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Noncommutative residue of projections in Boutet de Monvel's calculus

Anders Gaarde

Abstract. Employing results by Melo, Nest, Schick and Schrohe on the K-theory of Boutet de Monvel's calculus of boundary value problems, we show that the noncommutative residue introduced by Fedosov, Golse, Leichtnam and Schrohe vanishes on projections in the calculus.

This partially answers a question raised in a recent collaboration with Grubb, namely whether the residue is zero on sectorial projections for boundary value problems: This is confirmed to be true when the sectorial projection is in the calculus.

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1. Introduction

Boutet de Monvel [2] constructed a calculus, often called the Boutet de Monvel calculus (or algebra), of pseudodifferential boundary operators on a manifold with boundary. It includes the classical differential boundary value problems as well as the parametrices of the elliptic elements:

Let X be a compact n-dimensional manifold with boundary ∂X; we consider X as an embedded submanifold of a closed n-dimensional manifold $\widetilde{X}$. Denote by X° the interior of X. Let E and F be smooth complex vector bundles over X and ∂X, respectively, with E the restriction to X of a bundle $\widetilde{E}$ over $\widetilde{X}$.

An operator in Boutet de Monvel's calculus – a (polyhomogeneous) Green operator – is a map A acting on sections of E and F, given by a matrix

$$A = \begin{pmatrix} P_+ + G & K \\ T & S \end{pmatrix} : C^\infty(X, E) \oplus C^\infty(\partial X, F) \to C^\infty(X, E) \oplus C^\infty(\partial X, F), \tag{1.1}$$

where P is a pseudodifferential operator (ψdo) on $\widetilde{X}$ with the transmission property and P_+ is its truncation to X:

$$P_+ = r^+ P e^+, \quad r^+ \text{ restricts from } \widetilde{X} \text{ to } X^\circ,\ e^+ \text{ extends by } 0.$$

G is a singular Green operator, T a trace operator, K a Poisson operator, and S a ψdo on the closed manifold ∂X. See [2], Grubb [6], or Schrohe [15] for details.

Fedosov, Golse, Leichtnam and Schrohe [4] extended the notion of noncommutative residue known from closed manifolds (cf. Wodzicki [17], [18] and Guillemin [9]) to the algebra of Green operators. The noncommutative residue of A from (1.1) is defined to be

$$\begin{aligned}\operatorname{res}_X(A) &= \int_X \int_{S_x^*X} \operatorname{tr}_E\, p_{-n}(x,\xi)\, đS(\xi)\,dx \\ &\quad + \int_{\partial X}\int_{S_{x'}^*\partial X} [\operatorname{tr}_E(\operatorname{tr}_n g)_{1-n}(x',\xi') + \operatorname{tr}_F s_{1-n}(x',\xi')]\, đS(\xi')\,dx'.\end{aligned}$$

Here tr_E and tr_F are traces in $\operatorname{Hom}(E)$ and $\operatorname{Hom}(F)$, respectively; $đS(\xi)$ (resp. $đS(\xi')$) denotes the surface measure on the unit sphere of the cotangent bundle, divided by $(2\pi)^n$ (resp. $(2\pi)^{n-1}$); $\operatorname{tr}_n g$ is the symbol of $\operatorname{tr}_n G$ (the normal trace of G), a ψdo on ∂X; and the subscripts $-n$ and $1-n$ indicate that we consider only the homogeneous terms of degree $-n$ resp. $1-n$. Also, a sign error in [4] has been corrected, cf. Grubb and Schrohe [8], (1.5).

It is well known [17] that on a closed manifold, the noncommutative residue of a classical ψdo projection vanishes. In the present paper we show that the same holds in the case of Green operators:

Theorem 1.1. *The noncommutative residue of a projection in the Boutet de Monvel calculus is zero.*

In the proof, we use K-theoretic arguments (in a C*-algebra setting) to reduce the problem to the known case of closed manifolds. We rely on results on the K-theory of Boutet de Monvel's algebra by Melo, Nest and Schrohe [10] and Melo, Schick and Schrohe [11].

In our recent collaboration with Grubb [5] we studied certain spectral projections: For the realization $B = (P+G)_T$ of an elliptic boundary value problem $\{P_+ + G, T\}$ of order $m > 0$ with two spectral cuts at angles θ and φ, one can define the *sectorial projection* $\Pi_{\theta,\varphi}(B)$. It is a (not necessarily self-adjoint) projection whose range contains the generalized eigenspace of B for the sector $\Lambda_{\theta,\varphi} = \{re^{i\omega} \mid r > 0,\ \theta < \omega < \varphi\}$ and whose nullspace contains the generalized eigenspace for $\Lambda_{\varphi,\theta+2\pi}$. It was considered earlier by Burak [3], and in the boundaryless case by Wodzicki [17] and Ponge [13].

In general this operator is not in Boutet de Monvel's calculus, but we showed that it has a residue in a slightly more general sense. The question was posed whether this residue vanishes.

The question of the noncommutative residue of projections is particularly interesting in the context of zeta-invariants as discussed by Grubb [7] and in [5]: The *basic zeta value* $C_{0,\theta}(B)$ for the realization B of a boundary value problem is defined via a choice of spectral cut in the complex plane; the difference in the basic zeta value based on two spectral cut angles θ and φ is then given as the noncommutative residue of the corresponding sectorial projection:

$$C_{0,\theta}(B) - C_{0,\varphi}(B) = \frac{2\pi i}{m} \operatorname{res}_X(\Pi_{\theta,\varphi}(B)).$$

Our results here show that the dependence of $C_{0,\theta}(B)$ upon θ is trivial whenever the projection $\Pi_{\theta,\varphi}(B)$ lies in Boutet de Monvel's calculus.

It should be noted that the literature in functional analysis and PDE-theory often uses "projection" as a synonym for idempotent, while C*-algebraists furthermore require that projections are self-adjoint. We choose here the former terminology; that is, in this text projection and idempotent are synonymous.

2. Preliminaries and notation

We employ Blackadar's [1] approach to K-theory: A pre-C*-algebra B is called *local* if it, as a subalgebra of its C*-completion $\overline{B}$, is closed under holomorphic function calculus. (Blackadar also requires that all matrix algebras $\mathcal{M}_n(B)$ are closed under holomorphic function calculus, but this follows automatically, cf. Schweitzer [16].) Let $\mathcal{M}_\infty(B)$ denote the direct limit of the matrix algebras $\mathcal{M}_m(B)$, $m \in \mathbb{N}$. Define $\mathcal{IP}_\infty(B) = \operatorname{Idem}(\mathcal{M}_\infty(B))$ to be the set of all idempotent matrices with entries from B. Likewise, $\mathcal{IP}_m(B) = \operatorname{Idem}(\mathcal{M}_m(B))$ is the set of all $m \times m$ idempotents. Define the relation $\sim$ on $\mathcal{IP}_\infty(B)$ by

$$x \sim y \text{ if there exist } a, b \in \mathcal{M}_\infty(B) \text{ such that } x = ab \text{ and } y = ba.$$

If B has a unit, we define $K_0(B)$ to be the Grothendieck group of the semigroup $V(B) = \mathcal{IP}_\infty(B)/\sim$. If B has no unit, we consider the scalar map from the unitization – indicated with a tilde as in $\widetilde{B}$ or $B^\sim$ – of B to the complex numbers $s\colon \widetilde{B} \to \mathbb{C}$ defined by $s(b + \lambda 1_{\widetilde{B}}) = \lambda$, and then define $K_0(B)$ as the kernel of the induced map $s_*\colon K_0(\widetilde{B}) \to K_0(\mathbb{C})$.

A fact that we shall use several times is that if B is local, then, cf. [1], p. 28,

$$V(B) \cong V(\overline{B}) \text{ and hence } K_0(B) \cong K_0(\overline{B}). \tag{2.1}$$

Combined with *the standard picture of* K_0 this implies that

$$K_0(\overline{B}) = \{\, [x]_0 - [y]_0 \mid x, y \in \mathcal{IP}_m(B),\ m \in \mathbb{N} \,\} \tag{2.2}$$

in the case where B is unital, and

$$K_0(\overline{B}) = \{\, [x]_0 - [y]_0 \mid x, y \in \mathcal{IP}_m(\widetilde{B}) \text{ with } x \equiv y \bmod \mathcal{M}_m(B),\ m \in \mathbb{N} \,\} \tag{2.3}$$

in the non-unital case [1].

Let $\mathcal{A}$ denote the set of Green operators as in (1.1) of order and class zero; it is equipped with a Fréchet topology which makes it a Fréchet $*$-algebra (Schrohe [14]). Moreover, $\mathcal{A}$ is a $*$-subalgebra of the bounded operators on the Hilbert space $\mathcal{H} = L_2(X, E) \oplus H^{-1/2}(\partial X, F)$; we will denote by $\mathfrak{A}$ its C*-closure in $\mathcal{B}(\mathcal{H})$. $\mathcal{A}$ is local [14], so $K_0(\mathcal{A}) \cong K_0(\mathfrak{A})$. Note that the definitions of $K_0(\mathcal{A})$ are equivalent: whether we consider $\mathcal{A}$ as a Fréchet algebra or as a $*$-subalgebra of $\mathfrak{A}$, cf. Phillips [12].

We follow here the definition of order and class from [6], as opposed to the convention used in [11] where the operators are bounded on the Hilbert space $\mathcal{H}' = L_2(X, E) \oplus L_2(\partial X, F)$. It is explained in [10], 1.1, how the two approaches are equivalent for our purposes.

Furthermore, the K-theory of $\mathcal{A}$ is independent of the specific bundles [10], 1.5, so for simplicity we assume in this paper the simple case $E = X \times \mathbb{C}$ and $F = \partial X \times \mathbb{C}$.

$\mathcal{K}$ denotes the subalgebra of smoothing operators, $\mathfrak{K}$ its C*-closure (the ideal of compact operators). We let $\mathcal{I}$ denote the set of elements in $\mathcal{A}$ of the form

$$\begin{pmatrix} \varphi P \psi + G & K \\ T & S \end{pmatrix} \tag{2.4}$$

with $\varphi, \psi \in C_c^\infty(X^\circ)$, P a ψdo on $\widetilde{X}$ of order zero, and G, K, T and S of negative order and class zero. $\mathfrak{I}$ will be the C*-closure of $\mathcal{I}$ in $\mathfrak{A}$.

The noncommutative residue defined in [4] is a trace – a linear functional that vanishes on commutators – $\mathrm{res}\colon \mathcal{A} \to \mathbb{C}$. It is continuous with respect to the Fréchet topology in $\mathcal{A}$, and induces a group homomorphism $\mathrm{res}_*\colon K_0(\mathcal{A}) \to \mathbb{C}$ such that

$$\mathrm{res}_*([A]_0) = \mathrm{res}_X(A) \tag{2.5}$$

for any idempotent $A \in \mathcal{A}$. Our goal is to prove the vanishing of res_*, which obviously implies that $\mathrm{res}_X(A) = 0$ for all idempotent A.

The quotient map $q\colon \mathfrak{A} \to \mathfrak{A}/\mathfrak{K}$ induces an isomorphism $q_*\colon K_0(\mathfrak{A}) \to K_0(\mathfrak{A}/\mathfrak{K})$ [10], Prop. 13. The isomorphisms $K_0(\mathcal{A}) \cong K_0(\mathfrak{A}) \cong K_0(\mathfrak{A}/\mathfrak{K})$ allow us to extend the noncommutative residue: For each $[\mathcal{A} + \mathfrak{K}]_0$ in $K_0(\mathfrak{A}/\mathfrak{K})$ there is an $A \in \mathcal{IP}_\infty(\mathcal{A})$ such that $q_*[A]_0 = [\mathcal{A} + \mathfrak{K}]_0$. We define

$$\widetilde{\mathrm{res}}_*[\mathcal{A} + \mathfrak{K}]_0 = \mathrm{res}_*[A]_0 = \mathrm{res}_X(A). \tag{2.6}$$

So $\widetilde{\mathrm{res}}_*$ is just $\mathrm{res}_*\, q_*^{-1}$ and a group homomorphism $K_0(\mathfrak{A}/\mathfrak{K}) \to \mathbb{C}$.

3. K-theory and the residue

We employ results from Melo, Schick and Schrohe [11]: Theorem 1 there proves an isomorphism

$$K_0(\mathfrak{A}/\mathfrak{K}) \cong K_0(C(X)) \oplus K_1(C_0(T^*X^\circ)).$$

The intuitive interpretation of this isomorphism is that each K_0-class in $\mathfrak{A}/\mathfrak{K}$ is the sum of (the K_0-class of) a continuous function and (the K_0-class of) something vanishing on the boundary ∂X.

More precisely, we will use their observation

$$K_0(\mathfrak{A}/\mathfrak{K}) = q_* m_* K_0(C(X)) + i_* K_0(\mathfrak{I}/\mathfrak{K}). \tag{3.1}$$

Here $m\colon C(X) \to \mathfrak{A}$ sends f to the multiplication operator $\left(\begin{smallmatrix} f & 0 \\ 0 & 0 \end{smallmatrix}\right)$ and i is the inclusion $\mathfrak{I}/\mathfrak{K} \to \mathfrak{A}/\mathfrak{K}$; m_* and i_* are then the corresponding induced maps in K_0. We will in general suppress i and i_* to simplify notation.

We show that $\widetilde{\operatorname{res}}_*$ vanishes on both terms on the right-hand side of (3.1). The following lemma treats the first of these terms:

Lemma 3.1. $\widetilde{\operatorname{res}}_*$ *vanishes on* $q_* m_* K_0(C(X))$.

Proof. Recall that multiplication with a smooth function is a Green operator of order zero, whose noncommutative residue is clearly zero since it has no homogeneous term of order $-n$.

Let $f \in \mathcal{IP}_m(C^\infty(X))$; $m(f)$ acts by multiplication with a smooth (matrix) function and therefore lies in $\mathcal{IP}_m(\mathcal{A})$. Then $q_* m_* [f]_0 = q_*[m(f)]_0 = [m(f) + \mathfrak{K}]_0$, and according to (2.6)

$$\widetilde{\operatorname{res}}_*(q_* m_* [f]_0) = \operatorname{res}_*[m(f)]_0 = \operatorname{res}_X(m(f)) = 0.$$

Since $C^\infty(X)$ is local in $C(X)$ [1, 3.1.1-2], any element of $K_0(C(X))$ can be written as $[f]_0 - [g]_0$ for some $f, g \in \mathcal{IP}_m(C^\infty(X))$, cf. (2.2). The lemma follows from this. $\square$

We now turn to the second term of (3.1); our strategy is to show that the elements of $K_0(\mathfrak{I}/\mathfrak{K})$ correspond to ψdos with symbols supported in the interior of X. This allows us to construct certain projections for which the noncommutative residue is given as the residue of a projection on the closed manifold $\widetilde{X}$.

The principal symbol induces an isomorphism $\mathfrak{I}/\mathfrak{K} \cong C_0(S^*X^\circ)$ [10], Theorem 1. We denote the induced isomorphism in K_0 by σ_*, i.e.,

$$\sigma_*\colon K_0(\mathfrak{I}/\mathfrak{K}) \xrightarrow{\cong} K_0(C_0(S^*X^\circ)). \tag{3.2}$$

Like in Lemma 3.1 we wish to consider smooth functions instead of merely continuous functions; the following shows that instead of $C_0(S^*X^\circ)$, it suffices to look at smooth functions (symbols) compactly supported in the interior:

The algebra $C_c^\infty(S^*X^\circ)$, equipped with the sup-norm, is a local C*-algebra [1], 3.1.1-2, with completion $C_0(S^*X^\circ)$. It follows from (2.1) that the injection $C_c^\infty(S^*X^\circ) \to C_0(S^*X^\circ)$ induces an isomorphism

$$K_0(C_c^\infty(S^*X^\circ)) \cong K_0(C_0(S^*X^\circ)). \tag{3.3}$$

We now show that each compactly supported symbol in $K_0(C_c^\infty(S^*X^\circ))$ gives rise to a ψdo projection Π_+ on X, which is in fact the truncation of a ψdo projection on $\widetilde{X}$. This will allow us to calculate the residue of Π_+ from the residue of a projection on the closed manifold $\widetilde{X}$.

Lemma 3.2. *Let $p(x,\xi) \in \mathcal{IP}_m(C_c^\infty(S^*X^\circ)^\sim)$. There is a zero-order ψdo projection Π acting on $C^\infty(\widetilde{X},\mathbb{C}^m)$, such that its symbol is constant on a neighborhood of $\widetilde{X} \setminus X^\circ$, its truncation Π_+ is an idempotent in $\mathcal{M}_m(\mathcal{J}^\sim)$, and*

$$\sigma_* q_*([\Pi_+]_0) = [p]_0. \tag{3.4}$$

Proof. By definition of the unitization of $C_c^\infty(S^*X^\circ)$, we can write p as a sum

$$p(x,\xi) = \alpha(x,\xi) + \beta,$$

with $\alpha \in \mathcal{M}_m(C_c^\infty(S^*X^\circ))$ and $\beta \in \mathcal{M}_m(\mathbb{C})$. Note that β itself is idempotent, since $p = \beta$ outside the support of α.

We extend α by zero to obtain a smooth function $\tilde{\alpha}(x,\xi)$ on the closed manifold $S^*\widetilde{X}$. We get a ψdo symbol (also denoted $\tilde{\alpha}$) of order zero on $\widetilde{X}$ by requiring $\tilde{\alpha}$ to be homogeneous of degree zero in ξ. Let $\tilde{p}(x,\xi) = \tilde{\alpha}(x,\xi) + \beta$.

We now have an idempotent ψdo-symbol $\tilde{p}$ on $\widetilde{X}$; we then construct a ψdo projection on $\widetilde{X}$ that has $\tilde{p}$ as its principal symbol.

In [7], Chapter 3, Grubb constructed an operator that, for a suitable choice of atlas on the manifold, carries over to the Euclidean Laplacian in each chart, modulo smoothing operators. Hence, choose that particular atlas on $\widetilde{X}$ and let D denote this particular operator, i.e., with scalar symbol $d(x,\xi) = |\xi|^2$. Define the auxiliary second order ψdo $C = \mathrm{OP}(c(x,\xi))$, with symbol $c(x,\xi)$ given in the local coordinates of the specified charts as

$$c(x,\xi) = (2\tilde{p}(x,\xi) - I)d(x,\xi).$$

Since $\tilde{p}$ is idempotent, the eigenvalues of $2\tilde{p} - I$ are ± 1, cf. (A.2), so C is an elliptic second order operator and $c(x,\xi) - \lambda$ is parameter-elliptic for λ on each ray in $\mathbb{C} \setminus \mathbb{R}$.

Then we can define the sectorial projection, cf. [13], [5], $\Pi = \Pi_{\theta,\varphi}(C)$ with angles $\theta = -\frac{\pi}{2}, \varphi = \frac{\pi}{2}$,

$$\Pi = \frac{i}{2\pi} \int_{\Gamma_{\theta,\varphi}} \lambda^{-1} C (C - \lambda)^{-1} \, d\lambda.$$

Π is a ψdo projection [13] on $\widetilde{X}$ with symbol π given in local coordinates by

$$\pi(x,\xi) = \frac{i}{2\pi} \int_{\mathcal{C}(x,\xi)} q(x,\xi,\lambda) \, d\lambda,$$

where $q(x,\xi,\lambda)$ is the symbol with parameter for a parametrix of $c(x,\xi) - \lambda$, and $\mathcal{C}(x,\xi)$ is a closed curve encircling the eigenvalues of $c_2(x,\xi)$ – the principal symbol of C – in the $\{\mathrm{Re}\, z > 0\}$ half-plane.

The eigenvalues of $c_2(x,\xi) = (2\tilde{p}(x,\xi) - I)|\xi|^2$ are $\pm|\xi|^2$, so we can choose $\mathcal{C}(x,\xi)$ as the boundary of a small ball $B(|\xi|^2, r)$ around $+|\xi|^2$.

Then the principal symbol of $\pi(x,\xi)$ is

$$\begin{aligned}\pi_0(x,\xi) &= \frac{i}{2\pi}\int_{\mathcal{C}(x,\xi)} q_{-2}(x,\xi,\lambda)\,d\lambda\\ &= \frac{i}{2\pi}\int_{\partial B(|\xi|^2,r)} [(2\tilde{p}(x,\xi) - I)|\xi|^2 - \lambda]^{-1}\,d\lambda = \tilde{p}(x,\xi)\end{aligned}$$

according to Lemma A.1. So Π is a ψdo projection with principal symbol $\tilde{p}(x,\xi)$, as desired.

Observe that for x outside the support of $\tilde{\alpha}$ we have $c(x,\xi) = (2\beta - I)|\xi|^2$ and $q(x,\xi,\lambda) = q_{-2}(x,\xi,\lambda) = ((2\beta - I)|\xi|^2 - \lambda)^{-1}$, so $\pi(x,\xi) = \pi_0(x,\xi) = \beta$ there. (We cannot be sure that the full symbol of π equals $\tilde{p}$ inside the support, since coordinate-dependence will in general influence the lower order terms of the parametrix.) In particular, $\pi(x,\xi)$ is constantly equal to β for x outside $\tilde{\alpha}$'s support, i.e., in a neighborhood of $\widetilde{X} \setminus X^\circ$.

Now consider the truncation Π_+. We have

$$(\Pi_+)^2 = (\Pi^2)_+ - L(\Pi,\Pi) = \Pi_+ - L(\Pi,\Pi),$$

where the singular Green operator $L(P,Q)$ is defined as $(PQ)_+ - P_+Q_+$ for ψdos P and Q. Since $\pi(x,\xi)$ equals the constant matrix β in a neighborhood of the boundary ∂X, it follows, cf. [6], Theorem 2.7.5, that $L(\Pi,\Pi) = 0$, so $(\Pi_+)^2 = \Pi_+$.

The symbol of $\Pi - \beta$ is compactly supported within X°, so we can write $\Pi_+ = \varphi P\psi + \beta$ for some φ, ψ, P, as in (2.4); hence Π_+ is in $\mathcal{M}_m(\mathcal{I}^\sim)$. Technically, Π_+ lies in the algebra where the boundary bundle F is the zero-bundle, but inserting zeros into Π_+'s matrix form will clearly allow us to augment it to the present case with $F = \partial X \times \mathbb{C}$.

Finally we take a look at (3.4): Since Π_+ is an idempotent in $\mathcal{M}_m(\mathcal{I}^\sim)$, it defines a K_0-class $[\Pi_+]_0$ in $K_0(\mathcal{I}^\sim)$. Then $q_*[\Pi_+]_0$ defines a class in $K_0(\mathfrak{I}/\mathfrak{K}^\sim)$, a class defined by its principal symbol. Since the principal symbol is exactly the idempotent $p(x,\xi)$, we obtain (3.4) by definition. □

We now have all the tools to prove our main theorem:

Proof of Theorem 1.1. An idempotent Green operator necessarily has order and class zero, and thus lies in $\mathcal{A}$. So we need to show that $\mathrm{res}_X(A)$ is zero for any idempotent $A \in \mathcal{A}$. By (2.5) it suffices to show that res_* vanishes on $K_0(\mathcal{A})$. In turn, according to equation (3.1) and Lemma 3.1, we only need to show that $\widetilde{\mathrm{res}}_*$ vanishes on $K_0(\mathfrak{I}/\mathfrak{K})$.

So let $\omega \in K_0(\mathfrak{I}/\mathfrak{K})$. Employing (2.3), (3.2), and (3.3) we can find p, p' in $\mathcal{IP}_m(C_c^\infty(S^*X^\circ)^\sim)$ such that

$$\sigma_*\omega = [p]_0 - [p']_0. \tag{3.5}$$

Now, for p, p' we use Lemma 3.2 to find corresponding ψdos Π, Π' with the specific properties mentioned there. By (3.4) and (3.5) we see that

$$q_*[\Pi_+]_0 - q_*[\Pi'_+]_0 = \sigma_*^{-1}([p]_0 - [p']_0) = \omega.$$

Using equation (2.6) then gives us

$$\widetilde{\operatorname{res}}_* \, \omega = \operatorname{res}_X(\Pi_+) - \operatorname{res}_X(\Pi'_+).$$

Here

$$\operatorname{res}_X(\Pi_+) = \int_X \int_{S_x^* X} \operatorname{tr} \pi_{-n}(x,\xi) đS(\xi) dx.$$

By construction, $\pi(x,\xi)$ is constant equal to β outside X; in particular $\pi_{-n}(x,\xi)$ is zero for $x \in \widetilde{X} \setminus X$ and therefore

$$\int_X \int_{S_x^* X} \operatorname{tr} \pi_{-n}(x,\xi) đS(\xi) dx = \int_{\tilde{X}} \int_{S_x^* \tilde{X}} \operatorname{tr} \pi_{-n}(x,\xi) đS(\xi) dx.$$

In other words,

$$\operatorname{res}_X(\Pi_+) = \operatorname{res}_{\tilde{X}}(\Pi),$$

where the latter is the noncommutative residue of a ψdo projection on a closed manifold. It is well known [17], [18] that this always vanishes, so $\operatorname{res}_X(\Pi_+) = 0$. Likewise we obtain $\operatorname{res}_X(\Pi'_+) = 0$ and finally

$$\widetilde{\operatorname{res}}_* \, \omega = 0,$$

as desired. □

In [5], it was an open question whether the residue is zero on the sectorial projection for a boundary value problem. This theorem answers that question in the positive for the cases where the projection lies in $\mathcal{A}$.

It is not, at this time, clear for which boundary value problems this is true; however, we showed in [5] that there certainly are boundary value problems where the sectorial projection is not in $\mathcal{A}$.

Appendix

Lemma A.1. *Let $M \in \mathcal{IP}_m(\mathbb{C})$. Let $d > 0$ and let $\partial B(d,r)$ denote the closed curve in the complex plane along the boundary of the ball with center d and radius $0 < r < d$. Then*

$$\frac{i}{2\pi} \int_{\partial B(d,r)} [(2M - I)d - \lambda]^{-1} d\lambda = M. \tag{A.1}$$

Proof. A direct computation shows that, for $\lambda \neq \pm d$,

$$[(2M - I)d - \lambda]^{-1} = \frac{M}{d - \lambda} - \frac{I - M}{d + \lambda}. \tag{A.2}$$

The result in (A.1) then follows from the residue theorem. □

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A. Gaarde, Flinterenden 5, 4.th, 2300 Copenhagen S, Denmark

E-mail: agaarde@gmail.com