(\Delta^{-2}(\delta_2(k)k^{-1})\Delta^{1/2}(k^{-1}\delta_1(k)))\\
&+4\mathcal{D}_{3,1}(\Delta_{(1)},\Delta_{(2)})(\Delta^{-2}(\delta_2(k)k^{-1})(k^{-1}\delta_1(k)))\\
&+4\mathcal{D}_{3,1}(\Delta_{(1)},\Delta_{(2)})(\Delta^{-5/2}(\delta_2(k)k^{-1})\Delta^{1/2}(k^{-1}\delta_1(k)))\\
&+4\mathcal{D}_{3,1}(\Delta_{(1)},\Delta_{(2)})(\Delta^{-5/2}(\delta_2(k)k^{-1})(k^{-1}\delta_1(k)))\\
&-2\mathcal{D}_{1,2}(\Delta_{(1)},\Delta_{(2)})(\Delta^{-1/2}(\delta_2(k)k^{-1})\Delta^{1/2}(k^{-1}\delta_1(k)))\\
&-2\mathcal{D}_{1,2}(\Delta_{(1)},\Delta_{(2)})(\Delta^{-1/2}(\delta_2(k)k^{-1})(k^{-1}\delta_1(k)))\\
&-4\mathcal{D}_{2,1}(\Delta_{(1)},\Delta_{(2)})(\Delta^{-1}(\delta_2(k)k^{-1})\Delta^{1/2}(k^{-1}\delta_1(k)))\\
&-6\mathcal{D}_{2,1}(\Delta_{(1)},\Delta_{(2)})(\Delta^{-1}(\delta_2(k)k^{-1})(k^{-1}\delta_1(k)))\\
&-6\mathcal{D}_{2,1}(\Delta_{(1)},\Delta_{(2)})(\Delta^{-3/2}(\delta_2(k)k^{-1})\Delta^{1/2}(k^{-1}\delta_1(k)))
\end{aligned}
$$

$$
\begin{aligned}
&- 8\mathcal{D}_{2,1}(\Delta_{(1)}, \Delta_{(2)})(\Delta^{-3/2}(\delta_2(k)k^{-1})(k^{-1}\delta_1(k))) \\
&+ 2\mathcal{D}_{1,1}(\Delta_{(1)}, \Delta_{(2)})(\Delta^{-1/2}(\delta_2(k)k^{-1})\Delta^{1/2}(k^{-1}\delta_1(k))) \\
&+ 4\mathcal{D}_{1,1}(\Delta_{(1)}, \Delta_{(2)})(\Delta^{-1/2}(\delta_2(k)k^{-1})(k^{-1}\delta_1(k)))\Big).
\end{aligned}
$$

Putting together the latter terms with the ones obtained in (3), up to an overall factor of π, we find the following expression

$$
\begin{aligned}
& f_1(\Delta)(k^{-1}\delta_1^2(k)) + f_2(\Delta)(k^{-2}\delta_1(k)^2) \\
&\quad + F(\Delta_{(1)}, \Delta_{(2)})((\delta_1(k)k^{-1})(k^{-1}\delta_1(k))) \\
&\quad + |\tau|^2 f_1(\Delta)(k^{-1}\delta_2^2(k)) + |\tau|^2 f_2(\Delta)(k^{-2}\delta_2(k)^2) \\
&\quad + |\tau|^2 F(\Delta_{(1)}, \Delta_{(2)})((\delta_2(k)k^{-1})(k^{-1}\delta_2(k))) \\
&\quad + \tau_1 f_1(\Delta)(k^{-1}\delta_1\delta_2(k)) + \tau_1 f_2(\Delta)(k^{-2}\delta_1(k)\delta_2(k)) \\
&\quad + \tau_1 F(\Delta_{(1)}, \Delta_{(2)})((\delta_1(k)k^{-1})(k^{-1}\delta_2(k))) \\
&\quad + \tau_1 f_1(\Delta)(k^{-1}\delta_2\delta_1(k)) + \tau_1 f_2(\Delta)(k^{-2}\delta_2(k)\delta_1(k)) \\
&\quad + \tau_1 F(\Delta_{(1)}, \Delta_{(2)})((\delta_2(k)k^{-1})(k^{-1}\delta_1(k))),
\end{aligned} \tag{6}
$$

where as given by the formulas (4) and (5) we have

$$
\begin{aligned}
f_1(u) &= -\frac{u^{1/2}(2 - 2u + (1+u)\log u)}{(-1+u^{1/2})^3(1+u^{1/2})^2}, \\
f_2(u) &= 2\frac{-1 + u^2 - 2u\log u}{(-1+u)^3},
\end{aligned}
$$

and

$$
\begin{aligned}
F(u,v) :=\ & 2\mathcal{D}_{2,2}(u,v)u^{-1}v^{1/2} + 2\mathcal{D}_{2,2}(u,v)u^{-1} + 2\mathcal{D}_{2,2}(u,v)u^{-3/2}v^{1/2} \\
& + 2\mathcal{D}_{2,2}(u,v)u^{-3/2} + 4\mathcal{D}_{3,1}(u,v)u^{-2}v^{1/2} + 4\mathcal{D}_{3,1}(u,v)u^{-2} \\
& + 4\mathcal{D}_{3,1}(u,v)u^{-5/2}v^{1/2} + 4\mathcal{D}_{3,1}(u,v)u^{-5/2} - 2\mathcal{D}_{1,2}(u,v)u^{-1/2}v^{1/2} \\
& - 2\mathcal{D}_{1,2}(u,v)u^{-1/2} - 4\mathcal{D}_{2,1}(u,v)u^{-1}v^{1/2} - 6\mathcal{D}_{2,1}(u,v)u^{-1} \\
& - 6\mathcal{D}_{2,1}(u,v)u^{-3/2}v^{1/2} - 8\mathcal{D}_{2,1}(u,v)u^{-3/2} + 2\mathcal{D}_{1,1}(u,v)u^{-1/2}v^{1/2} \\
& + 4\mathcal{D}_{1,1}(u,v)u^{-1/2} \\
=\ & (2u(-(((-1+uv)(1+\sqrt{u}(-1-\sqrt{v}-(-2+\sqrt{u}+u)v+uv^{3/2}))))/ \\
& ((-1+\sqrt{u})(-1+\sqrt{v}))) + (\sqrt{u}\sqrt{v}(-1-\sqrt{u}+u \\
& + u(-2-\sqrt{u}+2u)\sqrt{v} + u(-1+\sqrt{u}+u)v + u^{5/2}v^{3/2})\log u)/ \\
& ((-1+\sqrt{u})^2(1+\sqrt{u})) + (\sqrt{v}(1-\sqrt{u}\sqrt{v}(-1-\sqrt{v}+v \\
& + uv(-1+\sqrt{v}+v) + \sqrt{u}(-2+\sqrt{v}+2v)))\log v)/ \\
& ((-1+\sqrt{v})^2(1+\sqrt{v}))))/(-1+uv)^3.
\end{aligned}
$$

Note that in (6), $\Delta_{(i)}$, i=1,2, signifies the action of Δ on the i-th factor of the product.

4.2. The computations for $\partial^* k^2 \partial$. In order to compute the second component of the scalar curvature corresponding to $\partial^* k^2 \partial$, '*the Laplacian on* $(1,0)$*-forms*', we first find the symbol of this operator:

Lemma 4.3. *The symbol of* $\partial^* k^2 \partial$ *is equal to* $c_2(\xi) + c_1(\xi)$ *where*

$$c_2(\xi) = \xi_1^2 k^2 + 2\tau_1 \xi_1 \xi_2 k^2 + |\tau|^2 \xi_2^2 k^2,$$
$$c_1(\xi) = (\delta_1(k^2) + \tau \delta_2(k^2))\xi_1 + (\bar{\tau}\delta_1(k^2) + |\tau|^2 \delta_2(k^2))\xi_2.$$

Proof. It follows easily from the composition rule explained in Proposition 3.3 and the fact that the symbols of ∂^*, left multiplication by k^2, and ∂ are equal to $\xi_1 + \tau\xi_2$, k^2, and $\xi_1 + \bar{\tau}\xi_2$. □

Following the same method as in Section 4.1, after a direct computation the corresponding b_2 term for the second half of the Laplacian $\triangle$, namely $\partial^* k^2 \partial$, is also given by a lengthy formula. It is interesting to note that unlike the corresponding term for the first half, we have now terms with complex coefficient i in front.

To integrate the second b_2 over the ξ-plane we use the change of variables (2), namely we let

$$\xi_1 = r\cos\theta - r\frac{\tau_1}{\tau_2}\sin\theta, \quad \xi_2 = \frac{r}{\tau_2}\sin\theta,$$

where θ ranges from 0 to 2π and r ranges from 0 to ∞.

After the integration with respect to θ, up to a factor of $\frac{r}{\tau_2}$ which is the Jacobian of the change of variables, one gets

$$\begin{aligned}
&4|\tau|^2\pi r^6 b_0^2 k^2 \delta_2(k) b_0^2 k^3 \delta_2(k) b_0 k + 4\tau_1 \pi r^6 b_0^2 k^2 \delta_2(k) b_0^2 k^3 \delta_1(k) b_0 k\\
&+ 4|\tau|^2\pi r^6 b_0^2 k^2 \delta_2(k) b_0^2 k^4 \delta_2(k) b_0 + 4\tau_1 \pi r^6 b_0^2 k^2 \delta_2(k) b_0^2 k^4 \delta_1(k) b_0\\
&+ 4\tau_1 \pi r^6 b_0^2 k^2 \delta_1(k) b_0^2 k^3 \delta_2(k) b_0 k + 4\pi r^6 b_0^2 k^2 \delta_1(k) b_0^2 k^3 \delta_1(k) b_0 k\\
&+ 4\tau_1 \pi r^6 b_0^2 k^2 \delta_1(k) b_0^2 k^4 \delta_2(k) b_0 + 4\pi r^6 b_0^2 k^2 \delta_1(k) b_0^2 k^4 \delta_1(k) b_0\\
&+ 4|\tau|^2\pi r^6 b_0^2 k^3 \delta_2(k) b_0^2 k^2 \delta_2(k) b_0 k + 4\tau_1 \pi r^6 b_0^2 k^3 \delta_2(k) b_0^2 k^2 \delta_1(k) b_0 k\\
&+ 4|\tau|^2\pi r^6 b_0^2 k^3 \delta_2(k) b_0^2 k^3 \delta_2(k) b_0 + 4\tau_1 \pi r^6 b_0^2 k^3 \delta_2(k) b_0^2 k^3 \delta_1(k) b_0\\
&+ 4\tau_1 \pi r^6 b_0^2 k^3 \delta_1(k) b_0^2 k^2 \delta_2(k) b_0 k + 4\pi r^6 b_0^2 k^3 \delta_1(k) b_0^2 k^2 \delta_1(k) b_0 k\\
&+ 4\tau_1 \pi r^6 b_0^2 k^3 \delta_1(k) b_0^2 k^3 \delta_2(k) b_0 + 4\pi r^6 b_0^2 k^3 \delta_1(k) b_0^2 k^3 \delta_1(k) b_0\\
&+ 8|\tau|^2\pi r^6 b_0^3 k^4 \delta_2(k) b_0 k \delta_2(k) b_0 k + 8\tau_1 \pi r^6 b_0^3 k^4 \delta_2(k) b_0 k \delta_1(k) b_0 k\\
&+ 8|\tau|^2\pi r^6 b_0^3 k^4 \delta_2(k) b_0 k^2 \delta_2(k) b_0 + 8\tau_1 \pi r^6 b_0^3 k^4 \delta_2(k) b_0 k^2 \delta_1(k) b_0\\
&+ 8\tau_1 \pi r^6 b_0^3 k^4 \delta_1(k) b_0 k \delta_2(k) b_0 k + 8\pi r^6 b_0^3 k^4 \delta_1(k) b_0 k \delta_1(k) b_0 k\\
&+ 8\tau_1 \pi r^6 b_0^3 k^4 \delta_1(k) b_0 k^2 \delta_2(k) b_0 + 8\pi r^6 b_0^3 k^4 \delta_1(k) b_0 k^2 \delta_1(k) b_0\\
&+ 8|\tau|^2\pi r^6 b_0^3 k^5 \delta_2(k) b_0 \delta_2(k) b_0 k + 8\tau_1 \pi r^6 b_0^3 k^5 \delta_2(k) b_0 \delta_1(k) b_0 k
\end{aligned}$$

$$
\begin{aligned}
&+ 8|\tau|^2\pi r^6 b_0^3 k^5 \delta_2(k) b_0 k \delta_2(k) b_0 + 8\tau_1 \pi r^6 b_0^3 k^5 \delta_2(k) b_0 k \delta_1(k) b_0 \\
&+ 8\tau_1 \pi r^6 b_0^3 k^5 \delta_1(k) b_0 \delta_2(k) b_0 k + 8\pi r^6 b_0^3 k^5 \delta_1(k) b_0 \delta_1(k) b_0 k \\
&+ 8\tau_1 \pi r^6 b_0^3 k^5 \delta_1(k) b_0 k \delta_2(k) b_0 + 8\pi r^6 b_0^3 k^5 \delta_1(k) b_0 k \delta_1(k) b_0 \\
&- 8|\tau|^2\pi r^4 b_0^2 k^2 \delta_2(k) b_0 k \delta_2(k) b_0 k - 8\tau_1 \pi r^4 b_0^2 k^2 \delta_2(k) b_0 k \delta_1(k) b_0 k \\
&- 8|\tau|^2\pi r^4 b_0^2 k^2 \delta_2(k) b_0 k^2 \delta_2(k) b_0 - 8\tau_1 \pi r^4 b_0^2 k^2 \delta_2(k) b_0 k^2 \delta_1(k) b_0 \\
&- 8\tau_1 \pi r^4 b_0^2 k^2 \delta_1(k) b_0 k \delta_2(k) b_0 k - 8\pi r^4 b_0^2 k^2 \delta_1(k) b_0 k \delta_1(k) b_0 k \\
&- 8\tau_1 \pi r^4 b_0^2 k^2 \delta_1(k) b_0 k^2 \delta_2(k) b_0 - 8\pi r^4 b_0^2 k^2 \delta_1(k) b_0 k^2 \delta_1(k) b_0 \\
&- 8|\tau|^2\pi r^4 b_0^2 k^3 \delta_2(k) b_0 \delta_2(k) b_0 k - 8\tau_1 \pi r^4 b_0^2 k^3 \delta_2(k) b_0 \delta_1(k) b_0 k \\
&- 8|\tau|^2\pi r^4 b_0^2 k^3 \delta_2(k) b_0 k \delta_2(k) b_0 - 8\tau_1 \pi r^4 b_0^2 k^3 \delta_2(k) b_0 k \delta_1(k) b_0 \\
&- 8\tau_1 \pi r^4 b_0^2 k^3 \delta_1(k) b_0 \delta_2(k) b_0 k - 8\pi r^4 b_0^2 k^3 \delta_1(k) b_0 \delta_1(k) b_0 k \\
&- 8\tau_1 \pi r^4 b_0^2 k^3 \delta_1(k) b_0 k \delta_2(k) b_0 - 8\pi r^4 b_0^2 k^3 \delta_1(k) b_0 k \delta_1(k) b_0 \\
&- 4\pi r^4 b_0^3 k^4 \delta_1^2(k) b_0 k - 4|\tau|^2 \pi r^4 b_0^3 k^4 \delta_2^2(k) b_0 k \\
&- 8\tau_1 \pi r^4 b_0^3 k^4 \delta_1 \delta_2(k) b_0 k - 8|\tau|^2 \pi r^4 b_0^3 k^4 \delta_2(k)^2 b_0 \\
&- 8\tau_1 \pi r^4 b_0^3 k^4 \delta_2(k) \delta_1(k) b_0 - 8\tau_1 \pi r^4 b_0^3 k^4 \delta_1(k) \delta_2(k) b_0 \\
&- 8\pi r^4 b_0^3 k^4 \delta_1(k)^2 b_0 - 4\pi r^4 b_0^3 k^5 \delta_1^2(k) b_0 \\
&- 4|\tau|^2 \pi r^4 b_0^3 k^5 \delta_2^2(k) b_0 - 8\tau_1 \pi r^4 b_0^3 k^5 \delta_1 \delta_2(k) b_0 \\
&- 2|\tau|^2 \pi r^4 b_0 \delta_2(k^2) b_0^2 k^2 \delta_2(k) b_0 k - 2(\tau_1 + i\tau_2) \pi r^4 b_0 \delta_2(k^2) b_0^2 k^2 \delta_1(k) b_0 k \\
&- 2|\tau|^2 \pi r^4 b_0 \delta_2(k^2) b_0^2 k^3 \delta_2(k) b_0 - 2(\tau_1 + i\tau_2) \pi r^4 b_0 \delta_2(k^2) b_0^2 k^3 \delta_1(k) b_0 \\
&- 2(\tau_1 - i\tau_2) \pi r^4 b_0 \delta_1(k^2) b_0^2 k^2 \delta_2(k) b_0 k - 2\pi r^4 b_0 \delta_1(k^2) b_0^2 k^2 \delta_1(k) b_0 k \\
&- 2(\tau_1 - i\tau_2) \pi r^4 b_0 \delta_1(k^2) b_0^2 k^3 \delta_2(k) b_0 - 2\pi r^4 b_0 \delta_1(k^2) b_0^2 k^3 \delta_1(k) b_0 \\
&- 2|\tau|^2 \pi r^4 b_0^2 k^2 \delta_2(k^2) b_0 \delta_2(k) b_0 k - 2(\tau_1 + i\tau_2) \pi r^4 b_0^2 k^2 \delta_2(k^2) b_0 \delta_1(k) b_0 k \\
&- 2|\tau|^2 \pi r^4 b_0^2 k^2 \delta_2(k^2) b_0 k \delta_2(k) b_0 - 2(\tau_1 + i\tau_2) \pi r^4 b_0^2 k^2 \delta_2(k^2) b_0 k \delta_1(k) b_0 \\
&- 2(\tau_1 - i\tau_2) \pi r^4 b_0^2 k^2 \delta_1(k^2) b_0 \delta_2(k) b_0 k - 2\pi r^4 b_0^2 k^2 \delta_1(k^2) b_0 \delta_1(k) b_0 k \\
&- 2(\tau_1 - i\tau_2) \pi r^4 b_0^2 k^2 \delta_1(k^2) b_0 k \delta_2(k) b_0 - 2\pi r^4 b_0^2 k^2 \delta_1(k^2) b_0 k \delta_1(k) b_0 \\
&- 2|\tau|^2 \pi r^4 b_0^2 k^2 \delta_2(k) b_0 k \delta_2(k^2) b_0 - 2(\tau_1 - i\tau_2) \pi r^4 b_0^2 k^2 \delta_2(k) b_0 k \delta_1(k^2) b_0 \\
&- 2(\tau_1 + i\tau_2) \pi r^4 b_0^2 k^2 \delta_1(k) b_0 k \delta_2(k^2) b_0 - 2\pi r^4 b_0^2 k^2 \delta_1(k) b_0 k \delta_1(k^2) b_0 \\
&- 2|\tau|^2 \pi r^4 b_0^2 k^3 \delta_2(k) b_0 \delta_2(k^2) b_0 - 2(\tau_1 - i\tau_2) \pi r^4 b_0^2 k^3 \delta_2(k) b_0 \delta_1(k^2) b_0 \\
&- 2(\tau_1 + i\tau_2) \pi r^4 b_0^2 k^3 \delta_1(k) b_0 \delta_2(k^2) b_0 - 2\pi r^4 b_0^2 k^3 \delta_1(k) b_0 \delta_1(k^2) b_0 \\
&+ 2\pi r^2 b_0^2 k^2 \delta_1^2(k) b_0 k + 2|\tau|^2 \pi r^2 b_0^2 k^2 \delta_2^2(k) b_0 k \\
&+ 4\tau_1 \pi r^2 b_0^2 k^2 \delta_1 \delta_2(k) b_0 k + 4|\tau|^2 \pi r^2 b_0^2 k^2 \delta_2(k)^2 b_0 \\
&+ 4\tau_1 \pi r^2 b_0^2 k^2 \delta_2(k) \delta_1(k) b_0 + 4\tau_1 \pi r^2 b_0^2 k^2 \delta_1(k) \delta_2(k) b_0 \\
&+ 4\pi r^2 b_0^2 k^2 \delta_1(k)^2 b_0 + 2\pi r^2 b_0^2 k^3 \delta_1^2(k) b_0
\end{aligned}
$$

$$
\begin{aligned}
&+2|\tau|^2\pi r^2 b_0^2 k^3 \delta_2^2(k) b_0 + 4\tau_1 \pi r^2 b_0^2 k^3 \delta_1\delta_2(k) b_0 \\
&+2|\tau|^2\pi r 2 b_0 \delta_2(k^2) b_0 \delta_2(k) b_0 k + 2(\tau_1 + i\tau_2)\pi r^2 b_0 \delta_2(k^2) b_0 \delta_1(k) b_0 k \\
&+2|\tau|^2\pi r^2 b_0 \delta_2(k^2) b_0 k \delta_2(k) b_0 + 2(\tau_1 + i\tau_2)\pi r^2 b_0 \delta_2(k^2) b_0 k \delta_1(k) b_0 \\
&+2(\tau_1 - i\tau_2)\pi r^2 b_0 \delta_1(k^2) b_0 \delta_2(k) b_0 k + 2\pi r^2 b_0 \delta_1(k^2) b_0 \delta_1(k) b_0 k \\
&+2(\tau_1 - i\tau_2)\pi r^2 b_0 \delta_1(k^2) b_0 k \delta_2(k) b_0 + 2\pi r^2 b_0 \delta_1(k^2) b_0 k \delta_1(k) b_0 \\
&+2\pi r^2 b_0^2 k^2 \delta_1^2(k^2) b_0 + 2|\tau|^2 \pi r^2 b_0^2 k^2 \delta_2^2(k^2) b_0 \\
&+4\tau_1 \pi r^2 b_0^2 k^2 \delta_1\delta_2(k^2) b_0 + 0 b_0 \delta_2(k^2) b_0 \delta_2(k^2) b_0 \\
&+0 b_0 \delta_2(k^2) b_0 \delta_1(k^2) b_0 + 0 b_0 \delta_1(k^2) b_0 \delta_2(k^2) b_0 \\
&+0 b_0 \delta_1(k^2) b_0 \delta_1(k^2) b_0,
\end{aligned}
$$

where

$$b_0 = (r^2k^2+1)^{-1}.$$

4.2.1. Terms with two factors of the form b_0^i, $i \geq 1$. These are the following terms:

$$
\begin{aligned}
&-4\pi r^4 b_0^3 k^4 \delta_1^2(k) b_0 k - 4|\tau|^2 \pi r^4 b_0^3 k^4 \delta_2^2(k) b_0 k - 8\tau_1 \pi r^4 b_0^3 k^4 \delta_1\delta_2(k) b_0 k \\
&-8|\tau|^2 \pi r^4 b_0^3 k^4 \delta_2(k)^2 b_0 - 8\tau_1 \pi r^4 b_0^3 k^4 \delta_2(k)\delta_1(k) b_0 \\
&-8\tau_1 \pi r^4 b_0^3 k^4 \delta_1(k)\delta_2(k) b_0 - 8\pi r^4 b_0^3 k^4 \delta_1(k)^2 b_0 - 4\pi r^4 b_0^3 k^5 \delta_1^2(k) b_0 \\
&-4|\tau|^2 \pi r^4 b_0^3 k^5 \delta_2^2(k) b_0 - 8\tau_1 \pi r^4 b_0^3 k^5 \delta_1\delta_2(k) b_0 + 2\pi r^2 b_0^2 k^2 \delta_1^2(k) b_0 k \\
&+2|\tau|^2 \pi r^2 b_0^2 k^2 \delta_2^2(k) b_0 k + 4\tau_1 \pi r^2 b_0^2 k^2 \delta_1\delta_2(k) b_0 k + 4|\tau|^2 \pi r^2 b_0^2 k^2 \delta_2(k)^2 b_0 \\
&+4\tau_1 \pi r^2 b_0^2 k^2 \delta_2(k)\delta_1(k) b_0 + 4\tau_1 \pi r^2 b_0^2 k^2 \delta_1(k)\delta_2(k) b_0 + 4\pi r^2 b_0^2 k^2 \delta_1(k)^2 b_0 \\
&+2\pi r^2 b_0^2 k^3 \delta_1^2(k) b_0 + 2|\tau|^2 \pi r^2 b_0^2 k^3 \delta_2^2(k) b_0 + 4\tau_1 \pi r^2 b_0^2 k^3 \delta_1\delta_2(k) b_0 \\
&+2\pi r^2 b_0^2 k^2 \delta_1^2(k^2) b_0 + 2|\tau|^2 \pi r^2 b_0^2 k^2 \delta_2^2(k^2) b_0 + 4\tau_1 \pi r^2 b_0^2 k^2 \delta_1\delta_2(k^2) b_0.
\end{aligned}
$$

Using Lemma 4.1, $\int_0^\infty \bullet\, r dr$ of these terms, up to an overall factor of π, is equal to

$$
\begin{aligned}
&-2\mathcal{D}_2(\Delta^{1/2}(k^{-1}\delta_1^2(k))) - 2|\tau|^2\mathcal{D}_2(\Delta^{1/2}(k^{-1}\delta_2^2(k))) \\
&-4\tau_1\mathcal{D}_2(\Delta^{1/2}(k^{-1}\delta_1\delta_2(k))) - 4|\tau|^2\mathcal{D}_2(k^{-2}\delta_2(k)^2) - 4\tau_1\mathcal{D}_2(k^{-2}\delta_2(k)\delta_1(k)) \\
&-4\tau_1\mathcal{D}_2(k^{-2}\delta_1(k)\delta_2(k)) - 4\mathcal{D}_2(k^{-2}\delta_1(k)^2) - 2\mathcal{D}_2(k^{-1}\delta_1^2(k)) \\
&-2|\tau|^2\mathcal{D}_2(k^{-1}\delta_2^2(k)) - 4\tau_1\mathcal{D}_2(k^{-1}\delta_1\delta_2(k)) + \mathcal{D}_1(\Delta^{1/2}(k^{-1}\delta_1^2(k))) \\
&+|\tau|^2\mathcal{D}_1(\Delta^{1/2}(k^{-1}\delta_2^2(k))) + 2\tau_1\mathcal{D}_1(\Delta^{1/2}(k^{-1}\delta_1\delta_2(k))) + 2|\tau|^2\mathcal{D}_1(k^{-2}\delta_2(k)^2) \\
&+2\tau_1\mathcal{D}_1(k^{-2}\delta_2(k)\delta_1(k)) + 2\tau_1\mathcal{D}_1(k^{-2}\delta_1(k)\delta_2(k)) + 2\mathcal{D}_1(k^{-2}\delta_1(k)^2) \\
&+\mathcal{D}_1(k^{-1}\delta_1^2(k)) + |\tau|^2\mathcal{D}_1(k^{-1}\delta_2^2(k)) + 2\tau_1\mathcal{D}_1(k^{-1}\delta_1\delta_2(k)) \\
&+\mathcal{D}_1(k^{-2}\delta_1^2(k^2)) + |\tau|^2\mathcal{D}_1(k^{-2}\delta_2^2(k^2)) + 2\tau_1\mathcal{D}_1(k^{-2}\delta_1\delta_2(k^2)).
\end{aligned}
$$

Therefore, up to an overall factor of π, the $\int_0^\infty \bullet r dr$ of the terms with two factors of powers of b_0 is equal to

$$\begin{aligned} &g_1(\Delta)(k^{-1}\delta_1^2(k)) + g_2(\Delta)(k^{-2}\delta_1(k)^2) \\ &\quad + |\tau|^2 g_1(\Delta)(k^{-1}\delta_2^2(k)) + |\tau|^2 g_2(\Delta)(k^{-2}\delta_2(k)^2) \\ &\quad + \tau_1 g_1(\Delta)(k^{-1}\delta_1\delta_2(k)) + \tau_1 g_2(\Delta)(k^{-2}\delta_1(k)\delta_2(k)) \\ &\quad + \tau_1 g_1(\Delta)(k^{-1}\delta_2\delta_1(k)) + \tau_1 g_2(\Delta)(k^{-2}\delta_2(k)\delta_1(k)), \end{aligned} \tag{7}$$

where

$$\begin{aligned} g_1(u) &:= -2\mathcal{L}_2(u)u^{1/2} - 2\mathcal{L}_2(u) + 2\mathcal{L}_1(u)u^{1/2} + 2\mathcal{L}_1(u) \\ &= \frac{-1 + u^2 - 2u\log u}{(-1 + u^{1/2})^3(1 + u^{1/2})^2} \end{aligned} \tag{8}$$

and

$$g_2(u) := -4\mathcal{L}_2(u) + 4\mathcal{L}_1(u) = 2\frac{-1 + u^2 - 2u\log u}{(-1 + u)^3}. \tag{9}$$

4.2.2. Terms with three factors of the form b_0^i, $i \geq 1$. These are the following terms:

$$\begin{aligned} &4|\tau|^2\pi r^6 b_0^2 k^2\delta_2(k)b_0^2 k^3\delta_2(k)b_0 k + 4\tau_1\pi r^6 b_0^2 k^2\delta_2(k)b_0^2 k^3\delta_1(k)b_0 k \\ &+ 4|\tau|^2\pi r^6 b_0^2 k^2\delta_2(k)b_0^2 k^4\delta_2(k)b_0 + 4\tau_1\pi r^6 b_0^2 k^2\delta_2(k)b_0^2 k^4\delta_1(k)b_0 \\ &+ 4\tau_1\pi r^6 b_0^2 k^2\delta_1(k)b_0^2 k^3\delta_2(k)b_0 k + 4\pi r^6 b_0^2 k^2\delta_1(k)b_0^2 k^3\delta_1(k)b_0 k \\ &+ 4\tau_1\pi r^6 b_0^2 k^2\delta_1(k)b_0^2 k^4\delta_2(k)b_0 + 4\pi r^6 b_0^2 k^2\delta_1(k)b_0^2 k^4\delta_1(k)b_0 \\ &+ 4|\tau|^2\pi r^6 b_0^2 k^3\delta_2(k)b_0^2 k^2\delta_2(k)b_0 k + 4\tau_1\pi r^6 b_0^2 k^3\delta_2(k)b_0^2 k^2\delta_1(k)b_0 k \\ &+ 4|\tau|^2\pi r^6 b_0^2 k^3\delta_2(k)b_0^2 k^3\delta_2(k)b_0 + 4\tau_1\pi r^6 b_0^2 k^3\delta_2(k)b_0^2 k^3\delta_1(k)b_0 \\ &+ 4\tau_1\pi r^6 b_0^2 k^3\delta_1(k)b_0^2 k^2\delta_2(k)b_0 k + 4\pi r^6 b_0^2 k^3\delta_1(k)b_0^2 k^2\delta_1(k)b_0 k \\ &+ 4\tau_1\pi r^6 b_0^2 k^3\delta_1(k)b_0^2 k^3\delta_2(k)b_0 + 4\pi r^6 b_0^2 k^3\delta_1(k)b_0^2 k^3\delta_1(k)b_0 \\ &+ 8|\tau|^2\pi r^6 b_0^3 k^4\delta_2(k)b_0 k\delta_2(k)b_0 k + 8\tau_1\pi r^6 b_0^3 k^4\delta_2(k)b_0 k\delta_1(k)b_0 k \\ &+ 8|\tau|^2\pi r^6 b_0^3 k^4\delta_2(k)b_0 k^2\delta_2(k)b_0 + 8\tau_1\pi r^6 b_0^3 k^4\delta_2(k)b_0 k^2\delta_1(k)b_0 \\ &+ 8\tau_1\pi r^6 b_0^3 k^4\delta_1(k)b_0 k\delta_2(k)b_0 k + 8\pi r^6 b_0^3 k^4\delta_1(k)b_0 k\delta_1(k)b_0 k \\ &+ 8\tau_1\pi r^6 b_0^3 k^4\delta_1(k)b_0 k^2\delta_2(k)b_0 + 8\pi r^6 b_0^3 k^4\delta_1(k)b_0 k^2\delta_1(k)b_0 \\ &+ 8|\tau|^2\pi r^6 b_0^3 k^5\delta_2(k)b_0\delta_2(k)b_0 k + 8\tau_1\pi r^6 b_0^3 k^5\delta_2(k)b_0\delta_1(k)b_0 k \\ &+ 8|\tau|^2\pi r^6 b_0^3 k^5\delta_2(k)b_0 k\delta_2(k)b_0 + 8\tau_1\pi r^6 b_0^3 k^5\delta_2(k)b_0 k\delta_1(k)b_0 \\ &+ 8\tau_1\pi r^6 b_0^3 k^5\delta_1(k)b_0\delta_2(k)b_0 k + 8\pi r^6 b_0^3 k^5\delta_1(k)b_0\delta_1(k)b_0 k \\ &+ 8\tau_1\pi r^6 b_0^3 k^5\delta_1(k)b_0 k\delta_2(k)b_0 + 8\pi r^6 b_0^3 k^5\delta_1(k)b_0 k\delta_1(k)b_0 \\ &- 8|\tau|^2\pi r^4 b_0^2 k^2\delta_2(k)b_0 k\delta_2(k)b_0 k - 8\tau_1\pi r^4 b_0^2 k^2\delta_2(k)b_0 k\delta_1(k)b_0 k \end{aligned}$$

$$
\begin{aligned}
& - 8|\tau|^2\pi r^4 b_0^2 k^2\delta_2(k)b_0k^2\delta_2(k)b_0 - 8\tau_1\pi r^4 b_0^2 k^2\delta_2(k)b_0k^2\delta_1(k)b_0 \\
& - 8\tau_1\pi r^4 b_0^2 k^2\delta_1(k)b_0k\delta_2(k)b_0k - 8\pi r^4 b_0^2 k^2\delta_1(k)b_0k\delta_1(k)b_0k \\
& - 8\tau_1\pi r^4 b_0^2 k^2\delta_1(k)b_0k^2\delta_2(k)b_0 - 8\pi r^4 b_0^2 k^2\delta_1(k)b_0k^2\delta_1(k)b_0 \\
& - 8|\tau|^2\pi r^4 b_0^2 k^3\delta_2(k)b_0\delta_2(k)b_0k - 8\tau_1\pi r^4 b_0^2 k^3\delta_2(k)b_0\delta_1(k)b_0k \\
& - 8|\tau|^2\pi r^4 b_0^2 k^3\delta_2(k)b_0k\delta_2(k)b_0 - 8\tau_1\pi r^4 b_0^2 k^3\delta_2(k)b_0k\delta_1(k)b_0 \\
& - 8\tau_1\pi r^4 b_0^2 k^3\delta_1(k)b_0\delta_2(k)b_0k - 8\pi r^4 b_0^2 k^3\delta_1(k)b_0\delta_1(k)b_0k \\
& - 8\tau_1\pi r^4 b_0^2 k^3\delta_1(k)b_0k\delta_2(k)b_0 - 8\pi r^4 b_0^2 k^3\delta_1(k)b_0k\delta_1(k)b_0 \\
& - 2|\tau|^2\pi r^4 b_0\delta_2(k^2)b_0^2k^2\delta_2(k)b_0k - 2(\tau_1+i\tau_2)\pi r^4 b_0\delta_2(k^2)b_0^2k^2\delta_1(k)b_0k \\
& - 2|\tau|^2\pi r^4 b_0\delta_2(k^2)b_0^2k^3\delta_2(k)b_0 - 2(\tau_1+i\tau_2)\pi r^4 b_0\delta_2(k^2)b_0^2k^3\delta_1(k)b_0 \\
& - 2(\tau_1-i\tau_2)\pi r^4 b_0\delta_1(k^2)b_0^2k^2\delta_2(k)b_0k - 2\pi r^4 b_0\delta_1(k^2)b_0^2k^2\delta_1(k)b_0k \\
& - 2(\tau_1-i\tau_2)\pi r^4 b_0\delta_1(k^2)b_0^2k^3\delta_2(k)b_0 - 2\pi r^4 b_0\delta_1(k^2)b_0^2k^3\delta_1(k)b_0 \\
& - 2|\tau|^2\pi r^4 b_0^2k^2\delta_2(k^2)b_0\delta_2(k)b_0k - 2(\tau_1+i\tau_2)\pi r^4 b_0^2k^2\delta_2(k^2)b_0\delta_1(k)b_0k \\
& - 2|\tau|^2\pi r^4 b_0^2k^2\delta_2(k^2)b_0k\delta_2(k)b_0 - 2(\tau_1+i\tau_2)\pi r^4 b_0^2k^2\delta_2(k^2)b_0k\delta_1(k)b_0 \\
& - 2(\tau_1-i\tau_2)\pi r^4 b_0^2k^2\delta_1(k^2)b_0\delta_2(k)b_0k - 2\pi r^4 b_0^2k^2\delta_1(k^2)b_0\delta_1(k)b_0k \\
& - 2(\tau_1-i\tau_2)\pi r^4 b_0^2k^2\delta_1(k^2)b_0k\delta_2(k)b_0 - 2\pi r^4 b_0^2k^2\delta_1(k^2)b_0k\delta_1(k)b_0 \\
& - 2|\tau|^2\pi r^4 b_0^2k^2\delta_2(k)b_0k\delta_2(k^2)b_0 - 2(\tau_1-i\tau_2)\pi r^4 b_0^2k^2\delta_2(k)b_0k\delta_1(k^2)b_0 \\
& - 2(\tau_1+i\tau_2)\pi r^4 b_0^2k^2\delta_1(k)b_0k\delta_2(k^2)b_0 - 2\pi r^4 b_0^2k^2\delta_1(k)b_0k\delta_1(k^2)b_0 \\
& - 2|\tau|^2\pi r^4 b_0^2k^3\delta_2(k)b_0\delta_2(k^2)b_0 - 2(\tau_1-i\tau_2)\pi r^4 b_0^2k^3\delta_2(k)b_0\delta_1(k^2)b_0 \\
& - 2(\tau_1+i\tau_2)\pi r^4 b_0^2k^3\delta_1(k)b_0\delta_2(k^2)b_0 - 2\pi r^4 b_0^2k^3\delta_1(k)b_0\delta_1(k^2)b_0 \\
& + 2|\tau|^2\pi r2 b_0\delta_2(k^2)b_0\delta_2(k)b_0k + 2(\tau_1+i\tau_2)\pi r^2 b_0\delta_2(k^2)b_0\delta_1(k)b_0k \\
& + 2|\tau|^2\pi r^2 b_0\delta_2(k^2)b_0k\delta_2(k)b_0 + 2(\tau_1+i\tau_2)\pi r^2 b_0\delta_2(k^2)b_0k\delta_1(k)b_0 \\
& + 2(\tau_1-i\tau_2)\pi r^2 b_0\delta_1(k^2)b_0\delta_2(k)b_0k + 2\pi r^2 b_0\delta_1(k^2)b_0\delta_1(k)b_0k \\
& + 2(\tau_1-i\tau_2)\pi r^2 b_0\delta_1(k^2)b_0k\delta_2(k)b_0 + 2\pi r^2 b_0\delta_1(k^2)b_0k\delta_1(k)b_0 \\
& + 0b_0\delta_2(k^2)b_0\delta_2(k^2)b_0 + 0b_0\delta_2(k^2)b_0\delta_1(k^2)b_0 \\
& + 0b_0\delta_1(k^2)b_0\delta_2(k^2)b_0 + 0b_0\delta_1(k^2)b_0\delta_1(k^2)b_0.
\end{aligned}
$$

Using Lemma 4.2 we compute $\int_0^\infty \bullet\, r dr$ of these terms, and the result, up to an overall factor of π, is equal to:

$$
\begin{aligned}
& 2\mathcal{D}_{2,2}(\Delta_{(1)},\Delta_{(2)})(\Delta^{-1}(\delta_1(k)k^{-1})\Delta^{1/2}(k^{-1}\delta_1(k))) \\
& + 2\mathcal{D}_{2,2}(\Delta_{(1)},\Delta_{(2)})(\Delta^{-1}(\delta_1(k)k^{-1})(k^{-1}\delta_1(k))) \\
& + 2\mathcal{D}_{2,2}(\Delta_{(1)},\Delta_{(2)})(\Delta^{-3/2}(\delta_1(k)k^{-1})\Delta^{1/2}(k^{-1}\delta_1(k))) \\
& + 2\mathcal{D}_{2,2}(\Delta_{(1)},\Delta_{(2)})(\Delta^{-3/2}(\delta_1(k)k^{-1})(k^{-1}\delta_1(k))) \\
& + 4\mathcal{D}_{3,1}(\Delta_{(1)},\Delta_{(2)})(\Delta^{-2}(\delta_1(k)k^{-1})\Delta^{1/2}(k^{-1}\delta_1(k)))
\end{aligned}
$$

$$
\begin{aligned}
&+ 4\mathcal{D}_{3,1}(\Delta_{(1)}, \Delta_{(2)})(\Delta^{-2}(\delta_1(k)k^{-1})(k^{-1}\delta_1(k)))\\
&+ 4\mathcal{D}_{3,1}(\Delta_{(1)}, \Delta_{(2)})(\Delta^{-5/2}(\delta_1(k)k^{-1})\Delta^{1/2}(k^{-1}\delta_1(k)))\\
&+ 4\mathcal{D}_{3,1}(\Delta_{(1)}, \Delta_{(2)})(\Delta^{-5/2}(\delta_1(k)k^{-1})(k^{-1}\delta_1(k)))\\
&- 4\mathcal{D}_{2,1}(\Delta_{(1)}, \Delta_{(2)})(\Delta^{-1}(\delta_1(k)k^{-1})\Delta^{1/2}(k^{-1}\delta_1(k)))\\
&- 4\mathcal{D}_{2,1}(\Delta_{(1)}, \Delta_{(2)})(\Delta^{-1}(\delta_1(k)k^{-1})(k^{-1}\delta_1(k)))\\
&- 4\mathcal{D}_{2,1}(\Delta_{(1)}, \Delta_{(2)})(\Delta^{-3/2}(\delta_1(k)k^{-1})\Delta^{1/2}(k^{-1}\delta_1(k)))\\
&- 4\mathcal{D}_{2,1}(\Delta_{(1)}, \Delta_{(2)})(\Delta^{-3/2}(\delta_1(k)k^{-1})(k^{-1}\delta_1(k)))\\
&- \mathcal{D}_{1,2}(\Delta_{(1)}, \Delta_{(2)})((\delta_1(k)k^{-1})\Delta^{1/2}(k^{-1}\delta_1(k)))\\
&- \mathcal{D}_{1,2}(\Delta_{(1)}, \Delta_{(2)})(\Delta^{-1/2}(\delta_1(k)k^{-1})\Delta^{1/2}(k^{-1}\delta_1(k)))\\
&- \mathcal{D}_{1,2}(\Delta_{(1)}, \Delta_{(2)})(\Delta^{-1/2}(\delta_1(k)k^{-1})(k^{-1}\delta_1(k)))\\
&- \mathcal{D}_{1,2}(\Delta_{(1)}, \Delta_{(2)})((\delta_1(k)k^{-1})(k^{-1}\delta_1(k)))\\
&- \mathcal{D}_{2,1}(\Delta_{(1)}, \Delta_{(2)})(\Delta^{-3/2}(\delta_1(k)k^{-1})\Delta^{1/2}(k^{-1}\delta_1(k)))\\
&- \mathcal{D}_{2,1}(\Delta_{(1)}, \Delta_{(2)})(\Delta^{-1}(\delta_1(k)k^{-1})\Delta^{1/2}(k^{-1}\delta_1(k)))\\
&- \mathcal{D}_{2,1}(\Delta_{(1)}, \Delta_{(2)})(\Delta^{-3/2}(\delta_1(k)k^{-1})(k^{-1}\delta_1(k)))\\
&- \mathcal{D}_{2,1}(\Delta_{(1)}, \Delta_{(2)})(\Delta^{-1}(\delta_1(k)k^{-1})(k^{-1}\delta_1(k)))\\
&- \mathcal{D}_{2,1}(\Delta_{(1)}, \Delta_{(2)})(\Delta^{-1}(\delta_1(k)k^{-1})(k^{-1}\delta_1(k)))\\
&- \mathcal{D}_{2,1}(\Delta_{(1)}, \Delta_{(2)})(\Delta^{-1}(\delta_1(k)k^{-1})\Delta^{1/2}(k^{-1}\delta_1(k)))\\
&- \mathcal{D}_{2,1}(\Delta_{(1)}, \Delta_{(2)})(\Delta^{-3/2}(\delta_1(k)k^{-1})(k^{-1}\delta_1(k)))\\
&- \mathcal{D}_{2,1}(\Delta_{(1)}, \Delta_{(2)})(\Delta^{-3/2}(\delta_1(k)k^{-1})\Delta^{1/2}(k^{-1}\delta_1(k)))\\
&+ \mathcal{D}_{1,1}(\Delta_{(1)}, \Delta_{(2)})(\Delta^{-1/2}(\delta_1(k)k^{-1})\Delta^{1/2}(k^{-1}\delta_1(k)))\\
&+ \mathcal{D}_{1,1}(\Delta_{(1)}, \Delta_{(2)})((\delta_1(k)k^{-1})\Delta^{1/2}(k^{-1}\delta_1(k)))\\
&+ \mathcal{D}_{1,1}(\Delta_{(1)}, \Delta_{(2)})(\Delta^{-1/2}(\delta_1(k)k^{-1})(k^{-1}\delta_1(k)))\\
&+ \mathcal{D}_{1,1}(\Delta_{(1)}, \Delta_{(2)})((\delta_1(k)k^{-1})(k^{-1}\delta_1(k)))\\
&+ |\tau|^2\Big(2\mathcal{D}_{2,2}(\Delta_{(1)}, \Delta_{(2)})(\Delta^{-1}(\delta_2(k)k^{-1})\Delta^{1/2}(k^{-1}\delta_2(k)))\\
&+ 2\mathcal{D}_{2,2}(\Delta_{(1)}, \Delta_{(2)})(\Delta^{-1}(\delta_2(k)k^{-1})(k^{-1}\delta_2(k)))\\
&+ 2\mathcal{D}_{2,2}(\Delta_{(1)}, \Delta_{(2)})(\Delta^{-3/2}(\delta_2(k)k^{-1})\Delta^{1/2}(k^{-1}\delta_2(k)))\\
&+ 2\mathcal{D}_{2,2}(\Delta_{(1)}, \Delta_{(2)})(\Delta^{-3/2}(\delta_2(k)k^{-1})(k^{-1}\delta_2(k)))\\
&+ 4\mathcal{D}_{3,1}(\Delta_{(1)}, \Delta_{(2)})(\Delta^{-2}(\delta_2(k)k^{-1})\Delta^{1/2}(k^{-1}\delta_2(k)))\\
&+ 4\mathcal{D}_{3,1}(\Delta_{(1)}, \Delta_{(2)})(\Delta^{-2}(\delta_2(k)k^{-1})(k^{-1}\delta_2(k)))\\
&+ 4\mathcal{D}_{3,1}(\Delta_{(1)}, \Delta_{(2)})(\Delta^{-5/2}(\delta_2(k)k^{-1})\Delta^{1/2}(k^{-1}\delta_2(k)))\\
&+ 4\mathcal{D}_{3,1}(\Delta_{(1)}, \Delta_{(2)})(\Delta^{-5/2}(\delta_2(k)k^{-1})(k^{-1}\delta_2(k)))
\end{aligned}
$$

$$
\begin{aligned}
&- 4\mathcal{D}_{2,1}(\Delta_{(1)}, \Delta_{(2)})(\Delta^{-1}(\delta_2(k)k^{-1})\Delta^{1/2}(k^{-1}\delta_2(k)))\\
&- 4\mathcal{D}_{2,1}(\Delta_{(1)}, \Delta_{(2)})(\Delta^{-1}(\delta_2(k)k^{-1})(k^{-1}\delta_2(k)))\\
&- 4\mathcal{D}_{2,1}(\Delta_{(1)}, \Delta_{(2)})(\Delta^{-3/2}(\delta_2(k)k^{-1})\Delta^{1/2}(k^{-1}\delta_2(k)))\\
&- 4\mathcal{D}_{2,1}(\Delta_{(1)}, \Delta_{(2)})(\Delta^{-3/2}(\delta_2(k)k^{-1})(k^{-1}\delta_2(k)))\\
&- \mathcal{D}_{1,2}(\Delta_{(1)}, \Delta_{(2)})((\delta_2(k)k^{-1})\Delta^{1/2}(k^{-1}\delta_2(k)))\\
&- \mathcal{D}_{1,2}(\Delta_{(1)}, \Delta_{(2)})(\Delta^{-1/2}(\delta_2(k)k^{-1})\Delta^{1/2}(k^{-1}\delta_2(k)))\\
&- \mathcal{D}_{1,2}(\Delta_{(1)}, \Delta_{(2)})(\Delta^{-1/2}(\delta_2(k)k^{-1})(k^{-1}\delta_2(k)))\\
&- \mathcal{D}_{1,2}(\Delta_{(1)}, \Delta_{(2)})((\delta_2(k)k^{-1})(k^{-1}\delta_2(k)))\\
&- \mathcal{D}_{2,1}(\Delta_{(1)}, \Delta_{(2)})(\Delta^{-3/2}(\delta_2(k)k^{-1})\Delta^{1/2}(k^{-1}\delta_2(k)))\\
&- \mathcal{D}_{2,1}(\Delta_{(1)}, \Delta_{(2)})(\Delta^{-1}(\delta_2(k)k^{-1})\Delta^{1/2}(k^{-1}\delta_2(k)))\\
&- \mathcal{D}_{2,1}(\Delta_{(1)}, \Delta_{(2)})(\Delta^{-3/2}(\delta_2(k)k^{-1})(k^{-1}\delta_2(k)))\\
&- \mathcal{D}_{2,1}(\Delta_{(1)}, \Delta_{(2)})(\Delta^{-1}(\delta_2(k)k^{-1})(k^{-1}\delta_2(k)))\\
&- \mathcal{D}_{2,1}(\Delta_{(1)}, \Delta_{(2)})(\Delta^{-1}(\delta_2(k)k^{-1})(k^{-1}\delta_2(k)))\\
&- \mathcal{D}_{2,1}(\Delta_{(1)}, \Delta_{(2)})(\Delta^{-1}(\delta_2(k)k^{-1})\Delta^{1/2}(k^{-1}\delta_2(k)))\\
&- \mathcal{D}_{2,1}(\Delta_{(1)}, \Delta_{(2)})(\Delta^{-3/2}(\delta_2(k)k^{-1})(k^{-1}\delta_2(k)))\\
&- \mathcal{D}_{2,1}(\Delta_{(1)}, \Delta_{(2)})(\Delta^{-3/2}(\delta_2(k)k^{-1})\Delta^{1/2}(k^{-1}\delta_2(k)))\\
&+ \mathcal{D}_{1,1}(\Delta_{(1)}, \Delta_{(2)})(\Delta^{-1/2}(\delta_2(k)k^{-1})\Delta^{1/2}(k^{-1}\delta_2(k)))\\
&+ \mathcal{D}_{1,1}(\Delta_{(1)}, \Delta_{(2)})((\delta_2(k)k^{-1})\Delta^{1/2}(k^{-1}\delta_2(k)))\\
&+ \mathcal{D}_{1,1}(\Delta_{(1)}, \Delta_{(2)})(\Delta^{-1/2}(\delta_2(k)k^{-1})(k^{-1}\delta_2(k)))\\
&+ \mathcal{D}_{1,1}(\Delta_{(1)}, \Delta_{(2)})((\delta_2(k)k^{-1})(k^{-1}\delta_2(k)))\Big)\\
&+ \tau_1\Big(2\mathcal{D}_{2,2}(\Delta_{(1)}, \Delta_{(2)})(\Delta^{-1}(\delta_1(k)k^{-1})\Delta^{1/2}(k^{-1}\delta_2(k)))\\
&+ 2\mathcal{D}_{2,2}(\Delta_{(1)}, \Delta_{(2)})(\Delta^{-1}(\delta_1(k)k^{-1})(k^{-1}\delta_2(k)))\\
&+ 2\mathcal{D}_{2,2}(\Delta_{(1)}, \Delta_{(2)})(\Delta^{-3/2}(\delta_1(k)k^{-1})\Delta^{1/2}(k^{-1}\delta_2(k)))\\
&+ 2\mathcal{D}_{2,2}(\Delta_{(1)}, \Delta_{(2)})(\Delta^{-3/2}(\delta_1(k)k^{-1})(k^{-1}\delta_2(k)))\\
&+ 4\mathcal{D}_{3,1}(\Delta_{(1)}, \Delta_{(2)})(\Delta^{-2}(\delta_1(k)k^{-1})\Delta^{1/2}(k^{-1}\delta_2(k)))\\
&+ 4\mathcal{D}_{3,1}(\Delta_{(1)}, \Delta_{(2)})(\Delta^{-2}(\delta_1(k)k^{-1})(k^{-1}\delta_2(k)))\\
&+ 4\mathcal{D}_{3,1}(\Delta_{(1)}, \Delta_{(2)})(\Delta^{-5/2}(\delta_1(k)k^{-1})\Delta^{1/2}(k^{-1}\delta_2(k)))\\
&+ 4\mathcal{D}_{3,1}(\Delta_{(1)}, \Delta_{(2)})(\Delta^{-5/2}(\delta_1(k)k^{-1})(k^{-1}\delta_2(k)))\\
&- 4\mathcal{D}_{2,1}(\Delta_{(1)}, \Delta_{(2)})(\Delta^{-1}(\delta_1(k)k^{-1})\Delta^{1/2}(k^{-1}\delta_2(k)))\\
&- 4\mathcal{D}_{2,1}(\Delta_{(1)}, \Delta_{(2)})(\Delta^{-1}(\delta_1(k)k^{-1})(k^{-1}\delta_2(k)))\\
&- 4\mathcal{D}_{2,1}(\Delta_{(1)}, \Delta_{(2)})(\Delta^{-3/2}(\delta_1(k)k^{-1})\Delta^{1/2}(k^{-1}\delta_2(k)))
\end{aligned}
$$

$$
\begin{aligned}
&- 4\mathcal{D}_{2,1}(\Delta_{(1)}, \Delta_{(2)})(\Delta^{-3/2}(\delta_1(k)k^{-1})(k^{-1}\delta_2(k)))\\
&- \mathcal{D}_{1,2}(\Delta_{(1)}, \Delta_{(2)})((\delta_1(k)k^{-1})\Delta^{1/2}(k^{-1}\delta_2(k)))\\
&- \mathcal{D}_{1,2}(\Delta_{(1)}, \Delta_{(2)})(\Delta^{-1/2}(\delta_1(k)k^{-1})\Delta^{1/2}(k^{-1}\delta_2(k)))\\
&- \mathcal{D}_{1,2}(\Delta_{(1)}, \Delta_{(2)})(\Delta^{-1/2}(\delta_1(k)k^{-1})(k^{-1}\delta_2(k)))\\
&- \mathcal{D}_{1,2}(\Delta_{(1)}, \Delta_{(2)})((\delta_1(k)k^{-1})(k^{-1}\delta_2(k)))\\
&- \mathcal{D}_{2,1}(\Delta_{(1)}, \Delta_{(2)})(\Delta^{-3/2}(\delta_1(k)k^{-1})\Delta^{1/2}(k^{-1}\delta_2(k)))\\
&- \mathcal{D}_{2,1}(\Delta_{(1)}, \Delta_{(2)})(\Delta^{-1}(\delta_1(k)k^{-1})\Delta^{1/2}(k^{-1}\delta_2(k)))\\
&- \mathcal{D}_{2,1}(\Delta_{(1)}, \Delta_{(2)})(\Delta^{-3/2}(\delta_1(k)k^{-1})(k^{-1}\delta_2(k)))\\
&- \mathcal{D}_{2,1}(\Delta_{(1)}, \Delta_{(2)})(\Delta^{-1}(\delta_1(k)k^{-1})(k^{-1}\delta_2(k)))\\
&- \mathcal{D}_{2,1}(\Delta_{(1)}, \Delta_{(2)})(\Delta^{-1}(\delta_1(k)k^{-1})(k^{-1}\delta_2(k)))\\
&- \mathcal{D}_{2,1}(\Delta_{(1)}, \Delta_{(2)})(\Delta^{-1}(\delta_1(k)k^{-1})\Delta^{1/2}(k^{-1}\delta_2(k)))\\
&- \mathcal{D}_{2,1}(\Delta_{(1)}, \Delta_{(2)})(\Delta^{-3/2}(\delta_1(k)k^{-1})(k^{-1}\delta_2(k)))\\
&- \mathcal{D}_{2,1}(\Delta_{(1)}, \Delta_{(2)})(\Delta^{-3/2}(\delta_1(k)k^{-1})\Delta^{1/2}(k^{-1}\delta_2(k)))\\
&+ \mathcal{D}_{1,1}(\Delta_{(1)}, \Delta_{(2)})(\Delta^{-1/2}(\delta_1(k)k^{-1})\Delta^{1/2}(k^{-1}\delta_2(k)))\\
&+ \mathcal{D}_{1,1}(\Delta_{(1)}, \Delta_{(2)})((\delta_1(k)k^{-1})\Delta^{1/2}(k^{-1}\delta_2(k)))\\
&+ \mathcal{D}_{1,1}(\Delta_{(1)}, \Delta_{(2)})(\Delta^{-1/2}(\delta_1(k)k^{-1})(k^{-1}\delta_2(k)))\\
&+ \mathcal{D}_{1,1}(\Delta_{(1)}, \Delta_{(2)})((\delta_1(k)k^{-1})(k^{-1}\delta_2(k)))\\
&+ 2\mathcal{D}_{2,2}(\Delta_{(1)}, \Delta_{(2)})(\Delta^{-1}(\delta_2(k)k^{-1})\Delta^{1/2}(k^{-1}\delta_1(k)))\\
&+ 2\mathcal{D}_{2,2}(\Delta_{(1)}, \Delta_{(2)})(\Delta^{-1}(\delta_2(k)k^{-1})(k^{-1}\delta_1(k)))\\
&+ 2\mathcal{D}_{2,2}(\Delta_{(1)}, \Delta_{(2)})(\Delta^{-3/2}(\delta_2(k)k^{-1})\Delta^{1/2}(k^{-1}\delta_1(k)))\\
&+ 2\mathcal{D}_{2,2}(\Delta_{(1)}, \Delta_{(2)})(\Delta^{-3/2}(\delta_2(k)k^{-1})(k^{-1}\delta_1(k)))\\
&+ 4\mathcal{D}_{3,1}(\Delta_{(1)}, \Delta_{(2)})(\Delta^{-2}(\delta_2(k)k^{-1})\Delta^{1/2}(k^{-1}\delta_1(k)))\\
&+ 4\mathcal{D}_{3,1}(\Delta_{(1)}, \Delta_{(2)})(\Delta^{-2}(\delta_2(k)k^{-1})(k^{-1}\delta_1(k)))\\
&+ 4\mathcal{D}_{3,1}(\Delta_{(1)}, \Delta_{(2)})(\Delta^{-5/2}(\delta_2(k)k^{-1})\Delta^{1/2}(k^{-1}\delta_1(k)))\\
&+ 4\mathcal{D}_{3,1}(\Delta_{(1)}, \Delta_{(2)})(\Delta^{-5/2}(\delta_2(k)k^{-1})(k^{-1}\delta_1(k)))\\
&- 4\mathcal{D}_{2,1}(\Delta_{(1)}, \Delta_{(2)})(\Delta^{-1}(\delta_2(k)k^{-1})\Delta^{1/2}(k^{-1}\delta_1(k)))\\
&- 4\mathcal{D}_{2,1}(\Delta_{(1)}, \Delta_{(2)})(\Delta^{-1}(\delta_2(k)k^{-1})(k^{-1}\delta_1(k)))\\
&- 4\mathcal{D}_{2,1}(\Delta_{(1)}, \Delta_{(2)})(\Delta^{-3/2}(\delta_2(k)k^{-1})\Delta^{1/2}(k^{-1}\delta_1(k)))\\
&- 4\mathcal{D}_{2,1}(\Delta_{(1)}, \Delta_{(2)})(\Delta^{-3/2}(\delta_2(k)k^{-1})(k^{-1}\delta_1(k)))\\
&- \mathcal{D}_{1,2}(\Delta_{(1)}, \Delta_{(2)})((\delta_2(k)k^{-1})\Delta^{1/2}(k^{-1}\delta_1(k)))\\
&- \mathcal{D}_{1,2}(\Delta_{(1)}, \Delta_{(2)})(\Delta^{-1/2}(\delta_2(k)k^{-1})\Delta^{1/2}(k^{-1}\delta_1(k)))
\end{aligned}
$$

$$
\begin{aligned}
&- \mathcal{D}_{1,2}(\Delta_{(1)}, \Delta_{(2)})(\Delta^{-1/2}(\delta_2(k)k^{-1})(k^{-1}\delta_1(k)))\\
&- \mathcal{D}_{1,2}(\Delta_{(1)}, \Delta_{(2)})((\delta_2(k)k^{-1})(k^{-1}\delta_1(k)))\\
&- \mathcal{D}_{2,1}(\Delta_{(1)}, \Delta_{(2)})(\Delta^{-3/2}(\delta_2(k)k^{-1})\Delta^{1/2}(k^{-1}\delta_1(k)))\\
&- \mathcal{D}_{2,1}(\Delta_{(1)}, \Delta_{(2)})(\Delta^{-1}(\delta_2(k)k^{-1})\Delta^{1/2}(k^{-1}\delta_1(k)))\\
&- \mathcal{D}_{2,1}(\Delta_{(1)}, \Delta_{(2)})(\Delta^{-3/2}(\delta_2(k)k^{-1})(k^{-1}\delta_1(k)))\\
&- \mathcal{D}_{2,1}(\Delta_{(1)}, \Delta_{(2)})(\Delta^{-1}(\delta_2(k)k^{-1})(k^{-1}\delta_1(k)))\\
&- \mathcal{D}_{2,1}(\Delta_{(1)}, \Delta_{(2)})(\Delta^{-1}(\delta_2(k)k^{-1})(k^{-1}\delta_1(k)))\\
&- \mathcal{D}_{2,1}(\Delta_{(1)}, \Delta_{(2)})(\Delta^{-1}(\delta_2(k)k^{-1})\Delta^{1/2}(k^{-1}\delta_1(k)))\\
&- \mathcal{D}_{2,1}(\Delta_{(1)}, \Delta_{(2)})(\Delta^{-3/2}(\delta_2(k)k^{-1})(k^{-1}\delta_1(k)))\\
&- \mathcal{D}_{2,1}(\Delta_{(1)}, \Delta_{(2)})(\Delta^{-3/2}(\delta_2(k)k^{-1})\Delta^{1/2}(k^{-1}\delta_1(k)))\\
&+ \mathcal{D}_{1,1}(\Delta_{(1)}, \Delta_{(2)})(\Delta^{-1/2}(\delta_2(k)k^{-1})\Delta^{1/2}(k^{-1}\delta_1(k)))\\
&+ \mathcal{D}_{1,1}(\Delta_{(1)}, \Delta_{(2)})((\delta_2(k)k^{-1})\Delta^{1/2}(k^{-1}\delta_1(k)))\\
&+ \mathcal{D}_{1,1}(\Delta_{(1)}, \Delta_{(2)})(\Delta^{-1/2}(\delta_2(k)k^{-1})(k^{-1}\delta_1(k)))\\
&+ \mathcal{D}_{1,1}(\Delta_{(1)}, \Delta_{(2)})((\delta_2(k)k^{-1})(k^{-1}\delta_1(k)))\Big)\\
&- i\tau_2\Big(- \mathcal{D}_{1,2}(\Delta_{(1)}, \Delta_{(2)})(\Delta^{-1/2}(\delta_1(k)k^{-1})\Delta^{1/2}(k^{-1}\delta_2(k)))\\
&- \mathcal{D}_{1,2}(\Delta_{(1)}, \Delta_{(2)})((\delta_1(k)k^{-1})\Delta^{1/2}(k^{-1}\delta_2(k)))\\
&- \mathcal{D}_{1,2}(\Delta_{(1)}, \Delta_{(2)})(\Delta^{-1/2}(\delta_1(k)k^{-1})(k^{-1}\delta_2(k)))\\
&- \mathcal{D}_{1,2}(\Delta_{(1)}, \Delta_{(2)})((\delta_1(k)k^{-1})(k^{-1}\delta_2(k)))\\
&- \mathcal{D}_{2,1}(\Delta_{(1)}, \Delta_{(2)})(\Delta^{-3/2}(\delta_1(k)k^{-1})\Delta^{1/2}(k^{-1}\delta_2(k)))\\
&- \mathcal{D}_{2,1}(\Delta_{(1)}, \Delta_{(2)})(\Delta^{-1}(\delta_1(k)k^{-1})\Delta^{1/2}(k^{-1}\delta_2(k)))\\
&- \mathcal{D}_{2,1}(\Delta_{(1)}, \Delta_{(2)})(\Delta^{-3/2}(\delta_1(k)k^{-1})(k^{-1}\delta_2(k)))\\
&- \mathcal{D}_{2,1}(\Delta_{(1)}, \Delta_{(2)})(\Delta^{-1}(\delta_1(k)k^{-1})(k^{-1}\delta_2(k)))\\
&+ \mathcal{D}_{2,1}(\Delta_{(1)}, \Delta_{(2)})(\Delta^{-1}(\delta_1(k)k^{-1})(k^{-1}\delta_2(k)))\\
&+ \mathcal{D}_{2,1}(\Delta_{(1)}, \Delta_{(2)})(\Delta^{-1}(\delta_1(k)k^{-1})\Delta^{1/2}(k^{-1}\delta_2(k)))\\
&+ \mathcal{D}_{2,1}(\Delta_{(1)}, \Delta_{(2)})(\Delta^{-3/2}(\delta_1(k)k^{-1})(k^{-1}\delta_2(k)))\\
&+ \mathcal{D}_{2,1}(\Delta_{(1)}, \Delta_{(2)})(\Delta^{-3/2}(\delta_1(k)k^{-1})\Delta^{1/2}(k^{-1}\delta_2(k)))\\
&+ \mathcal{D}_{1,1}(\Delta_{(1)}, \Delta_{(2)})(\Delta^{-1/2}(\delta_1(k)k^{-1/2})\Delta^{1/2}(k^{-1}\delta_2(k)))\\
&+ \mathcal{D}_{1,1}(\Delta_{(1)}, \Delta_{(2)})((\delta_1(k)k^{-1})\Delta^{1/2}(k^{-1}\delta_2(k)))\\
&+ \mathcal{D}_{1,1}(\Delta_{(1)}, \Delta_{(2)})(\Delta^{-1/2}(\delta_1(k)k^{-1})(k^{-1}\delta_2(k)))\\
&+ \mathcal{D}_{1,1}(\Delta_{(1)}, \Delta_{(2)})((\delta_1(k)k^{-1})(k^{-1}\delta_2(k)))\\
&+ \mathcal{D}_{1,2}(\Delta_{(1)}, \Delta_{(2)})(\Delta^{-1/2}(\delta_2(k)k^{-1})\Delta^{1/2}(k^{-1}\delta_1(k)))
\end{aligned}
$$

$$
\begin{aligned}
&+ \mathcal{D}_{1,2}(\Delta_{(1)}, \Delta_{(2)})((\delta_2(k)k^{-1})\Delta^{1/2}(k^{-1}\delta_1(k)))\\
&+ \mathcal{D}_{1,2}(\Delta_{(1)}, \Delta_{(2)})(\Delta^{-1/2}(\delta_2(k)k^{-1})(k^{-1}\delta_1(k)))\\
&+ \mathcal{D}_{1,2}(\Delta_{(1)}, \Delta_{(2)})((\delta_2(k)k^{-1})(k^{-1}\delta_1(k)))\\
&+ \mathcal{D}_{2,1}(\Delta_{(1)}, \Delta_{(2)})(\Delta^{-3/2}(\delta_2(k)k^{-1})\Delta^{1/2}(k^{-1}\delta_1(k)))\\
&+ \mathcal{D}_{2,1}(\Delta_{(1)}, \Delta_{(2)})(\Delta^{-1}(\delta_2(k)k^{-1})\Delta^{1/2}(k^{-1}\delta_1(k)))\\
&+ \mathcal{D}_{2,1}(\Delta_{(1)}, \Delta_{(2)})(\Delta^{-3/2}(\delta_2(k)k^{-1})(k^{-1}\delta_1(k)))\\
&+ \mathcal{D}_{2,1}(\Delta_{(1)}, \Delta_{(2)})(\Delta^{-1}(\delta_2(k)k^{-1})(k^{-1}\delta_1(k)))\\
&- \mathcal{D}_{2,1}(\Delta_{(1)}, \Delta_{(2)})(\Delta^{-1}(\delta_2(k)k^{-1})(k^{-1}\delta_1(k)))\\
&- \mathcal{D}_{2,1}(\Delta_{(1)}, \Delta_{(2)})(\Delta^{-1}(\delta_2(k)k^{-1})\Delta^{1/2}(k^{-1}\delta_1(k)))\\
&- \mathcal{D}_{2,1}(\Delta_{(1)}, \Delta_{(2)})(\Delta^{-3/2}(\delta_2(k)k^{-1})(k^{-1}\delta_1(k)))\\
&- \mathcal{D}_{2,1}(\Delta_{(1)}, \Delta_{(2)})(\Delta^{-3/2}(\delta_2(k)k^{-1})\Delta^{1/2}(k^{-1}\delta_1(k)))\\
&- \mathcal{D}_{1,1}(\Delta_{(1)}, \Delta_{(2)})(\Delta^{-1/2}(\delta_2(k)k^{-1/2})\Delta^{1/2}(k^{-1}\delta_1(k)))\\
&- \mathcal{D}_{1,1}(\Delta_{(1)}, \Delta_{(2)})((\delta_2(k)k^{-1})\Delta^{1/2}(k^{-1}\delta_1(k)))\\
&- \mathcal{D}_{1,1}(\Delta_{(1)}, \Delta_{(2)})(\Delta^{-1/2}(\delta_2(k)k^{-1})(k^{-1}\delta_1(k)))\\
&- \mathcal{D}_{1,1}(\Delta_{(1)}, \Delta_{(2)})((\delta_2(k)k^{-1})(k^{-1}\delta_1(k)))\Big).
\end{aligned}
$$

Putting together the latter with (7), up to an overall factor of π, we get

$$
\begin{aligned}
&g_1(\Delta)(k^{-1}\delta_1^2(k)) + g_2(\Delta)(k^{-2}\delta_1(k)^2)\\
&\quad + G(\Delta_{(1)}, \Delta_{(2)})((\delta_1(k)k^{-1})(k^{-1}\delta_1(k)))\\
&\quad + |\tau|^2 g_1(\Delta)(k^{-1}\delta_2^2(k)) + |\tau|^2 g_2(\Delta)(k^{-2}\delta_2(k)^2)\\
&\quad + |\tau|^2 G(\Delta_{(1)}, \Delta_{(2)})((\delta_2(k)k^{-1})(k^{-1}\delta_2(k)))\\
&\quad + \tau_1 g_1(\Delta)(k^{-1}\delta_1\delta_2(k)) + \tau_1 g_2(\Delta)(k^{-2}\delta_1(k)\delta_2(k))\\
&\quad + \tau_1 G(\Delta_{(1)}, \Delta_{(2)})((\delta_1(k)k^{-1})(k^{-1}\delta_2(k)))\\
&\quad + \tau_1 g_1(\Delta)(k^{-1}\delta_2\delta_1(k)) + \tau_1 g_2(\Delta)(k^{-2}\delta_2(k)\delta_1(k))\\
&\quad + \tau_1 G(\Delta_{(1)}, \Delta_{(2)})((\delta_2(k)k^{-1})(k^{-1}\delta_1(k)))\\
&\quad - i\tau_2 L(\Delta_{(1)}, \Delta_{(2)})((\delta_1(k)k^{-1})(k^{-1}\delta_2(k)))\\
&\quad + i\tau_2 L(\Delta_{(1)}, \Delta_{(2)})((\delta_2(k)k^{-1})(k^{-1}\delta_1(k))),
\end{aligned}
\tag{10}
$$

where as given by formulas (8) and (9)

$$
g_1(u) = \frac{-1 + u^2 - 2u\log u}{(-1 + u^{1/2})^3(1 + u^{1/2})^2},
$$

$$
g_2(u) = 2\frac{-1 + u^2 - 2u\log u}{(-1 + u)^3}.
$$

The function G is defined by

$$\begin{aligned}
G(u,v) :=\ & 2\mathcal{D}_{2,2}(u,v)u^{-1}v^{1/2} + 2\mathcal{D}_{2,2}(u,v)u^{-1} + 2\mathcal{D}_{2,2}(u,v)u^{-3/2}v^{1/2} \\
& + 2\mathcal{D}_{2,2}(u,v)u^{-3/2} + 4\mathcal{D}_{3,1}(u,v)u^{-2}v^{1/2} + 4\mathcal{D}_{3,1}(u,v)u^{-2} \\
& + 4\mathcal{D}_{3,1}(u,v)u^{-5/2}v^{1/2} + 4\mathcal{D}_{3,1}(u,v)u^{-5/2} - 4\mathcal{D}_{2,1}(u,v)u^{-1}v^{1/2} \\
& - 4\mathcal{D}_{2,1}(u,v)u^{-1} - 4\mathcal{D}_{2,1}(u,v)u^{-3/2}v^{1/2} - 4\mathcal{D}_{2,1}(u,v)u^{-3/2} \\
& - \mathcal{D}_{1,2}(u,v)v^{1/2} - \mathcal{D}_{1,2}(u,v)u^{-1/2}v^{1/2} - \mathcal{D}_{1,2}(u,v)u^{-1/2} \\
& - \mathcal{D}_{1,2}(u,v) - \mathcal{D}_{2,1}(u,v)u^{-3/2}v^{1/2} - \mathcal{D}_{2,1}(u,v)u^{-1}v^{1/2} \\
& - \mathcal{D}_{2,1}(u,v)u^{-3/2} - \mathcal{D}_{2,1}(u,v)u^{-1} - \mathcal{D}_{2,1}(u,v)u^{-1} \\
& - \mathcal{D}_{2,1}(u,v)u^{-1}v^{1/2} - \mathcal{D}_{2,1}(u,v)u^{-3/2} - \mathcal{D}_{2,1}(u,v)u^{-3/2}v^{1/2} \\
& + \mathcal{D}_{1,1}(u,v)u^{-1/2}v^{1/2} + \mathcal{D}_{1,1}(u,v)v^{1/2} \\
& + \mathcal{D}_{1,1}(u,v)u^{-1/2} + \mathcal{D}_{1,1}(u,v) \\
=\ & -(\sqrt{u}(u(-1+v)^2(-1+uv(-4+u(4+v)))\log(1/u) + (-1+u) \\
& ((1+u(-2+v))(-1+v)(-1+uv)(1+uv) + (-1+u)v \\
& (-1+u(-4+v(4+uv)))\log v)))/((-1+\sqrt{u})^2(1+\sqrt{u}) \\
& (-1+\sqrt{v})^2(1+\sqrt{v})(-1+uv)^3)
\end{aligned}$$

and

$$\begin{aligned}
L(u,v) :=\ & -\mathcal{D}_{1,2}(u,v)u^{-1/2}v^{1/2} - \mathcal{D}_{1,2}(u,v)v^{1/2} - \mathcal{D}_{1,2}(u,v)u^{-1/2} \\
& - \mathcal{D}_{1,2}(u,v) - \mathcal{D}_{2,1}(u,v)u^{-3/2}v^{1/2} - \mathcal{D}_{2,1}(u,v)u^{-1}v^{1/2} \\
& - \mathcal{D}_{2,1}(u,v)u^{-3/2} - \mathcal{D}_{2,1}(u,v)u^{-1} + \mathcal{D}_{2,1}(u,v)u^{-1} \\
& + \mathcal{D}_{2,1}(u,v)u^{-1}v^{1/2} + \mathcal{D}_{2,1}(u,v)u^{-3/2} + \mathcal{D}_{2,1}(u,v)u^{-3/2}v^{1/2} \\
& + \mathcal{D}_{1,1}(u,v)u^{-1/2}v^{1/2} + \mathcal{D}_{1,1}(u,v)v^{1/2} + \mathcal{D}_{1,1}(u,v)u^{-1/2} \\
& + \mathcal{D}_{1,1}(u,v) \\
=\ & (\sqrt{u}(u(-1+v)^2\log(1/u) + (-1+u)((-1+v)(-1+uv) \\
& + (v-uv)\log v)))/((-1+\sqrt{u})^2(1+\sqrt{u})(-1+\sqrt{v})^2 \\
& (1+\sqrt{v})(-1+uv)).
\end{aligned}$$

5. The scalar curvature in terms of $\log(k)$

In order to express the scalar curvature of $(\mathbb{T}_\theta^2, \tau, k)$ in terms of $\log k$ we need to find some identities that relate $k^{-1}\delta_i\delta_j(k)$ and $k^{-2}\delta_i(k)^2$, for $i, j = 1, 2$, to $\log k$. This is carried out in the following lemma.

Lemma 5.1. *For $i, j = 1, 2$, we have*

$$k^{-2}\delta_i(k)\delta_j(k) = 4\frac{\Delta - \Delta^{1/2}}{\log \Delta}(\delta_i(\log k))\frac{\Delta^{1/2} - 1}{\log \Delta}(\delta_j(\log k)). \tag{11}$$

Also we have

$$\begin{aligned} k^{-1}\delta_i\delta_j(k) = 2\frac{\Delta^{1/2} - 1}{\log \Delta}(\delta_i\delta_j(\log k)) + g(\Delta_{(1)}, \Delta_{(2)})(\delta_j(\log k)\delta_i(\log k)) \\ + g(\Delta_{(1)}, \Delta_{(2)})(\delta_i(\log k)\delta_j(\log k)), \end{aligned} \tag{12}$$

where

$$g(u, v) := 4\frac{(\sqrt{uv} - 1)\log u - (\sqrt{u} - 1)\log(uv)}{\log v \log u \log(uv)}$$

and $\Delta_{(i)}$ signifies the action of Δ on the i-th factor of the product.

Proof. We note that the following identity from [16] will be used in the proof of both identities:

$$k^{-1}\delta_j(k) = 2\frac{\Delta^{1/2} - 1}{\log \Delta}(\delta_j(\log k)).$$

First we prove (11):

$$\begin{aligned} k^{-2}\delta_i(k)\delta_j(k) &= k^{-1}2\frac{\Delta^{1/2} - 1}{\log \Delta}(\delta_i(\log k))\,\delta_j(k) \\ &= 4\frac{\Delta - \Delta^{1/2}}{\log \Delta}(\delta_i(\log k))\frac{\Delta^{1/2} - 1}{\log \Delta}(\delta_j(\log k)). \end{aligned}$$

To prove (12), we write

$$\begin{aligned} k^{-1}\delta_i\delta_j(k) &= \int_0^1 \Delta^{s/2}\delta_i\delta_j(\log k)\,ds \\ &\quad + \int_0^1 \Delta^{s/2}(\delta_j(\log k))2\frac{\Delta^{s/2} - 1}{\log \Delta}(\delta_i(\log k))\,ds \\ &\quad + \int_0^1 \Delta^{s/2}(\delta_i(\log k))2\frac{\Delta^{s/2} - 1}{\log \Delta}(\delta_j(\log k))\,ds \\ &= 2\frac{\Delta^{1/2} - 1}{\log \Delta}(\delta_i\delta_j(\log k)) + g(\Delta_{(1)}, \Delta_{(2)})(\delta_j(\log k)\delta_i(\log k)) \\ &\quad + g(\Delta_{(1)}, \Delta_{(2)})(\delta_i(\log k)\delta_j(\log k)). \end{aligned}$$

□

5.1. The terms corresponding to $k\partial^*\partial k$. Applying Lemma 5.1 to the local expression (6), we can write it in terms of $\log k$ as follows:

$$\begin{aligned}
(6) = {} & f_1(\Delta)(2\frac{\Delta^{1/2}-1}{\log\Delta}(\delta_1^2(\log k))) \\
& + f_1(\Delta)(2g(\Delta_{(1)},\Delta_{(2)})(\delta_1(\log k)\delta_1(\log k))) \\
& + f_2(\Delta)\Big(4\frac{\Delta-\Delta^{1/2}}{\log\Delta}(\delta_1(\log k))\frac{\Delta^{1/2}-1}{\log\Delta}(\delta_1(\log k))\Big) \\
& + F(\Delta_{(1)},\Delta_{(2)})\Big(\Big(-2\frac{\Delta^{-1/2}-1}{\log\Delta}(\delta_1(\log k))\Big)\Big(2\frac{\Delta^{1/2}-1}{\log\Delta}(\delta_1(\log k))\Big)\Big) \\
& + |\tau|^2 f_1(\Delta)\Big(2\frac{\Delta^{1/2}-1}{\log\Delta}(\delta_2^2(\log k))\Big) \\
& + |\tau|^2 f_1(\Delta)(2g(\Delta_{(1)},\Delta_{(2)})(\delta_2(\log k)\delta_2(\log k))) \\
& + |\tau|^2 f_2(\Delta)\Big(4\frac{\Delta-\Delta^{1/2}}{\log\Delta}(\delta_2(\log k))\frac{\Delta^{1/2}-1}{\log\Delta}(\delta_2(\log k))\Big) \\
& + |\tau|^2 F(\Delta_{(1)},\Delta_{(2)})\Big(\Big(-2\frac{\Delta^{-1/2}-1}{\log\Delta}(\delta_2(\log k))\Big)\Big(2\frac{\Delta^{1/2}-1}{\log\Delta}(\delta_2(\log k))\Big)\Big) \\
& + \tau_1 f_1(\Delta)\Big(2\frac{\Delta^{1/2}-1}{\log\Delta}(\delta_1\delta_2(\log k))\Big) \\
& + \tau_1 f_1(\Delta)(g(\Delta_{(1)},\Delta_{(2)})(\delta_2(\log k)\delta_1(\log k))) \\
& + \tau_1 f_1(\Delta)(g(\Delta_{(1)},\Delta_{(2)})(\delta_1(\log k)\delta_2(\log k))) \\
& + \tau_1 f_2(\Delta)\Big(4\frac{\Delta-\Delta^{1/2}}{\log\Delta}(\delta_1(\log k))\frac{\Delta^{1/2}-1}{\log\Delta}(\delta_2(\log k))\Big) \\
& + \tau_1 F(\Delta_{(1)},\Delta_{(2)})\Big(\Big(-2\frac{\Delta^{-1/2}-1}{\log\Delta}(\delta_1(\log k))\Big)\Big(2\frac{\Delta^{1/2}-1}{\log\Delta}(\delta_2(\log k))\Big)\Big) \\
& + \tau_1 f_1(\Delta)\Big(2\frac{\Delta^{1/2}-1}{\log\Delta}(\delta_2\delta_1(\log k))\Big) \\
& + \tau_1 f_1(\Delta)(g(\Delta_{(1)},\Delta_{(2)})(\delta_1(\log k)\delta_2(\log k))) \\
& + \tau_1 f_1(\Delta)(g(\Delta_{(1)},\Delta_{(2)})(\delta_2(\log k)\delta_1(\log k))) \\
& + \tau_1 f_2(\Delta)\Big(4\frac{\Delta-\Delta^{1/2}}{\log\Delta}(\delta_2(\log k))\frac{\Delta^{1/2}-1}{\log\Delta}(\delta_1(\log k))\Big) \\
& + \tau_1 F(\Delta_{(1)},\Delta_{(2)})\Big(\Big(-2\frac{\Delta^{-1/2}-1}{\log\Delta}(\delta_2(\log k))\Big)\Big(2\frac{\Delta^{1/2}-1}{\log\Delta}(\delta_1(\log k))\Big)\Big).
\end{aligned}$$

Now, writing the latter in terms of $\log\Delta$ and considering an overall factor of -1 (cf. Section 3.2), up to an overall factor of $\frac{\pi}{\tau_2}$, we obtain the following expression for the first component of the scalar curvature of the perturbed spectral triple attached to

$(\mathbb{T}_\theta^2, \tau, k)$:

$$K(\log\Delta)(\delta_1^2(\log k) + |\tau|^2\delta_2^2(\log k) + 2\tau_1\delta_1\delta_2(\log k))$$
$$+ H(\log\Delta_{(1)}, \log\Delta_{(2)})(\delta_1(\log k)\delta_1(\log k) + |\tau|^2\delta_2(\log k)\delta_2(\log k)$$
$$+ \tau_1\delta_1(\log k)\delta_2(\log k) + \tau_1\delta_2(\log k)\delta_1(\log k)),$$

where

$$K(x) := -2f_1(e^x)\frac{e^{x/2}-1}{x} = \frac{2e^{x/2}(2+e^x(-2+x)+x)}{(-1+e^x)^2 x},$$

and

$$H(s,t) := -2f_1(e^{s+t})g(e^s,e^t) - 4f_2(e^{s+t})\frac{e^s-e^{s/2}}{s}\frac{e^{t/2}-1}{t}$$
$$+ 4F(e^s,e^t)\frac{e^{-s/2}-1}{s}\frac{e^{t/2}-1}{t}$$
$$= -\frac{-t(s+t)\cosh s + s(s+t)\cosh t - (s-t)(s+t+\sinh s + \sinh t - \sinh(s+t))}{st(s+t)\sinh(s/2)\sinh(t/2)\sinh^2((s+t)/2)}.$$

5.2. The terms corresponding to $\partial^* k^2 \partial$. We also apply Lemma 5.1 to the local expression (10) and obtain the following:

$$(10) = g_1(\Delta)\Big(2\frac{\Delta^{1/2}-1}{\log\Delta}(\delta_1^2(\log k))\Big)$$
$$+ g_1(\Delta)(2g(\Delta_{(1)},\Delta_{(2)})(\delta_1(\log k)\delta_1(\log k)))$$
$$+ g_2(\Delta)\Big(4\frac{\Delta-\Delta^{1/2}}{\log\Delta}(\delta_1(\log k))\frac{\Delta^{1/2}-1}{\log\Delta}(\delta_1(\log k))\Big)$$
$$+ G(\Delta_{(1)},\Delta_{(2)})\bigg(\Big(-2\frac{\Delta^{-1/2}-1}{\log\Delta}(\delta_1(\log k))\Big)\Big(2\frac{\Delta^{1/2}-1}{\log\Delta}(\delta_1(\log k))\Big)\bigg)$$
$$+ |\tau|^2 g_1(\Delta)\Big(2\frac{\Delta^{1/2}-1}{\log\Delta}(\delta_2^2(\log k))\Big)$$
$$+ |\tau|^2 g_1(\Delta)(2g(\Delta_{(1)},\Delta_{(2)})(\delta_2(\log k)\delta_2(\log k)))$$
$$+ |\tau|^2 g_2(\Delta)\Big(4\frac{\Delta-\Delta^{1/2}}{\log\Delta}(\delta_2(\log k))\frac{\Delta^{1/2}-1}{\log\Delta}(\delta_2(\log k))\Big)$$
$$+ |\tau|^2 G(\Delta_{(1)},\Delta_{(2)})\bigg(\Big(-2\frac{\Delta^{-1/2}-1}{\log\Delta}(\delta_2(\log k))\Big)\Big(2\frac{\Delta^{1/2}-1}{\log\Delta}(\delta_2(\log k))\Big)\bigg)$$
$$+ \tau_1 g_1(\Delta)\Big(2\frac{\Delta^{1/2}-1}{\log\Delta}(\delta_1\delta_2(\log k))\Big)$$

$$
\begin{aligned}
&+ \tau_1 g_1(\Delta)(g(\Delta_{(1)}, \Delta_{(2)})(\delta_2(\log k)\delta_1(\log k))) \\
&+ \tau_1 g_1(\Delta)(g(\Delta_{(1)}, \Delta_{(2)})(\delta_1(\log k)\delta_2(\log k))) \\
&+ \tau_1 g_2(\Delta)\Big(4\frac{\Delta - \Delta^{1/2}}{\log\Delta}(\delta_1(\log k))\frac{\Delta^{1/2}-1}{\log\Delta}(\delta_2(\log k))\Big) \\
&+ \tau_1 G(\Delta_{(1)}, \Delta_{(2)})\Big(\Big(-2\frac{\Delta^{-1/2}-1}{\log\Delta}(\delta_1(\log k))\Big)\Big(2\frac{\Delta^{1/2}-1}{\log\Delta}(\delta_2(\log k))\Big)\Big)
\end{aligned}
$$

$$
\begin{aligned}
&\tau_1 g_1(\Delta)\Big(2\frac{\Delta^{1/2}-1}{\log\Delta}(\delta_2\delta_1(\log k))\Big) \\
&+ \tau_1 g_1(\Delta)(g(\Delta_{(1)}, \Delta_{(2)})(\delta_1(\log k)\delta_2(\log k))) \\
&+ \tau_1 g_1(\Delta)(g(\Delta_{(1)}, \Delta_{(2)})(\delta_2(\log k)\delta_1(\log k))) \\
&+ \tau_1 g_2(\Delta)\Big(4\frac{\Delta - \Delta^{1/2}}{\log\Delta}(\delta_2(\log k))\frac{\Delta^{1/2}-1}{\log\Delta}(\delta_1(\log k))\Big) \\
&+ \tau_1 G(\Delta_{(1)}, \Delta_{(2)})\Big(\Big(-2\frac{\Delta^{-1/2}-1}{\log\Delta}(\delta_2(\log k))\Big)\Big(2\frac{\Delta^{1/2}-1}{\log\Delta}(\delta_1(\log k))\Big)\Big) \\
&- i\tau_2 L(\Delta_{(1)}, \Delta_{(2)})\Big(\Big(-2\frac{\Delta^{-1/2}-1}{\log\Delta}(\delta_1(\log k))\Big)\Big(2\frac{\Delta^{1/2}-1}{\log\Delta}(\delta_2(\log k))\Big)\Big) \\
&+ i\tau_2 L(\Delta_{(1)}, \Delta_{(2)})\Big(\Big(-2\frac{\Delta^{-1/2}-1}{\log\Delta}(\delta_2(\log k))\Big)\Big(2\frac{\Delta^{1/2}-1}{\log\Delta}(\delta_1(\log k))\Big)\Big).
\end{aligned}
$$

Now we write the latter in terms of $\log\Delta$, and after considering an overall factor of -1, up to an overall factor of $\frac{\pi}{\tau_2}$, we obtain the following expression for the second component of the scalar curvature of the perturbed spectral triple attached to $(\mathbb{T}^2_\theta, \tau, k)$:

$$
\begin{aligned}
&S(\log\Delta)(\delta_1^2(\log k) + |\tau|^2\delta_2^2(\log k) + 2\tau_1\delta_1\delta_2(\log k)) \\
&\quad + T(\log\Delta_{(1)}, \log\Delta_{(2)})\big(\delta_1(\log k)\delta_1(\log k) + |\tau|^2\delta_2(\log k)\delta_2(\log k) \\
&\quad + \tau_1\delta_1(\log k)\delta_2(\log k) + \tau_1\delta_2(\log k)\delta_1(\log k)) \\
&\quad - i\tau_2 W(\log\Delta_{(1)}, \log\Delta_{(2)})(\delta_1(\log k)\delta_2(\log k) - \delta_2(\log k)\delta_1(\log k)).
\end{aligned}
$$

Here

$$
\begin{aligned}
S(x) &:= -2g_1(e^x)\frac{e^{x/2}-1}{x} = -\frac{4e^x(-x+\sinh x)}{(-1+e^{x/2})^2(1+e^{x/2})^2 x}, \\
T(s,t) &:= -2g_1(e^{s+t})g(e^s,e^t) - 4g_2(e^{s+t})\frac{e^s - e^{s/2}}{s}\frac{e^{t/2}-1}{t} + \\
&\qquad + 4G(e^s,e^t)\frac{e^{-s/2}-1}{s}\frac{e^{t/2}-1}{t} \\
&= -\cosh((s+t)/2) \\
&\qquad \cdot \frac{-t(s+t)\cosh s + s(s+t)\cosh t - (s-t)(s+t+\sinh s+\sinh t-\sinh(s+t))}{st(s+t)\sinh(s/2)\sinh(t/2)\sinh^2((s+t)/2)}
\end{aligned}
$$

and

$$\begin{aligned} W(s,t) &:= +4L(e^s,e^t)\frac{e^{-s/2}-1}{s}\frac{e^{t/2}-1}{t} \\ &= -4\frac{-s-t+t\cosh s+s\cosh t+\sinh s+\sinh t-\sinh(s+t)}{st(\sinh s+\sinh t-\sinh(s+t))} \\ &= \frac{-s-t+t\cosh s+s\cosh t+\sinh s+\sinh t-\sinh(s+t)}{st\sinh(s/2)\sinh(t/2)\sinh((s+t)/2)}. \end{aligned}$$

5.3. The scalar curvature. We collect the results of this paper in the following theorems. They are also independently proved by Connes and Moscovici in [15]. Note that in our final formulas we have considered an overall minus sign which comes from the change of sign initially considered in the Cauchy integral formula (1).

Theorem 5.2. *Let θ be an irrational number, τ a complex number in the upper half plane representing the conformal class of a metric on T^2_θ, and k an invertible positive element in A^∞_θ playing the role of the Weyl factor. Then the scalar curvature R of the perturbed spectral triple attached to (T^2_θ,τ,k), up to an overall factor of $-\frac{\pi}{\tau_2}$, is equal to*

$$\begin{aligned} &R_1(\log\Delta)\big(\delta_1^2(\log k)+|\tau|^2\delta_2^2(\log k)+2\tau_1\delta_1\delta_2(\log k)\big) \\ &\quad+R_2(\log\Delta_{(1)},\log\Delta_{(2)})(\delta_1(\log k)\delta_1(\log k)+|\tau|^2\delta_2(\log k)\delta_2(\log k) \\ &\quad+\tau_1(\delta_1(\log k)\delta_2(\log k)+\delta_2(\log k)\delta_1(\log k))) \\ &\quad-iW(\log\Delta_{(1)},\log\Delta_{(2)})(\tau_2(\delta_1(\log k)\delta_2(\log k)-\delta_2(\log k)\delta_1(\log k))), \end{aligned}$$

where

$$\begin{aligned} R_1(x) &:= K(x)+S(x) = -\frac{2\coth(x/4)}{x}+\frac{1}{2\sinh^2(x/4)} = \frac{\frac{1}{2}-\frac{\sinh(x/2)}{x}}{\sinh^2(x/4)}, \\ R_2(s,t) &:= H(s,t)+T(s,t) \\ &= -(1+\cosh((s+t)/2)) \\ &\quad\cdot\frac{-t(s+t)\cosh s+s(s+t)\cosh t-(s-t)(s+t+\sinh s+\sinh t-\sinh(s+t))}{st(s+t)\sinh(s/2)\sinh(t/2)\sinh^2((s+t)/2)} \end{aligned}$$

and

$$W(s,t) = \frac{-s-t+t\cosh s+s\cosh t+\sinh s+\sinh t-\sinh(s+t)}{st\sinh(s/2)\sinh(t/2)\sinh((s+t)/2)}.$$

Theorem 5.3. *Assuming the hypotheses of Theorem 5.2, the chiral scalar curvature R^γ of the perturbed graded spectral triple attached to (T^2_θ,τ,k), up to an overall* *factor of* $-\frac{\pi}{\tau_2}$*, is given by*

$$\begin{aligned}
&R_1^\gamma(\log\Delta)(\delta_1^2(\log k)+|\tau|^2\delta_2^2(\log k)+2\tau_1\delta_1\delta_2(\log k))\\
&\quad+R_2^\gamma(\log\Delta_{(1)},\log\Delta_{(2)})(\delta_1(\log k)\delta_1(\log k)+|\tau|^2\delta_2(\log k)\delta_2(\log k)\\
&\quad+\tau_1(\delta_1(\log k)\delta_2(\log k)+\delta_2(\log k)\delta_1(\log k)))\\
&\quad+iW(\log\Delta_{(1)},\log\Delta_{(2)})(\tau_2(\delta_1(\log k)\delta_2(\log k)-\delta_2(\log k)\delta_1(\log k))),
\end{aligned}$$

where

$$\begin{aligned}
R_1^\gamma(x) &:= K(x)-S(x)=\frac{x+2\sinh(x/2)}{x+x\cosh(x/2)}=\frac{\frac{1}{2}+\frac{\sinh(x/2)}{x}}{\cosh^2(x/4)},\\
R_2^\gamma(s,t) &:= H(s,t)-T(s,t)\\
&= -(1-\cosh((s+t)/2))\\
&\quad\cdot\frac{-t(s+t)\cosh s+s(s+t)\cosh t-(s-t)(s+t+\sinh s+\sinh t-\sinh(s+t))}{st(s+t)\sinh(s/2)\sinh(t/2)\sinh^2((s+t)/2)}
\end{aligned}$$

and

$$W(s,t)=\frac{-s-t+t\cosh s+s\cosh t+\sinh s+\sinh t-\sinh(s+t)}{st\sinh(s/2)\sinh(t/2)\sinh((s+t)/2)}.$$

Remark 5.4. We note that the above local expressions R and R^γ for the scalar curvature of $(\mathbb{T}_\theta^2,\tau,k)$ reduce to the scalar curvature of the ordinary two torus when $\theta=0$. That is, since

$$\lim_{x\to 0}R_1(x)=-\frac{1}{3},\quad \lim_{x\to 0}R_1^\gamma(x)=1,$$
$$\lim_{s,t\to 0}R_2(s,t)=\lim_{s,t\to 0}R_2^\gamma(s,t)=0,$$

and

$$\lim_{s,t\to 0}W(s,t)=-\frac{2}{3},$$

in the commutative case, the expressions for R and R^γ stated in the above theorems reduce to constant multiples of

$$\frac{1}{\tau_2}\delta_1^2(\log k)+\frac{|\tau|^2}{\tau_2}\delta_2^2(\log k)+2\frac{\tau_1}{\tau_2}\delta_1\delta_2(\log k).$$

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