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Group quasi-representations and almost flat bundles

Marius Dadarlat*

Abstract. We study the existence of quasi-representations of discrete groups G into unitary groups $U(n)$ that induce prescribed partial maps $K_0(C^*(G)) \to \mathbb{Z}$ on the K-theory of the group C*-algebra of G. We give conditions for a discrete group G under which the K-theory group of the classifying space BG consists entirely of almost flat classes.

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1. Introduction

The notions of almost flat bundle and group quasi-representation were introduced by Connes, Moscovici and Gromov [4] as tools for proving the Novikov conjecture for large classes of groups. The first example of a topologically nontrivial quasi-representation is due to Voiculescu for $G = \mathbb{Z}^2$, [27]. In this paper we use known results on the Novikov and the Baum–Connes conjectures to derive the existence of topologically nontrivial quasi-representations of certain discrete groups G, as well as the existence of nontrivial almost flat bundles on the classifying space BG, by employing the concept of quasidiagonality.

A discrete completely positive asymptotic representation of a C*-algebra A consists of a sequence $\{\pi_n : A \to \mathrm{M}_{k(n)}(\mathbb{C})\}_n$ of unital completely positive maps such that $\lim_{n\to\infty} \|\pi_n(aa') - \pi_n(a)\pi_n(a')\| = 0$ for all $a, a' \in A$. The sequence $\{\pi_n\}_n$ induces a unital $*$-homomorphism

$$A \to \prod_n \mathrm{M}_{k(n)}(\mathbb{C}) / \sum_n \mathrm{M}_{k(n)}(\mathbb{C})$$

and hence a group homomorphism $K_0(A) \to \prod_n \mathbb{Z}/\sum_n \mathbb{Z}$. This gives a canonical way to push forward an element $x \in K_0(A)$ to a sequence of integers $(\pi_{n\sharp}(x))$, which is well-defined up to tail equivalence; two sequences are tail equivalent, $(y_n) \equiv (z_n)$, if there is m such that $x_n = y_n$ for all $n \geq m$.

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In the first part of the paper we study the existence of discrete asymptotic representations of group C*-algebras that interpolate on K-theory a given group homomorphism $h\colon K_0(C^*(G)) \to \mathbb{Z}$. We rely heavily on results of Kasparov, Higson, Yu, Skandalis and Tu [15], [12], [29], [24], [16], [26]. For illustration, we have the following:

Theorem 1.1. *Let G be a countable, discrete, torsion-free group with the Haagerup property. Suppose that $C^*(G)$ is residually finite dimensional. Then, for any group homomorphism $h\colon K_0(C^*(G)) \to \mathbb{Z}$, there is a discrete completely positive asymptotic representation $\{\pi_n\colon C^*(G) \to \mathrm{M}_{k(n)}(\mathbb{C})\}_n$ such that $\pi_{n\sharp}(x) \equiv h(x)$ for all $x \in K_0(I(G))$.*

Here $I(G)$ is the kernel of the trivial representation $\iota\colon C^*(G) \to \mathbb{C}$. By contrast, any finite dimensional unitary representation of G induces the zero map on $K_0(I(G))$. The groups with the Haagerup property are characterized by the requirement that there exists a sequence of normalized continuous positive-definite functions which vanish at infinity on G and converge to 1 uniformly on finite subsets of G. The conclusion of Theorem 1.1 also holds if G is an increasing union of residually finite amenable groups, see Theorem 3.4. The class of groups considered in Theorem 1.1 contains all countable, torsion-free, amenable, residually finite groups (also the maximally periodic groups) and the surface groups [17]. Moreover, this class is closed under free products (see [10], [3]). If we impose a weaker condition, namely that $C^*(G)$ is quasidiagonal, then in general we need two asymptotic representations in order to interpolate h, see Theorem 3.3. Theorem 1.1 remains true if we replace the assumption that G has the Haagerup property by the requirements that G is uniformly embeddable in a Hilbert space and that the assembly map $\mu\colon \mathrm{RK}_0(\mathrm{B}G) \to K_0(C^*(G))$ is surjective. Let us recall that Hilbert space uniform embeddability of G implies that μ is split injective, as proven by Yu [29] if the classifying space BG is finite and by Skandalis, Yu and Tu [24] in the general case. We will also use a strengthening of this result by Tu [26] who showed that G has a gamma element. In conjunction with a theorem of Kasparov [15] this guarantees the surjectivity of the dual assembly map $\nu\colon K^0(C^*(G)) \to \mathrm{RK}^0(\mathrm{B}G)$ for countable, discrete, torsion-free groups which are uniformly embeddable in a Hilbert space.

The notion of almost flat K-theory class was introduced in [4] as a tool for proving the Novikov conjecture. In the second part of the paper we pursue a reverse direction. Namely, we use known results on the Baum–Connes and the Novikov conjectures to derive the existence of almost flat K-theory classes by employing the concept of quasidiagonality.

Theorem 1.2. *Let G be a countable, discrete, torsion-free group which is uniformly embeddable in a Hilbert space. Suppose that the classifying space* BG *is a finite simplicial complex and that the full group C*-algebra $C^*(G)$ is quasidiagonal. Then all the elements of $K^0(\mathrm{B}G)$ are almost flat.*

The class of groups considered in Theorem 1.2 is closed under free products, by [1] and [2]. If G can be written as a union of amenable residually finite groups (as is the case if G is a linear amenable group), then $C^*(G)$ is quasidiagonal. It is an outstanding open question if all discrete amenable groups have quasidiagonal C*-algebras [28].

Voiculescu has asked in [28] if there are invariants of a topological nature which can be used to describe the obstruction that a C*-algebra be quasidiagonal. One can view Theorem 1.2 as further evidence towards a topological nature of quasidiagonality, since it shows that the existence of non-almost flat classes in $K^0(\mathrm{B}G)$ represents an obstruction for the quasidiagonality of $C^*(G)$.

The fundamental connection between deformations of C*-algebras and K-theory was discovered by Connes and Higson [5]. They introduced the concept of asymptotic homomorphism of C*-algebras which formalizes the intuitive idea of deformations of C*-algebras. An asymptotic homomorphism is a family of maps $\varphi_t\colon A \to B$, $t \in [0,\infty)$, such that for each $a \in A$ the map $t \to \varphi_t(a)$ is continuous and bounded and the family $(\varphi_t)_{t\in[0,\infty)}$ satisfies asymptotically the axioms of $*$-homomorphisms. There is a natural notion of homotopy for asymptotic homomorphisms. E-theory is defined as homotopy classes of asymptotic homomorphisms from the suspension of A to the stable suspension of B, $E(A,B) = [\![C_0(\mathbb{R}) \otimes A, C_0(\mathbb{R}) \otimes B \otimes \mathcal{K}]\!]$. The introduction of the suspension and of the compact operators $\mathcal{K}$ yields an abelian group structure on $E(A,B)$. Connes and Higson showed that E-theory defines the universal half-exact C*-stable homotopy functor on separable C*-algebras. In particular the KK-theory of Kasparov factors through E-theory. A similar construction based on completely positive asymptotic homomorphisms gives a realization of KK-theory itself as shown by Larsen and Thomsen [13].

While E-theory gives in general maps of suspensions of C*-algebras it is often desirable to have interesting deformations of unsuspended C*-algebras. In joint work with Loring [8], [6], we proved a suspension theorem for commutative C*-algebras $A = C_0(X \setminus x_0)$, where X is a compact connected space and $x_0 \in X$ is a base point. Specifically, we showed that the reduced K-homology group $\widetilde{K}_0(X) = K_0(X, x_0)$ is isomorphic to the homotopy classes of asymptotic homomorphisms $[\![C_0(X \setminus x_0), \mathcal{K}]\!]$. One can replace the compact operators $\mathcal{K}$ by $\bigcup_{n=1}^{\infty} \mathrm{M}_n(\mathbb{C})$ and conclude that the reduced K-homology of X classifies the deformations of $C_0(X)$ into matrices. The case of $X = \mathbb{T}^2$ played an important role in the history of the subject. Indeed, Voiculescu [27] exhibited pairs of almost commuting unitaries $u, v \in U(n)$ whose properties reflect the non-triviality of $H^2(\mathbb{T}^2, \mathbb{Z})$. One can view such a pair as associated to a quasi-representation of $C^*(\mathbb{Z}^2) \cong C(\mathbb{T}^2)$. If the commutator $\|uv - vu\|$ is sufficiently small, then there is an induced pushforward of the Bott class that represents the obstruction for perturbing u, v to a pair of commuting unitaries, [27], [9]. It is therefore quite natural to investigate deformations of C*-algebras associated to non-commutative groups. In view of Theorem 1.1 we propose the following:

Conjecture. *If G is a discrete, countable, torsion-free, amenable group, then the natural map $[\![I(G),\mathcal{K}]\!] \to \mathrm{KK}(I(G),\mathcal{K}) \cong K^0(I(G))$ is an isomorphism of groups.*

This is verified if G is commutative. Indeed, $I(G) \cong C_0(\widehat{G} \setminus x_0)$ and $\widehat{G}$ is connected since G is torsion-free, so that we can apply the suspension result of [6].

Manuilov, Mishchenko and their co-authors have studied various aspects and applications of quasi-representations and asymptotic representations of discrete groups. The paper [18] is a very interesting survey of their contributions. The notion of quasi-representation of a group is used in the literature in several non-equivalent contexts, to mean several different things, see [22].

2. Quasi-representations and K-theory

Definition 2.1. Let A and B be unital C*-algebras. Let $F \subset A$ be a finite set and let $\varepsilon > 0$. A unital completely positive map $\varphi\colon A \to B$ is called an (F,ε)-*homomorphism* if $\|\varphi(aa') - \varphi(a)\varphi(a')\| < \varepsilon$ for all $a, a' \in F$. If B is the C*-algebra of bounded linear operators on a Hilbert space, then we say that φ is an (F,ε)-*representation* of A. We will use the term *quasi-representation* to refer to an (F,ε)-representation where F and ε are not necessarily specified.

An important method for turning K-theoretical invariants of A into numerical invariants is to use quasi-representations to pushforward projections in matrices over A to scalar projections. Consider a finite set of projections $\mathcal{P} \subset \mathrm{M}_m(A)$. We say that $(\mathcal{P}, F, \varepsilon)$ is a K_0-triple if for any (F,ε)-homomorphism $\varphi\colon A \to B$ and $p \in \mathcal{P}$, the element $b = (\mathrm{id}_m \otimes \varphi)(p)$ satisfies $\|b^2 - b\| < 1/4$ and hence the spectrum $\mathrm{sp}(b)$ of b is contained in $[0,1/2) \cup (1/2,1]$. We denote by q the projection $\chi(b)$, where χ is the characteristic function of the interval $(1/2,1]$. It is not hard to show that for any finite set of projections $\mathcal{P}$ there exist a finite set $F \subset A$ and $\varepsilon > 0$ such that $(\mathcal{P}, F, \varepsilon)$ is a K_0-triple. If $(\mathcal{P}, F, \varepsilon)$ is a K_0-triple, then any (F,ε)-homomorphism $\varphi\colon A \to B$ induces a map $\varphi_\sharp\colon \mathcal{P} \to K_0(B)$ defined by $\varphi_\sharp(p) = [q]$. Let $\mathrm{Proj}(A)$ denote the set of all projections in matrices over A. It is convenient to extend $\varphi_\sharp$ to $\mathrm{Proj}(A)$ by setting $\varphi_\sharp(p) = 0$ if $b = (\mathrm{id}_m \otimes \varphi)(p)$ does not satisfy $\|b^2 - b\| < 1/4$. If φ were a $*$-homomorphism, then φ would induce a map $\varphi_*\colon K_0(A) \to K_0(B)$. Intuitively, one may think of $\varphi_\sharp$ as a substitute for φ_*.

Two sequences (a_n) and (b_n) are called *tail-equivalent* if there is n_0 such that $a_n = b_n$ for $n \geq n_0$. Tail-equivalence is denoted by $(a_n) \equiv (b_n)$ or even $a_n \equiv b_n$, abusing the notation.

We will also work with discrete completely positive asymptotic morphisms $(\varphi_n)_n$. They consists of a sequence of contractive completely positive maps $\varphi_n\colon A \to B_n$ with $\lim_{n\to\infty} \|\varphi_n(aa') - \varphi_n(a)\varphi_n(a')\| = 0$ for all $a, a' \in A$. If in addition each B_n is a matricial algebra $B_n = \mathrm{M}_{k(n)}(\mathbb{C})$, then we call $(\varphi_n)_n$ a *discrete asymptotic representation* of A. A discrete completely positive asymptotic morphism $(\varphi_n)_n$

induces a sequence of maps $\varphi_{n\sharp}\colon \mathrm{Proj}(A) \to K_0(B_n)$. Note that if $p, q \in \mathrm{Proj}(A)$ have the same class in $K_0(A)$, then $\varphi_{n\sharp}(p) \equiv \varphi_{n\sharp}(q)$.

For any $x \in K_0(A)$, we fix projections $p, q \in \mathrm{Proj}(A)$ such that $x = [p] - [q]$ and set $\varphi_{n\sharp}(x) = \varphi_{n\sharp}(p) - \varphi_{n\sharp}(q) \in K_0(B_n)$. The sequence $(\varphi_{n\sharp}(x))$ depends on the particular projections that we use to represent x but only up to tail-equivalence. While in general the maps $\varphi_{n\sharp}\colon K_0(A) \to K_0(B_n)$ are not group homomorphisms, the sequence $(\varphi_{n\sharp}(x))$ does satisfy $(\varphi_{n\sharp}(x+y)) \equiv (\varphi_{n\sharp}(x) + \varphi_{n\sharp}(y))$ for all $x, y \in K_0(A)$.

A subset $B \subset L(H)$ is called *quasidiagonal* if there is an increasing sequence (p_n) of finite rank projections in $L(H)$ which converges strongly to 1_H and such that $\lim_{n\to\infty} \|[b, p_n]\| = 0$ for all $b \in B$. B is *block-diagonal* if there is a sequence (p_n) as above such that $[b, p_n] = 0$ for all $b \in B$ and $n \geq 1$. Let A be a separable C*-algebra. Let us recall that the elements of $\mathrm{KK}(A, \mathbb{C})$ can be represented by Cuntz pairs, i.e., by pair of $*$-representations $\varphi, \psi\colon A \to L(H)$ such that $\varphi(a) - \psi(a) \in K(H)$ for all $a \in A$.

Definition 2.2. Let A be a separable C*-algebra. An element $\alpha \in \mathrm{KK}(A, \mathbb{C})$ is called *quasidiagonal* if it can be represented by a Cuntz pair $(\varphi, \psi)\colon A \to L(H)$ with the property that the set $\psi(A) \subset L(H)$ is quasidiagonal. In this case let us note that the set $\varphi(A) \subset L(H)$ must be also quasidiagonal. Similarly, we say that α is *residually finite dimensional* if it can be represented by a Cuntz pair with the property that the set $\psi(A)$ is block-diagonal. We denote by $\mathrm{KK}_{\mathrm{qd}}(A, \mathbb{C})$ the subset of $\mathrm{KK}(A, \mathbb{C})$ consisting of quasidiagonal classes and by $\mathrm{KK}_{\mathrm{rfd}}(A, \mathbb{C})$ the subset of $\mathrm{KK}(A, \mathbb{C})$ consisting of residually finite dimensional classes. It is clear that $\mathrm{KK}_{\mathrm{rfd}}(A, \mathbb{C}) \subset \mathrm{KK}_{\mathrm{qd}}(A, \mathbb{C})$, that $\mathrm{KK}_{\mathrm{qd}}(A, \mathbb{C})$ is a subgroup of $\mathrm{KK}(A, \mathbb{C})$ and that $\mathrm{KK}_{\mathrm{rfd}}(A, \mathbb{C})$ is a subsemigroup.

We say that A is K-quasidiagonal if $\mathrm{KK}_{\mathrm{qd}}(A, \mathbb{C}) = \mathrm{KK}(A, \mathbb{C})$ and that A is K-residually finite dimensional if $\mathrm{KK}_{\mathrm{rfd}}(A, \mathbb{C}) = \mathrm{KK}(A, \mathbb{C})$.

Remark 2.3. Let A be a separable C*-algebra. It was pointed out by Skandalis [23] that for any given faithful $*$-representation $\pi\colon A \to L(H)$ such that $\pi(A) \cap K(H) = \{0\}$, one can represent all the elements of $\mathrm{KK}(A, \mathbb{C})$ by Cuntz pairs where the second map is fixed and equal to π. It follows that a separable quasidiagonal C*-algebra is K-quasidiagonal and a separable residually finite dimensional C*-algebra is K-residually finite dimensional. More generally, if A is homotopically dominated by B and B is K-quasidiagonal or K-residually finite dimensional then so is A. Let us note that the Cuntz algebra O_2 is K-residually finite dimensional while it is not quasidiagonal.

The following lemma and proposition are borrowed from [7]. For the sake of completeness, we review briefly some of the arguments from their proofs. Let B be a unital C*-algebra and let E be a right Hilbert B-module. If $e, f \in L_B(E)$ are projections such that $e - f \in K_B(E)$, we denote by $[e, f]$ the corresponding element of $\mathrm{KK}(\mathbb{C}, B) \cong K_0(B)$.

Lemma 2.4. *Let B be a unital C*-algebra and let E be a right Hilbert B-module. Let $e, f \in L_B(E)$ and $h \in K_B(E)$ be projections such that $e - f \in K_B(E)$ and $\|eh - he\| \le 1/9$, $\|fh - hf\| \le 1/9$, $\|(1-h)(e-f)(1-h)\| \le 1/9$. Then*

$$\mathrm{sp}(heh) \cup \mathrm{sp}(hfh) \subset [0, 1/2) \cup (1/2, 1],$$

$$[e, f] = [\chi(heh), \chi(hfh)] \in \mathrm{KK}(\mathbb{C}, B) \cong K_0(B).$$

Proof. One shows that if $e', f' \in L_B(E)$ are projections such that $e' - f' \in K_B(E)$ and $\|e - e'\| < 1/2$, $\|f - f'\| < 1/2$, then $[e, f] = [e', f']$. This is proved using the homotopy $(\chi(e_t), \chi(f_t))$ where $e_t = (1-t)e + te'$, $f_t = (1-t)f + tf'$, $0 \le t \le 1$. Then one applies this observation to conclude that

$$[e, f] = [\chi(x)+\chi(x'), \chi(y)+\chi(y')] = [\chi(x)+\chi(x'), \chi(y)+\chi(x')] = [\chi(x), \chi(y)],$$

where $x = heh$, $x' = (1-h)e(1-h)$, $y = hfh$, $y' = (1-h)f(1-h)$. □

Let A, B be separable C*-algebras. An element $\alpha \in \mathrm{KK}(A, \mathbb{C})$ induces a group homomorphism $\alpha_* \colon K_0(A \otimes B) \to K_0(B)$ via the cup product

$$\mathrm{KK}(\mathbb{C}, A \otimes B) \times \mathrm{KK}(A, \mathbb{C}) \to \mathrm{KK}(\mathbb{C}, B), \quad (x, \alpha) \mapsto x \circ (\alpha \otimes 1_B).$$

Here we work with the maximal tensor product.

Proposition 2.5. *Let A be a separable unital C*-algebra and $\alpha \in \mathrm{KK}_{\mathrm{qd}}(A, \mathbb{C})$. There exist two discrete asymptotic representations $(\varphi_n)_n$ and $(\psi_n)_n$ consisting of unital completely positive maps $\varphi_n \colon A \to \mathrm{M}_{k(n)}(\mathbb{C})$ and $\psi_n \colon A \to \mathrm{M}_{r(n)}(\mathbb{C})$ such that for any separable unital C*-algebra B, the map $\alpha_* \colon K_0(A \otimes B) \to K_0(B)$ has the property that*

$$\alpha_*(x) \equiv (\varphi_n \otimes \mathrm{id}_B)_\sharp(x) - (\psi_n \otimes \mathrm{id}_B)_\sharp(x)$$

for all $x \in K_0(A \otimes B)$. If $\alpha \in \mathrm{KK}_{\mathrm{rfd}}(A, \mathbb{C})$, then all ψ_n can be chosen to be $*$-representations.

Proof. Represent α by a Cuntz pair $\varphi, \psi \colon A \to L(H)$ with $\varphi(a) - \psi(a) \in K(H)$, for all $a \in A$, and such that the set $\psi(A)$ is quasidiagonal. Therefore there is an increasing approximate unit $(p_n)_n$ of $K(H)$ consisting of projections such that $(p_n)_n$ commutes asymptotically with both $\varphi(A)$ and $\psi(A)$. Let us define contractive completely positive maps $\varphi_n, \psi_n \colon A \to L(p_n H)$ by $\varphi_n(a) := p_n \varphi(a) p_n$ and $\psi_n(a) := p_n \psi(a) p_n$. Without any loss of generality we may assume that x is the class of a projection $e \in A \otimes B$. It follows from the definition of the Kasparov product that

$$\alpha_*(x) = [(\varphi \otimes \mathrm{id}_B)(e), (\psi \otimes \mathrm{id}_B)(e)] \in \mathrm{KK}(\mathbb{C}, B).$$

On the other hand, the sequence of projections $p_n \otimes 1_B \in K(H) \otimes B$ commutes asymptotically with both projections $(\varphi \otimes \mathrm{id}_B)(e)$ and $(\psi \otimes \mathrm{id}_B)(e)$ and moreover

$$\lim_{n\to\infty} \|p_n \otimes 1_B \left((\varphi \otimes \mathrm{id}_B)(e) - (\psi \otimes \mathrm{id}_B)(e)\right) p_n \otimes 1_B\| = 0,$$

since the sequence $(p_n \otimes 1_B)_n$ forms an approximative unit of $K(H) \otimes B$. Now it follows from Lemma 2.4 that

$$[(\varphi \otimes \mathrm{id}_B)(e), (\psi \otimes \mathrm{id}_B)(e)] = (\varphi_n \otimes \mathrm{id}_B)_\sharp(e) - (\psi_n \otimes \mathrm{id}_B)_\sharp(e)$$

for all sufficiently large n. It is standard to perturb φ_n and ψ_n to completely positive maps such that $\varphi_n(1)$ and $\psi_n(1)$ are projections. Finally, let us note that ψ_n is a $*$-homomorphism if p_n commutes with ψ. □

3. Asymptotic representations of group C*-algebras

We use the following notation for the Kasparov product:

$$\mathrm{KK}(A, B) \times \mathrm{KK}(B, C) \to \mathrm{KK}(A, C), \quad (y, x) \mapsto y \circ x.$$

In the case of the pairing $K_i(B) \times K^i(B) \to \mathbb{Z}$ we will also write $\langle y, x\rangle$ for $y \circ x$. We are mostly interested in the map

$$K^i(C^*(G)) \to \mathrm{Hom}(K_i(C^*(G)), \mathbb{Z}), \tag{1}$$

induced by the pairing above for $B = C^*(G)$. If G has the Haagerup property, then it was shown in [25] that $C^*(G)$ is KK-equivalent with a commutative C*-algebra and hence the map (1) is surjective. Assuming that G is a countable, discrete, torsion-free group that is uniformly embeddable in a Hilbert space, we are going to verify that the map (1) is split surjective whenever the assembly map $\mu\colon\ \mathrm{RK}_i(BG) \to K_i(C^*(G))$ is surjective.

Following Kasparov [15], for a locally compact, σ-compact, Hausdorff space X and $C_0(X)$-algebras A and B we consider the representable K-homology groups $\mathrm{RK}_i(X)$, the representable K-theory groups $\mathrm{RK}^i(X)$ and the bivariant theory $\mathcal{R}\mathrm{KK}_i(X; A, B)$. If Y is compact, then $\mathrm{RK}_i(Y) = \mathrm{KK}_i(C(Y), \mathbb{C})$ and $\mathrm{RK}^i(Y) = \mathrm{KK}_i(\mathbb{C}, C(Y))$. Suppose now that X is locally compact, σ-compact and Hausdorff. Then

$$\mathrm{RK}_i(X) \cong \varinjlim_{Y\subset X} \mathrm{RK}_i(Y) = \varinjlim_{Y\subset X} \mathrm{KK}_i(C(Y), \mathbb{C}),$$

where Y runs over the compact subsets of X. Kasparov [15], Prop. 2.20, has shown that

$$\mathrm{RK}^i(X) \cong \mathcal{R}\mathrm{KK}_i(X; C_0(X), C_0(X)).$$

Moreover, if $Y \subset X$ is a compact set, then the restriction map $\mathrm{RK}^i(X) \to \mathrm{RK}^i(Y)$ corresponds to the map

$$\mathcal{R}\mathrm{KK}_i(X; C_0(X), C_0(X)) \to \mathcal{R}\mathrm{KK}_i(Y; C(Y), C(Y)) \cong \mathrm{KK}_i(\mathbb{C}, C(Y)).$$

It is useful to introduce the group

$$LK^i(X) = \varprojlim_{Y \subset X} \mathrm{RK}^i(Y),$$

where Y runs over the compact subsets of X. If X is written as the union of an increasing sequence $(Y_n)_n$ of compact subspaces, then, as explained in the proof of Lemma 3.4 from [16], there is a Milnor $\varprojlim^1$ exact sequence:

$$0 \to \varprojlim{}^1 \mathrm{RK}^{i+1}(Y_n) \to \mathrm{RK}^i(X) \to \varprojlim \mathrm{RK}^i(Y_n) \to 0.$$

The morphism $\mathrm{RK}^i(X) \to \mathrm{Hom}(\mathrm{RK}_i(X), \mathbb{Z})$ induced by the pairing $\mathrm{RK}_i(X) \times \mathrm{RK}^i(X) \to \mathbb{Z}$ factors through the morphism

$$\begin{aligned}\varprojlim \mathrm{RK}^i(Y_n) = LK^i(X) \to \mathrm{Hom}(\mathrm{RK}_i(X), \mathbb{Z}) &= \mathrm{Hom}(\varinjlim \mathrm{RK}_i(Y_n), \mathbb{Z}) \\ &\cong \varprojlim \mathrm{Hom}(\mathrm{RK}_i(Y_n), \mathbb{Z})\end{aligned}$$

given by the projective limit of the morphisms $\mathrm{RK}^i(Y_n) \to \mathrm{Hom}(\mathrm{RK}_i(Y_n), \mathbb{Z})$.

If X is a locally finite separable CW-complex, then there is a Universal Coefficient Theorem [16], Lemma 3.4:

$$0 \to \mathrm{Ext}(\mathrm{RK}_{i+1}(X), \mathbb{Z}) \to \mathrm{RK}^i(X) \to \mathrm{Hom}(\mathrm{RK}_i(X), \mathbb{Z}) \to 0. \tag{2}$$

In particular, it follows that the map $LK^i(X) \to \mathrm{Hom}(\mathrm{RK}_i(X), \mathbb{Z})$ is surjective.

Let us recall the construction of the assembly map $\mu\colon \mathrm{RK}_i(\mathrm{B}G) \to K_i(C^*(G))$ and of the dual map $\nu\colon K^i(C^*(G)) \to \mathrm{RK}^i(\mathrm{B}G)$ as given in [15]. Kasparov considers a natural element

$$\beta_G \in \mathcal{R}\mathrm{KK}(\mathrm{B}G; C_0(\mathrm{B}G), C_0(\mathrm{B}G) \otimes C^*(G))$$

(which we denote here by ℓ as it corresponds to Mischenko's "line bundle" on $\mathrm{B}G$). If G is a discrete countable group then it is known [15], §6, that $\mathrm{E}G$ and $\mathrm{B}G$ can be realized as locally finite separable CW-complexes. Write $\mathrm{B}G$ as the union of an increasing sequence $(Y_n)_n$ of finite CW-subcomplexes. Let ℓ_n be the image of ℓ in

$$\mathcal{R}\mathrm{KK}(Y_n; C(Y_n), C(Y_n) \otimes C^*(G)) \cong \mathrm{KK}(\mathbb{C}, C(Y_n) \otimes C^*(G))$$

under the restriction map induced by the inclusion $Y_n \subset \mathrm{B}G$.

The map $\mu_n : \mathrm{RK}_i(Y_n) \to K_i(C^*(G))$ is defined as the cap product by ℓ_n:

$$\begin{gathered}\mathrm{KK}(\mathbb{C}, C(Y_n) \otimes C^*(G)) \times \mathrm{KK}_i(C(Y_n), \mathbb{C}) \to \mathrm{KK}_i(\mathbb{C}, C^*(G)),\\ (\ell_n, z) \mapsto \mu_n(z) = \ell_n \circ (z \otimes 1).\end{gathered}$$

The assembly map $\mu\colon \mathrm{RK}_i(\mathrm{B}G) \to K_i(C^*(G))$ is the inductive limit homomorphism $\mu := \varinjlim \mu_n$. The homomorphism $\nu\colon K^i(C^*(G)) \to \mathrm{RK}^i(\mathrm{B}G)$ is defined as the cap product by ℓ:

$$\begin{aligned} &\mathcal{R}\mathrm{KK}(\mathrm{B}G; C_0(\mathrm{B}G), C_0(\mathrm{B}G) \otimes C^*(G)) \times \mathrm{KK}_i(C^*(G), \mathbb{C}) \\ &\qquad \longrightarrow \mathcal{R}\mathrm{KK}_i(\mathrm{B}G; C_0(\mathrm{B}G), C_0(\mathrm{B}G)), \\ &\qquad (\ell, x) \mapsto \nu(x) = \ell \circ (1 \otimes x). \end{aligned}$$

Let $\nu_n\colon K^i(C^*(G)) \to \mathrm{RK}^i(Y_n)$ be obtained by composing ν with the restriction map $\mathrm{RK}^i(\mathrm{B}G) \to \mathrm{RK}^i(Y_n)$. Noting that ν_n is also given by the cap product by ℓ_n, Kasparov has shown that

$$\nu_n(x) \circ z = \mu_n(z) \circ x$$

for all $x \in K^i(C^*(G))$ and $z \in \mathrm{RK}_i(Y_n)$, [15], Lemma 6.2. The assembly map induces a homomorphism $\mu^*\colon \mathrm{Hom}(K_i(C^*(G)), \mathbb{Z}) \to \mathrm{Hom}(\mathrm{RK}_i(\mathrm{B}G), \mathbb{Z})$. Since

$$\mathrm{Hom}(\mathrm{RK}_i(\mathrm{B}G), \mathbb{Z}) \cong \mathrm{Hom}(\varinjlim \mathrm{RK}_i(Y_n), \mathbb{Z}) \cong \varprojlim \mathrm{Hom}(\mathrm{RK}_i(Y_n), \mathbb{Z})$$

and since the equalities $\nu_n(x) \circ z = x \circ \mu_n(z)$ are compatible with the maps induced by the inclusions $Y_n \subset Y_{n+1}$, we obtain that the following diagram is commutative:

$$\begin{array}{ccc} K^i(C^*(G)) & \longrightarrow & \mathrm{Hom}(K_i(C^*(G)), \mathbb{Z}) \\ \nu \big\downarrow & & \big\downarrow \mu^* \\ \mathrm{RK}^i(\mathrm{B}G) & \twoheadrightarrow & \mathrm{Hom}(\mathrm{RK}_i(\mathrm{B}G), \mathbb{Z}), \end{array}$$

where the horizontal arrows correspond to natural pairings of K-theory with K-homology. The map $\mathrm{RK}^i(\mathrm{B}G) \to \mathrm{Hom}(\mathrm{RK}_i(\mathrm{B}G), \mathbb{Z})$ is surjective by (2).

In view of the previous discussion, by combining results of Kasparov [15] and Tu [26], one derives the following.

Theorem 3.1. *Let G be a countable, discrete, torsion-free group. Suppose that G is uniformly embeddable in a Hilbert space. Then for any group homomorphism $h\colon K_i(C^*(G)) \to \mathbb{Z}$ there is $x \in K^i(C^*(G))$ such that $h(\mu(z)) = \langle \mu(z), x\rangle$ for all $z \in \mathrm{RK}_i(\mathrm{B}G)$.*

Proof. For a discrete group G which admits a uniform embedding into a Hilbert space it was shown in [26], Thm. 3.3, that G has a γ-element. Since G is torsion-free, we can take $\underline{\mathrm{B}}G = \mathrm{B}G$. If G has a γ-element, it follows by Theorem 6.5 and Lemma. 6.2 of [15] that the dual map $\nu\colon \mathrm{KK}_i(C^*(G), \mathbb{C}) \to \mathrm{RK}^i(\mathrm{B}G)$ is split surjective. Therefore, in the diagram above, the composite map $K^i(C^*(G)) \to \mathrm{Hom}(\mathrm{RK}_i(\mathrm{B}G), \mathbb{Z})$, $x \mapsto \langle \nu(x), \cdot\rangle$ is surjective. This shows that if $h\colon K_i(C^*(G)) \to \mathbb{Z}$ is a group homomorphism, then $\mu^*(h) = h \circ \mu = \langle \nu(x), \cdot\rangle$ for some $x \in K^i(C^*(G))$. Since the diagram above is commutative, we obtain that $h \circ \mu = \langle \nu(x), \cdot\rangle = \langle \mu(\cdot), x\rangle$. □

The following proposition is more or less known; for example, it is implicitly contained in [11]. Let ι be the trivial representation of G, $\iota(s) = 1$ for all $s \in G$.

Proposition 3.2. *Let* $\mu\colon \mathrm{RK}_0(\mathrm{B}G) \to K_0(C^*(G))$ *be the assembly map. Then* $\pi_* \circ \mu = m \cdot \iota_* \circ \mu$ *for any unital finite dimensional representation* $\pi\colon C^*(G) \to \mathrm{M}_m(\mathbb{C})$.

Proof. Write $\mathrm{B}G$ as the union of an increasing sequence $(Y_n)_n$ of finite CW-subcomplexes. Let $z \in \mathrm{RK}_0(Y_n)$ for some $n \geq 1$ and let $x = [\pi] \in K^0(C^*(G))$. The equality $\nu_n(x) \circ z = \mu_n(z) \circ x$ becomes $\langle \nu_n(x), z\rangle = \pi_*(\mu_n(z))$. The Chern character makes the following commutative:

$$\begin{array}{ccc} \mathrm{RK}^0(Y_n) \times \mathrm{RK}_0(Y_n) & \longrightarrow & \mathbb{Z} \\ \Big\downarrow{\scriptstyle \mathrm{ch}^* \times \mathrm{ch}_*} & & \Big\downarrow \\ H^{\mathrm{even}}(Y_n, \mathbb{Q}) \times H_{\mathrm{even}}(Y_n, \mathbb{Q}) & \longrightarrow & \mathbb{Q}. \end{array}$$

Thus $\langle \mathrm{ch}^*(\nu_n(x)), \mathrm{ch}_*(z)\rangle = \pi_*(\mu_n(z))$. Since x is the class of a unital finite dimensional representation $\pi\colon C^*(G) \to \mathrm{M}_n(\mathbb{C})$, it follows that $\nu_n(x)$ is simply the class of the flat complex vector bundle $[V] = \pi_*(\ell_n)$ over Y_n. On the other hand, if V is a flat vector bundle, then $\mathrm{ch}^*(V) = \mathrm{rank}(V) = m = \dim(\pi)$ by [14]. Therefore, for any unital m-dimensional representation π, $\pi_*(\mu_n(z)) = m \cdot \langle 1, \mathrm{ch}_*(z)\rangle$. By applying the same formula for the trivial representation $\iota\colon C^*(G) \to \mathbb{C}$, we get $\iota_*(\mu_n(z)) = \langle 1, \mathrm{ch}_*(z)\rangle$. It follows that $\pi_*(\mu_n(z)) = m \cdot \iota_*(\mu_n(z))$. □

Recall that we denote by $I(G)$ the kernel of the trivial representation $\iota\colon C^*(G) \to \mathbb{C}$. Since the extension $0 \to I(G) \to C^*(G) \to \mathbb{C} \to 0$ is split, $K_0(C^*(G)) \cong K_0(I(G)) \oplus \mathbb{Z}$.

Theorem 3.3. *Let G be a countable, discrete, torsion-free group that is uniformly embeddable in a Hilbert space. Let* $h\colon K_0(C^*(G)) \to \mathbb{Z}$ *be a group homomorphism.*

(i) *If $C^*(G)$ is K-quasidiagonal, then there exist two discrete completely positive asymptotic representations* $\{\pi_n\colon C^*(G) \to \mathrm{M}_{k(n)}(\mathbb{C})\}_n$ *and* $\{\gamma_n\colon C^*(G) \to \mathrm{M}_{r(n)}(\mathbb{C})\}_n$ *such that* $\pi_{n\sharp}(x) - \gamma_{n\sharp}(x) \equiv h(x)$ *for all* $x \in \mu(\mathrm{RK}_0(\mathrm{B}G))$.

(ii) *If $C^*(G)$ is K-residually finite dimensional, then there is a discrete completely positive asymptotic representation* $\{\pi_n\colon C^*(G) \to \mathrm{M}_{k(n)}(\mathbb{C})\}_n$ *such that* $\pi_{n\sharp}(x) \equiv h(x)$ *for all* $x \in K_0(I(G)) \cap \mu(K_0(\mathrm{B}G))$.

Proof. Part (i) follows from Theorem 3.1 and Proposition 2.5 for $A = C^*(G)$ and $B = \mathbb{C}$. For part (ii) we observe that if γ_n is a $*$-representation, then $\gamma_* = 0$ on $K_0(I(G))$ by Proposition 3.2. □

Theorem 3.4. *Let G be a countable, discrete, torsion-free group. Suppose that G satisfies either one of the conditions* (a) *or* (b) *below.*

(a) *G has the Haagerup property and $C^*(G)$ is K-residually finite dimensional.*

(b) *G is an increasing union of residually finite, amenable groups.*

Then for any group homomorphism $h\colon K_0(C^*(G)) \to \mathbb{Z}$ there is a discrete completely positive asymptotic representation $\{\pi_n\colon C^*(G) \to \mathrm{M}_{k(n)}(\mathbb{C})\}_n$ such that $\pi_{n\sharp}(x) \equiv h(x)$ for all $x \in K_0(I(G))$.

Proof. Recall that the assembly map is an isomorphism for groups with the Haagerup property by a result of Higson and Kasparov [12], and that these groups are also embeddable in a Hilbert space. Thus, if G satisfies (a), then the conclusion follows from Theorem 3.3(ii). Suppose now that G satisfies (b). Thus $G = \bigcup_i G_i$ where G_i are residually finite, amenable groups and $G_i \subset G_{i+1}$. Then $C^*(G) = \overline{\bigcup_i C^*(G_i)}$ and $K_0(C^*(G)) \cong \varinjlim K_0(C^*(G_i))$. Similarly, $I(G) = \overline{\bigcup_i I(G_i)}$ and $K_0(I(G)) = \varinjlim K_0(I(G_i))$. Let $\theta_i\colon K_0(C^*(G_i)) \to K_0(C^*(G))$ be the map induced by the inclusion $C^*(G_i) \subset C^*(G)$. Let h be given as in the statement of the theorem. By the first part of the theorem, for each i, there is a discrete completely positive asymptotic representation $(\pi_n^{(i)})_n$ of $C^*(G_i)$ such that $\pi_{n\sharp}^{(i)}(x) \equiv h(\theta_i(x))$ for all $x \in K_0(I(G_i))$. By Arveson's extension theorem, each $\pi_n^{(i)}$ extends to a unital completely positive map $\bar{\pi}_n^{(i)}$ on $C^*(G)$. Since $C^*(G)$ is separable, $K_0(I(G))$ is countable and $K_0(I(G)) = \varinjlim K_0(I(G_i))$, it follows that there is a sequence of natural numbers $r(1) < r(2) < \cdots$ such that $(\bar{\pi}_{r(i)}^{(i)})_i$ is a discrete completely positive asymptotic representation of $C^*(G)$ such that $\bar{\pi}_{r(i),\sharp}^{(i)}(x) \equiv h(x)$ for all $x \in K_0(I(G))$. □

4. Almost flat K-theory classes

In this section we use the dual assembly to derive the existence of almost flat K-theory classes on the classifying space BG if the group C*-algebra of G is quasidiagonal. It is convenient to work with an adaptation of the notion of almost flatness to simplicial complexes, see [19].

Definition 4.1. Let Y be a compact Hausdorff space and let $(U_i)_{i\in I}$ be a fixed finite open cover of Y. A complex vector bundle $E \in \mathrm{Vect}_m(Y)$ is called *ε-flat* if is represented by a cocycle $v_{ij} : U_i \cap U_j \to U(m)$ such that $\|v_{ij}(y) - v_{ij}(y')\| < \varepsilon$ for all $y, y' \in U_i \cap U_j$ and all $i, j \in I$. A K-theory class $\alpha \in K^0(Y)$ is called *almost flat* if for any $\varepsilon > 0$ there are ε-flat vector bundles E, F such that $\alpha = [E] - [F]$. This property does not depend on the cover $(U_i)_{i\in I}$.

Remark 4.2. The set of all almost flat elements of $K^0(Y)$ form a subring denoted by $K^0_{\mathrm{af}}(Y)$. If $f : Z \to Y$ is a continuous map, then $f^*(K^0_{\mathrm{af}}(Y)) \subset K^0_{\mathrm{af}}(Z)$.

The following proposition gives a method for producing ε-flat vector bundles. Let Y be a finite simplicial complex with universal cover $\widetilde{Y}$ and fundamental group G. Consider the flat line bundle ℓ with fiber $C^*(G)$, $\widetilde{Y} \times_G C^*(G) \to Y$, where $G \subset C^*(G)$ acts diagonally, and let P be the corresponding projection in $\mathrm{M}_m(\mathbb{C}) \otimes C(Y) \otimes C^*(G)$. Consider a discrete asymptotic representation $\{\varphi_n : C^*(G) \to \mathrm{M}_{k(n)}(\mathbb{C})\}_n$ and set $F_n = (\mathrm{id}_m \otimes \mathrm{id}_{C(Y)} \otimes \varphi_n)(P)$. Since $\|F_n^2 - F_n\| \to 0$ as $n \to \infty$, $E_n := \chi(F_n)$ is a projection in $\mathrm{M}_{mk(n)}(C(Y))$ such that $\|E_n - F_n\| \to 0$ as $n \to \infty$.

Proposition 4.3. *For any $\varepsilon > 0$ there is $n_0 > 0$ such that for any $n \geq n_0$ there is an ε-flat vector bundle on Y which is isomorphic to the vector bundle given by the idempotent E_n.*

Proof. We rely on a construction and results of Phillips and Stone from [20], [21], see also [18]. A simplicial complex is locally ordered by giving a partial ordering $\mathbf{o}$ of its vertices in which the vertices of each simplex are totally ordered. The first barycentric subdivision of any simplicial complex has a natural local ordering [21], §1.4. Thus we may assume that Y is endowed with a fixed local ordering $\mathbf{o}$. Let Y have vertices $I = \{1, 2, \ldots, m\}$. We denote by Y^k the set of k-simplices of Y. Given $r \geq 1$, a $U(r)$-valued lattice gauge field $\mathbf{u}$ on the simplicial complex Y is a function that assigns to each 1-simplex $\langle i, j\rangle$ of Y an element $u_{ij} \in U(r)$ subject to the condition that $u_{ji} = u_{ij}^{-1}$, see [21], Def. 3.2. Consider the cover of Y by dual cells $(V_i)_{i\in I}$ [21], A.1.

Phillips and Stone show that for a fixed locally ordered finite simplicial complex Y as above there is a function $h : [0, +\infty) \to [0, 1]$ with $\lim_{t\to\infty} h(t) = 0$ and which has the following property. Let $\mathbf{u}$ be a $U(r)$-valued lattice gauge field on Y for some $r \geq 1$. Suppose that

$$\|u_{ij}u_{jk} - u_{ik}\| \leq \delta \tag{3}$$

for all 2-simplices $\langle i, j, k\rangle$ (with vertices so $\mathbf{o}$-ordered). Then there is a cocycle $v_{ij} : V_i \cap V_j \to U(r)$, $\langle i, j\rangle \in Y^1$, such that

$$\sup_{x\in V_i\cap V_j} \|v_{ij}(x) - u_{ij}\| < h(\delta).$$

The functions $v_{ij}(x)$ are constructed by an iterative process, based on the skeleton of Y. At each stage of the construction one takes affine combinations of functions defined at a previous stage, starting with the constant matrices u_{ij}. It follows that for each $i \in I$ there exists a fixed small open tubular neighborhood U_i of V_i which is affinely homotopic to V_i, such that the cover $(U_i)_{i\in I}$ has the following property. For any $U(r)$-valued lattice gauge field $\mathbf{u}$ on Y that satisfies (3), there is a cocycle $v_{ij} : U_i \cap U_j \to U(r)$, $\langle i, j\rangle \in Y^1$, such that

$$\sup_{x\in U_i\cap U_j} \|v_{ij}(x) - u_{ij}\| < 2h(\delta).$$

We are going to use the asymptotic representation $(\varphi_n)_n$ as follows. Using trivializations of ℓ to U_i one obtains group elements $s_{ij} \in G$ for $\langle i, j \rangle \in Y^1$ giving a constant cocycle on $U_i \cap U_j$ that represents ℓ, so that $s_{ij}^{-1} = s_{ji}$ and $s_{ij} \cdot s_{jk} = s_{ik}$ whenever $\langle i, j, k \rangle \in Y^2$.

If $(\chi_i)_{i \in I}$ are positive continuous functions with χ_i supported in U_i and such that $\sum_{i \in I} \chi_i^2 = 1$, then ℓ is represented by an idempotent

$$P = \sum_{i,j \in I} e_{ij} \otimes \chi_i \chi_j \otimes s_{ij} \in \mathrm{M}_m(\mathbb{C}) \otimes C(Y) \otimes C^*(G).$$

Here $m = |I|$ and (e_{ij}) is the canonical matrix unit of $\mathrm{M}_m(\mathbb{C})$. It follows that for all n sufficiently large, $(\mathrm{id}_m \otimes \mathrm{id}_{C(Y)} \otimes \varphi_n)_\sharp(P)$ is given by the class of a projection E_n with $\|E_n - F_n\| < 1/2$, where $F_n = (\mathrm{id}_m \otimes \mathrm{id}_{C(Y)} \otimes \varphi_n)(P)$. We have

$$F_n = \sum_{i,j \in I} e_{ij} \otimes \chi_i \chi_j \otimes \varphi_n(s_{ij}) \in \mathrm{M}_m(\mathbb{C}) \otimes C(Y) \otimes \mathrm{M}_{k(n)}(\mathbb{C}).$$

For $v \in \mathrm{GL}_k(\mathbb{C})$ we denote by $w(v)$ the unitary $v(v^*v)^{-1/2}$. Fix n sufficiently large so that $\varphi_n(s_{ij}) \in \mathrm{GL}_{k(n)}(\mathbb{C})$. For each ordered edge $\langle i, j \rangle \in Y^1$ we set $u_{ij} = w(\varphi_n(s_{ij}))$ and $u_{ji} = u_{ij}^{-1}$. This will define a $U(k(n))$-valued lattice gauge field on the ordered simplicial complex Y. Fix $\varepsilon > 0$ such that $4m^2\varepsilon < 1/2$ and choose $\delta > 0$ such that $h(\delta) < \varepsilon/2$. Since $(\varphi_n)_n$ is an asymptotic representation, there is $n_0 > 0$ such that if $n \geq n_0$, then

$$\|\varphi_n(s_{ij}) - u_{ij}\| < \varepsilon/2 \tag{4}$$

for all $\langle i, j \rangle \in Y^1$ and $\|u_{ij}u_{jk} - u_{ik}\| \leq \delta$ for all 2-simplices $\langle i, j, k \rangle$. By the result of Phillips and Stone quoted above, there exists a cocycle $v_{ij} : U_i \cap U_j \to U(k(n))$ such that

$$\|v_{ij}(x) - u_{ij}\| < h(\delta) < \varepsilon/2 \tag{5}$$

for all $x \in U_i \cap U_j$. It follows that $\|v_{ij}(x) - v_{ij}(x')\| < \varepsilon$ for all $x, x' \in U_i \cap U_j$ and all $i, j \in I$ and hence the idempotent

$$e_n(x) = \sum_{i,j \in I} e_{ij} \otimes \chi_i(x)\chi_j(x)v_{ij}(x), \quad x \in Y,$$

gives an ε-flat vector bundle on Y. From (4) and (5) we have

$$\|v_{ij}(x) - \varphi_n(s_{ij})\| < \varepsilon \tag{6}$$

for all $x \in U_i \cap U_j$ and $\langle i, j \rangle \in Y^1$. Using (6) we see that $\|e_n - F_n\| \leq 2m^2\varepsilon < 1/2$ and hence $\|e_n - E_n\| \leq \|e_n - F_n\| + \|E_n - F_n\| < 1$. It follows that $E_n = we_nw^{-1}$ for some invertible element w. This shows that the isomorphism class of the vector bundle given the idempotent E_n is represented by an ε-flat vector bundle since we have seen that e_n has that property. □

Let Y be a finite simplicial complex with universal cover $\widetilde{Y}$ and fundamental group G and let ℓ be the corresponding flat line bundle with fiber $C^*(G)$. Recall that the Kasparov product $K_0(C(Y) \otimes C^*(G)) \times \mathrm{KK}(C^*(G), \mathbb{C}) \to K^0(Y)$ induces a map $\nu\colon \mathrm{KK}(C^*(G), \mathbb{C}) \to K^0(Y)$, $\nu(\alpha) = [\ell] \circ (\alpha \otimes 1)$.

Corollary 4.4. $\nu(\mathrm{KK}_{\mathrm{qd}}(C^*(G), \mathbb{C})) \subset K^0_{\mathrm{af}}(Y)$.

Proof. This follows from Propositions 2.5 and 4.3. □

Theorem 4.5. *Let G be a countable, discrete, torsion-free group which is uniformly embeddable in a Hilbert space. Suppose that the classifying space* BG *is a finite simplicial complex and that the full group C*-algebra $C^*(G)$ is K-quasidiagonal. Then all the elements of $K^0(\mathrm{B}G)$ are almost flat.*

Proof. We have seen in the proof of Theorem 3.1 that under the present assumptions on G the dual assembly map $\nu\colon \mathrm{KK}(C^*(G), \mathbb{C}) \to K^0(\mathrm{B}G)$ is surjective. Since $C^*(G)$ is K-quasidiagonal by hypothesis (this holds for instance if $C^*(G)$ is quasidiagonal as observed in Remark 2.3), we have $\mathrm{KK}(C^*(G), \mathbb{C}) = \mathrm{KK}_{\mathrm{qd}}(C^*(G), \mathbb{C})$. The result follows now from Corollary 4.4. □

From Theorem 4.5 one can derive potential obstructions to quasidiagonality of group C*-algebras.

Remark 4.6. Let G be a countable, discrete, torsion-free group which is uniformly embeddable in a Hilbert space and such that the classifying space BG is a finite simplicial complex. If not all elements of $K^0(\mathrm{B}G)$ are almost flat, then $C^*(G)$ is not quasidiagonal.

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M. Dadarlat, Department of Mathematics, Purdue University, 150 N. University Street, West Lafayette, IN 47907-2067, U.S.A.

E-mail: mdd@math.purdue.edu