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On the Hochschild and cyclic (co)homology of rapid decay group algebras

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Abstract. We show that the technical condition of solvable conjugacy bound, introduced in [JOR1], can be removed without affecting the main results of that paper. The result is a Burghelea-type description of the summands $\mathrm{HH}^t_*(\mathcal{H}_{\mathcal{B},L}(G))_{\langle x\rangle}$ and $\mathrm{HC}^t_*(\mathcal{H}_{\mathcal{B},L}(G))_{\langle x\rangle}$ for any bounding class $\mathcal{B}$, discrete group with word-length (G, L) and conjugacy class $\langle x\rangle \in \langle G\rangle$. We use this description to prove the conjecture $\mathcal{B}$-SrBC of [JOR1] for a class of groups that goes well beyond the cases considered in that paper. In particular, we show that the conjecture ℓ^1-SrBC (the Strong Bass Conjecture for the topological K-theory of $\ell^1(G)$) is true for all semihyperbolic groups which satisfy SrBC, a statement consistent with the rationalized Bost conjecture for such groups.

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1. Introduction

Given a bounding class $\mathcal{B}$ (the definition of which we recall below) and group with word-length function (G, L), one may construct the rapid decay algebra $\mathcal{H}_{\mathcal{B},L}(G)$. This algebra was introduced in [JOR1]; it is a Fréchet algebra whenever the bounding class $\mathcal{B}$ is equivalent to a countable class (there are no known cases when this doesn't occur). When $\mathcal{B} = \mathcal{P}$, the class of polynomial bounding functions, this algebra is precisely the rapid decay algebra $H^{1,\infty}_L(G)$ introduced by Jolissaint in [Jo1], [Jo2]. Because $\mathcal{H}_{\mathcal{B},L}(G)$ is a rapid decay algebra formed using weighted ℓ^1-norms one has, associated to each conjugacy class $\langle x\rangle \in \langle G\rangle$, summands $F^t_*(\mathcal{H}_{\mathcal{B},L}(G))_{\langle x\rangle}$ of $F^t_*(\mathcal{H}_{\mathcal{B},L}(G))$, where $F = \mathrm{HH}, \mathrm{HC}, \mathrm{HPer}$, and thus projection maps

$$p_{\langle x\rangle}\colon F^t_*(\mathcal{H}_{\mathcal{B},L}(G)) \twoheadrightarrow F^t_*(\mathcal{H}_{\mathcal{B},L}(G))_{\langle x\rangle}, \quad F = \mathrm{HH}, \mathrm{HC}, \mathrm{HPer},$$

and similarly for cohomology.

Recall that a conjugacy class $\langle x\rangle$ is *elliptic* if $\langle x\rangle$ has finite order, and *non-elliptic* if x has infinite order. As in [JOR1], one may then posit – for a given bounding class $\mathcal{B}$ and group with word-length (G, L) – the following generalization of the Strong Bass Conjecture:

Conjecture $\mathcal{B}$-SrBC. *For each non-elliptic conjugacy class $\langle x\rangle$, the image of the composition*

$$p_{\langle x\rangle}\circ \mathrm{ch}_*\colon K_*^t(\mathcal{H}_{\mathcal{B},L}(G)) \to \mathrm{HC}_*^t(\mathcal{H}_{\mathcal{B},L}(G)) \twoheadrightarrow \mathrm{HC}_*^t(\mathcal{H}_{\mathcal{B},L}(G))_{\langle x\rangle}$$

is zero.

When $\mathcal{B} = \mathcal{P}$, the subalgebra $\mathcal{H}_{\mathcal{P},L}(G) = H^{1,\infty}L(G)$ is smooth in $\ell^1(G)$ by [Jo1], [Jo2]; in this case the above conjecture can be restated as:

Conjecture ℓ^1-SrBC. *For each non-elliptic conjugacy class $\langle x\rangle$, the image of the composition*

$$p_{\langle x\rangle}\circ \mathrm{ch}_*\colon K_*^t(\ell^1(G)) \to \mathrm{HC}_*^t(H_L^{1,\infty}(G)) \twoheadrightarrow \mathrm{HC}_*^t(H_L^{1,\infty}(G))_{\langle x\rangle}$$

is zero.

The conjecture ℓ^1-SrBC is certainly the most important special case of the first more general conjecture; as shown in the appendix of [JOR1] these conjectures follow from the rational surjectivity of an appropriately defined Baum–Connes assembly map. Thus any counter-example to one of these conjectures would, in turn, provide a counterexample to a Baum–Connes type conjecture holding for the corresponding rapid decay algebra.

It is a result due to Burghelea that one has decompositions

$$\mathrm{HH}_*(\mathbb{C}[G]) \cong \bigoplus_{\langle x\rangle\in\langle G\rangle} \mathrm{HH}_*(\mathbb{C}[G])_{\langle x\rangle}, \quad \mathrm{HC}_*(\mathbb{C}[G]) \cong \bigoplus_{\langle x\rangle\in\langle G\rangle} \mathrm{HC}_*(\mathbb{C}[G])_{\langle x\rangle}$$

and for each conjugacy class $\langle x\rangle$ an isomorphism

$$\mathrm{HH}_*(\mathbb{C}[G])_{\langle x\rangle} \cong H_*(BG_x;\mathbb{C}),$$

where G_x denotes the centralizer of $x \in G$. This last fact allows one to derive corresponding descriptions of $\mathrm{HC}_*(\mathbb{C}[G])_{\langle x\rangle}$; specifically, for non-elliptic classes one has

$$\mathrm{HC}_*(\mathbb{C}[G])_{\langle x\rangle} \cong H_*(B(G_x/(x));\mathbb{C}),$$

where $(x) \subset G_x$ denotes the infinite cyclic subgroup generated by x and $G_x/(x)$ the quotient group. The standard way of arriving at this result is via the isomorphism of sets

$$G_x\backslash G \to S_{\langle x\rangle} \tag{1}$$

where $S_{\langle x\rangle} \subset G$ is the set of elements conjugate to x, and the correspondence is given by $G_x g \mapsto g^{-1}xg$. When G is equipped with word-length L, this map becomes a $\mathcal{B}$-G-morphism of weighted right $\mathcal{B}$-G-sets in the sense of [JOR1], but one whose inverse is in general unbounded. This leads to the notion of $\mathcal{B}$-solvable conjugacy bound, introduced in [JOR1] – the condition that the map in (1) is a $\mathcal{B}$-G-isomorphism –, which in turn allows for the identification of the summands $F_*^t(\mathcal{H}_{\mathcal{B},L}(G))_{\langle x\rangle}$ in terms of the rapid decay homology of BG_x or $B(G_x/(x))$. Unfortunately, this condition is very restrictive, as it requires the group to have a conjugacy problem which is solvable in $\mathcal{B}$-bounded time. In particular, when $\mathcal{B} = \mathcal{P}$, it is difficult to go much beyond hyperbolic groups with this restriction in place (the main result of [JOR1] was verification of a relative version of the ℓ^1-BC conjecture in the presence of relative hyperbolicity).

With this in mind, we may now state our main result, by which the solvable conjugacy bound constraint of [JOR1] is removed. The consequence is a complete characterization of the conjugacy class summands of the Hochschild and cyclic (co)-homology groups of $\mathcal{H}_{\mathcal{B},L}(G)$ for any group with word-length (G, L) and bounding class $\mathcal{B}$. Recall that the *distortion* of a subgroup with length function (H, L_H) of a group with length function (G, L_G) is the function $\mathrm{Dist}(H)\colon \mathbb{N} \to \mathbb{N}$ defined by $\mathrm{Dist}(H)(n) = \max\{L_H(h) \mid h \in H,\ L_H(h) \le n\}$.

Theorem A. *Let*[1] (G, L) *be a discrete group equipped with proper word-length function and $\mathcal{B}$ a countable bounding class. Then for each conjugacy class $\langle x\rangle \in \langle G\rangle$ there are isomorphisms in topological Hochschild* (*co*)*-homology*

$$\mathrm{HH}_t^*(\mathcal{H}_{\mathcal{B},L}(G))_{\langle x\rangle} \cong \mathcal{B}H^*(BG_x), \quad \mathrm{HH}_*^t(\mathcal{H}_{\mathcal{B},L}(G))_x \cong \mathcal{B}H_*(BG_x).$$

These isomorphisms imply the existence of isomorphisms in topological cyclic (*co*)*-homology*

$$\begin{aligned}
&\mathrm{HC}_t^*(\mathcal{H}_{\mathcal{B},L}(G))_{\langle x\rangle} \cong \mathcal{B}H^*(BG_x) \otimes \mathrm{HC}^*(\mathbb{C}),\ \langle x\rangle \textit{ elliptic},\\
&\mathrm{HC}_*^t(\mathcal{H}_{\mathcal{B},L}(G))_{\langle x\rangle} \cong \mathcal{B}H_*(BG_x) \otimes \mathrm{HC}_*(\mathbb{C}),\ \langle x\rangle \textit{ elliptic},\\
&\mathrm{HC}_t^*(\mathcal{H}_{\mathcal{B},L}(G))_{\langle x\rangle} \cong \mathcal{B}H^*(B(G_x/(x))),\ \langle x\rangle \textit{ non-elliptic and } \mathrm{Dist}((x)) \le \mathcal{B},\\
&\mathrm{HC}_*^t(\mathcal{H}_{\mathcal{B},L}(G))_{\langle x\rangle} \cong \mathcal{B}H_*(B(G_x/(x))),\ \langle x\rangle \textit{ non-elliptic and } \mathrm{Dist}((x)) \le \mathcal{B}.
\end{aligned}$$

Here the weighting on BG_x and $B(G_x/(x))$ comes from the induced word-length function on G_x and the corresponding quotient word-length function on $G_x/(x)$, while $\mathcal{B}H^*(\_)$, $\mathcal{B}H_*(\_)$ denote the $\mathcal{B}$-bounded (co)-homology groups as defined in [JOR2] (and reviewed below). For the cyclic groups, the $\mathbb{C}[u]$-module and comodule structures of the terms on the right are exactly as in the case of the group algebra for both elliptic and non-elliptic classes (compare [B1]). Finally, $\mathrm{Dist}((x))$ refers to the distortion of the infinite cyclic subgroup (x) of G; the condition $\mathrm{Dist}((x)) \le \mathcal{B}$ means that the distortion of (x) is bounded above by a function in $\mathcal{B}$.

This theorem allows for the construction of a class of groups satisfying the $\mathcal{B}$-SrBC conjecture, modeled on the class of groups originally considered by the first author in [Ji1] and independently by Chadha and Passi in [CP], and extended to

[1] The statement of this theorem amends Theorem 1.4.7 of [JOR1], where the effects of distortion were not taken into account.

a slightly larger class by Emmanouil in [E1]. It also allows for direct verification of the ℓ^1-SrBC conjecture for a large and important class of groups, as given by the following theorem. Recall that a group G satisfies the *nilpotency condition* if $S\colon \mathrm{HC}^*(\mathbb{C}[G])_{\langle x\rangle} \to \mathrm{HC}^{*+2}(\mathbb{C}[G])_{\langle x\rangle}$ is a nilpotent operator for all non-elliptic classes $\langle x\rangle \in \langle G\rangle$. Similarly, G satisfies the *$\mathcal{B}$-nilpotency condition* if $S\colon \mathrm{HC}_t^*(\mathcal{H}_{\mathcal{B},L}(G))_{\langle x\rangle} \to \mathrm{HC}^{*+2}(\mathcal{H}_{\mathcal{B},L}(G))_{\langle x\rangle}$ is a nilpotent operator for all non-elliptic classes $\langle x\rangle \in \langle G\rangle$. By [JOR1], if G satisfies the $\mathcal{B}$-nilpotency condition, then $\mathcal{B}$-SrBC is true for G. Finally, (G, L) is $\mathcal{B}$-isocohomological if the comparison map in cohomology $\mathcal{B}H^*(BG) \to H^*BG)$ is an isomorphism ([JOR2]).

Theorem B. *Let (G, L) be a semihyperbolic group in the sense of* [AB]. *If G satisfies the nilpotency condition, then it satisfies the $\mathcal{B}$-nilpotency condition for any bounding class $\mathcal{B}$ containing $\mathcal{P}$. In particular, if $G_x/(x)$ has finite cohomological dimension over $\mathbb{Q}$ for each non-elliptic class $\langle x\rangle$, then G satisfies $\mathcal{B}$-SrBC for all bounding classes $\mathcal{B}$ containing $\mathcal{P}$.*

Proof. By [AB] the centralizer subgroups G_h are quasi-convex subgroups of the (linearly) combable group G, hence each G_h is combable and quasi-isometrically embedded into G. By [O1], [M1], or [JOR2] each G_h is $\mathcal{B}$-isocohomological for any bounding class containing $\mathcal{P}$. Moreover, (x) is quasi-isometrically embedded in G_x – hence also in G – for each conjugacy class $\langle x\rangle \in \langle G\rangle$, implying its distortion in G is at most polynomial. Thus for each non-elliptic conjugacy class $\langle h\rangle \in \langle G\rangle$ we have isomorphisms

$$\mathrm{HC}_t^*(\mathcal{H}_{\mathcal{B},L}(G))_{\langle h\rangle} \cong \mathcal{B}H^*(B(G_h/(h))) \cong H^*(B(G_h/(h))).$$

By Burghelea's result together with Theorem A above, the nilpotency condition for G implies the $\mathcal{B}$-nilpotency condition for (G, L) whenever $\mathcal{B}$ is a bounding class containing $\mathcal{P}$. □

2. Preliminaries

Let $\mathcal{S}$ denote the set of non-decreasing functions $\{f \mid \mathbb{R}_+ \to \mathbb{R}^+\}$. Suppose that $\phi\colon \mathcal{S}^n \to \mathcal{S}$ is a function of sets and $\mathcal{B} \subset \mathcal{S}$. Then $\mathcal{B}$ is weakly closed under ϕ if for each $(f_1, f_2, \dots, f_n) \in \mathcal{B}^n$ there is $f \in \mathcal{B}$ with $\phi(f_1, f_2, \dots, f_n) \le f$. A *bounding class* is a subset $\mathcal{B} \subset \mathcal{S}$ satisfying the following:

(1) It contains the constant function 1.

(2) It is weakly closed under the operations of taking positive rational linear combinations.

(3) It is weakly closed under the operation $(f, g) \mapsto f \circ g$ for $g \in \mathcal{L}$, where $\mathcal{L}$ denotes the linear bounding class $\{f(x) = ax + b \mid a, b \in \mathbb{Q}_+\}$.

A bounding class is *composable* if it is weakly closed under the operation of composition of functions.

Naturally occurring classes besides $\mathcal{L}$ are $\mathcal{B}_{min} = \mathbb{Q}_+$, $\mathcal{P}$ = the set of polynomials with non-negative rational coefficients, and $\mathcal{E} = \{e^f \mid f \in \mathcal{L}\}$. Bounding classes were discussed in detail in [JOR1], [JOR2]. The bounding classes we will be considering will all contain $\mathcal{L}$.

Recall that a function $f: X \to \mathbb{R}_+$, with X discrete, is *proper* if the preimage of any bounded subset is finite. A *weighted set* (X, w) will refer to a countable discrete set X together with a proper function $w: X \to \mathbb{R}_+$. A morphism $\phi: (X, w) \to (X', w')$ between weighted sets is $\mathcal{B}$*-bounded* if

$$\text{for all } f \in \mathcal{B} \text{ there exists } f' \in \mathcal{B} \text{ such that } f(w'(\phi(x))) \leq f'(w(x))$$

for all $x \in X$. Given two weighted sets (X, w) and (X', w'), the Cartesian product $X \times X'$ carries a canonical weighted set structure, $(X \times X', r)$ with weight function $r(x, x') = w(x) + w'(x')$. Suppose further that X is a right G-set and X' is a left G-set. Let $X \times_G X'$ denote the quotient of $X \times X'$ by the relation $(x, gx') \sim (xg, x')$. There is a natural weight, ρ, on $X \times_G X'$ induced by the weight r on $X \times X'$ as follows:

$$\rho([x, x']) = \min_{g \in G} r(xg, g^{-1}x').$$

Given a weighted set (X, w) and $f \in \mathcal{S}$, the seminorm $|\cdot|_f$ on $\mathrm{Hom}(X, \mathbb{C})$ is given by $|\phi|_f := \sum_{x \in X} |\phi(x)| f(w(x))$. We will mainly be concerned with the case $(X, w) = (G, L)$ is a discrete group endowed with a length function L. The length function L is called a word-length function (with respect to a generating set S) if $L(1) = 1$ and there is a function $\phi: S \to \mathbb{R}^+$ with

$$L(g) = \min\{\textstyle\sum_{i=1}^n \phi(x_i) \mid x_i \in S,\ x_1 x_2 \ldots x_n = g\}.$$

When S is finite, taking $\phi = 1$ produces the standard word-length function on G.

Recall that a simplicial object in a category $\mathcal{C}$ is a covariant functor $B_\bullet: \Delta \to \mathcal{C}$, from the simplicial category into $\mathcal{C}$. That is, for each $k \geq 0$ $B_k \in \mathcal{C}^{(0)}$, and there exist degeneracy morphisms in $\mathrm{Hom}_{\mathcal{C}}(B_k, B_{k+1})$ and face morphisms in $\mathrm{Hom}_{\mathcal{C}}(B_k, B_{k-1})$ which satisfy the usual simplicial identities.

The augmented simplicial category, Δ^+, is obtained by adding another object $[-1]$ to Δ and a single face morphism in $\mathrm{Hom}_{\Delta^+}([-1], [0])$. An augmented simplicial object in a category $\mathcal{C}$ is a covariant functor $B_\bullet: \Delta^+ \to \mathcal{C}$. This consists of a simplicial set B_k, $k \geq 0$, as well as $B_{-1} \in \mathcal{C}^{(0)}$ with an augmentation $B_0 \twoheadrightarrow B_{-1}$.

An augmented simplicial group with word-length $(\Gamma_\bullet^+, L_\bullet)$ consists of an augmented simplicial group $\Gamma_\bullet^+$, with L_k a word-length function on Γ_k for all $k \geq -1$. If $\mathcal{B}$ is a bounding class, the augmented simplicial group with word-length $(\Gamma_\bullet^+, L_\bullet)$ is $\mathcal{B}$-bounded if all face and degeneracy maps, including the augmentation map, are $\mathcal{B}$-bounded with respect to the word-lengths, $\{L_k\}_{k \geq -1}$. Then $(\Gamma_\bullet^+, L_\bullet)$ is a *type* $\mathcal{B}$*-resolution* if i) (Γ_k^+, L_k) is a countably generated free group with $\mathbb{N}$-valued word-length metric L_k generated by a proper function on the set of generators for Γ_k for all

$k \geq 0$, and ii) $\Gamma_\bullet^+$ admits a simplicial set contraction $\tilde{s} = \{\tilde{s}_{k+1} \mid \Gamma_k \to \Gamma_{k+1}\}_{k \geq -1}$ which is a $\mathcal{B}$-bounded set map in each degree. Every countable discrete group G admits a type $\mathcal{B}$ resolution $\Gamma_\bullet^+$ with $G = \Gamma_{-1}$. In fact, starting with G, the resolution can always be constructed so that the face and degeneracy maps, the augmentation, and the simplicial contraction are all linearly bounded [O1], Appendix.

Example 1. The cyclic bar construction, $N_\bullet^{\mathrm{cy}}(G)$.

Let G be a countable group and let L be a word-length function on G. We define $N_\bullet^{\mathrm{cy}}(G) = \{[n] \mapsto G^{n+1}\}_{n \geq 0}$ with

$$\begin{aligned}
\partial_i(g_0, g_1, \ldots, g_n) &= (g_0, g_1, \ldots, g_i g_{i+1}, \ldots, g_n), \quad 0 \leq i \leq n-1, \\
\partial_n(g_0, g_1, \ldots, g_n) &= (g_n g_0, g_1, g_2, \ldots, g_{n-1}), \\
s_j(g_0, g_1, \ldots, g_n) &= (g_0, \ldots, g_j, 1, g_{j+1}, \ldots, g_n).
\end{aligned}$$

The simplicial weight is given by $w_n(g_0, \ldots, g_n) = \sum_{i=1}^n L(g_i)$. $N_\bullet^{\mathrm{cy}}(G)$ is a $\mathcal{B}$-bounded simplicial set.

Example 2. The bar resolution, $EG_\bullet$.

Recall that the non-homogeneous bar resolution of G is $EG_\bullet = \{[n] \mapsto G^{n+1}\}_{n \geq 0}$ with

$$\begin{aligned}
\partial_i[g_0, \ldots, g_n] &= [g_0, \ldots, g_i g_{i+1}, \ldots, g_n], \quad 0 \leq i \leq n-1, \\
\partial_n[g_0, \ldots, g_n] &= [g_0, \ldots, g_{n-1}], \\
s_j[g_0, \ldots, g_n] &= [g_0, \ldots, g_j, 1, g_{j+1}, \ldots, g_n].
\end{aligned}$$

The simplicial weight function on $EG_\bullet$ is given by $w([g_0, \ldots, g_n]) = \sum_{i=0}^n L(g_i)$. The left G-action is given, as usual, by $g[g_0, g_1, \ldots, g_n] = [gg_0, g_1, \ldots, g_n]$. Note that with respect to the given weight function and action of G, $EG_\bullet$ is a $\mathcal{B}$-bounded simplicial G-set for any $\mathcal{B}$.

For $\mathcal{B}$ a bounding class and (X, w) a weighted set, $\mathcal{B}C(X)$ will denote the collection of $\mathcal{B}$-bounded functions on X. That is, $\mathcal{B}C(X)$ consists of all functions $f : X \to \mathbb{C}$ such that there is $\phi \in \mathcal{B}$ with $|f(x)| \leq \phi(w(x))$ for all $x \in X$. Dually, $\mathcal{H}_{\mathcal{B},w}(X)$ will consist of all $f : X \to \mathbb{C}$ such that for all $\phi \in \mathcal{B}$, the sum $\sum_{x \in X} |f(x)|\phi(x)$ is finite. This is a completion of the collection of finitely supported chains on X with respect to a family of seminorms on $\mathcal{H}_{\mathcal{B},w}(X)$ given by

$$\|f\|_\phi := \sum_{x \in X} |f(x)|\phi(w(x))$$

for $\phi \in \mathcal{B}$. When (X, w) is (G, L), a discrete group with a word-length function, $\mathcal{H}_{\mathcal{B},L}(G)$ is the $\mathcal{B}$-Rapid Decay algebra of G. When $\mathcal{B}$ is equivalent to a countable bounding class the seminorms give $\mathcal{H}_{\mathcal{B},L}(G)$ the structure of a Fréchet algebra.

For $(X_\bullet, w_\bullet)$ a weighted simplicial set and $\mathcal{B}$ a bounding class, set $\mathcal{B}C_n(X_\bullet) = \mathcal{H}_{\mathcal{B},w_n}(X_n)$, the completion of $C_n(X_\bullet)$. If each of the face maps of $(X_\bullet, w_\bullet)$ are $\mathcal{B}$-bounded, then the boundary maps $d_n\colon C_n(X_\bullet) \to C_{n-1}(X_\bullet)$ extend to $\mathcal{B}$-bounded maps $d_n\colon \mathcal{B}C_n(X_\bullet) \to \mathcal{B}C_{n-1}(X_\bullet)$. The homology of the resulting bornological complex $\{\mathcal{B}C_n(X_\bullet), d_n\}_{n\geq 0}$ is the $\mathcal{B}$-bounded homology $\mathcal{B}H_*(X)$ of $(X_\bullet, w_\bullet)$. Similarly the cohomology of the bornological cochain complex $\{\mathcal{B}C^n(X_\bullet) = \mathcal{B}C(X_n), \partial\}$ is the $\mathcal{B}$-bounded cohomology $\mathcal{B}H^*(X)$ of $(X_\bullet, w_\bullet)$. In the case of the weighted cyclic bar construction, $N_\bullet^{\mathrm{cy}}(G)$, we have identifications from [JOR1],

$$\mathcal{B}H_*(N_\bullet^{\mathrm{cy}}(G)) \cong \mathrm{HH}_*^t(\mathcal{H}_{\mathcal{B},L}(G)),$$
$$\mathcal{B}H^*(N_\bullet^{\mathrm{cy}}(G)) \cong \mathrm{HH}_t^*(\mathcal{H}_{\mathcal{B},L}(G)).$$

A group with length function (G, L) has $\mathcal{B}$-cohomological dimension ($\mathcal{B}$-cd) $\leq n$ if there is a projective resolution of $\mathbb{C}$ over $\mathcal{H}_{\mathcal{B},L}(G)$ of length at most n. Here we do not require finite generation over $\mathcal{H}_{\mathcal{B},L}(G)$ in any degree, but as in [JOR2] we require a $\mathcal{B}$-bounded $\mathbb{C}$-linear contracting homotopy. We denote by $\mathrm{bExt}_{\mathcal{H}_{\mathcal{B},L}(G)}$ the Ext-functor on the category of bornological $\mathcal{H}_{\mathcal{B},L}(G)$-modules. That is, for a bornological $\mathcal{H}_{\mathcal{B},L}(G)$-module, $\mathrm{bExt}_{\mathcal{H}_{\mathcal{B},L}(G)}(\mathbb{C}, M)$ is obtained by taking a projective resolution of $\mathbb{C}$ over $\mathcal{H}_{\mathcal{B},L}(G)$, apply $\mathrm{Hom}^{\mathrm{bdd}}(\,\cdot\,, M)$, and take the cohomology of the resulting cochain complex.

We have the following analogue of Lemma VIII.2.1 of [Br1].

Lemma 1. *For a group with length function (G, L) and $\mathcal{B}$ a composable bounding class, the following are equivalent.*

(1) *$\mathcal{B}$-cd $(G, L) \leq n$.*

(2) $\mathrm{bExt}^i_{\mathcal{H}_{\mathcal{B},L}(G)}(\mathbb{C},\,\cdot\,) = 0$ *for all $i > n$.*

(3) $\mathrm{bExt}^{n+1}_{\mathcal{H}_{\mathcal{B},L}(G)}(\mathbb{C},\,\cdot\,) = 0$.

(4) *If $0 \to K \to P_{n-1} \to \cdots \to P_1 \to P_0 \to \mathbb{C} \to 0$ is any sequence of bornological $\mathcal{H}_{\mathcal{B},L}(G)$-modules admitting a $\mathcal{B}$-bounded $\mathbb{C}$-linear contracting homotopy with each P_j projective, then K is projective.*

(5) *If $0 \to K \to F_{n-1} \to \cdots \to F_1 \to F_0 \to \mathbb{C} \to 0$ is any sequence of bornological $\mathcal{H}_{\mathcal{B},L}(G)$-modules admitting a $\mathcal{B}$-bounded $\mathbb{C}$-linear contracting homotopy with each F_j free, then K is free.*

Proof. (1) iff (2) iff (3) iff (4) as in [Br1] needing virtually no modification. (4) iff (5) follows from the bornological Eilenberg swindle. □

3. Avoiding bounded conjugators, and the proof of Theorem A

There is always, for each conjugacy class $\langle x\rangle \in \langle G\rangle$, an isomorphism of simplicial sets

$$G_x\backslash G \times_G EG_\bullet \to N_\bullet^{\mathrm{cy}}(G)_x.$$

(Here G_x is the centralizer of the element $x \in G$.) This map induces an isomorphism

$$\coprod_{\langle x\rangle\in\langle G\rangle} G_x\backslash G \times_G EG_\bullet \to \coprod_{\langle x\rangle\in\langle G\rangle} N_\bullet^{\mathrm{cy}}(G)_x.$$

When x is not central in G, this isomorphism depends on the particular choice of representative for each conjugacy class. Recall from [JOR1] that a conjugacy class $\langle x\rangle$ has a *$\mathcal{B}$-bounded conjugator length* if there is an $f \in \mathcal{B}$ such that, for all $x, y \in S_{\langle x\rangle} = \{$elements in G conjugate to $x\}$, there is a $g \in G$ with $g^{-1}xg = y$ and $L_G(g) \le f(L_G(x)+L_G(y))$. This is equivalent to the statement that the natural map

$$\pi_x\colon G_x\backslash G \to S_x$$

defined by $G_x g \mapsto g^{-1}xg$ is a $\mathcal{B}$-bounded isomorphism. When $\langle x\rangle \in \langle G\rangle$ has a $\mathcal{B}$-bounded conjugator length, the map $G_x\backslash G \times_G EG_\bullet \to N_\bullet^{\mathrm{cy}}(G)_x$ is a $\mathcal{B}$-bounded isomorphism of weighted simplicial sets, which in turn induces an isomorphism of cohomology groups

$$\mathrm{HH}^*(\mathcal{H}_{\mathcal{B},L}(G))_{\langle x\rangle} \cong \mathcal{B}H^*(BG_x).$$

We are interested in the case where there is no such conjugacy bound (which is most of the time). In this case, the map

$$\coprod_{\langle x\rangle\in\langle G\rangle} (G_x\backslash G \times_G EG_\bullet) \to N_\bullet^{\mathrm{cy}}(G)$$

still exists as a $\mathcal{B}$-bounded map and an isomorphism of simplicial sets, however a choice of $\mathcal{B}$-bounded inverse may not exist.

Assume given a type $\mathcal{B}$ simplicial resolution $\Gamma_\bullet \twoheadrightarrow G$. Associated to this simplicial type resolution is a simplicial chain complex equipped with augmentation map

$$\mathcal{B}C_*(N_\bullet^{\mathrm{cy}}(\Gamma_\bullet)) := \{[n] \mapsto \mathcal{B}C_*(N_\bullet^{\mathrm{cy}}(\Gamma_n))\}_{n\ge 0} \twoheadrightarrow \mathcal{B}C_*(N_\bullet^{\mathrm{cy}}(G)).$$

By Theorem 1 of [O2], this simplicial complex is of resolution type. Consequently, the double complex associated to $\mathcal{B}C_*(N_\bullet^{\mathrm{cy}}(\Gamma_\bullet))$ maps via the augmentation map to $\mathcal{B}C_*(N_\bullet^{\mathrm{cy}}(G))$ by a map inducing an isomorphism in $\mathcal{B}$-bounded homology and cohomology.

Now consider the following diagram of simplicial sets:

$$\begin{tikzcd}
\vdots \arrow[d, shift left=3] \arrow[d, shift left=1] \arrow[d, shift right=1] \arrow[d, shift right=3] & \\
N^{\mathrm{cy}}_\bullet(\Gamma_1) \arrow[u, shift left=5] \arrow[u, shift left=7] \arrow[d, shift left=2] \arrow[d] \arrow[d, shift right=2] & \\
N^{\mathrm{cy}}_\bullet(\Gamma_0) \arrow[u, shift left=4] \arrow[d] & \coprod_{y\in\langle\Gamma_0\rangle} (\Gamma_0)_y\backslash\Gamma_0 \times_{\Gamma_0} E(\Gamma_0)_\bullet \arrow[l, "\phi"', "\cong"] \arrow[d] \\
N^{\mathrm{cy}}_\bullet(G) & \coprod_{\langle x\rangle\in\langle G\rangle} G_x\backslash G\times_G EG_\bullet. \arrow[l]
\end{tikzcd}$$

For the map on the right, explicit centralizer subgroups G_x and $(\Gamma_0)_y$ are chosen as follows: for each $\langle x\rangle \in \langle G\rangle$, fix a basepoint x of the set $S_{\langle x\rangle}$. Then for each $\langle y\rangle \in \langle\Gamma_0\rangle$, we choose a basepoint $y \in S_{\langle y\rangle}$ of minimal word-length so that the augmentation map $\varepsilon\colon \Gamma_0 \twoheadrightarrow G$ induces a map of basepointed sets

$$(S_{\langle y\rangle}, y) \xrightarrow{\varepsilon} (S_{\langle\varepsilon(y)\rangle}, \varepsilon(y)).$$

The map ϕ is a linearly bounded isomorphism with linearly bounded inverse (hence a $\mathcal{B}$-bounded isomorphism for all $\mathcal{B}$), as Γ_0 is a free group equipped with word-length metric. This isomorphism will allow us to construct a simplicial resolution of $\coprod_{\langle x\rangle\in\langle G\rangle} G_x\backslash G\times_G EG_\bullet$.

Define the bisimplicial set $X_{\bullet\bullet}$ to be

$$X_{\bullet n} = \begin{cases} \coprod_{\langle y\rangle\in\langle\Gamma_0\rangle} (\Gamma_0)_y\backslash\Gamma_0 \times_{\Gamma_0} E(\Gamma_0)_\bullet, & n = 0,\\ N^{\mathrm{cy}}_\bullet(\Gamma_n), & n > 0. \end{cases}$$

The face maps and degeneracy maps between $X_{\bullet n}$ and $X_{\bullet m}$, for $m, n > 0$ are given by the corresponding maps $\{\partial_i^{\Gamma_\bullet}, s_j^{\Gamma_\bullet}\}$ in $N^{\mathrm{cy}}_\bullet(\Gamma_\bullet)$ induced by the face and degeneracy maps of $\Gamma_\bullet$. The face and degeneracy maps involving $X_{\bullet 0}$ are determined by $\partial_i\colon X_{\bullet 1} \to X_{\bullet 0}$, $\partial_i = \phi^{-1}\circ\partial_i^{\Gamma_1}$ for $i = 0, 1$ and $s_0\colon X_{\bullet 0} \to X_{\bullet 1}$, $s_0 = s_0^{\Gamma_0}\circ\phi$.

These maps satisfy the simplicial identities and are all $\mathcal{B}$-bounded. In particular we have a $\mathcal{B}$-bounded bisimplicial G set $\{[n] \mapsto X_{\bullet n}\}$. There is a $\mathcal{B}$-bounded isomorphism of bisimplicial G-sets

$$\{[n] \mapsto X_{\bullet n}\}_{n\geq 0} \to \{[n] \mapsto N^{\mathrm{cy}}_\bullet(\Gamma_n)\}_{n\geq 0}.$$

As the augmentation $\varepsilon\colon \Gamma_0 \to G$ is surjective, the induced map

$$\varepsilon_0\colon \coprod_{y\in\langle\Gamma_0\rangle} (\Gamma_0)_y\backslash\Gamma_0 \times_{\Gamma_0} E(\Gamma_0)_\bullet \to \coprod_{\langle x\rangle\in\langle G\rangle} G_x\backslash G\times_G EG_\bullet$$

is a surjective morphism of weighted simplicial sets, which moreover is bounded. Our aim is to show that the induced and given weight functions on the set to the right are equivalent. To this end, we define a section $\tilde{s}$ of ε_0 by using a set-theoretic splitting $G \rightarrowtail \Gamma_0$ of the augmentation $\varepsilon\colon \Gamma_0 \twoheadrightarrow G$ which is length-minimizing among all basepoint-preserving maps; precisely, if $x \in S_{\langle x \rangle}$ is the basepoint of $S_{\langle x \rangle}$, then i) $\tilde{s}(x) \in S_{\langle \tilde{s}(x) \rangle}$ should be the basepoint of $S_{\langle \tilde{s}(x) \rangle}$, and ii) $\tilde{s}(x)$ should have minimal word-length among all basepoints $y \in S_{\langle y \rangle} \subset \Gamma_0$ for which $\varepsilon(y) = x$. Because of the manner in which the basepoints in Γ_0 were chosen, such a section always exists, and moreover it can be chosen so as to be length-preserving on basepoints. Of course, it can always be chosen so as to be length-preserving away from basepoints.

Lemma 2. *The section $\tilde{s}$ induces a splitting $\{\tilde{s}_n\}_{n \geq 0}$ of ε_0 on the level of graded sets which is $\mathcal{B}$-bounded.*

Proof. Denote by L_0 the proper length function on Γ_0 and by L the proper length function on G. Fix a conjugacy class $\langle y \rangle \in \langle \Gamma_0 \rangle$ and consider an element of $(\Gamma_0)_y \backslash \Gamma_0 \times_{\Gamma_0} E(\Gamma_0)_\bullet$, $[(\Gamma_0)_y \gamma \times (\gamma_0, \gamma_1, \ldots, \gamma_n)]$. We may assume $(\Gamma_0)_y \gamma \times (\gamma_0, \gamma_1, \ldots, \gamma_n)$ is a minimal weight representative of the class, and that γ is a minimal length element of the coset $(\Gamma_0)_y \gamma$, so the weight of $[(\Gamma_0)_y \gamma \times (\gamma_0, \gamma_1, \ldots, \gamma_n)]$ is $L_0(\gamma) + L_0(\gamma_0) + \cdots + L_0(\gamma_n)$.

Given that we have arranged the basepoints for the sets $\{S_{\langle y \rangle}\}$ so as to make the augmentation basepoint-preserving, the induced map ϵ_0 on the right has the form

$$\epsilon_0[(\Gamma_0)_y \gamma \times (\gamma_0, \gamma_1, \ldots, \gamma_n)] = [G_x \epsilon(\gamma) \times (\epsilon(\gamma_0), \epsilon(\gamma_1), \ldots, \epsilon(\gamma_n))]$$

where $x = \varepsilon(y)$. The weight of this class is no more than $L(\epsilon(\gamma)) + L(\epsilon(\gamma_0)) + \cdots + L(\epsilon(\gamma_n))$. If $\beta \in \mathcal{B}$ is a bounding function for ϵ, then we see that the weight of this class is no more than $\beta(L_0(\gamma)) + \beta(L_0(\gamma_0)) + \cdots + \beta(L_0(\gamma_n))$. Thus ϵ_0 is bounded by $(n+2)\beta \in \mathcal{B}$.

For each n-simplex $[G_x g \times (g_0, g_1, \ldots, g_n)] \in G_x \backslash G \times_G EG_n$, fix a minimal weight representative $G_x g \times (g_0, g_1, \ldots, g_n)$ with g of minimal length in the coset $G_x g$. The desired section on n-simplices is defined by

$$\tilde{s}_n[G_x g \times (g_0, g_1, \ldots, g_n)] = [(\Gamma_0)_{\tilde{s}(x)} \tilde{s}(g) \times (\tilde{s}(g_0), \tilde{s}(g_1), \ldots, \tilde{s}(g_n))].$$

Note that the collection of maps $\{\tilde{s}_n\}_{n \geq 0}$ will not, in general, define a map of simplicial sets, because the original section $\tilde{s}$ cannot be chosen to be a homomorphism unless G itself is free. However, it is clear that $\tilde{s}_n$ is a set-theoretic splitting of ϵ_0 on n-simplices which moreover is bounded. To see this, note that the weight of $\tilde{s}_n[G_x g \times (g_0, g_1, \ldots, g_n)]$ can be no greater than $L_0(\tilde{s}(g)) + L_0(\tilde{s}(g_0)) + \cdots + L_0(\tilde{s}(g_n))$. Thus if $\tilde{s}$ is bounded by $\beta' \in \mathcal{B}$, then $\tilde{s}_n$ is bounded by $(n+2)\beta' \in \mathcal{B}$. □

As indicated, there are two basic approaches to endowing $\coprod_{\langle x \rangle \in \langle G \rangle} G_x \backslash G \times_G EG_\bullet$ with a weight function. One could use the weight induced by the length structure on

G itself, or one could use the weights induced by the surjection ϵ_0. However, the existence of a bounded section guaranteed by the previous lemma shows that these two methods yield $\mathcal{B}$-equivalent weight structures.

Theorem 3. *If $\Gamma_\bullet \twoheadrightarrow G$ is a type $\mathcal{B}$ resolution of G, then $X_{\bullet\bullet}$ is a $\mathcal{B}$-bounded augmented simplicial set for which the bisimplicial set $\{[n] \to X_{\bullet,n}\}_{n\geq 0}$ yields isomorphisms in homology and cohomology*

$$\mathcal{B}H^*\Big(\coprod_{\langle x\rangle\in G} G_x\backslash G\times_G EG_\bullet\Big)\cong \mathcal{B}H^*(X_{\bullet\bullet}),$$
$$\mathcal{B}H_*\Big(\coprod_{\langle x\rangle\in G} G_x\backslash G\times_G EG_\bullet\Big)\cong \mathcal{B}H_*(X_{\bullet\bullet}).$$

Proof. The sections coming from the previous lemma, together with those arising from the type $\mathcal{B}$ resolution, imply that the augmented simplicial topological chain complex

$$[n]\mapsto \begin{cases}\mathcal{B}C_*(X_{\bullet n}), & n\geq 0,\\ \mathcal{B}C_*(\coprod_{\langle x\rangle\in G} G_x\backslash G\times_G EG_\bullet), & n=-1,\end{cases}$$

is of resolution type. The result then follows as in [O2], Theorem 2. □

Corollary 1. *There are isomorphisms in $\mathcal{B}$-bounded homology and cohomology*

$$\mathcal{B}H^*\Big(\coprod_{\langle x\rangle\in\langle G\rangle} G_x\backslash G\times_G EG_\bullet\Big)\cong \mathcal{B}H^*(N_\bullet^{\mathrm{cy}}(G)) = H_t^*(\mathcal{H}_{\mathcal{B},L}(G)),$$
$$\mathcal{B}H_*\Big(\coprod_{\langle x\rangle\in\langle G\rangle} G_x\backslash G\times_G EG_\bullet\Big)\cong \mathcal{B}H_*(N_\bullet^{\mathrm{cy}}(G)) = H_*^t(\mathcal{H}_{\mathcal{B},L}(G)).$$

Proof. The weighted bisimplicial set $X_{\bullet\bullet}$ is $\mathcal{B}$-boundedly isomorphic to the weighted bisimplicial set $\{[n]\to N_\bullet^{\mathrm{cy}}(\Gamma_n)\}_{n\geq 0}$, and the augmentation map

$$\varepsilon\colon \mathcal{B}C_*(N_\bullet^{\mathrm{cy}}(\Gamma_\bullet))\twoheadrightarrow \mathcal{B}C_*(N_\bullet^{\mathrm{cy}}(G))$$

induces an isomorphism in $\mathcal{B}$-bounded homology and cohomology. The result follows then from Theorem 3. □

Corollary 2. *There is an isomorphism in $\mathcal{B}$-bounded cohomology*

$$\mathrm{HH}_t^*(\mathcal{H}_{\mathcal{B},L}(G))\cong \mathcal{B}H^*\Big(\coprod_{\langle x\rangle\in\langle G\rangle} BG_x\Big).$$

Proof. By [JOR1], Proposition 1.4.5, there is an isomorphism

$$\mathcal{B}H^*\Big(\coprod_{\langle x\rangle\in\langle G\rangle} BG_x\Big)\cong \mathcal{B}H^*\Big(\coprod_{\langle x\rangle\in\langle G\rangle} G_x,\backslash G\times_G EG_\bullet\Big),$$

where each centralizer subgroup G_x is equipped with the induced word-length function coming from the embedding into G. The previous corollary gives the result. □

Corollary 3. *For*[2] *each non-elliptic conjugacy class* $\langle x\rangle \in \langle G\rangle$ *with* $\mathrm{Dist}((x)) \leq \mathcal{B}$, *there is an isomorphism*

$$\mathrm{HC}_t^*(\mathcal{H}_{\mathcal{B},L}(G))_{\langle x\rangle} \cong \mathcal{B}H^*(G_x/(x)),$$

where $G_x/(x)$ *is equipped with the word-length function induced by the projection* $G_x \twoheadrightarrow N_x = G_x/(x)$.

Proof. The isomorphism of Corollary 2 splits over conjugacy classes

$$\mathrm{HH}_t^*(\mathcal{H}_{\mathcal{B},L}(G))_{\langle x\rangle} \cong \mathcal{B}H^*(BG_x).$$

When $\mathrm{Dist}((x)) \leq \mathcal{B}$, the Connes–Gysin sequences in [JOR1], §1.4, for the summand $\mathrm{HC}_t^*(\mathcal{H}_{\mathcal{B},L}(G))_{\langle x\rangle}$ along with this isomorphism for $\mathrm{HH}^*(\mathcal{H}_{\mathcal{B},L}(G))_{\langle x\rangle}$ gives the result. □

In the case $\mathrm{Dist}((x)) > \mathcal{B}$, it is almost certainly true that $\mathrm{HC}_t^*(\mathcal{H}_{\mathcal{B},L}(G))_{\langle x\rangle} \cong \mathcal{B}H^*(G_x) \otimes \mathrm{HC}^*(\mathbb{C})$. To conclude this, however, one needs to know that if $\mathbb{Z}$ is embedded into a group with distortion greater than $\mathcal{B}$, then $\mathcal{B}H^*(\mathbb{Z})$ is trivial for $* \geq 1$ (where $\mathbb{Z}$ is equipped with the word-length function induced by the embedding). This is conjectured to be true, but is currently unverified.

Remark. As noted, the results of this section were previously known only for groups with conjugacy classes satisfying $\mathcal{B}$-bounded conjugacy length bounds of [JOR1]. They may be interpreted as saying that, even when the conjugacy problem for G cannot be solved in an appropriately bounded timeframe, or even solved at all, it can always be solved "up to bounded homotopy".

4. A class of groups satisfying $\mathcal{B}$-SrBC

We start with a technical lemma.

Lemma 3. *Suppose that* $N \rightarrowtail G \twoheadrightarrow Q$ *is an extension of groups with length functions, and let* $\mathcal{B}$ *be a composable bounding class. If* $\mathcal{B}$-cd $N < \infty$ *and* $\mathcal{B}$-cd $Q < \infty$ *then* $\mathcal{B}$-cd $G \leq \mathcal{B}$-cd $N + \mathcal{B}$-cd Q.

Proof. We follow the setup of [O1], Theorem 1.1.12. Denote by L_G the length function on G, which by restriction is the length function on N, and by L_Q the quotient length function on Q. As $\mathcal{B}$-cd $Q < \infty$, there is a finite length free resolution of $\mathbb{C}$ over $\mathcal{H}_{\mathcal{B},L_Q}(Q)$, with $\mathcal{B}$-bounded $\mathbb{C}$-linear splittings. Denote this resolution by R_*, let P_* be a free resolution of $\mathbb{C}$ over $\mathcal{H}_{\mathcal{B},L_G}(G)$, and set $S_q = \mathbb{C}\,\widehat{\otimes}_{\mathcal{H}_{\mathcal{B},L_G}(N)} P_q$.

[2] This result generalizes Theorem 2.4.4 of [JOR1].

Fix an $\mathcal{H}_{\mathcal{B},L_G}(G)$-module M and let $C^{p,q} = \mathcal{B}\operatorname{Hom}_{\mathcal{H}_{\mathcal{B},L_Q}(Q)}(R_p \hat{\otimes} S_q, M)$. The spectral sequence associated to the row filtration collapses at $E_2^{*,*}$ with $E_2^{0,q} \cong \mathcal{B}H^q(G; M)$ and $E_2^{p,q} = 0$ if $p > 0$.

Consider the spectral sequence associated to the column filtration. For $p > \mathcal{B}\text{-cd}\, Q$, $C^{p,q} = 0$, so $E_1^{p,q} = 0$ for $p > \mathcal{B}\text{-cd}\, Q$. Similarly, by the finiteness condition on $\mathcal{B}\text{-cd}\, N$, $E_1^{pq} = 0$ for $q > \mathcal{B}\text{-cd}\, N$. Consequently, $E_2^{p,q} = 0$ whenever $p + q > \mathcal{B}\text{-cd}\, Q + \mathcal{B}\text{-cd}\, N$. As a consequence, $\mathcal{B}H^n(G; M) = 0$ for $n > \mathcal{B}\text{-cd}\, Q + \mathcal{B}\text{-cd}\, N$ for all coefficients M. □

Let $\mathcal{C}$ be the collection of all countable groups G which satisfy the nilpotency condition, and let $\mathcal{B}$-$\mathcal{C}$ be the collection of all groups with length function (G, L) which satisfy the $\mathcal{B}$-nilpotency condition. Any group $(G, L) \in \mathcal{B}$-$\mathcal{C}$ satisfies $\mathcal{B}$-SrBC, just as any $G \in \mathcal{C}$ satisfies SBC. Moreover if (G, L) satisfies $\mathcal{B}$-SrBC, then G satisfies SrBC. Recall that a group with length function (G, L) is said to be $\mathcal{B}$-*isocohomological* ($\mathcal{B}$-IC) if the inclusion $\mathbb{C}[G] \to \mathcal{H}_{\mathcal{B},L}(G)$ induces isomorphisms $\mathcal{B}H^p(G; \mathbb{C}) \to H^p(G; \mathbb{C})$ for all p. The group is $\mathcal{B}$-*strongly isocohomological* ($\mathcal{B}$-SIC) if the maps $\mathcal{B}H^p(G; M) \to H^p(G; M)$ are isomorphisms for every p and every bornological $\mathcal{H}_{\mathcal{B},L}(G)$-module M.

Theorem 4. *Suppose that the discrete group G lies in $\mathcal{C}$ and L is a proper length function on G. If for each non-elliptic conjugacy class $\langle x \rangle \in \langle G \rangle$, the centralizer (G_x, L) is $\mathcal{B}$-IC and the embedding $\mathbb{Z} \cong (x) \rightarrowtail G_x$ is at most $\mathcal{B}$-distorted, then $(G, L) \in \mathcal{B}$-$\mathcal{C}$.*

Proof. By Theorem A, $\mathrm{HC}^*(\mathbb{C}[G])_{\langle x \rangle} \cong \mathrm{HC}_t^*(\mathcal{H}_{\mathcal{B},L}(G))_{\langle x \rangle}$. This isomorphism identifies periodicity operators. □

As shown in the introduction this class contains all semihyperbolic groups G with word-length function L, which satisfy the nilpotency condition. We remark that this, in turn, contains all word-hyperbolic and finitely generated abelian groups.

We now identify and examine a class of groups inside $\mathcal{B}$-$\mathcal{C}$.

Definition. Let $\mathcal{B}$-$\mathcal{E}$ be the collection of all countable groups with length function (G, L) which satisfy the following three properties.

(1) G has finite $\mathcal{B}$-cd.

(2) For every non-elliptic conjugacy class $\langle x \rangle \in \langle G \rangle$, N_x has finite $\mathcal{B}$-cd in the induced length function.

(3) For every non-elliptic conjugacy class $\langle x \rangle \in \langle G \rangle$, the distortion of (x) as a subgroup of G is $\mathcal{B}$-bounded.

Let $\mathcal{E}$ be the collection of all countable groups in $\mathcal{B}_{\max}$-$\mathcal{E}$.

It is clear that $\mathcal{B}$-$\mathcal{E} \subset \mathcal{B}$-$\mathcal{C}$ and $\mathcal{E} \subset \mathcal{C}$.

The following is clear from the definition of $\mathcal{B}$-SIC.

Lemma 4. *Suppose that G is a countable group in $\mathcal{E}$ with length function L. If (G, L) is $\mathcal{B}$-SIC and N_x is $\mathcal{B}$-SIC for each non-elliptic $\langle x\rangle \in \langle G\rangle$, then (G, L) is in $\mathcal{B}$-$\mathcal{E}$.*

Theorem 5. *The class $\mathcal{B}$-$\mathcal{E}$ is closed under the following operations when considered in the induced length functions:*

(1) *taking subgroups,*

(2) *taking extensions of groups with length functions,*

(3) *acting on trees with vertex and edge stabilizers in $\mathcal{B}$-$\mathcal{E}$.*

Proof. We follow the proof of Theorem 4.3 of [Ji1]. (1) follows from [Ji1] after noting that if (G, L) is a group with length function and $H < G$, any projective resolution of $\mathbb{C}$ over $\mathcal{H}_{\mathcal{B},L}(G)$ is also a projective resolution of $\mathbb{C}$ over $\mathcal{H}_{\mathcal{B},L}(H)$.

(2) follows as in [Ji1] using Lemma 3 to bound the $\mathcal{B}$-cd of the involved extensions.

(3) follows as in [Ji1], noting that each of the stabilizers involved are assumed to be in $\mathcal{B}$-$\mathcal{E}$ in the restricted length function. □

References

[AB] J. M. Alonso and M. R. Bridson, Semihyperbolic groups. *Proc. London Math. Soc.* (3) **70** (1995), 56–114. Zbl 0823.20035 MR 1300841

[Br1] K. S. Brown, *Cohomology of groups*. Graduate Texts in Math. 87, Springer-Verlag, New York 1994. Corrected reprint of the 1982 original. Zbl 0584.20036 MR 1324339

[B1] D. Burghelea, The cyclic homology of the group rings. *Comment. Math. Helv.* **60** (1985), 354–365. Zbl 0595.16022 MR 814144

[CP] G. K. Chadha and I. B. S. Passi, Centralizers and homological dimension. *Comm. Algebra* **22** (1994), 5703–5708. Zbl 0817.20053 MR 1298744

[E1] I. Emmanouil, On a class of groups satisfying Bass' conjecture. *Invent. Math.* **132** (1998), 307–330. Zbl 0913.20002 MR 1621420

[Ji1] R. Ji, Nilpotency of Connes' periodicity operator and the idempotent conjectures. *K-Theory* **9** (1995), 59–76. Zbl 0826.46063 MR 1340840

[JOR1] R. Ji, C. Ogle, and B. Ramsey, Relatively hyperbolic groups, rapid decay algebras and a generalization of the Bass conjecture. *J. Noncommut. Geom.* **4** (2010), 83–124. Zbl 1192.46066 MR 2575390

[JOR2] R. Ji, C. Ogle, and B. Ramsey, $\mathcal{B}$-bounded cohomology and applications. *Internat J. Algebra Comput.* **23** (2013) 147–204. Zbl 06156068 MR 3040807

[Jo1] P. Jolissaint, Les fonctions à décroissance rapide dans les C^*-algèbres résduites de groupes. Thèse, Univ. de Genéve, 1987.

[Jo2] P. Jolissaint, K-theory of reduced C^*-algebras and rapidly decreasing functions on groups. *K-Theory* **2** (1989), 723–735. Zbl 0692.46062 MR 1010979

[M1] R. Meyer, Combable groups have group cohomology of polynomial growth. *Quart. J. Math.* **57** (2006), 241–261. Zbl 1167.20028 MR 2237601

[O1] C. Ogle, Polynomially bounded cohomology and discrete groups. *J. Pure Appl. Algebra* **195** (2005), 173–209. Zbl 1071.18007 MR 2109110

[O2] C. Ogle, Polynomially bounded cohomology and the Novikov conjecture. Preprint 2010. arXiv:1004.4680

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