

The groupoid C^* -algebra of a rational map

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Abstract. This paper contains a quite detailed description of the C^* -algebra arising from the transformation groupoid of a rational map of degree at least two on the Riemann sphere. The algebra is decomposed stepwise via extensions of familiar C^* -algebras whose nature depend on the structure of the Julia set and the stable regions in the Fatou set, as well as on the behaviour of the critical points.

Mathematics Subject Classification (2010). 58B34, 46L85; 34A10, 26C15.

Keywords. C^* -algebra, groupoid, holomorphic dynamics, rational map.

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1. Introduction and presentation of results

In noncommutative geometry it is a basic principle, referred to as Connes' dictum in [Kh], that a quotient space should be replaced with a noncommutative algebra, preferably a C^* -algebra, in the cases where the topology of the quotient is ill-behaved. Following this dictum the procedure should go via an intermediate step which first produces a groupoid, and the noncommutative algebra should then arise as the convolution algebra of the groupoid. The structure of the resulting noncommutative algebra offers to compensate for the poor topology which the quotient very often is equipped with, and at the same time it encodes the equivalence relation defining the quotient space which is otherwise lost.

A standard example of the construction is the classical crossed product arising from a group acting on a locally compact Hausdorff space which in this picture is a noncommutative substitute for the quotient of the space under the orbit equivalence relation given by the action. There are several other examples of this type of construction arising from dynamical systems. The C^* -algebras introduced by D. Ruelle in [Ru] are noncommutative algebras representing the quotient space under the homoclinic equivalence relation arising from a hyperbolic homeomorphism, while the extension of Ruelle's approach by I. Putnam ([Pu]) also allows to consider the quotient by the heteroclinic equivalence relations of the same type of dynamics. The full orbit equivalence relation arising from a non-invertible continuous self map can also serve as input when the map is locally injective. For local homeomorphisms the construction was developed in stages by J. Renault ([Re]), V. Deaconu ([De]) and C. Anantharaman-Delaroche ([An]), while the extension to locally injective maps was carried out in [Th1]. In all cases, including the work of Ruelle and Putnam, a major problem is to equip the natural groupoid with a sufficiently nice topology which allows the construction of the convolution C^* -algebra. The best one can hope for is to turn the groupoid into a locally compact Hausdorff groupoid in such a way that the range map becomes a local homeomorphism. In this case the groupoid is said to be étale. These crucial properties come relatively cheap for the transformation groupoid of Renault, Deaconu and Anantharaman-Delaroche, while it is harder to obtain them for the groupoids in Putnam's construction ([PS]). In a recent work ([Th2]) it was shown that it is possible to formulate the definition of the transformation groupoid of a homeomorphism or a local homeomorphism in such a way that it not only makes sense for a larger class of continuous maps, but also retains the structure of a locally compact second countable Hausdorff groupoid with the important étale property. This class of continuous maps includes the non-constant holomorphic self-maps of a Riemann surface and for such maps it was shown in [Th2] that the convolution C^* -algebra of the transformation groupoid obtained in this way is equipped with a one-parameter group of automorphisms for which the KMS-states correspond to the conformal measures introduced in complex dynamics by D. Sullivan. Applied to a particular class of quadratic maps the result was systems for which the KMS states exhibit phase transition with spontaneous symmetry breaking in the sense of Bost and

Connes. This illustrates one application of Connes' dictum, obtaining new models in quantum statistical mechanics. In [Th2] the focus was on the one-parameter action with its KMS states, and the structure of the C^* -algebras carrying the action was not investigated. It is the purpose of the present paper to present a relatively detailed description of these C^* -algebras $C_r^*(R)$ when they arise from a rational map R of degree at least two acting on the Riemann sphere $\bar{\mathbb{C}}$. It is well known that the dynamics of such a map is highly complicated, exhibiting features that are both beautiful and fascinating. As we try to show here, the structure of the associated C^* -algebra is no less fascinating, although the appreciation of it may require a somewhat more specialized background of the observer than what is needed to admire the colorful pictures used to depict the dynamics of the maps.

The dynamics of a rational map is partitioned by two totally invariant subsets; the Julia set on which the map behaves chaotically under iteration and the Fatou set on which its iterates form an equicontinuous family. As one would expect from familiarity with crossed products, this division gives rise to a decomposition of $C_r^*(R)$ as an extension where the Fatou set, as the open subset, gives rise to an ideal $C_r^*(F_R)$ and the Julia set, as the closed subset, represents the corresponding quotient $C_r^*(J_R)$. Thus the first decomposition of $C_r^*(R)$ is given by an extension

$$0 \rightarrow C_r^*(F_R) \rightarrow C_r^*(R) \rightarrow C_r^*(J_R) \rightarrow 0$$

which reflects the partitioning of the $\bar{\mathbb{C}}$ by the Julia and Fatou sets. The two C^* -algebras $C_r^*(J_R)$ and $C_r^*(F_R)$ in this extension are of very different nature. The C^* -algebra $C_r^*(J_R)$ of the Julia set is always purely infinite, nuclear and satisfies the universal coefficient theorem (UCT) of Rosenberg and Schochet ([RS]), and it is often, but not always simple. Ideals in $C_r^*(J_R)$ arise from the possible presence of finite subsets of the Julia set invariant under the equivalence relation represented by the transformation groupoid which produces the C^* -algebra $C_r^*(R)$; we call this relation 'restricted orbit equivalence' and it is a relation which is slightly stronger than orbit equivalence. The finite subsets of the Julia set invariant under restricted orbit equivalence comprise the finite subsets considered by Makarov and Smirnov in their work on phase transition in the thermodynamic formalism associated to the dynamics ([MS1], [MS2]), and they are closely related to, but not identical with the subsets introduced in [GPRR] in connection with work on exceptional rational maps. The possible presence of such subsets of the Julia set implies that in general the structure of $C_r^*(J_R)$ must be decoded from an extension of the form

$$0 \rightarrow C_r^*(J_R \setminus E_R) \rightarrow C_r^*(J_R) \rightarrow B \rightarrow 0,$$

where $C_r^*(J_R \setminus E_R)$ is purely infinite and simple, while B is a finite direct sum of algebras of the form $M_n(\mathbb{C})$ for some $n \leq 3$ or $C(\mathbb{T}) \otimes M_n(\mathbb{C})$ for some $n \leq 4$.

In contrast to $C_r^*(J_R)$ the C^* -algebra $C_r^*(F_R)$ of the Fatou set is finite, and its ideal structure is typically much more complex than that of $C_r^*(J_R)$. This is partly due to the fact that the Fatou set is partitioned into classes of connected components,

the so-called stable regions which are termed super-attracting, attracting, parabolic, Siegel and Herman regions according to the asymptotic behaviour of their elements under iteration. This division of F_R results in a direct sum decomposition of the ideal $C_r^*(F_R)$ where each direct summand is further decomposed as an extension where the structure of the ideal depends on the type of the stable region and where the nature of the quotient is governed by the presence or absence of critical and periodic points in the region. Specifically,

$$C_r^*(F_R) = \bigoplus_{i=1}^N C_r^*(\Omega_i),$$

where $C_r^*(\Omega_i)$ is the C^* -algebra obtained by restricting attention to the stable region $\Omega_i \subseteq F_R$. It turns out that the nature of the algebra $C_r^*(\Omega_i)$ varies with the type of the stable region: If Ω_i is super-attractive, there is an extension

$$0 \rightarrow \mathbb{K} \otimes \text{MT}_d \rightarrow C_r^*(\Omega_i) \rightarrow B \rightarrow 0,$$

where B is a finite direct sum of algebras stably isomorphic to either \mathbb{C} or the continuous functions on the Cantor set. The algebra MT_d is the mapping torus of an endomorphism on a Bunce–Deddens algebra of type d^∞ where d is the product of the valencies of the elements in the critical orbit.

If Ω_i is attractive, there is an extension

$$0 \rightarrow \mathbb{K} \otimes C(\mathbb{T}^2) \rightarrow C_r^*(\Omega_i) \rightarrow B \rightarrow 0,$$

where B is a finite direct sum of algebras stably isomorphic to either \mathbb{C} or the continuous functions on the circle \mathbb{T} .

If Ω_i is parabolic, there is an extension

$$0 \rightarrow \mathbb{K} \otimes C(\mathbb{T}) \otimes C_0(\mathbb{R}) \rightarrow C_r^*(\Omega_i) \rightarrow B \rightarrow 0,$$

where B is a finite direct sum of algebras stably isomorphic to \mathbb{C} .

If Ω_i is of Siegel type, there is an extension

$$0 \rightarrow \mathbb{K} \otimes C_0(\mathbb{R}) \otimes A_\theta \rightarrow C_r^*(\Omega_i) \rightarrow B \rightarrow 0, \quad (1)$$

where B is a finite direct sum of algebras stably isomorphic to either \mathbb{C} or the continuous functions on the circle \mathbb{T} , and A_θ is the irrational rotation algebra corresponding to the angle of rotation in the Siegel domain inside Ω_i .

Finally, if Ω_i is of Herman type, there is an extension quite similar to (1). The only difference is that while the quotient algebra B must contain a summand stably isomorphic to $C(\mathbb{T})$ in the Siegel case, in the Herman case all summands are stably isomorphic to \mathbb{C} .

It almost goes without saying that the entire structure in the decomposition of $C_r^*(R)$ described above reflects easily identified structures of the dynamics of R . For

example, the difference between the structure of the summands in $C_r^*(F_R)$ coming from a Siegel and Herman region is due to the periodic point in a Siegel domain which is absent in a Herman ring.

From the point of view of operator algebra theory a study of a non-simple C^* -algebra often begins with a description of the primitive ideals and the corresponding irreducible quotients. In [CT] T. Carlsen and the author identified the primitive and maximal ideals of the C^* -algebras arising from the transformation groupoid of a locally injective surjection on a finite dimensional compact metric space. In the final section of the present paper the method from [CT] is carried over to the groupoid C^* -algebras of rational maps and we obtain in this way a description of the primitive ideals and primitive quotients. In particular, it is shown that the primitive ideal space of $C_r^*(R)$ is only Hausdorff in the hull-kernel topology when it has to be, i.e., when $C_r^*(R)$ is simple. This occurs only when $J_R = \overline{C}$ and there are no finite sets invariant under restricted orbit equivalence. In all other cases the primitive ideal space is not even T_0 .

While there is often a rich variety of primitive quotients, there are always very few types of simple quotients. The finite invariant subsets under restricted orbit equivalence give rise to maximal ideals, but the corresponding simple quotients are matrix algebras of size no more than 4. In most cases $C_r^*(J_R)$ is also a simple quotient, but only when there are no finite subsets of J_R invariant under restricted orbit equivalence. There are no other simple quotients. In particular, when there are finite subsets of J_R invariant under restricted orbit equivalence the only simple quotients of $C_r^*(R)$ are matrix algebras of size not exceeding 4.

There are other ways to associate a C^* -algebra to a rational map, and we refer to [DM] and [KW] for these. It would be interesting to find the precise relationship between the algebras investigated here and those of Kajiwara and Watatani. Presently it is only clear that they are generally very different.

Acknowledgement. This work was completed during a visit to the Institut Henri Poincaré as part of the Research in Paris program, and I take the opportunity to thank the IHP for support and for the exceptional working conditions.

2. Étale groupoids and C^* -algebras from dynamical systems

Let G be an étale second countable locally compact Hausdorff groupoid with unit space $G^{(0)}$. Let $r: G \rightarrow G^{(0)}$ and $s: G \rightarrow G^{(0)}$ be the range and source maps, respectively. For $x \in G^{(0)}$ put $G^x = r^{-1}(x)$, $G_x = s^{-1}(x)$ and $Is_x = s^{-1}(x) \cap r^{-1}(x)$. Note that Is_x is a group, the *isotropy group* at x . The space $C_c(G)$ of continuous compactly supported functions is a $*$ -algebra if the product is defined by

$$(f_1 f_2)(g) = \sum_{h \in G^{r(g)}} f_1(h) f_2(h^{-1}g)$$

and the involution by $f^*(g) = \overline{f(g^{-1})}$. Let $x \in G^{(0)}$. There is a representation π_x of $C_c(G)$ on the Hilbert space $l^2(G_x)$ of square-summable functions on G_x given by

$$\pi_x(f)\psi(g) = \sum_{h \in G^r(g)} f(h)\psi(h^{-1}g).$$

The *reduced groupoid C^* -algebra* $G_r^*(G)$ is the completion of $C_c(G)$ with respect to the norm

$$\|f\| = \sup_{x \in G^{(0)}} \|\pi_x(f)\|.$$

2.0.1. Stability of $C_r^*(G)$. We shall need the following sufficient condition for stability of $C_r^*(G)$. Recall that a C^* -algebra A is *stable* if $A \otimes \mathbb{K} \simeq A$, where \mathbb{K} denotes the C^* -algebra of compact operators on an infinite dimensional separable Hilbert space. Recall also that a *bi-section* in G is an open subset $U \subseteq G$ such that $r : U \rightarrow G^{(0)}$ and $s : U \rightarrow G^{(0)}$ both are injective.

Lemma 2.1. *Let G be a locally compact second countable étale groupoid. Assume that for every compact subset $K \subseteq G^{(0)}$ there is a finite collection $\{U_i\}_{i=1}^N$ of bi-sections in G such that*

- (i) $K \subseteq \bigcup_{i=1}^N s(U_i)$,
- (ii) $r(U_i) \cap r(U_j) = \emptyset$ if $i \neq j$, and
- (iii) $K \cap \bigcup_{i=1}^N r(U_i) = \emptyset$.

It follows that $C_r^(G)$ is a stable C^* -algebra.*

Proof. By Theorem 2.1 and (b) of Proposition 2.2 in [HR] it suffices to consider a positive element $a \in C_r^*(G)$ and an $\varepsilon > 0$ and show that there is an element $v \in C_r^*(G)$ such that $\|v^*v - a\| \leq \varepsilon$ and $v^2 = 0$. Write $a = a_0^*a_0$ for some $a_0 \in C_r^*(G)$. By approximating a_0 with an element $h \in C_c(G)$ and taking $b = h^*h$ we obtain an element $b \in C_c(G)$ which is positive in $C_r^*(G)$ and satisfies that $\|a - b\| \leq \varepsilon$. Let $\text{supp } b$ be the support of b in G and set $K = r(\text{supp } b)$. By assumption there is a finite collection $\{U_i\}_{i=1}^N$ of bi-sections such that (i)–(iii) hold. Let $\{h_i\} \subseteq C_c(G^{(0)})$ be a partition of unity on K subordinate to $\{s(U_i)\}_{i=1}^N$. For each i let $f_i \in C_c(G)$ be supported in U_i and satisfy $f_i = 1$ on $s^{-1}(\text{supp } h_i) \cap U_i$. Set $w = \sum_{i=1}^N f_i \sqrt{h_i} \in C_c(G)$ and note that $w^*wb = \sum_i h_i b = b$ while $bw = 0$. Set $v = w\sqrt{b}$. □

2.0.2. G -orbits and reductions. If W is a subset of $G^{(0)}$, we set

$$G_W = \{g \in G \mid r(g), s(g) \in W\},$$

which is a subgroupoid of G called a *reduction* of G . If W is an open subset of G , the reduction G_W will be an étale groupoid in the relative topology inherited from

G and there is an embedding $C_r^*(G_W) \subseteq C_r^*(G)$, cf., e.g., Proposition 1.9 in [Ph]. In fact, $C_r^*(G_W)$ is a hereditary C^* -subalgebra of $C_r^*(G)$. If $x \in G^{(0)}$, the set $Gx = \{r(g) \mid g \in G_x\}$ will be called the G -orbit of x . We say that $W \subseteq G^{(0)}$ is G -invariant if $x \in W$ implies $Gx \subseteq W$. If W is G -invariant and locally compact in the topology inherited from $G^{(0)}$, the reduction G_W is an étale locally compact groupoid in the topology inherited from G . If W is both open and G -invariant $C_r^*(G_W)$ is an ideal in $C_r^*(G)$. Similarly, if F is a closed subset of $G^{(0)}$ which is also G -invariant, then $C_r^*(G_F)$ is a quotient of $C_r^*(G)$. It is known that under a suitable amenability condition the kernel of the quotient map

$$\pi_F : C_r^*(G) \rightarrow C_r^*(G_F)$$

is $C_r^*(G_{G^{(0)} \setminus F})$. We shall avoid the amenability issue here and prove this equality directly in the cases we are interested in. See Section 3.

We shall need the following fact which follows straightforwardly from the definitions.

Lemma 2.2. *Assume that there is a finite partition $G^{(0)} = \bigsqcup_{i=1}^n W_i$ such that each W_i is open and G -invariant. It follows that*

$$C_r^*(G) \simeq \bigoplus_{i=1}^n C_r^*(G_{W_i}).$$

The following result was obtained by Muhly, Renault and Williams in [MRW].

Theorem 2.3. *Let G be an étale second countable locally compact Hausdorff groupoid and $W \subseteq G^{(0)}$ an open subset such that $Gx \cap W \neq \emptyset$ for all $x \in G^{(0)}$. It follows that $C_r^*(G)$ is stably isomorphic to $C_r^*(G_W)$.*

Proof. The set $\bigcup_{x \in W} G_x$ is a (G, G_W) -equivalence in the sense of [MRW] and hence Theorem 2.8 of [MRW] applies. □

2.0.3. Pure infiniteness of $C_r^*(G)$. Following [An] we say that an étale groupoid G is *essentially free* when the points x of the unit space $G^{(0)}$ for which the isotropy group Is_x is trivial (i.e., only consists of $\{x\}$) is dense in $G^{(0)}$. In the same vein we say that G is *locally contracting* if every open non-empty subset of $G^{(0)}$ contains an open subset V with the property that there is an open bisection S in G such that $\bar{V} \subseteq s(S)$ and $\alpha_S^{-1}(\bar{V}) \subsetneq V$ when $\alpha_S : r(S) \rightarrow s(S)$ is the homeomorphism defined such that $\alpha_S(x) = s(g)$, where $g \in S$ is the unique element with $r(g) = x$, cf. Definition 2.1 of [An] (but note that the source map is denoted by d in [An]).

We say that a C^* -algebra is *purely infinite* if every non-zero hereditary C^* -subalgebra of A contains an infinite projection. Proposition 2.4 of [An] then says the following.

Theorem 2.4. *Suppose that G be an étale second countable locally compact Hausdorff groupoid. Assume that G is essentially free and locally contracting. Then $C_r^*(G)$ is purely infinite.*

2.1. The transformation groupoid of a local homeomorphism. In this section we describe the construction of an étale second countable locally compact Hausdorff groupoid from a local homeomorphism of a locally compact Hausdorff space which was introduced in increasing generality by J. Renault [Re], V. Deaconu [De] and C. Anantharaman-Delaroche [An].

Let X be a second countable locally compact Hausdorff space and $\varphi: X \rightarrow X$ a local homeomorphism. Thus we assume that φ is open and locally injective, but not necessarily surjective. Set

$$G_\varphi = \{(x, k, y) \in X \times \mathbb{Z} \times X \mid \exists n, m \in \mathbb{N}, k = n - m, \varphi^n(x) = \varphi^m(y)\}.$$

This is a groupoid with the set of composable pairs being

$$G_\varphi^{(2)} = \{((x, k, y), (x', k', y')) \in G_\varphi \times G_\varphi \mid y = x'\}.$$

The multiplication and inversion are given by

$$(x, k, y)(y, k', y') = (x, k + k', y') \quad \text{and} \quad (x, k, y)^{-1} = (y, -k, x).$$

Note that the unit space of G_φ can be identified with X via the map $x \mapsto (x, 0, x)$. Under this identification the range map $r: G_\varphi \rightarrow X$ is the projection $r(x, k, y) = x$ and the source map the projection $s(x, k, y) = y$.

To turn G_φ into a locally compact topological groupoid, fix $k \in \mathbb{Z}$. For each $n \in \mathbb{N}$ such that $n + k \geq 0$, set

$$G_\varphi(k, n) = \{(x, l, y) \in X \times \mathbb{Z} \times X \mid l = k, \varphi^{k+n}(x) = \varphi^n(y)\}.$$

This is a closed subset of the topological product $X \times \mathbb{Z} \times X$ and hence a locally compact Hausdorff space in the relative topology. Since φ is locally injective, $G_\varphi(k, n)$ is an open subset of $G_\varphi(k, n + 1)$ and hence the union

$$G_\varphi(k) = \bigcup_{n \geq -k} G_\varphi(k, n)$$

is a locally compact Hausdorff space in the inductive limit topology. The disjoint union

$$G_\varphi = \bigcup_{k \in \mathbb{Z}} G_\varphi(k)$$

is then a locally compact Hausdorff space in the topology where each $G_\varphi(k)$ is an open and closed set. In fact, as is easily verified, G_φ is an étale groupoid, i.e., the range and source maps are local homeomorphisms. The groupoid G_φ will be called

the *transformation groupoid* of φ . To simplify notation we denote in the following the corresponding C^* -algebra by $C_r^*(\varphi)$; i.e., we set

$$C_r^*(G_\varphi) = C_r^*(\varphi).$$

Note that the G_φ -orbit $G_\varphi x$ of a point $x \in X$ is the *orbit* of x under φ , i.e.,

$$G_\varphi x = \bigcup_{n,m \in \mathbb{N}} \varphi^{-n}(\varphi^m(x)),$$

by some authors called the full or grand orbit to distinguish it from $\{\varphi^n(x) \mid n \in \mathbb{N}\}$, which we will call the *forward orbit*.

If the local homeomorphism φ is proper in the sense that inverse images of compact sets are compact, the C^* -algebra $C_r^*(\varphi)$ can be realised as the crossed product by an endomorphism, cf. [De],[An], in the following way: The subset

$$R_\varphi = G_\varphi(0) \simeq \{(x, y) \in X \times X \mid \varphi^n(x) = \varphi^n(y) \text{ for some } n \in \mathbb{N}\}$$

is an open subgroupoid of G_φ and an étale groupoid in itself. The reduced groupoid C^* -algebra $C_r^*(R_\varphi)$ is a C^* -subalgebra of $C_r^*(\varphi)$. If φ is proper, there is an endomorphism $\hat{\varphi}: C_r^*(R_\varphi) \rightarrow C_r^*(R_\varphi)$ defined such that

$$\hat{\varphi}(f)(x, y) = (\# \varphi^{-1}(\varphi(x)) \# \varphi^{-1}(\varphi(y)))^{-1/2} f(\varphi(x), \varphi(y))$$

when $f \in C_c(R_\varphi)$. We will refer to this endomorphism as *the Deaconu endomorphism*. As shown in [An], there is an isomorphism

$$C_r^*(\varphi) = C_r^*(R_\varphi) \rtimes_{\hat{\varphi}} \mathbb{N},$$

where the crossed product is a crossed product by an endomorphism both in the sense of Paschke ([P]) and the sense of Stacey ([St]).

2.2. The transformation groupoid of a rational map. In this section we describe an étale groupoid coming from a non-constant holomorphic map on a Riemann surface by a construction introduced in [Th2]. But since we shall focus on the Riemann sphere in this paper we restrict the presentation accordingly.

Let $\bar{\mathbb{C}}$ be the Riemann sphere and $R: \bar{\mathbb{C}} \rightarrow \bar{\mathbb{C}}$ a rational map of degree at least 2. Consider a subset $X \subseteq \bar{\mathbb{C}}$ which is locally compact in the topology inherited from $\bar{\mathbb{C}}$, without isolated points and *totally R -invariant* in the sense that $R^{-1}(X) = X$. Let \mathcal{P} be the pseudo-group on X of local homeomorphisms $\xi: U \rightarrow V$ between open subsets of X with the property that there are open subsets U_1, V_1 in $\bar{\mathbb{C}}$ and a bi-holomorphic map $\xi_1: U_1 \rightarrow V_1$ such that $U_1 \cap X = U, V_1 \cap X = V$ and $\xi = \xi_1$ on U . For each $k \in \mathbb{Z}$ we denote by $\mathcal{T}_k(X)$ the elements $\eta: U \rightarrow V$ of \mathcal{P} with the property that there are natural numbers n, m such that $k = n - m$ and

$$R^n(z) = R^m(\eta(z)) \quad \text{for all } z \in U.$$

The elements of $\mathcal{T} = \bigcup_{k \in \mathbb{Z}} \mathcal{T}_k(X)$ will be called *local transfers* for $R|_X$. We denote by $[\eta]_x$ the germ at a point $x \in X$ of an element $\eta \in \mathcal{T}$. Set

$$\mathcal{G}_X = \{(x, k, \eta, y) \in X \times \mathbb{Z} \times \mathcal{P} \times X \mid \eta \in \mathcal{T}_k(X), \eta(x) = y\}.$$

We define an equivalence relation \sim in \mathcal{G}_X such that $(x, k, \eta, y) \sim (x', k', \eta', y')$ if $x = x', y = y', k = k'$ and $[\eta]_x = [\eta']_x$. Let $[x, k, \eta, y]$ denote the equivalence class of $(x, k, \eta, y) \in \mathcal{G}_X$.¹ The quotient space $G_X = \mathcal{G}_X / \sim$ is a groupoid where two elements $[x, k, \eta, y]$ and $[x', k', \eta', y']$ are composable if $y = x'$ and their product is

$$[x, k, \eta, y][y, k', \eta', y'] = [x, k + k', \eta' \circ \eta, y'].$$

The inversion in G_X is defined such that $[x, k, \eta, y]^{-1} = [y, -k, \eta^{-1}, x]$. The unit space of G_X can be identified with X via the map $x \mapsto [x, 0, \text{id}, x]$, where id is the identity map on X . If $\eta \in \mathcal{T}_k(X)$ and U is an open subset of the domain of η , we set

$$U(\eta) = \{[z, k, \eta, \eta(z)] \mid z \in U\}. \tag{2}$$

It is straightforward to verify that by varying k, η and U the sets (2) constitute a base for a topology on G_X . A crucial point is that the topology is also Hausdorff since the local transfers are holomorphic. It follows that G_X is an étale second countable locally compact Hausdorff groupoid, cf. [Th2].

It is the possible presence of critical points of R in X which prevents us from using the procedure of Section 2.1 to get an étale groupoid. The additional feature, the local transfers, which is introduced to obtain a well-behaved étale groupoid is therefore not necessary when there are no critical points in X , and it is therefore reassuring that the two constructions coincide in the absence of critical points in X .

Proposition 2.5. *Assume that $Y \subseteq X$ is an open subset of X which does not contain any critical point of R and is R -invariant in the sense that $R(Y) \subseteq Y$. Then the reduction $(G_X)_Y$ is isomorphic, as a topological groupoid, to the transformation groupoid of the local homeomorphism $R|_Y : Y \rightarrow Y$.*

Proof. Define $\mu : (G_X)_Y \rightarrow G_{R|_Y}$ such that $\mu[x, k, \eta, y] = (x, k, y)$.

μ is surjective: If $(x, k, y) \in G_{R|_Y}$ there are $n, m \in \mathbb{N}$ such that $k = n - m$ and $R^n(x) = R^m(y)$. Note that $(R^n)'(x) \neq 0$ and $(R^m)'(y) \neq 0$ since Y does not contain critical points. It follows that there are open neighbourhoods U and V of x and y in $\bar{\mathbb{C}}$, respectively, such that $R^n : U \rightarrow R^n(U)$ and $R^m : V \rightarrow R^m(V)$ are univalent holomorphic maps. Let $R^{-m} : R^m(V) \rightarrow V$ be the inverse of $R^m : V \rightarrow R^m(V)$.

¹Since R is holomorphic and not globally periodic, the number k is actually determined by the germ $[\eta]_x$ and could therefore be suppressed in the notation. For several purposes, such as the comparison with the transformation groupoid of a local homeomorphism which is performed in Proposition 2.5 and the definition of the gauge action which will be a crucial tool in several places, it seems best to make the number visible.

Then $\eta_0 = R^{-m} \circ R^n : R^{-n}(R^m(V)) \cap U \rightarrow R^{-m}(R^n(U)) \cap V$ is bi-holomorphic. Set $\eta = \eta_0|_X$ and note that $[x, k, \eta, y] \in (G_X)_Y$.

μ is injective: Assume that $\mu[x, k, \eta, y] = \mu[x', k', \eta', y']$. Then $(x, k, y) = (x', k', y')$. Since $(R^m)'(y) \neq 0$, it follows that R^m is injective in a neighborhood of y . Since $R^m(\eta(z)) = R^n(z) = R^m(\eta'(z))$ for all z close to x , it follows that $\eta = \eta'$ in a neighborhood of x , i.e., $[x, k, \eta, y] = [x', k', \eta', y']$.

μ is continuous: Let $\mu[x, k, \eta, y] = (x, k, y)$ and consider an open neighborhood Ω of (x, k, y) in $G_{R|_Y}$. There is then an $m \in \mathbb{N}$ and a local transfer $\eta \in \mathcal{T}_k(X)$ such that $k+m \geq 0$, $\eta(x) = y$ and $R^{k+m}(z) = R^m(\eta(z))$ for all z in a neighborhood of x . By definition of the topology of $G_{R|_Y}$, the set $\Omega \cap G_{R|_Y}(k, m)$ is open in $G_{R|_Y}(k, m)$ when this set has the relative topology inherited from $X \times \mathbb{Z} \times X$. From this fact it follows that there is an open neighborhood W of x in the domain of η such that $(z, k, \eta(z)) \in \Omega \cap G_{R|_Y}(k, m)$ for all $z \in W$. Since Y is open in X , we can shrink W to ensure that $W \subseteq Y$ and $\eta(W) \subseteq Y$. Then $\{[z, k, \eta, \eta(z)] \mid z \in W\}$ is an open neighborhood of $[x, k, \eta, y]$ in $(G_X)_Y$ such that $\mu(\{[z, k, \eta, \eta(z)] \mid z \in W\}) \subseteq \Omega$.

μ is open: Let $\eta \in \mathcal{T}_k(X)$ be a local transfer. Let U be an open subset of the domain of η such that $U \subseteq Y$ and $\eta(U) \subseteq Y$. It suffices to show that

$$\eta(\{[z, k, \eta, \eta(z)] \mid z \in U\})$$

is open in $G_{R|_Y}$. To this end note that there is an $m \in \mathbb{N}$ such that $k+m \geq 0$ and $R^{k+m}(z) = R^m(\eta(z))$ for all $z \in U$. Consider a point $(z_1, k, z_2) \in \eta(\{[z, k, \eta, \eta(z)] \mid z \in U\})$. Then $(z_1, k, z_2) \in G_{R|_Y}(k, m)$. Let W be an open neighborhood of z_2 in X such that $W \subseteq \eta(U)$ and R^m is injective on W . If $(z'_1, k, z'_2) \in (\eta^{-1}(W) \times \{k\} \times W) \cap G_{R|_Y}(k, m)$, we have $R^m(z'_2) = R^{k+m}(z'_1) = R^m(\eta(z'_1))$, which implies that $z'_2 = \eta(z'_1)$. This shows that

$$(\eta^{-1}(W) \times \{k\} \times W) \cap G_{R|_Y}(k, m) \subseteq \eta(\{[z, k, \eta, \eta(z)] \mid z \in U\}). \quad \square$$

When we take the set X in the construction of G_X to be the whole Riemann sphere $\bar{\mathbb{C}}$ we obtain the groupoid $G_{\bar{\mathbb{C}}}$ and the corresponding C^* -algebra $C_r^*(G_{\bar{\mathbb{C}}})$. There is no good reason to emphasize $\bar{\mathbb{C}}$, so we will denote the groupoid $G_{\bar{\mathbb{C}}}$ by G_R instead and $C_r^*(G_{\bar{\mathbb{C}}})$ by $C_r^*(R)$. We call G_R the *transformation groupoid* of R . It follows from Lemma 4.1 in [Th2] that two elements $x, y \in \bar{\mathbb{C}}$ are in the same G_R -orbit if and only if there are natural numbers $n, m \in \mathbb{N}$ such that $R^n(x) = R^m(y)$ and $\text{val}(R^n, x) = \text{val}(R^m, y)$, where $\text{val}(R^n, x)$ is the valency of R^n at the point x . In particular, the G_R -orbit of x , which we will denote by $\text{RO}(x)$ in the following, is a subset of the orbit of x conditioned by an equality of valencies. Specifically,

$$\begin{aligned} \text{RO}(x) &= \{y \in \bar{\mathbb{C}} \mid R^n(y) = R^m(x) \text{ and} \\ &\quad \text{val}(R^n, y) = \text{val}(R^m, x) \text{ for some } n, m \in \mathbb{N}\}. \end{aligned}$$

We call $\text{RO}(x)$ the *restricted orbit* of x . In the following a subset Y of $\bar{\mathbb{C}}$ will be called *restricted orbit invariant*, or RO -invariant if $y \in Y$ implies $\text{RO}(y) \subseteq Y$. Note

that a totally R -invariant subset is RO-invariant, but the converse can fail when there are isolated points in Y .

We shall work with many different reductions of G_R . Recall that for any subset $Y \subseteq \bar{C}$ the *reduction* of G_R to Y is the set

$$G_Y = \{[x, k, \eta, y] \in G_R \mid x, y \in Y\}.$$

This is always a subgroupoid of G_R . If X is both locally compact in the topology inherited from \bar{C} and RO-invariant, G_X will be a locally compact Hausdorff étale groupoid in the topology inherited from G_R . If X is also totally R -invariant and has no isolated points, the reduction is equal to the groupoid G_X described above, showing that our notation is consistent. Furthermore, when X is open in the relative topology of a subset $Y \subseteq \bar{C}$ which is RO-invariant and locally compact in the topology inherited from \bar{C} , the reduction G_X is again a locally compact Hausdorff étale groupoid in the topology inherited from G_R . The reduction G_X of G_R by such a set is the most general type of reduction we shall need in this paper. To simplify notation we set

$$C_r^*(X) = C_r^*(G_X).$$

3. The Julia–Fatou extension

Let X be an RO-invariant subset of \bar{C} which is locally compact in the relative topology, or at least an open subset of such a set. The C^* -algebra $C_r^*(X)$ carries a natural action β by the circle group \mathbb{T} defined such that

$$\beta_\mu(f)[x, k, \eta, y] = \mu^k f[x, k, \eta, y] \tag{3}$$

when $f \in C_c(G_X)$ and $\mu \in \mathbb{T}$. To identify the fixed point algebra of β , set

$$G_X^0 = \{[x, k, \eta, y] \in G_X \mid k = 0\}.$$

Since G_X^0 is an open subgroupoid of G_X , there is an inclusion $C_c(G_X^0) \subseteq C_c(G_X)$ which extends to an embedding $C_r^*(G_X^0) \subseteq C_r^*(X)$ of C^* -algebras.

Lemma 3.1. $C_r^*(G_X^0)$ is the fixed point algebra $C_r^*(X)^\beta$ of the gauge action.

Proof. Let $a \in C_r^*(X)^\beta$. For any $\varepsilon > 0$ there is a function $f \in C_c(G_X)$ such that $\|a - f\| \leq \varepsilon$. We can then write f as a finite sum $f = \sum_{k \in \mathbb{Z}} f_k$, where f_k is supported in

$$\{[x, l, \eta, y] \in G_X \mid l = k\}.$$

Since $\int_{\mathbb{T}} \beta_\mu(f_k) d\mu = \int_{\mathbb{T}} \mu^k f_k d\mu = 0$ when $k \neq 0$ and $\|a - \int_{\mathbb{T}} \beta_\mu(f) d\mu\| \leq \varepsilon$, we deduce that $\|a - f_0\| \leq \varepsilon$. This shows that $C_r^*(X)^\beta \subseteq C_r^*(G_X^0)$; the reversed inclusion is trivial. □

Just as for local homeomorphisms, [An], [De], there is an inductive limit decomposition of $C_r^*(G_X^0)$ which throws some light on its structure. Let $n \in \mathbb{N}$. Set

$$G_X^0(n) = \{[x, 0, \eta, y] \in G_X \mid R^n(z) = R^n(\eta(z)) \text{ in a neighbourhood of } x\}.$$

Each $G_X^0(n)$ is an open subgroupoid of G_X^0 , $G_X^0(n) \subseteq G_X^0(n+1)$ for all n and $G_X^0 = \bigcup_n G_X^0(n)$. It follows that $C_r^*(G_X^0(n)) \subseteq C_r^*(G_X^0(n+1)) \subseteq C_r^*(G_X^0)$ for all n , and

$$C_r^*(G_X^0) = \overline{\bigcup_n C_r^*(G_X^0(n))}. \tag{4}$$

Assume now that X is an RO-invariant subset of $\overline{\mathbb{C}}$ which is locally compact in the relative topology, and not just an open subset of such a set. Let Y be a closed RO-invariant subset of X . Then $X \setminus Y$ is open in X and RO-invariant. Since Y and $X \setminus Y$ are locally compact in the topology inherited from $\overline{\mathbb{C}}$, we can consider the reduced groupoid C^* -algebras $C_r^*(Y)$ and $C_r^*(X \setminus Y)$. Furthermore, we have a surjective $*$ -homomorphism

$$\pi_Y : C_r^*(X) \rightarrow C_r^*(Y)$$

because Y is closed and RO-invariant in X .

Lemma 3.2. *Let X be a RO-invariant subset of $\overline{\mathbb{C}}$ which is locally compact in the relative topology. Let Y be a closed RO-invariant subset of X . The sequence*

$$0 \rightarrow C_r^*(X \setminus Y) \rightarrow C_r^*(X) \xrightarrow{\pi_Y} C_r^*(Y) \rightarrow 0$$

is exact.

Proof. It is clear that $C_r^*(X \setminus Y) \subseteq \ker \pi_Y$. To establish the reverse inclusion, let $a \in \ker \pi_Y$ and let $\varepsilon > 0$. Note first that the formula (3) also defines an action by \mathbb{T} on $C_r^*(Y)$ which makes π_Y equivariant. It follows that $\ker \pi_Y$ is left globally invariant by the gauge action, and there is therefore an approximate unit in $\ker \pi_Y$ consisting of elements fixed by the gauge action. It follows then from Lemma 3.1 that there is an element $u \in C_r^*(G_X^0) \cap \ker \pi_Y$ such that $\|ua - a\| \leq \varepsilon$. It follows from (4) that

$$C_r^*(G_X^0) \cap \ker \pi_Y = \overline{\bigcup_n C_r^*(G_X^0(n)) \cap \ker \pi_Y},$$

and there is therefore an $n \in \mathbb{N}$ and an element $v \in C_r^*(G_X^0(n)) \cap \ker \pi_Y$ such that $\|ua - va\| \leq \varepsilon$. Note now that $\#R^{-n}(x) \leq (\deg R)^n$ for all $x \in \overline{\mathbb{C}}$ if $\deg R$ is the degree of R . By definition of the norm on $C_r^*(G_X^0(n))$ this gives us the estimate

$$\|f\| \leq (\deg R)^n \sup_{\gamma \in G_X^0(n)} |f(\gamma)| \tag{5}$$

for all $f \in C_c(G_X^0(n))$. Set $\delta = \varepsilon((2(\deg R)^n + 1)(\|a\| + 1))^{-1}$ and choose $g \in C_c(G_X^0(n))$ such that $\|g - v\| \leq \delta$. Then $\|\pi_Y(g)\| \leq \delta$ and so

$$\sup_{\gamma \in G_X^0(n) \cap s^{-1}(Y)} |g(\gamma)| \leq \delta$$

by Proposition 4.2 of [Re], p. 99. We can therefore write $g = g_1 + g_2$ where $g_1 \in C_c(G_X^0(n))$ has support in $s^{-1}(X \setminus Y)$ and $\sup_{\gamma \in G_X^0(n)} |g_2(\gamma)| \leq 2\delta$. It follows then from (5) that $\|g_1 a - v a\| \leq \delta \|a\| + \|g_2 a\| \leq \varepsilon$ and so $\|g_1 a - a\| \leq 3\varepsilon$. Since $g_1 a \in C_r^*(X \setminus Y)$, it follows that $a \in C_r^*(X \setminus Y)$. \square

Since the Fatou set F_R and the Julia set J_R are totally R -invariant and hence also RO -invariant, we get the following.

Corollary 3.3. *The sequence*

$$0 \rightarrow C_r^*(F_R) \rightarrow C_r^*(R) \xrightarrow{\pi_{J_R}} C_r^*(J_R) \rightarrow 0.$$

is exact.

4. The structure of $C_r^*(J_R)$

4.1. Pure infiniteness of $C_r^*(J_R)$

Proposition 4.1. *G_{J_R} is essentially free and locally contractive, and $C_r^*(J_R)$ is purely infinite.*

For the proof of Proposition 4.1 we need a couple of lemmas.

Lemma 4.2. *Assume that $(R^n)'(x) \neq 0$. Then $R^n(x) \in RO(x)$.*

Proof. If x is not critical for R^n , there is an open neighbourhood U of x such that $R^n : U \rightarrow R^n(U)$ is a local transfer and $[x, n, R^n|_U, R^n(x)] \in G_R$. \square

In the following proof and in the rest of the paper Crit will denote the set of critical points of R .

Lemma 4.3. *Let Y be a closed RO -invariant subset of \bar{C} . If Y does not contain an isolated point which is periodic or critical, it follows that the elements of Y that are neither pre-periodic nor pre-critical are dense in Y .*

Proof. Assume that there is a non-empty open subset $U \subseteq Y$ consisting entirely of pre-critical and pre-periodic point. Let $\text{Per}_n R$ denote the set of n -periodic points of R . Since

$$U \subseteq \bigcup_{n,j} R^{-j}(\text{Per}_n R \cup \text{Crit})$$

by assumption, it follows from the Baire category theorem that there are $n, j \in \mathbb{N}$ and a non-empty open subset W of Y such that

$$W \subseteq U \cap R^{-j}(\text{Per}_n R \cup \text{Crit}).$$

Since R is neither periodic nor constant, it follows that $R^{-j}(\text{Per}_n R \cup \text{Crit})$ is finite. Hence W must contain a point z_0 which is isolated in Y . If $z_0 \notin \text{Crit}$ we conclude from Lemma 4.2 that $R(z_0) \in Y$ since Y is RO-invariant. By repeating this argument we either reach an $l < j$ such that $R^l(z_0) \in \text{Crit} \cap Y$ or conclude that $R^j(z_0) \in Y$. In the first case R^l is a local transfer in an open neighbourhood of z_0 and in the second R^j is. Hence $R^l(z_0)$ is isolated in Y in the first case, and $R^j(z_0)$ in the second. In any case we conclude that Y contains an isolated point which is either critical or periodic. This contradicts our assumption on Y . \square

We can then give the proof of Proposition 4.1:

Proof. By Proposition 4.4 of [Th2] the elements of J_R with trivial isotropy group in G_{J_R} are the points that are neither pre-periodic nor pre-critical. It is well known that J_R is closed, totally R -invariant and without isolated points. It follows therefore from Lemma 4.3 that G_{J_R} is essentially free.

To prove that G_{J_R} is also locally contracting we use that the repelling periodic points are dense in J_R by ii) of Theorem 14.1 in [Mi]. The argument is then essentially the same used in the proof of Lemma 4.2 in [Th3]: Let $U \subseteq J_R$ be an open non-empty set. There is a repelling periodic point $z_0 \in U \cap \mathbb{C}$, and there is an $n \in \mathbb{N}$, a positive number $\kappa > 1$ and an open neighbourhood $W \subseteq U \cap \mathbb{C}$ of z_0 such that $R^n(z_0) = z_0$, R^n is injective on W and

$$|R^n(y) - z_0| \geq \kappa|y - z_0| \tag{6}$$

for all $y \in W$. Let $\delta_0 > 0$ be so small that

$$\{y \in \mathbb{C} \mid |y - z_0| \leq \delta_0\} \subseteq R^n(W) \cap W. \tag{7}$$

Because z_0 is not isolated in J_R , there is an element $z_1 \in J_R \cap \mathbb{C}$ such that $0 < |z_1 - z_0| < \delta_0$. Choose δ strictly between $|z_1 - z_0|$ and δ_0 such that

$$\kappa|z_1 - z_0| > \delta. \tag{8}$$

Set $V = \{y \in J_R \cap \mathbb{C} \mid |y - z_0| < \delta\}$. Then

$$\bar{V} \subsetneq R^n(V). \tag{9}$$

Indeed, if $|y - z_0| \leq \delta$, then (7) implies that there is a $y' \in W$ such that $R^n(y') = y$, and (6) implies that $|y' - z_0| < \delta$. Since $y' \in J_R$ because $R^{-1}(J_R) = J_R$, it follows that $\bar{V} \subseteq R^n(V)$. On the other hand, it follows from (8) and (6) that $R^n(z_1) \notin \bar{V}$. This shows that (9) holds. Then

$$S = \{[z, n, R^n|_V, R^n(z)] \in G_{J_R} \mid z \in V\}$$

is an open bisection in G_{J_R} such that $\bar{V} \subseteq s(S)$ and

$$\alpha_{S^{-1}}(\bar{V}) \not\subseteq V,$$

where $\alpha_{S^{-1}} : s(S) \rightarrow r(S)$ is the homeomorphism defined by S , cf. Section 2.0.3. This shows that G_{J_R} is locally contracting, and it then follows from Theorem 2.4 that $C_r^*(J_R)$ is purely infinite. \square

4.2. Exposed points and finite quotients of $C_r^*(J_R)$

Lemma 4.4. *Let X be a totally R -invariant set which is locally compact in the topology inherited from $\bar{\mathbb{C}}$. Let $Y \subseteq X$ be a closed RO-invariant subset of X and let Y_0 be the subset of Y obtained by deleting the isolated points of Y . Then Y_0 is totally R -invariant, i.e., $R^{-1}(Y_0) = Y_0$.*

Proof. Let $y \in Y_0$. Let $n, m \in \mathbb{N}$ and consider a point $x \in X$ such that $R^n(x) = R^m(y)$. We must show that $x \in Y_0$. Since $R^{-n}(R^n(x))$ is a finite set, there is an open neighborhood U_0 of x such that

$$R^{-n}(R^m(y)) \cap \bar{U}_0 = \{x\}. \tag{10}$$

Because R^n and R^m are both open maps, we can also arrange that there is an open neighborhood V_0 of y such that $R^n(U_0) = R^m(V_0)$. Set $U = U_0 \cap X$ and $V = V_0 \cap X$ and note that $R^n(U) = R^m(V)$. Since $\bigcup_{j=0}^m R^{-j}(\text{Crit})$ and $R^{-m}(\bigcup_{j=0}^n R^j(\text{Crit}))$ are both finite sets and y is not isolated in Y , there is a sequence $\{y_k\}$ of mutually distinct elements in

$$Y \cap V \setminus (R^{-m}(\bigcup_{j=0}^n R^j(\text{Crit})) \cup \bigcup_{j=0}^m R^{-j}(\text{Crit}))$$

converging to y . Since $\#R^{-m}(z) \leq (\deg R)^m$ for all z , we can prune the sequence $\{y_k\}$ to arrange that $k \neq l$ implies that $R^m(y_k) \neq R^m(y_l)$. Let $\{x_k\} \subseteq U$ be points such that $R^n(x_k) = R^m(y_k)$ for all k . Then $x_k \in U \setminus \bigcup_{j=0}^n R^{-j}(\text{Crit})$ for all k . Passing to a subsequence we can arrange that $\{x_k\}$ converges in X , necessarily to x because of (10). Note that $\text{val}(R^m, y_k) = \text{val}(R^n, x_k) = 1$ for all k since $x_k \notin \bigcup_{j=0}^n R^{-j}(\text{Crit})$ and $y_k \notin \bigcup_{j=0}^m R^{-j}(\text{Crit})$. It follows therefore, either from Proposition 4.1 in [Th2] or from Lemma 4.2 above, that $x_k \in \text{RO}(y_k)$. Hence $x_k \in Y$ because $y_k \in Y$ and Y is RO-invariant. It follows that $x \in Y$ because Y is closed. Furthermore, since the x_k 's are distinct, $x \in Y_0$. This shows that Y_0 is totally R -invariant. \square

Lemma 4.5. *Let L be a closed RO-invariant subset of J_R . Then L is either finite or equal to J_R .*

Proof. It follows from Lemma 4.4 that we can write $L = L_0 \cup L_1$, where L_0 is closed and totally R -invariant while L_1 is discrete. If $L_0 \neq \emptyset$, it follows from Corollary 4.13 of [Mi] that $L_0 = J_R = L$. If $L_0 = \emptyset$, the compactness of L implies that it is finite. \square

4.2.1. Exposed points. The preceding lemma forces us to look for points in J_R , or more generally in $\bar{\mathbb{C}}$, whose restricted orbits are finite. In the following we say that a point $x \in \bar{\mathbb{C}}$ is *exposed* if the restricted orbit $RO(x)$ of x is finite. A non-empty subset $A \subseteq \bar{\mathbb{C}}$ is *exposed* if it is finite and RO -invariant. Clearly the so-called exceptional points, those with finite orbit, are exposed. There are at most two of them, and they are always elements of the Fatou set. See e.g. §4.1 in [B]. In contrast exposed points can occur in the Julia set as well.

Lemma 4.6. *Let $A \subseteq \bar{\mathbb{C}}$ be a finite subset with the property that*

$$R^{-1}(A) \setminus \text{Crit} \subseteq A. \tag{11}$$

Then A contains at most 4 elements, and at most 3 if it contains a critical point.

Proof. The proof is a repetition of the proof of Lemma 1 in [GPRR]. Set $d = \deg R$ and let $\text{val}(R, x)$ denote the valency of R at x . Then

$$\begin{aligned} d(\#A) &= \sum_{x \in R^{-1}(A)} \text{val}(R, x) \quad (\text{since } \sum_{x \in R^{-1}(y)} \text{val}(R, x) = d \text{ for all } y) \\ &= \#R^{-1}(A) + \sum_{x \in R^{-1}(A)} (\text{val}(R, x) - 1) \\ &\leq \#A + \#\text{Crit} + \sum_{x \in R^{-1}(A)} (\text{val}(R, x) - 1) \quad (\text{by (11)}) \\ &= \#A + \#\text{Crit} + 2d - 2 \quad (\text{by Theorem 2.7.1 of [B]}) \\ &\leq \#A + 4(d - 1) \quad (\text{by Corollary 2.7.2 of [B]}). \end{aligned}$$

It follows that $\#A \leq 4$. If A contains a critical point the first inequality above is strict and hence $\#A \leq 3$. □

It follows from Lemma 4.2 that an exposed subset satisfies (11). This gives us the following.

Corollary 4.7. *An exposed subset does not contain more than 4 elements. If it contains a critical point, it contains at most 3 elements.*

The upper bound on the number of elements in an exposed set can be improved if R is a polynomial since the number of critical points for a polynomial is at most $\deg R$, and ∞ is always one of them. Specifically, applied to a polynomial the proof of Lemma 4.6 yields the following.

Lemma 4.8. *Assume that R is a polynomial of degree at least 2 and A is an exposed subset of \mathbb{C} . Then $\#A \leq 2$, and $\#A \leq 1$ if A contains a critical point.*

We say that an exposed subset is of *type 1* if it does not contain a critical point, of *type 2* if it contains a pre-periodic critical point and *type 3* if it contains a critical point but no pre-periodic critical point.

Lemma 4.9. *Let A be a finite subset of $\bar{\mathbb{C}}$. Then A is an exposed subset of type 1 if and only if*

$$R^{-1}(A) \setminus \text{Crit} = A. \tag{12}$$

Proof. Assume first that A is an exposed subset of type 1. Since A does not contain a critical point, it follows from Lemma 4.2 that $R(A) \subseteq A$, i.e., $A \subseteq R^{-1}(A) \setminus \text{Crit}$. Combined with (11) this shows that (12) holds.

Conversely, it follows from (12) that $R(A) \subseteq A$ and that $A \cap \text{Crit} = \emptyset$. Hence $\text{val}(R^n, x) = 1$ for all $x \in A$ and all $n \in \mathbb{N}$. Thus if $x \in A$ and $y \in \text{RO}(x)$, we have both that $R^m(y) = R^n(x) \in A$ for some n, m and that $\text{val}(R^m, y) = \text{val}(R^n, x) = 1$. But then $R^j(y) \notin \text{Crit}$ for all $0 \leq j \leq m - 1$, and repeated use of (12) shows that $y \in A$, i.e., A is RO-invariant. \square

Finite subsets of the Julia set satisfying (12) were considered by Makarov and Smirnov in [MS1], and their work can be used to find examples of polynomials with exposed subsets of type 1 in the Julia set. In [MS2] a rational map with an exposed subset of type 1 was called exceptional. This notion was extended in [GPRR], where a rational map was called exceptional when the Julia set contains a finite forward invariant subset satisfying (11). There are exceptional rational maps, in the sense of [GPRR], without any exposed subsets, and conversely, there are non-exceptional rational maps with exposed subsets in the Julia set. Thus although there is of course a relationship between exposed subsets and the subsets used to define the exceptional maps in [GPRR], the two notions are not the same. Note that it follows from Corollary 4.7 that the total number of exposed points is at most 4.

4.2.2. Finite quotients of $C_r^*(J_R)$. Let E_R denote the union of the exposed subsets in $\bar{\mathbb{C}}$, a set with at most 4 elements. If $E_R \cap J_R \neq \emptyset$, the purely infinite C^* -algebra $C_r^*(J_R)$ will have $C_r^*(E_R \cap J_R)$ as a quotient. The corresponding ideal, however, is simple.

Proposition 4.10. *$C_r^*(J_R \setminus E_R)$ is simple.*

Proof. It follows from Lemma 4.3 that there are many points in $J_R \setminus E_R$ whose isotropy group in $G_{J_R \setminus E_R}$ is trivial. By Corollary 2.18 of [Th1] it suffices therefore to show that there are no non-trivial (relatively) closed RO-invariant subsets in $J_R \setminus E_R$. Assume that L is a non-empty RO-invariant subset of $J_R \setminus E_R$, which is closed in $J_R \setminus E_R$. It follows first that $L \cup (E_R \cap J_R)$ is closed and RO-invariant in J_R and then from Lemma 4.5 that $L = J_R \setminus E_R$, which is the desired conclusion. \square

In order to obtain a description of the quotient

$$C_r^*(J_R \cap E_R) \simeq C_r^*(J_R) / C_r^*(J_R \setminus E_R),$$

we consider a more general situation because the result can then be used to examine $C_r^*(F_R)$.

Lemma 4.11. *Let $x \in \bar{\mathbb{C}}$ and assume that the restricted orbit $\text{RO}(x)$ of x is discrete in $\bar{\mathbb{C}}$. There is an isomorphism*

$$C_r^*(\text{RO}(x)) \simeq C^*(\text{Is}_x) \otimes \mathbb{K}(l^2(\text{RO}(x))),$$

where $\mathbb{K}(l^2(\text{RO}(x)))$ denotes the C^* -algebra of compact operators on $l^2(\text{RO}(x))$ and Is_x the isotropy group $\text{Is}_x = \{g \in G_R \mid s(g) = r(g) = x\}$.

Proof. Note that the reduction $G_{\text{RO}(x)}$ is discrete in G_R and that $C_r^*(\text{RO}(x))$ is generated by $1_z, z \in G_{\text{RO}(x)}$, if 1_z denotes the characteristic function of the set $\{z\}$. For every $y \in \text{RO}(x)$ choose an element $\eta_y \in r^{-1}(y) \cap s^{-1}(x)$. For every $g \in \text{Is}_x$ set

$$u_g = \sum_{y \in \text{RO}(x)} 1_{\eta_y g \eta_y^{-1}}.$$

The sum converges in the strict topology and defines a unitary in the multiplier algebra of $C_r^*(\text{RO}(x))$. Note that $u_g u_h = u_{gh}$ so that u is a unitary representation of Is_x as multipliers of $C_r^*(\text{RO}(x))$. The elements $1_{\eta_u \eta_v^{-1}}, u, v \in \text{RO}(x)$ constitute a system of matrix units generating a copy of $\mathbb{K}(l^2(\text{RO}(x)))$ inside $C_r^*(\text{RO}(x))$. Since each u_g commutes with $1_{\eta_u \eta_v^{-1}}$ for all u, v , we obtain a $*$ -homomorphism $C^*(\text{Is}_x) \otimes \mathbb{K}(l^2(\text{RO}(x))) \rightarrow C_r^*(\text{RO}(x))$ sending $1_g \otimes 1_{\eta_u \eta_v^{-1}}$ to $u_g 1_{\eta_u \eta_v^{-1}}$. It is easy to see that this is an isomorphism. \square

Lemma 4.12. *Let A be a finite RO-orbit in J_R . Set $n = \#A$.*

(a) *Assume that A is of type 1. Then $n \leq 4$ and*

$$C_r^*(A) \simeq C(\mathbb{T}) \otimes M_n(\mathbb{C}).$$

(b) *Assume that A is of type 2. Then $n \leq 3$ and*

$$C_r^*(A) \simeq M_n(\mathbb{C}) \otimes C(\mathbb{T}) \otimes \mathbb{C}^d,$$

where $d = \lim_{k \rightarrow \infty} \text{val}(R^k, x)$ for any critical point $x \in A$.

(c) *Assume that A is of type 3. Then $n \leq 3$ and*

$$C_r^*(A) \simeq M_n(\mathbb{C}) \otimes \mathbb{C}^d,$$

where $d = \lim_{k \rightarrow \infty} \text{val}(R^k, x)$ for any critical point $x \in A$.

Proof. (a) It follows from Corollary 4.7 that $n \leq 4$. It follows from Lemma 4.9 that $A = \text{RO}(x)$ for some point $x \in J_R$ which is periodic and whose forward orbit is contained in A . Since A contains no critical points, Proposition 4.4 b) in [Th2] tells us that $\text{Is}_x \simeq \mathbb{Z}$. Then the conclusion follows from Lemma 4.11.

(b) If A contains a critical point, the last assertion in Corollary 4.7 says that $\#A \leq 3$. If x is a pre-periodic critical point in A , we can determine the isotropy group Is_x from Proposition 4.4 in [Th2]. Since J_R does not contain any periodic critical orbit, we are

in situation d2) of that proposition and therefore obtain the stated conclusion from Lemma 4.11.

(c) follows in the same way as (b). The only difference is that Is_x is determined by use of c) in Proposition 4.4 of [Th2]. □

Given a point $z \in \bar{\mathbb{C}}$ we call the limit $\lim_{k \rightarrow \infty} \text{val}(R^k, z)$ occurring in Lemma 4.12 the asymptotic valency of z . It can be infinite, but only if z is pre-periodic to a critical periodic orbit.

Theorem 4.13. *There is an extension*

$$0 \rightarrow C_r^*(J_R \setminus E_R) \rightarrow C_r^*(J_R) \xrightarrow{\pi_{E_R}} \bigoplus_A C_r^*(A) \rightarrow 0, \tag{13}$$

where the direct sum \bigoplus_A is over the (possibly empty) collection of finite RO-orbits A in J_R . Furthermore, $C_r^*(J_R \setminus E_R)$ is separable, purely infinite, nuclear, simple and satisfies the universal coefficient theorem of Rosenberg and Schochet, [RS]. If non-zero, the quotient $\bigoplus_A C_r^*(A)$ is isomorphic to a finite direct sum of matrix algebras $M_n(\mathbb{C})$ with $n \leq 3$ and circle algebras $C(\mathbb{T}) \otimes M_n(\mathbb{C})$ with $n \leq 4$.

Proof. The extension (13) is a special case of the extension from Lemma 3.2. The direct sum decomposition of the quotient follows from Lemma 2.2 and its description from Lemma 4.12. The pure infiniteness of the ideal follows from Proposition 4.1 because pure infiniteness is inherited by ideals. It is simple by Proposition 4.10. That $C_r^*(J_R \setminus E_R)$ is nuclear and satisfies the UCT will be shown in Section 4.3 below by making a connection to the work of Katsura, [Ka]. □

In view of Theorem 4.13 it seems appropriate to point out that the Julia set can contain exposed orbits of all three types. For an example of type 1 observe that for the Chebyshev polynomial $R(z) = z^2 - 2$ the set $E_R \cap J_R$ consists of the points $\{-2, 2\}$ which is a finite RO-orbit of type 1 in the Julia set $[-2, 2]$. Hence

$$\bigoplus_A C_r^*(A) \simeq C(\mathbb{T}) \otimes M_2(\mathbb{C}).$$

No other polynomial in the quadratic family $z^2 + c$ has an exposed point in the Julia set.

For an example of a finite RO-orbit of type 2 consider the rational map

$$R(z) = \frac{(z - 2)^2}{z^2}.$$

The Julia set is the entire sphere in this case, cf. §11.9 in [B], and the maximal exposed subset consists of the points $\{0, \infty, 1\}$, which is the union of the finite RO-orbit $\{1, \infty\}$ of type 1 and the finite RO-orbit $\{0\}$, which is of type 2. The asymptotic valency $\lim_{n \rightarrow \infty} \text{val}(R^n, 0)$ is 2 and hence

$$\bigoplus_A C_r^*(A) \simeq C(\mathbb{T}) \oplus C(\mathbb{T}) \oplus (C(\mathbb{T}) \otimes M_2(\mathbb{C})).$$

To give examples of finite RO-orbits of type 3 in the Julia set we use the work of M. Rees. She shows in Theorem 2 of [R] that for ‘many’ $\lambda \in \mathbb{C} \setminus \{0\}$ the rational map

$$R(z) = \lambda(1 - \frac{z}{2})^2$$

has a dense critical forward orbit. In particular, the Julia set J_R is the entire sphere. The critical points are 0 and 2, and $R^{-1}(0) = \{2\}$. Since the forward orbit of 0 is dense, it follows that $\{0\}$ is a finite RO-orbit of type 2. There are no other exposed points, i.e., $E_R = \{0\}$. Hence

$$\bigoplus_A C_r^*(A) \simeq \mathbb{C} \oplus \mathbb{C}$$

in this case because the asymptotic valency of 0 is 2.

4.3. Amenability and the UCT. Set

$$J'_R = J_R \setminus (E_R \cup \bigcup_{j=0}^3 R^{-j}(\text{Crit}))$$

and consider

$$\Gamma = \{[x, k, \eta, y] \in G_{J'_R} \mid k = 1\},$$

which is an open subset of $G_{J'_R}$. Let X_Γ be the closure of $C_c(\Gamma)$ in $C_r^*(J'_R)$. Since $X_\Gamma^* X_\Gamma \subseteq C_r^*(G_{J'_R}^0)$, we can consider X_Γ as a Hilbert $C_r^*(G_{J'_R}^0)$ -module with the ‘inner product’ $\langle a, b \rangle = a^*b$. Furthermore, since

$$C_r^*(G_{J'_R}^0)X_\Gamma \subseteq X_\Gamma,$$

we can consider any $a \in C_r^*(G_{J'_R}^0)$ as an adjointable operator $\varphi(a)$ on X_Γ . Then the pair (φ, X_Γ) is a C^* -correspondence in the sense of Katsura, [Ka], and we aim to show that the C^* -algebra \mathcal{O}_{X_Γ} introduced in [Ka] is a copy of $C_r^*(J'_R)$.

Lemma 4.14. φ is injective.

Proof. Assume that $\varphi(a) = 0$. To show that $a = 0$, consider the continuous function $j(a)$ on $G_{J'_R}^0$ defined by a , cf. Proposition 4.2 on p. 99 in [Re]. Consider an element $\gamma \in G_{J'_R}$ such that $s(\gamma) \notin R^{-4}(\text{Crit})$. It follows from Lemma 4.2 that there is a function $f \in C_c(\Gamma)$ supported in $\{[z, 1, R|_U, R(z)] \mid z \in U\}$ for some open neighborhood U of $s(\gamma)$ such that $ff^* \in C_c(J'_R)$ and $ff^*(s(\gamma)) = 1$. Since $a f f^* = 0$, it follows that $j(a)(\gamma) = 0$, i.e., $j(a) = 0$ on the set

$$\{\gamma \in G_{J'_R}^0 \mid s(\gamma) \notin R^{-4}(\text{Crit})\}.$$

Since this set is dense in $G_{J'_R}$, it follows first that $j(a) = 0$ and then that $a = 0$ because j is injective. □

Note that we can consider the inclusions $C_r^*(G_{J'_R}^0) \subseteq C_r^*(J'_R)$ and $X_\Gamma \subseteq C_r^*(J'_R)$ as an injective representation of the C^* -correspondence (φ, X_Γ) . It follows that there is an injective $*$ -homomorphism $\psi_t: \mathcal{K}(X_\Gamma) \rightarrow C_r^*(J'_R)$ such that

$$\psi_t(\Theta_{a,b}) = ab^*,$$

cf. Definition 2.3 and Lemma 2.4 in [Ka]. Note that the range of ψ_t is the ideal

$$X_\Gamma X_\Gamma^* = \overline{\text{Span}\{ab^* \mid a, b \in X_\Gamma\}}$$

in $C_r^*(G_{J'_R}^0)$. We are here and in the following lemma borrowing notation from [Ka].

Lemma 4.15. *The ideal $\{a \in C_r^*(G_{J'_R}^0) \mid \varphi(a) \in \mathcal{K}(X_\Gamma)\}$ is equal to $X_\Gamma X_\Gamma^*$, and $a = \psi_t(\varphi(a))$ for all $a \in J_{X_\Gamma}$.*

Proof. The inclusion

$$X_\Gamma X_\Gamma^* \subseteq \{a \in C_r^*(G_{J'_R}^0) \mid \varphi(a) \in \mathcal{K}(X_\Gamma)\}$$

is trivial, so we focus on the inverse. Let therefore $a \in C_r^*(G_{J'_R}^0)$ be an element such that $\varphi(a) \in \mathcal{K}(X_\Gamma)$. There is then a sequence

$$\sum_{i=1}^{N_n} \Theta_{a_i^n, b_i^n}, \quad n = 1, 2, 3, \dots,$$

where $a_i^n, b_i^n \in X_\Gamma$ for all i, n , which converges to $\varphi(a)$ in $\mathcal{K}(X_\Gamma)$. In particular,

$$af = \lim_{n \rightarrow \infty} \sum_{i=1}^{N_n} \Theta_{a_i^n, b_i^n} f = \lim_{n \rightarrow \infty} \sum_{i=1}^{N_n} a_i^n b_i^{n*} f \tag{14}$$

for all $f \in X_\Gamma$. By continuity of ψ_t it follows that the sequence $\psi_t(\sum_{i=1}^{N_n} \Theta_{a_i^n, b_i^n}) = \sum_{i=1}^{N_n} a_i^n b_i^{n*}$ converges in $X_\Gamma X_\Gamma^*$ to $\psi_t(\varphi(a))$. It follows from (14) that $af = \psi_t(\varphi(a))f$ for all $f \in X_\Gamma$, and as in the proof of Lemma 4.14 we deduce first that $j(a - \psi_t(\varphi(a))) = 0$ and then that $a = \psi_t(\varphi(a))$. \square

It follows from Lemma 4.15 that the representation of the C^* -correspondence X_Γ given by the inclusions $C_r^*(G_{J'_R}^0) \subseteq C_r^*(J'_R)$ and $X_\Gamma \subseteq C_r^*(J'_R)$ is covariant in the sense of Katsura, [Ka]. Combined with the presence of the gauge action on $C_r^*(G_{J'_R}^0)$ this allows us now to use Theorem 6.4 from [Ka] to conclude that the C^* -algebra \mathcal{O}_{X_Γ} defined from the C^* -correspondence X_Γ is isomorphic to the C^* -subalgebra of $C_r^*(J'_R)$ generated by $C_r^*(G_{J'_R}^0)$ and X_Γ . It remains to show that this is all of $C_r^*(J'_R)$.

Lemma 4.16. *For all $x \in J'_R$ there is an element $\gamma \in \Gamma$ such that $s(\gamma) = x$.*

Proof. For any $z \in J_R \setminus E_R$ and $n \geq 1$, set

$$D_n(z) = \{y \in R^{-n}(z) \mid y, R(y), \dots, R^{n-1}(y) \notin \text{Crit}\}.$$

Let $x \in J'_R$. If $R^{-1}(x) \not\subseteq \text{Crit}$, choose $y \in R^{-1}(x) \setminus \text{Crit}$. It follows then from Lemma 4.2 that $[y, 1, \eta, x] \in \Gamma$ for some local transfer η . Assume therefore that $R^{-1}(x) \subseteq \text{Crit}$. Then $R(x) \neq x$ since otherwise $\text{RO}(x) = \{x\}$, contradicting $x \notin E_R$. If $D_2(R(x)) \neq \emptyset$, choose an element $y \in D_2(R(x))$ and note that $[y, 1, \eta, x] \in \Gamma$ for some local transfer η . Thus assume now that $D_2(R(x)) = R^{-1}(x) \setminus \text{Crit} = \emptyset$. Then $R^2(x) \notin \{x, R(x)\}$ since otherwise $\text{RO}(x) \subseteq \{x, R(x)\} \cup D_1(R(x))$, contradicting $x \notin E_R$. If $D_3(R^2(x)) \neq \emptyset$, choose $y \in D_3(R^2(x))$ and note that $[y, 1, \eta, x] \in \Gamma$ for some local transfer η . Assume therefore that $D_3(R^2(x)) = D_2(R(x)) = R^{-1}(x) \setminus \text{Crit} = \emptyset$. Then $R^3(x) \notin \{x, R(x), R^2(x)\}$ since otherwise $\text{RO}(x) \subseteq \{x, R(x), R^2(x)\} \cup D_1(R(x)) \cup D_1(R^2(x)) \cup D_2(R^2(x))$, contradicting $x \notin E_R$. We claim that $D_4(R^3(x))$ cannot be empty. Indeed, if it is empty we have either $R^4(x) \in \{x, R(x), R^2(x), R^3(x)\}$ and then

$$\begin{aligned} \text{RO}(x) \subseteq & \{x, R(x), R^2(x), R^3(x)\} \cup D_1(R(x)) \cup D_1(R^2(x)) \\ & \cup D_2(R^2(x)) \cup D_1(R^3(x)) \cup D_2(R^3(x)) \cup D_3(R^3(x)), \end{aligned}$$

which is impossible since $x \in E_R$, or $R^4(x) \notin \{x, R(x), R^2(x), R^3(x)\}$, in which case

$$\begin{aligned} A = & \{x, R(x), R^2(x), R^3(x), R^4(x)\} \cup D_1(R(x)) \cup D_1(R^2(x)) \\ & \cup D_2(R^2(x)) \cup D_1(R^3(x)) \cup D_2(R^3(x)) \cup D_3(R^3(x)) \end{aligned}$$

is a set with more than 4 elements for which (11) holds, contradicting Lemma 4.6. Thus $D_4(R^3(x))$ is not empty. We choose $y \in D_4(R^3(x))$ and note that $[y, 1, \eta, x] \in \Gamma$ for some local transfer η . \square

Proposition 4.17. $\mathcal{O}_{X_\Gamma} \simeq C_r^*(J'_R)$, and $C_r^*(J_R \setminus E_R)$ is stably isomorphic to \mathcal{O}_{X_Γ} .

Proof. By the remarks preceding Lemma 4.16, to establish the isomorphism $\mathcal{O}_{X_\Gamma} \simeq C_r^*(J'_R)$ it suffices to show that $C_r^*(J'_R)$ is generated by $C_r^*(G_{J'_R}^0)$ and X_Γ . Let $k \geq 0$. Consider a bisection in $G_{J'_R}$ of the form

$$S = \{[z, k, \eta, \eta(z)] \mid z \in U\}$$

for some local transfer η and an open subset $U \subseteq J'_R$ in the domain of η such that $\eta(U) \subseteq J'_R$. By varying k, η and U , functions $f \in C_c(G_{J'_R})$ with support in a set of the form S generate $C_r^*(J'_R)$ as a C^* -algebra. It suffices therefore to show that such functions are elements of the $*$ -algebra generated by $C_r^*(G_{J'_R}^0)$ and X_Γ . We will prove this by induction in k , starting with the observation that the assertion is trivial if $k = 0$ and $k = 1$. Assume that the assertion is established for $k - 1$ and consider

$f \in C_c(G_{J'_R})$ supported in S . Let $x \in U$. It follows from Lemma 4.16 that we can find $\gamma \in \Gamma$ such that $s(\gamma) = \eta(x)$. This means that there is a bisection in Γ of the form $\{[z, 1, \xi, \xi(z)] \mid z \in V\}$ such that $\xi(y) = \eta(x)$ for some $y \in V$. Then

$$[z, k, \eta, \eta(z)] = [z, k - 1, \xi^{-1} \circ \eta, \xi^{-1}(\eta(z))][\xi^{-1}(\eta(z)), 1, \xi, \eta(z)]$$

for all z in a neighborhood $U_x \subseteq U$ of x . It follows that there are functions $h, g \in C_c(G_{J'_R})$ supported in

$$\{[z, k - 1, \xi^{-1} \circ \eta, \xi^{-1}(\eta(z))] \mid z \in U_x\} \quad \text{and} \quad \{[\xi^{-1}(\eta(z)), 1, \xi, \eta(z)] \mid z \in U_x\},$$

respectively, such that $f = hg$ in a neighborhood of $[x, k, \eta, \eta(x)]$. Note that h is then in the $*$ -algebra generated by $C_r^*(G_{J'_R}^0)$ and X_Γ by induction hypothesis and that $g \in X_\Gamma$. We choose functions $\psi_i \in C_c(J'_R)$ forming a finite partition of unity $\{\psi_i\}$ on $r(\text{supp } f)$ such that

$$f = \sum_i \psi_i h_i g_i,$$

where $g_i \in X_\Gamma$ and h_i is in the $*$ -algebra generated by $C_r^*(G_{J'_R}^0)$ and X_Γ . Since $\psi_i \in C_r^*(G_{J'_R}^0)$ for each i , this completes the induction step and hence the proof of the isomorphism $\mathcal{O}_{X_\Gamma} \simeq C_r^*(J'_R)$.

Since $G_{J'_R}$ is the reduction to the open subset J'_R of $J_R \setminus E_R$, the algebra $C_r^*(J'_R)$ is a hereditary C^* -subalgebra of $C_r^*(J_R \setminus E_R)$. The latter algebra is simple by Proposition 4.10, and it follows therefore from [Br] that the two algebras are stably isomorphic. \square

Corollary 4.18. $C_r^*(J_R \setminus E_R)$ is a nuclear C^* -algebra which satisfies the universal coefficient theorem of Rosenberg and Schochet, [RS].

Proof. Due to Proposition 4.17 and Lemma 4.15 the assertions follow from Corollary 7.4 and Proposition 8.8 in [Ka] provided we show that both $C_r^*(G_{J'_R}^0)$ and the ideal $X_\Gamma X_\Gamma^*$ are nuclear and satisfy the UCT. To this end we use the inductive limit decomposition

$$C_r^*(G_{J'_R}^0) = \overline{\bigcup_n C_r^*(G_{J'_R}^0(n))}, \tag{15}$$

cf. (4). We claim that $C_r^*(G_{J'_R}^0(n))$ is liminary for each n . In fact we will show that every irreducible representation of $C_r^*(G_{J'_R}^0(n))$ is finite dimensional for each n . By Theorem 6.2.3 in [Pe] this will show that $C_r^*(G_{J'_R}^0(n))$ is a type I C^* -algebra, and it follows from (15) that $C_r^*(G_{J'_R}^0)$ is nuclear and satisfies the UCT. Furthermore, since we also have the inductive limit decomposition

$$X_\Gamma X_\Gamma^* = \overline{\bigcup_n C_r^*(G_{J'_R}^0(n)) \cap X_\Gamma X_\Gamma^*}, \tag{16}$$

and since we know that every irreducible representation of $C_r^*(G_{J'_R}^0(n)) \cap X_\Gamma X_\Gamma^*$ is finite dimensional if this holds for $C_r^*(G_{J'_R}^0(n))$, we can at the same time conclude that also $X_\Gamma X_\Gamma^*$ is nuclear and satisfies the UCT.

To show that $C_r^*(G_{J'_R}^0(n))$ is liminary, consider an irreducible representation π of $C_r^*(G_{J'_R}^0(n))$. Every function $f \in C(\bar{\mathbb{C}})$ defines a multiplier $\psi(f)$ of $C_r^*(G_{J'_R}^0(n))$ such that

$$\psi(f)g[x, 0, \eta, y] = f(R^n(x))g[x, 0, \eta, y]$$

if $g \in C_c(G_{J'_R}^0(n))$. Note that $\psi(f)$ is central in the multiplier algebra and that ψ is a $*$ -homomorphism. Since π is irreducible, there is a point $z \in \bar{\mathbb{C}}$ such that $\pi(\psi(f)g) = f(z)\pi(g)$ for all $f \in C(\bar{\mathbb{C}})$ and all $g \in C_r^*(G_{J'_R}^0(n))$. Consequently, $\pi(g) = 0$ for every $g \in C_c(G_{J'_R}^0(n))$ whose support does not contain elements from $F = r^{-1}(R^{-n}(z))$. Since all the isotropy groups in $G_{J'_R}^0(n)$ are finite by Lemma 4.2 in [Th2], it follows that F is a finite set. Therefore $\pi(C_c(G_{J'_R}^0(n)))$ must be finite dimensional and the same is then necessarily true for π . \square

Note that the C^* -correspondance X_Γ represents an element

$$[X_\Gamma] \in KK(X_\Gamma X_\Gamma^*, C_r^*(G_{J'_R}^0)).$$

This element defines a homomorphism

$$[X_\Gamma]_*: K_*(X_\Gamma X_\Gamma^*) \rightarrow K_*(C_r^*(G_{J'_R}^0)),$$

which fits into the following six-term exact sequence, cf. Theorem 8.6 in [Ka].

Corollary 4.19. *There is an exact sequence*

$$\begin{array}{ccccc} K_0(X_\Gamma X_\Gamma^*) & \xrightarrow{\text{id}-[X_\Gamma]_*} & K_0(C_r^*(G_{J'_R}^0)) & \xrightarrow{\iota_*} & K_0(C_r^*(J_R \setminus E_R)) \\ \uparrow & & & & \downarrow \\ K_1(C_r^*(J_R \setminus E_R)) & \xleftarrow{\iota_*} & K_1(C_r^*(G_{J'_R}^0)) & \xleftarrow{\text{id}-[X_\Gamma]_*} & K_1(X_\Gamma X_\Gamma^*), \end{array}$$

where $\iota: C_r^*(G_{J'_R}^0) \rightarrow C_r^*(J_R \setminus E_R)$ is the inclusion $C_r^*(G_{J'_R}^0) \subseteq C_r^*(J'_R)$ followed by the stable isomorphism $C_r^*(J'_R) \simeq C_r^*(J_R \setminus E_R)$.

It may seem possible to calculate the K -theory of $C_r^*(J_R \setminus E_R)$ from Corollary 4.19 and the inductive limit decompositions (15) and (16). In practice, however, the task is very difficult, not only because the six-term exact sequence of Corollary 4.19 is less helpful than the one which is available for local homeomorphisms, [DM], [Th3], and which can be applied here if there are no critical points in the Julia set, but also because the topology of J_R is poorly understood in general.

5. The structure of $C_r^*(F_R)$

It is well known and not difficult to see that R takes a connected component W of F_R onto another connected component $R(W)$ of F_R . It follows that we can define an equivalence relation \sim in F_R such that $x \sim y$ if and only if there are $n, m \in \mathbb{N}$ such that $R^n(x)$ and $R^m(y)$ are contained in the same connected component of F_R . By Sullivan’s no-wandering-domain theorem, Theorem 16.4 in [Mi], and a result of Shishikura, [Sh], the set of equivalence classes F_R/\sim is finite, and in fact cannot have more than $2 \deg R - 2$ elements. We can therefore write

$$F_R = \bigsqcup_{i=1}^N \Omega_i,$$

with $N \leq 2 \deg R - 2$, such that each Ω_i is open, $R^{-1}(\Omega_i) = \Omega_i$ and $\Omega_i \cap \Omega_j = \emptyset$ if $i \neq j$. The Ω_i ’s will be called the *stable regions* of R . Since they are RO-invariant, it follows from Lemma 2.2 that

$$C_r^*(F_R) = \bigoplus_{i=1}^N C_r^*(\Omega_i).$$

The stable regions are divided into different types reflecting the fate of their elements under iteration.

Definition 5.1. Let U be an open subset of F_R and $p \in \mathbb{N}$.

(a) U is called a *super-attracting domain* of period p if $R^p(U) \subseteq U$, $R^i(U) \cap U = \emptyset$, $1 \leq i \leq p - 1$, and there are a natural number $d \geq 2$, an $r \in]0, 1[$ and a conformal conjugacy $\psi : U \rightarrow D_r = \{z \in \mathbb{C} \mid |z| < r\}$ such that

$$\begin{array}{ccc} U & \xrightarrow{R^p} & U \\ \psi \downarrow & & \downarrow \psi \\ D_r & \xrightarrow{z \mapsto z^d} & D_r \end{array}$$

commutes.

(b) U is called an *attracting domain* of period p if $R^p(U) \subseteq U$, $R^i(U) \cap U = \emptyset$, $1 \leq i \leq p - 1$, and there are a $\lambda \in \mathbb{C}$, $|\lambda| < 1$, an $r > 0$ and a conformal conjugacy $\psi : U \rightarrow D_r = \{z \in \mathbb{C} \mid |z| < r\}$ such that

$$\begin{array}{ccc} U & \xrightarrow{R^p} & U \\ \psi \downarrow & & \downarrow \psi \\ D_r & \xrightarrow{z \mapsto \lambda z} & D_r \end{array}$$

commutes.

(c) U is called a *parabolic domain* of period p if $R^p(U) \subseteq U$, $R^i(U) \cap U = \emptyset$, $1 \leq i \leq p-1$, and there is a conformal conjugacy $\alpha: U \rightarrow \mathbb{H} = \{z \in \mathbb{C} \mid \operatorname{Re} z > 0\}$ such that

$$\begin{array}{ccc} U & \xrightarrow{R^p} & U \\ \alpha \downarrow & & \downarrow \alpha \\ \mathbb{H} & \xrightarrow{z \mapsto z+1} & \mathbb{H} \end{array}$$

commutes.

(d) U is called a *Siegel disk* of period p if $R^p(U) = U$, $R^i(U) \cap U = \emptyset$, $1 \leq i \leq p-1$, and there are a $t \in \mathbb{R} \setminus \mathbb{Q}$ and a conformal conjugacy $\psi: U \rightarrow D_1 = \{z \in \mathbb{C} \mid |z| < 1\}$ such that

$$\begin{array}{ccc} U & \xrightarrow{R^p} & U \\ \psi \downarrow & & \downarrow \psi \\ D_1 & \xrightarrow{z \mapsto e^{2\pi i t} z} & D_1 \end{array}$$

commutes.

(e) U is called a *Herman ring* of period p if $R^p(U) = U$, $R^i(U) \cap U = \emptyset$, $1 \leq i \leq p-1$, and there are a $t \in \mathbb{R} \setminus \mathbb{Q}$ and a conformal conjugacy $\psi: U \rightarrow \mathbb{A} = \{z \in \mathbb{C} \mid 1 < |z| < 2\}$ such that

$$\begin{array}{ccc} U & \xrightarrow{R^p} & U \\ \psi \downarrow & & \downarrow \psi \\ \mathbb{A} & \xrightarrow{z \mapsto e^{2\pi i t} z} & \mathbb{A} \end{array}$$

commutes.

It follows from the classification of the periodic Fatou components, [Mi], that a stable region Ω contains a domain U of one of the types described in (a)–(e) of Definition 5.1 with the property that

$$\Omega = \bigcup_{n=0}^{\infty} R^{-n}(U). \tag{17}$$

We will say that Ω is a super-attractive, attractive, parabolic, Siegel or Herman region in accordance with the nature of the domain U which we will refer to as a *core* of Ω .

Lemma 5.2. *Let $x \in F_R$ be periodic. Then $\operatorname{RO}(x)$ is closed and discrete in F_R .*

Proof. Let W be a connected component of F_R and $K \subseteq W$ a compact subset. We must show that $K \cap \operatorname{RO}(x)$ is finite. Assume therefore that $W \cap \operatorname{RO}(x) \neq \emptyset$. Let Ω be the stable region containing W and U a core for Ω . It follows from (17) and the

compactness of K that there is an $l \in \mathbb{N}$ such that $R^l(K) \subseteq U$. If $y \in K \cap \text{RO}(x)$ there is a $k > l$ such that $R^k(y) = x$. Thus $R^l(y)$ is a pre-periodic element of U . By inspecting the possible types of U we see that U contains at most one point pre-periodic under R . Thus $R^l(y) = x$, proving that $K \cap \text{RO}(x) \subseteq R^{-l}(x)$, which is a finite set. \square

Lemma 5.3. $\bigcup_{c \in \text{Crit}} \text{RO}(c) = \bigcup_{n=0}^{\infty} R^{-n}(\text{Crit})$.

Proof. Assume that $x \in \bigcup_{c \in \text{Crit}} \text{RO}(c)$. There is a critical point $c \in \text{Crit}$ and $n, m \in \mathbb{N}$ such that $R^n(x) = R^m(c)$ and $\text{val}(R^n, x) = \text{val}(R^m, c)$. Since $\text{val}(R^m, c) \geq 2$, this implies that $\text{val}(R^n, x) = \text{val}(R, R^{n-1}(x)) \text{val}(R, R^{n-2}(x)) \dots \text{val}(R, x) \geq 2$ and hence that $x \in \bigcup_{j=0}^{n-1} R^{-j}(\text{Crit})$. Conversely, if $x \in \bigcup_{n=0}^{\infty} R^{-n}(\text{Crit})$, let $j \in \mathbb{N}$ be the least natural number such that $R^j(x) \in \text{Crit}$. It follows from Lemma 4.2 that $x \in \text{RO}(R^j(x)) \subseteq \bigcup_{c \in \text{Crit}} \text{RO}(c)$. \square

Lemma 5.4. *Suppose that x is a critical point in F_R . Then $\text{RO}(x)$ is closed and discrete in F_R .*

Proof. As in the proof of Lemma 5.2 we take a connected component W of F_R and show that $K \cap \text{RO}(x)$ is finite for any compact subset $K \subseteq W$. Let Ω be the stable region containing W and U a core for Ω of period p . There is an $l \in \mathbb{N}$ such that $R^l(K) \subseteq U$. By inspecting the possible types of U we see that there is at most one element of U which is pre-critical under R , and that element is already critical for R^p if it exists. It follows therefore from Lemma 5.3 that $K \cap \text{RO}(x) \subseteq \bigcup_{n=0}^{l+p} R^{-n}(\text{Crit})$ which is a finite set. \square

Since there are only finitely many periodic and critical points in F_R , the union of their restricted orbits, which we will denote by \mathcal{I} , is closed and discrete in F_R by Lemma 5.2 and Lemma 5.4. It follows therefore from Lemma 3.2 that for each stable region Ω in F_R there is an extension

$$0 \rightarrow C_r^*(\Omega \setminus \mathcal{I}) \rightarrow C_r^*(\Omega) \xrightarrow{\pi_{\mathcal{I}}} C_r^*(\Omega \cap \mathcal{I}) \rightarrow 0. \tag{18}$$

To study the ideal $C_r^*(\Omega \setminus \mathcal{I})$ we need the following lemma, which seems to be folklore for mathematicians working with rational maps. We sketch a proof for the benefit of the operator algebraists.

Lemma 5.5. *Let $U \subseteq \bar{\mathbb{C}}$ be an open simply connected subset such that*

$$U \cap \bigcup_{n=1}^{\infty} R^n(\text{Crit}) = \emptyset.$$

Let $d = \deg R$ be the degree of R . For each $n \in \mathbb{N}$ there are d^n open connected subsets $W_1^n, W_2^n, \dots, W_{d^n}^n$ and d^n holomorphic maps $\chi_i^n: U \rightarrow W_i^n, i = 1, 2, \dots, d^n$, such that

- (i) $R^n(\chi_i^n(z)) = z, z \in U,$
- (ii) $\chi_i^n(U) = W_i^n, i = 1, 2, \dots, d^n,$
- (iii) $W_i^n \cap W_j^n = \emptyset, i \neq j,$ and
- (iv) $R^{-n}(U) = \bigcup_{i=1}^{d^n} W_i^n.$

Proof. First observe that R^n is a d^n -fold covering of U by $R^{-n}(U)$ because $U \cap \bigcup_{j=1}^n R^j(\text{Crit}) = \emptyset$. Let $W_i, i = 1, 2, \dots, N,$ be the connected components of $R^{-n}(U)$. We claim that $R^n(W_i) = U$. To see this let $x \in W_i$ and let $y \in U$. Since $R^n: R^{-n}(U) \rightarrow U$ has the path-lifting property, we can lift a path in U connecting $R^n(x)$ to y in U to a path starting in x . This path must end in a point in W_i which maps to y under R^n , proving the claim. Then $R^n: W_i \rightarrow U$ is also a covering and since U is simply connected, it must be a homeomorphism. Let $\chi_i^n: U \rightarrow W_i$ be its inverse and note that χ_i^n is holomorphic since R^n is. It follows also that $N = d^n$ since $\#R^{-n}(y) = d^n$ for all $y \in U$. The proof is complete. \square

Lemma 5.6. *Let Ω be a stable region for R and $U \subseteq \Omega$ a core for Ω . Then*

$$C_r^*(\Omega \setminus \mathcal{I}) \simeq C_r^*(U \setminus \mathcal{I}) \otimes \mathbb{K}.$$

Proof. Let $x \in \Omega \setminus \mathcal{I}$. It follows from (17) that there is a $k \in \mathbb{N}$ such that $R^k(x) \in U$. Since $x \notin \mathcal{I}$, it follows from Lemma 5.3 and Lemma 4.2 that $R^k(x) \in \text{RO}(x)$. This shows that $\text{RO}(x) \cap (U \setminus \mathcal{I}) \neq \emptyset$, and it then follows from Theorem 2.3 that $C_r^*(\Omega \setminus \mathcal{I})$ is stably isomorphic to $C_r^*(U \setminus \mathcal{I})$. It suffices therefore now to show that $C_r^*(\Omega \setminus \mathcal{I})$ is stable. To this end we use Lemma 2.1 and consider therefore a compact subset K of $\Omega \setminus \mathcal{I}$. The construction of the required bi-sections will be performed differently for the different core domains.

Assume first that U is a Siegel disk or a Herman ring. It follows from Theorem 16.1 in [Mi] that U is a connected component of F_R in these cases. There is an $l \in \mathbb{N}$ such that $R^l(K) \subseteq U$. Note that if U is a Siegel disk, the periodic point at the center of U is not in $R^l(K)$. For each $z \in \text{Crit} \cap \Omega$, let $\chi(z)$ be the first element from U in the forward orbit of z , i.e., $\chi(z) \in U$ is the element determined by the condition that $R^m(z) = \chi(z)$, while $R^i(z) \notin U, i = 0, 1, 2, \dots, m - 1$. Then $\chi(\text{Crit})$ is a finite (possibly empty) set. Since the rotation in the core is irrational and $K \cap \bigcup_{j=0}^\infty R^{-j}(\text{Crit}) = \emptyset$, there is for each point x in K an open neighbourhood V_x of x such that R^i is injective on V_x for all $i \in \mathbb{N}$ and an $n_x \in \mathbb{N}$ with the property that $R^{n_x+l}(V_x) \cap \chi(\text{Crit}) = \emptyset$. We can also arrange that $R^{n_x+l}(V_x)$ is simply connected (e.g. a small disc). By compactness of K there is a finite collection $V_{x_i}, i = 1, 2, \dots, N,$ such that $K \subseteq \bigcup_{i=1}^N V_{x_i}$. Let $U_{-1}, U_{-2}, U_{-3}, \dots$ be a sequence of Fatou components such that $U_{-1} \notin \{U, R(U), R^2(U), \dots, R^{p-1}(U)\}$ and $R(U_{-i}) = U_{-i+1}, i \geq 2$. Such a sequence exists since $R^{-p}(U) \not\subseteq U$. Note that $U_{-i} \cap U_{-j} = \emptyset$ if $i \neq j$. Since K is compact, there is an $m \in \mathbb{N}$ such that

$$K \cap U_{-j} = \emptyset, \quad j \geq m.$$

Let $i \in \{1, 2, \dots, N\}$. By using Lemma 5.5, we can choose a connected subset $V'_i \subseteq U_{-i-m}$ such that R^{i+m} is a homeomorphism from V'_i onto $R^{n_{x_i}+l}(V_{x_i})$. Let $R^{-i-m} : R^{n_{x_i}+l}(V_{x_i}) \rightarrow V'_i$ denote its inverse. Then

$$S_i = \{(R^{-i-m} \circ R^{n_{x_i}+l}(z), i + m - n_{x_i} - l, (R^{-i-m} \circ R^{n_{x_i}+l}|_{V_{x_i}})^{-1}, z) \mid z \in V_i\},$$

$i = 1, 2, \dots, N$, is a collection of open bi-sections in $G_{\Omega \setminus \mathcal{I}}$ meeting the requirements in Lemma 2.1.

Assume instead that U is attracting or parabolic. We choose first L such that $R^L(K) \subseteq U$, and then a finite open and relatively compact cover V_1, V_2, \dots, V_N of K in $\Omega \setminus \mathcal{I}$ such that $R^L : V_i \rightarrow R^L(V_i) \subseteq U$ is injective for each i . Subsequently we choose $n_1, n_2, \dots, n_N \in \mathbb{N}$ such that $R^{n_i}(R^L(V_i)) \cap R^{n_j}(R^L(V_j)) = \emptyset$ if $i \neq j$ and

$$R^{n_i+L}(V_i) \cap K = \emptyset$$

for all i . Then

$$S_i = \{(R^{n_i+L}(z), -n_i - L, (R^{n_i+L}|_{V_i})^{-1}, z) \mid z \in V_i\},$$

$i = 1, 2, \dots, N$, is a collection of open bi-sections in $G_{\Omega \setminus \mathcal{I}}$ meeting the requirements in Lemma 2.1.

Finally, in the super-attractive case choose L such that $R^L(K) \subseteq U$ and $K \cap R^i(K) = \emptyset, i \geq L$. Let U_0 be an open subset of U such that $R^L(K) \subseteq U_0$ and \bar{U}_0 is a compact subset of $U \setminus \{x\}$, where x is the critical point in U . Set

$$Y = R^{-L}(\bigcup_{n=1}^{\infty} R^n(\text{Crit})) \cap K$$

and note that Y is a finite set. For every $z \in Y$ choose a small neighbourhood V_z of z and a natural number n_z such that R^{n_z+L} is injective on $\bar{V}_z, R^{n_z+L}(\bar{V}_z) \subseteq U \setminus \{x\}$ and

$$R^{n_z+L}(V_z) \cap K = \emptyset$$

for all z , and $R^{n_z+L}(\bar{V}_z) \cap R^{n_{z'}+L}(\bar{V}_{z'}) = \emptyset$ if $z \neq z'$. This is possible because $K \cap \mathcal{I} = \emptyset$, cf. Lemma 5.3. Then $R^L(K \setminus \bigcup_{z \in Y} V_z) \cap \bigcup_{n=1}^{\infty} R^n(\text{Crit}) = \emptyset$. We can therefore cover $K \setminus \bigcup_{z \in Y} V_z$ by a finite collection $W_i, i = 1, 2, \dots, N$, of open sets such that $R^L(W_i)$ is an open simply connected subset of $U_0 \setminus \bigcup_{n=1}^{\infty} R^n(\text{Crit})$. It follows then from Lemma 5.5 that for any collection $n_i, i = 1, 2, \dots, N$, of natural numbers we can find univalent holomorphic maps $\chi_i : R^L(W_i) \rightarrow R^{-n_i}(U_0)$ such that $R^{n_i} \circ \chi_i(z) = z$ for all $z \in R^L(W_i)$. Set $U_1 = U_0 \cup \bigcup_{z \in Y} R^{n_z+L}(V_z)$ and note that U_1 is a relatively compact subset of $U \setminus \{x\}$. There is therefore an $N_1 \in \mathbb{N}$ such that $R^i(U_1) \cap U_1 = \emptyset, i \geq N_1$. Thus, if we arrange that $N_1 + L \leq n_1$ and $n_i \geq n_{i-1} + N_1, i = 2, 3, \dots, N$, it follows that the sets

$$\chi_i \circ R^L(W_i), \quad i = 1, 2, \dots, N,$$

are mutually disjoint and are also disjoint from $K \cup U_1$. For each $z \in Y$ the set

$$T_z = \{R^{n_z+L}(z), -n_z - L, (R^{n_z+L}|_{V_z})^{-1}, z) \mid z \in V_z\}$$

is an open bi-section in $G_{\Omega \setminus \mathcal{I}}$. The same is true for

$$S_i = \{(\chi_i \circ R^L(z), n_i - L, (\chi_i \circ R^L|_{W_i})^{-1}, z) \mid z \in W_i\},$$

$i = 1, 2, \dots, N$. Taken together we have a collection of bi-sections in $G_{\Omega \setminus \mathcal{I}}$ with the properties required in Lemma 2.1. \square

5.1. Super-attractive stable regions. In this section we are concerned with the C^* -algebra $C_r^*(\Omega \setminus \mathcal{I})$ in the case where Ω is a super-attractive stable region. Let U be a core domain for Ω . Then $\mathcal{I} \cap U$ consists only of the super-attracting periodic point x at the center of U . It follows from Lemma 5.6 and Proposition 2.5 that $C_r^*(\Omega \setminus \mathcal{I}) \simeq C_r^*(\psi) \otimes \mathbb{K}$, where $\psi: D_r \setminus \{0\} \rightarrow D_r \setminus \{0\}$ for some $r \in]0, 1[$ is the local homeomorphism $\psi(z) = z^d$. Note that $d = \text{val}(R^p, x) \geq 2$, where p is the period of x . Let $D = \{z \in \mathbb{C} \mid 0 < |z| < 1\}$. Define $\alpha: D \rightarrow D$ such that $\alpha(z) = z^d$. Since $D = \bigcup_j \alpha^{-j}(D_r \setminus \{0\})$, it follows from Theorem 2.3 that $C_r^*(\psi)$ is stably isomorphic to $C_r^*(\alpha)$. Thus, $C_r^*(\Omega \setminus \mathcal{I}) \simeq C_r^*(\alpha) \otimes \mathbb{K}$ since $C_r^*(\Omega \setminus \mathcal{I})$ is stable by Lemma 5.6, and in this section we identify the stable isomorphism class of $C_r^*(\alpha)$.

First identify D with $]0, 1[\times \mathbb{T}$ via that map $(t, \lambda) \mapsto t\lambda$. In this picture

$$\alpha(t, \lambda) = (t^d, \lambda^d).$$

Map $]0, 1[\times \mathbb{T}$ to $\mathbb{R} \times \mathbb{T}$ using the map

$$(t, \lambda) \mapsto \left(\frac{\log(-\log t)}{\log d}, \lambda \right).$$

This gives us a conjugacy between (D, α) and $(\mathbb{R} \times \mathbb{T}, \tau \times \beta)$, where $\tau(t) = t + 1$ and $\beta(\lambda) = \lambda^d$. It follows that $C_r^*(\alpha) \simeq C_r^*(\tau \times \beta)$. Let S^1 be the one-point compactification of \mathbb{R} and let τ^+ be the continuous extension of τ to S^1 . To simplify notation set $\varphi = \tau^+ \times \beta$. It follows from Proposition 4.6 in [CT] that there is an extension

$$0 \rightarrow C_r^*(\tau \times \beta) \rightarrow C_r^*(\varphi) \rightarrow C_r^*(\beta) \rightarrow 0. \tag{19}$$

In the notation of Section 2, observe that $R_\varphi = S^1 \times R_\beta$ and that this decomposition gives rise to an isomorphism

$$C_r^*(R_\varphi) \simeq C(S^1) \otimes C_r^*(R_\beta). \tag{20}$$

Under this identification the Deaconu endomorphism $\hat{\varphi}$ of $C_r^*(R_\varphi)$ becomes the tensor product $\hat{\tau} \otimes \hat{\beta}$, where $\hat{\beta}: C_r^*(R_\beta) \rightarrow C_r^*(R_\beta)$ is the Deaconu endomorphism of $C_r^*(\beta)$ and $\hat{\tau}: C(S^1) \rightarrow C(S^1)$ is given by

$$\hat{\tau}(f)(x) = f(\tau^+(x)).$$

Let B_β be the inductive limit of the sequence

$$C_r^*(R_\beta) \xrightarrow{\hat{\beta}} C_r^*(R_\beta) \xrightarrow{\hat{\beta}} C_r^*(R_\beta) \xrightarrow{\hat{\beta}} C_r^*(R_\beta) \xrightarrow{\hat{\beta}} \dots \tag{21}$$

and let $\hat{\beta}_\infty$ be the automorphism of B_β induced by letting $\hat{\beta}$ act on all copies of $C_r^*(R_\beta)$ in the sequence (21). Similarly, we can consider the inductive limit B_φ of the sequence

$$C_r^*(R_\varphi) \xrightarrow{\hat{\varphi}} C_r^*(R_\varphi) \xrightarrow{\hat{\varphi}} C_r^*(R_\varphi) \xrightarrow{\hat{\varphi}} C_r^*(R_\varphi) \xrightarrow{\hat{\varphi}} \dots$$

Using (20) and the tensor product decomposition $\hat{\varphi} = \hat{\tau} \otimes \hat{\beta}$ it follows that $B_\varphi \simeq C(S^1) \otimes B_\beta$ under an isomorphism which turns $\hat{\varphi}_\infty$ into $\hat{\tau} \otimes \hat{\beta}_\infty$. It follows in this way from Theorem 4.8 in [Th1] that there are embeddings of $C_r^*(\tau^+ \times \beta)$ and $C_r^*(\beta)$ into full corners of $(C(S^1) \otimes B_\beta) \rtimes_{\hat{\tau} \otimes \hat{\beta}_\infty} \mathbb{Z}$ and $B_\beta \rtimes_{\hat{\beta}_\infty} \mathbb{Z}$, respectively. Together with the extension (19) this gives us a commuting diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & C_r^*(\tau \times \beta) & \longrightarrow & C_r^*(\tau^+ \times \beta) & \longrightarrow & C_r^*(\beta) \longrightarrow 0 \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & (C_0(\mathbb{R}) \otimes B_\beta) \rtimes_{\hat{\tau} \otimes \hat{\beta}_\infty} \mathbb{Z} & \longrightarrow & (C(S^1) \otimes B_\beta) \rtimes_{\hat{\tau} \otimes \hat{\beta}_\infty} \mathbb{Z} & \longrightarrow & B_\beta \rtimes_{\hat{\beta}_\infty} \mathbb{Z} \longrightarrow 0 \end{array}$$

with exact rows. Consequently the range of the embedding

$$C_r^*(\tau \times \beta) \rightarrow (C_0(\mathbb{R}) \otimes B_\beta) \rtimes_{\hat{\tau} \otimes \hat{\beta}_\infty} \mathbb{Z}$$

is a full corner, and we conclude therefore from [Br] that $C_r^*(\tau \times \beta)$ is stably isomorphic to $(C_0(\mathbb{R}) \otimes B_\beta) \rtimes_{\hat{\tau} \otimes \hat{\beta}_\infty} \mathbb{Z}$.

Recall that the *mapping torus* MT_γ of an endomorphism $\gamma: B \rightarrow B$ of a C^* -algebra B is the C^* -algebra

$$MT_\gamma = \{f \in C[0, 1] \otimes B \mid \gamma(f(0)) = f(1)\}.$$

We need the following lemma.

Lemma 5.7. *Let A be a C^* -algebra and $\alpha: A \rightarrow A$ an automorphism. It follows that the crossed product $(C_0(\mathbb{R}) \otimes A) \rtimes_{\hat{\tau} \otimes \alpha} \mathbb{Z}$ is isomorphic to the mapping torus $MT_{\text{id}_{\mathbb{K}} \otimes \alpha}$ of the automorphism $\text{id}_{\mathbb{K}} \otimes \alpha: \mathbb{K} \otimes A \rightarrow \mathbb{K} \otimes A$.*

Proof. For $f \in C_0(\mathbb{R})$ and $n \in \mathbb{Z}$, let $f_n \in C[0, 1]$ be the function $f_n(t) = f(n+t)$. Let e_{ij} , $i, j \in \mathbb{Z}$, be the standard matrix units in $\mathbb{K} = \mathbb{K}(l^2(\mathbb{Z}))$. Define a $*$ -homomorphism $\pi: C_0(\mathbb{R}) \otimes A \rightarrow C[0, 1] \otimes \mathbb{K} \otimes A$ such that

$$\pi(f \otimes a) = \sum_{n \in \mathbb{Z}} f_n \otimes e_{nn} \otimes \alpha^{-n}(a).$$

Let S be the two-sided shift on $l^2(\mathbb{K})$ such that $Se_{nn}S^* = e_{n-1,n-1}$. Then π maps into the mapping torus of $(\text{Ad } S) \otimes \alpha$ and $\pi \circ (\hat{\tau} \otimes \alpha) = \text{Ad } T \circ \pi$, where $T = 1_{C[0,1]} \otimes S \otimes 1_A$. It follows from the universal property of the crossed product that we get a $*$ -homomorphism

$$\Phi: (C_0(\mathbb{R}) \otimes A) \rtimes_{\hat{\tau} \otimes \alpha} \mathbb{Z} \rightarrow C[0, 1] \otimes \mathbb{K} \otimes A,$$

which is injective since its restriction to $C_0(\mathbb{R}) \otimes A$ clearly is. Its range is generated by elements in $C[0, 1] \otimes \mathbb{K} \otimes A$ of the form $T\pi(f \otimes a)$ and is therefore contained in the mapping torus of $(\text{Ad } S) \otimes \alpha$. To see that

$$B = \Phi((C_0(\mathbb{R}) \otimes A) \rtimes_{\hat{\tau} \otimes \alpha} \mathbb{Z})$$

actually is equal to this mapping torus, let $ev_t: C[0, 1] \otimes \mathbb{K} \otimes A \rightarrow \mathbb{K} \otimes A$ denote evaluation at $t \in [0, 1]$. Then $ev_t(B)$ is generated by elements of the form $e_{n-1,n} \otimes a$ for some $n \in \mathbb{Z}$ and some $a \in A$, and it is easy to see that this is all of $\mathbb{K} \otimes A$. Consider then a continuous function $g: [0, 1] \rightarrow \mathbb{K} \otimes A$ which is an element of $\text{MT}_{(\text{Ad } S) \otimes \alpha}$, i.e., has the property that

$$(\text{Ad } S) \otimes \alpha(g(0)) = g(1).$$

Let $\varepsilon > 0$. For each $t \in [0, 1]$, there is then an element $f_t \in B$ such that $g(t) = f_t(t)$. We can therefore choose intervals $I_j = [\frac{j}{M}, \frac{j+1}{M}]$ and elements $f_j \in B$ such that $\|g(t) - f_j(t)\| \leq \varepsilon, t \in I_j$, for all j , and such that

$$(\text{Ad } S) \circ \alpha(f_0(0)) = f_{M-1}(1).$$

Choose a partition of unity $h_j \in C[0, 1], j = 0, 1, 2, \dots, M - 1$, such that $h_0(0) = h_{M-1}(1) = 1$ and $\text{supp } h_j \subseteq I_j$ for all j . Then $f = \sum_{j=0}^{M-1} h_j f_j \in B$ because B is a module over $\{f \in C[0, 1] \mid f(0) = f(1)\}$. Since $\|f - g\| \leq \varepsilon$, this shows that B is equal to the entire mapping torus of $(\text{Ad } S) \otimes \alpha$. This mapping torus is isomorphic to that of $\text{id}_{\mathbb{K}} \otimes \alpha$ because the automorphism group of \mathbb{K} is connected, cf. Proposition 10.5.1 in [B1]. \square

It follows from Lemma 5.7 and the preceding considerations that $C_r^*(\alpha)$ is stably isomorphic to the mapping torus of $\hat{\beta}_\infty: B_\beta \rightarrow B_\beta$.

It is known that $C_r^*(R_\beta)$ is isomorphic to the Bunce–Deddens algebra $\text{BD}(d^\infty)$ of type d^∞ , cf. Example 3 in [De]. Thus $\text{BD}(d^\infty)$ is the unique simple unital AT-algebra with a unique trace state such that $K_1(\text{BD}(d^\infty)) \simeq \mathbb{Z}$ and $K_0(\text{BD}(d^\infty))$ is isomorphic, as a partially ordered group with order unit, to the group $\mathbb{Z}[1/d]$ of d -adic rationals if the latter has the order inherited from \mathbb{R} and the order unit 1. As shown in Example 3 of [De], the map $\hat{\beta}_*: K_1(\text{BD}(d^\infty)) \rightarrow K_1(\text{BD}(d^\infty))$ is the identity, while $\hat{\beta}_*: K_0(\text{BD}(d^\infty)) \rightarrow K_0(\text{BD}(d^\infty))$ is multiplication by $\frac{1}{d}$ on $\mathbb{Z}[1/d]$. To emphasise the number d , which is the determining input for the construction, we will denote the mapping torus of $\hat{\beta}$ by MT_d in the following.

Lemma 5.8. *The mapping torus $\text{MT}_{\hat{\beta}_\infty}$ of $\hat{\beta}_\infty$ is stably isomorphic to*

$$\text{MT}_d = \{f \in C[0, 1] \otimes C_r^*(\beta) \mid \hat{\beta}(f(0)) = f(1)\},$$

the mapping torus of the Deaconu endomorphism of $C_r^(R_\beta)$.*

Proof. Note that the mapping torus of $\hat{\beta}_\infty$ is isomorphic to the inductive limit

$$\text{MT}_d \xrightarrow{\text{id}_{C[0,1]} \otimes \hat{\beta}} \text{MT}_d \xrightarrow{\text{id}_{C[0,1]} \otimes \hat{\beta}} \text{MT}_d \xrightarrow{\text{id}_{C[0,1]} \otimes \hat{\beta}} \dots \quad (22)$$

Let $\rho_{\infty,1}: C_r(R_\beta) \rightarrow B_\beta$ be the canonical homomorphism out of the first copy of $C_r^*(R_\beta)$ in the sequence (21). As observed in [An], the isometry $V \in C_r^*(\beta)$ which implements the Deaconu endomorphism, in the sense that $\hat{\beta}(a) = VaV^*$, has the property that $V^*C_r^*(R_\beta)V \subseteq C_r^*(R_\beta)$. It follows that $\hat{\beta}(C_r^*(R_\beta)) = VV^*C_r^*(R_\beta)VV^*$ and that

$$\rho_{\infty,1}(C_r^*(R_\beta)) = qB_\beta q$$

where $q = \rho_{\infty,1}(1)$. It follows from the commuting diagram

$$\begin{array}{ccc} \text{MT}_d & \xrightarrow{\quad\quad\quad} & \text{MT}_{\hat{\beta}_\infty} \\ \downarrow & & \downarrow \\ C[0, 1] \otimes C_r^*(R_\beta) & \xrightarrow{\text{id}_{C[0,1]} \otimes \rho_{\infty,1}} & C[0, 1] \otimes B_\beta \\ \uparrow & & \uparrow \\ C_r^*(R_\beta) & \xrightarrow{\rho_{\infty,1}} & B_\beta \end{array}$$

that the image in the mapping torus $\text{MT}_{\hat{\beta}_\infty}$ of the first copy of MT_d from the sequence (22) is equal to

$$\{f \in C[0, 1] \otimes B_\beta \mid \hat{\beta}_\infty(f(0)) = f(1), qf(t) = f(t)q = f(t) \text{ for all } t\}, \quad (23)$$

which is visibly a hereditary C^* -subalgebra of the mapping cone of $\hat{\beta}_\infty$. Since B_β is simple (because $\text{BD}(d^\infty)$ is), it follows that an ideal in $\text{MT}_{\hat{\beta}_\infty}$ which contains the set (23) must have full fiber over every $t \in [0, 1]$. Then a standard partition of unity argument shows, much as in the proof of Lemma 5.7, that such an ideal must be all of $\text{MT}_{\hat{\beta}_\infty}$, i.e., (23) is both hereditary and full in $\text{MT}_{\hat{\beta}_\infty}$. The desired conclusion follows from Corollary 2.6 of [Br]. \square

We can now summarise with the following.

Proposition 5.9. *Let Ω be a super-attractive stable region. Then $C_r^*(\Omega \setminus \mathcal{I})$ is isomorphic to $\mathbb{K} \otimes \text{MT}_d$, where MT_d is the mapping torus of the Deaconu endomorphism on $\text{BD}(d^\infty)$.*

To describe the quotient $C_r^*(\Omega \cap \mathcal{I})$ in (18), we need to determine the restricted orbits in \mathcal{I} and find the isotropy groups of their elements. Note that every periodic point in Ω is RO-equivalent to a critical point in the critical periodic orbit. Hence every RO-equivalence class in $\Omega \cap \mathcal{I}$ is represented by a critical point z in Ω . If z is eventually periodic, it follows from Proposition 4.4 of [Th2] that Is_z is an infinite subgroup of \mathbb{Q}/\mathbb{Z} and hence $C_r^*(Is_z) \simeq C(\widehat{Is_x}) \simeq C(K)$, where K is the Cantor set. If z is not pre-periodic, it follows from Proposition 4.4 of [Th2] that $Is_z = \mathbb{Z}_v$, where v is the asymptotic valency of z . By using Lemma 2.2 and Lemma 4.11 we get in this way a complete description of $C_r^*(\Omega \cap \mathcal{I})$, and we can then put the information we have obtained into (18). To summarise our findings, we introduce the notation \mathbb{K}_x for the C^* -algebra of compact operators on the Hilbert space $l^2(RO(x))$. Thus

$$\mathbb{K}_x = \begin{cases} \mathbb{K} & \text{if } x \text{ is not exposed, and} \\ M_n(\mathbb{C}), & \text{where } n = \#RO(x) \leq 4 \text{ when } x \text{ is exposed.} \end{cases}$$

Theorem 5.10. *Let Ω be a super-attractive stable region and c_1, c_2, \dots, c_{n+m} critical points in Ω such that $\Omega \cap \mathcal{I} = \bigsqcup_{i=1}^{n+m} RO(c_i)$, and c_1, c_2, \dots, c_n are pre-periodic, while $c_{n+1}, c_{n+2}, \dots, c_{n+m}$ are not. Let v_i be the asymptotic valency of c_i , $n + 1 \leq i \leq n + m$. There is an extension*

$$0 \rightarrow \mathbb{K} \otimes MT_d \rightarrow C_r^*(\Omega) \rightarrow \left(\bigoplus_{i=1}^n C(K) \otimes \mathbb{K}_{c_i}\right) \oplus \left(\bigoplus_{i=n+1}^{n+m} \mathbb{C}^{v_i} \otimes \mathbb{K}_{c_i}\right) \rightarrow 0,$$

where K is the Cantor set and MT_d is the mapping torus of the Deaconu endomorphism on the Bunce–Deddens algebra of type d^∞ .

5.2. Attractive stable regions. Let now Ω be an attractive stable region. Let q be an element of the periodic orbit in Ω . The number $\lambda = (R^p)'(q)$, where p is the period of q , is the multiplier of q . It agrees with the number λ from (b) of Definition 5.1. Let now α be the local homeomorphism of D_1 defined such that $\alpha(z) = \lambda z$. By the method used in the previous section, we find that $C_r^*(\Omega \setminus \mathcal{I}) \simeq \mathbb{K} \otimes C_r^*(\alpha)$. Write $\lambda = |\lambda|e^{2\pi i\theta}$, where $\theta \in [0, 1[$, so that α can be realised as the map on $]0, 1[\times \mathbb{T}$ given by

$$(t, \mu) \mapsto (|\lambda|t, \mu e^{2\pi i\theta}).$$

Via the map $(t, \mu) \rightarrow \left(\frac{\log t}{\log |\lambda|}, \mu\right)$ we see that α is conjugate to the map $(t, \mu) \mapsto (t + 1, \mu e^{2\pi i\theta})$ on $\mathbb{R}_+ \times \mathbb{T}$. The transformation groupoid of the last map is a reduction of the transformation groupoid of the homeomorphism $(t, \lambda) \mapsto (t + 1, \lambda e^{2\pi i\theta})$ on $\mathbb{R} \times \mathbb{T}$. Hence $C_r^*(\alpha)$ is stably isomorphic to the corresponding crossed product $C_0(\mathbb{R} \times \mathbb{T}) \rtimes \mathbb{Z}$ by Theorem 2.3. It follows from Lemma 5.7 that the latter crossed product is isomorphic to $\mathbb{K} \otimes C(\mathbb{T}^2)$. In this way we obtain the following.

Proposition 5.11. *Let Ω be an attractive stable region. Then*

$$C_r^*(\Omega \setminus \mathcal{I}) \simeq \mathbb{K} \otimes C(\mathbb{T}^2).$$

It is also straightforward to adopt the methods from the preceding section to obtain a description of $C_r^*(\Omega \cap \mathcal{I})$. The periodic points lie in the same restricted orbit and the isotropy group of any of its members is a copy of \mathbb{Z} by Proposition 4.4 of [Th2]. The restricted orbits of the critical points are divided according to whether or not they are pre-periodic. Since the periodic orbit is not critical, the isotropy group of a critical pre-periodic point is now $\mathbb{Z} \oplus \mathbb{Z}_d$, where d is the asymptotic valency by Proposition 4.4 of [Th2]. This leads to the following description of $C_r^*(\Omega)$.

Theorem 5.12. *Let Ω be an attractive stable region and q a periodic point in Ω . Let c_1, c_2, \dots, c_{n+m} be critical points in Ω such that*

$$\Omega \cap \mathcal{I} = \text{RO}(q) \sqcup \bigsqcup_{i=1}^{n+m} \text{RO}(c_i),$$

and c_1, c_2, \dots, c_n are pre-periodic while $c_i, i \geq n + 1$, are not. Let v_i be the asymptotic valency of c_i . There is an extension

$$0 \rightarrow \mathbb{K} \otimes C(\mathbb{T}^2) \rightarrow C_r^*(\Omega) \rightarrow A \rightarrow 0,$$

where

$$A = (C(\mathbb{T}) \otimes \mathbb{K}_q) \oplus \left(\bigoplus_{i=1}^n \mathbb{C}^{v_i} \otimes C(\mathbb{T}) \otimes \mathbb{K}_{c_i}\right) \oplus \left(\bigoplus_{i=n+1}^{n+m} \mathbb{C}^{v_i} \otimes \mathbb{K}_{c_i}\right).$$

5.3. Parabolic stable regions. The remaining cases corresponding to stable regions of parabolic, Siegel or Herman type can be handled by similar methods. Since the considerations are simpler than those involved in the attractive cases, we merely state the results.

Theorem 5.13. *Let Ω be parabolic stable region. Let $c_i, i = 1, 2, \dots, N$, be representatives for the restricted orbits of the critical points in Ω and let v_i be the asymptotic valency of c_i . There is an extension*

$$0 \rightarrow \mathbb{K} \otimes C(\mathbb{T}) \otimes C_0(\mathbb{R}) \rightarrow C_r^*(\Omega) \rightarrow \bigoplus_{i=1}^N \mathbb{C}^{v_i} \otimes \mathbb{K}_{c_i} \rightarrow 0.$$

5.4. Stable regions of Siegel type. Let $\theta \in [0, 1] \setminus \mathbb{Q}$. The corresponding irrational rotation algebra is the universal C^* -algebra generated by two unitaries U, V satisfying the relation $UV = e^{2\pi i \theta} VU$. See [EE] for more on its structure.

Theorem 5.14. *Let Ω be a stable region of Siegel type. Let q be a periodic point in Ω . Let c_1, c_2, \dots, c_{n+m} be critical points in Ω such that*

$$\Omega \cap \mathcal{I} = \text{RO}(q) \sqcup \bigsqcup_{i=1}^{n+m} \text{RO}(c_i),$$

and c_1, c_2, \dots, c_n are pre-periodic, while $c_i, i \geq n + 1$, are not. Let v_i be the asymptotic valency of c_i . There is an extension

$$0 \rightarrow \mathbb{K} \otimes C_0(\mathbb{R}) \otimes A_\theta \rightarrow C_r^*(\Omega) \rightarrow B \rightarrow 0,$$

where A_θ is the irrational rotation algebra corresponding to the rotation by the angle $2\pi\theta$ in the core domain and

$$B = (C(\mathbb{T}) \otimes \mathbb{K}_q) \oplus \left(\bigoplus_{i=1}^n \mathbb{C}^{v_i} \otimes C(\mathbb{T}) \otimes \mathbb{K}_{c_i}\right) \oplus \left(\bigoplus_{i=n+1}^{n+m} \mathbb{C}^{v_i} \otimes \mathbb{K}_{c_i}\right).$$

5.5. Stable regions of Herman type

Theorem 5.15. *Let Ω be a stable region of Herman type. Let $c_i, i = 1, 2, \dots, N$, be representatives for the restricted orbits of the critical points in Ω and let v_i be the asymptotic valency of c_i . There is an extension*

$$0 \rightarrow \mathbb{K} \otimes C_0(\mathbb{R}) \otimes A_\theta \rightarrow C_r^*(\Omega) \rightarrow \bigoplus_{i=1}^N \mathbb{C}^{v_i} \otimes \mathbb{K}_{c_i} \rightarrow 0,$$

where A_θ is the irrational rotation algebra corresponding to the rotation by the angle $2\pi\theta$ in the core domain.

5.6. A square of six extensions. It is possible to combine the extensions from the last sections into an exact square of 6 extensions in the following way. Let \mathcal{I}_p be the union of the RO-orbits containing a non-critical periodic orbit in F_R and $\mathcal{I}_c = \mathcal{I} \setminus \mathcal{I}_p$ its complement in \mathcal{I} . Several applications of Lemma 3.2 gives us the following commuting diagram with exact rows and columns.

$$\begin{array}{ccccccc}
 & & 0 & & 0 & & 0 \\
 & & \downarrow & & \downarrow & & \downarrow \\
 0 & \longrightarrow & C_r^*(F_R \setminus \mathcal{I}) & \longrightarrow & C_r^*(F_R \setminus \mathcal{I}_c) & \longrightarrow & C_r^*(\mathcal{I}_p) \longrightarrow 0 \\
 & & \downarrow & & \downarrow & & \downarrow \\
 0 & \longrightarrow & C_r^*(F_R \setminus \mathcal{I}_p) & \longrightarrow & C_r^*(R) & \longrightarrow & C_r^*(J_R \cup \mathcal{I}_p) \longrightarrow 0 \\
 & & \downarrow & & \downarrow & & \downarrow \\
 0 & \longrightarrow & C_r^*(\mathcal{I}_c) & \longrightarrow & C_r^*(J_R \cup \mathcal{I}_c) & \longrightarrow & C_r^*(J_R) \longrightarrow 0 \\
 & & \downarrow & & \downarrow & & \downarrow \\
 & & 0 & & 0 & & 0
 \end{array} \tag{24}$$

The algebras in the corners $C_r^*(F_R \setminus \mathcal{I})$, $C_r^*(\mathcal{I}_p)$, $C_r^*(\mathcal{I}_c)$ and $C_r^*(J_R)$ can all be identified from the preceding sections. Specifically, $C_r^*(J_R)$ is either nuclear, simple and purely infinite, or an extension of such an algebra by a finite direct sum of circle and matrix algebras, cf. Theorem 4.13. $C_r^*(F_R \setminus \mathcal{I})$ is a finite direct sum of algebras, each of which is the stabilization of MT_d , $C(\mathbb{T}^2)$, $C(\mathbb{T}) \otimes C_0(\mathbb{R})$ or $C_0(\mathbb{R}) \otimes A_\theta$. Which of the four types are present, depends on the nature of the stable regions in F_R . The algebra $C_r^*(\mathcal{I}_p)$ is a finite direct sum of algebras stably isomorphic to $C(\mathbb{T})$, while $C_r^*(\mathcal{I}_c)$ is a finite direct sum of algebras stably isomorphic to \mathbb{C} , $C(\mathbb{T})$ or

$C(K)$. Which summands occur, depends on the behaviour under iteration of the critical points in F_R .

It should be noted that the decomposition of $C_r^*(R)$ depicted in (24) is not the only possible. In fact there is a commuting square of the form (24) for any RO-invariant partitioning of \mathcal{I} , not just for the partition $\mathcal{I} = \mathcal{I}_p \sqcup \mathcal{I}_p$ chosen above.

6. Primitive ideals and primitive quotients

In the following an ideal in a C^* -algebra is a closed two-sided and proper ideal. Recall that an ideal I is *primitive* if it is the kernel of an irreducible non-zero representation, and *prime* if it has the property that $I_1 I_2 \subseteq I$ implies that $I_1 \subseteq I$ or $I_2 \subseteq I$ when I_1 and I_2 are also ideals. Since we shall only deal with separable C^* -algebras, the primitive ideals will be the same as the prime ideals, cf. e.g. [RW].

6.1. The primitive ideals. If I is an ideal in $C_r^*(R)$, we set

$$\rho(I) = \{x \in \overline{C} \mid f(x) = 0 \text{ for all } f \in C(\overline{C}) \cap I\}.$$

We call $\rho(I)$ the *co-support* of I .

Lemma 6.1. $\rho(I)$ is a closed non-empty RO-invariant subset of \overline{C} .

Proof. See Lemma 4.5 in [CT]. □

Lemma 6.2. Let I be an ideal in $C_r^*(R)$ and let A be a closed RO-invariant subset of \overline{C} . If $\rho(I) \subseteq A$, then $\ker \pi_A \subseteq I$.

Proof. See Lemma 4.8 in [CT]. □

If $I \subseteq C_r^*(R)$ is an ideal, we let $q_I : C_r^*(R) \rightarrow C_r^*(R)/I$ denote the corresponding quotient map. Note that it follows from Lemma 6.2 that q_I factorises through $C_r^*(\rho(I))$, i.e., there is a $*$ -homomorphism $C_r^*(\rho(I)) \rightarrow C_r^*(R)/I$ such that

$$\begin{array}{ccc}
 C_r^*(R) & \xrightarrow{q_I} & C_r^*(R)/I \\
 \searrow \pi_{\rho(I)} & & \nearrow \\
 & C_r^*(\rho(I)) &
 \end{array}
 \tag{25}$$

commutes.

A non-empty closed RO-invariant subset $A \subseteq \overline{C}$ is *prime* if the implication

$$A \subseteq B \cup C \implies A \subseteq B \text{ or } A \subseteq C$$

holds for all closed RO-invariant subsets B and C of \overline{C} .

Lemma 6.3. *Assume that I is a primitive ideal in $C_r^*(R)$. It follows that $\rho(I)$ is prime.*

Proof. See Proposition 4.10 in [CT]. □

Lemma 6.4. *Let Y be a prime subset of \overline{C} . Assume that $x \in Y$ is isolated in Y . Then all elements of $\text{RO}(x)$ are isolated in Y and $Y = \overline{\text{RO}(x)}$.*

Proof. It is clear that all elements of $\text{RO}(x)$ are isolated in Y since x is. Set

$$B = \overline{\{z \in Y \mid z \notin \text{RO}(x)\}}.$$

Since $Y \subseteq \overline{\text{RO}(x)} \cup B$, the primeness of Y implies $Y \subseteq \overline{\text{RO}(x)}$ or $Y \subseteq B$. Note that $x \notin B$ since x is isolated in Y . It follows that $Y \subseteq \overline{\text{RO}(x)}$. □

In the following we denote by $\text{Orb}(x)$ the (full) orbit of x , i.e.,

$$\text{Orb}(x) = \{y \in \overline{C} \mid R^n(x) = R^m(y) \text{ for some } n, m \in \mathbb{N}\}.$$

Lemma 6.5. *Let Y be a prime subset of \overline{C} . Assume Y has no isolated points. It follows that there is a point $x \in Y \setminus \bigcup_{j=0}^{\infty} R^{-j}(\text{Crit})$ such that $Y = \overline{\text{RO}(x)} = \overline{\text{Orb}(x)}$.*

Proof. The proof is largely the same as the proof of Proposition 4.9 in [CT], but with a few crucial modifications. It follows from Lemma 4.4 that Y is totally R -invariant and hence in particular that $\overline{\text{RO}(x)} \subseteq \overline{\text{Orb}(x)} \subseteq Y$ for all $x \in Y$. It suffices therefore to find an $x \in Y \setminus \bigcup_{j=0}^{\infty} R^{-j}(\text{Crit})$ such that $Y \subseteq \overline{\text{RO}(x)}$. Let $\{U_k\}_{k=1}^{\infty}$ be a basis for the topology of Y . We will by induction construct compact sets $\{C_k\}_{k=0}^{\infty}$ and $\{C'_k\}_{k=0}^{\infty}$ with non-empty interiors in Y and positive integers $(n_k)_{k=0}^{\infty}$ and $(n'_k)_{k=0}^{\infty}$ such that

- (i) $C_k \subseteq U_k$,
- (ii) $C'_k \subseteq R^{n_{k-1}}(C_{k-1}) \cap R^{n'_{k-1}}(C'_{k-1})$ if $k \geq 1$, and
- (iii) $C'_k \cap (\bigcup_{j=0}^{n'_0+n'_1+\dots+n'_{k-1}} R^j(\text{Crit}) \cup \bigcup_{j=0}^{n_{k-1}} R^j(\text{Crit})) = \emptyset$ if $k \geq 1$.

Let $C_0 = C'_0$ be any compact subset of Y with non-empty interior in Y . Assume that $k \geq 1$ and that $C_1, \dots, C_k, C'_1, \dots, C'_k, n_0, \dots, n_{k-1}$ and n'_0, \dots, n'_{k-1} satisfying the conditions above have been chosen. Choose non-empty open subsets $V_k \subseteq C_k$ and $V'_k \subseteq C'_k$. Then

$$\bigcup_{l,m=0}^{\infty} R^{-l}(R^m(V_k)) \quad \text{and} \quad \bigcup_{l,m=0}^{\infty} R^{-l}(R^m(V'_k))$$

are non-empty open and totally R -invariant subsets of Y , and hence

$$Y \setminus \bigcup_{l,m=0}^{\infty} R^{-l}(R^m(V_k)) \quad \text{and} \quad Y \setminus \bigcup_{l,m=0}^{\infty} R^{-l}(R^m(V'_k)) \tag{26}$$

are closed and totally R -invariant subsets of Y . Since Y is prime and not contained in either of the sets from (26), it is also not contained in their union. That is,

$$\left(\bigcup_{l,m=0}^{\infty} R^{-l}(R^m(V_k))\right) \cap \left(\bigcup_{l,m=0}^{\infty} R^{-l}(R^m(V'_k))\right) \neq \emptyset.$$

It follows that there are positive integers n_k and n'_k such that $R^{n_k}(V_k) \cap R^{n'_k}(V'_k)$ is non-empty. We can therefore choose a non-empty compact set C_{k+1} with non-empty interior such that $C_{k+1} \subseteq U_{k+1}$, and a non-empty compact set C'_{k+1} with non-empty interior such that $C'_{k+1} \subseteq R^{n_k}(V_k) \cap R^{n'_k}(V'_k)$. Since

$$\bigcup_{j=0}^{n'_0+n'_1+\dots+n'_k} R^j(\text{Crit}) \cup \bigcup_{j=0}^{n_k} R^j(\text{Crit})$$

is a finite set and Y contains no isolated points, we can arrange that

$$C'_{k+1} \cap \left(\bigcup_{j=0}^{n'_0+n'_1+\dots+n'_k} R^j(\text{Crit}) \cup \bigcup_{j=0}^{n_k} R^j(\text{Crit})\right) = \emptyset.$$

This completes the induction step.

It follows from (ii) that

$$C'_k = R^{n'_0+n'_1+\dots+n'_{k-1}}(C'_0 \cap R^{-n'_0}(C'_1) \cap \dots \cap R^{-n'_0-n'_1-\dots-n'_{k-1}}(C'_k))$$

for all k and hence

$$C'_0 \cap R^{-n'_0}(C'_1) \cap \dots \cap R^{-n'_0-\dots-n'_k}(C'_{k+1}), \quad k = 0, 1, \dots,$$

is a decreasing sequence of non-empty compact sets. Let

$$x \in \bigcap_{k=0}^{\infty} R^{-n'_0-\dots-n'_k}(C'_{k+1}) \cap C'_0.$$

By construction there is for each k an element $u_k \in U_k$ such that $R^{n'_0+\dots+n'_k}(x) = R^{n_k}(u_k)$ and

$$\text{val}(R^{n'_0+\dots+n'_k}, x) = \text{val}(R^{n_k}, u_k) = 1.$$

Since this implies that $u_k \in \text{RO}(x)$, we have that $\text{RO}(x)$ is dense in Y . Furthermore, it also implies that $\text{val}(R, R^j(x)) = 1$ for all j , i.e., $x \notin \bigcup_{j=0}^{\infty} R^{-j}(\text{Crit})$. \square

Corollary 6.6. *Let $Y \subseteq \overline{\mathbb{C}}$ be a closed RO-invariant subset. Then Y is prime if and only if there is a point $x \in \overline{\mathbb{C}}$ such that $Y = \overline{\text{RO}(x)}$.*

Proof. If Y is prime, it follows from Lemma 6.4 and Lemma 6.5 that there is an element $x \in \overline{\mathbb{C}}$ such that $Y = \overline{\text{RO}(x)}$. This proves the necessity of the condition. Sufficiency follows immediately from the definitions. \square

Let \mathcal{M} be the set of prime subsets of $\overline{\mathbb{C}}$. Let \mathcal{M}_{ex} denote the collection of elements $Y \in \mathcal{M}$ with the property that Y contains an isolated point which is either periodic or critical.

Lemma 6.7. *Let $Y \in \mathcal{M} \setminus \mathcal{M}_{\text{ex}}$. It follows that $\ker \pi_Y$ is the only ideal I in $C_r^*(R)$ with $\rho(I) = Y$, and $\ker \pi_Y$ is a primitive ideal in $C_r^*(R)$.*

Proof. Let I be an ideal in $C_r^*(R)$ with $\rho(I) = Y$. Then $\ker \pi_Y \subseteq I$ by Lemma 6.2. To conclude that $I = \ker \pi_Y$ it suffices therefore to show that $\pi_Y(I) = \{0\}$ in $C_r^*(Y)$. To this end note first that $\pi_Y(I) \cap C(Y) = \{0\}$. Indeed, if $h \in \pi_Y(I) \cap C(Y)$, let $g \in C(\overline{\mathbb{C}})$ be a function such that $g|_Y = h$ and let $a \in I$ be an element such that $\pi_Y(a) = h$. Then $\pi_Y(a - g) = 0$ and hence $a - g \in \ker \pi_Y \subseteq I$. It follows that $g = a - (a - g) \in I \cap C(\overline{\mathbb{C}})$ and so $g(y) = 0$ for all $y \in \rho(I) = Y$. Thus $h = 0$, proving that $\pi_Y(I) \cap C(Y) = \{0\}$. To conclude from this that $\pi_Y(I) = 0$, note first that the elements of Y with non-trivial isotropy in G_Y are dense in Y . This follows from Lemma 4.3 because a point $y \in Y$ with non-trivial isotropy in G_Y must be pre-periodic or pre-critical for R by Proposition 4.4 a) in [Th2]. It then follows from Lemma 2.15 of [Th1] that $P(\pi_Y(I)) = \{0\}$ if $P : C_r^*(Y) \rightarrow C(Y)$ denotes the conditional expectation. Since P is faithful, this shows that $\pi_Y(I) = 0$ and hence that $I = \ker \pi_Y$.

To show that $\ker \pi_Y$ is primitive we may as well show that $C_r^*(Y)$ is a prime C^* -algebra. Consider therefore two ideals $I_j \subseteq C_r^*(Y)$, $j = 1, 2$, such that $I_1 I_2 = \{0\}$. Then

$$\begin{aligned} & \{y \in Y \mid f(y) = 0 \text{ for all } f \in I_1 \cap C(Y)\} \\ & \cup \{y \in Y \mid f(y) = 0 \text{ for all } f \in I_2 \cap C(Y)\} = Y. \end{aligned}$$

By Corollary 6.6, there is an element $x \in Y$ such that $Y = \overline{\text{RO}(x)}$. Then x must be in $\{y \in Y \mid f(y) = 0 \text{ for all } f \in I_j \cap C_0(Y)\}$ for either $j = 1$ or $j = 2$. Assume without loss of generality that $x \in \{y \in Y \mid f(y) = 0 \text{ for all } f \in I_1 \cap C(Y)\}$. The latter set is both closed and RO-invariant, so we conclude that

$$Y = \overline{\text{RO}(x)} = \{y \in Y \mid f(y) = 0 \text{ for all } f \in I_1 \cap C(Y)\},$$

i.e., $I_1 \cap C(Y) = \{0\}$. As above we conclude that $I_1 = \{0\}$ due to Lemma 2.15 of [Th1]. □

Let $Y \in \mathcal{M}_{\text{ex}}$ and let $y \in Y$ be an isolated point which is either periodic or critical. By Proposition 4.4 of [Th2], the isotropy group Is_y is abelian and in fact either \mathbb{Z} , a non-zero subgroup of \mathbb{Q}/\mathbb{Z} or isomorphic to $\mathbb{Z} \oplus \mathbb{Z}_d$ for some $d \in \mathbb{N}$. Let $\widehat{\text{Is}}_y$ be its Pontryagin dual group. Since y is isolated in Y , every element $\xi \in \text{Is}_y$ is isolated in G_Y and hence the characteristic function 1_ξ of the set $\{\xi\}$ is an element of $C_c(G_Y) \subseteq C_r^*(Y)$. For each $\omega \in \widehat{\text{Is}}_y$ set

$$I(y, \omega) = \pi_Y^{-1}(I_0(y, \omega)),$$

where $I_0(y, \omega)$ is the ideal in $C_r^*(Y)$ generated by the elements

$$1_{[y,0,\text{id},y]} - \overline{\omega(\xi)}1_\xi, \quad \xi \in \text{Is}_y.$$

By adopting the proof of Proposition 4.15 from [CT] in a straightforward way we obtain the following.

Lemma 6.8. *Let $Y \in \mathcal{M}_{\text{ex}}$ and let $y \in Y$ be an isolated point. Then the map $\widehat{\text{Is}}_y \ni \omega \mapsto I(y, \omega)$ is a bijection from $\widehat{\text{Is}}_y$ onto the collection of primitive ideals I in $C_r^*(R)$ with $\rho(I) = Y$.*

In particular, it follows from Lemma 6.8 and Lemma 6.7 that every prime subset of $\widehat{\mathbb{C}}$ is the co-support of a primitive ideal in $C_r^*(R)$. By combining Lemma 6.8 with Lemma 6.7 we get the following.

Lemma 6.9. *For each $A \in \mathcal{M}_{\text{ex}}$ choose an isolated point y_A in A which is either periodic or critical. Then the set of primitive ideals in $C_r^*(R)$ is the disjoint union*

$$\{\ker \pi_B \mid B \in \mathcal{M} \setminus \mathcal{M}_{\text{ex}}\} \cup \bigcup_{A \in \mathcal{M}_{\text{ex}}} \{I(y_A, \omega) \mid \omega \in \widehat{\text{Is}}_{y_A}\}.$$

Lemma 6.10. *Let $A \in \mathcal{M}_{\text{ex}}$. There is either an exposed point $x \in E_R$ such that $A = \text{RO}(x)$, or a critical or a periodic point $x \in F_R \setminus E_R$ such that $A = \text{RO}(x) \sqcup J_F$.*

Proof. Let x be periodic or critical point such that x is isolated in A and $A = \text{RO}(x)$. If $x \in J_R$, it follows from Lemma 4.5 that $\text{RO}(x)$ is finite since J_R has no isolated points, i.e., x is exposed. Assume $x \in F_R$. It follows from Lemma 5.2 and Lemma 5.4 that $\overline{\text{RO}(x)} \setminus \text{RO}(x) \subseteq J_R$. Since $\overline{\text{RO}(x)} \setminus \text{RO}(x)$ is closed, RO -invariant and has no isolated points, it follows from Lemma 4.5 that $\overline{\text{RO}(x)} \setminus \text{RO}(x) = \emptyset$ or $\overline{\text{RO}(x)} \setminus \text{RO}(x) = J_R$. In the first case x is exposed and in the second we have $A = \text{RO}(x) \sqcup J_F$. □

Lemma 6.11. *Let $A \in \mathcal{M} \setminus \mathcal{M}_{\text{ex}}$. Then either $A = J_R$ or $A = \overline{\text{RO}(x)}$ for some $x \in F_R \setminus \mathcal{I}$. In the last case, $J_R \subseteq A$.*

Proof. Let $x \in A$ such that $\overline{\text{RO}(x)} = A$. If $x \in J_R$, it follows from Lemma 4.5 that $A = J_R$ or A is finite. In the last case $A = \text{RO}(x)$ is an exposed RO -orbit which must contain either a periodic point or a critical point, cf. Section 4.2. Since this is impossible when $A \notin \mathcal{M}_{\text{ex}}$, we must have $A = J_R$.

Assume that $x \in F_R$. If A contains an isolated point y , it follows from Lemma 6.4 that $A = \overline{\text{RO}(y)}$. Note that $y \in F_R$ because $y \in \text{RO}(x)$ and F_R is totally R -invariant. It follows that $y \notin \mathcal{I}$ since $A \notin \mathcal{M}_{\text{ex}}$. To prove that $J_R \subseteq A$ note that $R^n(y) \in \text{RO}(y)$ for all $n \in \mathbb{N}$ since $y \notin \bigcup_{i=0}^\infty R^{-i}(\text{Crit})$. Thus y cannot be pre-periodic since this would contradict that $A \notin \mathcal{M}_{\text{ex}}$. Now an argument from the proof of Lemma 7.3 in

[Th2] shows that there is an $n \in \mathbb{N}$ such that the backward orbit of $R^n(y)$ contains no critical points. Then the backward orbit of $R^n(y)$ is contained in $\text{RO}(y)$, and since $R^n(y)$ is not exceptional, it follows therefore from Theorem 4.2.5 in [B] that $J_R \subseteq \overline{\text{RO}(x)}$.

If A has no isolated points, it follows from Lemma 6.5 that there is a point $y \in A \setminus \bigcup_{j=0}^{\infty} R^{-j}(\text{Crit})$ such that $A = \overline{\text{RO}(y)}$. Note that y cannot be pre-periodic because $A \notin \mathcal{M}_{\text{ex}}$. Since the asymptotic valency of y is 1, it follows that $y \notin \mathcal{I}$, i.e., $y \in F_R \setminus \mathcal{I}$. As above it follows that there is an $n \in \mathbb{N}$ such that the backward orbit of $R^n(y)$ is in $\text{RO}(y)$ and again $J_R \subseteq A$. \square

We can now show that the primitive ideal space of $C_r^*(R)$ is not Hausdorff, or even T_0 , in the hull-kernel topology unless $J_R = \overline{C}$ and there are no exposed points. Indeed, if $J_R \neq \overline{C}$ it follows from Lemma 4.3 that F_R contains a point y which is neither pre-critical nor pre-periodic or exposed. Then Lemma 6.10 combined with Lemma 5.4 and Lemma 5.2 shows that $A = \overline{\text{RO}(y)} \in \mathcal{M} \setminus \mathcal{M}_{\text{ex}}$. Furthermore, $J_R \subseteq A$ by Lemma 6.11. If $J_R = \overline{C}$ and there is an exposed point, its restricted orbit will be an element $B \in \mathcal{M}_{\text{ex}}$ such that $B \subsetneq J_R$. In the first case it follows from Lemma 6.2 that $\ker \pi_A \subsetneq \ker \pi_{J_R}$ so that $\ker \pi_{J_R}$ is in the closure of $\{\ker \pi_A\}$ with respect to the hull-kernel topology. In the second case $\{0\} = \ker \pi_{J_R} \subsetneq \ker \pi_B \subseteq I(y_B, \omega)$ for any $y_B \in B$ and any $\omega \in \widehat{\text{Is}}_{y_B}$, and then $I(y_B, \omega)$ is a primitive ideal in the closure of $\{\ker \pi_{J_R}\}$. In both cases we conclude that the primitive ideal spectrum is not T_0 . Note that if $J_R = \overline{C}$ and there are no exposed points, $C_r^*(R)$ is simple by Proposition 4.10 and the primitive ideal spectrum reduces to one point.

6.2. The primitive quotients. It follows from Lemma 6.10 and Lemma 6.11 that we can divide the primitive ideals I of $C_r^*(R)$ into four types, according to the nature of their co-supports:

- (i) $\rho(I) = J_R$,
- (ii) $\rho(I) = \text{RO}(x)$ for some exposed point x ,
- (iii) $\rho(I) = \text{RO}(x) \sqcup J_R$ for some $x \in \mathcal{I} \cap F_R \setminus E_R$, and
- (iv) $\rho(I) = \overline{\text{RO}(x)}$ for some $x \in F_R \setminus \mathcal{I}$.

If $\rho(I) = J_R$, the quotient $C_r^*(R)/I$ is $C_r^*(J_R)$, whose structure was elucidated in Section 4. If $\rho(I) = \text{RO}(x)$ for some exposed point, it follows from (25), Lemma 4.11 and Corollary 4.7 that $C_r^*(R)/I \simeq M_n(\mathbb{C})$ for some $n \leq 4$. In the case (iii) it follows first from Lemma 3.2 and Lemma 4.11 that there is an extension

$$0 \rightarrow C^*(\text{Is}_x) \otimes \mathbb{K} \rightarrow C_r^*(\rho(I)) \rightarrow C_r^*(J_R) \rightarrow 0$$

and from (25) and Lemma 6.8 that there is an extension

$$0 \rightarrow \mathbb{K} \rightarrow C_r^*(R)/I \rightarrow C_r^*(J_R) \rightarrow 0.$$

It remains to describe the primitive quotient $C_r^*(R)/I$ in case (iv). The result depends very much on which stable region the point $x \in F_R \setminus \mathcal{I}$ that generates $\rho(I)$ comes from. We consider the different possibilities in the following subsections.

6.2.1. The super-attractive and attractive stable regions. Assume x is contained in a super-attracting stable region Ω . It follows from Lemma 6.7 that $C_r^*(R)/I \simeq C_r^*(\rho(I))$. Since $I \cap C_r^*(\Omega \setminus \mathcal{I})$ is a primitive ideal in $C_r^*(\Omega \setminus \mathcal{I})$, it follows that

$$C_r^*(\rho(I) \cap \Omega \setminus \mathcal{I}) = C_r^*(\Omega \setminus \mathcal{I})/I$$

is a primitive quotient of $C_r^*(\Omega \setminus \mathcal{I})$ and hence isomorphic to the stabilised Bunce–Deddens algebra $\mathbb{K} \otimes \text{BD}(d^\infty)$ by Proposition 5.9. When we apply the method from Section 5.6 to $C_r^*(\rho(I))$ rather than to $C_r^*(R)$, we thus obtain the following commuting diagram with exact rows and columns because there are no periodic non-critical orbits.

$$\begin{array}{ccccccc}
 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \\
 & & \text{BD}(d^\infty) \otimes \mathbb{K} & \xlongequal{\quad} & \text{BD}(d^\infty) \otimes \mathbb{K} & & \\
 & & \downarrow & & \downarrow & & \\
 0 & \longrightarrow & C_r^*(\rho(I) \cap F_R) & \longrightarrow & C_r^*(R)/I & \longrightarrow & C_r^*(J_R) \longrightarrow 0 \\
 & & \downarrow & & \downarrow & & \parallel \\
 0 & \longrightarrow & A & \longrightarrow & C_r^*(J_R \cup (\mathcal{I}_c \cap \rho(I))) & \longrightarrow & C_r^*(J_R) \longrightarrow 0 \\
 & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & &
 \end{array}$$

Here

$$A = C_r^*(\mathcal{I}_c \cap \rho(I)) = \left(\bigoplus_{i=1}^n C(K) \otimes \mathbb{K}_{c_i}\right) \oplus \left(\bigoplus_{i=n+1}^{n+m} \mathbb{C}^{v_i} \otimes \mathbb{K}_{c_i}\right),$$

where c_1, c_2, \dots, c_{n+m} are critical points in Ω such that $\rho(I) \cap \mathcal{I}_c = \bigsqcup_{i=1}^{n+m} \text{RO}(c_i)$, and c_1, \dots, c_n are pre-periodic, while c_{n+1}, \dots, c_{n+m} are not. As usual v_i is the asymptotic valency of c_i and K is the Cantor set.

If Ω is attractive with periodic point p , we get by the same reasoning the following

diagram.

$$\begin{array}{ccccccc}
 & & 0 & & 0 & & 0 \\
 & & \downarrow & & \downarrow & & \downarrow \\
 0 & \longrightarrow & \mathbb{K} & \longrightarrow & C_r^*(\text{RO}(p)) & \longrightarrow & C(\mathbb{T}) \otimes \mathbb{K}_p \longrightarrow 0 \\
 & & \downarrow & & \downarrow & & \downarrow \\
 0 & \longrightarrow & C_r^*(\rho(I) \cap F_R \setminus \text{RO}(p)) & \longrightarrow & C_r^*(R)/I & \longrightarrow & C_r^*(J_R \cup \text{RO}(p)) \longrightarrow 0 \\
 & & \downarrow & & \downarrow & & \downarrow \\
 0 & \longrightarrow & A & \longrightarrow & C_r^*(J_R \cup (J_c \cap \rho(I))) & \longrightarrow & C_r^*(J_R) \longrightarrow 0 \\
 & & \downarrow & & \downarrow & & \downarrow \\
 & & 0 & & 0 & & 0
 \end{array}$$

Here

$$A = C_r^*(J_c \cap \rho(I)) = \left(\bigoplus_{i=1}^n C(\mathbb{T}) \otimes \mathbb{C}^{v_i} \otimes \mathbb{K}_{c_i} \right) \oplus \left(\bigoplus_{i=n+1}^{n+m} \mathbb{C}^{v_i} \otimes \mathbb{K}_{c_i} \right)$$

where c_1, c_2, \dots, c_{n+m} are critical points in Ω such that $\rho(I) \cap J_c = \bigsqcup_{i=1}^{n+m} \text{RO}(c_i)$, and c_1, \dots, c_n are pre-periodic, while c_{n+1}, \dots, c_{n+m} are not.

6.2.2. Parabolic stable regions. Assume now that x is contained in a parabolic stable region Ω . In this case there is no periodic point in Ω and we get the following diagram.

$$\begin{array}{ccccccc}
 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \\
 & & \mathbb{K} & \xlongequal{\quad\quad\quad} & \mathbb{K} & & \\
 & & \downarrow & & \downarrow & & \\
 0 & \longrightarrow & C_r^*(\rho(I) \cap F_R) & \longrightarrow & C_r^*(R)/I & \longrightarrow & C_r^*(J_R) \longrightarrow 0 \\
 & & \downarrow & & \downarrow & & \parallel \\
 0 & \longrightarrow & \bigoplus_{i=1}^N \mathbb{C}^{v_i} \otimes \mathbb{K}_{c_i} & \longrightarrow & C_r^*(J_R \cup (J_c \cap \rho(I))) & \longrightarrow & C_r^*(J_R) \longrightarrow 0 \\
 & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & &
 \end{array}$$

Here $c_i, i = 1, 2, \dots, N$, are critical points in Ω such that $\mathcal{I}_c \cap \rho(I) = \bigsqcup_{i=1}^N \text{RO}(c_i)$.

6.2.3. Stable regions of Siegel or Herman type. Assume now that x is contained in a stable region Ω of Siegel type. In this case there is a periodic point in Ω with a non-critical orbit, but since $x \notin \mathcal{I}$, this orbit is not in $\rho(I)$. Therefore the picture is the same as in the case of a Herman type stable region and we get in both cases a diagram similar to the parabolic case. The only difference is that the algebra \mathbb{K} in the last diagram is exchanged with the stabilised irrational rotation algebra A_θ corresponding to the rotation by the angle $2\pi\theta$ in the core of Ω .

This completes the list of primitive quotients of $C_r^*(R)$. Note that only very few of the primitive quotients are simple. In fact, the simple quotients of $C_r^*(R)$ are all matrix algebras $M_n(\mathbb{C})$ with $n \leq 4$, together with $C_r^*(J_R)$ if there are no exposed points in J_R .

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Received February 17, 2012

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