Bost–Connes systems associated with function fields

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Abstract. With a global function field K with constant field $\mathbb{F}_q$, a finite set S of primes in K and an abelian extension L of K, finite or infinite, we associate a C*-dynamical system. The systems, or at least their underlying groupoids, defined earlier by Jacob using the ideal action on Drinfeld modules and by Consani–Marcolli using commensurability of K-lattices are isomorphic to particular cases of our construction. We prove a phase transition theorem for our systems and show that the unique KMS_β-state for every $0 < \beta \le 1$ gives rise to an ITPFI-factor (ITPFI stands for "infinite tensor product of finite type I factors") of type $\mathrm{III}_{q^{-\beta n}}$, where n is the degree of the algebraic closure of $\mathbb{F}_q$ in L. Therefore for $n = +\infty$ we get a factor of type III_0. Its flow of weights is a scaled suspension flow of the translation by the Frobenius element on $\mathrm{Gal}(\overline{\mathbb{F}}_q/\mathbb{F}_q)$.

Introduction

In 1995 Bost and Connes [3] constructed a quantum statistical mechanical system with remarkable connections to number theory. Among other properties, this system has phase transition with spontaneous symmetry breaking: the symmetry group of the unique equilibrium state for large temperatures contains $\mathrm{Gal}(\mathbb{Q}^{\mathrm{ab}}/\mathbb{Q})$, while for small temperatures T the extremal equilibrium states have partition function $\zeta(T^{-1})$ and no symmetries in the Galois group. After numerous and only partially successful attempts by a number of people, the accepted correct analogue of this system for an arbitrary number field was defined by Ha and Paugam in 2005 [11]. Soon after that analogues of the Bost–Connes system for function fields were proposed by Jacob [14] and Consani–Marcolli [6]. Their constructions are not directly comparable, since Jacob works within the traditional C*-algebraic setting, while Consani and Marcolli develop quantum statistical mechanics over fields of non-zero characteristic. In both cases, however, the algebras of observables are defined using groupoids, and these can be compared. Both constructions are motivated by the explicit class field theory

for function fields. The standard formulation of the latter fixes a prime ∞ in a function field K and describes either the maximal abelian extension $K^{\mathrm{ab},\infty}$ of K that is completely split at infinity, or a slightly larger extension $\mathbb{K}$ that is totally ramified over $K^{\mathrm{ab},\infty}$ at infinity. The fields $\mathbb{K}$ and $K^{\mathrm{ab},\infty}$ play the role of $\mathbb{Q}^{\mathrm{ab}}$ in the constructions of Jacob and Consani–Marcolli, respectively. The present paper emerged from our attempt to answer the following two natural questions. Is there an analogue of the Bost–Connes system for K corresponding to the maximal abelian extension rather than to $\mathbb{K}$ or $K^{\mathrm{ab},\infty}$? What is the exact relation between the groupoids of Jacob and Consani–Marcolli?

In Section 1 we define a C*-dynamical system $(A_{L,S}, \sigma)$ for every abelian extension L of K and a finite set S of primes in K, and, following the strategy in [16], show that its equilibrium states have the expected properties. The case $L = K^{\mathrm{ab}}$ and $S = \emptyset$ is our answer to the first question. The definition of $(A_{L,S}, \sigma)$ looks straightforward if one compares it with the system of Ha and Paugam as presented in [16]. A "new" simple idea is to consider all possible abelian extensions instead of a fixed one and to allow removing a finite set of primes. This brings a lot of flexibility into the analysis of the systems and allows us to use the standard number-theoretic strategy: reduce a proof of a given property to the situation where the extension L/K is finite and S contains all primes in K that ramify in L, in which case the system $(A_{L,S}, \sigma)$ has a very transparent structure.

In Section 2 we compute the type of the unique equilibrium state of $(A_{L,S}, \sigma)$ at inverse temperature $\beta = 1/T$ for $0 < \beta \le 1$. The spectrum of the corresponding modular operator consists of powers of q^β, where q is the number of elements in the constant field of K. A natural guess is, therefore, that the type is $\mathrm{III}_{q^{-\beta}}$. This would indeed be the case if we could show that the centralizer of the state is a factor. It turns out that this is the case if and only if the extension L/K is geometric. In the general case the type is $\mathrm{III}_{q^{-\beta n}}$, where q^n is the number of elements in the constant field of L. After a number of reductions the proof boils down to a computation of the asymptotic ratio set for a sequence of measures. This computation, in turn, relies on a version of the Chebotarev density theorem. When $n = +\infty$, the type is III_0, and we show that the corresponding flow of weights is a suitably scaled suspension flow of the translation by the Frobenius element on $\mathrm{Gal}(\overline{\mathbb{F}}_q/\mathbb{F}_q)$.

In the remaining part of the paper we show that the systems of Jacob and Consani–Marcolli fit into our framework, with $L = \mathbb{K}$ or $L = K^{\mathrm{ab},\infty}$ and $S = \{\infty\}$. In case of the Consani–Marcolli groupoid this is quite clear from results in [6]. But Jacob's system requires some work.

In Section 3 we summarize results on Drinfeld modules that are needed later. In Section 4 we then prove that Jacob's system is isomorphic to our system corresponding to $L = \mathbb{K}$ and $S = \{\infty\}$. The isomorphism is non-canonical, but any two isomorphisms in the class we define differ by an element of $\mathrm{Gal}(\mathbb{K}/K)$ acting on one of the system. Our approach to Bost–Connes systems therefore provides, in the case $L = \mathbb{K}$ and $S = \{\infty\}$, an alternative route to results of Jacob and corrects his statement about the types of the equilibrium states in the critical region.

Returning to our second motivating question, the isomorphism established in Section 4 shows that the groupoid of Consani–Marcolli is non-canonically isomorphic to the quotient of Jacob's groupoid by the action of $\mathrm{Gal}(\mathbb{K}/K^{\mathrm{ab},\infty})$. Furthermore, a small modification of the construction of Consani–Marcolli leads to a groupoid that is isomorphic to Jacob's groupoid.

In an appendix we collect some general results on the ratio set of an equivalence relation that we need for the computations in Section 2.

1. Systems associated with abelian extensions

Let K be a global function field with constant field $\mathbb{F}_q$. For every prime $\mathfrak{p}$ in K denote by $K_{\mathfrak{p}}$ the corresponding completion of K, and by $\mathcal{O}_{\mathfrak{p}} \subset K_{\mathfrak{p}}$ its maximal compact subring. Denote by $\mathcal{D}$ the group of divisors of K, by $\mathbb{A}$ the adele ring of K, and by $\mathcal{O}_{\mathbb{A}} \subset \mathbb{A}$ its maximal compact subring. So $\mathcal{D}$ is the free abelian group with generators $\mathfrak{p}$, $\mathbb{A}$ is the restricted product $\prod' K_{\mathfrak{p}}$ of the fields $K_{\mathfrak{p}}$ with respect to $\mathcal{O}_{\mathfrak{p}} \subset K_{\mathfrak{p}}$, and $\mathcal{O}_{\mathbb{A}} = \prod_{\mathfrak{p}} \mathcal{O}_{\mathfrak{p}}$. For a finite set S of primes in K denote by $\mathcal{D}_S \subset \mathcal{D}$ the subgroup consisting of divisors with support in the complement S^{c} of S. Similarly, define $\mathbb{A}_S = \prod'_{\mathfrak{p} \in S^{\mathrm{c}}} K_{\mathfrak{p}}$ and $\mathcal{O}_{\mathbb{A},S} = \prod_{\mathfrak{p} \in S^{\mathrm{c}}} \mathcal{O}_{\mathfrak{p}}$.

Let L/K be an abelian extension, finite or infinite, and S be a finite, possibly empty, set of primes in K. Consider the space

$$X_{L,S} = \mathrm{Gal}(L/K) \times_{\mathcal{O}_{\mathbb{A},S}^*} \mathbb{A}_S.$$

Here the action of $\mathcal{O}_{\mathbb{A},S}^*$ on $\mathrm{Gal}(L/K)$ is defined using the Artin map $r_{L/K} \colon \mathbb{A}^* \to \mathrm{Gal}(L/K)$. Thus $X_{L,S}$ is the quotient of $\mathrm{Gal}(L/K) \times \mathbb{A}_S$ by the action $g(x,y) = (x r_{L/K}(g)^{-1}, gy)$ of $\mathcal{O}_{\mathbb{A},S}^*$. Identify $\mathcal{D}_S$ with $\mathbb{A}s^*/\mathcal{O}_{\mathbb{A},S}^*$. Then the diagonal action of $\mathbb{A}_S^*$ on $\mathrm{Gal}(L/K) \times \mathbb{A}_S$, given again by $g(x,y) = (x r_{L/K}(g)^{-1}, gy)$, defines an action of $\mathcal{D}_S$ on $X_{L,S}$. Put $Y_{L,S} = \mathrm{Gal}(L/K) \times_{\mathcal{O}_{\mathbb{A},S}^*} \mathcal{O}_{\mathbb{A},S} \subset X_{L,S}$. Consider the C*-algebra

$$A_{L,S} = \mathbb{1}_{Y_{L,S}} (C_0(X_{L,S}) \rtimes \mathcal{D}_S) \mathbb{1}_{Y_{L,S}}.$$

We can also write $A_{L,S}$ as the semigroup crossed product $C(Y_{L,S}) \rtimes \mathcal{D}_S^+$, where $\mathcal{D}_S^+ \subset \mathcal{D}_S$ is the subsemigroup of effective divisors.

The action of $\mathrm{Gal}(L/K)$ by translations on itself defines an action of $\mathrm{Gal}(L/K)$ on $X_{L,S}$, which in turn defines an action on $A_{L,S}$.

Define a dynamics σ on $A_{L,S}$. It is easier to explain its extension to the multiplier algebra of the whole crossed product $C_0(X_{L,S}) \rtimes \mathcal{D}_S$, which we continue to denote by σ. We put $\sigma_t(f) = f$ for $f \in C_0(X_{L,S})$ and $\sigma_t(u_D) = N(D)^{it} u_D$ for $D \in \mathcal{D}_S$, where $N(D) = q^{\deg D}$ is the norm of D.

Recall that a KMS-state for σ at inverse temperature $\beta \in \mathbb{R}$, or a σ-KMS$_\beta$-state, is a σ-invariant state φ such that $\varphi(ab) = \varphi(b\sigma_{i\beta}(a))$ for a and b in a set of σ-analytic elements with dense linear span. A σ-invariant state φ is called a ground state if the

holomorphic function $z \mapsto \varphi(a\sigma_z(b))$ is bounded on the upper half-plane for a and b in a set of σ-analytic elements spanning a dense subspace. If a state φ is a weak* limit point of a sequence of states $\{\varphi_n\}_n$ such that φ_n is a σ-KMS$_{\beta_n}$-state and $\beta_n \to +\infty$ as $n \to \infty$, then φ is a ground state. Such ground states are called σ-KMS$_\infty$-states.

Theorem 1.1. *For the system $(A_{L,S}, \sigma)$ we have:*

(i) *for $\beta < 0$ there are no KMS$_\beta$-states;*

(ii) *for every $0 < \beta \le 1$ there is a unique KMS$_\beta$-state;*

(iii) *for every $1 < \beta < \infty$ the extremal KMS$_\beta$-states are indexed by the points of the subset*

$$Y^0_{L,S} = \mathrm{Gal}(L/K) \times_{\mathcal{O}^*_{\mathbb{A},S}} \mathcal{O}^*_{\mathbb{A},S} \cong \mathrm{Gal}(L/K)$$

of $Y_{L,S}$, with the state corresponding to $w \in Y^0_{L,S}$ given by

$$\varphi_{\beta,w}(\mathbb{1}_{Y_{L,S}} f u_D \mathbb{1}_{Y_{L,S}}) = \frac{\delta_{D,0}}{\zeta_{K,S}(\beta)} \sum_{D' \in \mathcal{D}^+_S} N(D')^{-\beta} f(D'w),$$

where $\zeta_{K,S}(\beta) = \sum_{D \in \mathcal{D}^+_S} N(D)^{-\beta}$; furthermore, every extremal KMS$_\beta$-state $\varphi_{\beta,w}$ is of type I_∞ and its partition function is $\zeta_{K,S}(\beta)$;

(iv) *the extremal ground states are indexed by $Y^0_{L,S}$, with the state corresponding to $w \in Y^0_{L,S}$ given by $\varphi_{\infty,w}(\mathbb{1}_{Y_{L,S}} f u_D \mathbb{1}_{Y_{L,S}}) = \delta_{D,0} f(w)$, and all ground states are KMS$_\infty$-states.*

Proof. Since the proof is essentially identical to that of [16], Theorem 2.1, we will only describe the main steps. Along the way we will make a few remarks that will be important later.

First, one shows that classifying KMS$_\beta$-states for $\beta \in \mathbb{R} \setminus \{0\}$ is equivalent to classifying measures μ on $X_{L,S}$ such that $\mu(Y_{L,S}) = 1$ and $\mu(DZ) = N(D)^{-\beta}\mu(Z)$ for any Borel set $Z \subset X_{L,S}$ and any divisor $D \in \mathcal{D}_S$. Namely, the state corresponding to μ is given by the composition of the canonical conditional expectation $A_{L,S} \to C(Y_{L,S})$ with the state on $C(Y_{L,S})$ defined by the measure μ. Since $DY_{L,S} \subset Y_{L,S}$ for $D \in \mathcal{D}^+_S$, there are no such measures for $\beta < 0$.

Next, one checks that if μ is a measure defining a KMS$_\beta$-state and $\beta > 1$, then $Y^0_{L,S}$ is a fundamental domain (modulo a set of measure zero) for the action of $\mathcal{D}_S$ on $(X_{L,S}, \mu)$, so the measure is completely determined by its restriction to $Y^0_{L,S}$. The case $\beta = +\infty$ is dealt with in a similar manner.

The most interesting and non-trivial region is $0 < \beta \le 1$. Let us see that a measure μ_β defining a KMS$_\beta$-state indeed exists. We define μ_β as the image of the measure

$$\lambda_{L/K} \times \prod_{\mathfrak{p} \in S^c} \mu_{\beta,\mathfrak{p}}$$

on $\mathrm{Gal}(L/K) \times \mathbb{A}_S$ under the quotient map $\mathrm{Gal}(L/K) \times \mathbb{A}_S \to X_{L,S}$, where $\lambda_{L/K}$ is the normalized Haar measure on $\mathrm{Gal}(L/K)$ and $\mu_{\beta,\mathfrak{p}}$ is the measure on $K_\mathfrak{p}$ defined as follows. The measure $\mu_{1,\mathfrak{p}}$ is the Haar measure on $K_\mathfrak{p}$ such that $\mu_{1,\mathfrak{p}}(\mathcal{O}_\mathfrak{p}) = 1$. The measure $\mu_{\beta,\mathfrak{p}}$ is absolutely continuous with respect to $\mu_{1,\mathfrak{p}}$ and

$$\frac{d\mu_{\beta,\mathfrak{p}}}{d\mu_{1,\mathfrak{p}}}(a) = \frac{1 - N(\mathfrak{p})^{-\beta}}{1 - N(\mathfrak{p})^{-1}} \|a\|_\mathfrak{p}^{\beta-1}.$$

Assume now that μ is a measure defining a KMS_β-state for some $0 < \beta \leq 1$. In order to see that $\mu = \mu_\beta$, it is convenient to make two reductions.

Given an intermediate finite extension E of K we can identify $X_{E,S}$ with $X_{L,S}/\mathrm{Gal}(L/E)$. Since any continuous function with support in $Y_{L,S}$ can be approximated by a $\mathrm{Gal}(L/E)$-invariant function with support in $Y_{L,S}$ for sufficiently large E, it follows that to prove that $\mu = \mu_\beta$ it suffices to consider the systems corresponding to finite extensions.

For the second reduction take a finite set S' of primes in K containing S. The subset

$$X_{L,S,S'} = \mathrm{Gal}(L/K) \times_{\mathcal{O}_{\mathbb{A},S}^*} \left(\textstyle\prod_{\mathfrak{p} \in S' \setminus S} \mathcal{O}_\mathfrak{p}^* \times \mathbb{A}_{S'} \right)$$

of $X_{L,S}$ is $\mathcal{D}_{S'}$-invariant, it is a fundamental domain (modulo a set of measure zero) for the action of the group generated by primes in $S' \setminus S$ on $(X_{L,S}, \mu)$, and the measure of

$$Y_{L,S,S'} = \mathrm{Gal}(L/K) \times_{\mathcal{O}_{\mathbb{A},S}^*} \left(\textstyle\prod_{\mathfrak{p} \in S' \setminus S} \mathcal{O}_\mathfrak{p}^* \times \mathcal{O}_{\mathbb{A},S'} \right)$$

is equal to $\prod_{\mathfrak{p} \in S' \setminus S}(1 - N(\mathfrak{p})^{-\beta})$. It follows that classifying measures μ on $X_{L,S}$ defining KMS_β-states is the same as classifying measures ν on $X_{L,S,S'}$ such that $\nu(Y_{L,S,S'}) = \prod_{\mathfrak{p} \in S' \setminus S}(1 - N(\mathfrak{p})^{-\beta})$ and $\nu(DZ) = N(D)^{-\beta}\nu(Z)$ for any Borel set $Z \subset X_{L,S,S'}$ and any divisor $D \in \mathcal{D}_{S'}$. But $X_{L,S,S'}$ can be identified with $X_{L,S'}$. Therefore, if we have uniqueness of KMS_β-states for S', we have it also for S.

To summarize, in proving the uniqueness of KMS_β-states for $0 < \beta \leq 1$ we may assume that the extension L/K is finite and the set S is as large as we want. In particular, we may assume that S contains all primes in K that ramify in L. In this case the kernel of the Artin map $\mathbb{A}_S^* \to \mathrm{Gal}(L/K)$ contains $\mathcal{O}_{\mathbb{A},S}^*$, so this map factors through $\mathcal{D}_S$,

$$X_{L,S} = \mathrm{Gal}(L/K) \times \mathbb{A}_S/\mathcal{O}_{\mathbb{A},S}^*,$$

and the action of $\mathcal{D}_S$ on $X_{L,S}$ is diagonal. In order to prove that $\mu = \mu_\beta$ one then computes the projection P in $L^2(Y_{L,S}, d\mu)$ onto the subspace of $\mathcal{D}_S^+$-invariant functions, which turns out to consist only of constants. Consider a continuous function on $Y_{L,S}$ that is supported on $Y_{L,S,S'} \cong Y_{L,S'} = \mathrm{Gal}(L/K) \times \mathcal{O}_{\mathbb{A},S'}/\mathcal{O}_{\mathbb{A},S'}^*$ for some $S' \supset S$ and whose value at $a \in Y_{L,S,S'}$ depends only on the coordinate of a in $\mathrm{Gal}(L/K)$. It can be shown that if f is defined by a nontrivial character χ of

Gal(L/K), then the L^2-norm of Pf is not larger than

$$\prod_{\mathfrak{p}\in S'^c}\left|\frac{1-N(\mathfrak{p})^{-\beta}}{1-\chi(\mathfrak{p})N(\mathfrak{p})^{-\beta}}\right|.$$

Since the Artin L-function $L(s,\chi)$ does not have a pole at $s=1$, the above product diverges to zero (for $0<\beta\leq 1$). Hence $Pf=0$, so that $\int fd\mu=0=\int fd\mu_\beta$. From this it is easy to deduce that $\mu=\mu_\beta$. $\qquad\square$

The case $\beta=0$ is special, as then there are KMS-states that do not factor through the conditional expectation $A_{L,S}\to C(Y_{L,S})$. In order to classify them, consider the subfield $L_S^{\mathrm{un}}\subset L$ such that Gal$(L/L_S^{\mathrm{un}})=r_{L/K}(\mathcal{O}_{\mathbb{A},S}^*)$. This is the maximal subextension of L/K unramified at all primes in S^c. Let $\mathcal{D}_{L,S}$ be the kernel of the Artin homomorphism $\mathcal{D}_S\to$ Gal(L_S^{un}/K), and $\mathcal{D}_{L,S}^0\subset\mathcal{D}_{L,S}$ be the subgroup of divisors of degree zero. Denote by μ_0 the unique Gal(L/K)-invariant measure on $Y_{L,S}$ concentrated on the image of Gal$(L/K)\times\{0\}\subset$ Gal$(L/K)\times\mathcal{O}_{\mathbb{A},S}$ in $Y_{L,S}$.

Proposition 1.2. *There is a one-to-one correspondence between extremal KMS$_0$-states on $A_{L,S}$ and characters of $\mathcal{D}_{L,S}^0$. Namely, the state τ_χ corresponding to a character χ is given by*

$$\tau_\chi(\mathbb{1}_{Y_{L,S}}fu_D\mathbb{1}_{Y_{L,S}})=\begin{cases}\chi(D)\int fd\mu_0 & \text{if }D\in\mathcal{D}_{L,S}^0,\\ 0 & \text{otherwise.}\end{cases}$$

Proof. We will use a general classification result from [19], although the present case is not difficult to deal with directly. By [19, Theorem 1.3] there is a one-to-one correspondence between KMS$_0$-states on $A_{L,S}$ and pairs $(\mu,\{\tau_x\}_x)$ consisting of a $\mathcal{D}_S$-invariant measure μ on $X_{L,S}$ with $\mu(Y_{L,S})=1$ and a μ-measurable field of states τ_x on $C^*(G_x)$, where $G_x\subset\mathcal{D}_S$ is the stabilizer of x, such that $\tau_{Dx}=\tau_x$ for μ-a.e. x and τ_x factors through the canonical conditional expectation $C^*(G_x)\to C^*(G_x^0)$, where $G_x^0\subset G_x$ is the subgroup of divisors of degree zero. Assume that $(\mu,\{\tau_x\}_x)$ is such a pair. Since the intersection of the sets $DY_{L,S}$, $D\in\mathcal{D}_S$, coincides with the image Z of Gal$(L/K)\times\{0\}$ in $Y_{L,S}$, the measure μ is concentrated on Z. The set Z can be identified with Gal$(L/K)/r_{L/K}(\mathcal{O}_{\mathbb{A},S}^*)=$ Gal(L_S^{un}/K). Since the image of $\mathcal{D}_S$ in Gal(L_S^{un}/K) is dense, the Haar measure on Gal(L_S^{un}/K) is the unique $\mathcal{D}_S$-invariant measure. Therefore $\mu=\mu_0$ and the action of $\mathcal{D}_S$ on $(X_{L,S},\mu)$ is ergodic. It follows that the field $\{\tau_x\}_x$ is essentially constant. The stabilizer of every point in Z is $\mathcal{D}_{L,S}$. Therefore we conclude that there is a one-to-one correspondence between KMS$_0$-states on $A_{L,S}$ and states on $C^*(\mathcal{D}_{L,S}^0)$. Then extremal KMS$_0$-states correspond to characters of $\mathcal{D}_{L,S}^0$. $\qquad\square$

2. Types of KMS-states in the critical region

We continue to use the notation of the previous section, so L/K is an abelian extension of a global function field K and S is a finite set of primes in K. Take $0 < \beta \le 1$ and denote by φ_β the unique σ-KMS$_\beta$-state on $A_{L,S}$.

Theorem 2.1. *Let* $\mathbb{F}_{q^n}$, $n \in \mathbb{N} \cup \{+\infty\}$, *be the algebraic closure of the constant field* $\mathbb{F}_q \subset K$ *in* L. *Then the von Neumann algebra* $\pi_{\varphi_\beta}(A_{L,S})''$ *is an injective factor of type* III$_{q^{-\beta n}}$.

Since φ_β is extremal, the von Neumann algebra $\pi_{\varphi_\beta}(A_{L,S})''$ is a factor. It is the reduction of the von Neumann algebra $L^\infty(X_{L,S}, \mu_\beta) \rtimes \mathcal{D}_S$ by the projection $\mathbb{1}_{Y_{L,S}}$, where μ_β is the measure on $X_{L,S}$ defined in Section 1. Since $\mathbb{1}_{Y_{L,S}}$ is a full projection in $A_{L,S}$, the von Neumann algebra $L^\infty(X_{L,S}, \mu_\beta) \rtimes \mathcal{D}_S$ is also a factor, hence the action of $\mathcal{D}_S$ on $(X_{L,S}, \mu_\beta)$ is ergodic. The equivalent formulation of the above theorem is therefore that this action (more precisely, the corresponding orbit equivalence relation) is of type III$_{q^{-\beta n}}$, see Appendix 4. Note that the injectivity of the factor $L^\infty(X_{L,S}, \mu_\beta) \rtimes \mathcal{D}_S$ is obvious, since $\mathcal{D}_S$ is abelian.

Our computation of the ratio set will rely on the following version of the Chebotarev density theorem for function fields, see [9], Proposition 6.4.8.

Theorem 2.2. *Let* K *be a function field with constant field* $\mathbb{F}_q$, L *a finite Galois extension of* K *with constant field* $\mathbb{F}_{q^n}$, $\mathcal{C}$ *a conjugacy class in* Gal(L/K) *consisting of* c *elements. Let* $0 \le a < n$ *be such that the restriction of every element in* $\mathcal{C}$ *to* $\mathbb{F}_{q^n}$ *is the* a-*th power of the Frobenius automorphism of* $\mathbb{F}_{q^n}/\mathbb{F}_q$. *Then every prime* $\mathfrak{p}$ *such that* $(\mathfrak{p}, L/K) = \mathcal{C}$ *has the property* deg $\mathfrak{p} \equiv a \mod n$, *and*

$$\#\{\mathfrak{p} \mid (\mathfrak{p}, L/K) = \mathcal{C} \text{ and } \deg \mathfrak{p} = kn + a\}$$
$$= \frac{cn}{(kn + a)[L : K]} q^{kn+a} + O(q^{kn/2}) \quad as \, k \to +\infty.$$

We are interested in the case when the extension L/K is abelian. Denote by Spl(L/K) the set of primes in K that are completely split in L.

Corollary 2.3. *Assume that* L/K *is a finite abelian extension and* S *contains the set of primes in* K *that ramify in* L. *Then the degree of every element in the kernel of the Artin map* $\mathcal{D}_S \to$ Gal(L/K) *is divisible by* n, *and*

$$\#\{\mathfrak{p} \mid \mathfrak{p} \in \text{Spl}(L/K) \text{ and } \deg \mathfrak{p} = kn\} = \frac{1}{k[L : K]} q^{kn} + O(q^{kn/2}) \quad as \, k \to +\infty.$$

Under the assumptions of the previous corollary consider the restricted product $X'_{L,S}$ of the spaces $K_\mathfrak{p}/\mathcal{O}^*_\mathfrak{p}$ with respect to $\mathcal{O}_\mathfrak{p}/\mathcal{O}^*_\mathfrak{p}$ over all $\mathfrak{p} \in S^c \cap \text{Spl}(L/K)$, and consider the measure $\mu'_\beta = \prod_{\mathfrak{p} \in S^c \cap \text{Spl}(L/K)} \mu_{\beta,\mathfrak{p}}$ on $X'_{L,S}$. Denote by $\mathcal{D}'_S$ the subgroup of $\mathcal{D}_S$ generated by primes in $S^c \cap \text{Spl}(L/K)$.

Lemma 2.4. *The action of $\mathcal{D}'_S$ on $(X'_{L,S}, \mu'_\beta)$ is of type* $\mathrm{III}_{q^{-\beta n}}$.

Proof. For $L = K$ the result is proved in [14], Lemma 4.5.1. The general case is similar but uses Corollary 2.3 instead of the distribution of prime ideals. Note also that an analogous statement for the field $\mathbb{Q}$ of rational numbers and $L = K$ goes back to [2]. For the reader's convenience we will nevertheless sketch a proof.

Since the degree of every prime $\mathfrak{p} \in S^c \cap \mathrm{Spl}(L/K)$ is divisible by n and the measure μ'_β has the property $\mu'_\beta(D \cdot) = N(D)^{-\beta} \mu'_\beta$, the ratio set is contained in the set $\{0\} \cup \{q^{-\beta n k} \mid k \in \mathbb{Z}\}$. Therefore it suffices to show that $q^{-\beta n}$ lies in the ratio set. Instead of the orbit equivalence relation on $X'_{L,S}$ we may consider the relation induced on the subset $Y'_{L,S} = \prod_{\mathfrak{p} \in S^c \cap \mathrm{Spl}(L/K)} \mathcal{O}^\times_\mathfrak{p}/\mathcal{O}^*_\mathfrak{p}$. But this is exactly the relation discussed at the end of Appendix 4: two points are equivalent if and only if their coordinates coincide outside a finite set of primes. The measure $\mu_{\beta,\mathfrak{p}}$ on $\mathcal{O}^\times_\mathfrak{p}/\mathcal{O}^*_\mathfrak{p}$ is given by $\mu_{\beta,\mathfrak{p}}(\pi^k_\mathfrak{p} \mathcal{O}^*_\mathfrak{p}) = N(\mathfrak{p})^{-\beta k}(1 - N(\mathfrak{p})^{-\beta})$, where $\pi_\mathfrak{p}$ is a uniformizer in $\mathcal{O}_\mathfrak{p}$.

Let m_k be the number of primes in $S^c \cap \mathrm{Spl}(L/K)$ of degree kn. By Corollary 2.3 we have $m_k \sim [L : K]^{-1} q^{kn}/k$. In particular, for sufficiently large k we have $m_{2k} < m_{2k+1}$. Hence for every prime $\mathfrak{p}$ of degree $2kn$ we can choose a prime $\mathfrak{p}'$ of degree $(2k+1)n$ in such a way that the map $\mathfrak{p} \mapsto \mathfrak{p}'$ is injective. Now as the sets I_j required by the definition of the asymptotic ratio set we take the sets $\{\mathfrak{p}, \mathfrak{p}'\}$, and as the sets K_j and L_j we take the single-point sets $\{(\pi_\mathfrak{p} \mathcal{O}^*_\mathfrak{p}, \mathcal{O}^*_{\mathfrak{p}'})\}$ and $\{(\mathcal{O}^*_\mathfrak{p}, \pi_{\mathfrak{p}'} \mathcal{O}^*_{\mathfrak{p}'})\}$, respectively. Since $\sum_k q^{-2\beta kn} m_{2k} = \infty$, we conclude that $q^{-\beta n}$ belongs to the asymptotic ratio set. $\qquad\square$

Proof of Theorem 2.1. Assume first that the extension L/K is finite. Since the action of $\mathcal{D}_S$ on $(X_{L,S}, \mu_\beta)$ is ergodic, in computing the ratio set instead of the orbit equivalence relation on $X_{L,S}$ we may consider the relation induced on any subset of positive measure. In particular, for every $S' \supset S$ we may consider the subset $X_{L,S,S'}$ introduced in Section 1. As we discussed there, it can be identified with $X_{L,S'}$, and the equivalence relation we get on $X_{L,S'}$ is exactly the one defined by the action of the group $\mathcal{D}_{S'}$. Therefore the type of the action of $\mathcal{D}_S$ on $(X_{L,S}, \mu_\beta)$ does not depend on S. Hence we may assume that S includes all primes that ramify in L. Then $X_{L,S} = \mathrm{Gal}(L/K) \times \mathbb{A}_S/\mathcal{O}^*_{\mathbb{A},S}$.

Consider the subset $\{e\} \times \mathbb{A}_S/\mathcal{O}^*_{\mathbb{A},S}$ of $X_{L,S}$. The equivalence relation induced on it is the one given by the action of the kernel G of the Artin map $\mathcal{D}_S \to \mathrm{Gal}(L/K)$ on $\mathbb{A}_S/\mathcal{O}^*_{\mathbb{A},S}$. By the first (obvious) part of Corollary 2.3 the degree of every element in G is divisible by n. Hence the ratio set of the action of G on $(\mathbb{A}_S/\mathcal{O}^*_{\mathbb{A},S}, \mu_\beta)$ is contained in $\{0\} \cup \{q^{-\beta n k} \mid k \in \mathbb{Z}\}$. On the other hand, by Lemma 2.4 the number $q^{-\beta n}$ is contained in the ratio set of the action of $\mathcal{D}'_S \subset G$ on $(X'_{L,S}, \mu'_\beta)$. Hence, by Proposition A.2, it is contained in the ratio set of the action of $\mathcal{D}'_S$ on $(\mathbb{A}_S/\mathcal{O}^*_{\mathbb{A},S}, \mu_\beta)$. This proves the theorem when L/K is finite.

Now consider an arbitrary abelian extension L/K. By Proposition A.3 the nonzero part of the ratio set of the action of $\mathcal{D}_S$ on $(X_{L,S}, \mu_\beta)$ is equal to the intersection of

the nonzero parts of the ratio sets of the actions of $\mathcal{D}_S$ on $(X_{L,S}/\mathrm{Gal}(L/E), \mu_\beta)$ for all finite intermediate extensions E/K. Since $X_{L,S}/\mathrm{Gal}(L/E) = X_{E,S}$, from the first part of the proof we conclude that if $n < +\infty$, then the nonzero part of the ratio set of the action of $\mathcal{D}_S$ on $(X_{L,S}, \mu_\beta)$ equals $\{q^{-\beta n k} \mid k \in \mathbb{Z}\}$, hence the action is of type $\mathrm{III}_{q^{-\beta n}}$.

In the case $n = +\infty$ we can only conclude that the ratio set is either $\{1\}$ or $\{0, 1\}$, so the factor $M_L = \pi_{\varphi_\beta}(A_{L,S})''$ is either semifinite or of type III_0. Since $M_K = M_L^{\mathrm{Gal}(L/K)}$, there exists a normal conditional expectation $M_L \to M_K$. As M_K is of type III, by [20], Proposition 10.21, it follows that M_L is also of type III, hence it is of type III_0. $\qquad\square$

For every $\lambda \in (0, 1]$ there exists a unique injective factor of type III_λ with separable predual. Therefore for $n < +\infty$ the above result completely describes the von Neumann algebra $\pi_{\varphi_\beta}(A_{L,S})''$. For type III_0 factors a complete invariant is the flow of weights. The Galois group $\mathrm{Gal}(L/K)$ acts on the flow of weights of $\pi_{\varphi_\beta}(A_{L,S})''$ and the quotient by $\mathrm{Gal}(L/E)$ gives the flow of weights of $\pi_{\varphi_\beta}(A_{E,S})''$. Since the flows corresponding to finite extensions are periodic, it is not difficult to see that this is enough to completely describe the flow of weights of $\pi_{\varphi_\beta}(A_{L,S})''$, but we will do everything explicitly.

The flow of weights of a crossed product algebra, or of the von Neumann algebra of an equivalence relation, can be described as follows, see [4]. Assume that a countable group Γ acts ergodically by non-singular transformations on a standard measure space (X, μ). Let λ be the Lebesgue measure on $\mathbb{R}$. Define a new action of Γ on $(\mathbb{R} \times X, \lambda \times \mu)$ by

$$g(s, x) = \left(s - \log \frac{dg\mu}{d\mu}(gx), gx \right).$$

Consider the measure theoretic quotient $\widetilde{X}$ of $(\mathbb{R} \times X, \lambda \times \mu)$ by this action. So $\widetilde{X}$ is a standard Borel space with a measure class $[\tilde{\mu}]$ such that $L^\infty(\widetilde{X}, \tilde{\mu}) = L^\infty(\mathbb{R} \times X, \lambda \times \mu)^\Gamma$. The action of $\mathbb{R}$ on $\mathbb{R} \times X$ given by $t(s, x) = (s + t, x)$ induces an ergodic flow $\{F_t\}_{t \in \mathbb{R}}$ on $(\widetilde{X}, \tilde{\mu})$. This flow depends, up to isomorphism, only on the measure class of μ and the orbit equivalence relation $\mathcal{R}$ on X defined by the action of Γ. It is the flow of weights of the factor $W^*(\mathcal{R})$. It is known that the action of Γ on (X, μ) is of type III_0 if and only if the flow is non-transitive, that is, there is no orbit of positive measure.

Returning to the Bost–Connes systems, consider an abelian extension L/K. As before, let $\mathbb{F}_{q^n}$, $n \in \mathbb{N} \cup \{+\infty\}$, be the algebraic closure of $\mathbb{F}_q$ in L.

Define a continuous map $X_{L,S} \to \mathrm{Gal}(\mathbb{F}_{q^n}/\mathbb{F}_q)$ as the composition of the quotient map

$$X_{L,S} \to X_{L,S}/\mathrm{Gal}(L/\mathbb{F}_{q^n}K) = X_{\mathbb{F}_{q^n}K,S} = \mathrm{Gal}(\mathbb{F}_{q^n}K/K) \times \mathbb{A}_S/\mathcal{O}_{\mathbb{A},S}^*,$$

where we have used that any finite constant field extension of K is unramified at every prime, with the projection

$$\mathrm{Gal}(\mathbb{F}_{q^n} K/K) \times \mathbb{A}_S/\mathcal{O}^*_{\mathbb{A},S} \to \mathrm{Gal}(\mathbb{F}_{q^n} K/K) = \mathrm{Gal}(\mathbb{F}_{q^n}/\mathbb{F}_q).$$

Taking the direct product with the identity map on $\mathbb{R}$ we get a continuous map

$$\mathbb{R} \times X_{L,S} \to \mathbb{R} \times \mathrm{Gal}(\mathbb{F}_{q^n}/\mathbb{F}_q).$$

This map becomes $\mathcal{D}_S$-equivariant if we define an action of $\mathcal{D}_S$ on $\mathbb{R} \times \mathrm{Gal}(\mathbb{F}_{q^n}/\mathbb{F}_q)$ by

$$D(s, g) = (s - \beta \log N(D), g \, \mathrm{res}_{\mathbb{F}_{q^n}} (r_{\mathbb{F}_{q^n} K/K}(D))^{-1}).$$

Note that $\mathrm{res}_{\mathbb{F}_{q^n}} (r_{\mathbb{F}_{q^n} K/K}(D))$ is simply the Frobenius automorphism raised to the power $\deg D$. Denote by λ_n the normalized Haar measure on $\mathrm{Gal}(\mathbb{F}_{q^n}/\mathbb{F}_q)$. Then the map $\mathbb{R} \times X_{L,S} \to \mathbb{R} \times \mathrm{Gal}(\mathbb{F}_{q^n}/\mathbb{F}_q)$ gives us a $\mathcal{D}_S$-equivariant embedding

$$L^\infty(\mathbb{R} \times \mathrm{Gal}(\mathbb{F}_{q^n}/\mathbb{F}_q), \lambda \times \lambda_n) \hookrightarrow L^\infty(\mathbb{R} \times X_{L,S}, \lambda \times \mu_\beta).$$

Lemma 2.5. *We have* $L^\infty(\mathbb{R} \times X_{L,S}, \lambda \times \mu_\beta)^{\mathcal{D}_S} = L^\infty(\mathbb{R} \times \mathrm{Gal}(\mathbb{F}_{q^n}/\mathbb{F}_q), \lambda \times \lambda_n)^{\mathcal{D}_S}.$

Proof. It suffices to check that the subalgebras of $\mathrm{Gal}(L/E)$-invariant elements on both sides coincide for all intermediate finite extensions E of K. This means that it is enough to prove the lemma for finite extensions since $X_{L,S}/\mathrm{Gal}(L/E) = X_{E,S}$.

Choose a finite set S' which contains S and all primes that ramify in L. Consider the set $X_{L,S,S'}$ introduced in Section 1. It can be identified with $X_{L,S'} = \mathrm{Gal}(L/K) \times \mathbb{A}_{S'}/\mathcal{O}^*_{\mathbb{A},S'}$. Consider the subset $\{e\} \times \mathbb{A}_{S'}/\mathcal{O}^*_{\mathbb{A},S'}$ of $X_{L,S'}$. We claim that if $f \in L^\infty(\mathbb{R} \times X_{L,S}, \lambda \times \mu_\beta)^{\mathcal{D}_S}$, then the restriction of f to $\mathbb{R} \times \{e\} \times \mathbb{A}_{S'}/\mathcal{O}^*_{\mathbb{A},S'}$ depends only on the first coordinate and therefore defines a function $f_1 \in L^\infty(\mathbb{R}, \lambda)$, and the function f_1 is $\beta n \log q$-periodic.

Indeed, let G be the kernel of the Artin map $\mathcal{D}_{S'} \to \mathrm{Gal}(L/K)$. By the proof of Theorem 2.1 the nonzero part of the ratio set of the action of G on $\{e\} \times \mathbb{A}_{S'}/\mathcal{O}^*_{\mathbb{A},S'}$ coincides with the set of values of the Radon–Nikodym derivatives. By Proposition A.4 it follows that the subgroup $G_0 \subset G$ of divisors of degree zero acts ergodically on $\{e\} \times \mathbb{A}_{S'}/\mathcal{O}^*_{\mathbb{A},S'}$. Since the group G_0 acts trivially on $\mathbb{R}$, the G_0-invariant function f on $\mathbb{R} \times \{e\} \times \mathbb{A}_{S'}/\mathcal{O}^*_{\mathbb{A},S'}$ depends only on the first coordinate and therefore defines a function $f_1 \in L^\infty(\mathbb{R}, \lambda)$. Since f is G-invariant and the homomorphism $\deg: G \to n\mathbb{Z}$ is surjective, the function f_1 is $\beta n \log q$-periodic.

The function f_1 defines, in turn, a $\mathcal{D}_S$-invariant function f_2 on $\mathbb{R} \times \mathrm{Gal}(\mathbb{F}_{q^n}/\mathbb{F}_q) \cong \mathbb{R} \times \mathbb{Z}/n\mathbb{Z}$ such that $f_2(s, 0) = f_1(s)$. Namely, $f_2(s, m) = f_1(s - \beta m \log q)$. Since the $\mathcal{D}_S$-orbit of almost every point in $\mathbb{R} \times X_{L,S}$ intersects $\mathbb{R} \times \{e\} \times \mathbb{A}_{S'}/\mathcal{O}^*_{\mathbb{A},S'}$, every $\mathcal{D}_S$-invariant function is completely determined by its values on this set. Hence $f = f_2 \in L^\infty(\mathbb{R} \times \mathrm{Gal}(\mathbb{F}_{q^n}/\mathbb{F}_q), \lambda \times \lambda_n)^{\mathcal{D}_S}$. $\square$

Assume now that $n = +\infty$, so that $\mathbb{F}_{q^n} = \overline{\mathbb{F}}_q$. We can identify the Galois group $\mathrm{Gal}(\overline{\mathbb{F}}_q/\mathbb{F}_q)$ with the group $\varprojlim \mathbb{Z}/k\mathbb{Z} = \hat{\mathbb{Z}}$. The action of $\mathcal{D}_S$ on $\mathbb{R} \times \hat{\mathbb{Z}}$ that we get is given by

$$D(s, a) = (s - \beta \deg D \log q, a - \deg D).$$

We can instead consider the action $D(s, a) = (s - \deg D, a - \deg D)$ by rescaling the first coordinate. We can now formulate a refinement of Theorem 2.1 for $n = +\infty$.

Theorem 2.6. *Assume that the algebraic closure of $\mathbb{F}_q$ in L is infinite. Then the von Neumann algebra $\pi_{\varphi_\beta}(A_{L,S})''$ is an ITPFI factor of type III_0. Its flow of weights is the flow on the compact group $(\mathbb{R} \times \hat{\mathbb{Z}})/\mathbb{Z}$ defined by*

$$F_t(s, a) = \left(s + \frac{t}{\beta \log q}, a\right).$$

Proof. As follows from Lemma 2.5 and the subsequent discussion, the flow of weights of the factor $L^\infty(X_{L,S}, \mu_\beta) \rtimes \mathcal{D}_S$ has the form given in the formulation of the theorem. We already know that this factor is of type III_0, but this is also clear from our description of the flow since the flow is obviously non-transitive. The flow is, however, approximately transitive [5], hence the factor is ITPFI. Since $\pi_{\varphi_\beta}(A_{L,S})''$ is a reduction of $L^\infty(X_{L,S}, \mu_\beta) \rtimes \mathcal{D}_S$, the same assertions hold for $\pi_{\varphi_\beta}(A_{L,S})''$, moreover, the two factors are isomorphic. $\qquad\square$

Next we will determine the center of the centralizer of the state φ_β on $\pi_{\varphi_\beta}(A_{L,S})''$. This centralizer is the reduction of the von Neumann algebra $L^\infty(X_{L,S}, \mu_\beta) \rtimes \mathcal{D}_S^0$ by the projection $\mathbb{1}_{Y_{L,S}}$. The center of $L^\infty(X_{L,S}, \mu_\beta) \rtimes \mathcal{D}_S^0$ is $L^\infty(X_{L,S}, \mu_\beta)^{\mathcal{D}_S^0}$. If the field $\mathbb{F}_q$ is not algebraically closed in L, then the ratio set of the action of $\mathcal{D}_S$ on $(X_{L,S}, \mu_\beta)$ is strictly smaller than the essential range of the Radon–Nikodym cocycle, so by Proposition A.4 the action of $\mathcal{D}_S^0$ on $(X_{L,S}, \mu_\beta)$ cannot be ergodic.

Consider the map $X_{L,S} \to \mathrm{Gal}(\mathbb{F}_{q^n}/\mathbb{F}_q)$ constructed before Lemma 2.5. It gives us an embedding of $L^\infty(\mathrm{Gal}(\mathbb{F}_{q^n}/\mathbb{F}_q), \lambda_n)$ into $L^\infty(X_{L,S}, \mu_\beta)$.

Proposition 2.7. *If L/K is an abelian extension and $\mathbb{F}_{q^n}$, $n \in \mathbb{N} \cup \{+\infty\}$, is the algebraic closure of $\mathbb{F}_q$ in L, then $L^\infty(X_{L,S}, \mu_\beta)^{\mathcal{D}_S^0} = L^\infty(\mathrm{Gal}(\mathbb{F}_{q^n}/\mathbb{F}_q), \lambda_n)$.*

Proof. As in the proof of Lemma 2.5, we may assume that L/K is finite. Consider the open subset

$$Z = \mathrm{Gal}(L/\mathbb{F}_{q^n} K) \times_{\mathcal{O}_{\mathbb{A},S}^*} \mathbb{A}_S \subset X_{L,S}.$$

The equivalence relation on Z induced by the action of $\mathcal{D}_S$ on $X_{L,S}$ is the orbit equivalence relation defined by the action of the kernel H of the map $\mathcal{D}_S \to \mathrm{Gal}(\mathbb{F}_{q^n} K/K) = \mathrm{Gal}(\mathbb{F}_{q^n}/\mathbb{F}_q)$. The group H is simply the subgroup of divisors in $\mathcal{D}_S$ of degree divisible by n. Since the action of $\mathcal{D}_S$ on $(X_{L,S}, \mu_\beta)$ is of type $\mathrm{III}_{q^{-\beta n}}$, by Proposition A.4 we conclude that the action of $\mathcal{D}_S^0 \subset H$ on (Z, μ_β) is ergodic.

It follows that any $\mathcal{D}_S^0$-invariant measurable subset of $X_{L,S}$ coincides, modulo a set of measure zero, with the union of translations of Z by elements of $\mathrm{Gal}(L/K)$. The set Z is nothing else but the pre-image of the unit element $e \in \mathrm{Gal}(\mathbb{F}_{q^n}/\mathbb{F}_q)$ under the map $X_{L,S} \to \mathrm{Gal}(\mathbb{F}_{q^n}/\mathbb{F}_q)$. Hence any $\mathcal{D}_S^0$-invariant measurable subset of $X_{L,S}$ is the pre-image of a subset of $\mathrm{Gal}(\mathbb{F}_{q^n}/\mathbb{F}_q)$. $\square$

We finish the section by making a few remarks about similarities and differences between the function field and number field [18] cases. In both cases one can say that the key part is to compute the types of KMS-states for finite extensions (although this may not be immediately obvious from the argument for number fields in [18]). If we have a finite extension L/K and an intermediate extension E/K, then $\pi_{\varphi_\beta}(A_{E,S})''$ is a finite index subfactor of $\pi_{\varphi_\beta}(A_{L,S})''$. For number fields the smallest factor $\pi_{\varphi_\beta}(A_{K,S})''$ is of type III_1, and to prove this one only needs to know the distribution of prime ideals. This automatically implies that all other factors are of type III_1 as well [17]. For function fields the smallest factor $\pi_{\varphi_\beta}(A_{K,S})''$ is of type $\mathrm{III}_{q^{-\beta}}$. Although this gives some restrictions on the type of $\pi_{\varphi_\beta}(A_{L,S})''$ [17], the exact value of the type still has to be computed, and the computation requires the Chebotarev density theorem.

3. An extremely brief introduction to Drinfeld modules

In this section we will give a brief summary of the facts about Drinfeld modules that are required to describe Jacob's system [14]. For a more thorough exposition see for instance [10], [13], where proofs of the results of this section can be found.

Let K be a global function field with constant field $\mathbb{F}_q$. Fix a distinguished prime ∞ in K. Denote by $\mathcal{O} \subset K$ the subring of functions having no pole away from ∞. Let $\mathbb{F}_\infty = \mathbb{F}_{q^{d_\infty}}$ be the residue field of K_∞. We will also consider it as a subfield of K_∞. The valuation on K_∞ extends uniquely to $\bar{K}_\infty$, its algebraic closure. Let $\mathbb{C}_\infty$ be the completion of $\bar{K}_\infty$ with respect to this valuation. Then $\mathbb{C}_\infty$ is algebraically closed and complete with respect to the valuation.

Let $\mathbb{C}_\infty\{\tau\}$ be the ring of twisted polynomials in τ, with product satisfying the commutation relation $\tau a = a^q \tau$ for $a \in \mathbb{C}_\infty$. It can be identified with the ring of $\mathbb{F}_q$-linear polynomial maps $\mathbb{C}_\infty \to \mathbb{C}_\infty$, with $\tau(x) = x^q$ and multiplication given by composition of maps. Define a map $D : \mathbb{C}_\infty\{\tau\} \to \mathbb{C}_\infty$ by sending a polynomial in τ to its constant term.

Definition 3.1. A *Drinfeld module* over $\mathbb{C}_\infty$ is a homomorphism $\phi : \mathcal{O} \to \mathbb{C}_\infty\{\tau\}$, $a \mapsto \phi_a$, of $\mathbb{F}_q$-algebras whose image is not contained in $\mathbb{C}_\infty$ and which satisfies $D(\phi_a) = a$. If the degree of ϕ_a (considered as a polynomial in τ) is equal to $-d_\infty \mathrm{ord}_\infty(a)$ for all $a \in \mathcal{O}^\times$, then ϕ is said to be of *rank one*.

By a *lattice* in $\mathbb{C}_\infty$ we mean a finitely generated discrete $\mathcal{O}$-submodule of $\mathbb{C}_\infty$.

Rank-one lattices have the form $\xi\mathfrak{a}$, where $\xi \in \mathbb{C}_\infty^\times$ and $\mathfrak{a}$ is a nonzero ideal in $\mathcal{O}$. Any such lattice Λ defines a rank-one Drinfeld module as follows.

First define the exponential function associated to Λ by

$$e_\Lambda(x) = x \prod_{\alpha \in \Lambda \setminus \{0\}} (1 - x/\alpha).$$

This is an entire function on $\mathbb{C}_\infty$ that induces an $\mathbb{F}_q$-linear isomorphism $\mathbb{C}_\infty/\Lambda \cong \mathbb{C}_\infty$. Next form a map

$$\phi_a^\Lambda(x) = ax \prod_{0 \neq \alpha \in a^{-1}\Lambda/\Lambda} (1 - x/e_\Lambda(\alpha))$$

for $a \in \mathcal{O}$. The definition of ϕ_a^Λ is motivated by the identity $e_\Lambda(ax) = \phi_a^\Lambda(e_\Lambda(x))$. The map ϕ_a^Λ is $\mathbb{F}_q$-linear and polynomial, so ϕ_a^Λ is an element of $\mathbb{C}_\infty\{\tau\}$. Therefore $a \mapsto \phi_a^\Lambda \in \mathbb{C}_\infty\{\tau\}$ is a rank-one Drinfeld module.

This way we get a one-to-one correspondence between rank-one lattices in $\mathbb{C}_\infty$ and rank-one Drinfeld modules over $\mathbb{C}_\infty$.

For a non-zero ideal $\mathfrak{a}$ in $\mathcal{O}$ and a Drinfeld module ϕ, consider the left ideal of $\mathbb{C}_\infty\{\tau\}$ generated by the ϕ_a for $a \in \mathfrak{a}$. This ideal can be shown to be principal, so there is a unique monic $\phi_\mathfrak{a} \in \mathbb{C}_\infty\{\tau\}$ generating it. One can show that for each $b \in \mathcal{O}$ there is a unique $\phi_b' \in \mathbb{C}_\infty\{\tau\}$ such that

$$\phi_\mathfrak{a}\phi_b = \phi_b'\phi_\mathfrak{a}.$$

The map $b \mapsto \phi_b'$ is a Drinfeld module, which we denote by $\mathfrak{a} * \phi$. This gives an action of the nonzero ideals of $\mathcal{O}$ on the set of Drinfeld modules. On the level of lattices, the Drinfeld module $\mathfrak{a} * \phi^\Lambda$ corresponds to the lattice $D(\phi_\mathfrak{a})\mathfrak{a}^{-1}\Lambda$.

We say that two lattices Λ and Λ' are *homothetic* if $\Lambda' = \xi\Lambda$ for some $\xi \in \mathbb{C}_\infty^\times$. The corresponding Drinfeld modules are said to be isomorphic, and we obtain $\xi\phi^\Lambda\xi^{-1} = \phi^{\Lambda'}$.

Definition 3.2. A *sign function* on K_∞ is a homomorphism $\operatorname{sgn}\colon K_\infty^\times \to \mathbb{F}_\infty^\times$ which coincides with the reduction modulo ∞ on $\mathcal{O}_\infty^*$.

From now on we fix a sign function. An element of $K_\infty^\times$ is called *positive* if it belongs to the kernel of sgn. Denote by $K_+^\times \subset K^\times$ the subgroup of positive elements in K.

Given a Drinfeld module ϕ, write μ_ϕ for the function mapping $a \in \mathcal{O}^\times$ to the leading coefficient of ϕ_a. We say that ϕ is *normalized* if μ_ϕ has image contained in $\mathbb{F}_\infty^\times$, and that ϕ is sgn-normalized if there is some $\sigma \in \operatorname{Gal}(\mathbb{F}_\infty/\mathbb{F}_q)$ (depending on ϕ) such that $\mu_\phi(a) = \sigma(\operatorname{sgn}(a))$ for all a.

If ϕ is a sgn-normalized Drinfeld module, then by definition $\phi_{(a)} = \phi_a$ for every positive $a \in \mathcal{O}^\times$. This in particular implies that the semigroup $\mathcal{P}^+$ of principal ideals in $\mathcal{O}$ with positive generators acts trivially on ϕ. Let I be the semigroup of nonzero ideals in $\mathcal{O}$. Put $\operatorname{Pic}^+(\mathcal{O}) = I/\mathcal{P}^+$.

Theorem 3.3. *Every rank-one Drinfeld module over $\mathbb{C}_\infty$ is isomorphic to a* sgn-*normalized Drinfeld module. The action by ideals of $\mathcal{O}$ preserves the set of* sgn-*normalized rank-one Drinfeld modules and is transitive on this set, and if $\mathfrak{a}*\phi = \mathfrak{b}*\phi$ for some ideals $\mathfrak{a}$ and $\mathfrak{b}$ and a* sgn-*normalized rank-one Drinfeld module ϕ, then $\mathfrak{a}$ and $\mathfrak{b}$ define the same class in* $\mathrm{Pic}^+(\mathcal{O})$.

In particular, the set of sgn-normalized rank-one Drinfeld modules consists of

$$\#\mathrm{Pic}^+(\mathcal{O}) = h(\mathcal{O})\frac{q^{d_\infty} - 1}{q - 1} = h(K)d_\infty\frac{q^{d_\infty} - 1}{q - 1}$$

elements, where $h(\mathcal{O})$ is the class number of $\mathcal{O}$ and $h(K)$ is that of K.

Given a finite abelian extension L/K and a finite prime $\mathfrak{p}$ in K unramified in L, we write $\sigma_\mathfrak{p}$ for the Frobenius automorphism $(\mathfrak{p}, L/K) \in \mathrm{Gal}(L/K)$. More generally, we write $\sigma_\mathfrak{a}$ for the automorphism corresponding to every ideal $\mathfrak{a}$ in $\mathcal{O}$ whose prime factors are unramified in L.

Let ϕ be a fixed sgn-normalized rank-one Drinfeld module, $y \in \mathcal{O}$ be any non-constant element, and H^+ be the field generated over K by the coefficients of ϕ_y.

Theorem 3.4. *The extension H^+/K is finite, abelian and unramified at every finite prime $\mathfrak{p}$ in K. It is independent of the choice of ϕ and y. The Artin map $I \to \mathrm{Gal}(H^+/K)$ defines an isomorphism* $\mathrm{Pic}^+(\mathcal{O}) \cong \mathrm{Gal}(H^+/K)$*, and for any $a \in \mathcal{O}$ and a nonzero ideal $\mathfrak{a} \subset \mathcal{O}$ we have*

$$\sigma_\mathfrak{a}(\phi_a) = (\mathfrak{a} * \phi)_a.$$

It follows that H^+ contains the Hilbert class field H of $\mathcal{O}$, which is the maximal abelian extension of K that is unramified at every finite prime and completely split at infinity. The Galois group of H/K is $\mathrm{Pic}(\mathcal{O}) = I/\mathcal{P}$, where $\mathcal{P}$ is the semigroup of nonzero principal ideals in $\mathcal{O}$.

Given a sgn-normalized rank-one Drinfeld module ϕ we can define an $\mathcal{O}$-module structure on $\mathbb{C}_\infty$ using the maps ϕ_a. For a nonzero proper ideal $\mathfrak{m}$ in $\mathcal{O}$ denote by $\phi[\mathfrak{m}] \subset \mathbb{C}_\infty$ the set of $\mathfrak{m}$-torsion points with respect to this structure. If $\phi = \phi^\Lambda$, then the exponential map e_Λ defines an $\mathcal{O}$-module isomorphism $\mathfrak{m}^{-1}\Lambda/\Lambda \cong \phi[\mathfrak{m}]$.

Put $K_\mathfrak{m} = H^+(\phi[\mathfrak{m}])$. Denote by $I_\mathfrak{m}$ be the semigroup of ideals of $\mathcal{O}$ which are relatively prime to $\mathfrak{m}$, and let

$$\mathcal{P}_\mathfrak{m}^+ = \{(a) \mid a \in \mathcal{O}^\times, \ \mathrm{sgn}(a) = 1, \ a \equiv 1 \bmod \mathfrak{m}\} \subset I_\mathfrak{m}.$$

The quotient $\mathrm{Pic}_\mathfrak{m}^+(\mathcal{O}) = I_\mathfrak{m}/\mathcal{P}_\mathfrak{m}^+$ is called the *narrow ray class group* modulo $\mathfrak{m}$ relative to sgn. Define also

$$\mathcal{P}_\mathfrak{m} = \{(a) \mid a \in \mathcal{O}^\times, \ a \equiv 1 \bmod \mathfrak{m}\} \subset I_\mathfrak{m}.$$

Theorem 3.5. *The extension $K_{\mathfrak{m}}/K$ is finite, abelian and unramified away from ∞ and the prime ideals dividing $\mathfrak{m}$. It is independent of the choice of ϕ. The Artin map $I_{\mathfrak{m}} \to \mathrm{Gal}(K_{\mathfrak{m}}/K)$ defines an isomorphism $\mathrm{Pic}_{\mathfrak{m}}^+(\mathcal{O}) \cong \mathrm{Gal}(K_{\mathfrak{m}}/K)$, and for any $\mathfrak{a} \in I_{\mathfrak{m}}$ and $\lambda \in \phi[\mathfrak{m}]$ we have*

$$\sigma_{\mathfrak{a}}(\lambda) = \phi_{\mathfrak{a}}(\lambda).$$

The subfield $K_{\mathfrak{m}}^+ \subset K_{\mathfrak{m}}$ of elements fixed by the semigroup $\mathcal{P}_{\mathfrak{m}}$ is contained in K_{∞}, so the extension $K_{\mathfrak{m}}^+/K$ is completely split at infinity. If we continue to denote by ∞ the extension of the place ∞ of K to $K_{\mathfrak{m}}^+$ defined by the inclusion $K_{\mathfrak{m}}^+ \subset K_{\infty}$, then $K_{\mathfrak{m}}/K_{\mathfrak{m}}^+$ is totally ramified at ∞.

Clearly, the Galois group of $K_{\mathfrak{m}}^+/K$ is $\mathrm{Pic}_{\mathfrak{m}}(\mathcal{O}) = I_{\mathfrak{m}}/\mathcal{P}_{\mathfrak{m}}$. Note also that the Galois group of $K_{\mathfrak{m}}/H^+$ is isomorphic to $(I_{\mathfrak{m}} \cap \mathcal{P}^+)/\mathcal{P}_{\mathfrak{m}}^+$. Since $\mathcal{O}^* = \mathbb{F}_q^{\times}$, the only positive element in $\mathcal{O}^*$ is 1, hence any ideal in $\mathcal{P}^+$ has a unique positive generator. We therefore have a well-defined map $I_{\mathfrak{m}} \cap \mathcal{P}^+ \to (\mathcal{O}/\mathfrak{m})^*$, $(a) \mapsto a \bmod \mathfrak{m}$ for positive a. It is not difficult to see that it is surjective, whence

$$\mathrm{Gal}(K_{\mathfrak{m}}/H^+) \simeq (\mathcal{O}/\mathfrak{m})^*. \tag{3.1}$$

Let $\mathbb{K} \subset \mathbb{C}_{\infty}$ be the union of the fields $K_{\mathfrak{m}}$ over non-zero proper ideals $\mathfrak{m}$ of $\mathcal{O}$, and $K^{\mathrm{ab},\infty} \subset K_{\infty}$ be the union of the fields $K_{\mathfrak{m}}^+$. These are abelian extensions of K and

$$\mathrm{Gal}(\mathbb{K}/K) \cong \varprojlim \mathrm{Pic}_{\mathfrak{m}}^+(\mathcal{O}) \quad \text{and} \quad \mathrm{Gal}(K^{\mathrm{ab},\infty}/K) \cong \varprojlim \mathrm{Pic}_{\mathfrak{m}}(\mathcal{O}),$$

where $\mathfrak{m}$ runs over the nonzero ideals of $\mathcal{O}$ ordered by divisibility. It follows that the Artin map $\mathbb{A}_f^* \to \mathrm{Gal}(\mathbb{K}/K)$, where $\mathbb{A}_f = \mathbb{A}_{\{\infty\}}$, defines isomorphisms

$$\mathrm{Gal}(\mathbb{K}/K) \cong \mathbb{A}_f^*/K_+^{\times} \quad \text{and} \quad \mathrm{Gal}(K^{\mathrm{ab},\infty}/K) \cong \mathbb{A}_f^*/K^{\times}.$$

The field $K^{\mathrm{ab},\infty}$ is the maximal abelian extension of K in which ∞ splits completely.

Later we will need to know the constant fields of $\mathbb{K}$ and $K^{\mathrm{ab},\infty}$ in order to calculate the types of KMS_{β}-states in the critical region.

Lemma 3.6. *The constant fields of $K_{\mathfrak{m}}$ and $K_{\mathfrak{m}}^+$ are both equal to $\mathbb{F}_{q^{d_\infty}}$.*

Proof. First consider $K_{\mathfrak{m}}^+$. On the one hand, the residue field of $K_{\mathfrak{m}}^+$ at infinity is $\mathbb{F}_{q^{d_\infty}}$, since $K_{\mathfrak{m}}^+$ is completely split at infinity. On the other hand, $K_{\mathfrak{m}}^+$ contains the Hilbert class field H. Since $\mathbb{F}_{q^{d_\infty}}K/K$ is unramified at every prime and completely split at infinity, we have $\mathbb{F}_{q^{d_\infty}}K \subset H \subset K_{\mathfrak{m}}^+$. Therefore $\mathbb{F}_{q^{d_\infty}}$ is both the constant field of $K_{\mathfrak{m}}^+$ and the residue field of $K_{\mathfrak{m}}^+$ at infinity. Since $K_{\mathfrak{m}}/K_{\mathfrak{m}}^+$ is totally ramified at infinity, the residue field of $K_{\mathfrak{m}}$ at infinity is $\mathbb{F}_{q^{d_\infty}}$. Hence the constant field of $K_{\mathfrak{m}}$ is also $\mathbb{F}_{q^{d_\infty}}$. $\qquad\square$

Corollary 3.7. *The algebraic closures of $\mathbb{F}_q$ in $\mathbb{K}$ and $K^{\mathrm{ab},\infty}$ are equal to $\mathbb{F}_{q^{d_\infty}}$.*

Finally, note that the isomorphisms (3.1) imply that the Artin map

$$\mathbb{A}_f^* \to \mathrm{Gal}(\mathbb{K}/K)$$

induces an isomorphism

$$\mathrm{Gal}(\mathbb{K}/H^+) \cong \hat{\mathcal{O}}^*,$$

where $\hat{\mathcal{O}} = \mathcal{O}_{\mathbb{A},\{\infty\}}$. This is also clear from the isomorphisms $\mathrm{Gal}(H^+/K) \cong \mathrm{Pic}^+(\mathcal{O}) \cong \mathbb{A}_f^*/K_+^\times \hat{\mathcal{O}}^*$ and $\mathrm{Gal}(\mathbb{K}/K) \cong \mathbb{A}_f^*/K_+^\times$.

4. Dynamical systems arising from Drinfeld modules

Inspired by the work of Drinfeld [7] and Hayes [12] on explicit class field theory for function fields, Jacob [14] defined a dynamical system associated to a function field K. The main goal of this section is to show that Jacob's system fits into our framework.

For a sgn-normalized rank-one Drinfeld module ϕ, write $\phi(\mathbb{C}_\infty)^{\mathrm{tor}}$ for the set $\bigcup_{\mathfrak{m}} \phi[\mathfrak{m}]$ of torsion points of the $\mathcal{O}$-action on $\mathbb{C}_\infty$ given by $a\xi = \phi_a(\xi)$. If $\phi = \phi^\Lambda$, then e_Λ defines an $\mathcal{O}$-module isomorphism $K\Lambda/\Lambda \cong \phi(\mathbb{C}_\infty)^{\mathrm{tor}}$. Let X_ϕ be the group of characters of $\phi(\mathbb{C}_\infty)^{\mathrm{tor}}$. Put $X = \bigsqcup_\phi X_\phi$, where the union is taken over the set of sgn-normalized rank-one Drinfeld modules. Since the set of such modules is finite and each X_ϕ is a profinite group, X is compact.

Define an action of I on X by letting an ideal $\mathfrak{a}$ map the character $\chi \in X_\phi$ to the character $\chi^{\mathfrak{a}} = \chi \circ (\mathfrak{a}^{-1} * \phi)_{\mathfrak{a}} \in X_{\mathfrak{a}^{-1}*\phi}$, where by $\mathfrak{a}^{-1} * \phi$ we, of course, mean the unique sgn-normalized Drinfeld module such that $\mathfrak{a} * (\mathfrak{a}^{-1} * \phi) = \phi$. This is a semigroup action. There is also an action of $\mathrm{Gal}(\mathbb{K}/K)$ on X given by $g\chi = \chi \circ g \in X_{g^{-1}(\phi)}$, where $g^{-1}(\phi)$ is the Drinfeld module defined by $g^{-1}(\phi)_a = g^{-1}(\phi_a)$. This action commutes with the ideal action.

The action of I on X defines a partially defined action of the group of fractional ideals of $\mathcal{O}$ on X, which gives rise to a transformation groupoid $\mathcal{G}_J$. The C*-algebra $C_{K,\infty}$ underlying Jacob's system is the C*-algebra of this groupoid. Alternatively, one can say that $C_{K,\infty}$ is the semigroup crossed product $C(X) \rtimes I$ with respect to the action

$$(\mathfrak{a}f)(\chi) = \begin{cases} f(\chi') & \text{if } \chi'^{\mathfrak{a}} = \chi, \\ 0 & \text{if no such } \chi' \text{ exists.} \end{cases}$$

The one-parameter family $\mathfrak{a} \mapsto N(\mathfrak{a})^{it}$ of characters of I defines a one-parameter group of automorphisms σ_t of $C_{K,\infty}$.

We want to show that the system $(C_{K,\infty}, \sigma)$ is isomorphic to the system $(A_{\mathbb{K},\{\infty\}}, \sigma)$ introduced in Section 1. The latter system was defined using the action of $I = \mathcal{D}_{\{\infty\}}^+$

on $Y_{\mathbb{K},\{\infty\}} = \mathrm{Gal}(\mathbb{K}/K) \times_{\widehat{\mathcal{O}}^*} \widehat{\mathcal{O}}$. Define an action of $\mathrm{Gal}(\mathbb{K}/K)$ on $Y_{\mathbb{K},\{\infty\}}$ by $g(x, y) = (g^{-1}x, y)$.

Theorem 4.1. *There is a* $\mathrm{Gal}(\mathbb{K}/K)$*- and* I*-equivariant homeomorphism* $\pi\colon X \to Y_{\mathbb{K},\{\infty\}}$. *In particular, the* C^**-dynamical systems* $(C_{K,\infty}, \sigma)$ *and* $(A_{\mathbb{K},\{\infty\}}, \sigma)$ *are isomorphic.*

Proof. Fix a sgn-normalized rank-one Drinfeld module ϕ^0. Recall that the semigroup $\mathcal{P}^+$ of principal ideals with positive generators acts trivially on ϕ^0, so we have an action of $\mathcal{P}^+$ on X_{ϕ^0}. The semigroup $\mathcal{P}^+$ can be identified with $\mathcal{O}_+^\times$. The latter semigroup acts on $\widehat{\mathcal{O}}$ by multiplication. Let us show first that there exists a $\mathcal{P}^+$-equivariant continuous isomorphism $\pi^0\colon X_{\phi^0} \to \widehat{\mathcal{O}}$.

The $\mathcal{O}$-module $\phi^0(\mathbb{C}_\infty)^{\mathrm{tor}}$ is isomorphic to $K/\mathcal{O}$. Indeed, if ϕ^0 is defined by a lattice Λ, then $\phi^0(\mathbb{C}_\infty)^{\mathrm{tor}} \cong K\Lambda/\Lambda$. The lattice Λ has the form $\xi\mathfrak{a}$ for some $\xi \in \mathbb{C}_\infty^\times$ and $\mathfrak{a} \subset \mathcal{O}$. Then $K\Lambda/\Lambda \cong K/\mathfrak{a}$. Next, the closure of $\mathfrak{a}$ in $\widehat{\mathcal{O}}$ has the form $g\widehat{\mathcal{O}}$ for some $g \in \mathbb{A}_f^* \cap \widehat{\mathcal{O}}$. Therefore,

$$K/\mathfrak{a} = \mathbb{A}_f/g\widehat{\mathcal{O}} \cong \mathbb{A}_f/\widehat{\mathcal{O}} = K/\mathcal{O},$$

and hence $\phi^0(\mathbb{C}_\infty)^{\mathrm{tor}} \cong K/\mathcal{O}$.

It is well known that the additive group $\mathbb{A}_f$ is self-dual. Furthermore, the pairing on $\mathbb{A}_f \times \mathbb{A}_f$ can be defined as $(a, b) \mapsto \omega(ab)$ for a character ω, and with an appropriate choice of ω the annihilator of $\widehat{\mathcal{O}}$ is $\widehat{\mathcal{O}}$. It follows that there exists an $\mathcal{O}$-module isomorphism

$$\widehat{K/\mathcal{O}} = \widehat{\mathbb{A}_f/\widehat{\mathcal{O}}} \cong \widehat{\mathcal{O}},$$

where the $\mathcal{O}$-module structure on $\widehat{K/\mathcal{O}}$ is defined by $a\chi = \chi(a \cdot)$.

Combining the isomorphisms $\phi^0(\mathbb{C}_\infty)^{\mathrm{tor}} \cong K/\mathcal{O}$ and $\widehat{K/\mathcal{O}} \cong \widehat{\mathcal{O}}$, we get the required $\mathcal{O}$-module isomorphism

$$\pi^0\colon X_{\phi^0} = \widehat{\phi^0(\mathbb{C}_\infty)^{\mathrm{tor}}} \to \widehat{\mathcal{O}},$$

where the $\mathcal{O}$-module structure on $\widehat{\phi^0(\mathbb{C}_\infty)^{\mathrm{tor}}}$ is defined by $a\chi = \chi \circ \phi_a^0$. Since $\phi_{(a)}^0 = \phi_a^0$ for positive a, this isomorphism is $\mathcal{P}^+$-equivariant.

Consider now the extension H^+/K defined in the previous section. As we observed there, the Artin map $\mathbb{A}_f^* \to \mathrm{Gal}(\mathbb{K}/K)$ gives an isomorphism of $\widehat{\mathcal{O}}^*$ onto $\mathrm{Gal}(\mathbb{K}/H^+)$, so the open subset $Y^0 = \mathrm{Gal}(\mathbb{K}/H^+) \times_{\widehat{\mathcal{O}}^*} \widehat{\mathcal{O}} \subset Y_{\mathbb{K},\{\infty\}}$ can be identified with $\widehat{\mathcal{O}}$. Hence the map π^0 can be considered to be a homeomorphism of X_{ϕ^0} onto Y^0.

We want to extend π^0 to a map $\pi\colon X \to Y_{\mathbb{K},\{\infty\}}$. Let ϕ be a sgn-normalized rank-one Drinfeld module, and $\chi \in X_\phi$ be a character. There exists an ideal $\mathfrak{a}$ such

that $\phi = \mathfrak{a} * \phi^0$. Then $\chi^{\mathfrak{a}} = \chi \circ \phi_{\mathfrak{a}}^0 \in X_{\phi^0}$. By definition of $\phi_{\mathfrak{a}}^0$ the kernel of $\phi_{\mathfrak{a}}^0 \colon \phi^0(\mathbb{C}_\infty)^{\mathrm{tor}} \to \phi(\mathbb{C}_\infty)^{\mathrm{tor}}$ is exactly $\phi^0[\mathfrak{a}]$. Therefore the kernel of $\chi^{\mathfrak{a}}$ contains $\phi^0[\mathfrak{a}]$. Hence, under our isomorphism of $\phi^0(\mathbb{C}_\infty)^{\mathrm{tor}}$ with $K/\mathcal{O}$ the kernel of $\chi^{\mathfrak{a}}$ contains $\mathfrak{a}^{-1}\mathcal{O}/\mathcal{O}$. Since the annihilator of $\mathfrak{a}^{-1}\widehat{\mathcal{O}}$ in $\mathbb{A}_f$ is $\mathfrak{a}\widehat{\mathcal{O}}$, we conclude that $\pi^0(\chi^{\mathfrak{a}}) \in \mathfrak{a}\widehat{\mathcal{O}}$. In particular, $\pi^0(\chi^{\mathfrak{a}}) \in \mathfrak{a} Y_{\mathbb{K},\{\infty\}}$, so we can define

$$\pi(\chi) = \mathfrak{a}^{-1}\pi^0(\chi^{\mathfrak{a}}) \in Y_{\mathbb{K},\{\infty\}}.$$

Since π^0 is $\mathcal{P}^+$-equivariant, this definition does not depend on the choice of $\mathfrak{a}$ (such that $\phi = \mathfrak{a} * \phi^0$).

We have therefore extended π^0 to a continuous map $\pi \colon X \to Y_{\mathbb{K},\{\infty\}}$. By construction this map is equivariant with respect to the ideal action.

Next let us show that π is $\mathrm{Gal}(\mathbb{K}/K)$-equivariant. For every nonzero proper ideal $\mathfrak{m} \subset \mathcal{O}$ consider the finite sets

$$X_{\mathfrak{m}} = \bigsqcup_{\phi} \widehat{\phi[\mathfrak{m}]}$$

and

$$Y_{\mathfrak{m}} = \mathrm{Gal}(K_{\mathfrak{m}}/K) \times_{\widehat{\mathcal{O}}^*} (\widehat{\mathcal{O}}/\mathfrak{m}\widehat{\mathcal{O}}) = \mathrm{Gal}(K_{\mathfrak{m}}/K) \times_{(\mathcal{O}/\mathfrak{m})^*} (\mathcal{O}/\mathfrak{m}).$$

Similarly to X and $Y_{\mathbb{K},\{\infty\}}$ these sets carry actions of $\mathrm{Gal}(\mathbb{K}/K)$ and I; note, however, that if $\mathfrak{a}$ is not prime to $\mathfrak{m}$, then the actions by $\mathfrak{a}$ are defined by non-injective maps. Our isomorphism $\widehat{\phi^0(\mathbb{C}_\infty)^{\mathrm{tor}}} \simeq \widehat{\mathcal{O}}$ induces an isomorphism $\widehat{\phi^0[\mathfrak{m}]} \cong \widehat{\mathcal{O}}/\mathfrak{m}\widehat{\mathcal{O}} = \mathcal{O}/\mathfrak{m}$. Since $\mathrm{Gal}(K_{\mathfrak{m}}/H^+) \simeq (\mathcal{O}/\mathfrak{m})^*$, we can identify $Y_{\mathfrak{m}}^0 = \mathrm{Gal}(K_{\mathfrak{m}}/H^+) \times_{(\mathcal{O}/\mathfrak{m})^*} (\mathcal{O}/\mathfrak{m}) \subset Y_{\mathfrak{m}}$ with $\mathcal{O}/\mathfrak{m}$. Hence, similarly to the construction of π, by choosing for every ϕ an ideal $\mathfrak{a}_\phi \in I_{\mathfrak{m}}$ such that $\phi = \mathfrak{a}_\phi * \phi^0$ we can extend the isomorphism $\widehat{\phi^0[\mathfrak{m}]} \cong \mathcal{O}/\mathfrak{m}$ to a map $\pi_{\mathfrak{m}} \colon X_{\mathfrak{m}} \to Y_{\mathfrak{m}}$. Clearly, the diagram

$$
\begin{array}{ccc}
X & \xrightarrow{\ \pi\ } & Y \\
\downarrow & & \downarrow \\
X_{\mathfrak{m}} & \xrightarrow{\ \pi_{\mathfrak{m}}\ } & Y_{\mathfrak{m}},
\end{array}
$$

where the vertical arrows are the obvious quotient maps, is commutative. It follows that $\pi_{\mathfrak{m}}$ is independent of the choice of $\mathfrak{a}_\phi$ and is I-equivariant.

Take a sgn-normalized rank-one Drinfeld module ϕ, a character $\chi \in \widehat{\phi[\mathfrak{m}]}$, and $\mathfrak{a} \in I_{\mathfrak{m}}$. For $\lambda \in (\mathfrak{a}^{-1} * \phi)[\mathfrak{m}] = \sigma_{\mathfrak{a}}^{-1}(\phi)[\mathfrak{m}]$ we have

$$\sigma_{\mathfrak{a}}(\lambda) = (\mathfrak{a}^{-1} * \phi)_{\mathfrak{a}}(\lambda),$$

so $\sigma_{\mathfrak{a}}\chi = \chi \circ \sigma_{\mathfrak{a}} = \chi \circ (\mathfrak{a}^{-1} * \phi)_{\mathfrak{a}} = \chi^{\mathfrak{a}}$. Hence the Galois action of $\mathrm{Gal}(K_{\mathfrak{m}}/K)$ on $X_{\mathfrak{m}}$ corresponds to the ideal action of $I_{\mathfrak{m}}$ via the Artin map.

Under the map $\pi_{\mathfrak{m}}$ above, this Galois action is then transported to $Y_{\mathfrak{m}}$ by

$$pi_{\mathfrak{m}}(\sigma_{\mathfrak{a}}\chi) = \pi_{\mathfrak{m}}(\chi^{\mathfrak{a}}) = \mathfrak{a}\pi_{\mathfrak{m}}(\chi).$$

Recall now that the action of α on $Y_{\mathfrak{m}}$ is defined using the action $g(x, y) = (xr_{K_{\mathfrak{m}}/K}(g)^{-1}, gy)$ on $\mathrm{Gal}(K_{\mathfrak{m}}/K) \times (\hat{\mathcal{O}}/\mathfrak{m}\hat{\mathcal{O}})$, where $g \in \mathbb{A}_f^* \cap \hat{\mathcal{O}}$ is any element such that $\alpha = g\hat{\mathcal{O}} \cap \mathcal{O}$. Since α is prime to $\mathfrak{m}$, we can take g such that $g_{\mathfrak{p}} = 1$ for all primes $\mathfrak{p}$ dividing $\mathfrak{m}$. Then the action of g on $\hat{\mathcal{O}}/\mathfrak{m}\hat{\mathcal{O}}$ is trivial and $r_{K_{\mathfrak{m}}/K}(g) = \sigma_{\alpha}$. Therefore, $\alpha\pi_{\mathfrak{m}}(\chi) = \sigma_{\alpha}\pi_{\mathfrak{m}}(\chi)$, so that

$$\pi_{\mathfrak{m}}(\sigma_{\alpha}\chi) = \sigma_{\alpha}\pi_{\mathfrak{m}}(\chi).$$

Since the Artin map $I_{\mathfrak{m}} \to \mathrm{Gal}(K_{\mathfrak{m}}/K)$ is surjective, we conclude that $\pi_{\mathfrak{m}}$ is $\mathrm{Gal}(K_{\mathfrak{m}}/K)$-equivariant.

Now note that $X = \varprojlim X_{\mathfrak{m}}$ and $Y = \varprojlim Y_{\mathfrak{m}}$. Hence the $\mathrm{Gal}(\mathbb{K}/K)$-equivariance of $\pi: X \to Y$ follows from the $\mathrm{Gal}(K_{\mathfrak{m}}/K)$-equivariance of $\pi_{\mathfrak{m}}$.

It remains to show that π is a homeomorphism. The space X is the disjoint union of the open sets gX_{ϕ^0}, where g runs over representatives of $\mathrm{Gal}(\mathbb{K}/K)/\mathrm{Gal}(\mathbb{K}/H^+) \cong \mathrm{Gal}(H^+/K) \cong \mathrm{Pic}^+(\mathcal{O})$. Similarly, $Y_{\mathbb{K},\{\infty\}}$ is the disjoint union of the sets gY^0. Since π is $\mathrm{Gal}(\mathbb{K}/K)$-equivariant and defines a homeomorphism of X_{ϕ^0} onto Y^0, we conclude that π is a homeomorphism. $\qquad\square$

Remark 4.2. The map π depends on the choice of a sgn-normalized rank-one Drinfeld module ϕ^0, an $\mathcal{O}$-module isomorphism $\phi^0(\mathbb{C}_{\infty})^{\mathrm{tor}} \cong K/\mathcal{O}$, and a character ω of $\mathbb{A}_f$ defining a pairing on $\mathbb{A}_f \times \mathbb{A}_f$ such that $\hat{\mathcal{O}}^{\perp} = \hat{\mathcal{O}}$. It is not difficult to see that if $\pi': X \to Y_{\mathbb{K},\{\infty\}}$ is another $\mathrm{Gal}(\mathbb{K}/K)$- and I-equivariant homeomorphism, e.g. one constructed using a different choice of the above three ingredients, then $\pi'(\chi) = g\pi(\chi)$ for a uniquely defined $g \in \mathrm{Gal}(\mathbb{K}/K)$.

Applied to the system $(A_{\mathbb{K},\{\infty\}}, \sigma)$, our Theorem 1.1 summarizes [14], Theorems 4.3.10, 4.4.15. Furthermore, by Theorem 2.1 and Corollary 3.7, the type of the unique KMS_{β}-state of Jacob's system for $\beta \in (0, 1]$ is $\mathrm{III}_{q^{-\beta d_{\infty}}}$. This corrects a mistake in [14], Theorem 4.5.8, which asserts that the type is $\mathrm{III}_{q^{-\beta}}$, a mistake partially caused by a wrong formulation of the Chebotarev density theorem.[1]

Another approach to defining a Bost–Connes system for function fields is that of Consani and Marcolli [6]. Their setting is different, as they develop a theory of dynamical systems for algebras of $\mathbb{C}_{\infty}$-valued functions. However, these algebras arise from groupoids, which makes it natural to consider the relationship between these groupoids and the ones we consider in this paper.

A one-dimensional K-lattice in K_{∞} is a pair (Λ, φ) where $\Lambda \subset K_{\infty}$ is a rank-one lattice and $\varphi: K/\mathcal{O} \to K\Lambda/\Lambda$ is an $\mathcal{O}$-module map. Two one-dimensional K-lattices

---

[1]The computation in [14] relies on this theorem, but in a way that is different from ours. Unfortunately, the strategy in [14] does not work even for $d_{\infty} = 1$, when the formulation of the Chebotarev density theorem becomes correct. The mistake is in the proof of the crucial Lemma 4.5.5, the same lemma where the Chebotarev density theorem is used, which does not take into account that the elements $\mu_{\mathfrak{p}}$ and $\mu_{\mathfrak{q}}$ do not belong to $M[\mathfrak{d}]$. In fact, the assertion of that lemma is not correct, it can be shown that already the center of $M[\mathfrak{d}]$ has elements that are not $\mathrm{Gal}(K_{\mathfrak{d}}/k)$-invariant.

(Λ_1, φ_1) and (Λ_2, φ_2) are called commensurable if Λ_1 and Λ_2 are commensurable (equivalently, $K\Lambda_1 = K\Lambda_2$) and the maps $K/\mathcal{O} \to K\Lambda_i/(\Lambda_1 + \Lambda_2)$ defined by φ_1 and φ_2 coincide.

The one-dimensional K-lattices in K_∞ can be parametrized (see [6], [16]) by the set

$$K_\infty^\times \times_{K^\times} \mathbb{A}_f^* \times_{\widehat{\mathcal{O}}*} \widehat{\mathcal{O}}.$$

There is a partial action of $\mathbb{A}_{K,f}^*$ on this space given by

$$g(\xi, x, y) = (\xi, xg^{-1}, gy)$$

as long as $gy \in \widehat{\mathcal{O}}$ and which is undefined if this is not the case. This action descends to a partial action of fractional ideals of $\mathcal{O}$, and the corresponding orbit equivalence relation is exactly the relation of commensurability. The action defines a groupoid $\tilde{\mathcal{G}}$ with the set of one-dimensional K-lattices in K_∞ as its object space.

Let $\mathcal{G}_{\mathrm{CM}}$ be the quotient groupoid obtained by identifying elements (ξ, x, y) and $(\zeta\xi, x, y)$ in the object space for $\zeta \in K_\infty^\times$. This is the main groupoid considered in [6]. It has the object space

$$\mathcal{G}_{\mathrm{CM}}^0 = K^\times \backslash \mathbb{A}_f^* \times_{\widehat{\mathcal{O}}*} \widehat{\mathcal{O}} \simeq \mathrm{Gal}(K^{\mathrm{ab},\infty}/K) \times_{\widehat{\mathcal{O}}*} \widehat{\mathcal{O}} = Y_{K^{\mathrm{ab},\infty},\{\infty\}}.$$

This identification respects the actions of the semigroup $I = \mathcal{D}_{\{\infty\}}^+$ of ideals, so the groupoid considered in [6] gives rise to the dynamical system $(A_{K^{\mathrm{ab},\infty},\{\infty\}}, \sigma)$. As for Jacob's system above, the type of the unique KMS_β-state for this system for $\beta \in (0, 1]$ is $\mathrm{III}_{q^{-\beta d_\infty}}$.

Theorem 4.1 clarifies the relation between the groupoids $\mathcal{G}_{\mathrm{CM}}$ and $\mathcal{G}_{\mathrm{J}}$ of Consani–Marcolli and Jacob: $\mathcal{G}_{\mathrm{CM}}$ is isomorphic to the quotient of $\mathcal{G}_{\mathrm{J}}$ by the action of $\mathrm{Gal}(\mathbb{K}/K^{\mathrm{ab},\infty}) \cong K^\times/K_+^\times \cong \mathbb{F}_\infty^\times$. Also note that in the case of the Bost–Connes system for $\mathbb{Q}$ we only divide out by scaling by positive reals. The natural analogue of the positive reals in our setting is the subgroup $K_\infty^+ \subset K_\infty^\times$ of positive elements. The groupoid $K_\infty^+ \backslash \tilde{\mathcal{G}}$ has the object space

$$K_\infty^+ \backslash K_\infty^\times \times_{K^\times} \mathbb{A}_f^* \times_{\widehat{\mathcal{O}}*} \widehat{\mathcal{O}} \simeq \mathbb{F}_\infty^\times \times_{K^\times} \mathbb{A}_f^* \times_{\widehat{\mathcal{O}}*} \widehat{\mathcal{O}} \simeq \mathrm{Gal}(\mathbb{K}/K) \times_{\widehat{\mathcal{O}}*} \widehat{\mathcal{O}},$$

and the ideal action is identical to the one on $Y_{\mathbb{K},\{\infty\}}$. Therefore, the modification of the construction of Consani–Marcolli obtained by considering K-lattices in K_∞ up to scaling by positive elements gives rise to a groupoid isomorphic to Jacob's groupoid $\mathcal{G}_{\mathrm{J}}$.

We finish by reformulating the classification of KMS_β-states of $(A_{\mathbb{K},\{\infty\}}, \sigma)$ for $\beta > 0$ in terms of relatively invariant measures on finite adeles. For number fields such a reformulation was used in [18], [19].

Theorem 4.3. *Consider measures μ on $\mathbb{A}_f$ such that*

$$\mu(\widehat{\mathcal{O}}) = 1 \quad and \quad \mu(a \cdot) = N(a)^{-\beta}\mu \;\; for\; all\; a \in K_+^\times, \tag{4.1}$$

where $N(a) = N(a\mathcal{O}) = q^{-d_\infty \operatorname{ord}_\infty(a)}$. Then

(i) *for every* $0 < \beta \leq 1$ *there exists a unique measure* μ_β *satisfying* (4.1); *furthermore, the action of* $K_+^\times$ *on* $(\mathbb{Z} \times \mathbb{A}_f, \lambda \times \mu_\beta)$ *given by* $a(n, x) = (\mathrm{ord}_\infty(a) + n, ax)$, *where* λ *is the counting measure, is ergodic;*

(ii) *for every* $\beta > 1$ *and* $g \in \mathbb{A}_f^*$ *there exists a unique measure satisfying* (4.1) *that is concentrated on* $K_+^\times g \subset \mathbb{A}_f^*$; *this way we get, for every* $\beta > 1$, *a one-to-one correspondence between extremal measures satisfying* (4.1) *and points of the compact group* $\mathbb{A}_f^*/K_+^\times \cong \mathrm{Gal}(\mathbb{K}/K)$.

Proof. As we already remarked in the proof of Theorem 1.1, classifying KMS_β-states of $(A_{\mathbb{K}, \{\infty\}}, \sigma)$ for $\beta \neq 0$ is equivalent to classifying measures ν on $X_{\mathbb{K}, \{\infty\}} = \mathrm{Gal}(\mathbb{K}/K) \times_{\hat{\mathcal{O}}^*} \mathbb{A}_f$ such that $\nu(Y_{\mathbb{K}, \{\infty\}}) = 1$ and $\nu(\mathfrak{a} \cdot) = N(\mathfrak{a})^{-\beta} \nu$ for all $\mathfrak{a} \in I$. Consider the subset $X^0 = \mathrm{Gal}(\mathbb{K}/H^+) \times_{\hat{\mathcal{O}}}^* \mathbb{A}_f \subset X_{\mathbb{K}, \{\infty\}}$ and note that X^0 can be identified with $\mathbb{A}_f$ since $\mathrm{Gal}(\mathbb{K}/H^+) \cong \hat{\mathcal{O}}^*$. The set X^0 is $\mathcal{P}^+$-invariant, $I X^0 = X_{\mathbb{K}, \{\infty\}}$ and $\mathfrak{a} X^0 \cap X^0 \neq \emptyset$ only for $\mathfrak{a} \in \mathcal{P}^+$. It is easy to see that this implies that any Radon measure μ on X^0 such that $\mu(\mathfrak{a} \cdot) = N(\mathfrak{a})^{-\beta} \mu$ for all $\mathfrak{a} \in \mathcal{P}^+$ extends uniquely to a Radon measure ν on $X_{\mathbb{K}, \{\infty\}}$ such that $\nu(\mathfrak{a} \cdot) = N(\mathfrak{a})^{-\beta} \nu$ for all $\mathfrak{a} \in I$. It follows that classifying measures ν on $X_{\mathbb{K}, \{\infty\}}$ as above is equivalent to classifying measures on $\mathbb{A}_f$ satisfying (4.1). Therefore all the statements of theorem, with the exception of the second part of (i), are indeed a reformulation of Theorem 1.1.

Since the unique KMS_β-state of $(A_{\mathbb{K}, \{\infty\}}, \sigma)$ for $\beta \in (0, 1]$ is of type $\mathrm{III}_{q^{-\beta d_\infty}}$, the action of $K_+^\times$ on $(\mathbb{A}_f, \mu_\beta)$ is of type $\mathrm{III}_{q^{-\beta d_\infty}}$. At the same time the norms of elements of $K^\times$ are integral powers of q^{d_∞}. By Proposition A.4, we conclude that the action of the subgroup of elements of $K_+^\times$ of norm one, that is, of elements of order zero at infinity, on $(\mathbb{A}_f, \mu_\beta)$ is ergodic. This is equivalent to the second statement in (i). $\square$

Appendix: Ratio set

Let (X, μ) be a standard measure space with σ-finite measure μ, and $\mathcal{R} \subset X \times X$ be a non-singular countable measurable equivalence relation on (X, μ) [8]. The non-singularity means that if $A \subset X$ is a set of measure zero, then the minimal $\mathcal{R}$-invariant subset X containing A also has measure zero. Thanks to this assumption there is a measurable (with respect to a measure class on $\mathcal{R}$ whose projection onto X coincides with $[\mu]$) map $c_\mu \colon \mathcal{R} \to \mathbb{R}_+^*$ such that $c_\mu(x, z) = c_\mu(x, y) c_\mu(y, z)$ if $x \sim_\mathcal{R} y \sim_\mathcal{R} z$ and for any measurable bijective map $T \colon A \to B$ with graph in $\mathcal{R}$ we have

$$\frac{d T^{-1} \mu}{d \mu}(x) = c_\mu(x, Tx) \quad \text{for } \mu\text{-a.e. } x \in A,$$

where $T^{-1}\mu$ is the measure on A defined by $(T^{-1}\mu)(Z) = \mu(TZ)$. The map c_μ is called the Radon–Nikodym cocycle.

By definition ([15], [8]) the ratio set $r(\mathcal{R}, \mu)$ is the intersection of the essential ranges of the restrictions of c_μ to $\mathcal{R} \cap (Z \times Z)$ for all measurable subsets $Z \subset X$ of positive measure. By the essential range we mean the smallest closed set which contains the value of almost every point.

Equivalently we can say that the ratio set is defined as follows. For a measurable set $Z \subset X$ of positive measure and $\varepsilon > 0$ denote by $r_{Z,\varepsilon}(\mathcal{R}, \mu)$ the set of numbers $\lambda \geq 0$ such that there exist measurable subsets $A, B \subset Z$ of positive measure and a measurable bijective map $T \colon A \to B$ with graph in $\mathcal{R}$ such that $|c_\mu(x, Tx) - \lambda| < \varepsilon$ for all $x \in A$. Then

$$r(\mathcal{R}, \mu) = \bigcap_{\mu(Z)>0, \ \varepsilon>0} r_{Z,\varepsilon}(\mathcal{R}, \mu).$$

Denote by $\bar{r}_{Z,\varepsilon}(\mathcal{R}, \mu)$ the subset of $\lambda \in r_{Z,\varepsilon}(\mathcal{R}, \mu)$ for which we can choose A and B as above such that $\mu(Z \setminus (A \cup B)) = 0$. Using Zorn's lemma it is easy to see that $r(\mathcal{R}, \mu) \subset \bar{r}_{Z,\varepsilon}(\mathcal{R}, \mu)$. Furthermore, since $c_\mu(x, x) = 1$ for all x, for every $\lambda \in r(\mathcal{R}, \mu) \setminus \{1\}$ we can choose the sets A and B such that $A \cap B = \emptyset$.

Clearly, $r(\mathcal{R}, \mu)$ is a closed subset of $[0, +\infty)$. Furthermore, $r(\mathcal{R}, \mu) \setminus \{0\}$ is a subgroup of $\mathbb{R}_+^*$. If we replace μ by an equivalent measure, the ratio set remains the same.

Lemma A.1. *For every $\lambda > 0$, there exist $\varepsilon_0 > 0$ and $\delta_0 > 0$ such that if Z and Z' are measurable subsets of X of finite positive measure, $0 < \varepsilon < \varepsilon_0$, $0 < \delta < \delta_0$, $\lambda \in \bar{r}_{Z',\varepsilon}(\mathcal{R}, \mu)$ and $\mu(Z \Delta Z') < \delta \mu(Z')$, then $\lambda \in r_{Z,\varepsilon}(\mathcal{R}, \mu)$.*

Proof. Let $A, B \subset Z'$ and $T \colon A \to B$ be as required in the definition of $\bar{r}_{Z',\varepsilon}(\mathcal{R}, \mu)$. If

$$\mu(Z \cap A \cap T^{-1}(Z \cap B)) > 0,$$

then clearly $\lambda \in r_{Z,\varepsilon}(\mathcal{R}, \mu)$. So assume that $Z \cap A \cap T^{-1}(Z \cap B)$ has measure zero. Then

$$\mu(Z \cap A) + \mu(T^{-1}(Z \cap B)) \leq \mu(A). \tag{A.1}$$

For every measurable subset $\Omega \subset A$ we have

$$|\mu(T\Omega) - \lambda \mu(\Omega)| \leq \varepsilon \mu(\Omega).$$

In particular,

$$|\mu(Z \cap B) - \lambda \mu(T^{-1}(Z \cap B))| \leq \varepsilon \mu(T^{-1}(Z \cap B)) \leq \varepsilon \mu(A).$$

Multiplying (A.1) by λ we then get

$$\lambda \mu(Z \cap A) + \mu(Z \cap B) \leq (\lambda + \varepsilon)\mu(A).$$

Since $A, B \subset Z'$ and $\mu(Z \Delta Z') < \delta \mu(Z')$, the above inequality yields

$$\lambda \mu(A) + \mu(B) \leq (\lambda + \varepsilon)\mu(A) + \delta(\lambda + 1)\mu(Z'),$$

hence
$$\mu(B) \leq \varepsilon\mu(A) + \delta(\lambda + 1)\mu(Z') \leq (\varepsilon + \delta(\lambda + 1))\mu(Z'). \qquad \text{(A.2)}$$

On the other hand, from (A.1) we also get
$$|\mu(B) - \lambda\mu(A)| \leq \varepsilon\mu(A).$$

Hence $\mu(B) \geq (\lambda - \varepsilon)\mu(A)$ and
$$\mu(Z') \leq \mu(A) + \mu(B) \leq \mu(A) + (\lambda - \varepsilon)\mu(A),$$

so that
$$\mu(B) \geq \frac{\lambda - \varepsilon}{1 + \lambda + \varepsilon}\mu(Z').$$

This inequality contradicts (A.2) if $\varepsilon + \delta(\lambda + 1) < (\lambda - \varepsilon)/(1 + \lambda + \varepsilon)$. $\qquad \square$

Proposition A.2. *Let $\mathcal{R}$ be a non-singular countable measurable equivalence relation on a standard measure space (X, μ). Assume that (Y, ν) is another standard measure space and define an equivalence relation $\mathcal{R} \times \mathrm{id}$ on $X \times Y$ such that $(x_1, y_1) \sim_{\mathcal{R}\times\mathrm{id}} (x_2, y_2)$ if and only if $x_1 \sim_{\mathcal{R}} x_2$ and $y_1 = y_2$. Then*
$$r(\mathcal{R} \times \mathrm{id}, \mu \times \nu) \setminus \{0\} = r(\mathcal{R}, \mu) \setminus \{0\}.$$

Proof. The inclusion $r(\mathcal{R} \times \mathrm{id}, \mu \times \nu) \subset r(\mathcal{R}, \mu)$ is obvious. Take $\lambda \in r(\mathcal{R}, \mu) \setminus \{0\}$ and $\varepsilon > 0$. Since for every measurable set $U \subset X$ of positive measure we have $\lambda \in \bar{r}_{U,\varepsilon}(\mathcal{R}, \mu)$, we get $\lambda \in \bar{r}_{Z,\varepsilon}(\mathcal{R} \times \mathrm{id}, \mu \times \nu)$ for every measurable rectangular set $Z = U \times V$ of positive measure. But then the same is true for every set Z of positive measure in the algebra generated by measurable rectangular sets. Since every measurable set in $X \times Y$ can be approximated by sets in this algebra, by the previous lemma we conclude that $\lambda \in r_{Z,\varepsilon}(\mathcal{R} \times \mathrm{id}, \mu \times \nu)$ for every measurable set Z of positive measure. $\qquad \square$

Another related immediate consequence of Lemma A.1 is the following continuity result.

Proposition A.3. *Let $\mathcal{R}$ be the orbit equivalence relation defined by an action of a countable group Γ on a standard measure space (X, μ) by non-singular transformations. Assume that $\{\xi_n\}_{n=1}^{\infty}$ is an increasing sequence of Γ-invariant measurable partitions such that $\vee_n\xi_n$ is the partition into points. Also assume that the measure μ_n defined by μ on $X_n = X/\xi_n$ is σ-finite and the Radon–Nikodym cocycle c_μ is ξ_n-measurable. Let $\mathcal{R}_n$ be the orbit equivalence relation defined by the action of Γ on X_n. Then*
$$r(\mathcal{R}, \mu) \setminus \{0\} = \bigcap_n r(\mathcal{R}_n, \mu_n) \setminus \{0\}.$$

Now assume that we have an ergodic countable measurable equivalence relation $\mathcal{R}$ on (X, μ), meaning that for every $\mathcal{R}$-invariant measurable set either the set itself or its complement has measure zero. Since $r(\mathcal{R}, \mu) \setminus \{0\}$ is a closed subgroup of $\mathbb{R}_+^*$, we have the following possibilities for the ratio set: $\{1\}$, $\{0, 1\}$, $\{0\} \cup \{\lambda^n \mid n \in \mathbb{Z}\}$ for some $\lambda \in (0, 1)$, and $[0, +\infty)$. In the last three cases the equivalence relation is said to be of type III_0, III_λ, and III_1, respectively.

If $\mathcal{R}$ is ergodic, it is easy to see that for every set $Z \subset X$ of positive measure the ratio set of the equivalence relation on Z induced by $\mathcal{R}$ is the same as the ratio set of $\mathcal{R}$.

Denote by $\mathcal{R}^0$ the kernel of c_μ, that is, the set of points $(x, y) \in \mathcal{R}$ such that $c_\mu(x, y) = 1$. It is a measurable equivalence relation, and the measure μ is $\mathcal{R}^0$-invariant. The next proposition follows from a general result on type III factors, see [20], Corollary 29.12, but it is much more elementary.

Proposition A.4. *Let $\mathcal{R}$ be an ergodic non-singular countable measurable equivalence relation on a standard measure space (X, μ). Assume that $\mathcal{R}^0$ is ergodic. Then $r(\mathcal{R}, \mu)$ coincides with the essential range of c_μ.*

Conversely, assume that $r(\mathcal{R}, \mu)$ coincides with the essential range of c_μ. Also assume that $r(\mathcal{R}, \mu) \setminus \{0\}$ is discrete, so $\mathcal{R}$ is not of type III_1. Then $\mathcal{R}^0$ is ergodic.

Proof. Let us prove the second statement, since it is the only one used in the main text.

Suppose $\mathcal{R}^0$ is not ergodic, so there exists an $\mathcal{R}^0$-invariant measurable subset $A \subset X$ such that both A and $A^c = X \setminus A$ have positive measure. By ergodicity of $\mathcal{R}$, there exist subsets $A_1 \subset A$ and $B_1 \subset A^c$ of positive measure and a measurable bijective map $T \colon A_1 \to B_1$ with graph in $\mathcal{R}$. Since the nonzero essential values of c_μ form a discrete set, by replacing A_1 by a smaller set we may assume that there exists $\lambda > 0$ such that $c_\mu(x, Tx) = \lambda$ for all $x \in A_1$. Since λ^{-1} lies in the ratio set and it is isolated in the set of essential values of c_μ, we can find subsets $B \subset B_1$ and $C \subset B_1$ of positive measure and a measurable bijective map $S \colon B \to C$ with graph in $\mathcal{R}$ such that $c_\mu(x, Sx) = \lambda^{-1}$ for all $x \in B$. Put $A_2 = T^{-1}B$. Then A_2 is a set of positive measure, $A_2 \subset A$, $STA_2 \subset A^c$ and $c_\mu(x, STx) = c_\mu(x, Tx)c_\mu(Tx, STx) = 1$ for all $x \in A_2$. This contradicts the $\mathcal{R}^0$-invariance of A. $\qquad\square$

Now consider a particular class of equivalence relations. Let $\{(X_n, \mu_n)\}_{n=1}^\infty$ be a sequence of at most countable probability spaces. Put $(X, \mu) = \prod_n (X_n, \mu_n)$, and define an equivalence relation $\mathcal{R}$ on X by

$$x \sim y \text{ if } x_n = y_n \text{ for all } n \text{ large enough.}$$

This equivalence relation is non-singular and ergodic.

For a finite subset $I \subset \mathbb{N}$ and $a \in \prod_{n \in I} X_n$ put

$$Z(a) = \{x \in X \mid x_n = a_n \text{ for } n \in I\}.$$

The asymptotic ratio set $r_\infty(\mathcal{R}, \mu)$ consists by definition [1] of all numbers $\lambda \geq 0$ such that for any $\varepsilon > 0$ there exist a sequence $\{I_n\}_{n=1}^\infty$ of mutually disjoint finite subsets of $\mathbb{N}$, disjoint subsets $K_n, L_n \subset \prod_{k \in I_n} X_k$ and bijections $\varphi_n \colon K_n \to L_n$ such that

$$\left| \frac{\mu(Z(\varphi_n(a)))}{\mu(Z(a))} - \lambda \right| < \varepsilon \text{ for all } a \in K_n \text{ and } n \geq 1, \text{ and } \sum_{n=1}^\infty \sum_{a \in K_n} \mu(Z(a)) = \infty.$$

Clearly, the asymptotic ratio set is a closed subset of $[0, +\infty)$.

Proposition A.5 ([15]). *We have $r_\infty(\mathcal{R}, \mu) \setminus \{0\} = r(\mathcal{R}, \mu) \setminus \{0\}$.*

The nontrivial part here is that $1 \in r_\infty(\mathcal{R}, \mu)$. What is not difficult to see, and this is more than enough for our purposes, is that

$$r(\mathcal{R}, \mu) \setminus \{1\} \subset r_\infty(\mathcal{R}, \mu) \quad \text{and} \quad r_\infty(\mathcal{R}, \mu) \setminus \{0\} \subset r(\mathcal{R}, \mu).$$

Let us sketch a proof of these inclusions. First of all note that the Radon–Nikodym cocycle is given by

$$c_\mu(x, y) = \prod_n \frac{\mu_n(y_n)}{\mu_n(x_n)}.$$

Take $\lambda \in r(\mathcal{R}, \mu) \setminus \{1\}$ and $\varepsilon > 0$. We can find measurable sets A and B and a measurable bijective map $T \colon A \to B$ with graph in $\mathcal{R}$ such that $\mu(X \setminus (A \cup B)) = 0$ and $|c_\mu(x, Tx) - \lambda| < \varepsilon$ for all $x \in A$. As we already noted, since $\lambda \neq 1$, we can choose A and B to be disjoint. Then the measure of A is close to $(1 + \lambda)^{-1}$. There exist $m \geq 1$ and disjoint measurable subsets $A_1, \ldots, A_N$ of A such that $\bigcup_i A_i$ is close to A and the restriction of T to A_i is such that Tx is obtained from $x \in A_i$ by applying a transformation φ_{1i} to the first m coordinates of x. Approximate the sets A_i by unions of cylindrical sets $Z(a)$, $a \in K_1 \subset \prod_{n \leq n_1} X_n$, where $n_1 \geq m$. We may assume that if $Z(a)$ is used in the approximation of A_i, then the ratio $\mu(Z(a) \cap A_i)/\mu(Z(a))$ is close to 1. In particular, for every $a \in K_1$ the cylindrical set $Z(a)$ is used in approximating only one of the sets A_i. Then the transformations φ_{1i} define an injective map $\varphi_1 \colon K_1 \to \prod_{n \leq n_1} X_n$ such that $K_1 \cap \varphi_1(K_1) = \emptyset$. By construction $\sum_{a \in K_1} \mu(Z(a))$ is close to $(1 + \lambda)^{-1}$.

Next replace X by $\prod_{n > n_1} X_n$ and repeat the same argument. Note that the ratio set for the new product-space remains the same, since $\mathcal{R}$ is ergodic and $\prod_{n > n_1} X_n$ can be identified with the cylindrical subset $Z(a)$ of X for some $a \in \prod_{n \leq n_1} X_n$. Continuing this process we gradually construct the required sets K_n and maps φ_n and conclude that $\lambda \in r_\infty(\mathcal{R}, \mu)$.

Now take $\lambda \in r_\infty(\mathcal{R}, \mu) \setminus \{0\}$. Then for every $\varepsilon > 0$ we can find measurable sets A_n and B_n and measurable bijective maps $T_n \colon A_n \to B_n$ with graphs in $\mathcal{R}$ such that $|c_\mu(x, T_n x) - \lambda| < \varepsilon$ for all $x \in A_n$, $A_n \cap B_n = \emptyset$, T_n maps $A_n \cap A_m$ onto

$B_n \cap A_m$ and $A_n \cap B_m$ onto $B_n \cap B_m$, the sets $A_n \cup B_n$ are mutually independent and $\sum_n \mu(A_n) = \infty$. Consider the sets

$$A = \bigcup_n \left(A_n \setminus \bigcup_{m<n}(A_m \cup B_m) \right) \quad \text{and} \quad B = \bigcup_n \left(B_n \setminus \bigcup_{m<n}(A_m \cup B_m) \right).$$

Then the maps T_n define a measurable bijective map $T\colon A \to B$, $A \cap B = \emptyset$ and $\mu(A \cup B) = 1$. Therefore $\lambda \in \bar{r}_{X,\varepsilon}(\mathcal{R}, \mu)$. Since the same is true if we replace X by $\prod_{n \geq m} X_n$, we conclude that $\lambda \in \bar{r}_{Z,\varepsilon}(\mathcal{R}, \mu)$ for any Z which is a finite union of cylindrical sets $Z(a)$. By Lemma A.1 it follows that $\lambda \in r_{Z,\varepsilon}(\mathcal{R}, \mu)$ for any measurable set Z of positive measure. Hence $\lambda \in r(\mathcal{R}, \mu)$.

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