The resolvent cocycle in twisted cyclic cohomology and a local index formula for the Podleś sphere

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Abstract. We continue the investigation of twisted homology theories in the context of dimension drop phenomena. This work unifies previous equivariant index calculations in twisted cyclic cohomology. We do this by proving the existence of the resolvent cocycle, a finitely summable analogue of the JLO cocycle, under weaker smoothness hypotheses and in the more general setting of 'modular' spectral triples. As an application we show that using our twisted resolvent cocycle, we can obtain a local index formula for the Podleś sphere. The resulting twisted cyclic cocycle has non-vanishing Hochschild class which is in dimension 2.

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1. Introduction

This paper proves a residue index formula in noncommutative geometry for 'modular spectral triples', which are analogues of spectral triples with twisted traces. This is the appropriate setting for examples arising from q-deformations which typically experience 'dimension drop' in homology, [H], [HK], [NT], [SW], [W]. The main results are as follows.

1) We show that for finitely summable modular spectral triples the resolvent cocycle exists, is continuous and is an index cocycle under weaker smoothness conditions than have previously been used. In particular we do not need the pseudodifferential calculus to establish these facts, so that we are free to replace the usual pseudodifferential calculus by other schemes later, in order to obtain local index formulae.

2) We show that modular spectral triples have a well-defined pairing with equivariant K-theory. In the finitely summable and weakly smooth case we show that this pairing can be computed using the resolvent cocycle, which defines a twisted cyclic cocycle.

3) We apply the results of 1) and 2) to prove a local index formula for the Podleś sphere in twisted cyclic cohomology. This index formula puts the results of several authors into a common framework, [H], [KW], [W]. In particular, the twisted Hochschild class of our residue cocycle is an explicit constant multiple of the fundamental Hochschild cocycle for the Podleś sphere, [H], [KW], and our explicit index pairings can be compared to those in [W].

The computations in 3) are similar to what was done in [NT], however they used the twisting by the modular automorphism, rather than the inverse of the modular automorphism. While the summability is the same in both cases, the twisted Hochschild homology for the modular automorphism is trivial in dimension 2, while the inverse of the modular automorphism avoids the dimension drop. Thus the cocycle obtained in [NT] is cohomologous to a 0-cocycle, while ours is not. We also note that in [NT] the starting point was the JLO cocycle in entire cyclic cohomology rather than the resolvent cocycle.

The exposition is as follows. In Section 2 we introduce the basic definitions for modular spectral triples, including smoothness and summability. We then show that a modular spectral triple defines an equivariant KK-class, and so gives us a well-posed K-theory valued index problem. The remainder of Section 2 demonstrates that together with a representative of an equivariant K-theory class, we obtain a well-posed numerical index problem. The aim of Section 3 is then to show that these notions are compatible.

We address the existence, continuity and index properties of the resolvent cocycle in Section 3. We begin by looking at our weak smoothness condition, and proving some basic results that follow from this assumption. Then we prove the existence and continuity of the resolvent cocycle, which originated in [CPRS2], and show that it computes the numerical index. Finally we show, using results from [KNR], that this numerical index is compatible with the K-theory valued index in a precise way.

In Section 4 we show that the spectral triple introduced by [DS] defines a 2-dimensional modular spectral triple, which is weakly smooth in our sense. Numerous results of [KW], [NT], [SW], [W] are incorporated into this statement. We employ Neshveyev and Tuset's modification of the pseudodifferential calculus to obtain a version of the local index formula for the Podleś sphere. Thus we see that with a suitable pseudodifferential calculus, our resolvent index formula can be extended to a full local index formula as in [CPRS2], [CM], [Hig]. We conclude by computing some explicit index pairings, and as a corollary see that the degree two term in the residue index cocycle is not a coboundary.

2. Modular spectral triples and equivariant K-theory

We begin this section by defining modular spectral triples, a generalisation of semi-finite spectral triples, [BeF], [CP2], [CPRS2], allowing for twisted traces (weights) in place of traces. We then consider the index pairings defined by modular spectral triples.

The strategy to study index pairings is almost the same as in [CPRS2], [CPRS3]. Given a representative of an equivariant K-theory class for an algebra $\mathcal{A}$, we show that a modular spectral triple over $\mathcal{A}$ allows us to formulate a well-defined (semi-finite) index problem. By following the strategy of [CPRS2], [CPRS3], we find that the index can be computed by pairing a cocycle with the Chern character of the K-theory class.

2.1. Modular spectral triples. Let $\mathcal{N}$ be a semi-finite von Neumann algebra acting on a Hilbert space $\mathcal{H}$, and fix a faithful normal semi-finite weight ϕ. We denote the modular automorphism group of ϕ by σ_t^ϕ. Then as ϕ is σ_t^ϕ invariant, we see that for all $T \in \operatorname{dom}\phi \subset \mathcal{N}$ and $t \in \mathbb{R}$,

$$\phi(T) = \phi(\sigma_t^\phi(T)).$$

Suppose further that the modular group σ_t^ϕ, which is inner since $\mathcal{N}$ is semi-finite, is periodic, and let α be the (least) period of σ_t^ϕ. Then

$$\phi(T) = \frac{1}{\alpha}\int_0^\alpha \phi(\sigma_t^\phi(T))dt = \phi\left(\frac{1}{\alpha}\int_0^\alpha \sigma_t^\phi(T)dt\right) =: (\phi \circ \Psi)(T),$$

where $\Psi\colon \mathcal{N} \to \mathcal{M} := \mathcal{N}^{\sigma^\phi}$ is the expectation onto the fixed point algebra $\mathcal{M}$ defined by the integral. Then the restriction of ϕ to $\mathcal{M}$ is a faithful normal trace.

The restriction of ϕ to $\mathcal{M}$ is also semi-finite if and only if ϕ is *strictly* semi-finite, meaning that ϕ is the sum of normal positive linear functionals whose supports are mutually orthogonal, [T], p. 105. In everything that follows, we suppose that ϕ is strictly semi-finite.

Given a faithful normal semi-finite trace τ on a von Neumann algebra $\mathcal{N}$, we define the ideal of τ-compact operators $\mathcal{K}(\mathcal{N}, \tau)$ to be the norm closure of the ideal generated by the projections p with finite trace, $\tau(p) < \infty$.

Definition 2.1. Let $\mathcal{N}$ be a semi-finite von Neumann algebra acting on a Hilbert space $\mathcal{H}$, and fix a faithful normal strictly semi-finite weight ϕ. Suppose further that the modular group σ_t^ϕ is periodic. Then we say that $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ is a *unital modular spectral triple* with respect to $(\mathcal{N}, \phi)$ if

(1) $\mathcal{A}$ is a separable unital $*$-subalgebra of $\mathcal{N}$ with norm closure A;

(2) $\mathcal{A}$ is invariant under σ^ϕ, $\mathcal{A}$ consists of analytic vectors for σ^ϕ, and $\sigma^\phi|_A$ is a strongly continuous action;

(3) $\mathcal{D}$ is a self-adjoint operator affiliated to the fixed point algebra $\mathcal{M} := \mathcal{N}^{\sigma^\phi}$;

(4) $[\mathcal{D}, a]$ extends to a bounded operator in $\mathcal{N}$ for all $a \in \mathcal{A}$;

(5) $(1 + \mathcal{D}^2)^{-\frac{1}{2}} \in \mathcal{K}(\mathcal{M}, \phi|_{\mathcal{M}})$.

The triple is even if there exists $\gamma = \gamma^* \in \mathcal{M}$ with $\gamma^2 = 1$, $\gamma a = a\gamma$ for all $a \in \mathcal{A}$ and $\gamma \mathcal{D} + \mathcal{D}\gamma = 0$. Otherwise the triple is odd.

We say that the triple is finitely summable with spectral dimension $p \geq 1$ if p is the least number such that

$$\phi((1 + \mathcal{D}^2)^{-\frac{s}{2}}) < \infty \quad \text{for all } \mathrm{Re}(s) > p.$$

Just as for ordinary spectral triples, there is a notion of smoothness and pseudodifferential operators for QC^∞ modular spectral triples, just as in [CPRS2], [CM], which we recall here.

Definition 2.2. A modular spectral triple $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ relative to $(\mathcal{N}, \phi)$ is QC^k for $k \geq 1$ (Q for quantum) if for all $a \in \mathcal{A}$ the operators a and $[\mathcal{D}, a]$ are in the domain of δ_1^k, where $\delta_1(T) = [(1 + \mathcal{D}^2)^{\frac{1}{2}}, T]$ is the partial derivation on $\mathcal{N}$ defined by $(1 + \mathcal{D}^2)^{\frac{1}{2}}$. We say that $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ is QC^∞ if it is QC^k for all $k \geq 1$. Equivalently, [CPRS2], Proposition 6.5, and [CM], Lemma B.2, $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ is QC^∞ if for all $a \in \mathcal{A}$ we have $a, [\mathcal{D}, a] \in \bigcap_{k,l \geq 0} \mathrm{dom}\, L_1^k \circ R_1^l$, where L, R are defined by

$$L(T) = (1 + \mathcal{D}^2)^{-\frac{1}{2}}[\mathcal{D}^2, T] =: (1 + \mathcal{D}^2)^{-\frac{1}{2}} T^{(1)}$$

and

$$R(T) = [\mathcal{D}^2, T](1 + \mathcal{D}^2)^{-\frac{1}{2}} =: T^{(1)}(1 + \mathcal{D}^2)^{-\frac{1}{2}}.$$

Here we have introduced the notation $T^{(1)} := [\mathcal{D}^2, T]$, and for $n > 1$ we recursively set $T^{(n)} := [\mathcal{D}^2, T^{(n-1)}]$.

Definition 2.3. Let $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ be a modular spectral triple relative to $(\mathcal{N}, \phi)$. For $r \in \mathbb{R}$ put

$$\mathrm{OP}^r = (1 + \mathcal{D}^2)^{\frac{r}{2}} \left(\bigcap_{n \geq 0} \mathrm{dom}\, \delta_1^n \right).$$

If $T \in \mathrm{OP}^r$ we say that T is a pseudodifferential operator and that the order of T is (at most) r. The definition is actually symmetric, since for r an integer (at least) we have by [CPRS2], Lemma 6.2,

$$\begin{aligned}
\mathrm{OP}^r &= (1 + \mathcal{D}^2)^{\frac{r}{2}} \left(\bigcap \mathrm{dom}\, \delta_1^n \right) \\
&= (1 + \mathcal{D}^2)^{\frac{r}{2}} \left(\bigcap \mathrm{dom}\, \delta_1^n \right) (1 + \mathcal{D}^2)^{-\frac{r}{2}} (1 + \mathcal{D}^2)^{\frac{r}{2}} \\
&\subseteq \left(\bigcap \mathrm{dom}\, \delta_1^n \right) (1 + \mathcal{D}^2)^{\frac{r}{2}}.
\end{aligned}$$

From this we easily see that $\mathrm{OP}^r \cdot \mathrm{OP}^s \subseteq \mathrm{OP}^{r+s}$. Finally, we note that if $b \in \mathrm{OP}^r$ for $r \geq 0$, then since $b = (1 + \mathcal{D}^2)^{\frac{r}{2}} a$ for some $a \in \mathrm{OP}^0$, we get $[(1 + \mathcal{D}^2)^{\frac{1}{2}}, b] = (1 + \mathcal{D}^2)^{\frac{r}{2}} [(1 + \mathcal{D}^2)^{\frac{1}{2}}, a] = (1 + \mathcal{D}^2)^{\frac{r}{2}} \delta_1(a)$, so $[(1 + \mathcal{D}^2)^{\frac{1}{2}}, b] \in \mathrm{OP}^r$.

Remarks. 1) An operator $T \in \mathrm{OP}^r$ if and only if $(1 + \mathcal{D}^2)^{-\frac{r}{2}} T \in \bigcap_{n \geq 0} \mathrm{dom}\, \delta_1^n$. Observe that operators of order at most zero are bounded.

2) We will need a weaker notion of smoothness, introduced in Section 3, for modular spectral triples, as Definition 2.2 is not satisfied for our main example, the Podleś sphere.

Example. A semi-finite spectral triple is a modular spectral triple with ϕ a semi-finite normal trace (and so $\mathcal{M} = \mathcal{N}$).

Example. Given a circle action on a unital C*-algebra A, every state on A which is KMS for this circle action gives rise to a modular spectral triple of dimension 1. Explicit examples are the Cuntz algebra with its usual gauge action, [CPR2], and the quantum group $\mathrm{SU}_q(2)$ with its Haar state, [CRT]. All these examples are QC^∞ (or regular or smooth) when we use the algebra of analytic vectors $\mathcal{A} \subset A$ for the circle action. More examples arising from a topological version of the group-measure space construction are presented in [CPPR]. In all these examples the von Neumann algebra is a proper subalgebra of $\mathcal{B}(\mathcal{H})$, where $\mathcal{H}$ is the Hilbert space of the modular spectral triple.

Example. The only other unital example (known to the authors) is the Podleś sphere, which provides an example of a modular spectral triple of dimension 2. This was first presented in [DS], and has been studied in numerous subsequent works by various authors. The paper [W] provides a good summary. This example is not QC^∞, but a replacement for the pseudodifferential calculus was developed in [NT]. This example is our main motivation for weakening the QC^∞ condition, and this example will be presented in detail in Section 4.

Nonunital examples. We have chosen to work in the unital case for simplicity, but there are nonunital examples, [CNNR], [CMR]. However, to simplify the discussion of the local index formula, we will restrict to the unital case. To handle the nonunital case in general, we would need to modify the definition of modular spectral triple in order to utilise (analogues of) the results of [CGRS2], where the local index formula is proved in the nonunital case.

2.2. Equivariant KK-theory and modular spectral triples. An odd modular spectral triple $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ with respect to $(\mathcal{N}, \phi)$ defines an equivariant Kasparov module, and so a class $[\mathcal{D}] \in \mathrm{KK}^{1,\mathbb{T}}(A, \mathcal{K}_{\mathcal{N}})$, where we recall that $A = \overline{\mathcal{A}}$. The construction of the Kasparov module associated to a modular spectral triple begins with the definition of a suitable ideal. We will deal explicitly with the odd case here, just stating the analogous results in the even case.

Definition 2.4. Given a modular spectral triple $(\mathcal{A}, \mathcal{H}, \mathcal{D}, \mathcal{N}, \phi)$, let

$$J_\phi := \{SkT \mid S, T \in \mathcal{N}, k \in \mathcal{K}(\mathcal{M}, \phi|_{\mathcal{M}})\}$$

denote the norm closed two-sided ideal in $\mathcal{N}$ generated by $\mathcal{K}(\mathcal{M}, \phi|_{\mathcal{M}})$.

The ideal J_ϕ is a right Hilbert module over itself, and $\mathcal{A}$ acts on the left of J_ϕ by multiplication. The axioms of a modular spectral triple imply that $(1 + \mathcal{D}^2)^{-\frac{1}{2}} \in J_\phi$. With a little effort we can show, as in [KNR], Theorem 4.1, that the pair $(J_\phi, \mathcal{D}(1 + \mathcal{D}^2)^{-\frac{1}{2}})$ is a Kasparov module, except that the module J_ϕ may not be countably generated.

To deal with this problem, we recall the following construction from [KNR], Theorem 5.3.

Definition 2.5. Let $(\mathcal{A}, \mathcal{H}, \mathcal{D}, \mathcal{N}, \phi)$ be a modular spectral triple, where we recall that $\mathcal{A}$ is separable. Write $F_\mathcal{D} := \mathcal{D}(1 + \mathcal{D}^2)^{-\frac{1}{2}}$ and let B_ϕ be the smallest C*-algebra in $\mathcal{N}$ containing the elements

$$F_\mathcal{D}[F_\mathcal{D}, a], \quad b[F_\mathcal{D}, a], \quad F_\mathcal{D}b[F_\mathcal{D}, a], \quad a\varphi(\mathcal{D})$$

for all $a, b \in \mathcal{A}$ and $\varphi \in C_0(\mathbb{R})$. Then B_ϕ is separable, and so σ-unital, and contained in J_ϕ.

Proposition 2.6. *A modular spectral triple $(\mathcal{A}, \mathcal{H}, \mathcal{D}, \mathcal{N}, \phi)$ defines an equivariant KK-theory class $[\mathcal{D}] = [B_\phi, F_\mathcal{D}] \in \mathrm{KK}^{1,\mathbb{T}}(A, B_\phi)$, where $F_\mathcal{D} := \mathcal{D}(1 + \mathcal{D}^2)^{-\frac{1}{2}}$.*

Proof. A modular spectral triple is automatically a von Neumann spectral triple with respect to J_ϕ in the sense of [KNR]. Then [KNR], Theorem 5.3, shows that B_ϕ is a countably generated right C* B_ϕ-module, and that the pair $(B_\phi, F_\mathcal{D})$ is a Kasparov module. The equivariance is immediate. $\qquad\square$

Having obtained an equivariant Kasparov module, and so a KK-class, the Kasparov product defines a $K_0^{\mathbb{T}}(B_\phi)$-valued index pairing between a modular spectral triple and equivariant K-theory. That is,

$$K_1^{\mathbb{T}}(A) \times \mathrm{KK}^{1,\mathbb{T}}(A, B_\phi) \to K_0^{\mathbb{T}}(B_\phi).$$

See [B], Theorem 18.4.4, for example. We now seek an analytic formula to compute this index, and in Section 3 we obtain such a formula, the resolvent index formula. The first step is the construction of a semi-finite spectral triple which encodes the index pairing between a modular spectral triple and an equivariant K-theory class. This is necessary to obtain a well-defined numerical index problem. We now describe this procedure.

Given a modular spectral triple $(\mathcal{A}, \mathcal{H}, \mathcal{D}, \mathcal{N}, \phi)$ and a class $[u] \in K_1^{\mathbb{T}}(A)$, there is a unitary $u \in M_n(\mathcal{A})$ and a representation $V \colon \mathbb{T} \to M_n(\mathbb{C})$ such that u is $\sigma^\phi \otimes \mathrm{Ad}\, V$ invariant, [B], [CNNR]. In particular, if $n = 1$ then u is σ^ϕ invariant.

We can diagonalise the representation $V_t = \bigoplus_{j=1}^n \lambda_j^{it}$, $\lambda_j \in [1, \infty)$, and in this basis it is clear that

(1) u_{ij} transforms under $\mathrm{Ad}\, V_t$ by $\lambda_i^{it} \lambda_j^{-it}$;

(2) u_{ij} transforms under σ_t^ϕ by $\lambda_i^{-it} \lambda_j^{it}$;

(3) V_t extends to an action of $\mathbb{C}$ which is not a $*$-action but satisfies $V_z^* = V_{-\bar{z}}$.

We define a positive functional $G \colon M_n(\mathbb{C}) \to \mathbb{C}$ by setting

$$G(T) = \mathrm{Tr}(V_{-i} T), \quad T \in M_n(\mathbb{C}).$$

Then G is a KMS_1 functional on $M_n(\mathbb{C})$, [BR], for the action $\mathrm{Ad}\, V$, but is not a state as it is not normalised.

Now consider the fixed point algebra $\mathcal{M}_n = (M_n(\mathcal{N}))^{\sigma^\phi \otimes \mathrm{Ad}\, V}$, which is the centralizer of the weight $\phi \otimes G$, [T], Proposition 4.3. Then $\phi \otimes G$ restricts to a faithful normal semi-finite trace on $\mathcal{M}$ and moreover $u \in \mathcal{M}_n$. The latter statement follows from the definition of u. The former follows since the strict semi-finiteness of ϕ implies the strict semi-finiteness of $\phi \otimes G$.

Lemma 2.7. *Let $(\mathcal{A}, \mathcal{H}, \mathcal{D}, \mathcal{N}, \phi)$ be a modular spectral triple which is finitely summable and $u \in M_n(\mathcal{A})$ a σ^ϕ equivariant unitary, with associated representation $V \colon \mathbb{T} \to M_n(\mathbb{C})$. Then*

$$(C^\infty(u), \mathcal{H} \otimes \mathbb{C}^n, \mathcal{D} \otimes \mathrm{Id}_n, \mathcal{M}_n, \phi \otimes G)$$

is a finitely summable semi-finite spectral triple. Here $C^\infty(u)$ is the algebra of all $f(u) \in C^*(u)$ with f a C^∞ function on the spectrum of u. Let $B_{\phi \otimes G} \subseteq \mathcal{K}(\mathcal{M}_n, \phi \otimes G)$ be defined as in Definition 2.5. Then this semi-finite spectral triple defines a Kasparov class in $\mathrm{KK}^{1,\mathbb{T}}(C^*(u), B_{\phi \otimes G})$.

Proof. The statement that we obtain a semi-finite spectral triple follows from the construction, and that we get a Kasparov module follows from Proposition 2.6. □

Thus given $[u, V] \in K_1^{\mathbb{T}}(\mathcal{A})$, we apply [KNR], Theorem 6.9, to compute the spectral flow, [Ph]. Let $i : B_{\phi \otimes G} \hookrightarrow \mathcal{K}(\mathcal{M}_n, \phi \otimes G)$ be the inclusion, and let $i_* : K_0(B_{\phi \otimes G}) \rightarrow K_0(\mathcal{K}(\mathcal{M}_n, \phi \otimes G))$. Then [KNR], Theorem 6.9, allows us to compute the spectral flow as

$$sf_{\phi \otimes G}(\mathcal{D} \otimes \mathrm{Id}_n, u(\mathcal{D} \otimes \mathrm{Id}_n)u^*) = (\phi \otimes G)_*(i_*([u] \otimes_{C^*(u)} [\mathcal{D} \otimes \mathrm{Id}_n])). \quad (2.1)$$

At this point, we have obtained an index problem which a priori depends on the representative u of the equivariant K-theory class (through the use of $C^*(u)$). To show that we do indeed have a well-defined pairing with $K_1^{\mathbb{T}}(\mathcal{A})$, we will show, via the resolvent index formula, that the index can be computed in terms of the Chern character of u, which is independent of the chosen representative of the class $[u]$. Finally, we show that the original index pairing between a modular spectral triple and equivariant K-theory can be described by the spectral flow above.

3. The resolvent index formula in twisted cyclic cohomology

In this section we express the spectral flow from eq. (2.1) in terms of the pairing between a twisted cyclic cocycle dependent only on the modular spectral triple and the Chern character of the equivariant unitary. In order to achieve this without invoking the QC^∞ property, we make a technical improvement on the work of [CPRS2] by using a weaker smoothness condition. This is necessary for our application, as the Podleś sphere modular spectral triple is not QC^∞.

3.1. Weakly QC^∞ modular spectral triples.
We weaken the QC^∞ condition with the aim of justifying a resolvent expansion, used in the proof of our index formulae, without recourse to the pseudodifferential calculus. There are two basic reasons for doing this.

The first is that the example of the Podleś sphere shows that we do not always have the QC^∞ property for modular spectral triples.

The second reason is that, conceptually, the use of the pseudodifferential calculus to prove existence and continuity of the resolvent cocycle is overkill, requiring us to invoke much more smoothness than is necessary for the statement of existence and continuity.

Definition 3.1. Let $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ be a modular spectral triple relative to $(\mathcal{N}, \phi)$. For $T \in \mathcal{N}$ mapping the domain of $\mathcal{D}^2$ to itself, define

$$WL(T) := (1 + \mathcal{D}^2)^{-1}[\mathcal{D}^2, T] = (1 + \mathcal{D}^2)^{-1}T(1 + \mathcal{D}^2) - T,$$
$$WR(T) := [\mathcal{D}^2, T](1 + \mathcal{D}^2)^{-1} = (1 + \mathcal{D}^2)T(1 + \mathcal{D}^2)^{-1} - T.$$

We say that $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ is weakly QC^∞ if

$$\mathcal{A} \subset OP^0 \subset \mathcal{N} \quad \text{and} \quad [\mathcal{D}, \mathcal{A}] \subset \text{w-}OP^0 := \bigcap_{k,l \geq 0} \text{dom}(WL)^k(WR)^l \subset \mathcal{N}.$$

The analogous definition of weak QC^k is awkward, since in Definition 2.2, QC^k is defined in terms of commutators with $|\mathcal{D}|$ or $(1 + \mathcal{D}^2)^{\frac{1}{2}}$. We will leave aside these questions and just work with weak QC^∞. Also, QC^∞ implies weak QC^∞ by the boundedness of $(1 + \mathcal{D}^2)^{-\frac{1}{2}}$.

While we do not have a pseudodifferential calculus for a weakly QC^∞ modular spectral triple $(\mathcal{A}, \mathcal{H}, \mathcal{D})$, we may consider the weak pseudodifferential operators of order $s \in \mathbb{R}$ given by

$$\text{w-}OP^s := (1 + \mathcal{D}^2)^{\frac{s}{2}}(\bigcap_{k,l} \text{dom } WL^k \circ WR^l).$$

This definition is symmetric, in the sense that

$$\text{w-}OP^s = (\bigcap_{k,l} \text{dom } WL^k \circ WR^l)(1 + \mathcal{D}^2)^{\frac{s}{2}},$$

since for all $s \in \mathbb{R}$, w-OP^s is preserved by $T \mapsto (1 + \mathcal{D}^2)^{\pm s}T(1 + \mathcal{D}^2)^{\mp s}$, by Lemma 3.2 below. Observe also that we have $OP^s \subset \text{w-}OP^s$.

It follows from the definitions that if $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ is a weakly QC^∞ modular spectral triple and $u \in M_n(\mathcal{A})$ is an equivariant unitary, then the associated semi-finite spectral triple $(C^\infty(u), \mathcal{H} \otimes \mathbb{C}^n, \mathcal{D} \otimes \text{Id}_n, \mathcal{M}_n, \phi \otimes G)$ is also weakly QC^∞.

The next few lemmas record some basic properties of the maps WL and WR.

Lemma 3.2. *Let* $\mathcal{D} \colon \text{dom}\,\mathcal{D} \subset \mathcal{H} \to \mathcal{H}$ *be an unbounded self-adjoint operator. Then* $T \in \mathcal{B}(\mathcal{H})$ *belongs to*

$$\bigcap_{k,l \geq 0} \text{dom}(WL)^k(WR)^l$$

if and only if $(1 + \mathcal{D}^2)^{\frac{s}{2}}T(1 + \mathcal{D}^2)^{-\frac{s}{2}}$ *extends to a bounded operator for all* $s \in \mathbb{R}$.

Proof. It follows from Definition 3.1 that $T \in \bigcap_{k,l \geq 0} \text{dom}(WL)^k(WR)^l$ if and only if $(1+\mathcal{D}^2)^k T(1+\mathcal{D}^2)^{-k}$ is a bounded operator for all $k \in \mathbb{Z}$. It is also immediate that if $(1+\mathcal{D}^2)^{\frac{s}{2}}T(1+\mathcal{D}^2)^{-\frac{s}{2}}$ is bounded for all $s \in \mathbb{R}$, then $T \in \bigcap_{k,l \geq 0} \text{dom}(WL)^k(WR)^l$. So let $0 < s < 1$, and recall that

$$(1 + \mathcal{D}^2)^{-s} = \frac{\sin(s\pi)}{\pi} \int_0^\infty \lambda^{-s}(1 + \lambda + \mathcal{D}^2)^{-1}d\lambda.$$

Then for $T \in \bigcap_{k,l \geq 0} \mathrm{dom}(WL)^k(WR)^l$ we have

$$(1 + \mathcal{D}^2)^s T (1 + \mathcal{D}^2)^{-s}$$
$$= (1 + \mathcal{D}^2)^s T \frac{\sin(s\pi)}{\pi} \int_0^\infty \lambda^{-s}(1 + \lambda + \mathcal{D}^2)^{-1} d\lambda$$
$$= (1 + \mathcal{D}^2)^s \frac{\sin(s\pi)}{\pi} \int_0^\infty \lambda^{-s}((1 + \lambda + \mathcal{D}^2)^{-1} T$$
$$\qquad + (1 + \lambda + \mathcal{D}^2)^{-1}[\mathcal{D}^2, T](1 + \lambda + \mathcal{D}^2)^{-1}) d\lambda$$
$$= (1 + \mathcal{D}^2)^s \frac{\sin(s\pi)}{\pi} \int_0^\infty \lambda^{-s}(1 + \lambda + \mathcal{D}^2)^{-1}$$
$$\qquad \cdot (T + [\mathcal{D}^2, T](1 + \mathcal{D}^2)^{-1}(1 + \mathcal{D}^2)(1 + \lambda + \mathcal{D}^2)^{-1}) d\lambda$$
$$= T + (1 + \mathcal{D}^2)^s \frac{\sin(s\pi)}{\pi} \int_0^\infty \lambda^{-s}(1 + \lambda + \mathcal{D}^2)^{-1}[\mathcal{D}^2, T](1 + \mathcal{D}^2)^{-1}$$
$$\qquad \cdot (1 + \mathcal{D}^2)(1 + \lambda + \mathcal{D}^2)^{-1} d\lambda.$$

An application of the functional calculus now shows that the integral is norm convergent, but in order to show that $(1 + \mathcal{D}^2)^s$ times the integral is bounded, we must work a little harder. We write

$$(1 + \mathcal{D}^2)(1 + \lambda + \mathcal{D}^2)^{-1} = 1 - \lambda(1 + \lambda + \mathcal{D}^2)^{-1}$$

so that the integral can be written, with $B = [\mathcal{D}^2, T](1 + \mathcal{D}^2)^{-1}$, as

$$\int_0^\infty \lambda^{-s}(1 + \lambda + \mathcal{D}^2)^{-1}[\mathcal{D}^2, T](1 + \mathcal{D}^2)^{-1}(1 + \mathcal{D}^2)(1 + \lambda + \mathcal{D}^2)^{-1} d\lambda$$
$$= \int_0^\infty \lambda^{-s}(1 + \lambda + \mathcal{D}^2)^{-1} B d\lambda$$
$$\qquad - \int_0^\infty \lambda^{-s}\lambda(1 + \lambda + \mathcal{D}^2)^{-1} B(1 + \lambda + \mathcal{D}^2)^{-1} d\lambda.$$

The first integral on the right hand side converges in norm to $\frac{\pi}{\sin(s\pi)}(1 + \mathcal{D}^2)^{-s} B$. For the second integral on the right hand side, we suppose first that B is self-adjoint. Then

$$\lambda(1+\lambda+\mathcal{D}^2)^{-1} B(1+\lambda+\mathcal{D}^2)^{-1} \leq \|B\| \, \lambda(1+\lambda+\mathcal{D}^2)^{-2} \leq \|B\| \, (1+\lambda+\mathcal{D}^2)^{-1}.$$

Thus for B self-adjoint, the second integral converges in norm to an operator which is bounded above by $\frac{\pi}{\sin(s\pi)}(1 + \mathcal{D}^2)^{-s}\|B\|$. By decomposing B into its real and imaginary parts, this is true for any bounded B. Thus $(1 + \mathcal{D}^2)^s T (1 + \mathcal{D}^2)^{-s}$ is bounded for $0 < s < 1$, and a similar argument shows that $(1 + \mathcal{D}^2)^{-s} T (1 + \mathcal{D}^2)^s$ is bounded. $\qquad \square$

In all the following, we define $R_s(\lambda) := (\lambda - (1 + s^2 + \mathcal{D}^2))^{-1}$ for $s \geq 0$ and λ in the vertical line

$$l := \{a + iv \mid -\infty < v < \infty\}$$

for some fixed $0 < a < 1/2$.

Lemma 3.3. *Let $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ be a weakly QC^∞ modular spectral triple relative to $(\mathcal{N}, \phi)$. Then $R_s(\lambda)[\mathcal{D}^2, T]$ is uniformly bounded on the line l independent of s, λ for all $T \in \mathcal{A} \cup [\mathcal{D}, \mathcal{A}]$. For all $T \in \mathcal{A} \cup [\mathcal{D}, \mathcal{A}]$, the function $\lambda \mapsto R_s(\lambda)TR_s(\lambda)^{-1}$ is uniformly bounded and differentiable on the line l with derivative $-R_s(\lambda)^2[\mathcal{D}^2, T]$, which vanishes as $\lambda \to a \pm i\infty$.*

Proof. First $R_s(\lambda)[\mathcal{D}^2, T] = R_s(\lambda)(1+\mathcal{D}^2)(1+\mathcal{D}^2)^{-1}[\mathcal{D}^2, T]$ and $R_s(\lambda)(1+\mathcal{D}^2)$ is uniformly bounded. Then $R_s(\lambda)TR_s(\lambda)^{-1} = R_s(\lambda)[\mathcal{D}^2, T] + T$ is uniformly bounded on l. For the differentiability, we form the difference quotients

$$R_s(\lambda + \varepsilon)TR_s(\lambda + \varepsilon)^{-1} - R_s(\lambda)TR_s(\lambda)^{-1}$$
$$= (R_s(\lambda + \varepsilon) - R_s(\lambda))TR_s(\lambda + \varepsilon)^{-1} + R_s(\lambda)T(R_s(\lambda + \varepsilon) - R_s(\lambda)^{-1})$$
$$= -\varepsilon R_s(\lambda + \varepsilon)R_s(\lambda)TR_s(\lambda + \varepsilon)^{-1} + \varepsilon R_s(\lambda)T,$$

where ε is chosen so that $\lambda + \varepsilon$ lies in a small ball centred on $\lambda = a + iv$. Now the uniform boundedness of $R_s(\lambda)TR_s(\lambda)^{-1}$ and the boundedness of $R_s(\lambda)T$ show that after dividing by ε, the norm limit as $\varepsilon \to 0$ exists and is given by

$$R_s(\lambda)T - R_s(\lambda)^2TR_s(\lambda)^{-1} = -R_s(\lambda)^2[\mathcal{D}^2, T].$$

This is not only bounded but goes to zero as $|\lambda| \to \infty$ along the line $l = a + iv$. $\qquad\square$

We recall the notation $T^{(n)} = [\mathcal{D}^2, T^{(n-1)}]$ for $n \geq 1$ and $T^{(0)} := T$ for T mapping $\mathcal{H}_\infty$ to itself.

Lemma 3.4. *With $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ as above and $T \in \mathcal{A} \cup [\mathcal{D}, \mathcal{A}]$, we have the formula*

$$R_s(\lambda)^n TR_s(\lambda)^{-n} = T + \sum_{j=1}^{n} (n - j + 1)R_s(\lambda)^j T^{(j)}.$$

Proof. Induction and the formula $R_s(\lambda)TR_s(\lambda)^{-1} = T + R_s(\lambda)[\mathcal{D}^2, T]$. $\qquad\square$

Corollary 3.5. *The function $\lambda \mapsto R_s(\lambda)^n TR_s(\lambda)^{-n}$ is norm differentiable for all $T \in \mathcal{A} \cup [\mathcal{D}, \mathcal{A}]$. The derivative goes to zero in norm as $\lambda \to a \pm i\infty$ and is given by*

$$\frac{d}{d\lambda} R_s(\lambda)^n TR_s(\lambda)^{-n} = -R_s(\lambda) \sum_{j=1}^{n} j(n - j + 1)R_s(\lambda)^j T^{(j)}. \qquad (3.1)$$

We now prove the main technical result we require, which weakens the smoothness hypotheses of [CPRS2], Lemma 7.2.

Lemma 3.6. *Let* $(\mathcal{A}, \mathcal{H}, \mathcal{D}, \mathcal{N}, \phi)$ *be a weakly* QC^∞ *modular spectral triple of dimension* $p \geq 1$. *Let* m *be a non-negative integer and* $j = 0, \ldots, m$.

(1) Let $A_j \in$ *w-OP*^{k_j}, $k_j \geq 0$, *with the product* $A_0 A_1 \ldots A_m$ *being* σ_t^ϕ*-invariant and affiliated to* $\mathcal{M} = \mathcal{N}^{\sigma^\phi}$. *Then the map*

$$r \mapsto B^r(s) = \frac{1}{2\pi i} \int_l \lambda^{-\frac{p}{2}-r} A_0 R_s(\lambda) A_1 R_s(\lambda) A_2 \ldots R_s(\lambda) A_m R_s(\lambda) d\lambda,$$

is an analytic function with values in $\mathrm{dom}(\phi)$ *for* $r \in \{z \in \mathbb{C} \mid \mathrm{Re}(z) > \frac{|k|}{2} - m, \ z \notin \mathbb{N} - \frac{p}{2}\}$, *where* $|k| = k_0 + k_1 + \cdots + k_m$. *For* $\alpha > 0$, *the function* $s \mapsto s^\alpha \times \phi(|B^r(s)|)$ *is integrable on* $[0, \infty)$ *when in addition we have* $1 + \alpha + |k| - 2m < 2\mathrm{Re}(r)$.

(2) Define $\widehat{R}_s(\lambda) = (\lambda - (1 + s^2 + \mathcal{D}^2 + sK))^{-1}$ *for an operator* $K = K^*$ *with* $\|K\|_\infty \leq \sqrt{2}$. *For* $a_j \in \mathcal{A}$ *with* $a_0 a_1 \ldots a_m \in \mathcal{M}$ *and* $r \in \mathbb{C}$ *with* $\mathrm{Re}(r) > 0$ *the operator*

$$\widetilde{B}^r(s) = \frac{1}{2\pi i} \int_l \lambda^{-\frac{p}{2}-r} a_0 R_s(\lambda)[\mathcal{D}, a_1] R_s(\lambda)[\mathcal{D}, a_2] \ldots R_s(\lambda)[\mathcal{D}, a_m] \widehat{R}_s(\lambda) d\lambda$$

is in $\mathrm{dom}(\phi)$, *and the function* $s \mapsto s^m \times \phi(|\widetilde{B}^r(s)|)$ *is integrable on* $[0, \infty)$ *when* $p < 1 + m$ *and* $1 < m + 2\mathrm{Re}(r)$.

Proof. The restriction of ϕ to the fixed point algebra $\mathcal{M} := \mathcal{N}^{\sigma^\phi}$ is a semi-finite trace. By assumption, we have $(1 + \mathcal{D}^2)^{-\frac{1}{2}} \in \mathcal{M}$, so $R_s(\lambda) \in \mathcal{M}$, and $A_0 A_1 \ldots A_m$ is affiliated to $\mathcal{M}$. Hence, the estimates in this proof will be done in the von Neumann algebra $\mathcal{M}$, and we denote the trace norm, with respect to ϕ on $\mathcal{M}$, by $\| \cdot \|_1$.

To prove statement (1), the strategy is to use the fundamental theorem of calculus, at first just doing norm convergence of integrals and norm differentiability. We abbreviate $R := R_s(\lambda)$, fix $k_0, \ldots, k_m$ as in the statement, and with $\mathrm{Re}(r) > 0$ sufficiently large, we have, for any integer $M > m$,

$$\frac{1}{2\pi i} \int_l \lambda^{-\frac{p}{2}-r} A_0 R A_1 R \ldots R A_m R d\lambda$$

$$= \frac{1}{2\pi i} \int_l \lambda^{-\frac{p}{2}-r} A_0 R A_1 R^{-1} R^2 A_2 R^{-2} \ldots R^m A_m R^{-m} R^{m+1} d\lambda$$

$$= \frac{(-1)^{M-m}}{2\pi i (\frac{p}{2} + r - 1)(\frac{p}{2} + r - 2) \ldots (\frac{p}{2} + r - (M-m))}$$

$$\cdot \int_l \frac{d^{M-m}}{d\lambda^{M-m}} (\lambda^{-\frac{p}{2}-r+(M-m)}) A_0 R A_1 R^{-1} R^2 A_2 R^{-2} \ldots$$

$$\ldots R^m A_m R^{-m} R^{m+1} d\lambda$$

$$= \frac{\Gamma(\frac{p}{2} + r - (M - m))}{2\pi i \, \Gamma(\frac{p}{2} + r)} \int_l \lambda^{-\frac{p}{2} - r + (M - m)} \sum_{j=0}^{M-m} \binom{M - m}{j}$$

$$\cdot \frac{d^j}{d\lambda^j} (A_0 R A_1 R^{-1} R^2 A_2 R^{-2} \dots R^m A_m R^{-m}) \frac{d^{M-m-j}}{d\lambda^{M-m-j}} (R^{m+1}) d\lambda.$$

$$\tag{3.2}$$

Iterating the derivative $\frac{d}{d\lambda}(R^{m+1}) = -(m+1)R^{m+2}$ yields

$$\frac{d^{M-m-j}}{d\lambda^{M-m-j}} (R^{m+1}) = (-1)^{M-m-j} \left(\prod_{n=m+1}^{M+1-j} n \right) R^{M+1-j}. \tag{3.3}$$

Now we consider

$$\frac{d^j}{d\lambda^j} (A_0 R A_1 R^{-1} R^2 A_2 R^{-2} \dots R^m A_m R^{-m}).$$

We would like to apply Lemma 3.3 to this term, however recall that each $A_j \in \text{w-OP}^{k_j}$ and not w-OP^0. So we rewrite

$$A_0 R A_1 R^{-1} R^2 A_2 R^{-2} \dots R^m A_m R^{-m}$$
$$= A_0 (1 + \mathcal{D}^2)^{-k_0/2} (R(1 + \mathcal{D}^2)^{k_0/2} A_1 (1 + \mathcal{D}^2)^{-(k_0+k_1)/2} R^{-1})$$
$$\cdot (R^2 (1 + \mathcal{D}^2)^{(k_0+k_1)/2} A_2 (1 + \mathcal{D}^2)^{-(k_0+k_1+k_2)/2} R^{-2}) \dots$$
$$\dots (R^m (1 + \mathcal{D}^2)^{(|k|-k_m)/2} A_m (1 + \mathcal{D}^2)^{-\frac{|k|}{2}} R^{-m})(1 + \mathcal{D}^2)^{\frac{|k|}{2}}.$$

By definition we have $A_j (1 + \mathcal{D}^2)^{-k_j/2} \in \text{w-OP}^0$, so using Lemma 3.2, we now find that $(1 + \mathcal{D}^2)^s A_j (1 + \mathcal{D}^2)^{-s-k_j/2} \in \text{w-OP}^0$ for all $s \in \mathbb{R}$. Hence, we define

$$A'_j := (1 + \mathcal{D}^2)^{\frac{1}{2} \sum_{n=0}^{j-1} k_n} A_j (1 + \mathcal{D}^2)^{-\frac{1}{2} \sum_{n=0}^{j} k_n} \in \text{w-OP}^0,$$

so that

$$A_0 R A_1 R^{-1} R^2 A_2 R^{-2} \dots R^m A_m R^{-m}$$
$$= A'_0 R A'_1 R^{-1} R^2 A'_2 R^{-2} \dots R^m A'_m R^{-m} (1 + \mathcal{D}^2)^{\frac{|k|}{2}}.$$

The purpose of introducing A'_j is to move all the w-OP^{k_j} behaviour into the factor $(1 + \mathcal{D}^2)^{\frac{|k|}{2}}$ on the right.

We now invoke Corollary 3.5 and find that each factor $R^j A'_j R^{-j}$ is norm differentiable with respect to λ. Indeed, by eq. (3.1) we have

$$\frac{d^n}{d\lambda^n} R^j A'_j R^{-j} = R^n B(s, \lambda) \tag{3.4}$$

for $n \geq 0$, and some operator $B(s, \lambda)$ uniformly bounded in s, λ. So we apply the chain rule to

$$\frac{d^j}{d\lambda^j}(A_0' R A_1' R^{-1} R^2 A_2' R^{-2} \ldots R^m A_m' R^{-m})(1 + \mathcal{D}^2)^{\frac{|k|}{2}}$$

$$= \frac{d^j}{d\lambda^j}(A_0 R A_1 R^{-1} R^2 A_2 R^{-2} \ldots R^m A_m R^{-m})$$

and use eq. (3.4) to compute the derivatives. Then Lemma 3.4 allows us to move each resolvent R^n arising from eq. (3.4) to the left, which gives

$$\frac{d^j}{d\lambda^j}(A_0 R A_1 R^{-1} R^2 A_2 R^{-2} \ldots R^m A_m R^{-m}) = R^j B_j(A_0', \ldots, A_m')(1 + \mathcal{D}^2)^{\frac{|k|}{2}},$$

where $B_j(A_0', \ldots, A_m') \in \text{w-OP}^0$ is uniformly bounded in s, λ.

Now we absorb the constants $(-1)^{M-m-j}\left(\prod_{n=m+1}^{M+1-j} n\right)$ from eq. (3.3) into $B_j(A_0', \ldots, A_m')$ and apply the derivative computations to eq. (3.2), which yields

$$\frac{1}{2\pi i} \int_l \lambda^{-\frac{p}{2}-r} A_0 R A_1 R \ldots R A_m R \, d\lambda$$

$$= \frac{\Gamma(\frac{p}{2} + r - (M-m))}{2\pi i \, \Gamma(\frac{p}{2} + r)} \int_l \lambda^{-\frac{p}{2}-r+(M-m)} \sum_{j=0}^{M-m}$$

$$\cdot \binom{M-m}{j} R^j B_j(A_0', \ldots, A_m')(1 + \mathcal{D}^2)^{\frac{|k|}{2}} R^{M+1-j} \, d\lambda$$

$$= \frac{\Gamma(\frac{p}{2} + r - (M-m))}{2\pi i \, \Gamma(\frac{p}{2} + r)} \int_l \lambda^{-\frac{p}{2}-r+(M-m)} \sum_{j=0}^{M-m}$$

$$\cdot \binom{M-m}{j} R^j B_j(A_0', \ldots, A_m') R^{-j}(1 + \mathcal{D}^2)^{\frac{|k|}{2}} R^{\frac{|k|}{2}} R^{M+1-\frac{|k|}{2}} \, d\lambda,$$

where the square roots use the principal branch of log.

For each j, the operator $R^j B_j(A_0', \ldots, A_m') R^{-j}$ is uniformly bounded in s, λ by Lemma 3.4 and the uniform boundedness of $B_j(A_0', \ldots, A_m')$. Also, the operator $(1 + \mathcal{D}^2)^{\frac{|k|}{2}} R^{\frac{|k|}{2}}$ is uniformly bounded in s, λ, so we are left with estimating $R^{M+1-\frac{|k|}{2}}$. The trace estimate for the resolvent in [CPRS2], Lemma 5.3, states that for M large enough and all $\varepsilon > 0$, there is a constant $C_\varepsilon > 0$ such that

$$\|R^{M+1-\frac{|k|}{2}}\|_1 \leq C_\varepsilon((1/2 + s^2 - a)^2 + v^2)^{-(M+1-\frac{|k|}{2})/2+(p/4+\varepsilon)}. \tag{3.5}$$

This estimate and the uniform boundedness of each $R^j B_j(A_0', \ldots, A_m') R^{-j}$, implies that

$$\frac{1}{2\pi i} \int_l \lambda^{-\frac{p}{2}-r} A_0 R A_1 R \ldots R A_m R \, d\lambda \in \text{dom}(\phi)$$

for $|k| - 2m + \varepsilon < 2\mathrm{Re}(r)$, for all $\varepsilon > 0$. We may apply this estimate only when $r \neq (M - j) - \frac{p}{2}$ as the prefactor

$$\frac{\Gamma(\frac{p}{2} + r - (M - m))}{2\pi i \, \Gamma(\frac{p}{2} + r)} = \frac{1}{2\pi i (\frac{p}{2} + r - 1)(\frac{p}{2} + r - 2) \ldots (\frac{p}{2} + r - (M - m))}$$

has a pole at these points. So now we estimate

$$\int_0^\infty s^\alpha \phi\left(\frac{1}{2\pi i} \int_l \lambda^{-\frac{p}{2} - r} A_0 R A_1 R \ldots R A_m R d\lambda\right) ds$$

in trace norm (recall that we regard ϕ as a trace on the fixed point algebra $\mathcal{M}$). The calculations above show that the trace norm is bounded by

$$\frac{|\Gamma(\frac{p}{2} + r - (M - m))|}{2\pi |\Gamma(\frac{p}{2} + r)|} \int_0^\infty s^\alpha \int_{-\infty}^\infty \sqrt{a^2 + v^2}^{-\frac{p}{2} - \mathrm{Re}(r) + (M - m)}$$

$$\cdot \sum_{j=0}^{M-m} \binom{M - m}{j} \|R^j B_j(A_0', A_1', \ldots, A_m') R^{-j}\|_\infty \|R^{M+1-\frac{|k|}{2}}\|_1 \, dv ds$$

$$\leq \sum_{j=0}^{M-m} \binom{M - m}{j} \frac{|\Gamma(\frac{p}{2} + r - (M - m))|}{2\pi |\Gamma(\frac{p}{2} + r)|} C_\varepsilon' \int_0^\infty s^\alpha$$

$$\cdot \int_{-\infty}^\infty \sqrt{a^2 + v^2}^{(M-m) - \frac{p}{2} - \mathrm{Re}(r)} \sqrt{(1/2 + s^2 - a)^2 + v^2}^{\frac{|k|}{2} - M - 1 + (p+\varepsilon)/2} \, dv ds,$$

where the constant C_ε' incorporates the constant from the estimate in eq. (3.5) and the constant coming from $\|R^j B_j(A_0', A_1', \ldots, A_m') R^{-j}\|_\infty \leq C$. Now by [CPRS2], Lemma 5.4, the double integral converges for

$$(\alpha + |k| - M) + (p + \varepsilon - M) < 1 \quad \text{and} \quad (\alpha + |k|) - 2m + \varepsilon - 2\mathrm{Re}(r) < -1.$$

The first constraint can always be satisfied by taking M sufficiently large. The second holds precisely when $\alpha + |k| + 1 - 2m < 2\mathrm{Re}(r)$, by choosing ε small enough.

Statement (2) of the lemma is proved just as above, with the extra $\hat{R}_s(\lambda)$ just estimated in operator norm, using [CPRS2], Lemma 5.1,

$$\|\hat{R}_s(\lambda)\|_\infty \leq (v^2 + (1 + s^2 - a - s\|K\|_\infty)^2)^{-\frac{1}{2}},$$

and the general integral estimate [CPRS2], Lemma 5.4. $\qquad\square$

3.2. Existence of the resolvent cocycle for weakly QC^∞ modular spectral triples.

First we explicitly define the resolvent cocycle associated to a modular spectral triple, again just working in the odd case. The definitions in the even case can be deduced from [CPRS3].

Definition 3.7. Let $(\mathcal{A}, \mathcal{H}, \mathcal{D}, \mathcal{N}, \phi)$ be a weakly QC^∞ odd modular spectral triple of dimension $p \geq 1$. Let $N = [\frac{p}{2}] + 1$ be the least integer strictly greater than $\frac{p}{2}$. Let m be an odd integer, $1 \leq m \leq 2N - 1$, and let $A_j \in$ w-OP^{k_j}, $j = 0, \ldots, m$, be operators whose product $A_0 A_1 \ldots A_m$ is σ_t^ϕ-invariant and affiliated to $\mathcal{M}$. For $2\mathrm{Re}(r) > (k_0 + \cdots + k_m) + 1 - m$, $r \notin \mathbb{N} - \frac{p}{2}$, define

$$\langle A_0, \ldots, A_m \rangle_{m,s,r} := \frac{1}{2\pi i} \phi\left(\int_l \lambda^{-\frac{p}{2}-r} A_0 R_s(\lambda) A_1 \ldots A_m R_s(\lambda) d\lambda \right).$$

The resolvent cocycle $(\Phi_m^r)_{m=1,3,\ldots,2N-1}$ is defined to be

$$\Phi_m^r(a_0, a_1, \ldots, a_m) := \frac{-2\sqrt{2\pi i}}{\Gamma((m+1)/2)} \int_0^\infty s^m \langle a_0, [\mathcal{D}, a_1], \ldots, [\mathcal{D}, a_m] \rangle_{m,s,r} ds$$

for $a_i \in \mathcal{A}$ satisfying $a_0 a_1 \ldots a_m \in \mathcal{M}$. For brevity we introduce the notation $da := [\mathcal{D}, a]$ for $a \in \mathcal{A}$.

We observe that Φ_m^r is finite for $\mathrm{Re}(r) > (1 - m)/2$, by Lemma 3.6. In this subsection we show that for weakly smooth modular spectral triples, $(\Phi_m^r)_{m=1,\ldots,2N-1}$ defines a twisted b, B cocycle modulo functions holomorphic in the half-plane $r > (1 - p)/2$.

We start by presenting the s- and λ-tricks, which are the main tools needed to prove continuity of the resolvent cocycle. These tricks appeared in [CPRS2], [CPRS3], [CPRS4] without appropriate justification for the convergence of the derivatives in trace norm. In [CGRS2] the justification was given with the aid of the pseudo-ifferential calculus. Here we present a different proof using only the weak QC^∞ hypothesis.

Lemma 3.8 (*s*-trick). *Let $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ be a weakly QC^∞ odd modular spectral triple relative to $(\mathcal{N}, \phi)$ of dimension $p \geq 1$. For any integers $m \geq 0, k \geq 1$ and operators $A_0, \ldots, A_m$ with $A_j \in$ w-OP^{k_j}, and $2\mathrm{Re}(r) > k + 2\sum k_j - 2m$, $r \notin \mathbb{N} - \frac{p}{2}$, we may choose r with $\mathrm{Re}(r)$ sufficiently large such that*

$$k \int_0^\infty s^{k-1} \langle A_0, \ldots, A_m \rangle_{m,s,r} ds$$

$$= -2 \sum_{j=0}^m \int_0^\infty s^{k+1} \langle A_0, \ldots, A_j, 1, A_{j+1}, \ldots, A_m \rangle_{m+1,s,r} ds.$$

Proof. The only thing that needs justification is the trace norm derivative formula

$$\frac{d}{ds} \langle A_0, \ldots, A_m \rangle_{m,s,r} = 2s \sum_{k=0}^m \langle A_0, \ldots, A_k, 1, A_{k+1}, \ldots, A_m \rangle_{m+1,s,r}$$

for suitable m, s, r and weak pseudodifferential operators A_j. So start with the difference quotient leading to one of the terms on the right hand side.

$$\frac{1}{2\pi i} \int_l \lambda^{-\frac{p}{2}-r} A_0 R \ldots R A_k \Big(\frac{R_{s+\varepsilon}-R_s}{\varepsilon} \Big) A_{k+1} R \ldots R A_m R d\lambda$$

$$= (2s+\varepsilon) \frac{1}{2\pi i} \int_l \lambda^{-\frac{p}{2}-r} A_0 R \ldots R A_k R_{s+\varepsilon} R_s A_{k+1} R \ldots R A_m R d\lambda.$$

Now repeat the trick of Lemma 3.6, giving

$$= (2s+\varepsilon) \frac{1}{2\pi i (\frac{p}{2}+r-1)(\frac{p}{2}+r-2)\ldots(\frac{p}{2}+r-(2M-1-m))}$$

$$\cdot \int_l \lambda^{-\frac{p}{2}-r+(2M-1-m)} \sum_{j=0}^{2M-1-m} \binom{2M-1-m}{j} \frac{d^j}{d\lambda^j} (A_0 R A_1 R^{-1} R^2 A_2 R^{-2} \ldots$$

$$\ldots R^k A_k R^{-k} R_{s+\varepsilon} R^{k+1} A_{k+1} R^{-k-1} \ldots R^m A_m R^{-m}) \frac{d^{2M-1-m-j}}{d\lambda^{2M-1-m-j}} (R^{m+1}) d\lambda.$$

Performing the derivatives yields a formula similar to that in the proof of Lemma 3.6, but in place of the uniformly bounded B_j's, we have uniformly bounded operators *and* one extra resolvent. Thus the same trace norm estimates apply and we see that the difference quotients converge in trace norm. Thus $\langle A_0, \ldots, A_m \rangle_{m,s,r}$ is trace norm differentiable in s, and the derivative goes to zero as $\lambda \to a \pm i\infty$. The proof is completed by applying the fundamental theorem of calculus to

$$\frac{d}{ds} (s^k \langle A_0, \ldots, A_m \rangle_{m,s,r}). \qquad \square$$

A completely analogous argument using the fundamental theorem of calculus for the parameter λ proves the following 'λ-trick' with our weak smoothness hypotheses.

Lemma 3.9 (λ-trick). *Let* $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ *be a weakly* QC^∞ *odd modular spectral triple relative to* $(\mathcal{N}, \phi)$ *of dimension* $p \geq 1$. *For any integer* $m \geq 0$, *operators* $A_j \in$ w-OP^{k_j}, $j = 0, \ldots, m$, *and* r *such that* $2\mathrm{Re}(r) > 2\sum k_j - 2m$, $r \notin \mathbb{N} - \frac{p}{2}$, *we have*

$$-(\tfrac{p}{2}+r)\langle A_0, \ldots, A_m \rangle_{m,s,r+1} = \sum_{k=0}^{m} \langle A_0, \ldots, A_k, 1, A_{k+1}, \ldots, A_m \rangle_{m+1,s,r}.$$

Proposition 3.10. *Let* $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ *be a weakly* QC^∞ *odd modular spectral triple relative to* $(\mathcal{N}, \phi)$ *of dimension* $p \geq 1$. *Let* $m = 1, 3, \ldots, 2N - 1$. *Let* $\mathcal{A} \otimes \mathcal{A}^{\otimes m}$ *have the projective tensor product topology coming from the seminorms* $a \mapsto \| WR^k \circ WL^l(a) \|_\infty + \| WR^k \circ WL^l([\mathcal{D}, a]) \|_\infty$ *on* $\mathcal{A}$, *and restrict this topology to the subspace* $(\mathcal{A} \otimes \mathcal{A}^{\otimes m})^{\sigma^\phi}$ *of* σ^ϕ *invariant tensors. (This can be called the weak* QC^∞-*topology). Then the maps*

$$(\mathcal{A} \otimes \mathcal{A}^{\otimes m})^{\sigma^\phi} \ni a_0 \otimes \cdots \otimes a_m \mapsto [r \mapsto \Phi_m^r(a_0, \ldots, a_m)]$$

are continuous multilinear maps from $(\mathcal{A} \otimes \mathcal{A}^{\otimes m})^{\sigma^\phi}$ to the functions holomorphic in $\{z \in \mathbb{C} \mid \mathrm{Re}(z) > (1-m)/2,\ z \notin \mathbb{N} - \frac{p}{2}\}$, with the topology of uniform convergence on compacta.

Proof. Let us first fix $r \in \mathbb{C}$ with $\mathrm{Re}(r) > (1 - m)/2$, and set $M = 2N - 1$. Lemma 3.6 ensures that our functionals are finite for these values of r, and it is an exercise (see [CPRS2], Lemma 7.4) to show that these functionals are holomorphic there. Thus all that we need to do is to improve the estimates to prove continuity. We do this, following [CPRS4], Proposition 5.18, using the s- and λ-tricks. We recall that we have defined $M = 2N - 1$. By applying successively the s- and λ-tricks (which commute) $(M - m)/2$ times each, we obtain

$$\Phi_m^r(a_0, \ldots, a_m)$$
$$= 2^{(M-m)/2}(M-m)! \prod_{l_1=1}^{(M-m)/2} \frac{1}{\frac{p}{2} + r - l_1} \prod_{l_2=1}^{(M-m)/2} \frac{1}{m + l_2}$$
$$\cdot \sum_{|k|=M-m} \int_0^\infty s^M \langle a_0, 1^{k_0}, da_1, 1^{k_1}, \ldots, da_m, 1^{k_m} \rangle_{M,s,r-(M-m)/2} ds,$$

where $1^{k_i} = 1, 1, \ldots, 1$ with k_i entries. Since $M \le p + 1$, the poles associated to the prefactors are outside the region $\{z \in \mathbb{C} \mid \mathrm{Re}(z) > (1 - m)/2\}$. Ignoring the prefactors, setting $n_i = k_i + 1$ and $R := R_{s,t}(\lambda)$, we need to deal with the integrals

$$\int_0^\infty s^M \phi \left(\int_l \lambda^{-\frac{p}{2} - r - (M-m)/2} a_0 R^{n_0} da_1 R^{n_1} \ldots da_m R^{n_m} d\lambda \right) ds, \quad |n| = M + 1,$$

where l is the vertical line $l = \{a + iv \mid v \in \mathbb{R}\}$ with $a = 1/2$.

To estimate the trace norm (using the trace given by restricting ϕ to the invariant subalgebra $\mathcal{N}^{\sigma^\phi}$) we first write

$$a_0 R^{n_0} da_1 R^{n_1} \ldots da_m R^{n_m} = a_0 (R^{n_0} da_1 R^{-n_0})(R^{n_0+n_1} da_2 R^{-(n_0+n_1)}) \ldots$$
$$\ldots (R^{n_0+\cdots+n_{m-1}} da_m R^{-(n_0+\cdots+n_{m-1})}) R^{n_0+\cdots+n_m}.$$

Then, using [CPRS2], Lemma 5.2, and the fact that $|n| = M + 1$, for each $\varepsilon > 0$ we obtain $C_\varepsilon > 0$ such that

$$\|a_0 R^{n_0} da_1 R^{n_1} \ldots da_m R^{n_m}\|_1$$
$$\le \|a_0 (R^{n_0} da_1 R^{-n_0})(R^{n_0+n_1} da_2 R^{-(n_0+n_1)}) \ldots$$
$$\ldots (R^{n_0+\cdots+n_{m-1}} da_m R^{-(n_0+\cdots+n_{m-1})})\|_\infty \|R^{M+1}\|_1$$
$$\le \|a_0 R^{n_0} da_1 R^{n_1} \ldots da_m R^{n_m} R^{-(M+1)}\|_\infty$$
$$\cdot C_\varepsilon ((s^2 + a^2) + v^2)^{-(M+1)/2 + (p+\varepsilon)/4}.$$

The operator norm of the product yields a constant $C(a_0, a_1, \ldots, a_m)$ depending on $a_0, a_1, \ldots, a_m$, which varies continuously as the a_j vary in a weak QC^∞ continuous way. Integrating now shows that

$$|\Phi_m^r(a_0, \ldots, a_m)| \leq |f(r)| \, C_{\varepsilon, M, m} C(a_0, a_1, \ldots, a_m)$$

for a function f continuous for $\mathrm{Re}(r) > (1 - m)/2$, $r \notin \mathbb{N} - \frac{p}{2}$ (coming from the prefactor and the integral) and some constant $C_{\varepsilon, M, m}$. $\qquad\square$

Proposition 3.11. *Let $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ be a weakly QC^∞ odd modular spectral triple relative to $(\mathcal{N}, \phi)$ of dimension $p \geq 1$, and let $N = [\frac{p}{2}] + 1$. The collection of functionals $\Phi^r = \{\Phi_m^r\}_{m=1}^{2N-1}$, m odd, is such that*

$$(B^\sigma \Phi_{m+2}^r + b^\sigma \Phi_m^r)(a_0, \ldots, a_{m+1}) = 0, \quad m = 1, 3, \ldots, 2N - 3,$$
$$(B^\sigma \Phi_1^r)(a_0) = 0,$$

where the $a_i \in \mathcal{A}$, $\sigma = \sigma_i^\phi$ and b^σ, B^σ are the twisted coboundary operators of cyclic cohomology. Moreover, there is a δ', $0 < \delta' < 1$, such that $b^\sigma \Phi_{2N-1}^r(a_0, \ldots, a_{2N})$ is a holomorphic function of r for $\mathrm{Re}(r) > -\frac{p}{2} + \frac{\delta'}{2}$.

Proof. The proof is just as in [CPRS2], Proposition 7.10, using the formulae for the twisted coboundaries b^σ, B^σ,

$$(b^\sigma \Phi_m^r)(a_0, a_1, \ldots, a_n, a_{m+1}) = \sum_{k=0}^{n} (-1)^k \Phi_m^r(a_0, \ldots, a_k a_{k+1}, \ldots, a_{m+1})$$
$$+ (-1)^{m+1} \Phi_m^r(\sigma(a_{m+1}) a_0, a_1, \ldots, a_n),$$
$$(B^\sigma \Phi_m^r)(a_0, a_1, \ldots, a_{m-1}) = \sum_{j=0}^{m-1} (-1)^{(m-1)j} \Phi_m^r(1, \sigma(a_j), \sigma(a_{j+1}), \ldots$$
$$\ldots, \sigma(a_{m-1}), a_0, \ldots, a_{j-1}),$$

and the twisted tracial property of ϕ. This yields $B^\sigma \Phi_{m+2}^r + b^\sigma \Phi_m^r = 0$ for $m = 1, 3, \ldots, 2N - 3$.

In particular, computing the Hochschild coboundary uses

$$-\langle A_0, \ldots, [\mathcal{D}^2, A_j], \ldots, A_m \rangle_{m,s,r}$$
$$= \langle A_0, \ldots, A_{j-1} A_j, \ldots, A_m \rangle_{m-1,s,r} - \langle A_0, \ldots, A_j A_{j+1}, \ldots, A_m \rangle_{m-1,s,r},$$

and, for $k \geq 1$,

$$\int_0^\infty s^k \langle \mathcal{D} A_0, A_1, \ldots, A_m \rangle_{m,s,r} \, ds = \int_0^\infty s^k \langle A_0, A_1, \ldots, A_m \mathcal{D} \rangle_{m,s,r} \, ds.$$

The first identity follows from observing that

$$-[\mathcal{D}^2, A_j] = R_s(\lambda)^{-1} A_j - A_j R_s(\lambda)^{-1}$$

and cancelling neighbouring $R_s(\lambda)$'s. The second follows by applying the twisted cyclicity of the expectation (see [CPRS2], Lemma 7.7, for the tracial version).

Employing these computations for the top degree term Φ^r_{2N-1} yields

$$(b^\sigma \Phi^r_{2N-1})(a_0, \ldots, a_{2N})$$
$$= \frac{-2\sqrt{2\pi i}}{\Gamma(N)} \sum_{j=1}^{2N} (-1)^j \int_0^\infty s^m \langle a_0, [\mathcal{D}, a_1], \ldots, [\mathcal{D}^2, a_j], \ldots, [\mathcal{D}, a_{2N}]\rangle_{2N,s,r}\, ds.$$

Since $\mathcal{A} \subset \mathrm{OP}^0$ we have $[\mathcal{D}^2, \mathcal{A}] \subset \mathrm{OP}^1$, and then the holomorphicity of $b^\sigma \Phi^r_{2N-1}$ follows just as in [CPRS2], Proposition 7.10. $\qquad\square$

We now specialise to the semi-finite case so that we may relate the resolvent cocycle to the index problem (that is, to compute spectral flow). Proposition 3.11 establishes that the resolvent cocycle is almost a cocycle, so we have the following theorem, proven just as in [CPRS2].

Theorem 3.12. *Let $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ be a weakly QC^∞ odd semi-finite spectral triple relative to $(\mathcal{N}, \tau)$ of dimension $p \geq 1$. Let $N = [\frac{p}{2}] + 1$ be the least positive integer strictly greater than $\frac{p}{2}$ and let $u \in \mathcal{A}$ be unitary. Then*

$$sf_\tau(\mathcal{D}, u^*\mathcal{D}u) = \frac{1}{\sqrt{2\pi i}} \operatorname{Res}_{r=(1-p)/2}\left(\sum_{m=1,\,\mathrm{odd}}^{2N-1} \Phi^r_m(\mathrm{Ch}_m(u))\right),$$

where $\mathrm{Ch}_m(u)$ is defined to be

$$\mathrm{Ch}_m(u) = (-1)^{(m-1)/2}\,((m-1)/2)!\,u^* \otimes u \otimes \cdots \otimes u^* \otimes u, \quad (m+1) \text{ entries.}$$

Proof. This 'resolvent index formula' is proved as in [CPRS2], where the differences for the weak QC^∞ assumption are detailed above. $\qquad\square$

Remark. In the even case we have a similar statement with $N = [(p+1)/2]$ and the sum runs over even integers from $m = 0$ to $2N$; see [CPRS3] for the QC^∞ case and [S] for the weakly QC^∞ case. We provide a more precise statement for the general modular case in Theorem 3.14 below.

3.3. The resolvent index formula for modular spectral triples. Let $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ be a modular spectral triple relative to $(\mathcal{N}, \phi)$ with modular group σ^ϕ, of spectral dimension $p \geq 1$, and weakly QC^∞ so that

$$\mathcal{A} \subset \mathrm{OP}^0, \quad [\mathcal{D}, \mathcal{A}] \subset \mathrm{w\text{-}OP}^0.$$

Let $u \in M_n(\mathcal{A})$ be unitary, $V \colon \mathbb{T} \to M_n(\mathbb{C})$ a representation and suppose that u is $\sigma^\phi \otimes \mathrm{Ad}\, V$ invariant.

Lemma 2.7 constructs a semi-finite spectral triple from $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ and u. The semi-finite resolvent index formula, Theorem 3.12, then shows that the resolvent cocycle defined using the trace $\phi \otimes G$ is 'almost' a b, B cocycle, and computes the spectral flow from $\mathcal{D}$ to $u\mathcal{D}u^*$.

With $N = [\frac{p}{2}] + 1$, we have

$$sf_{\phi \otimes G}(\mathcal{D} \otimes \mathrm{Id}_n, u(\mathcal{D} \otimes \mathrm{Id}_n)u^*) = \frac{1}{\sqrt{2\pi i}} \mathrm{Res}_{r=\frac{1-p}{2}} \sum_{m=1,\,\mathrm{odd}}^{2N-1} (\Phi_G)_m^r(\mathrm{Ch}_m(u)),$$

(3.6)

where $(\Phi_G)_m^r$ is the resolvent cocycle defined using the trace $\phi \otimes G$. In particular the sum on the right hand side of (3.6) analytically continues to a deleted neighbourhood of $r = (1-p)/2$ with *at worst* a simple pole at $r = (1-p)/2$.

We will compute the G part of the trace, leaving us with a functional defined in terms of ϕ.

The Chern character of u is defined to be the (infinite) sum $\bigoplus_j \mathrm{Ch}_{2j+1}(u) \in \mathrm{HE}_{2j+1}(M_N(\mathcal{A}))$, the entire cyclic homology, with

$$\mathrm{Ch}_{2j+1}(u) = (-1)^j \, j! \, u^* \otimes u \otimes \cdots \otimes u^* \otimes u \qquad (2j+2) \text{ entries.}$$

Now in [W], Lemma 4.1, Wagner has shown, in a slightly different context, that the map

$$G_* : \bigoplus_j \mathrm{HE}_{2j+1}(M_N(\mathcal{A})^{\sigma \otimes \mathrm{Ad}\, V}) \to \bigoplus_j \mathrm{HE}_{2j+1}^\sigma(\mathcal{A})$$

to σ-twisted cyclic homology given on chains by

$$
\begin{aligned}
&G_*(T_0 \otimes \cdots \otimes T_{2j+1}) \\
&\quad = \sum_{i_0, i_1, \ldots, i_{2j+2}} (V_{-i})_{i_{2j+2}, i_0}(T_0)_{i_0, i_1} \otimes (T_1)_{i_1, i_2} \otimes \cdots \otimes (T_{2j+1})_{i_{2j+1}, i_{2j+2}}
\end{aligned}
$$

is an isomorphism. Here V_{-i} is the extension of the representation of $\mathbb{T}$ to $-i$, as discussed in Section 2.2.

Now each equivariant unitary with class $[u] \in K_1^{\mathbb{T}}(A)$ is equivariant for its own representation of the circle. So it makes sense to regard the representation V as part of the data, so $[u] = [u, V]$. We define $\mathrm{Ch}_{2j+1}([u, V]) \in \mathrm{HE}_{2j+1}^\sigma(\mathcal{A})$ by

$$
\begin{aligned}
&\mathrm{Ch}_{2j+1}([u, V]) \\
&\quad = (-1)^j \, j! \sum_I (V_{-i})_{i_{2j+2}, i_0}(u^*)_{i_0, i_1} \otimes (u)_{i_1, i_2} \otimes \cdots \otimes (u_{2j+1})_{i_{2j+1}, i_{2j+2}}.
\end{aligned}
$$

Then it is straightforward to check that this does indeed define an entire twisted cyclic

cycle. Moreover it is immediate from the definitions that

$$sf_{\phi \otimes G}(\mathcal{D} \otimes \mathrm{Id}_n, u(\mathcal{D} \otimes \mathrm{Id}_n)u^*) = \frac{1}{\sqrt{2\pi i}} \operatorname{Res}_{r=\frac{1-p}{2}} \sum_{m=1, \, \mathrm{odd}}^{2N-1} (\Phi_G)_m^r(\mathrm{Ch}_m(u)),$$

$$= \frac{1}{\sqrt{2\pi i}} \operatorname{Res}_{r=\frac{1-p}{2}} \sum_{m=1, \, \mathrm{odd}}^{2N-1} \Phi_m^r(\mathrm{Ch}_m([u, V])).$$

Here Φ_m^r is the resolvent cocycle given by the modular spectral triple.

We now collect these observations into a statement describing the resolvent index formula for weakly smooth modular spectral triples.

Theorem 3.13. *For a weakly* QC^∞ *odd modular spectral triple* $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ *relative to* $(\mathcal{N}, \phi)$ *of spectral dimension* $p \geq 1$, *and with* $N = [\frac{p}{2}] + 1$, *the function valued cochain* $(\Phi_m^r)_{m=1,\ldots,2N-1}$ *is a twisted cyclic cocycle modulo cochains with values in functions holomorphic in a half-plane containing* $(1-p)/2$. *Moreover, for* $[u, V] \in K_1^{\mathbb{T}}(\mathcal{A})$ *with representative* $u \in M_n(\mathcal{A})$ *we have*

$$sf_{\phi \otimes G}(\mathcal{D} \otimes \mathrm{Id}_n, u^*(\mathcal{D} \otimes \mathrm{Id}_n)u) = \frac{1}{\sqrt{2\pi i}} \operatorname{Res}_{r=\frac{1-p}{2}} \Big(\sum_{m=1, \, odd}^{2N-1} \Phi_m^r(\mathrm{Ch}_m([u, V])) \Big).$$

In particular, there is a well-defined map

$$K_1^{\mathbb{T}}(\mathcal{A}) \mapsto \mathbb{R}, \quad [u, V] \mapsto sf_{\phi \otimes G}(\mathcal{D} \otimes \mathrm{Id}_n, u^*(\mathcal{D} \otimes \mathrm{Id}_n)u).$$

Though we have not proved it here, a similar result is true in the even case; see [S].

Theorem 3.14. *For a weakly* QC^∞ *even modular spectral triple* $(\mathcal{A}, \mathcal{H}, \mathcal{D}, \gamma)$ *relative to* $(\mathcal{N}, \phi)$ *of spectral dimension* $p \geq 1$, *and with* $M = [(p+1)/2]$, *the function valued cochain* $(\Phi_m^r)_{m=0,\ldots,2M}$ *is a twisted cyclic cocycle modulo cochains with values in functions holomorphic in a half-plane containing* $(1-p)/2$. *Moreover, for* $[P, V] \in K_0^{\mathbb{T}}(\mathcal{A})$ *with representative* $P \in M_n(\mathcal{A})$ *and* $\mathcal{D}_+ = \frac{1}{4}(1-\gamma)\mathcal{D}(1+\gamma)$ *we have*

$$\mathrm{Index}_{\phi \otimes G}(P(\mathcal{D}_+ \otimes \mathrm{Id}_n)P) = \frac{1}{\sqrt{2\pi i}} \operatorname{Res}_{r=(1-p)/2} \Big(\sum_{m=0, \, \mathrm{even}}^{2M} \Phi_m^r(\mathrm{Ch}_m([P, V])) \Big).$$

In particular, there is a well-defined map

$$K_0^{\mathbb{T}}(\mathcal{A}) \mapsto \mathbb{R}, \quad [P, V] \mapsto \mathrm{Index}_{\phi \otimes G}(P(\mathcal{D}_+ \otimes \mathrm{Id}_n)P).$$

Remark. The Chern character of an equivariant projection is

$$\mathrm{Ch}_0([P, V]) = \mathrm{Tr}(V_{-i} P),$$

$$\mathrm{Ch}_{2k}([P, V]) = (-1)^k \frac{(2k)!}{k!} \sum (V_{-i}(P - \tfrac{1}{2}))_{i_0 i_1} \otimes P_{i_1 i_2} \otimes \cdots \otimes P_{i_{2k} i_0}. \tag{3.7}$$

Finally, the next two results relate the even index given by the resolvent index formula above back to the K-theory valued index pairing between the KK-class defined by the modular spectral triple and equivariant K-theory.

Lemma 3.15. *Let $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ be a modular spectral triple relative to $(\mathcal{N}, \phi)$. Let $J_\phi \subset \mathcal{N}$ be the ideal from Definition 2.4 and $J_\phi^\sim$ its unitisation. Let $E \in M_k(J_\phi^\sim)$ be a $\sigma^\phi \otimes \mathrm{Ad}\, W$-invariant projection for the associated representation $W : \mathbb{T} \to M_k(\mathbb{C})$ so that $[E, W] \in K_0^{\mathbb{T}}(J_\phi^\sim)$. Define*

$$\phi_*([E, W]) := (\phi \otimes G_W)(E) \in [0, \infty],$$

where $G_W(T) = \mathrm{Tr}(W_{-i} T)$ for $T \in M_k(\mathbb{C})$. Then ϕ_* is a well-defined map on the semigroup of Murray–von Neumann equivalence classes of equivariant projections in $J_\phi^\sim \otimes \mathcal{K}$, where $\mathcal{K}$ is the compact operators. The Grothendieck group of the sub-semigroup for which ϕ_* takes finite values is (isomorphic to) a subgroup of $K_0^{\mathbb{T}}(J_\phi)$, and we call this the domain of ϕ_*.

Proof. Let $W_1 : \mathbb{T} \to M_n(\mathbb{C})$ and $W_2 : \mathbb{T} \to M_m(\mathbb{C})$ be representations. Let $E_1 \in M_n(J_\phi)$ denote a $\sigma \otimes \mathrm{Ad}\, W_1$ projection, and let $E_2 \in M_m(J_\phi)$ denote a $\sigma^\phi \otimes \mathrm{Ad}\, W_2$ projection. Suppose that $[E_1, W_1]$ and $[E_2, W_2]$ are equivariantly Murray–von Neumann equivalent ([W], Definition 3.1), meaning there exists some $S \in M_{m \times n}(J_\phi)$ such that

$$S^* S = E_1, \quad S S^* = E_2 \quad \text{and} \quad W_{2,z} S = S W_{1,z} \quad \text{for all } z \in \mathbb{T}.$$

Then we compute

$$\begin{aligned}
\phi_*([E_1, W_1]) &= (\phi \otimes G_{W_1})(E_1) \\
&= \phi(\mathrm{Tr}_n(W_{1,-i} E_1)) \\
&= \phi(\mathrm{Tr}_n(W_{1,-i} S^* S)) \\
&= \phi(\mathrm{Tr}_n(S W_{1,-i} S^*)).
\end{aligned}$$

Now, by analytically continuing, $S W_{1,-i} = W_{2,-i} S$, so

$$\phi_*([E_1, W_1]) = \phi(\mathrm{Tr}_m(W_{2,-i} S S^*)) = \phi(\mathrm{Tr}_m(W_{2,-i} E_2)) = \phi_*([E_2, W_2]).$$

Using the universal property of the Grothendieck group, we see that the Grothendieck group of equivalence classes for which ϕ_* takes finite values may be regarded as a subgroup of $K_0^{\mathbb{T}}(J_\phi)$. On this subgroup, ϕ_* is well defined. $\square$

Theorem 3.16. *Let $(\mathcal{A}, \mathcal{H}, \mathcal{D}, \gamma)$ be a weakly QC^∞ even modular spectral triple relative to $(\mathcal{N}, \phi)$ of spectral dimension $p \geq 1$, and $[P, V] \in K_0^{\mathbb{T}}(\mathcal{A})$. Let $B_\phi \subset J_\phi$ be as in Definition 2.5, and let $i : B_\phi \to J_\phi$ be the inclusion. Then $i_*([P, V] \otimes_A [(B_\phi, F_{\mathcal{D}})]) \in K_0^{\mathbb{T}}(J_\phi)$ is in the domain of ϕ_*. Furthermore,*

$$\phi_*(i_*([P, V] \otimes_A [(B_\phi, F_{\mathcal{D}})])) = \mathrm{Res}_{r=(1-p)/2} \Big(\sum_{m=0,\,\mathrm{even}}^{2N} \Phi_m^r(\mathrm{Ch}_m([P, V])) \Big).$$

Proof. Given the modular spectral triple $(\mathcal{A}, \mathcal{H}, \mathcal{D}, \gamma, \mathcal{N}, \phi)$, we define $[(B_\phi, F_\mathcal{D})] \in$ $\mathrm{KK}^{0,\mathbb{T}}(A, B_\phi)$. Also, let $V \colon \mathbb{T} \to M_n(\mathbb{C})$ be a representation and $P \in M_n(\mathcal{A})$ a projection which is $\sigma^\phi \otimes \mathrm{Ad}\,V$ invariant, so that we obtain a class $[P, V] \in K_0^{\mathbb{T}}(A)$.

Define the projections

$$N_\pm := \ker(P(\mathcal{D} \otimes \mathrm{Id}_n)^\pm P)$$

so that

$$\mathrm{Ind}_{\phi \otimes G}(P(\mathcal{D} \otimes \mathrm{Id}_n)^+ P) = (\phi \otimes G)(N_+) - (\phi \otimes G)(N_-).$$

By the construction of the semi-finite spectral triple $(C^\infty(P), \mathcal{H} \otimes \mathbb{C}^n, \mathcal{D} \otimes \mathrm{Id}_n, \mathcal{M}_n,$ $\phi \otimes G)$, we have $N_\pm \in \mathcal{K}((M_n(\mathcal{N}))^{\sigma^\phi \otimes \mathrm{Ad}\,V}, \phi \otimes G)$, since the $N_\pm$ are kernel projections and

$$N_\pm \leq (P + (P(\mathcal{D} \otimes \mathrm{Id}_n)P)^2)^{-1}. \tag{3.8}$$

Also, the $\sigma^\phi \otimes \mathrm{Ad}\,V$-invariance of P implies the same invariance for $N_\pm$.

We now want to show that we also have $N_\pm \in M_n(B_\phi)$ so that they define classes in $K_0^{\mathbb{T}}(B_\phi)$. We do this by proving that the operator $(P + (P(\mathcal{D} \otimes \mathrm{Id}_n)P)^2)^{-1} \in$ $M_n(B_\phi)$, then applying eq. (3.8) again to see that $N_\pm \in M_n(B_\phi)$.

Let $\mathcal{D}_n := \mathrm{Id}_n \otimes \mathcal{D}$. Consider the operator $(P + (P\mathcal{D}_n P)^2)^{-1} \colon P(\mathbb{C}^n \otimes B_\phi) \to P(\mathbb{C}^n \otimes B_\phi)$. The inverse exists because P acts as the identity on $P(\mathbb{C}^n \otimes B_\phi)$ and $(P\mathcal{D}_n P)^2 \geq 0$. The adjointable endomorphisms on $P(\mathbb{C}^n \otimes B_\phi)$ are $PM_n(M(B_\phi))P$, where $M(B_\phi)$ is the multiplier algebra, while the compact operators are $PM_n(B_\phi)P$. A priori, we know only that $(P + (P\mathcal{D}_n P)^2)^{-1}$ is bounded on $P(\mathbb{C}^n \otimes B_\phi)$.

To show the compactness of $(P + (P\mathcal{D}_n P)^2)^{-1}$, we compute

$$\begin{aligned}
(P + (P\mathcal{D}_n P)^2)^{-1} &= (P + P[\mathcal{D}_n, P]\mathcal{D}_n P + P\mathcal{D}_n^2 P)^{-1} \\
&= (P + P\mathcal{D}_n^2 P)^{-1} + [(P + P[\mathcal{D}_n, P][\mathcal{D}_n, P]P \\
&\quad + P\mathcal{D}_n^2 P)^{-1} - (P + P\mathcal{D}_n^2 P)^{-1}],
\end{aligned}$$

where the last line follows from the observation

$$P[\mathcal{D}_n, P]P = P[\mathcal{D}_n, P^2]P = P(P[\mathcal{D}_n, P] + [\mathcal{D}_n, P]P)P = 2P[\mathcal{D}_n, P]P = 0,$$

so that $P[\mathcal{D}_n, P][\mathcal{D}_n, P]P = P[\mathcal{D}_n, P]\mathcal{D}_n P$. The algebraic result $\alpha^{-1} - \beta^{-1} = \beta^{-1}(\beta - \alpha)\alpha^{-1}$ yields

$$\begin{aligned}
(P &+ P[\mathcal{D}_n, P][\mathcal{D}_n, P]P + P\mathcal{D}_n^2 P)^{-1} - (P + P\mathcal{D}_n^2 P)^{-1} \\
&= -(P + P\mathcal{D}_n^2 P)^{-1}(P[\mathcal{D}_n, P][\mathcal{D}_n, P]P) \\
&\quad \cdot (P + P[\mathcal{D}_n, P][\mathcal{D}_n, P]P + P\mathcal{D}_n^2 P)^{-1}.
\end{aligned}$$

Hence

$$(P + (P\mathcal{D}_n P)^2)^{-1} = (P + P\mathcal{D}_n^2 P)^{-1} B(P),$$

where $B(P)$ is a bounded operator given by

$$B(P) = 1 - (P[\mathcal{D}_n, P][\mathcal{D}_n, P]P)(P + P[\mathcal{D}_n, P][\mathcal{D}_n, P]P + P\mathcal{D}_n^2 P)^{-1}.$$

Now consider $(1 + \mathcal{D}_n)^{-1} : \mathbb{C}^n \otimes B_\phi \to \mathbb{C}^n \otimes B_\phi$. Then we have

$$
\begin{aligned}
(P + P\mathcal{D}_n^2 P)P(1 + \mathcal{D}_n)^{-1}P &= P + P[\mathcal{D}_n^2, P](1 + \mathcal{D}_n^2)^{-1}P \\
&= P + PC(P)(1 + \mathcal{D}_n^2)^{-\frac{1}{2}}P.
\end{aligned}
\tag{3.9}
$$

Here $C(P)$ is bounded since $P \in M_n(\mathcal{A}) \subset \mathrm{OP}^0$ (where OP^0 is defined using $\mathcal{D}_n$).

Now $(1 + \mathcal{D}^2)^{-\frac{1}{2}} \in B_\phi$ by definition, so $(1 + \mathcal{D}_n^2)^{-\frac{1}{2}} \in M_n(B_\phi)$. Hence

$$P[\mathcal{D}_n^2, P](1 + \mathcal{D}_n^2)^{-1}P \in PM_n(B_\phi)P,$$

so eq. (3.9) now implies that

$$(P + P\mathcal{D}_n^2 P)^{-1} \in PM_n(B_\phi)P.$$

We know B_ϕ is an ideal in the endomorphisms, so eq. (3.8) now implies that $N_\pm \in M_n(B_\phi)$. By the $\sigma^\phi \otimes \mathrm{Ad}\, V$-invariance of $N_\pm$, we have $[N_\pm, V] \in K_0^{\mathbb{T}}(B_\phi)$. Then

$$(\phi \otimes G)(N_+) - (\phi \otimes G)(N_-) = \phi_*(i_*([N_+, V] - [N_-, V])).
\tag{3.10}$$

In order to compare eq. (3.10) to the Kasparov product $[P, V] \otimes_A [(B_\phi, F_\mathcal{D})]$, we rewrite the classes $[N_\pm, V]$ as Kasparov modules. We have

$$[N_+, V] - [N_-, V] = \left[(N_+(\mathbb{C}^n \otimes B_\phi) \oplus N_-(\mathbb{C}^n \otimes B_\phi), 0, \begin{pmatrix} N_+ & 0 \\ 0 & -N_- \end{pmatrix}, V \oplus V \right],$$

where $N_+(\mathbb{C}^n \otimes B_\phi) \oplus N_-(\mathbb{C}^n \otimes B_\phi)$ is the right Hilbert B_ϕ-module, 0 is the operator, $\begin{pmatrix} N_+ & 0 \\ 0 & -N_- \end{pmatrix}$ is the grading and $V \oplus V$ is the $\mathbb{T}$-action giving the equivariance.

Now the operator $P(\mathrm{Id}_n \otimes F_\mathcal{D})^+ P$ gives an isomorphism from $(1 - N_+)(\mathbb{C}^n \otimes B_\phi)$ to $(1 - N_-)(\mathbb{C}^n \otimes B_\phi)$. Hence, the Kasparov module constructed from $(1 - N_\pm)(\mathbb{C}^n \otimes B_\phi)$ and $P(\mathrm{Id}_n \otimes F_\mathcal{D})^+ P$ has trivial class. Consequently,

$$
\begin{aligned}
\left[(N_+(\mathbb{C}^n \otimes B_\phi) \oplus N_-(\mathbb{C}^n \otimes B_\phi), 0, \begin{pmatrix} N_+ & 0 \\ 0 & -N_- \end{pmatrix}, V \oplus V \right] \\
= [(\mathbb{C}^n \otimes B_\phi, P(\mathrm{Id}_n \otimes F_\mathcal{D})P, \mathrm{Id}_n \otimes \gamma, V)].
\end{aligned}
$$

Finally, observe that, see [B] for example, we have an explicit representative of the Kasparov product

$$[P, V] \otimes_A [(B_\phi, F_\mathcal{D})] = [(\mathbb{C}^n \otimes B_\phi, P(\mathrm{Id}_n \otimes F_\mathcal{D})P, \mathrm{Id}_n \otimes \gamma, V)].$$

Reiterating the above results, we have proved that

$$
\begin{aligned}
\mathrm{Ind}_{\phi \otimes G}(P(\mathcal{D} \otimes \mathrm{Id}_n)^+ P) &= (\phi \otimes G)(N_+) - (\phi \otimes G)(N_-) \\
&= \phi_*(i_*([N_+, V] - [N_-, V])) \\
&= \phi_*(i_*([(\mathbb{C}^n \otimes B_\phi, P(\mathrm{Id}_n \otimes F_\mathcal{D})P, \mathrm{Id}_n \otimes \Gamma, V)])) \\
&= \phi_*(i_*([P, V] \otimes_A [(B_\phi, F_\mathcal{D})])). \qquad \square
\end{aligned}
$$

4. The local index formula for the Podleś sphere

In this section we will explicitly compute a twisted b, B cocycle for the (modular) spectral triple over the Podleś sphere first investigated in [DS]. We do this by applying the modified pseudodifferential calculus of [NT] to the twisted resolvent cocycle of the previous section. Having done this, we construct some equivariant projections for a circle action arising from the Haar state and compute the index pairing via a residue formula, yielding a local index formula.

4.1. The modular spectral triple for the Podleś sphere. We first recall (see [KS]) that the quantum algebra $\mathcal{A} = \mathcal{O}(\mathrm{SU}_q(2))$ for $q \in [0, 1]$ is generated by elements a, b, c, d modulo the relations

$$ab = qba, \quad ac = qca, \quad bd = qdb, \quad cd = qdc, \quad bc = cb$$
$$ad = 1 + qbc, \quad da = 1 + q^{-1}bc$$
$$a^* = d, \quad b^* = -qc, \quad c^* = -q^{-1}b, \quad d^* = a.$$

The Podleś sphere, which we denote by $\mathcal{B}$, is (isomorphic to) the unital $*$-subalgebra of $\mathcal{O}(\mathrm{SU}_q(2))$ generated by $q^{-1}ab$, $-cd$ and $-q^{-1}bc$.

Recall that for each $l \in \frac{1}{2}\mathbb{N}_0$, there is a unique (up to unitary equivalence) irreducible corepresentation V_l of the coalgebra $\mathcal{A}$ of dimension $2l + 1$, and that $\mathcal{A}$ is cosemisimple. That is, if we fix a vector space basis in each of the V_l and denote by $t_{r,s}^l \in \mathcal{A}$ the corresponding matrix coefficients, then we have the following analogue of the Peter–Weyl theorem.

Theorem 4.1 ([KS], Theorem 4.13). *Let $I_l := \{-l, -l + 1, \ldots, l - 1, l\}$. Then the set $\{t_{r,s}^l \mid l \in \frac{1}{2}\mathbb{N}_0, \ r, s \in I_l\}$ is a vector space basis of $\mathcal{A}$.*

This will be referred to as the Peter–Weyl basis. With a suitable choice of basis in $V_{\frac{1}{2}}$, one has

$$a = t_{-\frac{1}{2},-\frac{1}{2}}^{\frac{1}{2}}, \quad b = t_{-\frac{1}{2},\frac{1}{2}}^{\frac{1}{2}}, \quad c = t_{\frac{1}{2},-\frac{1}{2}}^{\frac{1}{2}}, \quad d = t_{\frac{1}{2},\frac{1}{2}}^{\frac{1}{2}}.$$

The expressions for the Peter–Weyl basis elements as linear combinations of the polynomial basis elements can be found in [KS], Section 4.2.4.

The algebra $\mathcal{A}$ has a useful direct sum decomposition. For $m, n \in \mathbb{Z}$ where $m - n$ is even, define

$$\mathcal{A}[m, n] := \mathrm{span}\{a^{\frac{1}{2}(m+n)} b^{k+\frac{1}{2}(m-n)} c^k,$$
$$b^{k+\frac{1}{2}(m-n)} c^k d^{-\frac{1}{2}(m+n)} \mid k + \min\{0, \tfrac{1}{2}(m - n)\} \in \mathbb{N}_0\},$$

and for $m - n$ odd let $\mathcal{A}[m, n] := \{0\}$. Then

$$\mathcal{A} = \bigoplus_{m,n\in\mathbb{Z}} \mathcal{A}[m, n] \quad \text{and} \quad \mathcal{A}[m_1, n_1] \cdot \mathcal{A}[m_2, n_2] \subseteq \mathcal{A}[m_1 + m_2, n_1 + n_2].$$

With this notation, we have $\mathcal{B} = \bigoplus_{m \in \mathbb{Z}} \mathcal{A}[m, 0]$.

Let h be the Haar state on the universal C^*-completion of the $*$-algebra $\mathcal{A}$, whose value on the Peter–Weyl basis is $h(t_{r,s}^l) = \delta_{l,0}$. Define an automorphism ϑ on $\mathcal{A}$ by

$$\vartheta(a) = q^2 a, \quad \vartheta(b) = b, \quad \vartheta(c) = c, \quad \vartheta(d) = q^{-2} d.$$

Then ϑ is the modular automorphism for the Haar state, in the sense that $h(\alpha\beta) = h(\vartheta(\beta)\alpha)$ for all $\alpha, \beta \in \mathcal{A}$. For all $n \in \mathbb{Z}$ define

$$\mathcal{H}_n := L^2(\mathrm{span}\{t_{r,\frac{n}{2}}^l \mid l \in \tfrac{n}{2} + \mathbb{N}_0, \, r \in I_l\}, h).$$

The left action of the dual Hopf algebra to $\mathcal{A}$ provides the unbounded operators $\partial_e \colon \mathcal{H}_n \to \mathcal{H}_{n+2}$ and $\partial_f \colon \mathcal{H}_n \to \mathcal{H}_{n-2}$ given by

$$\partial_e(t_{r,s}^l) = \sqrt{[l + \tfrac{1}{2}]_q^2 - [s + \tfrac{1}{2}]_q^2} \, t_{r,s+1}^l, \quad \partial_f(t_{r,s}^l) = \sqrt{[l + \tfrac{1}{2}]_q^2 - [s - \tfrac{1}{2}]_q^2} \, t_{r,s-1}^l,$$

where our definition of the q-number $[a]_q$ is

$$[a]_q := \frac{q^{-a} - q^a}{q^{-1} - q} = Q(q^{-a} - q^a) \quad \text{for any } a \in \mathbb{C},$$

and we abbreviated $Q := (q^{-1} - q)^{-1} \in (0, \infty)$. Finally, we define unbounded linear operators Δ_R, ∂_k on $\mathcal{A} \subset \bigoplus \mathcal{H}_n$ by

$$\Delta_R(t_{r,s}^l) := q^{2r} t_{r,s}^l, \quad \partial_k \triangleright t_{i,j}^l := q^j t_{i,j}^l.$$

The left actions of e and f are twisted derivations in the sense that

$$\partial_e(\alpha\beta) = \partial_e(\alpha)\partial_k(\beta) + \partial_k^{-1}(\alpha)\partial_e(\beta),$$
$$\partial_f(\alpha\beta) = \partial_f(\alpha)\partial_k(\beta) + \partial_k^{-1}(\alpha)\partial_f(\beta)$$

for $\alpha, \beta \in \mathcal{A}$.

Definition 4.2. Define the Hilbert space $\mathcal{H} := \mathcal{H}_1 \oplus \mathcal{H}_{-1}$, and represent $\mathcal{B}$ on $\mathcal{H}$ by left multiplication. The Hilbert space $\mathcal{H}$ is graded by $\gamma := \left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$. Define the weight Ψ_R on $\mathcal{B}(\mathcal{H})$ by $\Psi_R(T) := \mathrm{Tr}(\Delta_R^{-\frac{1}{2}} T \Delta_R^{-\frac{1}{2}})$. Finally, on a suitable domain in $\mathcal{H}$, define the self-adjoint operator $\mathcal{D} := \left(\begin{smallmatrix} 0 & \partial_e \\ \partial_f & 0 \end{smallmatrix}\right)$.

In fact $(\mathcal{B}, \mathcal{H}, \mathcal{D}, \gamma)$ defines an honest spectral triple, [DS], (i.e., a modular spectral triple with von Neumann algebra $\mathcal{B}(\mathcal{H})$ and weight given by the operator trace) which is ε-summable for all $\varepsilon > 0$.

Lemma 4.3. *The data $(\mathcal{B}, \mathcal{H}, \mathcal{D}, \mathcal{B}(\mathcal{H}), \Psi_R)$ defines a weakly QC^∞ even modular spectral triple, which is finitely summable with spectral dimension* 2.

Proof. We first show that the data produces a modular spectral triple. Certainly $\mathcal{B}$ is a separable $*$-subalgebra of $\mathcal{B}(\mathcal{H})$, and by construction the modular automorphism group of Ψ_R is ϑ_t^{-1}, and $\mathcal{B}$ consists of analytic vectors for ϑ_t^{-1}.

Also, the commutators $[\mathcal{D}, \beta]$ extend to bounded operators for all $\beta \in \mathcal{B}$, given by

$$d\beta := [\mathcal{D}, \beta] = \begin{pmatrix} 0 & q^{-\frac{1}{2}} \partial_e(\beta) \\ q^{\frac{1}{2}} \partial_f(\beta) & 0 \end{pmatrix}. \tag{4.1}$$

We also observe that $\gamma = \gamma^*$ and $\gamma^2 = I$, and by construction $\gamma \mathcal{D} + \mathcal{D}\gamma = 0$.

Now, for $T \in \mathcal{B}(\mathcal{H})$ set $T^+ = (1+\gamma)T(1+\gamma)/4$ and $T^- = (1-\gamma)T(1-\gamma)/4$. From the definition of the operator trace, and using the normalised Peter–Weyl basis $\xi_{r,j}^l := t_{r,j}^l / \|t_{r,j}^l\|$, we find for $T \geq 0$ that

$$\Psi_R(T) = \sum_{l,r} q^{-2r} (\langle \xi_{r,1/2}^l, T^+ \xi_{r,1/2}^l \rangle + \langle \xi_{r,-\frac{1}{2}}^l, T^- \xi_{r,-\frac{1}{2}}^l \rangle).$$

We first observe from the above formula that the finite rank operators are in the domain of Ψ_R, so Ψ_R is semi-finite. Next, we see that Ψ_R is a sum of vector states with orthogonal support, as the Peter–Weyl basis is orthogonal. Hence Ψ_R is strictly semi-finite.

The Peter–Weyl basis elements can be used to construct a common eigenbasis for $\mathcal{D}$ and Δ_R on $\mathcal{H}$, so the spectral projections of $\mathcal{D}$ and Δ_R commute. We conclude that $\mathcal{D}$ is affiliated to the fixed point algebra $\mathcal{M} := \mathcal{B}(\mathcal{H})^{\vartheta^{-1}}$. All that remains to be proved is that $(1 + \mathcal{D}^2)^{-\frac{1}{2}} \in \mathcal{K}(\mathcal{M}, \Psi_R|_\mathcal{M})$. To establish this, we observe that $\mathcal{D}^2$ has the following spectral projections

$$\mathcal{P}_l \begin{pmatrix} t_{r,\frac{1}{2}}^k \\ 0 \end{pmatrix} := \delta_{l,k} \begin{pmatrix} t_{r,\frac{1}{2}}^k \\ 0 \end{pmatrix}, \quad \mathcal{P}_l \begin{pmatrix} 0 \\ t_{r,-\frac{1}{2}}^k \end{pmatrix} := \delta_{l,k} \begin{pmatrix} 0 \\ t_{r,-\frac{1}{2}}^k \end{pmatrix},$$

for $l = 1/2, 3/2, \ldots$, which correspond to the eigenvalues $[l+\frac{1}{2}]_q^2$. Now $\Psi_R(\mathcal{P}_l) = \sum_{r=-l}^{l} q^{-2r} = [2l+1]_q$, and the sum $\sum_{l=\frac{1}{2},\frac{3}{2},\ldots} (1 + [l+\frac{1}{2}]_q^2)^{-\frac{1}{2}} < \infty$ implies that

$$(1 + \mathcal{D}^2)^{-\frac{1}{2}} = \sum_{l=\frac{1}{2},\frac{3}{2},\ldots} (1 + [l+\frac{1}{2}]_q^2)^{-\frac{1}{2}} \mathcal{P}_l$$

is norm convergent. It follows that $(1 + \mathcal{D}^2)^{-\frac{1}{2}} \in \mathcal{K}(\mathcal{M}, \Psi_R|_\mathcal{M})$ and so $(\mathcal{B}, \mathcal{H}, \mathcal{D}, \gamma, \mathcal{B}(\mathcal{H}), \Psi_R)$ is a modular spectral triple. The spectral dimension is shown to be 2 in [KW].

We now prove that $\mathcal{B} \subset \text{OP}^0$, $[\mathcal{D}, \mathcal{B}] \subset \text{w-OP}^0$ so that the modular spectral triple is weakly QC^∞. The first statement is proved in [NT], Proposition 3.2. To prove the second statement we show that for all $\beta \in \mathcal{B}$ and $z \in \mathbb{C}$, the operators $(1 + \mathcal{D}^2)^{-z}[\mathcal{D}, \beta](1 + \mathcal{D}^2)^z \in \mathcal{B}(\mathcal{H})$, as per Lemma 3.2. We begin by observing

that $\mathcal{D}^2$ has eigenbasis given by

$$\mathcal{D}^2 \begin{pmatrix} t^l_{r,\frac{1}{2}} \\ 0 \end{pmatrix} = [l + \tfrac{1}{2}]^2 \begin{pmatrix} t^l_{r,\frac{1}{2}} \\ 0 \end{pmatrix}, \quad \mathcal{D}^2 \begin{pmatrix} 0 \\ t^l_{r,-\frac{1}{2}} \end{pmatrix} = [l + \tfrac{1}{2}]^2 \begin{pmatrix} 0 \\ t^l_{r,-\frac{1}{2}} \end{pmatrix}.$$

Now we consider $\beta \in \mathcal{B}$ to be of the form $t^p_{r,0}$ (as finite linear combinations of these elements span $\mathcal{B}$). Then the commutator $[\mathcal{D}, t^p_{r,0}] = \begin{pmatrix} 0 & \kappa_1 t^p_{r,1} \\ \kappa_2 t^p_{r,-1} & 0 \end{pmatrix}$ for some coefficients κ_1, κ_2. We expand the product $t^p_{r,0} t^l_{s,\frac{1}{2}}$ using the Clebsch–Gordan coefficients (see [DLSSV], [KS]), giving

$$(1 + \mathcal{D}^2)^{-z} t^p_{r,0} (1 + \mathcal{D}^2)^z \begin{pmatrix} t^l_{s,\frac{1}{2}} \\ 0 \end{pmatrix}$$

$$= (1 + [l + \tfrac{1}{2}]^2_q)^z \sum_{k=|l-p|}^{l+p} (1 + [k + \tfrac{1}{2}]^2_q)^{-z} c^{p,l,k}_{s,r} \begin{pmatrix} t^k_{s+r,\frac{1}{2}} \\ 0 \end{pmatrix},$$

where $c^{p,l,k}_{s,r}$ is some product of Clebsch–Gordan coefficients that will be subsumed later.

The norm of $(1 + \mathcal{D}^2)^{-z} t^p_{r,0} (1 + \mathcal{D}^2)^z \begin{pmatrix} t^l_{s,\frac{1}{2}} \\ 0 \end{pmatrix}$ can be computed using the orthogonality of the Peter–Weyl basis, so

$$\left\| (1 + \mathcal{D}^2)^{-z} t^p_{r,0} (1 + \mathcal{D}^2)^z \begin{pmatrix} t^l_{s,\frac{1}{2}} \\ 0 \end{pmatrix} \right\|^2$$

$$= \sum_{k=|l-p|}^{l+p} \left(\frac{1 + [l + \tfrac{1}{2}]^2}{1 + [k + \tfrac{1}{2}]^2} \right)^{2\mathrm{Re}(z)} |c^{p,l,k}_{s,r}|^2 \left\| \begin{pmatrix} t^k_{s+r,\frac{1}{2}} \\ 0 \end{pmatrix} \right\|^2.$$

Let $M_{l,p} := \max_{|l-p| \le k \le l+p} \{((1 + [l + \tfrac{1}{2}]^2)/(1 + [k + \tfrac{1}{2}]^2))^{2\mathrm{Re}(z)}\}$. Then

$$\left\| (1 + \mathcal{D}^2)^{-z} t^p_{r,0} (1 + \mathcal{D}^2)^z \begin{pmatrix} t^l_{s,\frac{1}{2}} \\ 0 \end{pmatrix} \right\|^2 \le M_{l,p} \sum_{k=|l-p|}^{l+p} |c^{p,l,k}_{s,r}|^2 \left\| \begin{pmatrix} t^k_{s+r,\frac{1}{2}} \\ 0 \end{pmatrix} \right\|^2$$

$$= M_{l,p} \left\| t^p_{r,0} \begin{pmatrix} t^l_{s,\frac{1}{2}} \\ 0 \end{pmatrix} \right\|^2.$$

It remains to show that there exist finite M_p such that $M_{l,p} \le M_p$ for all $l \ge 0$. Let $\varepsilon_k = Q(1 - q^{2k})$ so that $[k]_q = q^{-k} \varepsilon_k$. Then for all $l \ge p + \tfrac{1}{2}$: if

$$\frac{1 + [l + \tfrac{1}{2}]^2}{1 + [|l - p| + \tfrac{1}{2}]^2} = \frac{\varepsilon^2_{l+\frac{1}{2}} + q^{2l+1}}{q^{2p} \varepsilon^2_{l-p+\frac{1}{2}} + q^{2l+1}},$$

then

$$\frac{\varepsilon_1^2}{1 + q^{2p} Q^2} \le \frac{1 + [l + \frac{1}{2}]^2}{1 + [|l - p| + \frac{1}{2}]^2} \le \frac{Q^2 + 1}{q^{2p} \varepsilon_1^2}.$$

It follows that the operator $(1 + \mathcal{D}^2)^{-1} t_{r,0}^p (1 + \mathcal{D}^2)$ is bounded on the set of vectors of the form $\begin{pmatrix} t_{s,\frac{1}{2}}^l \\ 0 \end{pmatrix}$. The same calculation can be performed for the vectors $\begin{pmatrix} 0 \\ t_{s,-\frac{1}{2}}^l \end{pmatrix}$, and again for the operators $\begin{pmatrix} 0 & t_{r,1}^p \\ 0 & 0 \end{pmatrix}$ and $\begin{pmatrix} 0 & 0 \\ t_{r,-1}^p & 0 \end{pmatrix}$, completing the proof.   □

4.2. The residue cocycle for the Podleś sphere. Lemma 4.3 shows that the modular spectral triple $(\mathcal{B}, \mathcal{H}, \mathcal{D})$ satisfies the hypotheses of Theorem 3.14. Hence we can employ the resolvent cocycle to compute index pairings with equivariant K-theory, or at least those classes which can be represented as projections over $\mathcal{B}$. As Δ_R implements the modular automorphism ϑ, then it follows that the weight Ψ_R is ϑ^{-1}-twisted. The resolvent cocycle, which we denote by $(\phi_m^r)_{m=0,2}$, therefore lives in ϑ^{-1}-twisted cohomology.

To simplify the computation of the resolvent cocycle, we would like to have a version of the pseudodifferential calculus. A simple replacement for the pseudodifferential calculus for this example was presented in [NT].

Lemma 4.4 ([NT], Corollary 3.3). *Define $\chi := \begin{pmatrix} q^{-1} & 0 \\ 0 & q \end{pmatrix}$ on $\mathcal{H}_1 \oplus \mathcal{H}_{-1}$. For any $\beta \in \mathcal{B}$ there exists an analytic function $z \mapsto M(z) \in$ w-OP$^0 \subset \mathcal{B}(\mathcal{H})$ with at most linear growth on vertical strips such that*

$$|D|^{-z} d\beta = d\beta \chi^z |D|^{-z} + M(z)|D|^{-z-1} = \chi^{-z} d\beta |D|^{-z} + M(z)|D|^{-z-1}.$$

We can now use this pseudodifferential calculus to simplify the computation of the resolvent cocycle, (ϕ_0^r, ϕ_2^r), and arrive at a twisted version of the local index formula.

The first simplification we make is to discard the 1 from the resolvent, replacing $R_s(\lambda) = (\lambda - (1 + s^2 + \mathcal{D}^2))^{-1}$ with $R_s(\lambda) = (\lambda - (s^2 + \mathcal{D}^2))^{-1}$. This is possible because $\mathcal{D}$ is invertible in this example, so we can employ the method of [CPRS4], Section 5.3, in particular [CPRS4], Proposition 5.20. (The transgression cochain defined there is well defined for weakly QC^∞ modular spectral triples since $\mathcal{D} \in$ OP^1, by essentially the same arguments as we employed for the resolvent cocycle). Removing the 1 from the resolvents modifies the resolvent cocycle by coboundaries and cochains holomorphic at $r = -\frac{1}{2}$.

Before proceeding, we recall the detailed summability properties of the spectral triple $(\mathcal{B}, \mathcal{H}, \mathcal{D})$ computed in [KW].

Lemma 4.5 ([KW], Proposition 1). *The function $r \mapsto \text{Tr}(\Delta_R^{-1} \frac{1}{2}(1 \pm \gamma)|\mathcal{D}|^{-3-2r})$ has a meromorphic continuation to the complex plane which is holomorphic for*

$\mathrm{Re}(r) > -\frac{1}{2}$ *and has a simple pole at $r = -\frac{1}{2}$. Furthermore, for all $\beta \in \mathcal{B}$ we have the equality*

$$\mathrm{Res}_{r=-\frac{1}{2}} \mathrm{Tr}(\Delta_R^{-1}\tfrac{1}{2}(1 \pm \gamma)\beta|\mathcal{D}|^{-3-2r}) = \frac{(q^{-1} - q)}{2 \ln q^{-1}}\varepsilon(\beta),$$

where ε is the counit of $\mathcal{A}$ restricted to $\mathcal{B}$ satisfying $\varepsilon(t_{i,0}^l) = \delta_{i,0}$.

The degree zero component ϕ_0^r of the resolvent cocycle is computed from the definition using the Cauchy formula and [CPRS3]. This yields the formula, for $a_0 \in \mathcal{B}$,

$$\phi_0^r(a_0) = 2 \int_0^\infty \mathrm{Tr}\left(\Delta_R^{-1}\gamma \frac{1}{2\pi i} \int_l \lambda^{-1-r} a_0 R_s(\lambda) d\lambda\right) ds$$

$$= \frac{\Gamma(\frac{1}{2})\Gamma(r + \frac{1}{2})}{\Gamma(r + 1)} \mathrm{Tr}(\Delta_R^{-1}\gamma a_0|\mathcal{D}|^{-2r-1}).$$

Since $\mathrm{Tr}(\Delta_R^{-1}\gamma|\mathcal{D}|^{-2r-1}) = 0$ for all sufficiently large $r \in \mathbb{R}$, then taking the trivial continuation to the whole real line gives $\phi_0^r(I) = 0$ for all $r \in \mathbb{R}$. This is the only evaluation of ϕ_0^r needed to compute the index pairing later on.

We can compute $\mathrm{Res}_{r=-\frac{1}{2}} \phi_0^r$ explicitly, but as the calculation is quite lengthy and we do not require this full computation for computing the index pairing, we just quote the result; see [S] for full details.

The functional $\phi_0 := \mathrm{Res}_{r=-\frac{1}{2}} \phi_0^r$ is supported on the span of the powers $(bc)^k$, $k = 0, 1, 2 \dots$. We have seen that $\phi_0(I) = 0$. For the remaining values we have

$$\mathrm{Res}_{r=-\frac{1}{2}} \phi_0^r(bc) = \frac{1}{2}\left(1 - \frac{\gamma}{\ln q^{-1}}\right) - qQ,$$

where γ is Euler's constant. For $k = 0, 1, 2, \dots$ and with h the Haar state, we have

$$\mathrm{Res}_{r=-\frac{1}{2}} \phi_0^r((bc)^{k+2}) = \frac{(-1)^{k+1}q^{k+1}}{1 - q^{2k+2}} = \frac{(-1)^{k+1}}{q^{-k-1} - q^{k+1}} = -\frac{h((bc)^k)}{q^{-1} - q}.$$

We now compute the degree two term ϕ_2^r of the resolvent cocycle starting with the definition,

$$\phi_2^r(a_0, a_1, a_2)$$

$$= 4 \int_0^\infty s^2 \mathrm{Tr}\left(\Delta_R^{-1}\gamma \frac{1}{2\pi i} \int_l \lambda^{-1-r} a_0 R_s(\lambda) da_1 R_s(\lambda) da_2 R_s(\lambda) d\lambda\right) ds.$$

We proceed by employing the pseudodifferential calculus described in Lemma 4.4 in order to rewrite the expression $a_0 R_s(\lambda) da_1 R_s(\lambda) da_2 R_s(\lambda)$ by moving all the

resolvents to the right. From Lemma 4.4, for each $\beta \in \mathcal{B}$ there exist bounded operators M_1, M_2 such that

$$(\lambda - s^2 - \mathcal{D}^2)d\beta = d\beta(x\lambda - s^2 - \chi^{-2}\mathcal{D}^2) + M_1|\mathcal{D}|,$$
$$(\lambda - s^2 - \chi^{-2}\mathcal{D}^2)d\beta = d\beta(\lambda - s^2 - \mathcal{D}^2) + M_2|\mathcal{D}|.$$

This gives the formulae

$$R_s(\lambda)d\beta = d\beta(\lambda - s^2 - \chi^{-2}\mathcal{D}^2)^{-1} - R_s(\lambda)M_1|\mathcal{D}|(\lambda - s^2 - \chi^{-2}\mathcal{D}^2)^{-1},$$
$$d\beta R_s(\lambda) = (\lambda - s^2 - \chi^{-2}\mathcal{D}^2)^{-1}d\beta + (\lambda - s^2 - \chi^{-2}\mathcal{D}^2)^{-1}M_2|\mathcal{D}|R_s(\lambda). \qquad (4.2)$$

Observe that the operators $R_s(\lambda)M_1|\mathcal{D}|(\lambda - s^2 - \chi^{-2}\mathcal{D}^2)^{-1}$ and $(\lambda - s^2 - \chi^{-2}\mathcal{D}^2)^{-1}M_2|\mathcal{D}|R_s(\lambda)$ are in w-OP[-3] by Lemma 4.4. Using this observation, and eq. (4.2), we can move all the resolvents to the right, and in doing so we only introduce errors which are functions holomorphic at $r = -\frac{1}{2}$. More precisely, for any a_0, a_1, $a_2 \in \mathcal{B}$, we obtain the formula

$$\phi_2^r(a_0, a_1, a_2) = 4\int_0^\infty s^2 \operatorname{Tr}\left(\Delta_R^{-1}\gamma a_0 da_1 da_2 \frac{1}{2\pi i}\int_l \lambda^{-1-r}R_s(\lambda)(\lambda - s^2\right.$$
$$\left. - \chi^{-2}\mathcal{D}^2)^{-1}R_s(\lambda)d\lambda\right)ds$$
$$= 4\int_0^\infty s^2 \operatorname{Tr}\left(\Delta_R^{-1}\gamma a_0 da_1 da_2 \frac{1}{2\pi i}\int_l \lambda^{-1-r}(\lambda - s^2\right.$$
$$\left. - \chi^{-2}\mathcal{D}^2)^{-1}R_s(\lambda)^2 d\lambda\right)ds$$

modulo functions holomorphic at $r = -\frac{1}{2}$. The integral

$$\int_l \lambda^{-1-r}(\lambda - s^2 - \chi^{-2}\mathcal{D}^2)^{-1}R_s(\lambda)^2 d\lambda \qquad (4.3)$$

is evaluated on the spectra of the operators $(\lambda - s^2 - \chi^{-2}\mathcal{D}^2)^{-1}$ and $R_s(\lambda)^2$. We want to use the Cauchy integral formula, however because there are two poles to consider, $\lambda = s^2 + \mathcal{D}^2$ and $\lambda = s^2 + \chi^{-2}\mathcal{D}^2$, we outline the process.

First note that χ and $\mathcal{D}^2$ are commuting operators with discrete spectra, and they can be simultaneously diagonalised with respect to the direct sum $\mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_{-1}$. Indeed, the integrand in eq. (4.3) has the eigenbasis

$$\left\{\begin{pmatrix} t^l_{i,\frac{1}{2}} \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ t^l_{i,-\frac{1}{2}} \end{pmatrix} \mid l - \tfrac{1}{2} \in \mathbb{N}_0, \ i \in \{-l, -l+1, \ldots, l\}\right\}, \qquad (4.4)$$

on which χ^{-2} simply acts via multiplication by the scalar $q^{\pm 2} \neq 1$. We specialise to the eigenbasis in $\mathcal{H}_1$, where χ^{-2} acts via multiplication by q^2. The argument we now present can be applied analogously to the remaining eigenbasis elements.

On each eigenvector, the integral in eq. (4.3) reduces to a scalar integral over λ, where we may apply the usual Cauchy integral formula. The integrand of this scalar integral has two poles; on the eigenbasis elements in $\mathcal{H}_1$ described in eq. (4.4) these poles are $\lambda_1 = s^2 + q^2[l + \frac{1}{2}]_q^2$ and $\lambda_2 = s^2 + [l + \frac{1}{2}]_q^2$, with $\lambda_1 < \lambda_2$. The contour of integration l is a vertical line to the left of the spectrum for all $s \geq 0$.
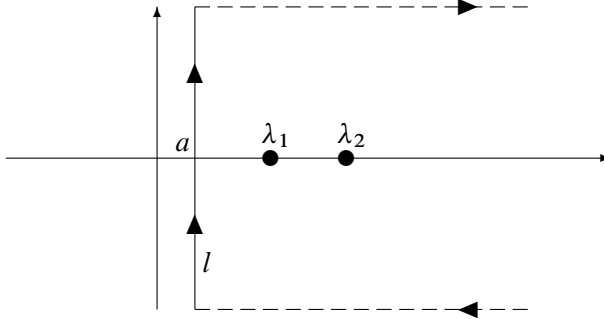
In order to apply the Cauchy integral formula, we modify the contour l by adding a vertical line $l' = \{a' + iv \mid \lambda_1 < a' < \lambda_2, v \in \mathbb{R}\}$ between the poles λ_1 and λ_2. We integrate along this line in both directions, allowing us to split the integral into two parts.

We denote by Γ_1 the contour obtained by going up along l and down along l', and denote by Γ_2 the remaining integration along l'. Lemma 3.6 shows that the horizontal dashed integrals go to zero.
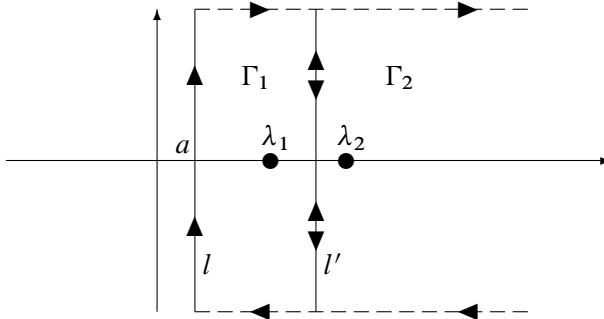
Define

$$f_1(\lambda) := \lambda^{-1-r} R_s(\lambda) R_s(\lambda), \quad f_2(\lambda) := \lambda^{-1-r}(\lambda - s^2 - \chi^{-2}\mathcal{D}^2)^{-1}.$$

By construction, the function f_1 is holomorphic on the domain defined by the contour Γ_1, while f_2 is holomorphic on the domain defined by Γ_2. Therefore, we may apply

the (scalar) Cauchy integral formula for each contour Γ_1 and Γ_2, so we write

$$\int_l \lambda^{-1-r}(\lambda - s^2 - \chi^{-2}\mathcal{D}^2)^{-1}R_s(\lambda)^2 d\lambda$$

$$= \int_{\Gamma_1} \frac{f_1(\lambda)}{(\lambda - s^2 - \chi^{-2}\mathcal{D}^2)}d\lambda + \int_{\Gamma_2} \frac{f_2(\lambda)}{(\lambda - s^2 - \mathcal{D}^2)^2}d\lambda$$

This yields

$$\frac{1}{2\pi i}\int_l \lambda^{-1-r}(\lambda - s^2 - \chi^{-2}\mathcal{D}^2)^{-1}R_s(\lambda)^2 d\lambda$$

$$= f_1(s^2 + \chi^{-2}\mathcal{D}^2) + f_2'(s^2 + \mathcal{D}^2)$$

$$= (s^2 + \chi^{-2}\mathcal{D}^2)^{-1-r}(\chi^{-2} - 1)^{-2}\mathcal{D}^{-4} - (1+r)(s^2 + \mathcal{D}^2)^{-2-r}$$

$$\cdot (1 - \chi^{-2})^{-1}\mathcal{D}^{-2} - (s^2 + \mathcal{D}^2)^{-1-r}(1 - \chi^{-2})^{-2}\mathcal{D}^{-4}.$$

Inserting the result of the Cauchy integral into our previous formula for ϕ_2^r and evaluating the s-integrals (see for example [CPRS3], Lemma 5.9) yields

$$\phi_2^r(a_0, a_1, a_2)$$

$$= \frac{\sqrt{\pi}\Gamma(r - \frac{1}{2})}{\Gamma(r + 1)} \operatorname{Tr}(\Delta_R^{-1}\gamma a_0 da_1 da_2 \chi^{2r-1}|\mathcal{D}|^{-2r+1}(\chi^{-2} - 1)^{-2}\mathcal{D}^{-4})$$

$$- \frac{\sqrt{\pi}\Gamma(r + \frac{1}{2})}{\Gamma(r + 1)} \operatorname{Tr}(\Delta_R^{-1}\gamma a_0 da_1 da_2 |\mathcal{D}|^{-2r-1}(1 - \chi^{-2})^{-1}\mathcal{D}^{-2})$$

$$- \frac{\sqrt{\pi}\Gamma(r - \frac{1}{2})}{\Gamma(r + 1)} \operatorname{Tr}(\Delta_R^{-1}\gamma a_0 da_1 da_2 |\mathcal{D}|^{-2r+1}(1 - \chi^{-2})^{-2}\mathcal{D}^{-4}),$$

modulo functions holomorphic at $r = -\frac{1}{2}$. Writing $\Gamma(r + \frac{1}{2}) = (r - \frac{1}{2})\Gamma(r - \frac{1}{2})$ and collecting terms, $\phi_2^r(a_0, a_1, a_2)$ is given by

$$\frac{\sqrt{\pi}\Gamma(r - \frac{1}{2})}{\Gamma(r + 1)} \operatorname{Tr}(\Delta_R^{-1}\gamma a_0 da_1 da_2 |\mathcal{D}|^{-2r-3}(1 - \chi^{-2})^{-2}$$

$$\cdot (\chi^{2r-1} - (r - \frac{1}{2})(1 - \chi^{-2}) - 1)).$$

Observe that

$$\Gamma(r - \tfrac{1}{2})(\chi^{2r-1} - (r - \tfrac{1}{2})(1 - \chi^{-2}) - 1)$$

$$= \Gamma(r - \tfrac{1}{2})(\chi^{-2}(\chi^{2r+1} - 1) - (r + \tfrac{1}{2})(1 - \chi^{-2}))$$

$$= (r + \tfrac{1}{2})\Gamma(r - \tfrac{1}{2})(\chi^{-2}(1 + \ln\chi^2) - 1) + \Gamma(r - \tfrac{1}{2})\chi^{-2}\sum_{n=2}^{\infty} \frac{(\ln\chi^2)^n}{n!}(r + \tfrac{1}{2})^n.$$

$$(4.5)$$

Now $\Gamma(r - \frac{1}{2})$ has a simple pole at $r = -\frac{1}{2}$, so the function in eq. (4.5) is holomorphic at $r = -\frac{1}{2}$ with constant term $1 - \chi^{-2}(1 + \ln \chi^2)$. Therefore

$$\phi_2(a_0, a_1, a_2) := \mathrm{Res}_{r=-\frac{1}{2}} \, \phi_2^r(a_0, a_1, a_2)$$
$$= \mathrm{Res}_{r=-\frac{1}{2}} \, \mathrm{Tr}(\Delta_R^{-1} \gamma a_0 da_1 da_2 |\mathcal{D}|^{-2r-3} C),$$

where $C = (1 - \chi^{-2}(1 + \ln \chi^2))(1 - \chi^{-2})^{-2} = (\chi^2 - 1 - \ln \chi^2)(\chi - \chi^{-1})^{-2}$ is a constant diagonal matrix. Finally, eq. (4.1) yields

$$a_0 da_1 da_2 = \begin{pmatrix} a_0 \partial_e(a_1) \partial_f(a_2) & 0 \\ 0 & a_0 \partial_f(a_1) \partial_e(a_2) \end{pmatrix},$$

and so invoking Lemma 4.5 gives the formula

$$\phi_2(a_0, a_1, a_2) = \frac{1}{2(q^{-1} - q) \ln q^{-1}}((q^{-2} - 1 - \ln q^{-2})\varepsilon(a_0 \partial_e(a_1) \partial_f(a_2)) \tag{4.6}$$
$$- (q^2 - 1 - \ln q^2)\varepsilon(a_0 \partial_f(a_1) \partial_e(a_2))).$$

4.3. Some equivariant projections and their Chern characters.

Our aim is to construct representatives in the equivariant K-theory $K_0^{\mathbb{T}}(\mathcal{B})$ for the action of the modular automorphism group Ψ_R, which is given by $\sigma_t^{\Psi_R} = \vartheta_t^{-1}$. These representatives will take the form of projections $p \in M_{N \times N}(\mathcal{B})$ together with a representation $V: \mathbb{T} \to M_{N \times N}(\mathbb{C})$ such that p is $\vartheta^{-1} \otimes \mathrm{Ad}(V)$-invariant. See [W] for similar constructions.

For $n \in \frac{1}{2}\mathbb{Z}$, define

$$T_n^l := \begin{pmatrix} t_{l,n}^l \\ t_{l-1,n}^l \\ \vdots \\ t_{-l,n}^l \end{pmatrix} \quad \text{and} \quad P_n := T_n^{|n|} T_n^{|n|*}.$$

More explicitly,

$$P_n = \begin{pmatrix} t_{|n|,n}^{|n|} t_{|n|,n}^{|n|*} & t_{|n|,n}^{|n|} t_{|n|-1,n}^{|n|*} & \cdots & t_{|n|,n}^{|n|} t_{-|n|,n}^{|n|*} \\ t_{|n|-1,n}^{|n|} t_{|n|,n}^{|n|*} & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \vdots \\ t_{-|n|,n}^{|n|} t_{|n|,n}^{|n|*} & \cdots & \cdots & t_{-|n|,n}^{|n|} t_{-|n|,n}^{|n|*} \end{pmatrix},$$

$$(P_n)_{r,s} = t_{|n|-r+1,n}^{|n|} t_{|n|-s+1,n}^{|n|*}.$$

By construction, $P_n^* = P_n$ and $P_n \in M_{(2|n|+1) \times (2|n|+1)}(\mathcal{B})$. Furthermore,

$$T_n^{l*} T_n^l = \sum_{p=-l}^{l} t_{p,n}^{l*} t_{p,n}^l = \mathrm{Id}_n$$

and hence $P_n^2 = P_n$, so P_n is a projection. Now define $\lambda_j = q^{-2j+2} \in [1, \infty)$ for $j \in \{1, 2, \ldots, 2|n| + 1\}$ and define $V_n \colon \mathbb{T} \to M_{2|n|+1}(\mathbb{C})$ by

$$V_{n,t} := \begin{pmatrix} \lambda_1^{it} & 0 & \ldots & 0 \\ 0 & \lambda_2^{it} & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \ldots & 0 & \lambda_{2|n|+1}^{it} \end{pmatrix}.$$

While we have defined V_n to be a real action on $M_{(2|n|+1) \times (2|n|+1)}(\mathbb{C})$, the action is periodic and so induces a circle action. Observe that

$$\sigma_t((P_n)_{r,s}) = q^{-2it(|n|-r+1-(|n|-s+1))} t^{|n|}_{|n|-r+1,n} t^{|n|*}_{|n|-s+1,n} = q^{2it(r-s)}(P_n)_{r,s}$$

and

$$\mathrm{Ad}(V_{n,t})(P_n)_{r,s} = (V_{n,t} P_n V_{n,t}^{-1})_{r,s} = \lambda_r^{it}(P_n)_{r,s}(\lambda_s^{it})^{-1} = q^{2it(s-r)}(P_n)_{r,s}.$$

So P_n is $\vartheta^{-1} \otimes \mathrm{Ad}(V_n)$-invariant. We define the weight $G \colon M_{(2|n|+1) \times (2|n|+1)}(\mathbb{C}) \to \mathbb{C}$ by

$$G(X) := \mathrm{Tr}(V_{n,-i} X)$$

for $X \in M_{(2|n|+1) \times (2|n|+1)}(\mathbb{C})$ and $(V_{n,-i})_{k,m} = \delta_{k,m} q^{-2k+2}$.

We have demonstrated that P_n is an equivariant projection for the circle action represented by V_n, and therefore defines a class in $K_0^{\mathbb{T}}(\mathcal{B})$. We now write down the Chern character of this representative, eq. (3.7).

Lemma 4.6. *The Chern character of $[P_n, V_n]$ is*

$$\mathrm{Ch}_0([P_n, V_n]) = q^{2(n-|n|)} I,$$

$$\mathrm{Ch}_2([P_n, V_n]) = -2 \sum_{k_0, k_1, k_2 = 0}^{2|n|} q^{-2k_0}(t^{|n|}_{|n|-k_0,n} t^{|n|*}_{|n|-k_1,n} - \tfrac{1}{2}\delta_{k_0,k_1})$$

$$\otimes t^{|n|}_{|n|-k_1,n} t^{|n|*}_{|n|-k_2,n} \otimes t^{|n|}_{|n|-k_2,n} t^{|n|*}_{|n|-k_0,n}.$$

Proof. Using eq. (3.7) we have

$$\mathrm{Ch}_0([P_n, V_n]) = \sum_{k_0, k_1 = 1}^{2|n|+1} (V_{n,-i})_{k_1,k_0}(P_n)_{k_0,k_1}$$

$$= \sum_{k_0, k_1 = 1}^{2|n|+1} \delta_{k_1,k_0} q^{-2k_0+2} t^{|n|}_{|n|-k_0+1,n} t^{|n|*}_{|n|-k_1+1,n}.$$

Next we apply the formulae for adjoints, $(t^l_{i,j})^* = (-q)^{j-i} t^l_{-i,-j}$ and $t^l_{i,j} = (-q)^{j-i}(t^l_{-i,-j})^*$, along with the unitary relations for the Peter–Weyl basis elements,

[KS], Proposition 16, Chapter 4, to obtain

$$\mathrm{Ch}_0([P_n, V_n]) = \sum_{k=0}^{2|n|} q^{-2k} q^{2(n-|n|+k)} t_{k-|n|,-n}^{|n|*} \, t_{k-|n|,-n}^{|n|} = q^{2(n-|n|)} I.$$

Finally,

$$\mathrm{Ch}_2([P_n, V_n])$$

$$= -\frac{2!}{1!} \sum_{k_0,k_1,k_2,k_3=1}^{2|n|+1} (V_{n,-i})_{k_3,k_0} (P_n - \tfrac{1}{2})_{k_0,k_1} \otimes (P_n)_{k_1,k_2} \otimes (P_n)_{k_2,k_3}$$

$$= -2 \sum_{k_0,k_1,k_2=1}^{2|n|+1} (V_{n,-i})_{k_0,k_0} (P_n - \tfrac{1}{2})_{k_0,k_1} \otimes (P_n)_{k_1,k_2} \otimes (P_n)_{k_2,k_0}$$

$$= -2 \sum_{k_0,k_1,k_2=0}^{2|n|} q^{-2k_0} (t_{|n|-k_0,n}^{|n|} t_{|n|-k_1,n}^{|n|*} - \tfrac{1}{2}\delta_{k_0,k_1})$$

$$\otimes t_{|n|-k_1,n}^{|n|} t_{|n|-k_2,n}^{|n|*} \otimes t_{|n|-k_2,n}^{|n|} t_{|n|-k_0,n}^{|n|*}. \qquad \square$$

4.4. The index pairing. The resolvent index formula established in Section 3.3 demonstrates that the index pairing defined by the modular spectral triple $(\mathcal{B}, \mathcal{H}, \mathcal{D}, \mathcal{B}(\mathcal{H}), \Psi_R)$ and the equivariant K-theory class defined by the projection P_n is given by the formula

$$\mathrm{Ind}_{\Psi_R \otimes G}(P_n(\mathcal{D} \otimes \mathrm{Id}_{2|n|+1})^+ P_n) = \phi_2(\mathrm{Ch}_2([P_n, V_n])) + \phi_0(\mathrm{Ch}_0([P_n, V_n])).$$

Now that we have explicit formulae for the cocycle (ϕ_0, ϕ_2) and the cycle $\mathrm{Ch}_*([P_n, V_n])$, the computation is straightforward.

Proposition 4.7. *The evaluation of ϕ_2 on $\mathrm{Ch}_2([P_n, V_n])$ is*

$$\phi_2(\mathrm{Ch}_2([P_n, V_n])) = q^{-2|n|}[2n]_q.$$

Proof. Recalling the formula for ϕ_2 from eq. (4.6) and from Lemma 4.6 the expression for $\mathrm{Ch}_2([P_n, V_n])$, we compute

$$\varepsilon((t_{|n|-k_0,n}^{|n|} t_{|n|-k_1,n}^{|n|*} - \tfrac{1}{2}\delta_{k_0,k_1}) \partial_e (t_{|n|-k_1,n}^{|n|} t_{|n|-k_2,n}^{|n|*}) \partial_f (t_{|n|-k_2,n}^{|n|} t_{|n|-k_0,n}^{|n|*}))$$

$$= (\delta_{|n|-k_0,n}\delta_{|n|-k_1,n} - \tfrac{1}{2}\delta_{k_0,k_1}) \varepsilon(\partial_e (t_{|n|-k_1,n}^{|n|} t_{|n|-k_2,n}^{|n|*})) \varepsilon(\partial_f (t_{|n|-k_2,n}^{|n|} t_{|n|-k_0,n}^{|n|*}))$$

$$= \delta_{k_0,k_1} (\delta_{k_0,|n|-n} - \tfrac{1}{2}) \varepsilon(\partial_e (t_{|n|-k_1,n}^{|n|} t_{|n|-k_2,n}^{|n|*})) \varepsilon(\partial_f (t_{|n|-k_2,n}^{|n|} t_{|n|-k_1,n}^{|n|*})).$$

We observe that this expression is zero for the case $n = 0$, because $\partial_e(I) = \partial_f(I) = 0$. So for the remainder we consider only nonzero n. Observe that $t_{|n|-k_2,n}^{|n|} t_{|n|-k_1,n}^{|n|*} = (t_{|n|-k_1,n}^{|n|} t_{|n|-k_2,n}^{|n|*})^*$. Now we use the property that $(g \triangleright \alpha)^* =$

$S(g)^* \rhd \alpha^*$ for all $g \in \mathcal{U}_q(su_2)$ and $\alpha \in \mathcal{A}$, so that $\varepsilon(\partial_e(\alpha^*)) = -q\varepsilon(\partial_f(\alpha)^*)$ and $\varepsilon(\partial_f(\alpha^*)) = -q^{-1}\varepsilon(\partial_e(\alpha)^*)$. Then

$$\varepsilon((t^{|n|}_{|n|-k_0,n}t^{|n|*}_{|n|-k_1,n} - \tfrac{1}{2}\delta_{k_0,k_1})\partial_e(t^{|n|}_{|n|-k_1,n}t^{|n|*}_{|n|-k_2,n})\partial_f(t^{|n|}_{|n|-k_2,n}t^{|n|*}_{|n|-k_0,n}))$$
$$= -q^{-1}\delta_{k_0,k_1}(\delta_{k_0,|n|-n} - \tfrac{1}{2})\varepsilon(\partial_e(t^{|n|}_{|n|-k_1,n}t^{|n|*}_{|n|-k_2,n}))^2,$$

and similarly

$$\varepsilon((t^{|n|}_{|n|-k_0,n}t^{|n|*}_{|n|-k_1,n} - \tfrac{1}{2}\delta_{k_0,k_1})\partial_f(t^{|n|}_{|n|-k_1,n}t^{|n|*}_{|n|-k_2,n})\partial_e(t^{|n|}_{|n|-k_2,n}t^{|n|*}_{|n|-k_0,n}))$$
$$= -q^{-1}\delta_{k_0,k_1}(\delta_{k_0,|n|-n} - \tfrac{1}{2})\varepsilon(\partial_e(t^{|n|}_{|n|-k_2,n}t^{|n|*}_{|n|-k_1,n}))^2.$$

Using the twisted derivation property of ∂_e on $\mathcal{A}$, we find, for $r, s \in \{0, \ldots, 2|n|\}$,

$$\varepsilon(\partial_e(t^{|n|}_{|n|-r,n}t^{|n|*}_{|n|-s,n}))^2$$
$$= \varepsilon(\partial_e(t^{|n|}_{|n|-r,n})\partial_k(t^{|n|*}_{|n|-s,n}) + \partial_k^{-1}(t^{|n|}_{|n|-r,n})\partial_e(t^{|n|*}_{|n|-s,n}))^2$$
$$= q^{-2n}(\varepsilon(\partial_e(t^{|n|}_{|n|-r,n}))\delta_{s,|n|-n} + \delta_{r,|n|-n}\varepsilon(\partial_e(t^{|n|*}_{|n|-s,n})))^2$$
$$= q^{-2n}(\varepsilon(\partial_e(t^{|n|}_{|n|-r,n}))\delta_{s,|n|-n} - q\delta_{r,|n|-n}\varepsilon(\partial_f(t^{|n|}_{|n|-s,n})))^2$$
$$= q^{-2n}(\varepsilon(\kappa^{|n|}_{n+1}t^{|n|}_{|n|-r,n+1})\delta_{s,|n|-n} - q\delta_{r,|n|-n}\varepsilon(\kappa^{|n|}_n t^{|n|}_{|n|-s,n-1}))^2$$
$$= q^{-2n}(\kappa^{|n|}_{n+1}\delta_{r,|n|-n-1}\delta_{s,|n|-n} - q\kappa^{|n|}_n\delta_{r,|n|-n}\delta_{s,|n|-n+1})^2$$
$$= q^{-2n}((\kappa^{|n|}_{n+1})^2\delta_{r,|n|-n-1}\delta_{s,|n|-n} + q^2(\kappa^{|n|}_n)^2\delta_{r,|n|-n}\delta_{s,|n|-n+1}),$$

where $\kappa^l_j = ([l + j]_q[l - j + 1]_q)^{\frac{1}{2}}$. Combining these results with the formula for $\mathrm{Res}_{r=-\frac{1}{2}} \phi_2^r$ and the expression for $\mathrm{Ch}_2([P_n, V])$ gives

$$\phi_2(\mathrm{Ch}_2([P_n, V_n]))$$

$$= \frac{-2}{2(q^{-1} - q)\ln q^{-1}} \sum_{k_0,k_1,k_2=0}^{2|n|} q^{-2k_0}(-q^{-1}\delta_{k_0,k_1}(\delta_{k_0,|n|-n} - \tfrac{1}{2}))$$
$$\cdot ((q^{-2} - 1 - \ln q^{-2})\varepsilon(\partial_e(t^{|n|}_{|n|-k_1,n}t^{|n|*}_{|n|-k_2,n}))^2$$
$$- (q^2 - 1 - \ln q^2)\varepsilon(\partial_e(t^{|n|}_{|n|-k_2,n}t^{|n|*}_{|n|-k_1,n}))^2)$$

$$= \frac{q^{-1}}{(q^{-1} - q)\ln q^{-1}} \sum_{k_1,k_2=0}^{2|n|} q^{-2k_1}(\delta_{k_1,|n|-n} - \tfrac{1}{2})q^{-2n}$$
$$\cdot ((q^{-2} - 1 - \ln q^{-2})((\kappa^{|n|}_{n+1})^2\delta_{k_1,|n|-n-1}\delta_{k_2,|n|-n}$$
$$+ q^2(\kappa^{|n|}_n)^2\delta_{k_1,|n|-n}\delta_{k_2,|n|-n+1})$$
$$- (q^2 - 1 - \ln q^2)((\kappa^{|n|}_{n+1})^2\delta_{k_2,|n|-n-1}\delta_{k_1,|n|-n}$$
$$+ q^2(\kappa^{|n|}_n)^2\delta_{k_2,|n|-n}\delta_{k_1,|n|-n+1})).$$

Using $(\delta_{k_1,|n|-n} - \frac{1}{2})\delta_{k_1,|n|-n} = \frac{1}{2}\delta_{k_1,|n|-n}$ and $(\delta_{k_1,|n|-n} - \frac{1}{2})\delta_{k_1,|n|-n\pm 1} = -\frac{1}{2}\delta_{k_1,|n|-n\pm 1}$ yields

$$\operatorname{Res}_{r=-\frac{1}{2}} \phi_2^r(\operatorname{Ch}_2([P_n, V_n]))$$

$$= \frac{q^{-1}}{(q^{-1} - q)\ln q^{-1}} \sum_{k_1,k_2=0}^{2|n|} \frac{1}{2}q^{-2k_1-2n}((q^{-2} - 1 - \ln q^{-2})(-(\kappa_{n+1}^{|n|})^2$$

$$\cdot \delta_{k_1,|n|-n-1}\delta_{k_2,|n|-n} + q^2(\kappa_n^{|n|})^2\delta_{k_1,|n|-n}\delta_{k_2,|n|-n+1})$$

$$- (q^2 - 1 - \ln q^2)((\kappa_{n+1}^{|n|})^2\delta_{k_1,|n|-n}\delta_{k_2,|n|-n-1}$$

$$- q^2(\kappa_n^{|n|})^2\delta_{k_1,|n|-n+1}\delta_{k_2,|n|-n})).$$

We can reduce the different summations over k_1 and k_2 down to two distinct sums, either

$$\sum_{k=0}^{2|n|} \delta_{k,|n|-n-1} = \delta_{n,-|n|} \quad \text{or} \quad \sum_{k=0}^{2|n|} \delta_{k,|n|-n+1} = \delta_{n,|n|}.$$

Hence

$$\phi_2(\operatorname{Ch}_2([P_n, V_n]))$$

$$= \frac{q^{-1}}{2(q^{-1} - q)\ln q^{-1}}((q^{-2} - 1 - \ln q^{-2})(-(\kappa_{n+1}^{|n|})^2\delta_{n,-|n|}q^{-2(|n|-n-1)-2n}$$

$$+ q^2(\kappa_n^{|n|})^2\delta_{n,|n|}q^{-2(|n|-n)-2n}) - (q^2 - 1 - \ln q^2)((\kappa_{n+1}^{|n|})^2\delta_{n,-|n|}$$

$$\cdot q^{-2(|n|-n)-2n} - q^2(\kappa_n^{|n|})^2\delta_{n,|n|}q^{-2(|n|-n+1)-2n}))$$

$$= \frac{q^{-1}}{2(q^{-1} - q)\ln q^{-1}}((q^{-2} - 1 - \ln q^{-2})(-(\kappa_{1-|n|}^{|n|})^2\delta_{n,-|n|}q^{-2|n|+2}$$

$$+ q^2(\kappa_{|n|}^{|n|})^2\delta_{n,|n|}q^{-2|n|}) - (q^2 - 1 - \ln q^2)((\kappa_{1-|n|}^{|n|})^2\delta_{n,-|n|}q^{-2|n|}$$

$$- q^2(\kappa_{|n|}^{|n|})^2\delta_{n,|n|}q^{-2|n|-2})).$$

Observe that $(\kappa_{1-|n|}^{|n|})^2 = (\kappa_{|n|}^{|n|})^2 = [2|n|]_q$ as $[1]_q = 1$, and so

$$\phi_2(\operatorname{Ch}_2([P_n, V_n]))$$

$$= \frac{q^{-1}}{2(q^{-1} - q)\ln q^{-1}}[2|n|]_q q^{-2|n|}$$

$$\cdot ((q^{-2} - 1 - \ln q^{-2})q^2(\delta_{n,|n|} - \delta_{n,-|n|}) - (q^2 - 1 - \ln q^2)(\delta_{n,-|n|} - \delta_{n,|n|}))$$

$$= \frac{q^{-2|n|-1}[2|n|]_q}{2(q^{-1} - q)\ln q^{-1}}(\delta_{n,|n|} - \delta_{n,-|n|})((q^{-2} - 1 - \ln q^{-2})q^2 + (q^2 - 1 - \ln q^2))$$

$$= \frac{q^{-2|n|-1}[2|n|]_q}{2(q^{-1} - q)\ln q^{-1}}(\delta_{n,|n|} - \delta_{n,-|n|})(-q^2 \ln q^{-2} - \ln q^2)$$

$$= \frac{q^{-2|n|-1}[2|n|]_q}{2(q^{-1}-q)\ln q^{-1}}(\delta_{n,|n|} - \delta_{n,-|n|})(1-q^2)\ln q^{-2}$$
$$= q^{-2|n|}[2|n|]_q(\delta_{n,|n|} - \delta_{n,-|n|}).$$

Considering $n \neq 0$, then $(\delta_{n,|n|} - \delta_{n,-|n|}) = \text{sgn}(n)$ and $\text{sgn}(n)[2|n|]_q = [2n]_q$. As $[0]_q = 0$, then for all $n \in \frac{1}{2}\mathbb{Z}$ we have

$$\phi_2(\text{Ch}_2([P_n, V_n])) = q^{-2|n|}[2n]_q. \qquad \square$$

We can now write down the index pairing and compute the classical limit as $q \to 1$.

Theorem 4.8. *For $N \in \mathbb{Z}$, the index pairing of the modular spectral triple $(\mathcal{B}, \mathcal{H}, \mathcal{D}, \mathcal{B}(\mathcal{H}), \Psi_R)$ with the equivariant projections $P_{N/2}$ is*

$$\text{Ind}(P_{N/2}(\mathcal{D} \otimes \text{Id}_{|N|+1})^+ P_{N/2}) = q^{-|N|}[N]_q.$$

The classical limit of the index as $q \to 1$ is

$$\lim_{q \to 1} \text{Ind}(P_{N/2}(\mathcal{D} \otimes \text{Id}_{|N|+1})^+ P_{N/2}) = N.$$

Proof. First, the degree zero contribution is $\phi_0(\text{Ch}_0([P_{N/2}, V_{N/2}])) = 0$. This follows from $\phi_0(I) = 0$, and from Lemma 4.6, which gives $\text{Ch}_0([P_{N/2}, V_{N/2}]) = q^{(N-|N|)}I$. Thus the index pairing is computed just with the degree 2 part, which comes from Proposition 4.7. To compute the classical limit of the index we recall that $\lim_{q \to 1}[N]_q = N$ (see for example [KS]). $\qquad \square$

We conclude with some comments on how the index depends on the representation in the equivariant K-theory class. For any $\mathbb{T}$-algebra A, the group $K_*^{\mathbb{T}}(A)$ is a module over the group ring $\mathbb{Z}[\mathbb{T}]$. This allows us to hold the projection or unitary constant, but vary the representation. This does change the equivariant K-theory class, and this can be detected by the index pairing.

In particular, if the V_n used here is rescaled by $q^{2|n|it}$ to obtain a new representation V_n', then the index is likewise rescaled by $q^{2|n|}$, and we obtain an index of $[2n]_q$. This observation helps in comparing our results with other authors' results, particularly Wagner's, [W]. We thank the referee for pointing this out to us.

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