J. Noncommut. Geom. 8 (2014), 789–836
DOI 10.4171/JNCG/171

Journal of Noncommutative Geometry

Leafwise homotopies and Hilbert–Poincaré complexes I. Regular HP complexes and leafwise pull-back maps

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Abstract. In the first part of our series of papers, we prove the leafwise homotopy invariance of K-theoretic signatures of foliations and laminations, using the formalism of Hilbert–Poincaré complexes as revisited by Higson and Roe. We use a generalization of their methods to give a homotopy equivalence of de Rham–Hilbert–Poincaré complexes associated with leafwise homotopy equivalence of foliations. In particular, we obtain an explicit path connecting the signature classes in K-theory, up to isomorphism induced by Morita equivalence. Applications of this path on the stability properties of rho-invariants à la Keswani will be carried out in the later parts of this series.

Mathematics Subject Classification (2010). 53C12, 55P10; 58B34, 46L80, 46L08.

Keywords. Foliations, laminations, leafwise homotopy equivalence, K-theoretic signatures, Morita equivalence, K-theory of C*-algebras, maximal Baum–Connes conjecture, imprimitivity bimodules, rho-invariants.

Contents

1. Introduction

This paper is the first of a series of three papers which study some secondary homotopy invariants for laminations. More precisely, we build up a suitable framework for the study of leafwise signature invariants which allows to deduce important consequences for the leafwise homotopy classification of laminations. In the third paper of this series, and under a usual Baum–Connes assumption, the second author deduces for instance that the type II Baum–Connes rho invariant associated with the leafwise signature operator on an odd dimensional lamination, which was introduced in [BePi:09], is a leafwise oriented homotopy invariant. This result generalizes the case of closed odd dimensional manifolds [Ke:99], [We:88], [ChWe:03], [Ch:04], [Ma:92], [PiSch1:07], [HiRo:10], [BR:14] and also the partial results for foliated topological bundles obtained in [BePi:09], and we believe that the techniques involved are deep enough to enjoy their own interest.

In their work on mapping surgery to analysis, Nigel Higson and John Roe have systematically studied the so-called Hilbert–Poincaré (abbreviated HP) complexes [MiFo:80] and deduced interesting topological consequences (see [HiRoI:05], [HiRoII:05], [HiRoIII:05]). They defined an HP complex as a complex of finitely generated projective Hilbert C^*-modules on a C^*-algebra A with adjointable differentials, and an additional structure of adjointable Poincaré duality operators that induce isomorphism on cohomology from the original complex to its dual complex. Associated with an HP complex there is a canonically defined class in $K_1(A)$, called the *signature* of the HP complex. It is shown in [HiRoI:05] that a homotopy equivalence of such complexes leaves the signature class invariant and yields an explicit homotopy, a path between the corresponding representatives of the signature classes. This explicit path was used in [Ke:00] to construct a controlled path of operators joining the signature operators on homotopy equivalent manifolds. When trying to extend the results of Keswani to general laminations, and in fact already to general smooth foliations, we faced the following difficulties:

(1) It is necessary to work with complete transversals and with the (maximal) C^*-algebras associated with the monodromy groupoids associated with the transversals. Therefore the HP complexes that naturally arose were not defined over isomorphic C^*-algebras but only Morita equivalent ones.

(2) For Galois coverings which correspond to a lamination with one leaf, the Whitney isomorphism allows to reduce the study of the de Rham complex to a finitely generated projective HP complex as introduced in [HiRoI:05]. For general foliations on closed manifolds, this reduction becomes highly involved, see [HeLa:91] for appropriate techniques to define the Whitney isomorphism in this context. It is thus natural to extend the Higson–Roe formalism to countably generated HP complex with regular operators.

(3) Since an oriented leafwise homotopy equivalence induces a Morita equivalence of the C^*-algebras, the explicit path joining the leafwise signature operators on

the equivalent laminations needs to be rethought up to an explicit imprimitivity bimodule.

(4) The construction of the Keswani loop of unitaries is done by concatenating three paths. The first one is the "spectral flow" path which can be defined easily and whose Fuglede–Kadison log-determinant has to be related with the measured Baum–Connes rho invariant [BePi:09]. The second path uses the above mentioned path of operators associated with the leafwise homotopy and is called the Large Time Path (LTP) while a third path called the Small Time Path (STP) needs a suitable description of the Baum–Connes map for laminations. This latter use of the Baum–Connes map turns out to be the hardest part of this work.

All these problems are solved in this paper, the second paper [RoII] by the second author and the third paper in preparation also by the second author [RoIII]. In this first paper, we begin by extending the results of [HiRoI:05] to deal with regular (unbounded) operators and more importantly to take into account Morita equivalence of underlying C^*-algebras. Given an oriented leafwise map, satisfying some natural assumption fulfilled by leafwise homotopies, between leafwise oriented laminations on compact spaces in the sense of [MoSc:06], we construct a pull-back morphism between the leafwise de Rham HP complexes and prove the expected (up to Morita equivalence) functoriality. When two leafwise oriented laminations are leafwise homotopy equivalent, our construction allows to deduce an explicit path joining the leafwise signature operators and hence the LTP path for laminations. So this first paper does not use any measure theory and is only concerned with the C^*-algebraic constructions associated with leafwise homotopies. In the second paper [RoII] of this series, holonomy invariant transverse measures are introduced and a semi-finite von Neumann algebra associated with the leafwise homotopy equivalence between the laminations is constructed. Moreover, the second author shows there that the measured laminated Baum–Connes rho invariant [BePi:09] associated with the leafwise signature operator is a foliated diffeomorphism invariant which does not depend on the leafwise metric used to define it, extending the result of Baum–Connes [ChGr:85]. Moreover, using the above mentioned von Neumann algebra with its trace, he proves that the measured Fuglede–Kadison determinant of the LTP cancels out in the large time limit. The third paper [RoIII] exploits a nice description of the Baum–Connes map to construct the STP which, when concatenated with the two previous paths, yields the allowed loop of unitaries. The Fuglede–Kadison determinant of the STP defined using again the above von Neumann algebra and its trace, is also proved to cancel out in the small time limit. So combining the results of this series, we obtain the leafwise homotopy invariance of the measured rho invariant of the lamination.

Let us now briefly explain the main results of this first paper.

With a lamination $(M, \mathcal{F})$ on a compact space M and a chosen complete transversal X in the sense of [MoSc:06], we associate the HP complex

$$\mathcal{E}_{X,0} \xrightarrow{d_X} \mathcal{E}_{X,1} \xrightarrow{d_X} \cdots \xrightarrow{d_X} \mathcal{E}_{X,p},$$

where $\mathcal{E}_{X,k}$, $0 \le k \le p$, is the completion of $C_c^{\infty,0}(\mathcal{G}_X, r^*(\bigwedge^k T_{\mathbb{C}}^*\mathcal{F}))$ with respect to a $C^*(\mathcal{G}_X^X)$-valued inner product (see Section 2 for the notations). The Poincaré duality operator, denoted by T_X, is induced on $\mathcal{E}_X^k$ by the lift of the leafwise Hodge $*$-operator on $\mathcal{G}_X$, and d_X is the regular operator induced by the lift of the leafwise de Rham differential to $\mathcal{G}_X$. Now consider a leafwise homotopy equivalence between two laminations $f : (M, \mathcal{F}) \to (M', \mathcal{F}')$ and let X' be similarly a complete transversal on $(M', \mathcal{F}')$. Then we use the explicit imprimitivity bimodule which implements the Morita equivalence to reduce to leafwise de Rham HP complexes over the same C*-algebra $C^*(\mathcal{G}_{X'}^{X'})$. Consequently, we define out of a leafwise homotopy equivalence a homotopy equivalence between the leafwise de Rham HP complexes of the two laminations. As a consequence, we deduce the well-known equality of the K_1 signature classes of the two foliations, and more importantly an explicit path whose determinant will play a fundamental part in [RoII] as explained above.

The contents of the present paper are as follows. Section 2 is devoted to the generalization of the definitions and properties of HP complexes to the setting of regular (unbounded) operators. In Section 3, we construct the pull-back chain map, associated with an oriented leafwise map satisfying a convenient condition fulfilled by leafwise homotopies, between the corresponding leafwise de Rham HP complexes, and prove its functoriality. Section 4 deals with oriented leafwise homotopy equivalences and proves that such an equivalence induces a homotopy between the HP complexes and an explicit path between the leafwise signature operators.

The main difficulties appear in the transition from smooth manifolds to smooth foliations. The extension of our results from smooth foliations to leafwise smooth laminations, although highly important for applications, is almost cosmetic. Therefore we have eventually decided to avoid unnecessary heavy notations which would mislead the reader and we have restricted to smooth foliations.

Acknowledgements. This work arose from discussions with P. Piazza and we would like to thank him for his helpful advices during the preparation of this work. We are also indebted to B. Bekka, A. Gorokhovsky, J. Heitsch, N. Higson, J. Hilgert, V. Mathai, J. Roe and T. Schick for several helpful discussions. Part of this work was done while the second author was visiting the department of mathematics of the university La Sapienza in Rome, and while the first author was visiting the IHP in Paris. Both authors are grateful to the two institutions for their support and for the warm hospitality.

2. Review of Hilbert–Poincaré complexes

We review in this section some basic properties of a so-called Hilbert–Poincaré complex and collect some results that will be used in the sequel. We mainly extend some results proved in [HiRoI:05] to encompass closed operators. More precisely, we put forward the necessary information required to define the signature of an HP complex, the notion of homotopy equivalence of two HP complexes and a crucial theorem, originally due to Higson and Roe and adapted here to our context, which states that homotopy equivalent HP complexes have the same signature. We refer to [La:95], [Ku:97], [Pa:99] for the detailed properties of Hilbert modules and regular operators that will be used here.

For a C*-algebra A and Hilbert right A-modules $\mathcal{E}$ and $\mathcal{E}'$, we denote by $\mathcal{L}_A(\mathcal{E}, \mathcal{E}')$ the set of adjointable operators from $\mathcal{E}$ to $\mathcal{E}'$, i.e., of maps $T: \mathcal{E} \to \mathcal{E}$ such that there exists a map $S: \mathcal{E} \to \mathcal{E}$ with the property

$$\langle T(v), v' \rangle = \langle v, S(v') \rangle \in A \quad \text{for } v \in \mathcal{E} \text{ and } v' \in \mathcal{E}'.$$

It is then easy to see that such S is unique, A-linear and bounded. It will be called the *adjoint* of T and is denoted by T^* as usual. A trivial consequence of the definition of an adjointable operator is that it is automatically A-linear and bounded. Moreover, $\mathcal{L}_A(\mathcal{E}, \mathcal{E}')$ is naturally endowed with the structure of a Banach space. When $\mathcal{E}' = \mathcal{E}$, we simply denote $\mathcal{L}_A(\mathcal{E}, \mathcal{E}')$ by $\mathcal{L}_A(\mathcal{E})$, which is then a unital C*-algebra.

An example of adjointable operator is given by finite rank operators and A-compact operators which we recall now for convenience. A rank one operator $\theta_{v,v'}$ between $\mathcal{E}$ and $\mathcal{E}'$ for nonzero elements $v \in \mathcal{E}$ and $v' \in \mathcal{E}'$ is defined by the formula

$$\theta_{v,v'}(z) := v\langle v', z \rangle \quad \text{for } z \in \mathcal{E}.$$

Then $\theta_{v,v'} \in \mathcal{L}_A(\mathcal{E}, \mathcal{E}')$, and finite rank operators will be finite sums of such rank one operators. The closure in $\mathcal{L}_A(\mathcal{E}, \mathcal{E}')$ of the subspace of finite rank operators is the space of A-compact operators.

Finally, recall that an operator t from a Hilbert A-module E to a Hilbert A-module F is called regular if t is closed and densely defined with densely defined adjoint t^* such that the operator $1 + t^*t$ has dense image in F. See [Pa:99] for more details.

2.1. Regular HP complexes. Unless otherwise specified, our C*-algebras will always be unital.

Definition 2.1. An n-dimensional *Hilbert–Poincaré complex* (abbreviated HP complex) over a C*-algebra A is a complex $(\mathcal{E}, b, T)$ of countably generated Hilbert right A-modules

$$\mathcal{E}_0 \xrightarrow{b_0} \mathcal{E}_1 \xrightarrow{b_1} \cdots \xrightarrow{b_n} \mathcal{E}_n,$$

where each b_i is a densely defined closed regular operator with a densely defined regular adjoint $b_\bullet^*: \mathcal{E}_{\bullet+1} \to \mathcal{E}_\bullet$ such that successive operators in the complex are composable (i.e., the image of one is contained in the domain of the other) and $b_{i+1} \circ b_i = 0$, together with adjointable operators $T\colon \mathcal{E}_\bullet \to \mathcal{E}_{n-\bullet}$ satisfying the following properties:

(1) For $v \in \mathcal{E}_p$, $T^* v = (-1)^{(n-p)p} T v$.

(2) T maps $\mathrm{Dom}(b^*)$ to $\mathrm{Dom}(b)$, and we have

$$T b^*_{p-1} v + (-1)^{p+1} b_{n-p} T v = 0$$

for $v \in \mathrm{Dom}(b^*) \subset \mathcal{E}_p$.

(3) T induces an isomorphism between the cohomology of the dual complex $(\mathcal{E}, b^*)$,

$$\mathcal{E}_n \xrightarrow{b^*_{n-1}} \mathcal{E}_{n-1} \xrightarrow{b^*_{n-2}} \cdots \xrightarrow{b^*_0} \mathcal{E}_0$$

and that of the complex $(\mathcal{E}, b)$, i.e., the induced map $T_*\colon H^k(\mathcal{E}, b^*) \to H^k(\mathcal{E}, b)$ is an isomorphism.

(4) The closed operator $B := b + b^*\colon \mathcal{E} \to \mathcal{E}$ satisfies $(B \pm i)^{-1} \in \mathcal{K}_A(\mathcal{E})$.

We have denoted by $\mathcal{E}$ the direct sum $\bigoplus_{0 \le i \le n} \mathcal{E}_i$, by $b = \bigoplus_{0 \le i \le n} b_i$ and similarly for b^*. The fourth item implies that B is a regular Fredholm operator, i.e., it has an inverse modulo compact operators in the direct sum Hilbert module.

Remark 2.2. Notice that if the *duality operator* T is bijective, then we have from condition (2): $b^* T^{-1} \pm T^{-1} b = 0$, therefore T^{-1} induces a map $T_*^{-1}\colon H^k(\mathcal{E}, b) \to H^k(\mathcal{E}, b^*)$, which can be shown to be the inverse of the map T_*. Hence condition (2) implies condition (3) of Definition 2.1. In particular, a duality operator T satisfying $T^2 = \pm 1$ (which will be the case in our geometric situation) fulfills this condition. We thank the referee for pointing this out to us.

It is worth pointing out for further use that a regular operator t is Fredholm if and only if it has a pseudo-left inverse and a pseudo-right inverse. A pseudo-left inverse for t is an operator $G \in \mathcal{L}_A(\mathcal{E})$ such that Gt is closable, $\overline{Gt} \in \mathcal{L}_A(\mathcal{E})$, and $\overline{Gt} = 1 \bmod \mathcal{K}_A(\mathcal{E})$. Similarly a pseudo-right inverse for t is an operator $G' \in \mathcal{L}_A(\mathcal{E})$ such that tG' is closable, $\overline{tG'} \in \mathcal{L}_A(\mathcal{E})$, and $\overline{tG'} = 1 \bmod \mathcal{K}_A(\mathcal{E})$. The cohomology of the complex $(\mathcal{E}, b)$ is defined here to be the unreduced one given by

$$H^k(\mathcal{E}, b) := \frac{\mathrm{Ker}(b_k)}{\mathrm{Im}(b_{k-1})}.$$

Remark 2.3. The complex $(\mathcal{E}, b, T)$ given in the definition is understood as a two-sided infinite complex with finitely many non-zero entries.

Definition 2.4 ([HiRoII:05], Definition 5.4). Let $\dim \mathcal{E} = n = 2l + 1$ be odd. On $\mathcal{E}_p$ define

$$S = i^{p(p-1)+l} T \quad \text{and} \quad D = iBS.$$

Then we call D the *signature operator* of the HP complex $(\mathcal{E}, b, T)$.

We have the following proposition from [HiRoI:05], which is valid in our setting.

Proposition 2.5 ([HiRoI:05], Lemma 3.4). *With the above notations, we have* $S^* = S$ *and* $bS + Sb^* = 0$.

Recall that a regular operator t is adjointably invertible if there exists an adjointable operator s such that $st \subseteq ts = 1$. Notice that when t is self-adjoint, this is equivalent to the surjectivity of t, see [Ku:97]. We now extend Proposition 2.1 in the first paper of [HiRoI:05] to our setting. By definition, an HP complex is acyclic if its cohomology groups are all zero.

Proposition 2.6. *An HP complex is acyclic if and only if the operator* B *is adjointably invertible. Moreover, in this case* $B^{-1} \in \mathcal{K}_{\mathcal{A}}(\mathcal{E})$.

Proof. We adapt the proof given in [HiRoI:05]. Assume that B is invertible. Then for $v \in \mathrm{Ker}(b)$, there exists $w \in \mathrm{Dom}(B)$ such that $v = Bw$ and

$$\|b^*w\|^2 = \|\langle b^*w, b^*w\rangle\| = \|\langle w, bb^*w\rangle\| = \|\langle w, bBw\rangle\| = 0,$$

hence $v = Bw = bw \in \mathrm{Im}(b)$. Therefore $\mathrm{Ker}(b) = \mathrm{Im}(b)$ and thus the complex is acyclic.

Conversely, let the HP complex be acyclic. To prove that B is adjointably invertible, it suffices to show that B is surjective. Since all the cohomologies are trivial, $\mathrm{Im}(b) = \mathrm{Ker}(b)$, so the range of b is closed. Since the differentials b_k, $k = 0, 1 \ldots, n$, are regular operators, $Q(b) := b(1 + b^*b)^{-1/2}$ is a bounded adjointable operator and we have by a classical argument (see also [Sk:96], Remarques (a), p. 74),

$$\mathrm{Im}(b) = \mathrm{Im}(Q(b)), \;\; \mathrm{Ker}(b) = \mathrm{Ker}(Q(b)).$$

Then by the Open Mapping Theorem, $Q(b)Q(b^*)$ is bounded below on $\mathrm{Im}(Q(b))$ and therefore $\mathrm{Im}(Q(b)) \subseteq \mathrm{Im}(Q(b)Q(b^*))$, see Theorem 3.2 in [La:95] . Similarly, $\mathrm{Im}(Q(b^*)) \subseteq \mathrm{Im}(Q(b^*)Q(b))$.

Now, as $\mathrm{Im}(Q(b))$ is closed, $\mathrm{Ker}(Q(b))$ is an orthocomplemented submodule with $\mathrm{Ker}(Q(b))^{\perp} = \mathrm{Im}(Q(b)^*) = \mathrm{Im}(Q(b^*))$. Hence we have $\mathcal{E} = \mathrm{Im}(Q(b)) \oplus \mathrm{Im}(Q(b^*))$. So for any $v \in \mathcal{E}$, we have $v = Q(b)v_1 + Q(b^*)v_2$ for some $v_1, v_2 \in \mathcal{E}$. However $Q(b)v_1 = Q(b)Q(b^*)w_1$, and $Q(b^*)v_2 = Q(b^*)Q(b)w_2$ for some $w_1, w_2 \in \mathcal{E}$. Hence we have

$$v = Q(b)Q(b^*)w_1 + Q(b^*)Q(b)w_2$$

for any $v \in \mathcal{E}$. But we have by Lemma 2.7 below

$$Q(b)^2 = Q(b^*)^2 = 0 \text{ and } Q(b + b^*) = Q(b) + Q(b^*).$$

So we have

$$v = (Q(b) + Q(b^*))(Q(b^*)w_1 + Q(b)w_2),$$

which shows that $Q(b) + Q(b^*)$ is surjective and hence so is $Q(b + b^*)$. However, $\mathrm{Im}(Q(b + b^*)) = \mathrm{Im}(B)$ and hence B is surjective and thus invertible. □

Lemma 2.7. *We have*

$$Q(b)^2 = Q(b^*)^2 = 0 \text{ and } Q(b + b^*) = Q(b) + Q(b^*).$$

Proof. As in the proof of Proposition 2.6, we have

$$\mathrm{Ker}(Q(b)) = \mathrm{Ker}(b) \quad \text{and} \quad \mathrm{Im}(Q(b)) = \mathrm{Im}(b).$$

This shows that $Q(b)^2 = 0$. The same argument applied to b^* gives $Q(b^*)^2 = 0$.

Let now $f = Q(b)$. Then we have $b = f(1 - f^*f)^{-1/2}$, $(1 + b^*b)^{-1/2} = (1-f^*f)^{1/2}$, and since $fp(f^*f) = p(ff^*)f$ for any polynomial p, by continuity it also holds for any $p \in C([0, 1])$. So in particular we have

$$f(1 - f^*f)^{1/2} = (1 - ff^*)^{1/2}f.$$

Notice now that the operators $(1+b^*b)(1+bb^*)$, $(1+bb^*)(1+b^*b)$ and $1+(b+b^*)^2$, defined on $\mathrm{Dom}(b^*b) \cap \mathrm{Dom}(bb^*)$, coincide. Therefore, setting $A = (1 + b^*b)^{-1/2}$ and $B = (1 + bb^*)^{-1/2}$ and using the spectral theorem, it is easy to see that we have the following equality of bounded operators:

$$AB = BA = [(1 + b^*b)(1 + bb^*)]^{-1/2}.$$

Hence

$$\begin{aligned} Q(b + b^*) &= (b + b^*)[1 + (b + b^*)^2]^{-1/2} \\ &= (b + b^*)[(1 + b^*b)(1 + bb^*)]^{-1/2} \\ &= (b + b^*)AB \\ &= bAB + b^*BA \\ &= fB + f^*A. \end{aligned}$$

We note that for any polynomial p with $p(0) = 1$, we have $fp(ff^*) = f$ since $f^2 = 0$. So the equality also holds by continuity for any $p \in C([0, 1])$ for which $p(0) = 1$. In particular we note that

$$fB = f(1 - ff^*)^{1/2} = f.$$

Similarly $f^*A = f^*$ and we finally get

$$Q(b + b^*) = fB + f^*A = f + f^* = Q(b) + Q(b)^*.$$

The above computation is justified by the facts that

$$\mathrm{Im}(b) \subseteq \mathrm{Dom}(b) \quad \text{and} \quad \mathrm{Im}(1 + b^*b)^{-1/2} \subseteq \mathrm{Dom}(b).$$ □

Proposition 2.8 ([HiRoI:05], Lemma 3.5). *The self-adjoint operators* $B \pm S : \mathcal{E} \to \mathcal{E}$ *are invertible.*

Proof. Consider the mapping cone complex of the chain map $S : (\mathcal{E}, b) \to (\mathcal{E}, -b^*)$ where we have changed b^* to $-b^*$. Its differential is

$$d_S = \begin{pmatrix} b & 0 \\ S & b^* \end{pmatrix}.$$

Since S is an isomorphism on cohomology, its mapping cone complex is acyclic, i.e., all the cohomology groups are zero. Therefore the operator $B_S = d_S + d_S^*$ is invertible on $\mathcal{E} \oplus \mathcal{E}$. Now $B_S = \left(\begin{smallmatrix} B & S \\ S & B \end{smallmatrix}\right)$ preserves the $+1$ eigenspace of the involution which interchanges the copies of $\mathcal{E}$ and identifies on this subspace with $B + S$ on $\mathcal{E}$. Hence $B + S$ is invertible. Considering the -1 eigenspace similarly, we deduce that and $B - S$ is also invertible. □

Definition 2.9. Let $(\mathcal{E}, b, T)$ be an odd-dimensional Hilbert–Poincaré complex and S be the operator associated to T as in Definition 2.4. Then the *signature* of $(\mathcal{E}, b, T)$ is defined as the class of the self-adjoint invertible operator $(B + S)(B - S)^{-1} \in K_1(\mathcal{K}_A(\mathcal{E}))$. We denote this class by $\sigma(\mathcal{E}, b, T)$.

Now let A, B be σ-unital C^*-algebras which are Morita-equivalent, with Morita bimodule ${}_A\mathcal{E}_B$. So we have $A \cong \mathcal{K}_B({}_A\mathcal{E}_B)$, and so there is a $*$-homomorphism $\phi: A \to \mathcal{L}({}_A\mathcal{E}_B)$. Let now $(\mathcal{E}, b, T)$ be a Hilbert–Poincaré complex of countably generated A-modules. For the de Rham complexes which admit the Hodge $*$-operator, the operator S is an involution. In this case, it is easy to relate $\sigma(\mathcal{E}, b, T)$ with the usual definition of the signature for the operator D, i.e.,

$$\begin{aligned} (D + iI)(D - iI)^{-1} &= (iBS + iI)(iBS - iI)^{-1} \\ &= (B + S)SS^{-1}(B - S)^{-1} \\ &= (B + S)(B - S)^{-1}. \end{aligned}$$

We assume for the rest of this section that $S^2 = 1$. We then form a Hilbert–Poincaré complex $(\mathcal{E} \otimes {}_A\mathcal{E}_B, b \otimes I)$:

$$\mathcal{E}_0 \otimes {}_A\mathcal{E}_B \xrightarrow{b\otimes I} \mathcal{E}_1 \otimes {}_A\mathcal{E}_B \xrightarrow{b\otimes I} \mathcal{E}_2 \otimes {}_A\mathcal{E}_B \ldots \xrightarrow{b\otimes I} \mathcal{E}_n \otimes {}_A\mathcal{E}_B.$$

Let $\mathcal{M}: K_1(A) \to K_1(B)$ be the isomorphism induced by the Morita equivalence between A and B.

Proposition 2.10. *With the above assumptions, it follows that* $\mathcal{M}[\sigma(\mathcal{E}, b, T)] = \sigma(\mathcal{E} \otimes {}_A\mathcal{E}_B, b \otimes I, T \otimes I)$.

Proof. Let $U(D) = (D + iI)(D - iI)^{-1}$ and $\mathcal{E}_A := \bigoplus_p \mathcal{E}_p$. Then $U(D) = (B+S)(B-S)^{-1}$. The class of $U(D)$ in $K_1(A)$ can be identified with the class of the KK-cycle in $\mathrm{KK}(\mathbb{C}, A)$ given by $(\mathcal{E}_A, \lambda, U(D))$, where λ is the scalar multiplication by complex numbers on the left. Then, $\mathcal{M}[\sigma(\mathcal{E}, b, T)]$ can be identified with an element of $\mathrm{KK}(\mathbb{C}, B)$ which will be given by the Kasparov product of $(\mathcal{E}_A, \lambda, U(D))$ with the Morita KK-cycle $({}_A\mathcal{E}_B, \phi, 0)$.

But this Kasparov product is given by the KK-cycle $(\mathcal{E}_A \otimes_A \mathcal{E}_B, \lambda \otimes I, U(D) \otimes I)$ [CoSk:84], [Ka:75]. Since D is a self-adjoint regular operator, by the uniqueness of the functional calculus we have $U(D) \otimes I = U(D \otimes I)$. But then we can identify the class of $U(D \otimes I)$ in $K_1(B)$ with the cycle $(\mathcal{E}_A \otimes {}_A\mathcal{E}_B, \lambda \otimes I, U(D) \otimes I)$ in $\mathrm{KK}(\mathbb{C}, B)$. This finishes the proof. □

2.2. Homotopy of HP complexes. We can now define the notion of homotopy equivalence of HP complexes

Definition 2.11. A *homotopy equivalence* between two HP complexes $(\mathcal{E}, b, T)$ and $(\mathcal{E}', b', T')$ is a chain map $A\colon (\mathcal{E}, b) \to (\mathcal{E}', b')$ which induces an isomorphism on cohomology and for which the maps ATA^* and T' between the complex $(\mathcal{E}', b')$ and its dual $(\mathcal{E}', b'^*)$ induce the same map on cohomology.

Definition 2.12. Let $(\mathcal{E}, b)$ be a complex of Hilbert-modules. An *operator homotopy* between Hilbert–Poincaré complexes $(\mathcal{E}, b, T_1)$ and $(\mathcal{E}, b, T_2)$ is a norm-continuous family of adjointable operators $T_s, s \in [0, 1]$ such that each $(\mathcal{E}, b, T_s)$ is a Hilbert–Poincaré complex.

Theorem 2.13 ([HiRoI:05], Theorem 4.3). *If two odd-dimensional HP complexes $(\mathcal{E}, b, T)$ and $(\mathcal{E}', b', T')$ are homotopy equivalent, then their signatures are equal in $K_1(\mathcal{K}_A(\mathcal{E}))$.*

Proof. The proof given in [HiRoI:05] works word by word in this case. Namely, it is shown that the signature of the complex $(\mathcal{E} \oplus \mathcal{E}', b \oplus b', T \oplus -T')$ is zero. This is achieved by using the chain map A in the definition of homotopy equivalence to construct an explicit path that connects the operator $T \oplus -T'$ to an operator which is in turn operator homotopic to its additive inverse. More precisely, the operator path is given very briefly as follows:

- First, the operator path

$$\begin{pmatrix} T & 0 \\ 0 & (s-1)T' - sATA^* \end{pmatrix}, \quad 0 \le s \le 1,$$

  connects the duality operators for the direct sum HP complex $T \oplus -T'$ to $T \oplus -ATA^*$.

- Next, the operator $T \oplus -ATA^*$ is connected to $\begin{pmatrix} 0 & TA^* \\ AT & 0 \end{pmatrix}$ via the path

$$\begin{pmatrix} \cos(s)T & \sin(s)TA^* \\ \sin(s)AT & -\cos(s)ATA^* \end{pmatrix}, \quad 0 \leq s \leq \frac{\pi}{2}.$$

- Finally, the operator $\begin{pmatrix} ccc0 & TA^* \\ AT & 0 \end{pmatrix}$ is connected to its additive inverse using the path

$$\begin{pmatrix} 0 & \exp(is)TA^* \\ \exp(-is)AT & 0 \end{pmatrix}, \quad 0 \leq s \leq \pi.$$

Thus, using Lemma 2.14 proved below, Remark 2.15, and the fact that $\sigma(\mathcal{E} \oplus \mathcal{E}', b \oplus b', T \oplus -T') = \sigma(\mathcal{E}, b, T) - \sigma(\mathcal{E}', b', T')$, the proof is complete. □

So we need to prove the following

Lemma 2.14 ([HiRoI:05], Lemma 4.5). *Operator homotopic HP complexes have the same signature.*

Proof. We adapt the proof of [HiRoI:05] to our setting.

Let $(\mathcal{E}, b)$ be a complex of Hilbert-modules and T_s, $s \in [0,1]$ be a norm-continuous family of duality operators acting on $(\mathcal{E}, b)$ and S_s be the self-adjoint operators defined from T_s as in definition of the operator S. First we note from Result 5.22 in [Ku:97] that for a regular operator t the map $\mathbb{C} \supseteq \rho(t) \ni \lambda \mapsto (t-\lambda)^{-1}$ is continuous. Here $\rho(t)$ denotes the resolvent set of t, $\rho(t) := \{c \in \mathbb{C} \mid t - c1 \text{ is adjointably invertible}\}$. Since $(B + S)$ is an invertible self-adjoint regular operator, the path of operators $(B + S + i\mu)^{-1}$ is norm-continuous for $\mu \in [0,1]$. Now for a fixed $\mu \in \mathbb{R}$ and any $s_1, s_2 \in \mathbb{R}$, the resolvent identity holds:

$$(B+S_{s_1}+i\mu)^{-1}-(B+S_{s_2}+i\mu)^{-1} = (B+S_{s_1}+i\mu)^{-1}(S_{s_2}-S_{s_1})(B+S_{s_2}+i\mu)^{-1}.$$

One can use techniques in Theorem VI.5 of [ReSiIV:80] to show that the above identity implies that $(B + S_s + i\mu)^{-1}$ is norm-continuous in $s \in [0,1]$.

Now consider the norm-continuous path $(B+S_0+is)(B+S_0+is)^{-1}$ that can be concatenated with the norm-continuous path $(B+S_s+i)(B+S_s+i)^{-1}$ and then again concatenated with the norm-continuous path $(B+S_1+(1-s)i)(B+S_1+(1-s)i)^{-1}$. This yields a well-defined norm-continuous path joining $(B + S_0)(B - S_0)^{-1}$ and $(B + S_1)(B - S_1)^{-1}$. Therefore, these operators lie in the same K_1-class. We thank the referee for suggesting this concatenation of paths in place of our first more complicated proof. □

Remark 2.15. From the proof of the previous lemma, it is easy to check that if the duality operator T is operator homotopic to $-T$, then the signature of the HP complex is zero. See [[HiRoI:05], Lemma 4.6] for more details.

2.3. The leafwise de Rham HP complex. We are mainly interested in HP complexes arising from the study of homotopy invariants constructed out of the signature operator on smooth foliations [Ch:04], [HiSk:83], [KaMi:85], [ChWe:03], [We:88], [Ma:92], [PiSch1:07], [HeLa:91], and we proceed now to explain this "paradigm example". Let $(V, \mathcal{F})$ be an oriented smooth foliation on a closed Riemannian manifold (V, g). The leafwise tangent space $T\mathcal{F}$ is then endowed with a Euclidean structure which allows to induce the complex Grassmann bundles $\bigwedge^i T^*_{\mathbb{C}}\mathcal{F}$ with Hermitian structures. Assume that the dimension of V is n and that the dimension of the leaves is p and set $q = n - p$ for the codimension of the foliation. We restrict to odd dimensional foliations as this is not as well understood as is the even dimensional situation, see for instance [BH:04], [BH:11], [LaMi:89], [MiFo:80], [Ne:79] where higher signatures play a fundamental part in the even case. Denote by $\mathcal{G}$ the monodromy (we could as well use holonomy) groupoid of the foliation and let $\lambda = (\lambda_x)_{x\in V}$ be a right-invariant smooth Haar system on $\mathcal{G}$. The space $\mathcal{G}^{(1)}$ of arrows is the space of homotopy classes of paths drawn in the leaves of $(V, \mathcal{F})$, and we make as usual the convenient confusion between $\mathcal{G}^{(1)}$ and $\mathcal{G}$. So two paths whose ranges are contained in a given leaf L define the same class in $\mathcal{G}$ if they start and end at the same points and if they are homotopic through paths drawn in the same leaf L and with fixed end points. Notice that concatenation of paths endows $\mathcal{G}$ with the structure of a smooth groupoid, which is also a foliated manifold. We denote as usual by $s\colon \mathcal{G} \to V$ and $r\colon \mathcal{G} \to V$ the source and range maps and we use the following standard notation. For subsets X, Y of the manifold V, we set

$$\mathcal{G}_X := s^{-1}(X), \quad \mathcal{G}^Y := r^{-1}(Y), \quad \mathcal{G}_X^Y := r^{-1}(Y) \cap s^{-1}(X).$$

Notice that when $Y = X$, the subspace $\mathcal{G}_X^X$ is a subgroupoid of $\mathcal{G}$. Let X be a complete smooth transversal of the foliation. The subspace $\mathcal{G}_X$ is a smooth submanifold of $\mathcal{G}$, which is foliated by the pull-back foliation $\mathcal{F}_X$ under the range map $r\colon \mathcal{G}_X \to V$. We set $\mathcal{E}^c_{X,i} := C_c^\infty(\mathcal{G}_X, \bigwedge^i T^*_{\mathbb{C}}\mathcal{F}_X)$ with the $\mathcal{A}_c^X := C_c^\infty(\mathcal{G}_X^X)$-valued inner product given by the formula

$$\langle \xi_1, \xi_2\rangle(u) = \int_{v\in\mathcal{G}_{r(u)}} \langle \xi_1(v), \xi_2(vu)\rangle \, d\lambda_{r(u)}(v) \tag{2.1}$$

for $\xi_1, \xi_2 \in \mathcal{E}^i_c$, $u \in \mathcal{G}_X^X$. The space $\mathcal{E}^c_{X,i}$ is a right $\mathcal{A}_c^X$-module and the formula for this action is given by

$$(\xi f)(\gamma) = \sum_{\gamma'\in\mathcal{G}^X_{r(\gamma)}} f(\gamma'\gamma)\xi(\gamma'^{-1})$$

for $f \in \mathcal{A}_c^X$, $\xi \in \mathcal{E}^i_c$, $\gamma \in \mathcal{G}_X$. A classical computation shows that $\mathcal{E}^c_{X,i}$ is then a pre-Hilbert module over the pre-C*-algebra $\mathcal{A}_c^X$. By taking the completion of $\mathcal{A}_c^X$ with respect to the maximal C*-norm and then completing the above pre-Hilbert module we obtain a Hilbert $C^*(\mathcal{G}_X^X)$-module $\mathcal{E}_{X,i}$ for $0 \le i \le p = \dim \mathcal{F}$.

Consider now the leafwise de Rham differential $d = (d_x)_{x\in V}$ on $(V,\mathcal{F})$ and for each $x \in V$ denote the $\mathcal{G}_x^x$- equivariant lift of d_x to $\mathcal{G}_x$ by $\tilde{d}_x$. Let $\tilde{d}$ denote the family of operators $(\tilde{d}_x)_{x\in V}$ acting on $\mathcal{E}^c_{X,i}$. Then $\tilde{d}^2 = 0$ and we get the de Rham complex on $\mathcal{G}_X$:

$$\mathcal{E}^c_{X,0} \xrightarrow{\tilde{d}} \mathcal{E}^c_{X,1} \xrightarrow{\tilde{d}} \cdots \xrightarrow{\tilde{d}} \mathcal{E}^c_{X,p}.$$

The operator $\tilde{d}$ is thus a densely defined (unbounded) operator from $\mathcal{E}_{X,i}$ to $\mathcal{E}_{X,i+1}$ which obviously extends to a *closed* operator that we denote by d_X.

We also consider the leafwise Hodge $\star$ operator along the leaves of $(V,\mathcal{F})$ associated with the fixed orientation of $T\mathcal{F}$ and denote its lift to $\mathcal{G}_X$ by

$$T_X : C_c^\infty(\mathcal{G}_X, r^*(\textstyle\bigwedge^i T^*_{\mathbb{C}}\mathcal{F})) \longrightarrow C_c^\infty(\mathcal{G}_X, r^*(\textstyle\bigwedge^{p-i} T^*_{\mathbb{C}}\mathcal{F})).$$

The formal adjoint of $\tilde{d}$ is easily seen to be the operator $\tilde{\delta} \colon \mathcal{E}^c_{X,i} \to \mathcal{E}^c_{X,i-1}$ given by the formula

$$\tilde{\delta} := (-1)^{p(i+1)+1} T_X \tilde{d} T_X.$$

Then $\tilde{\delta}$ extends to a closed densely defined $C^*(\mathcal{G}_X^X)$-linear operator $\delta_X : \mathcal{E}_{X,\bullet} \to \mathcal{E}_{X,\bullet-1}$ such that $d_X^* = \delta_X$.

Proposition 2.16. *The closed unbounded operators d_X and δ_X are regular operators.*

Proof. This is a classical result that we sketch for completeness, see for instance [HiSk:92]. Let $\Delta = \tilde{d}\tilde{\delta} + \tilde{\delta}\tilde{d}$ on $\mathcal{E}^c_{X,k}$. Then we have on $\mathcal{E}^c_{X,k}$:

$$(1+\tilde{d}\tilde{\delta})(1+\tilde{\delta}\tilde{d}) = (1+\tilde{d}\tilde{\delta}+\tilde{\delta}\tilde{d}) = (1+\Delta).$$

Moreover, the leafwise elliptic operator Δ extends to a regular operator Δ_X on the corresponding 2-Sobolev space $\mathrm{Dom}(\Delta_X)$ on which $(1+\Delta_X)$ is surjective, see for instance [VaI:06]. Since the Sobolev space $\mathrm{Dom}(\Delta_X)$ is contained in $\mathrm{Dom}(d_X\delta_X) \cap \mathrm{Dom}(\delta_X d_X)$, we deduce that for any element $z \in \mathrm{Dom}(\Delta_X)$, the element $z + \delta_X d_X z$ makes sense and belongs to $\mathrm{Dom}(d_X\delta_X)$ since $\mathrm{Im}(\delta_X) \subset \mathrm{Ker}(\delta_X)$. So we deduce that for any $t \in \mathcal{E}_{X,*}$, there exists $z \in \mathrm{Dom}(\Delta_X)$ such that $z + \Delta_X z = t$ and considering $u = z + \delta_X d_X z$, we deduce that u belongs to $\mathrm{Dom}(d_X\delta_X)$ and satisfies $u + d_X\delta_X u = t$. This shows that $1 + d_X\delta_X$ is surjective and hence that d_X is regular. A similar proof works for δ_X. □

Proposition 2.17. *The complex $(\mathcal{E}_X, d_X)$,*

$$\mathcal{E}_{X,0} \xrightarrow{d_X} \mathcal{E}_{X,1} \xrightarrow{d_X} \mathcal{E}_{X,2} \cdots \xrightarrow{d_X} \mathcal{E}_{X,p},$$

together with the operator T_X is an HP complex over the C^*-algebra $C^*(\mathcal{G}_X^X)$.

Proof. Since T_X is the lift of the leafwise Hodge $\star$ operator, we have on smooth compactly supported forms:

$$\langle T_X\omega_1, T_X\omega_2\rangle = \langle\omega_1,\omega_2\rangle \quad \text{and} \quad T_X(T_X\omega) = (-1)^{k(p-k)}\omega \quad \text{for } \omega \in \mathcal{E}^c_{X,k}.$$

Hence we get $T_X^* T_X = 1$ and $T_X^* = (-1)^{k(p-k)} T_X$ on $\mathcal{E}^c_{X,k}$. Therefore T_X extends to an adjointable operator on $\mathcal{E}_{X,k}$ which satisfies (1) of Definition 2.1. Denote as before by δ_X the adjoint of d_X. Then we have $T_X \circ \delta_X = (-1)^k d_X \circ T_X$ on smooth k-forms and hence as closed operators on the maximal completions. This shows condition (2) of Definition 2.1.

To see that condition (3) is verified, we first note that due to condition (2) and Remark 2.2 the map T_X takes $\mathrm{Im}(\delta_X)$ to $\mathrm{Im}(d_X)$ and therefore the induced map $(T_X)_*\colon H^*(\mathcal{E}_X,\delta_X) \to H^*(\mathcal{E}_X, d_X)$ is well defined.

Let $z \in \mathcal{E}_{X,k}$ be d_X-closed and such that $[T_X z] = 0 \in H^{p-k}(\mathcal{E}_X,\delta_X)$. Then there exists a $y \in \mathrm{Dom}(\delta_X) \subset \mathcal{E}_{X,p-k+1}$ such that $T_X z = \delta_X y$ and we have

$$z = \pm T_X(\delta_X y) = d_X(\pm T_X y).$$

Thus $z \in \mathrm{Im}(b)$. Therefore the induced map $(T_X)_*$ is injective. Surjectivity of $(T_X)_*$ follows from the relations $T_X \circ \delta_X = (-1)^k d_X \circ T_X$ and $T_X^2 = \pm 1$. Hence $(T_X)_*$ is an isomorphism.

Finally, to check condition (4) in Definition 2.1 we remark that $\tilde{d} + \tilde{\delta}$ is an elliptic $\mathcal{G}$-operator and therefore extends to a regular Fredholm operator on the Hilbert module, and the extension of $\tilde{d} + \tilde{\delta}$ coincides with $d_X + \delta_X$ (cf. [VaI:06], [Va:01]). Moreover, since $(\tilde{d} + \tilde{\delta} \pm i)^{-1}$ is a pseudo-differential $\mathcal{G}$-operator of negative order, its extension to the Hilbert module is a compact operator [Co:79]. This extension coincides again with $(d_X + \delta_X \pm i)^{-1}$. □

3. Hilbert modules and leafwise homotopy equivalence

We review in this section some classical properties of Hilbert modules associated with leafwise maps that will be used in the subsequent sections. We fix two smooth foliations $(V,\mathcal{F})$ and $(V',\mathcal{F}')$ together with a leafwise map $f\colon (V,\mathcal{F}) \to (V',\mathcal{F}')$. So f is a smooth map which sends leaves to leaves. Denote by $\mathcal{G}$ and $\mathcal{G}'$ the monodromy groupoids of $(V,\mathcal{F})$ and $(V',\mathcal{F}')$, respectively. The leafwise map f naturally induces a well-defined map still denoted by $f\colon \mathcal{G} \to \mathcal{G}'$ which is clearly a groupoid morphism. In the sequel and for simplicity, we will use the same notation r and s for the range and the source maps on the groupoids $\mathcal{G}$ and $\mathcal{G}'$. We are only interested in leafwise homotopy equivalences, we shall therefore make the following simplifying assumption

Assumption 3.1. For any leaf L' of $(V',\mathcal{F}')$ and any transverse submanifold X to $(V,\mathcal{F})$, the intersection $f^{-1}(L') \cap X$ is (at most) a countable subset of X.

Notice that Assumption 3.1 is satisfied when f satisfies that $f^{-1}(L')$ is a finite union of leaves of $(V,\mathcal{F})$, for any given leaf L' of $(V',\mathcal{F}')$. For a leafwise homotopy equivalence, this inverse image is a single leaf. In the whole present section, leafwise map means smooth leafwise map satisfying Assumption 3.1.

3.1. The reduced Hilbert bimodule of a leafwise map. We now introduce the reduced graph $(\mathcal{G}_{W'}^{W}(f), r_f, s_f)$ associated with the subspaces W and W' of V and V', respectively, by setting

$$\begin{array}{ccc} & \mathcal{G}_{W'}^{W}(f) := \{(w,\gamma') \in W \times \mathcal{G}'_{W'} \mid f(w) = r(\gamma')\} & \\ \swarrow^{r_f} & & \searrow^{s_f} \\ W & & W', \end{array}$$

where $r_f(w,\gamma') = w$ and $s_f(w,\gamma') = s(\gamma')$. Let X (resp. X') be a complete smooth transversal in $(V,\mathcal{F})$ (resp. in $(V',\mathcal{F}')$). We shall be mainly interested in the case $W = X$ and $W' = X'$ and in the reduced graph $\mathcal{G}_{X'}^{X}(f)$. The groupoid $\mathcal{G}'^{X'}_{X'}$ acts on the right on $\mathcal{G}_{X'}^{X}(f)$ as follows. If $\alpha' \in \mathcal{G}'^{X'}_{X'}$ is such that $r(\alpha') = s_f(x,\gamma')$, then $(x,\gamma')\alpha' := (x,\gamma'\alpha')$. Note that $r(\gamma'\alpha') = f(x)$, so this action is well defined. It is easy to see that since $\mathcal{G}'^{X'}_{X'}$ acts properly and freely on $\mathcal{G}'_{X'}$, it also acts properly and freely on $\mathcal{G}_{X'}^{X}(f)$.

The space $C_c(\mathcal{G}_{X'}^{X}(f))$ of compactly supported continuous complex-valued functions on $\mathcal{G}_{X'}^{X}(f)$, is thus endowed with the structure of a right $C_c(\mathcal{G}'^{X'}_{X'})$-module. For $\xi \in C_c(\mathcal{G}_{X'}^{X}(f))$ and $\phi' \in C_c(\mathcal{G}'^{X'}_{X'})$ the module structure is defined by the formula

$$(\xi\phi')(x,\gamma') = \sum_{\alpha' \in \mathcal{G}'^{X'}_{s(\gamma')}} \xi(x,\gamma'\alpha'^{-1})\phi'(\alpha').$$

On the other hand, the groupoid $\mathcal{G}_X^X$ acts on the left on $\mathcal{G}_{X'}^{X}(f)$ through the formula

$$\alpha(x,\gamma') = (r(\alpha), f(\alpha)\gamma')$$

for $(x,\gamma') \in \mathcal{G}_{X'}^{X}(f)$ and $\alpha \in \mathcal{G}_x^X$. The left action of $\mathcal{G}_X^X$ on $\mathcal{G}_{X'}^{X}(f)$ induces a representation π_f of the algebra $C_c(\mathcal{G}_X^X)$ on the $C_c(\mathcal{G}_{X'}^{X'})$-module $C_c(\mathcal{G}_{X'}^{X}(f))$ given for $\phi \in C_c(\mathcal{G}_X^X)$ and $\xi \in C_c(\mathcal{G}_{X'}^{X}(f))$ by the formula

$$\pi_f(\phi)\xi(x,\gamma') = \sum_{\alpha \in \mathcal{G}_X^x} \phi(\alpha)\xi(s(\alpha), f(\alpha^{-1})\gamma').$$

We define the $C_c(\mathcal{G}'^{X'}_{X'})$-valued inner product by the formula

$$\langle \xi, \eta\rangle(\gamma') = \sum_{\gamma'_1 \in \mathcal{G}'^{f(X)}_{r(\gamma')}} \sum_{\{x \in X, f(x) = r(\gamma'_1)\}} \overline{\xi(x,\gamma'_1)}\eta(x,\gamma'_1\gamma')$$

for any $\xi, \eta \in C_c(\mathcal{G}_{X'}^X(f))$, $\gamma' \in \mathcal{G}'^{X'}_{X'}$. Since X is a transversal and by Assumption 3.1, the space $\{x \in X, f(x) = r(\gamma'_1)\}$ is a countable subset of the leaf $f^{-1}(L')$, where L' is the leaf which contains (the representatives of) γ'. It is then easy to check, with obvious notations, that

$$\langle \xi, \eta\phi' \rangle = \langle \xi, \eta \rangle \phi', \quad \langle \xi, \eta \rangle^* = \langle \eta, \xi \rangle \quad \text{and} \quad \langle \xi, \xi \rangle \geq 0 \quad \text{in } C^*(\mathcal{G}'^{X'}_{X'}).$$

Completing $C_c(\mathcal{G}_{X'}^X(f))$ with respect to the maximal C*-algebra norm on $C_c(\mathcal{G}'^{X'}_{X'})$, we end up with a Hilbert C*-bimodule over the maximal C*-algebras $C^*(\mathcal{G}_X^X)$ and $C^*(\mathcal{G}'^{X'}_{X'})$, which we denote by $\mathcal{E}_{X'}^X(f)$.

Remark 3.2. The choice of maximal completion is dictated to us by the construction of measured determinants and rho invariants in part II of this series of papers [RoII]. Similar results hold with other completions.

Now let $(V, X, \mathcal{F})$, $(V', X', \mathcal{F}')$ and $(V'', X'', \mathcal{F}'')$ be foliated manifolds with complete transversals X, X' and X'', respectively. Let

$$(V, \mathcal{F}) \xrightarrow{f} (V', \mathcal{F}') \xrightarrow{g} (V'', \mathcal{F}'')$$

be leafwise maps. We define the space $\mathcal{G}_{X'}^X(f) \times_{\mathcal{G}'^{X'}_{X'}} \mathcal{G}_{X''}^{X'}(g)$ as the fibered product defined as the quotient of

$$\{((x, \gamma'); (x', \gamma'')) \in \mathcal{G}_{X'}^X(f) \times \mathcal{G}_{X''}^{X'}(g), x' = s(\gamma')\}$$

under the equivalence relation

$$((x, \gamma'); (x', \gamma'')) \sim ((x, \gamma')\alpha'; \alpha'^{-1}(x', \gamma''))$$

for $\alpha' \in \mathcal{G}'^{X'}_{X'})$, $r(\alpha') = s(\gamma') = x'$. The equivalence class of $((x, \gamma'); (x', \gamma''))$ is denoted by $[(x, \gamma'); (x', \gamma'')]$.

Proposition 3.3. (1) *The space* $\mathcal{G}_{X'}^X(f) \times_{\mathcal{G}'^{X'}_{X'}} \mathcal{G}_{X''}^{X'}(g)$ *is a smooth manifold which is diffeomorphic to* $\mathcal{G}_{X''}^X(g \circ f)$.

(2) *The map* $C_c(\mathcal{G}_{X'}^X(f)) \otimes_{C_c(\mathcal{G}'^{X'}_{X'})} C_c(\mathcal{G}_{X''}^{X'}(g) \to C_c(\mathcal{G}_{X''}^X(g \circ f))$ *which assigns to* $\xi_f \otimes \eta_g$ *for* $\xi_f \in C_c(\mathcal{G}_{X'}^X(f))$ *and* $\eta_g \in C_c(\mathcal{G}_{X''}^{X'}(g))$ *the function* $\xi_f * \eta_g$ *given by*

$$\xi_f * \eta_g(x, \alpha'') := \sum_{\alpha'_1 \in \mathcal{G}_{X'}^{f(x)}} \xi_f(x, \alpha'_1)\eta_g(s(\alpha'_1), g(\alpha'^{-1}_1)\alpha'')$$

for $(x, \alpha'') \in \mathcal{G}_{X''}^X(g \circ f)$ *is well defined.*

Proof. (1) We define a map $\chi\colon \mathcal{G}^X_{X'}(f)\times_{\mathcal{G}'^{X'}_{X'}}\mathcal{G}^{X'}_{X''}(g)\to\mathcal{G}^X_{X''}(g\circ f)$ in the following way:

$$\chi([(x,\gamma');(x',\gamma'')])=(x,g(\gamma')\gamma'')$$

We note that $g\circ f(x)=g(r(\gamma'))=r(g(\gamma'))$, so $(x,g(\gamma')\gamma'')\in\mathcal{G}^X_{X''}(g\circ f)$. It is easy to see that this map is well defined and smooth since the map χ_0 given by $\chi_0((x,\gamma');(x',\gamma''))=(x,g(\gamma')\gamma'')$ is clearly smooth. The relation

$$\chi([(x_1,\gamma'_1);(x'_1,\gamma''_1)])=\chi([(x_2,\gamma'_2);(x'_2,\gamma''_2)])$$

implies that

$$x_1=x_2\quad\text{and}\quad g(\gamma'_1)\gamma''_1=g(\gamma'_2)\gamma''_2.$$

Now setting $\alpha'=\gamma'^{-1}_2\gamma'_1\in\mathcal{G}'^{X'}_{X'}$ we get

$$((x_2,\gamma'_2)\alpha';\alpha'^{-1}(x'_2,\gamma''_2))=((x_1,\gamma'_1);(x'_1,\gamma''_1)).$$

So χ is injective. Surjectivity is also clear and uses that X' is a complete transversal. The rest of the proof of the first item is also clear.

(2) If we use the identification χ defined in the first item, then the formula for $\xi_f*\eta_g$ becomes

$$\xi_f*\eta_g[(x,\gamma');(x',\gamma'')]:=\sum_{\alpha'\in\mathcal{G}'^{s(\gamma')}_{X'}}\xi_f(x,\gamma'\alpha')\eta_g(s(\alpha'),g(\alpha'^{-1})\gamma'').$$

A direct inspection shows that $\xi_f*\eta_g$ is well defined on $\mathcal{G}^X_{X''}(g\circ f)$, is compactly supported and is continuous. Moreover, for $\phi'\in C_c(\mathcal{G}'^{X'}_{X'})$ we compute

$$\begin{aligned}\xi_f\phi'*\eta_g&[(x,\gamma');(x',\gamma'')]\\&=\sum_{\alpha'\in\mathcal{G}'^{s(\gamma')}_{X'}}(\xi_f\phi')(x,\gamma'\alpha')\eta_g(\alpha'^{-1}x',g(\alpha'^{-1})\gamma'')\\&=\sum_{\alpha'\in\mathcal{G}'^{s(\gamma')}_{X'}}\sum_{\alpha'_1\in\mathcal{G}'^{X'}_{s(\alpha')}}(\xi_f)(x,\gamma'\alpha'\alpha'^{-1}_1)\phi'(\alpha'_1)\eta_g(\alpha'^{-1}x',g(\alpha'^{-1})\gamma'').\end{aligned}\tag{3.1}$$

On the other hand, we also have

$$
\begin{aligned}
&\xi_f * \pi_g(\phi')\eta_g[(x,\gamma');(x',\gamma'')] \\
&\quad= \sum_{\beta'\in\mathcal{G}'^{s(\gamma')}_{X'}} \xi_f(x,\gamma'\beta')[\pi_g(\phi')\eta_g](\beta'^{-1}x', g(\beta'^{-1})\gamma'') \\
&\quad= \sum_{\beta'\in\mathcal{G}'^{s(\gamma')}_{X'}} \xi_f(x,\gamma'\beta')[\pi_g(\phi')\eta_g](s(\beta'), g(\beta'^{-1})\gamma'') \\
&\quad= \sum_{\beta'\in\mathcal{G}'^{s(\gamma')}_{X'}} \xi_f(x,\gamma'\beta') \textstyle\sum_{\beta_1'\in\mathcal{G}'^{s(\beta')}_{X'}} \eta_g(s(\beta_1'), g(\beta_1'^{-1})g(\beta'^{-1})\gamma'')\phi'(\beta_1') \\
&\quad= \sum_{\beta'\in\mathcal{G}'^{s(\gamma')}_{X'}} \xi_f(x,\gamma'\beta') \textstyle\sum_{\beta_2'\in\mathcal{G}'^{r(\beta')}_{X'}} \eta_g(s(\beta_2'), g(\beta_2'^{-1})\gamma'')\phi'(\beta'^{-1}\beta_2') \\
&\quad= \sum_{\beta_2'\in\mathcal{G}'^{s(\gamma')}_{X'}} \textstyle\sum_{\beta'\in\mathcal{G}'^{s(\gamma')}_{X'}} \xi_f(x,\gamma'\beta')\eta_g(s(\beta_2'), g(\beta_2'^{-1})\gamma'')\phi'(\beta'^{-1}\beta_2') \\
&\quad= \sum_{\beta_2'\in\mathcal{G}'^{s(\gamma')}_{X'}} \textstyle\sum_{\beta_3'\in\mathcal{G}'^{X'}_{s(\beta_2')}} \xi_f(x,\gamma'\beta_2'\beta_3'^{-1})\eta_g(s(\beta_2'), g(\beta_2'^{-1})\gamma'')\phi'(\beta_3').
\end{aligned}
\tag{3.2}
$$

Comparing (3.1) and (3.2) gives $\xi_f\phi' * \eta_g = \xi_f * \pi_g(\phi')\eta_g$. □

Remark 3.4. We shall show in Proposition 4.1, under the simplifying assumption that f is a leafwise homotopy equivalence that the map defined in the previous proposition extends to an isometric isomorphism of Hilbert modules.

Proposition 3.5. *If f is an oriented leafwise homotopy equivalence, then π_f induces a C^*-algebra isomorphism between $C^*(\mathcal{G}^X_X)$ and $\mathcal{K}_{C^*(\mathcal{G}'^{X'}_{X'})}(\mathcal{E}^X_{X'}(f))$, the algebra of adjointable compact operators on the Hilbert C^*-module $\mathcal{E}^X_{X'}(f)$.*

Proof. Since smooth, compactly supported functions on $\mathcal{G}^X_X$ yield compact operators on $\mathcal{E}^X_{X'}(f)$ (see for instance [HiSk:83]), π_f is well defined and takes values in $\mathcal{K}_{C^*(\mathcal{G}'^{X'}_{X'})}(\mathcal{E}^X_{X'}(f))$. Let $\eta_1,\eta_2 \in \mathcal{E}^{X'}_X(f)$ and denote by θ_{η_1,η_2} the corresponding compact operator of $\mathcal{E}^X_{X'}(f)$ given by

$$\theta_{\eta_1,\eta_2}\zeta := \eta_1\langle\eta_2,\zeta\rangle.$$

The isomorphism of C*-algebras follows from the fact that since the Hilbert-module $\mathcal{E}^X_{X'}(f)$ is compatible with the Connes–Skandalis bimodule (see Proposition 4.5), it is an imprimitivity bimodule, using the corresponding result from [HiSk:83]. It can also be proved directly and we proceed now to do it for surjectivity. The direct proof of injectivity is similar. A straightforward computation gives for $(x,\gamma') \in \mathcal{G}^X_{X'}(f)$:

$$\begin{aligned}
\theta_{\eta_1,\eta_2}\zeta \underbrace{(x,\gamma')}_{f(x)=r(\gamma')}
&= (\eta_1 \cdot \langle \eta_2, \zeta\rangle)(x,\gamma')\\
&= \sum_{\alpha'\in \mathcal{G}'^{X'}_{s(\gamma')}} \eta_1(x,\gamma'\alpha'^{-1})\langle \eta_2,\zeta\rangle(\alpha')\\
&= \sum_{\alpha'\in \mathcal{G}'^{X'}_{s(\gamma')}} \eta_1(x,\gamma'\alpha'^{-1}) \sum_{x_1\in X\cap L_{s(\alpha')}} \sum_{\gamma'_1\in \mathcal{G}'^{f(x_1)}_{r(\alpha')}} \overline{\eta_2(x_1,\gamma'_1)}\zeta(x_1,\gamma'_1\alpha')\\
&= \sum_{\alpha'\in \mathcal{G}'^{X'}_{s(\gamma')}} \eta_1(x,\gamma'\alpha'^{-1}) \sum_{x_1\in X\cap L_{s(\alpha')}} \sum_{\gamma'_2\in \mathcal{G}'^{f(x_1)}_{s(\alpha')}} \overline{\eta_2(x_1,\gamma'_2\alpha'^{-1})}\zeta(x_1,\gamma'_2)\\
&= \sum_{\alpha'\in \mathcal{G}'^{X'}_{s(\gamma')}} \eta_1(x,\gamma'\alpha'^{-1}) \sum_{x_1\in X\cap L_{s(\gamma')}} \sum_{\gamma'_3\in \mathcal{G}'^{f(x)}_{f(x_1)}} \overline{\eta_2(x_1,\gamma_3'^{-1}\gamma'\alpha'^{-1})}\zeta(x_1,\gamma_3'^{-1}\gamma')\\
&= \sum_{x_1\in X\cap L_{s(\gamma')}} \sum\nolimits_{\gamma'_3\in \mathcal{G}'^{f(x)}_{f(x_1)}} \sum_{\alpha'\in \mathcal{G}'^{X'}_{s(\gamma')}} \eta_1(x,\gamma'\alpha'^{-1})\overline{\eta_2(x_1,\gamma_3'^{-1}\gamma'\alpha'^{-1})}\zeta(x_1,\gamma_3'^{-1}\gamma')\\
&= \sum_{x_1\in X\cap L_{s(\gamma')}} \sum\nolimits_{\gamma_3\in \mathcal{G}^x_{x_1}} \sum_{\alpha'\in \mathcal{G}'^{X'}_{s(\gamma')}} \eta_1(x,\gamma'\alpha'^{-1})\\
&\qquad \overline{\eta_2(x_1,f(\gamma_3^{-1})\gamma'\alpha'^{-1})}\zeta(s(\gamma_3),f(\gamma_3^{-1})\gamma')\\
&\qquad \text{(since there exists a unique } \gamma_3\in\mathcal{G}^x_{x_1} \text{ such that } f(\gamma_3)=\gamma'_3)\\
&= \sum_{\gamma_3\in \mathcal{G}^x_X} \sum_{\alpha'\in \mathcal{G}'^{X'}_{s(\gamma')}} \eta_1(x,\gamma'\alpha'^{-1})\overline{\eta_2(s(\gamma_3),f(\gamma_3^{-1})\gamma'\alpha'^{-1})}\zeta(s(\gamma_3),f(\gamma_3^{-1})\gamma')\\
&= \sum_{\gamma_3\in \mathcal{G}^x_X} \sum_{\alpha'_1\in \mathcal{G}'^{f(x)}_{X'}} \eta_1(x,\alpha'_1)\overline{\eta_2(s(\gamma_3),f(\gamma_3^{-1})\alpha_1'^{-1})}\zeta(s(\gamma_3),f(\gamma_3^{-1})\gamma')\\
&= \sum_{\gamma_3\in \mathcal{G}^x_X} \Big(\sum_{\alpha'_1\in \mathcal{G}'^{f(x)}_{X'}} \eta_1(r(\gamma_3),\alpha'_1)\overline{\eta_2(s(\gamma_3),f(\gamma_3^{-1})\alpha_1'^{-1})}\Big)\zeta(s(\gamma_3),f(\gamma_3^{-1})\gamma')\\
&= \sum_{\gamma_3\in \mathcal{G}^x_X} (\eta_1\star\eta_2)(\gamma_3)\zeta(s(\gamma_3),f(\gamma_3^{-1})\gamma')\\
&= \pi_f(\eta_1\star\eta_2)\zeta(x,\gamma').
\end{aligned}$$

Here we denoted by $\eta_1\star\eta_2$ the function

$$(\eta_1\star\eta_2)(\alpha) := \sum_{\alpha'_1\in \mathcal{G}'^{f(r(\alpha))}_{X'}} \eta_1(r(\alpha),\alpha'_1)\overline{\eta_2(s(\alpha),f(\alpha^{-1})\alpha'_1)}$$

for any $\alpha\in\mathcal{G}^X_X$. Thus we get

$$\theta_{\eta_1,\eta_2} = \pi_f(\eta_1\star\eta_2).$$

This finishes the proof of surjectivity by classical arguments. □

3.2. Pull-back maps on Hilbert modules. Let as before $(V, X, \mathcal{F})$ and $(V', X', \mathcal{F}')$ be closed foliated (oriented) manifolds with complete transversals X and X', respectively. Let again $f: (V, \mathcal{F}) \to (V', \mathcal{F}')$ be a leafwise oriented smooth leafwise map. The goal of the present section is to prove that f induces a well-defined operator, the pull-back f_ϕ^*, which is functorial and is moreover a chain map between the corresponding de Rham HP complexes which, see Theorem 3.17 and Theorem 4.7. Let $E \to V$ and $E' \to V'$ be given Hermitian vector bundles. Our main interest concerns leafwise Grassmann bundles, over $(V, \mathcal{F})$ and $(V', \mathcal{F}')$ respectively.

Since the C*-algebras $C^*(\mathcal{G}_X^X)$ and $C^*(\mathcal{G}'^{X'}_{X'})$ are only Morita equivalent, our goal is to define an adjointable homomorphism

$$f_\phi^*: \mathcal{E}_{X',E'} \longrightarrow \mathcal{E}_{X,E} \otimes_{C^*(\mathcal{G}_X^X)} \mathcal{E}_{X'}^X(f),$$

which will be associated with some nice cutoff function ϕ and which will be a chain map between the corresponding HP complexes [HiRoI:05]. Here $\mathcal{E}_{X',E'}$ and $\mathcal{E}_{X,E}$ are appropriate Hilbert C*-modules (see definitions below) associated with the foliations $(V', \mathcal{F}')$ and $(V, \mathcal{F})$, respectively. We later on prove a Poincaré Lemma when f is an oriented leafwise homotopy equivalence.

Notice that the manifold $\mathcal{G}_{X'}^V(f)$ is naturally diffeomorphic to the manifold $\mathcal{G}_X \times_{\mathcal{G}_X^X} \mathcal{G}_{X'}^X(f)$, which is the quotient, under the free and proper action of $\mathcal{G}_X^X$, of the fibered product

$$\{(\gamma; (x, \gamma')) \in \mathcal{G}_X \times \mathcal{G}_{X'}^X(f) \mid x = s(\gamma)\}.$$

Recall that $((\gamma; (x, \gamma')) \sim (\gamma\alpha; \alpha^{-1}(x, \gamma'))$ for $\alpha \in \mathcal{G}_X^X$ with $r(\alpha) = s(\gamma)$. More precisely, let

$$\phi_0: \{(\gamma; (x, \gamma')) \in \mathcal{G}_X \times \mathcal{G}_{X'}^X(f) \mid x = s(\gamma)\} \to \mathcal{G}_{X'}^V(f)$$

be defined by $\phi_0(\gamma; (x, \gamma')) = (r(\gamma), f(\gamma)\gamma')$. Then it is easy to check that ϕ_0 induces a well-defined map

$$\phi: \mathcal{G}_X \times_{\mathcal{G}_X^X} \mathcal{G}_{X'}^X(f) \to \mathcal{G}_{X'}^V(f),$$

which is a diffeomorphism.

Define $\pi_1: \mathcal{G}_{X'}^V(f) \to V$ and $\pi_2: \mathcal{G}_{X'}^V(f) \to \mathcal{G}'_{X'}$ by projecting onto the first and second factor, respectively. Then the Hermitian bundle $E \to V$ allows to define the pre-Hilbert module $C_c^\infty(\mathcal{G}_X, r^*E)$ over the pre-C*-algebras $C_c^\infty(\mathcal{G}_X^X)$. The maximal completion of $C_c^\infty(\mathcal{G}_X, r^*E)$ will be denoted by $\mathcal{E}_{X,E}$; it is a Hilbert module over the maximal C*-algebra $C^*(\mathcal{G}_X^X)$. We define similarly the Hilbert module $\mathcal{E}_{X',E'}$ over $C^*(\mathcal{G}_{X'}^{X'})$ associated with the Hermitian bundle $E' \to V'$.

Let pr_1 be the projection onto the first factor in $\mathcal{G}_X \times_{\mathcal{G}_X^X} \mathcal{G}_{X'}^X(f)$. We define a map $\Phi_f: C_c^\infty(\mathcal{G}_{X'}^V(f), \pi_1^*E) \to C_c^\infty(\mathcal{G}_X \times_{\mathcal{G}_X^X} \mathcal{G}_{X'}^X(f), (r \circ \mathrm{pr}_1)^*E)$ as follows:

$$\Phi_f(\xi)[\gamma; (s(\gamma), \gamma')] = \xi(r(\gamma), f(\gamma)\gamma') \quad \text{for } \xi \in C_c^\infty(\mathcal{G}_{X'}^V(f), \pi_1^*E).$$

We thus have $\Phi_f(\xi)[\gamma;(s(\gamma),\gamma')] \in E_{r(\gamma)}$. In the same way we define the map

$$\nu_f : C_c^\infty(\mathcal{G}_X, r^*E) \otimes C_c^\infty(\mathcal{G}_{X'}^X(f)) \longrightarrow C_c(\mathcal{G}_X \times_{\mathcal{G}_X^X} \mathcal{G}_{X'}^X(f), (r\circ\pi_1)^*E)$$

by setting

$$\nu_f(\xi\otimes\eta)[\gamma;(x,\gamma')] := \sum_{\alpha\in\mathcal{G}_X^{s(\gamma)}} \xi(\gamma\alpha)\eta(\alpha^{-1}x', f(\alpha^{-1})\gamma').$$

The spaces $C_c^\infty(\mathcal{G}_{X'}^V(f), \pi_1^*E)$ and $C_c^\infty(\mathcal{G}_X \times_{\mathcal{G}_X^X} \mathcal{G}_{X'}^X(f), (r\circ\pi_1)^*E)$ are naturally endowed with the structure of right $C_c^\infty(\mathcal{G}'^{X'}_{X'})$-modules. Using the diffeomorphism $\phi\colon \mathcal{G}_X \times_{\mathcal{G}_X^X} \mathcal{G}_{X'}^X(f) \xrightarrow{\cong} \mathcal{G}_{X'}^V(f)$ described above we have $\Phi_f(\xi) = \xi\circ\phi$. The inner product on $C_c^\infty(\mathcal{G}_X \times_{\mathcal{G}_X^X} \mathcal{G}_{X'}^X(f), (r\circ \mathrm{pr}_1)^*E)$ is defined by

$$\langle \xi_1, \xi_2\rangle := \langle \xi_1\circ\phi, \xi_2\circ\phi\rangle,$$

the inner product on the right-hand side is the one defined on $C_c^\infty(\mathcal{G}_{X'}^V(f), \pi_1^*E)$ by the formula

$$\langle \xi,\eta\rangle(\gamma') := \int_{(v,\gamma_1')\in V\times\mathcal{G}'_{r(\gamma')}, f(v)=r(\gamma_1')} \langle \xi(v,\gamma_1'), \eta(v,\gamma_1'\gamma)\rangle_{E_v}\, d\alpha(v).$$

Here we assume for simplicity and since we shall only be interested in leafwise homotopy equivalences that the inverse image of a leaf by f is a finite union of leaves and that $d\alpha$ is the fixed Borel measure on the leaves of $(V,\mathcal{F})$. The general case introduces some tedious technicalities that we do not address here. When f is an oriented leafwise homotopy equivalence, it is clear that the inverse image of a leaf is a leaf, and then the scalar product becomes

$$\langle \xi,\eta\rangle(\gamma') := \int_{v\in L_{\gamma'}} \sum_{\gamma_1'\in\mathcal{G}'^{f(v)}_{r(\gamma')}} \langle \xi(v,\gamma_1'), \eta(v,\gamma_1'\gamma)\rangle_{E_v}\, d\alpha(v).$$

Here $L_{\gamma'}$ is the leaf in V such that $f(L_{\gamma'}) = L'_{r(\gamma')}$.

We denote the completion of $C_c^\infty(\mathcal{G}_{X'}^V(f), \pi_1^*E)$ with respect to the maximal C*-norm by $\mathcal{E}_{X',E}(f)$. The completion of $C_c^\infty(\mathcal{G}_X \times_{\mathcal{G}_X^X} \mathcal{G}_{X'}^X(f), (r\circ\pi_1)^*E)$, again with respect to the maximal norm, is denoted by $\mathcal{E}_{X,X';E}(f)$.

Proposition 3.6. *The above maps Φ_f and ν_f induce isomorphisms of Hilbert modules over the C^*-algebra $C^*(\mathcal{G}'^{X'}_{X'})$:*

$$\Phi_f : \mathcal{E}_{X',E}(f) \to \mathcal{E}_{X,X';E}(f) \quad \text{and} \quad \nu_f : \mathcal{E}_{X,E} \otimes_{C^*(\mathcal{G}_X^X)} \mathcal{E}_{X'}^X(f) \to \mathcal{E}_{X,X';E}(f).$$

Proof. The map Φ_f is clearly an isometry and Φ_f is obviously surjective with inverse given by the map induced by ϕ^{-1}. Therefore Φ_f is an isometric isomorphism of Hilbert modules.

Also we can follow the proof of Proposition 3.3 to deduce that ν_f extends to an isometric isomorphism of Hilbert modules. Notice that an isometric isomorphism is obviously adjointable with the adjoint given by the inverse. □

Definition 3.7. The composition map $\nu_f^{-1} \circ \Phi_f$ will be denoted by ϵ_f. So ϵ_f is an isomorphism of Hilbert modules over $C^*({\mathcal{G}'}_{X'}^{X'})$:

$$\epsilon_f : \mathcal{E}_{X',E}(f) \to \mathcal{E}_{X,E} \otimes_{C^*(\mathcal{G}_X^X)} \mathcal{E}_{X'}^X(f).$$

Proposition 3.8. *Let ξ be an element of $C_c^\infty(\mathcal{G}_{X'}^V(f), \pi_1^* E)$, then we have*

$$(\epsilon_f \circ \tilde{d}^f)(\xi) = ((\tilde{d} \otimes \mathrm{id}) \circ \epsilon_f)(\xi).$$

More precisely, Φ_f and ν_f are both chain maps.

Proof. The identification ν_f is clearly a chain map and it will be forgotten in this proof. Recall that we have a diffeomorphism

$$\phi\colon \mathcal{G}_X \times_{\mathcal{G}_X^X} \mathcal{G}_{X'}^X(f) \to \mathcal{G}_{X'}^V(f) \quad \text{given by} \quad \phi([\gamma, (x, \gamma')]) = (r(\gamma), f(\gamma)\gamma').$$

Moreover, the map $r \circ \pi_1\colon \mathcal{G}_X \times \mathcal{G}_{X'}^X(f) \to V$ induces a smooth covering map $r_1\colon \mathcal{G}_X \times_{\mathcal{G}_X^X} \mathcal{G}_{X'}^X(f) \to V$ such that $\pi_1 \circ \phi = r_1$. Here projections on the first factor are denoted by π_1. The differential $\tilde{d}^f$ is by definition the pull-back differential of the leafwise de Rham differential d under the covering map $\pi_1\colon \mathcal{G}_{X'}^V(f) \to V$; it can be denoted by the suggestive notation $\pi_1^* d$. The differential $\tilde{d} \otimes \mathrm{id}$ is by definition the differential induced on the quotient manifold $\mathcal{G}_X \times_{\mathcal{G}_X^X} \mathcal{G}_{X'}^X(f)$ by the pull-back under $r \circ \pi_1\colon \mathcal{G}_X \times \mathcal{G}_{X'}^X(f) \to V$ of the same leafwise de Rham differential d. Thus,

$$\tilde{d}^f = \pi_1^* d \text{ and } \tilde{d} \otimes \mathrm{id} \text{ is induced by } (r \circ \pi_1)^* d.$$

Denote by $p\colon \mathcal{G}_X \times \mathcal{G}_{X'}^X(f) \to \mathcal{G}_X \times_{\mathcal{G}_X^X} \mathcal{G}_{X'}^X(f)$ the covering map. If η is a smooth leafwise form on V. Then we can write

$$r_1^*(d\eta) = (\tilde{d} \otimes \mathrm{id})(r_1^*\eta) \quad \text{and} \quad \pi_1^*(d\eta) = \tilde{d}^f(\pi_1^*\eta).$$

Therefore,

$$\begin{aligned}(\Phi_f \circ \tilde{d}^f)(\pi_1^*\eta) &= (\phi^* \circ \pi_1^*) d\eta \\ &= (\pi_1 \circ \phi)^* \phi^* d\eta \\ &= r_1^* d\eta \\ &= (\tilde{d} \otimes \mathrm{id})(r_1^*\eta) \\ &= (\tilde{d} \otimes \mathrm{id})\phi^*(\pi_1^*\eta).\end{aligned}$$

So $\Phi_f \circ \tilde{d}^f = (\tilde{d} \otimes \mathrm{id}) \circ \Phi_f$ on sections of the form $\pi_1^* \eta$. Now the statement being local, a classical argument again allows to deduce the allowed relation for any smooth form on $\mathcal{G}_{X'}^V(f)$. □

We assume from now on that $E := \bigwedge^* T_{\mathbb{C}}^* \mathcal{F}$ *and* $E' := \bigwedge^* T_{\mathbb{C}}^* \mathcal{F}'$ *are the leafwise Grassmannian bundles with their Hermitian structures inherited from the leafwise metrics.*

We proceed first to define a smooth pull-back map induced by f between the corresponding spaces of smooth differential forms. Let d' denote the longitudinal de Rham differential along the leaves of $(V', \mathcal{F}')$, and let $\tilde{d}' = (\tilde{d}'_{x'})_{x' \in V'}$ be its lift under the covering map r to the fibers of the monodromy groupoid $\mathcal{G}'$. Similarly, let d be the de Rham differential on the leaves of $(V, \mathcal{F})$ and $\tilde{d}$ its lift by r the fibers of $\mathcal{G}$. We also set $\tilde{d}^f = (\tilde{d}^f_{x'})_{x' \in X'}$ where $\tilde{d}^f_{x'}$ is the lift of d, by the first projection π_1, to $\mathcal{G}_{x'}^V(f) := \{(v, \gamma') \in V \times \mathcal{G}_{x'} \mid r(\gamma') = f(v)\}$.

Definition 3.9. For $\omega' \in C_c^\infty(\mathcal{G}'_{X'}, r^* E')$, we define

$$\Psi_f(\omega')(v, \gamma') = ({}^t f_{*v})[(\pi_2^! \omega')(v, \gamma')], \quad (v, \gamma') \in \mathcal{G}_{X'}^V(f),$$

where ${}^t f_{*v} : E'_{f(v)} \to E_v$ is induced by the transpose of the tangent map $f_{*,v} : T_v \mathcal{F} \to T_{f(v)} \mathcal{F}'$ and $\pi_2^!$ is the pullback via π_2 of elements of $C_c^\infty(\mathcal{G}'_{X'}, r^* E')$. Therefore, $(\pi_2^! \omega')(v, \gamma') = \omega'(\gamma') \in E'_{r(\gamma') = f(v)}$.

When f is a leafwise homotopy equivalence, it is uniformly proper on the different foliated spaces, see [HeLa:91], [BH:11]. Therefore in this case the support of $\Psi_f(\omega')$ is also compact and we get in this way a map

$$\Psi_f : C_c^\infty(\mathcal{G}'_{X'}, r^* E') \longrightarrow C_c^\infty(\mathcal{G}_{X'}^V(f), \pi_1^* E).$$

We also denote by Ψ_f the same map acting on smooth, not necessarily compactly supported, sections. Denote by $f^* \alpha'$ the usual pull-back by f of a differential form α' on V'.

Proposition 3.10. *Let $x' \in X'$. We have the following properties:*

(1) *For any $\alpha' \in C_c^\infty(L'_{x'}, E')$, $\pi_1^!(f^* \alpha') = \Psi_f(r^! \alpha')$.*

(2) $\tilde{d}^f_{x'} \circ \Psi_f = \Psi_f \circ \tilde{d}'_{x'}$ *on* $C_c^\infty(\mathcal{G}'_{x'}, r^* E')$.

Here we denoted, as for π_2 above, pullbacks via π_1 and r by $\pi_1^!$ and $r^!$ respectively.

Proof. Notice that for $(v, \gamma') \in \mathcal{G}_{x'}^V(f)$ we have

$$f \circ \pi_1(v, \gamma') = f(v) = r(\gamma') = r \circ \pi_2(v, \gamma').$$

Therefore, we compute

$$\begin{aligned}\pi_1^!(f^*\alpha')(v,\gamma') &= (f^*\alpha')(v)\\ &= {}^tf_{*v}(\alpha'_{f(v)})\\ &= {}^tf_{*v}[\alpha'_{(f\circ\pi_1)(v,\gamma')}]\\ &= {}^tf_{*v}[(f\circ\pi_1)^!(\alpha'_{(v,\gamma')})]\\ &= {}^tf_{*v}[(r\circ\pi_2)^!(\alpha'_{(v,\gamma')})]\\ &= {}^tf_{*v}[\pi_2^!\circ r^!(\alpha'_{(v,\gamma')})]\\ &= \Psi_f(r^!\alpha')(v,\gamma'),\end{aligned}$$

hence the first item.

For the second item, we fix $s \in C_c^\infty(\mathcal{G}'_{x'}, r^*E')$. Since the statement is local in the leaf $L'_{x'}$, we can assume that our section s can be written in the form

$$s = \sum_i h_i r^!\alpha'_i, \quad \text{where } h_i \in C_c^\infty(\mathcal{G}'_{x'}) \text{ and } \alpha'_i \in C_c^\infty(L'_{x'}, E').$$

Now notice that for any $\alpha' \in C^\infty(L'_{x'}, E')$ the relation

$$(\tilde{d}^f_{x'}\circ\Psi_f)(r^!\alpha') = (\Psi_f\circ\tilde{d}_{x'})(r^!\alpha')$$

holds. Indeed, since $\Psi_f(r^!\alpha') = \pi_1^!(f^*\alpha')$ and $\tilde{d}^f_{x'}$ is the pull-back operator of d_x under π_1, this result is a consequence of the fact that $(f^*\circ d'_{x'})(\alpha') = (d_x\circ f^*)(\alpha')$. The differential $\tilde{d}^f_{x'}$ can also be described as the de Rham differential on the manifold $\mathcal{G}^V_{x'}(f)$ since this latter is the total space of a covering over L_x given by the projection π_1. So we compute

$$\begin{aligned}(\tilde{d}^f_{x'}\circ\Psi_f)(\sum_i h_i r^!\alpha'_i) &= \tilde{d}^f_{x'}[\sum_i(\pi_2^!h_i)(\Psi_f r^!\alpha')]\\ &= \sum_i(\tilde{d}^f_{x'}\pi_2^!h_i\wedge\Psi_f r^!\alpha'_i + \pi_2^!h_i\tilde{d}^f_{x'}\Psi_f r^!\alpha'_i) \qquad (3.3)\\ &= \sum_i(\tilde{d}^f_{x'}\pi_2^!h_i\wedge\Psi_f r^!\alpha'_i + \pi_2^!h_i\Psi_f(\tilde{d}_{x'}r^!\alpha'_i)).\end{aligned}$$

We also have

$$\begin{aligned}(\Psi_f\circ\tilde{d}'_{x'})(\sum_i h_i r^!\alpha'_i) &= \Psi_f[\sum_i \tilde{d}'_{x'}h_i\wedge r^!\alpha'_i + h_i\tilde{d}'_{x'}r^!\alpha'_i]\\ &= \sum_i(\Psi_f(\tilde{d}'_{x'}h_i\wedge r^!\alpha'_i) + \Psi_f(h_i\tilde{d}'_{x'}r^!\alpha'_i)).\end{aligned} \qquad (3.4)$$

It thus remains to show that for a given smooth function h on $\mathcal{G}'_{x'}$ the following relation holds:

$$(\Psi_f\circ\tilde{d}'_{x'})(h) = (\tilde{d}^f_{x'}\circ\pi_2^!)(h).$$

Since $\tilde{d}'_{x'}$ is a differential operator, this is again a local statement and we can use the covering $\mathcal{G}'_{x'} \to L'_{x'}$ to reduce to an open submanifold $\widetilde{U}$ of $\mathcal{G}'_{x'}$ which is diffeomorphic through r to an open submanifold U of $L'_{x'}$. Therefore, we can suppose that h is the pull-back $r^! h_0 = r^* h_0$ of a smooth function h_0 on U. But then

$$(\Psi_f \circ \tilde{d}'_{x'})(h) = \Psi_f(r^* d'_{x'} h_0)$$

and

$$(\tilde{d}^f_{x'} \circ \pi_2^!)(h) = \pi_1^!(d_x f^* h_0) = (\pi_1^! \circ f^*)(d_x h_0).$$

Evaluating at $(v, \gamma') \in \mathcal{G}^V_{x'}(f)$ we thus get

$$(\Psi_f \circ \tilde{d}'_{x'})(h)(v, \gamma') = {}^t f_{*,v}(d_x h_0)_{r(\gamma')},$$

while

$$(\tilde{d}^f_{x'} \circ \pi_2^!)(h)(v, \gamma') = {}^t f_{*,v}(d_x h_0)_{f(v)}.$$

Since $f(v) = r(\gamma')$, the proof is finished by using (3.3) and (3.4). □

Remark 3.11. The map Ψ_f is not bounded in general (and hence not adjointable). In fact, it is even not regular in general!

We denote by Δ and Δ' the Laplace operator along the leaves of the monodromy groupoids $\mathcal{G}$ and $\mathcal{G}'$ respectively. So $\Delta = (\tilde{d} + \tilde{d}^*)^2$ where $\tilde{d}^*$ is the formal adjoint and $\Delta = (\Delta_v)_{v \in V}$ where Δ_v is the Laplace operator on differential forms of $\mathcal{G}_x$. Therefore, and since Δ is a differential operator, it yields a linear map

$$\Delta \colon C_c^\infty(\mathcal{G}_X, r^* E) \longrightarrow C_c^\infty(\mathcal{G}_X, r^* E).$$

Lemma 3.12 ([VaI:06]). *The operator Δ is $C_c^\infty(\mathcal{G}_X^X)$-linear and extends to a regular self-adjoint operator, still denoted by Δ, on the Hilbert $C^*(\mathcal{G}_X^X)$-module $\mathcal{E}_{X,E}$. A similar statement holds of course for Δ'.*

Using the continuous functional calculus theorem for regular self-adjoint operators, we define for any continuous bounded function ϕ on $\mathbb{R}$ a bounded operator $\phi(\Delta)$ on the Hilbert $C^*(\mathcal{G}_X^X)$-module $\mathcal{E}_{X,E}$.

Definition 3.13. Let ϕ be a function on $\mathbb{R}$ which is the Fourier transform of an element of $C_c^\infty(\mathbb{R})$ and such that $\phi(0) = 1$. We define

$$\Psi_f^\phi := \Psi_f \circ \phi(\Delta') \colon C_c^\infty(\mathcal{G}'_{X'}, r^* E') \longrightarrow C^\infty(\mathcal{G}^V_{X'}(f), \pi_1^* E).$$

Proposition 3.14. *Suppose that f is uniformly proper (cf.* [BH:12]*). Then the $C_c^\infty(\mathcal{G}_X^X)$-linear map Ψ_f^ϕ extends to an adjointable operator*

$$\Psi_f^\phi \colon \mathcal{E}_{X',E'} \to \mathcal{E}_{X',E}(f).$$

Proof. To see that $\Psi_f^\phi := \Psi_f \circ \phi(\Delta')$ extends to an adjointable operator on Hilbert modules, we compute its Schwartz kernel. Let k_ϕ denote the Schwartz kernel of $\phi(\Delta')$. Since ϕ is chosen such that its Fourier transform is smooth compactly supported, the kernel k_ϕ is a smooth section with compact support over $\mathcal{G}'$ [RoI:87]. We set

$$K_f^\phi(v,\gamma') = {}^t f_{*,v} \circ k_\phi(\gamma') \in \operatorname{Hom}(E'_{s(\gamma')}, E_v).$$

Then for any $\omega' \in C_c^\infty(\mathcal{G}'_{X'}, r^* E')$ we have

$$\Psi_f^\phi(\omega')(v,\gamma') = \int_{\mathcal{G}'_{s(\gamma')}} K_f^\phi(v, \gamma'\gamma_1'^{-1})\omega'(\gamma_1'^{-1}) d\lambda_{s(\gamma')}(\gamma_1').$$

Here $d\lambda_{x'}$ is the $\mathcal{G}'$-invariant Haar system pulled back from the Borel measure α' on the leaves. Since k_ϕ is smooth with compact support in $\mathcal{G}'$, K_ϕ^f also is smooth with compact support in $\mathcal{G}_{X'}^V(f)$. Then classical arguments show that Ψ_ϕ^f extends to an adjointable operator as claimed. □

Recall from [BH:12] that a leafwise homotopy equivalence is always uniformly proper. We are now in position to define a pull-back map associated with the leafwise oriented leafwise smooth map $f: (V,\mathcal{F}) \to (V',\mathcal{F}')$ (satisfying 3.1) and with respect to ϕ and to the leafwise metric defining Δ'.

Definition 3.15. Let ϕ be the Fourier transform of an element of $C_c^\infty(\mathbb{R})$ such that $\phi(0) = 1$. Let as before X and X' be complete transversals for $(V,\mathcal{F})$ and $(V',\mathcal{F}')$ respectively. Then the *pull-back map* by f associated with ϕ is the adjointable $C^*(\mathcal{G}'^{X'}_{X'})$-linear operator

$$f_\phi^* := \epsilon_f \circ \Psi_f^\phi = \nu_f^{-1} \circ \Phi_f \circ \Psi_f \circ \phi(\Delta') \colon \mathcal{E}_{X',E'} \longrightarrow \mathcal{E}_{X,E} \otimes_{C^*(\mathcal{G}_X^X)} \mathcal{E}_{X'}^X(f).$$

Lemma 3.16. (1) *For any function ϕ which is the Fourier transform of an element of $C_c^\infty(\mathbb{R})$ and which satisfies $\phi(0) = 1$, the operator $\phi(\Delta)\colon \mathcal{E}_{X,E} \to \mathcal{E}_{X,E}$ is an adjointable chain map which induces the identity on cohomology.*

(2) *The two adjointable chain maps f_ϕ^* and $(\phi(\Delta) \otimes \mathrm{id}) \circ \epsilon_f \circ \Psi_f$ induce the same map on cohomologies.*

Proof. (1) As ϕ has compactly supported Fourier transform, it is easy to check that $\operatorname{Im}(\phi(\Delta_X)) \subseteq \operatorname{Dom}(d_X)$. Furthermore, since the Fourier transform of a compactly supported smooth function is an entire function whose restriction to $\mathbb{R}$ is Schwartz, we get that ϕ is entire. Then, following the arguments in [HeLa:91], we consider the holomorphic functional calculus for the self-adjoint regular operator Δ_X, which makes sense as the resolvent map $z \mapsto (zI - \Delta_X)^{-1}$ is analytic on the resolvent of Δ_X in $\mathbb{C}$ (cf. Result 5.23 in [Ku:97]). Therefore, choosing a curve γ in $\mathbb{C}$ that does not intersect $\mathbb{R}^+$ and surrounds it, as in [HeLa:91], one can write

$$\phi(\Delta_X) = \frac{1}{2\pi i}\int_\gamma \phi(z)(zI - \Delta_X)^{-1} dz.$$

Now, for $z \in \mathbb{C}$ in the resolvent of Δ_X, we have $(zI - \Delta_X)^{-1} d_X = d_X (zI - \Delta_X)^{-1}$ and thus $\phi(\Delta_X) d_X = d_X \phi(\Delta_X)$, more precisely, the image of the adjointable operator $\phi(\Delta_X)$ is contained in the domain of d_X and the two operators $\phi(\Delta_X) d_X$ and $d_X \phi(\Delta_X)$ extend to adjointable operators which coincide on the domain of d_X and hence coincide. Similar arguments show that $\phi(\Delta_X)\delta_X = \delta_X \phi(\Delta_X)$.

Now to show that $\phi(\Delta_X)$ induces the identity map on cohomology we proceed as follows. As ϕ is entire with $\phi(0) = 1$, the function ψ given by $\psi(x) = \frac{\phi(x)-1}{x}$ is also entire and in particular smooth on $\mathbb{R}$. Now, there exists a sequence of Schwartz functions with compactly supported Fourier transforms $(\alpha_n)_{n\in\mathbb{N}}$ such that $\alpha_n \xrightarrow{n\to\infty} \psi$ in the $\|\bullet\|_\infty$ norm. Consequently, if $v \in \mathcal{E}^c_{X,k}$ for $k = 0, 1, 2, \dots, p$, we get

$$\alpha_n(\Delta_X)v \xrightarrow{n\to\infty} \psi(\Delta_X)v$$

and

$$d_X(\alpha_n(\Delta_X)v) = \alpha_n(\Delta_X)(d_X v) \xrightarrow{n\to\infty} \psi(\Delta_X)(d_X v).$$

Therefore $\psi(\Delta_X)v \in \mathrm{Dom}(d_X)$ and $\psi(\Delta_X)d_X = d_X\psi(\Delta_X)$ on $\mathcal{E}^c_{X,k}$.

Now let $\omega \in \mathrm{Ker}(d_X)$. Then there exists a sequence $(\omega_n)_{n\geq 0}$ such that each ω_n is a compactly supported smooth form, $\omega_n \xrightarrow{n\to\infty} \omega$, and $d_X\omega_n \to 0$ in the Hilbert modules. We then have

$$\begin{aligned}
\phi(\Delta_X)\omega_n - \omega_n &= \psi(\Delta_X)\Delta_X(\omega_n) \\
&= \psi(\Delta_X)(d_X\delta_X\omega_n) + \psi(\Delta_X)(\delta_X d_X \omega_n) \\
&\quad \text{(notice that } \mathrm{Im}(d_X|_{\mathcal{E}^c_{X,k}}) \subseteq \mathcal{E}^c_{X,k+1}, \mathrm{Im}(\delta_X|_{\mathcal{E}^c_{X,k}}) \subseteq \mathcal{E}^c_{X,k-1}) \\
&= d(\psi(\Delta_X)\delta_X\omega_n) + \psi(\Delta_X)\delta_X(d_X\omega_n). \qquad (3.5)
\end{aligned}$$

But on compactly supported smooth forms we have

$$\begin{aligned}
\psi(\Delta_X)\circ\delta_X &= \psi(\Delta_X)[(I+\Delta_X)(I+\Delta_X)^{-1}]\delta_X \\
&= [\psi(\Delta_X)(I+\Delta_X)][(I+\Delta_X)^{-1}\delta_X] \\
&\quad \text{(since } \mathrm{Im}(I+\Delta_X)^{-1} \subseteq \mathrm{Dom}(I+\Delta_X)) \\
&= [\psi(\Delta_X) + \phi(\Delta_X) - I][(I+\Delta_X)^{-1}\delta_X].
\end{aligned}$$

Now $\psi(\Delta_X) + \phi(\Delta_X) - I$ is clearly adjointable as ϕ and ψ are bounded smooth functions and $(I+\Delta_X)^{-1}\delta_X$ is adjointable because it is a pseudo-differential operator of negative order [VaI:06]. Hence $\psi(\Delta_X)\circ\delta_X$ is an adjointable operator.

We thus get

$$\psi(\Delta_X)\delta_X(\omega_n) \xrightarrow{n\to\infty} \psi(\Delta_X)(\delta_X\omega) \text{ and } \psi(\Delta_X)\delta_X(d\omega_n) \xrightarrow{n\to\infty} 0.$$

Hence by (3.5) we obtain

$$\psi(\Delta_X)\delta_X(\omega_n) \xrightarrow{n\to\infty} \psi(\Delta_X)(\delta_X\omega)$$

and

$$d(\psi(\Delta_X)\delta_X\omega_n)) = \phi(\Delta_X)\omega_n - \omega_n - \psi(\Delta_X)\delta_X(d_X\omega_n) \xrightarrow{n\to\infty} \phi(\Delta)\omega - \omega.$$

Thus the above two limits together imply

$$\psi(\Delta_X)\delta_X\omega \in \mathrm{Dom}(d_X) \quad \text{and} \quad \phi(\Delta_X)\omega - \omega = d_X(\psi(\Delta)\delta\omega) \subseteq \mathrm{Im}(d_X).$$

So $\phi(\Delta_X)\omega - \omega = 0$ on cohomology and $\phi(\Delta_X)$ is the identity map on cohomology.

(2) We may compose the adjointable chain map f_ϕ^* on the left by the chain map $\phi(\Delta) \otimes \mathrm{id}$ and get an adjointable chain map which induces by (1) the same map as f_ϕ^* on cohomlogies. In the same way, we can compose $(\phi(\Delta) \otimes \mathrm{id}) \circ \epsilon_f \circ \Psi_f$ on the right by the chain map $\phi(\Delta')$ and get an adjointable chain map which again by (1) induces the same map as $(\phi(\Delta) \otimes \mathrm{id}) \circ \epsilon_f \circ \Psi_f$ on cohomologies. This completes the proof of (2). □

We summarize our results in the following proposition.

Theorem 3.17. *The pull-back map f_ϕ^* is an adjointable operator which is a chain map between the de Rham Hilbert–Poincaré complexes. Moreover, the map induced by f_ϕ^* on cohomology does not depend on the choices of ϕ and of the leafwise metric on $(V', \mathcal{F}')$.*

Proof. Only the last part of the statement needs to be proved. From the second item of Lemma 3.16, we see that the map induced by f_ϕ^* does not depend on the leafwise metric on $(V', \mathcal{F}')$. Now assume that ψ is another function which is the Fourier transform of an element of $C_c^\infty(\mathbb{R})$ and which satisfies $\psi(0) = 1$. Then $\phi\psi$ satisfies the same conditions as ϕ and ψ and we have

$$f_\phi^* \circ \psi(\Delta') = f_\psi^* \circ \phi(\Delta').$$

By the first item of Lemma 3.16, $\psi(\Delta')$ and $\phi(\Delta')$ are chain maps which induce the identity on cohomologies, therefore f_ϕ^* and f_ψ^* induce the same map on cohomologies. □

3.3. Functoriality of the pull-back. Recall that $f: (V, \mathcal{F}) \to (V', \mathcal{F}')$ is a leafwise oriented smooth map which satisfies Assumption 3.1. We have fixed complete transversals X and X' for the foliations $(V, \mathcal{F})$ and $(V', \mathcal{F}')$ respectively. Let $g: (V', \mathcal{F}') \to (V'', \mathcal{F}'')$ be another leafwise oriented smooth map, also satisfying Assumption 3.1, and let X'' be a complete transversal in $(V'', \mathcal{F}'')$. In order to work with adjointable operators, we shall also assume that f and g are uniformly proper maps between the two foliations. We defined the pull-back adjointable operators (cf. Definition 3.15) associated with a fixed nice cutoff function ϕ:

$$f_\phi^*: \mathcal{E}_{X',E'} \to \mathcal{E}_{X,E} \otimes \mathcal{E}_{X'}^{X}(f) \quad \text{and} \quad g_\phi^*: \mathcal{E}_{X'',E''} \to \mathcal{E}_{X',E'} \otimes \mathcal{E}_{X''}^{X'}(g).$$

We also proved that f_ϕ^* and g_ϕ^* are chain maps between the de Rham Hilbert–Poincaré complexes. The goal of this section is to prove the expected functoriality property.

We denote by $\Xi_{g,f}^{X'}$ the isomorphism

$$\Xi_{g,f}^{X'} : \mathcal{E}_{X''}^{X}(g \circ f) \longrightarrow \mathcal{E}_{X'}^{X}(f) \otimes_{C^*(\mathcal{G}'^{X'}_{X'})} \mathcal{E}_{X''}^{X'}(g),$$

described in Proposition 3.3. Notice that this is a bimodule isomorphism. Then we can state:

Theorem 3.18. *The following diagram of adjointable chain maps induces a commutative diagram between the corresponding $C^*(\mathcal{G}'^{X'}_{X'})$ Hilbert–Poincaré de Rham cohomologies.*

$$\begin{array}{ccc}
\mathcal{E}_{X'',E''} & \xrightarrow{\quad g_\phi^* \quad} & \mathcal{E}_{X',E'} \otimes_{C^*(\mathcal{G}'^{X'}_{X'})} \mathcal{E}_{X''}^{X'}(g) \\
\Big\downarrow {\scriptstyle (g\circ f)_\phi^*} & & \Big\downarrow {\scriptstyle f_\phi^* \otimes_{C^*(\mathcal{G}'^{X'}_{X'})} \mathrm{id}} \\
\mathcal{E}_{X,E} \otimes_{C^*(\mathcal{G}^{X}_{X})} \mathcal{E}_{X''}^{X}(g \circ f) & \xrightarrow{\mathrm{id} \otimes_{C^*(\mathcal{G}^X_X)} \Xi^X_{g,f}} & \mathcal{E}_{X,E} \otimes_{C^*(\mathcal{G}^{X}_{X})} \mathcal{E}_{X'}^{X}(f) \otimes_{C^*(\mathcal{G}'^{X'}_{X'})} \mathcal{E}_{X''}^{X'}(g)
\end{array}$$

We devote the rest of this section to the proof of this theorem. Indeed, we shall prove more precisely that the map $((\epsilon_g \circ \Psi_g) \otimes_{C_c^\infty(\mathcal{G}^X_X)} \mathrm{id}) \circ (\epsilon_f \circ \Psi_f)$ coincides exactly with $\epsilon_{f\circ g} \circ \Psi_{f \circ g}$ on smooth compactly supported sections, when we identify $\mathcal{E}_{X'}^{X}(f) \otimes \mathcal{E}_{X''}^{X'}(g)$ with $\mathcal{E}_{X''}^{X}(g \circ f)$. For simplicity, we shall denote for an operator T and for an algebra A by $T \otimes \mathrm{id}$ the expression $T \otimes_A \mathrm{id}$ when the algebra A is clearly understood and no confusion can occur. Even if the maps Ψ_f and Ψ_g are not adjointable, we shall restrict to smooth compactly supported sections and by using eventually the regularization $\phi(\Delta')$ or $\phi(\Delta)$, we shall easily deduce the corresponding results at the level of Hilbert module completions.

Proposition 3.19. *There exists an isometric isomorphism of Hilbert modules*

$$\mathcal{E}_{X',E}(f) \otimes_{C^*(\mathcal{G}'^{X'}_{X'})} \mathcal{E}_{X''}^{X'}(g) \longrightarrow \mathcal{E}_{X'',E}(g \circ f).$$

Proof. The proof is a straightforward generalization of the easier proof corresponding to the case where the Hermitian bundles E and E'' are the trivial line bundles. In this latter case, we give below the proof of Proposition 4.1, which adapts immediately to the replacement of X by V and we thus leave the details of the precise modifications as an exercise. □

Denote by

$$\Pi_{g,f} : \mathcal{E}_{X'',E}(g \circ f) \longrightarrow \mathcal{E}_{X',E}(f) \otimes_{C^*(\mathcal{G}'^{X'}_{X'})} \mathcal{E}_{X''}^{X'}(g)$$

the isomorphism described in Proposition 3.19. Recall that $\Pi_{g,f}$ is the extension of a $C_c^\infty(\mathcal{G}''^{X''}_{X''})$-linear map, still denoted by $\Pi_{g,f}$, between the smooth compactly supported sections.

Lemma 3.20. *The relation*

$$\Pi_{g,f} \circ \Psi_{g\circ f} = (\Psi_f \otimes \mathrm{id}) \circ \epsilon_g \circ \Psi_g$$

holds on $C_c^\infty(\mathcal{G}^{V}_{X''}, r^*E'')$.

Proof. First consider the diffeomorphism $\lambda\colon \mathcal{G}^V_{X'}(f) \times_{\mathcal{G}^{X'}_{X'}} \mathcal{G}^{X'}_{X''}(g) \to \mathcal{G}^V_{X''}(g\circ f)$ given by $\lambda[(v,\gamma');(s(\gamma'),\gamma'')] = (v, g(\gamma')\gamma'')$. (The proof of this is analogous to that of Proposition 3.3.) Then λ induces the map

$$\Lambda_{g,f}\colon C_c^\infty(\mathcal{G}^V_{X''}(g\circ f), \pi_1^*E) \longrightarrow C_c^\infty(\mathcal{G}^V_{X'}(f) \times_{\mathcal{G}^{X'}_{X'}} \mathcal{G}^{X'}_{X''}(g), (\pi_1\circ \mathrm{pr}_1)^*E).$$

By arguing as in the proof of Proposition 3.3, $\Lambda_{g,f}$ is an isometry which extends to an isometric isomorphism between the Hilbert modules. Similar arguments allow to construct, as for ν_f in the previous section, an isometric isomorphism μ which restricts to

$$\mu_{g,f}\colon C_c^\infty(\mathcal{G}^V_{X'}(f) \times_{\mathcal{G}^{X'}_{X'}} \mathcal{G}^{X'}_{X''}(g), (\pi_1\circ \mathrm{pr}_1)^*E) \\ \longrightarrow C_c^\infty(\mathcal{G}^V_{X'}(f), \pi_1^*E) \otimes_{C_c^\infty(\mathcal{G}^{X'}_{X'})} C_c^\infty(\mathcal{G}^{X'}_{X''}(g)).$$

A direct inspection shows that

$$\mu_{g,f} \circ \Lambda_{g,f} = \Pi_{g,f}.$$

On the other hand, if $\alpha \in C_c^\infty(\mathcal{G}_{X'} \times_{\mathcal{G}^{X'}_{X'}} \mathcal{G}^{X'}_{X''}(g)), (r\circ \mathrm{pr}_1)^*E')$, then we set

$$\hat{\Psi}_f(\alpha)[(v,\gamma');(s(\gamma'),\gamma'')] =^t f_{*,v}(\alpha[\gamma';(s(\gamma'),\gamma'')])$$

and define in this way

$$\hat{\Psi}_f\colon C_c^\infty(\mathcal{G}_{X'} \times_{\mathcal{G}^{X'}_{X'}} \mathcal{G}^{X'}_{X''}(g), (r\circ\pi_1)^*E') \\ \longrightarrow C_c^\infty(\mathcal{G}^V_{X'}(f) \times_{\mathcal{G}^{X'}_{X'}} \mathcal{G}^{X'}_{X''}(g), (\pi_1\circ \mathrm{pr}_1)^*E),$$

which corresponds to $\Psi_f \otimes \mathrm{id}$ through the isomorphisms. More precisely, the following diagram commutes.

$$\begin{array}{ccc} C_c^\infty(\mathcal{G}_{X'} \times_{\mathcal{G}^{X'}_{X'}} \mathcal{G}^{X'}_{X''}(g), (r\circ\pi_1)^*E') & \xrightarrow{\hat{\Psi}_f} & C_c^\infty(\mathcal{G}^V_{X'}(f) \times_{\mathcal{G}^{X'}_{X'}} \mathcal{G}^{X'}_{X''}(g), (\pi_1\circ \mathrm{pr}_1)^*E) \\ \Big\downarrow {\scriptstyle (\nu)_g^{-1}} & & \Big\downarrow {\scriptstyle \mu_{g,f}} \\ C_c^\infty(\mathcal{G}_{X'}, r^*E') \otimes_{C_c^\infty(\mathcal{G}^{X'}_{X'})} C_c^\infty(\mathcal{G}^{X'}_{X''}(g)) & \xrightarrow{\Psi_f\otimes \mathrm{id}} & C_c^\infty(\mathcal{G}^V_{X'}(f), \pi_1^*E) \otimes_{C_c^\infty(\mathcal{G}^{X'}_{X'})} C_c^\infty(\mathcal{G}^{X'}_{X''}(g)) \end{array}$$

Thus it remains to show that, on smooth compactly supported forms, we have

$$\Lambda_{g,f} \circ \Psi_{g\circ f} = \hat{\Psi}_f \circ \Phi_g \circ \Psi_g.$$

Now, for $\alpha \in C_c^\infty(\mathcal{G}_{X''}^V(g\circ f), \pi_1^* E)$, we have

$$(\Lambda_{g,f}\alpha)[(v,\gamma');(s(\gamma'),\gamma'')] := \alpha(v, g(\gamma')\gamma'') \in E_v.$$

Let then $\beta'' \in C_c^\infty(\mathcal{G}_{X''}'', r^*E'')$. Then we have

$$\begin{aligned}(\hat{\Psi}_f\circ\Phi_g\circ\Psi_g)(\beta'')[(v,\gamma');(s(\gamma'),\gamma'')] &= {}^t f_{*,v}[\Psi_g(\beta'')(r(\gamma'), g(\gamma')\circ\gamma'')]\\ &= {}^t f_{*,v}[{}^t g_{*,r(\gamma')}(\beta''(g(\gamma')\gamma''))]\\ &= {}^t(g_{*,f(v)}\circ f_{*,v})(\beta''(g(\gamma')\gamma''))\\ &= ({}^t(g\circ f)_{*,v})(\beta''(g(\gamma')\gamma'')).\end{aligned}$$

On the other hand,

$$\begin{aligned}(\Lambda_{g,f}\circ\Psi_{g\circ f})(\beta'')[(v,\gamma');(s(\gamma'),\gamma')] &= \Psi_{g\circ f}(\beta'')(v, g(\gamma')\gamma'')\\ &= ({}^t(g\circ f)_{*,v})(\beta''(g(\gamma')\gamma'')),\end{aligned}$$

which completes the proof of the lemma. ☐

Next we prove the following

Lemma 3.21. *With the above notations, the following diagram commutes.*

$$\begin{array}{ccc} C_c^\infty(\mathcal{G}_{X''}^V(g\circ f),\pi_1^*E) & \xrightarrow{\Pi_{g,f}} & C_c^\infty(\mathcal{G}_{X'}^V(f),\pi_1^*E)\otimes C_c^\infty(\mathcal{G}_{X''}^{X'}(g)) \\ \Big\downarrow{\scriptstyle \epsilon_{g\circ f}} & & \Big\downarrow{\scriptstyle \epsilon_f\otimes \mathrm{id}} \\ C_c^\infty(\mathcal{G}_X, r^*E)\otimes C_c^\infty(\mathcal{G}_{X''}^X(g\circ f)) & \xrightarrow{\mathrm{id}\otimes\Xi_{g,f}^{X'}} & C_c^\infty(\mathcal{G}_X, r^*E)\otimes C_c^\infty(\mathcal{G}_{X'}^X(f))\otimes C_c^\infty(\mathcal{G}_{X''}^{X'}(g)) \end{array}$$

Proof. We see the elements of $C_c^\infty(\mathcal{G}_X, r^*E)\otimes C_c^\infty(\mathcal{G}_{X'}^X(f))\otimes C_c^\infty(\mathcal{G}_{X''}^{X'}(g))$ as sections over

$$\mathcal{G}_X \times_{\mathcal{G}_X^X} \mathcal{G}_{X'}^X(f) \times_{\mathcal{G}_{X'}^{X'}} \mathcal{G}_{X''}^{X''}(g).$$

Then a straightforward computation gives for $\alpha \in C_c^\infty(\mathcal{G}_{X''}^V, \pi_1^*E)$

$$\begin{aligned}&((\epsilon_f\otimes\mathrm{id})\circ\Pi_{g,f})(\alpha)[\gamma_1;[(s(\gamma_1),\gamma');(s(\gamma'),\gamma'')]]\\ &\quad= \alpha(r(\gamma_1), g(f(\gamma_1)\gamma')\gamma'')\\ &\quad= \alpha(r(\gamma_1'), (g\circ f)(\gamma_1)g(\gamma')\gamma''),\end{aligned} \tag{3.6}$$

while

$$\begin{aligned}&((\mathrm{id}\otimes\Xi_{g,f}^{X'})\circ\epsilon_{g\circ f})(\alpha)[\gamma_1;[(s(\gamma_1),\gamma');(s(\gamma'),\gamma'')]]\\ &\quad= (\epsilon_{g\circ f}\alpha)[\gamma_1;(s(\gamma_1), g(\gamma')\gamma'')]\\ &\quad= \alpha(r(\gamma_1), (g\circ f)(\gamma_1)g(\gamma')\gamma'').\end{aligned} \tag{3.7}$$

Thus from eqs. (3.6) and (3.7) we get the desired equality. ☐

From the previous two lemmas we can deduce:

Proposition 3.22. *The following diagram is commutative.*

$$\begin{array}{ccc}
C_c^\infty(\mathcal{G}_{X''}, r^*E'') & \xrightarrow{\epsilon_g\Psi_g} & C_c^\infty(\mathcal{G}_{X'}, r^*E') \otimes C_c^\infty(\mathcal{G}_{X''}^{X'}(g)) \\
\Big\downarrow{\scriptstyle \epsilon_{g\circ f}\Psi_{g\circ f}} & & \Big\downarrow{\scriptstyle \epsilon_f\Psi_f\otimes \mathrm{id}} \\
C_c^\infty(\mathcal{G}_X, r^*E) \otimes C_c^\infty(\mathcal{G}_{X''}^{X}(g\circ f)) & \xrightarrow{\mathrm{id}\otimes \Xi_{g,f}^{X'}} & C_c^\infty(\mathcal{G}_X, r^*E) \otimes C_c^\infty(\mathcal{G}_{X'}^{X}(f)) \otimes C_c^\infty(\mathcal{G}_{X''}^{X''}(g))
\end{array}$$

Proof. We know from Lemma 3.20 that

$$\Pi_{g,f}\circ\Psi_{g\circ f} = (\Psi_f\otimes\mathrm{id})\circ\epsilon_g\circ\Psi_g.$$

Therefore,

$$(\epsilon_f\Psi_f\otimes\mathrm{id})\circ\epsilon_g\Psi_g = (\epsilon_f\otimes\mathrm{id})\circ\Pi_{g,f}\circ\Psi_{g\circ f}.$$

Now from Lemma 3.21 we deduce that

$$(\epsilon_f\otimes\mathrm{id})\circ\Pi_{g,f} = (\mathrm{id}\otimes\Xi_{g,f}^{X'})\circ\epsilon_{g\circ f}.$$

Hence, we finally get

$$(\epsilon_f\Psi_f\otimes\mathrm{id})\circ\epsilon_g\Psi_g = (\mathrm{id}\otimes\Xi_{g,f}^{X'})\circ\epsilon_{g\circ f}\circ\Psi_{g\circ f}. \qquad \square$$

We finish this section by deducing the proof of Theorem 3.18.

The composite map $(f_\phi^*\otimes\mathrm{id})\circ g_\phi^*$ is given by

$$(f_\phi^*\otimes\mathrm{id})\circ g_\phi^* = (\epsilon_f\circ\Psi_f\circ\phi(\Delta)\otimes\mathrm{id})\circ\epsilon_g\circ\Psi_g\circ\phi(\Delta'').$$

But the adjointable chain map $\epsilon_f\circ\Psi_f\circ\phi(\Delta')\otimes\mathrm{id}$ induces the same map on cohomology as the adjointable chain map $\phi(\Delta)\circ\epsilon_f\circ\Psi_f\otimes\mathrm{id}$. Therefore, $(f_\phi^*\otimes\mathrm{id})\circ g_\phi^*$ induces the same map on cohomology as

$$(\phi(\Delta)\otimes\mathrm{id}\otimes\mathrm{id})\circ(\epsilon_f\circ\Psi_f\otimes\mathrm{id})\circ\epsilon_g\circ\Psi_g\circ\phi(\Delta'').$$

Now by Proposition 3.22 we have on smooth compactly supported forms

$$\begin{aligned}
&(\phi(\Delta)\otimes\mathrm{id}\otimes\mathrm{id})\circ(\epsilon_f\circ\Psi_f\otimes\mathrm{id})\circ\epsilon_g\circ\Psi_g\circ\phi(\Delta'') \\
&\qquad = (\phi(\Delta)\otimes\mathrm{id}\otimes\mathrm{id})\circ(\mathrm{id}\otimes\Xi_{g,f}^{X'})\circ(g\circ f)_\phi^*.
\end{aligned}$$

Since this is an adjointable operator, this relation still holds on the Hilbert module $\mathcal{E}_{X'',E''}$. Moreover, the operator $(\mathrm{id}\otimes\Xi_{g,f}^{X'})\circ(g\circ f)_\phi^*$ is an adjointable chain map and $\phi(\Delta)\otimes\mathrm{id}\otimes\mathrm{id}$ induces the identity on cohomologies, whence $(f_\phi^*\otimes\mathrm{id})\circ g_\phi^*$ induces the same map on cohomologies as the map $(\mathrm{id}\otimes\Xi_{g,f}^{X'})\circ(g\circ f)_\phi^*$.

4. Leafwise homotopy equivalence and HP complexes

This section is devoted to the main result of this paper, namely that any leafwise homotopy equivalence induces a homotopy of the corresponding HP complexes, and hence an explicit homotopy between the corresponding leafwise signature operators.

4.1. Leafwise homotopy equivalences.

Proposition 4.1. *Assume that $(V,\mathcal{F})$ and $(V',\mathcal{F}')$ are closed oriented foliated manifolds and that f is an oriented leafwise homotopy equivalence between $(V,\mathcal{F})$ and $(V',\mathcal{F}')$ with homotopy inverse g. We fix as before complete transversals X and X' in $(V,\mathcal{F})$ and $(V',\mathcal{F}')$ respectively, and we also consider another complete transversal X'' in $(V,\mathcal{F})$. Then we have an isomorphism of Hilbert modules*

$$\mathcal{E}_{X'}^{X}(f)\otimes_{C^*(\mathcal{G}'^{X'}_{X'})}\mathcal{E}_{X''}^{X'}(g)\cong\mathcal{E}_{X''}^{X}(g\circ f).$$

Proof. We use the previous notations in the proof of Proposition 3.3 and compute for $\xi_f\in C_c(\mathcal{G}_{X'}^{X}(f))$ and $\eta_g\in C_c(\mathcal{G}_{X''}^{X'}(g)$, $<\xi_f*\eta_g,\xi_f*\eta_g>$ as follows. Let us thus denote by L, L' and L'' the leaves in V, V' and V'' corresponding to a fixed $\gamma''\in\mathcal{G}''^{X''}_{X''}$, see comment after the proof. Then we can write

$$\langle\xi_f*\eta_g,\xi_f*\eta_g\rangle(\gamma'')=\sum_{x\in L\cap X}\sum_{\gamma_1''\in\mathcal{G}''^{gf(x)}_{r(\gamma'')}}\overline{\xi_f*\eta_g(x,\gamma_1'')}\xi_f*\eta_g(x,\gamma_1''\gamma'').$$

Replacing $\xi_f*\eta_g$ by its definition, we get

$$\langle\xi_f*\eta_g,\xi_f*\eta_g\rangle(\gamma'')=\sum_{x\in L\cap X}\sum_{\gamma_1''\in\mathcal{G}''^{gf(x)}_{r(\gamma'')}}\sum_{\gamma_1'\in\mathcal{G}'^{f(x)}_{X'}}\overline{\xi_f(x,\gamma_1')\eta_g(s(\gamma_1'),g(\gamma'^{-1}_1\gamma_1'')}\sum_{\gamma_2'\in\mathcal{G}'^{f(x)}_{X'}}\xi_f(x,\gamma_2')\eta_g(s(\gamma_2'),g(\gamma'^{-1}_2)\gamma_1''\gamma'').$$

We hence get

$$\langle\xi_f*\eta_g,\xi_f*\eta_g\rangle(\gamma'')=\sum_{x\in L\cap X}\sum_{x',y'\in L'\cap X'}\sum_{\gamma_1''\in\mathcal{G}''^{g(x')}_{r(\gamma'')}}\sum_{\gamma_2'\in\mathcal{G}'^{f(x)}_{x'}}\sum_{\alpha'\in\mathcal{G}'^{x'}_{y'}}\overline{\xi_f(x,\gamma_2')\eta_g(x',\gamma_1'')}\eta_g(y',g(\alpha')^{-1}\gamma_1''\gamma'')\xi_f(x,\gamma_2'\alpha').$$

In the sum over α' we set $\gamma_3'=\gamma_2'\alpha'$ and thus get

$$\langle\xi_f*\eta_g,\xi_f*\eta_g\rangle(\gamma'')=\sum_{x\in L\cap X}\sum_{x',y'\in L'\cap X'}\sum_{\gamma_1''\in\mathcal{G}''^{g(x')}_{r(\gamma'')}}\sum_{\gamma_2'\in\mathcal{G}'^{f(x)}_{x'}}\sum_{\gamma_3'\in\mathcal{G}'^{f(x)}_{y'}}\overline{\xi_f(x,\gamma_2')\eta_g(x',\gamma_1'')}\eta_g(y',g(\gamma_3')^{-1}g(\gamma_2')\gamma_1''\gamma'')\xi_f(x,\gamma_3').$$

Computing similarly $\langle \eta_g, \pi_g(\langle \xi_f, \xi_f \rangle)(\eta_g)\rangle(\gamma'')$ we get exactly the same expression.

We now need to check surjectivity. But Proposition 3.5 shows that the representation

$$\pi_{g\circ f} : C^*(\mathcal{G}_X^X) \longrightarrow \mathcal{K}_{C^*(\mathcal{G}_{X''}^{X''})}(\mathcal{E}_{X''}^X(g\circ f))$$

is a C*-algebra isomorphism. On the other hand, for any $\xi \in C_c(\mathcal{G}_{X''}^X(g\circ f))$, we have

$$\pi_{g\circ f}(\eta_1 \star \eta_2)\xi = \eta_1 * (\eta_2 \bullet \xi),$$

where

$$\eta_2 \bullet \xi(x', \gamma'') := \sum_{x\in L\cap X} \sum_{\gamma' \in \mathcal{G}'^{f(x)}_{x'}} \overline{\eta_2(x,\gamma')}\xi(x, g(\gamma')\gamma'').$$

We compute the left-hand side as follows:

$$\begin{aligned}
&\pi_{g\circ f}(\eta_1 \star \eta_2)\xi \\
&\quad = \sum_{\alpha\in \mathcal{G}_X^x} (\eta_1 \star \eta_2)(\alpha)\xi(s(\alpha), g\circ f(\alpha^{-1})\gamma'') \\
&\quad = \sum_{\alpha\in \mathcal{G}_X^x} \sum_{\alpha_1' \in \mathcal{G}'^{f(x)}_{X'}} \eta_1(x,\alpha_1')\overline{\eta_2(s(\alpha), f(\alpha^{-1})\alpha_1')}\xi(s(\alpha), g\circ f(\alpha^{-1})\gamma'').
\end{aligned} \tag{4.1}$$

Now computing the right-hand side we have, for any $\gamma' \in \mathcal{G}'^{f(x)}_{X'}$,

$$\begin{aligned}
&[\eta_1 * (\eta_2 \bullet \xi)](x, \gamma'') \\
&\quad = [\eta_1 * (\eta_2 \bullet \xi)][(x,\gamma'); (s(\gamma'), g(\gamma'^{-1})\gamma'')] \\
&\quad = \sum_{\alpha' \in \mathcal{G}'^{s(\gamma')}_{X'}} \eta_1(x, \gamma'\alpha')(\eta_2 * \xi)(s(\alpha'), g(\alpha'^{-1}\gamma'^{-1})\gamma'') \\
&\quad = \sum_{\alpha_1' \in \mathcal{G}'^{f(x)}_{X'}} \eta_1(x, \alpha_1')(\eta_2 * \xi)(s(\alpha'), g(\alpha_1'^{-1})\gamma'') \\
&\quad = \sum_{\alpha_1' \in \mathcal{G}'^{f(x)}_{X'}} \eta_1(x,\alpha_1') \sum_{x_1 \in X\cap L_{s(\alpha_1')}} \sum_{\gamma_1' \in \mathcal{G}'^{f(x_1)}_{s(\alpha_1')}} \overline{\eta_2(x_1,\gamma_1')}\xi(x_1, g(\gamma_1'\alpha_1'^{-1})\gamma'') \\
&\quad = \sum_{\alpha_1' \in \mathcal{G}'^{f(x)}_{X'}} \eta_1(x,\alpha_1') \sum_{x_1 \in X\cap L_{s(\alpha_1')}} \sum_{\gamma_2' \in \mathcal{G}'^{f(x_1)}_{f(x)}} \overline{\eta_2(x_1,\gamma_2'\alpha_1')}\xi(x_1, g(\gamma_2')\gamma'') \\
&\quad = \sum_{\alpha_1' \in \mathcal{G}'^{f(x)}_{X'}} \eta_1(x,\alpha_1') \sum_{x_1 \in X\cap L_x} \textstyle\sum_{\gamma_2 \in \mathcal{G}_x^{x_1}} \overline{\eta_2(x_1, f(\gamma_2)\alpha_1')}\xi(x_1, g\circ f(\gamma_2))\gamma'') \\
&\quad = \sum_{\alpha_1' \in \mathcal{G}'^{f(x)}_{X'}} \eta_1(x,\alpha_1') \sum_{\gamma_2 \in \mathcal{G}_x^X} \overline{\eta_2(r(\gamma_2), f(\gamma_2)\alpha_1')}\xi(r(\gamma_2), g\circ f(\gamma_2))\gamma'') \\
&\quad = \sum_{\alpha_1' \in \mathcal{G}'^{f(x)}_{X'}} \eta_1(x,\alpha_1') \sum_{\gamma \in \mathcal{G}_X^x} \overline{\eta_2(s(\gamma), f(\gamma^{-1})\alpha_1')}\xi(s(\gamma), g\circ f(\gamma^{-1}))\gamma'')
\end{aligned}$$

$$= \sum_{\alpha'_1 \in \mathcal{G}'^{f(x)}_{X'}} \sum_{\gamma \in \mathcal{G}^x_X} \eta_1(x, \alpha'_1) \overline{\eta_2(s(\gamma), f(\gamma^{-1})\alpha'_1)} \xi(s(\gamma), g \circ f(\gamma^{-1}))\gamma''). \quad (4.2)$$

Comparing (4.1) and (4.2) gives the desired equality. □

We have used in the previous proof the following standard results (see also [RoyPhD:10]):

- Let f be an oriented homotopy equivalence between the oriented closed manifolds V and V'. Then f is surjective.
- Assume that f is a leafwise homotopy equivalence between $(V, \mathcal{F})$ and $(V', \mathcal{F}')$. Then for any $(x, y) \in V^2$ lying in the same leaf the induced map between the monodromy groupoids $\mathcal{G}$ and $\mathcal{G}'$ restricts to a bijection between $\mathcal{G}^y_x$ and $\mathcal{G}'^{f(y)}_{f(x)}$.
- Assume that f is a leafwise homotopy equivalence between $(V, \mathcal{F})$ and $(V', \mathcal{F}')$. Then the inverse image of a given leaf is a single leaf.

Corollary 4.2. *Under the assumption of Proposition 4.1 we have an isomorphism*

$$\mathcal{E}^X_{X'}(f) \otimes_{C^*(\mathcal{G}'^{X'}_{X'})} \mathcal{E}^{X'}_X(g) \cong C^*(\mathcal{G}^X_X)$$

of Hilbert $C^*(\mathcal{G}^X_X)$-modules. In the same way we have an isomorphism

$$\mathcal{E}^{X'}_X(g) \otimes_{C^*(\mathcal{G}^X_X)} \mathcal{E}^X_{X'}(f) \cong C^*(\mathcal{G}'^{X'}_{X'})$$

of Hilbert $C^*(\mathcal{G}'^{X'}_{X'})$-modules.

Proof. We only prove the first isomorphism. Recall that $g \circ f$ is then leaf-preserving and that there exists a smooth leaf-preserving homotopy $H : V \times [0,1] \to V$ such that

$$H(0, \bullet) = \mathrm{id} \quad \text{and} \quad H(1, \bullet) = g \circ f.$$

Here the foliation on $V \times [0,1]$ is the one with leaves $L \times [0,1]$ where L is a leaf of $(V, \mathcal{F})$. For any $x \in V$ the homotopy class of the path $(H(t,x))_{0 \le t \le 1}$ is denoted by γ_x, so $\gamma_x \in \mathcal{G}^{gf(x)}_x$. We then consider the map

$$\Lambda : C_c(\mathcal{G}^X_X(g \circ f)) \to C_c(\mathcal{G}^X_X) \quad \text{given by} \quad \Lambda\xi(\gamma) := \xi(r(\gamma), \gamma_{r(\gamma)}\gamma).$$

A straightforward computation shows that Λ allows to identify the Hilbert $C^*(\mathcal{G}^X_X)$-modules $\mathcal{E}^X_X(g \circ f)$ and $C^*(\mathcal{G}^X_X)$. Applying Proposition 4.1, we conclude. □

For $0 \le s \le 1$, let $h_s := h \circ i_s$, where $i_s : V \hookrightarrow [0,1] \times V$ is the map $i_s(v) = (s, v)$. Then the arguments in the proof of Corollary 4.2 above give

Proposition 4.3. *$\mathcal{E}^X_X(h_s)$ is isomorphic to $C^*(\mathcal{G}^X_X)$ as a Hilbert module for all $0 \le s \le 1$.*

Proof. Since the techniques are the same, we shall be brief. Denote more generally for $0 \leq s \leq 1$ and for $x \in V$ by γ_x^s the homotopy class of the path $t \to h(t, x)$, $0 \leq t \leq s$. We define a map $\theta_h^s \colon C_c(\mathcal{G}_X^X) \to C_c(\mathcal{G}_X^X(h_s))$ by the formula

$$\theta_h^s(\xi)(x, \gamma) = \xi((\gamma_x^s)^{-1}\gamma) \quad \text{for } \xi \in C_c(\mathcal{G}_X^X).$$

Then a direct inspection shows that θ_h^s is $C_c(\mathcal{G}_X^X)$-linear. Moreover, θ_h^s is easily proved to be an isometry. To finish the proof, we set, for $\eta \in C_c(\mathcal{G}_X^X(g \circ f))$ and for $\gamma \in \mathcal{G}_X^X$,

$$(\theta_h^s)^*\eta(\gamma) = \eta(r(\gamma), \gamma_{r(\gamma)}^s\gamma).$$

Then $(\theta_h^s)^*\eta \in C_c(\mathcal{G}_X^X)$ and we have

$$\theta_h^s((\theta_h^s)^*\eta)(x, \gamma) = ((\theta_h^s)^*\eta)((\gamma_x^s)^{-1}\gamma) = \eta(x, \gamma_x^s(\gamma_x^s)^{-1}\gamma) = \eta(x, \gamma).$$

Thus, θ_h^s is surjective. □

Remark 4.4. The map $\mathcal{G}_X^X(\mathrm{id}_V) \xrightarrow{\pi_2} \mathcal{G}_X^X$ induced by the projection onto the second factor is a diffeomorphism.

We point out that for any Hilbert $C^*(\mathcal{G}_X^X)$-module $\mathcal{E}$ the Hilbert module $\mathcal{E} \otimes_{C^*(\mathcal{G}_X^X)} C^*(\mathcal{G}_X^X)$ is canonically isomorphic to $\mathcal{E}$. We denote this canonical isomorphism generically by δ.

Assume again that f is an oriented leafwise homotopy equivalence with homotopy inverse g and let us briefly describe the relation between our Morita Hilbert modules and those introduced by Connes and Skandalis. In [Co:81], [CoSk:84], the graph $\mathcal{G}(f)$ and a corresponding Hilbert module are defined. More precisely, the Connes–Skandalis graph is given by

$$\mathcal{G}(f) := \{(v, \alpha') \mid v \in V, \alpha' \in \mathcal{G}' \text{ and } f(x) = r(\alpha')\}.$$

$\mathcal{G}(f)$ is a right principal $\mathcal{G}'$-bundle, and it also has an action of $\mathcal{G}$ on the left. One then defines a Hilbert module $\mathcal{E}(f)$ that we shall call the Connes–Skandalis module. Let $\mathcal{E}_X$ be the Hilbert $C^*(\mathcal{G}_X^X)$-module which is the completion of $C_c^\infty(\mathcal{G}_X)$ with the inner product given by the same formula as in equation (2.1), p. 800, where the inner product inside the integral is now simply the $\mathbb{C}$-valued standard inner product. We define in the same way the Hilbert $C^*(\mathcal{G}'^{X'}_{X'})$-module $\mathcal{E}_{X'}$. Then the expected relation between the four Hilbert modules can be proved, see Corollary 5.1.10 of [RoyPhD:10]:

Proposition 4.5. *We have the following isomorphism of Hilbert $C^*(\mathcal{G}'^{X'}_{X'})$-modules*

$$\mathcal{E}_X \otimes_{C^*(\mathcal{G}_X^X)} \mathcal{E}_{X'}^X(f) \cong \mathcal{E}(f) \otimes_{C^*(\mathcal{G}')} \mathcal{E}_{X'}.$$

Proof. From Proposition 3.6 we have $\mathcal{E}_X \otimes_{C^*(\mathcal{G}_X^X)} \mathcal{E}_{X'}^X(f) \cong \mathcal{E}_{X'}(f)$. It suffices to show that $\mathcal{E}(f) \otimes_{C^*(\mathcal{G}')} \mathcal{E}_{X'} \cong \mathcal{E}_{X'}(f)$. We define a map $\upsilon\colon C_c^\infty(\mathcal{G}(f)) \otimes_{C_c^\infty(\mathcal{G}_X^X)} C_c^\infty(\mathcal{G}_{X'}) \to C_c^\infty(\mathcal{G}_{X'}(f))$ by

$$\upsilon(\xi \otimes \eta)(v, \gamma') := \int_{\alpha' \in \mathcal{G}'^{r(\gamma')}} \xi(v, \alpha')\eta(\alpha'^{-1}\gamma')d\lambda^{r(\gamma')}(\alpha')$$

for $\xi \in C_c^\infty(\mathcal{G}(f)), \eta \in C_c^\infty(\mathcal{G}_{X'})$.

One can check that the above map is well defined and the following property holds (see [RoyPhD:10]):

$$\langle \upsilon(\xi \otimes \eta), \upsilon(\xi \otimes \eta)\rangle = \langle \eta, \chi_m(\langle \xi, \xi\rangle)\eta\rangle.$$

Here $\chi_m\colon C^*(\mathcal{G}') \to \mathcal{K}(\mathcal{E}_{X'})$ is an isomorphism (cf. [HiSk:83]). The above property implies that υ induces an isometry on the level of Hilbert-modules $\upsilon\colon \mathcal{E}(f) \otimes_{C^*(\mathcal{G}_X^X)} \mathcal{E}_{X'} \to \mathcal{E}_{X'}(f)$.

Lastly, in order to prove surjectivity of υ, we proceed as follows (cf. [CoSk:84], Proposition 4.5). Since $\mathcal{E}(f)$ implements the Morita equivalence between $C^*(\mathcal{G})$ and $C^*(\mathcal{G}')$, we have an isomorphism $\pi_f : C^*(\mathcal{G}) \to \mathcal{K}(\mathcal{E}(f))$ ([HiSk:83], Corollary 7) given by the formula

$$\pi_f(\phi)\xi(v, \gamma') = \int_{\alpha \in \mathcal{G}^v} \phi(\alpha)\psi(s(\alpha), f(\alpha^{-1})\gamma')d\lambda^v(\alpha)$$

for $\phi \in C_c^\infty(\mathcal{G})$, $\psi \in C_c^\infty(\mathcal{G}(f))$. Let $\xi_1, \xi_2 \in C_c^\infty(\mathcal{G}(f))$. Let $\xi_1 \star \xi_2$ denote the function on $\mathcal{G}$ given by

$$\xi_1 \star \xi_2(\gamma) = \int_{\alpha' \in \mathcal{G}'^{f(r(\gamma))}} \xi_1(r(\gamma), \alpha')\overline{\xi_2(s(\gamma), f(\gamma^{-1})\alpha')}d\lambda^{f(r(\gamma))}(\alpha').$$

Denote by θ_{ξ_1,ξ_2} the operator in $\mathcal{K}(\mathcal{E}(f))$ given by $\theta_{\xi_1,\xi_2}\zeta := \xi_1\langle \xi_2, \zeta\rangle$. Then a direct calculation shows that $\theta_{\xi_1,\xi_2} = \pi_f(\xi_1 \star \xi_2)$. Now we also have a representation $\pi(f)\colon C^*(\mathcal{G}) \to \mathcal{K}(\mathcal{E}_{X'}(f))$ defined by

$$[\pi(f)(\phi)\psi](v, \gamma') = \int_{\alpha \in \mathcal{G}^v} \phi(\alpha)\psi(s(\alpha), f(\alpha^{-1})\gamma')d\lambda^v(\alpha)$$

for $\phi \in C_c^\infty(\mathcal{G})$, $\psi \in C_c^\infty(\mathcal{G}_{X'}(f))$. Now, as in Proposition 3.5, we can easily prove the equality

$$\theta_{\phi_1,\phi_2}\psi = \pi(f)(\phi_1 \star \phi_2)\psi \quad \text{for } \phi_1, \phi_2, \psi \in C_c^\infty(\mathcal{G}_{X'}(f)),$$

where

$$\phi_1 \star \phi_2(\alpha) = \sum_{\alpha_1' \in \mathcal{G}_{X'}'^{f(r(\alpha))}} \phi_1(r(\alpha), \alpha_1')\overline{\phi_2(s(\alpha), f(\alpha^{-1})\alpha_1)} \quad \text{for } \alpha \in \mathcal{G}.$$

This implies that $\pi(f)(C^*(\mathcal{G}))\mathcal{E}_{X'}(f)$ is dense in $\mathcal{E}_{X'}(f)$. Therefore, to prove surjectivity it suffices to show that an element of the form $\pi(f)(h)\xi$ lies in the range of υ for all $h \in C^*(\mathcal{G})$ and $\xi \in \mathcal{E}_{X'}(f)$. This is done by a straightforward calculation to prove that

$$\pi(f)(\xi_1 \star \xi_2)\kappa = \upsilon(\xi_1 \otimes (\xi_2 \bullet \kappa))$$

for $\xi_1, \xi_2 \in C_c^\infty(\mathcal{G}(f))$, $\kappa \in C_c^\infty(\mathcal{G}_{X'}(f))$, where

$$\xi_2 \bullet \kappa(\gamma') = \int_{v \in L_{\gamma'}} \sum_{\gamma_1' \in \mathcal{G}'^{f(v)}_{r(\gamma')}} \overline{\xi_2(v, \gamma_1')} \kappa(v, \gamma_1'\gamma') d\lambda^{L_{\gamma'}}(v) \quad \text{for } \gamma' \in \mathcal{G}'^{X'}_{X'}.$$

We refer the reader to Corollary 5.1.10 of [RoyPhD:10] for the detailed computations. □

Corollary 4.6. *$\mathcal{E}^X_{X'}(f)$ is an imprimitivity bimodule.*

Proof. We note that there is a canonical diffeomorphism $\theta : \mathcal{G}^X \times_{\mathcal{G}} \mathcal{G}_X \cong \mathcal{G}^X_X$, where

$$\mathcal{G}^X \times_{\mathcal{G}} \mathcal{G}_X := \{(\alpha, \beta), \alpha \in \mathcal{G}^X, \beta \in \mathcal{G}_X \mid s(\alpha) = r(\beta)\}/\sim$$

and the equivalence relation is given by $(\alpha, \beta) \sim (\alpha\gamma, \gamma^{-1}\beta)$ for any $\gamma \in \mathcal{G}$ such that $r(\gamma) = s(\alpha)$. The map θ above is given by

$$\theta([\alpha, \beta]) := \alpha\beta.$$

One can easily check that θ is bijective and smooth with smooth inverse. Moreover, this diffeomorphism induces an isomorphism of Hilbert C*-modules $\mathcal{E}^X \otimes_{C^*(\mathcal{G})} \mathcal{E}_X \cong C^*(\mathcal{G}^X_X)$. Therefore by tensoring with $\mathcal{E}^X$ on both sides of eq. (4.5) we get

$$C^*(\mathcal{G}^X_X) \otimes_{C^*(\mathcal{G}^X_X)} \mathcal{E}^X_{X'}(f) \cong \mathcal{E}^X \otimes_{C^*(\mathcal{G})} \mathcal{E}(f) \otimes_{C^*(\mathcal{G}^{X'}_{X'})} \mathcal{E}_{X'}.$$

Using the fact that $C^*(\mathcal{G}^X_X)$ is a unital C*-algebra, we have an isomorphism $C^*(\mathcal{G}^X_X) \otimes_{C^*(\mathcal{G}^X_X)} \mathcal{E}^X_{X'}(f) \cong \mathcal{E}^X_{X'}(f)$. Therefore we finally obtain

$$\mathcal{E}^X_{X'}(f) \cong \mathcal{E}^X \otimes_{C^*(\mathcal{G})} \mathcal{E}(f) \otimes_{C^*(\mathcal{G}^{X'}_{X'})} \mathcal{E}_{X'},$$

which concludes the proof since $\mathcal{E}^X$, $\mathcal{E}(f)$ and $\mathcal{E}_{X'}$ are known to be imprimitivity bimodules (cf. [HiSk:83], Proposition 4.7 of [La:95]). This also means that $\pi_f : C^*(\mathcal{G}^X_X) \to \mathcal{K}(\mathcal{E}^X_{X'}(f))$ is an isomorphism, thus giving an alternative proof of the statement in Proposition 3.5. □

4.2. Poincaré lemma for foliations. Again $f\colon (V,\mathcal{F}) \to (V',\mathcal{F}')$ is a smooth oriented leafwise homotopy equivalence with homotopy inverse g. We denote by $h\colon [0,1]\times V \to V$ and $h'\colon [0,1]\times V' \to V'$ the oriented smooth leaf-preserving homotopies between id_V and $g\circ f$ on the one hand, and between $\mathrm{id}_{V'}$ and $f\circ g$ on the other hand. Thus,

$$h(0,\bullet) = \mathrm{id}_V, \quad h(1,\bullet) = g\circ f, \quad h'(0,\bullet) = \mathrm{id}_{V'} \quad \text{and} \quad h'(1,\bullet) = f\circ g.$$

The goal of the present subsection is to prove the following

Theorem 4.7. *The maps f^*_ϕ and g^*_ϕ induce isomorphisms in cohomology, which are inverse of each other when we identify cohomologies using the Morita isomorphisms described in Corollary 4.2. Said differently, an oriented leafwise homotopy equivalence induces a homotopy of the corresponding HP complexes.*

We set, as before, $h_s(v) := h(s,v)$ and $h'_s(v') := h'(s,v')$ for any $s\in[0,1]$. Recall also that we have defined isometric isomorphisms of Hilbert $C^*(\mathcal{G}^X_X)$-modules (resp. Hilbert $C^*(\mathcal{G}'^{X'}_{X'})$-modules):

$$\theta^s_h\colon C^*(\mathcal{G}^X_X) \to \mathcal{E}^X_X(h_s) \quad (\text{resp. } \theta^s_{h'}\colon C^*(\mathcal{G}'^{X'}_{X'}) \to \mathcal{E}^{X'}_{X'}(h'_s)).$$

The notations θ_h and $\theta_{h'}$ stand for θ^1_h and $\theta^1_{h'}$, respectively.

Lemma 4.8. *The map θ^s_h is equivariant with respect to the left representation of the C^*-algebra $C^*(\mathcal{G}^X_X)$ and the well defined composite map*

$$\rho^s_h := \delta\circ(I\otimes_{C^*(\mathcal{G}^X_X)}(\theta^s_h)^{-1})\colon \mathcal{E}_{X,E}\otimes_{C^*(\mathcal{G}^X_X)}\mathcal{E}^X_X(h_s) \longrightarrow \mathcal{E}_{X,E}$$

commutes with the differential on $\mathcal{E}_{X,E}$.

Proof. Let $\phi,\xi\in C_c(\mathcal{G}^X_X)$. Then

$$\theta^s_h(\phi*\xi)(x,\gamma) = (\phi*\xi)((\gamma^s_x)^{-1}\gamma) = \sum_{\gamma_1\in\mathcal{G}^x_X}\phi(\gamma_1)\xi(\gamma_1^{-1}(\gamma^s_x)^{-1}\gamma).$$

On the other hand,

$$\pi(\phi)\theta^s_h(\xi)(x,\gamma) = \sum_{\alpha\in\mathcal{G}^x_X}\phi(\alpha)\xi((\gamma^s_{s(\alpha)})^{-1}\circ(f\circ g)(\alpha^{-1})\circ\gamma).$$

Putting $\gamma_1 = (\gamma^s_x)^{-1}(f\circ g)(\alpha)\gamma^s_{s(\alpha)}$ we see from the proof of Proposition 5.1.7 of [RoyPhD:10] that $\gamma_1=\alpha$ and so

$$\pi(\phi)\theta^s_h(\xi)(x,\gamma) = \sum_{\gamma_1\in\mathcal{G}^x_X}\phi(\gamma_1)\xi(\gamma_1^{-1}(\gamma^s_x)^{-1}\gamma).$$

To see that ρ_h^s is a chain map, we compute

$$\rho_h^s \circ (\tilde{d} \otimes I) = \delta \circ (I \otimes (\theta_h^s)^{-1}) \circ (\tilde{d} \otimes I) = \delta \circ (\tilde{d} \otimes I) \circ (I \otimes (\theta_h^s)^{-1}).$$

But since $\tilde{d}$ is $C_c^\infty(\mathcal{G}_X^X)$-linear, we have

$$\delta \circ (\tilde{d} \otimes I) = \tilde{d} \circ \delta.$$

Therefore from the two computations above we get the desired result. □

Let $X_0 := \{0\} \times X \subset [0,1] \times V$ be the complete transversal of the product foliation on $[0,1] \times V$. We denote for simplicity this foliation by $[0,1] \times \mathcal{F}$. Let $\widehat{\mathcal{G}}$ be the monodromy groupoid of the foliation $([0,1] \times V, [0,1] \times \mathcal{F})$. Then $\widehat{\mathcal{G}}$ can be, and will be, identified as a smooth groupoid with the cartesian product $\mathcal{G} \times [0,1]^2$ of the groupoid $\mathcal{G}$ with the product groupoid $[0,1]^2$.

Define maps ϵ_h and Ψ_h for the leafwise map h, as we did in Section 4 for f. We note that using a similar proof as for Lemma 4.3, one has an isomorphism

$$\rho_h \colon \mathcal{E}_{X_0,\hat{E}} \otimes_{C^*(\hat{\mathcal{G}}_{X_0}^{X_0})} \mathcal{E}_X^{X_0}(h) \longrightarrow \mathcal{E}_{X_0,\hat{E}}$$

where $\hat{E} := \bigwedge^* T_{\mathbb{C}}^*([0,1] \times \mathcal{F})$. We set $H^* := \rho_h \circ \epsilon_h \circ \Psi_h$, so this is the map from $\mathcal{E}_{X,E}$ to $\mathcal{E}_{X_0,\hat{E}}$ given by

$$\mathcal{E}_{X,E} \xrightarrow{\Psi_h} \mathcal{E}_{X,\hat{E}}(h) \xrightarrow{\epsilon_h} \mathcal{E}_{X_0,\hat{E}} \otimes_{C^*(\hat{\mathcal{G}}_{X_0}^{X_0})} \mathcal{E}_X^{X_0}(h) \xrightarrow{\rho_h} \mathcal{E}_{X_0,\hat{E}}.$$

Notice that $\widehat{\mathcal{G}}_{X_0}$ is identified with $[0,1] \times \mathcal{G}_X$ while $\widehat{\mathcal{G}}_{X_0}^{X_0}$ is identified with $\mathcal{G}_X^X$. We finally get as in Section 4 a well-defined adjointable chain map

$$H_\phi^\sharp := H^* \circ \phi(\Delta) = \rho_h \circ H_\phi^* \colon \mathcal{E}_{X,E} \longrightarrow \mathcal{E}_{X_0,\hat{E}}.$$

We define in the same way $(h_s)_\phi^\sharp \colon \mathcal{E}_{X,E} \to \mathcal{E}_{X,E}$ for any $s \in [0,1]$. Notice that integration over $(0,1)$ yields a well-defined degree -1 linear map

$$\int_{(0,1)} \colon C_c^\infty(\hat{\mathcal{G}}_{X_0}, r^*\hat{E}) \to C_c^\infty(\mathcal{G}_X, r^*E).$$

It is clear from its very definition that $\int_{(0,1)}$ is $C_c^\infty(\mathcal{G}_X^X)$-linear when we identify as we did $\hat{\mathcal{G}}_{X_0}^{X_0}$ with $\mathcal{G}_X^X$. Moreover, a direct inspection shows that $\int_{(0,1)}$ extends to an adjointable operator

$$\int_{(0,1)} \colon \mathcal{E}_{X_0,\hat{E}} \to \mathcal{E}_{X,E},$$

with the adjoint given by the extension of $\eta \mapsto \eta \wedge dt$ where t is the variable in [0, 1] and η is understood via its usual pull-back. We set

$$K_\phi^\sharp := \int_{(0,1)} \circ H_\phi^\sharp \colon \mathcal{E}_{X,E} \to \mathcal{E}_{X,E}.$$

So $K_\phi^\sharp$ is an adjointable operator.

Lemma 4.9. *The relation*

$$(g \circ f)_\phi^\sharp - \phi(\Delta) = K_\phi^\sharp \circ d_X + d_X \circ K_\phi^\sharp$$

holds.

Proof. First notice that for any $x \in X$, the map

$$H_x^* \colon C_c^\infty(\mathcal{G}_x, r^* E) \longrightarrow C_c^\infty(\widehat{\mathcal{G}}_{(0,x)}, r^* \hat{E}) = C_c^\infty([0,1] \times \mathcal{G}_x, r^* \hat{E}).$$

coincides with the composite map

$$\begin{aligned} C_c^\infty(\mathcal{G}_X, r^* E) &\xrightarrow{\Psi_h} C_c^\infty(\mathcal{G}_X^{[0,1]\times V}(h), (r \circ \mathrm{pr}_1)^* \hat{E}) \\ &\xrightarrow{\Phi_h} C_c^\infty(\widehat{\mathcal{G}}_{X_0} \times_{\widehat{\mathcal{G}}_{X_0}^{X_0}} \mathcal{G}_X^{X_0}(h), (r \circ \mathrm{pr}_1)^* \hat{E}) \xrightarrow{\hat{\rho}_h} C_c^\infty(\widehat{\mathcal{G}}_{X_0}, \hat{E}), \end{aligned}$$

where we set $\hat{\rho}_h = \rho_h \circ \nu_h^{-1}$ and use the notations of the previous section. We compute

$$\begin{aligned} (\Phi_h \circ \Psi_h)(\xi)[(t,\gamma),(s(t,\gamma),\alpha)] &= \Psi_h(\xi)(r(t,\gamma), h(t,\gamma)\alpha) \\ &= ({}^t h_*)_{(t,r(\gamma))}(\xi(h(t,\gamma)\alpha)). \end{aligned}$$

Therefore, since $\theta_h \colon \mathcal{G}_X^{X_0}(h) \to \mathcal{G}_{X_0}^{X_0}$ is given by $((0,x),\gamma) \mapsto (0,0,\gamma)$, we get

$$\begin{aligned} (I \otimes \theta_h^{-1}) \circ (\epsilon_h \circ \Psi_h)(\xi)[(t,\gamma),(0,\alpha)] &= (\Phi_h \circ \Psi_h)(\xi)[(t,\gamma),(s(t,\gamma),(0,\alpha))] \\ &= ({}^t h_*)_{(t,r(\gamma)}(\xi(h(t,\gamma)\alpha)). \end{aligned}$$

Hence,

$$\begin{aligned} H^*(\xi)(t,\gamma) &= (I \otimes \theta_h^{-1})(\Phi_h \circ \Psi_h)(\xi)[(t,\gamma), 1_{(0,\gamma)}] \\ &= ({}^t h_*)_{(t,r(\gamma))}(\xi(h(t,\gamma)1_{0,\gamma})) \\ &= ({}^t h_*)_{(t,r(\gamma))}(\xi(\theta(t,\gamma)\gamma_{s(\gamma)}^t)). \end{aligned}$$

We have used the following relations

$$\begin{gathered} h(t,\gamma) := h(t,0,\gamma) = \gamma_{r(\gamma)}^t \gamma, \quad h(t,s,\gamma) = \gamma_{r(\gamma)}^t \gamma (\gamma_{s(\gamma)}^s)^{-1}, \\ \theta(t,\gamma) := \gamma_{r(\gamma)}^t \gamma (\gamma_{s(\gamma)}^t)^{-1}, \quad \gamma_{r(\gamma)}^s \gamma = h_t(\gamma) \gamma_{s(\gamma)}^s. \end{gathered}$$

Therefore, H_x^* is simply given by the formula

$$H_x^*(\xi)(t,\gamma) = ({}^t h_*)_{(t,r(\gamma)}(\xi(\theta(t,\gamma)\circ\gamma_{s(\gamma)}^t)), \tag{4.3}$$

which explains the notation. Now, computing in local coordinates, we check that the usual Poincaré equation holds. More precisely, in our setting, we have

$$\tilde{d}_x \int_{(0,1)} \xi + \int_{(0,1)} d_{\hat{\mathcal{G}}_{(0,x)}} \xi = \xi_1 \circ i_1 - \xi_1 \circ i_0$$

for $\xi = \xi_1 + dt \wedge \xi_2$ with $\xi_1, \xi_2 \in C_c^\infty(\hat{\mathcal{G}}_x, r^*\hat{E})$. Here $i_s : \mathcal{G}_x \hookrightarrow \hat{\mathcal{G}}_{(0,x)}$ is the map $i_s(\gamma) := (s, 0, \gamma)$. Applying this relation to $\xi = H_x^*(\eta)$ for $\eta \in C_c^\infty(\mathcal{G}_x, r^*E)$, we get

$$\int_{(0,1)} d_{\hat{\mathcal{G}}_{(0,x)}} H_x^*(\eta) + \tilde{d}_x\Big(\int_{(0,1)} H_x^*(\eta)\Big) = H^*(\eta)_1|_{\{1\}\times\mathcal{G}_x} - H^*(\eta)_1|_{\{0\}\times\mathcal{G}_x}.$$

Here we have used the notation $H^*\eta = (H^*\eta)_1 + dt \wedge (H^*\eta)_2$.

But we know from eq. (4.3) that

$$H^*(\eta)(1,\gamma) = ({}^t h_*)_{(1,r(\gamma))}[\eta(\theta(1,\gamma)\circ\gamma_{s(\gamma)})]$$

and

$$h_1^*(\eta)(\gamma) = ({}^t h_{1,*})_{r(\gamma)}[\eta(h_1(\gamma)\circ\gamma_{s(\gamma)})].$$

To finish the proof, we thus only need to check that $H^*(\eta)_1|_{(1,\gamma)} = h_1^*(\eta)(\gamma)$, for then we deduce

$$h_1^*(\eta) - h_0^*(\eta) = \int_{(0,1)} d_{\hat{\mathcal{G}}_{(0,x)}} H_x^*(\eta) + \tilde{d}_x\Big(\int_{(0,1)} H_x^*(\eta)\Big),$$

and composing with $\phi(\Delta)$ on the right would end the proof. Note that as H^* is a chain map, we also have $d_{\hat{\mathcal{G}}_{(0,x)}} \circ H_x^* = H_x^* \circ \tilde{d}_x$.

Now to prove that $H^*(\eta)_1|_{(1,\gamma)} = h_1^*(\eta)(\gamma)$, we first notice that if $\xi = \xi_1 + \xi_2 \wedge dt$ then $\xi_1 = ({}^t i_{1,*})\xi$. As $h_1 = h \circ i_1$, we have $({}^t i_{1,*})_x \circ ({}^t h_*)_{(1,x)} = ({}^t h_{1,*})_x$, and so

$$\begin{aligned} H^*(\eta)_1|_{(1,\gamma)} &= ({}^t i_{1,*})_{r(\gamma)}(H^*(\eta)(1,\gamma)) \\ &= ({}^t i_{1,*})_{r(\gamma)} \circ ({}^t h_*)_{(1,r(\gamma))}(\eta(h(1,\gamma)\gamma_{s(\gamma)}) \\ &= ({}^t h_{1,*})_{r(\gamma)}(\eta(h(1,\gamma)\gamma_{s(\gamma)}) \\ &= h_1^*(\eta)(\gamma), \end{aligned}$$

and the proof is thus complete. □

Corollary 4.10. *$(g\circ f)_\phi^\sharp$ and $(f\circ g)_\phi^\sharp$ induce the identity maps on the cohomologies of the complexes $(\mathcal{E}_X, d_X)$ and $(\mathcal{E}'_{X'}, d'_{X'})$, respectively.*

Proof. The first assertion is immediate from the previous proposition and the fact that $\phi(\Delta)$ induces the identity on cohomology, while $K_\phi^\sharp \circ d_X + d_X \circ K_\phi^\sharp$ is zero on cohomology. The second one is a consequence of the first one permuting the roles of f and g. □

4.3. Compatibility with Poincaré duality. In this section we prove the compatibility of the pullback map f_ϕ^* with the Poincaré duality operators $T'_{X'}$ and $T_X \otimes \mathrm{id}$ of the HP complexes $(\mathcal{E}_{X',E'}, d'_{X'})$ and $(\mathcal{E}_{X,E} \otimes \mathcal{E}^X_{X'}(f), d_X \otimes \mathrm{id})$, respectively, in order to achieve the proof that the map f_ϕ^* is indeed a homotopy equivalence of HP complexes, as per Definition 2.11. Let $(f_\phi^*)^\sharp$ be the adjoint of f_ϕ^*. We will show that

Proposition 4.11. *The maps* $f_\phi^* T'_{X'} (f_\phi^*)^\sharp$ *and* $T_X \otimes \mathrm{id}$ *induce the same map on cohomology.*

Proof. Recall that $f_\phi^* = \epsilon_f \Psi_f^\phi$ with $\Psi_f^\phi = \Psi_f \circ \phi(\Delta') \colon \mathcal{E}_{X',E'} \to \mathcal{E}_{X',E}(f)$. We note that the map $\Psi_f^\phi \colon C_c^\infty(\mathcal{G}'_{X'}, r^*E') \to C_c^\infty(\mathcal{G}^V_{X'}(f), \pi_1^* E)$ can also be described as $\pi_{2,f}^* \circ \phi(\Delta')$, where $\pi_{2,f}^*$ is the leafwise pullback map associated with $\pi_{2,f} = [(\pi_{2,f})_{x'} \colon \mathcal{G}^V_{x'}(f) \to \mathcal{G}'_{x'}]$, where $(\pi_{2,f})_{x'}(v, \gamma') = \gamma'$. Similarly, we define $\pi_{2,g}$ and its induced map $\pi_{2,g}^*$ associated with the map g. We shall denote the composition $\pi_{2,f}^* \circ \phi(\Delta')$ as $\pi_{2,f,\phi}^*$.

Now $\pi_{2,f}$ is a leafwise homotopy equivalence for the induced foliations on $\mathcal{G}^V_{X'}(f)$ and $\mathcal{G}'_{X'}$ with a homotopy inverse given for instance by $\lambda_f \colon \gamma' \mapsto (g(r(\gamma')), (f \circ g)(\gamma'))$. As a consequence, we have

$$\int_{\mathcal{G}^V_{x'}(f)} \Psi_f(\omega') = \int_{\mathcal{G}^V_{x'}(f)} (\pi_{2,f}^*)_{x'}(\omega') = \int_{\mathcal{G}'_{x'}} \omega',$$

for any leafwise top dimensional closed differential form ω' on $\mathcal{G}'_{x'}$. The map λ_f induces a pullback on forms, which we denote by λ_f^*. Similarly, one can define a map λ_g and its induced map λ_g^* associated with the map g. We shall denote the composition $\phi(\Delta') \circ \lambda_f^*$ as $\lambda_{f,\phi}^*$ and $\phi(\Delta) \circ \lambda_g^*$ as $\lambda_{g,\phi}^*$.

Consider the following map $\Gamma \colon \mathcal{E}_{X,E} \otimes \mathcal{E}^X_{X'}(f) \to \mathcal{E}_{X',E'}$ given by the composition

$$\begin{aligned}\mathcal{E}_{X,E} \otimes \mathcal{E}^X_{X'}(f) &\xrightarrow{T_X \otimes \mathrm{id}} \mathcal{E}_{X,E} \otimes \mathcal{E}^X_{X'}(f) \\ &\xrightarrow{g_\phi^* \otimes \mathrm{id}} \mathcal{E}_{X',E'} \otimes \mathcal{E}^{X'}_X(g) \otimes \mathcal{E}^X_{X'}(f) \xrightarrow{\Lambda'} \mathcal{E}_{X',E'} \xrightarrow{T'_{X'}} \mathcal{E}_{X',E'},\end{aligned}$$

where $\Lambda' = \rho^1_{h'} \circ (\mathrm{id} \times (\Xi^X_{f,g})^{-1})$ and $\rho^s_{h'}$ is the isometric isomorphism of Hilbert modules defined analogous to ρ^s_h for $s \in [0,1]$, given in Lemma 4.8.

We claim that

$$\Gamma = (-1)^{k(p-k)} (f_\phi^*)^\sharp \quad \text{on degree } k\text{-cohomology.} \tag{4.4}$$

Assuming the claim (4.4), it is easy to conclude the proof. Indeed, we then have on cohomology of degree k:

$$\begin{aligned} f_\phi^* T'_{X'} (f_\phi^*)^\sharp &= f_\phi^* T'_{X'} [(-1)^{k(p-k)} \Gamma] \\ &= (-1)^{k(p-k)} f_\phi^* T'_{X'} T'_{X'} \rho^1_{h'} (\mathrm{id} \times (\Xi^X_{f,g})^{-1}) (g_\phi^* \otimes \mathrm{id})(T_X \otimes \mathrm{id}). \end{aligned}$$

Since on cohomology we have, by Theorem 3.18, $(g_\phi^*\otimes \mathrm{id}) f_\phi^* = (\mathrm{id}\otimes \Xi_{f,g}^X)\circ(f\circ g)_\phi^*$, so $g_\phi^*\otimes\mathrm{id} = (\mathrm{id}\otimes\Xi_{f,g}^X)\circ(f\circ g)_\phi^*\circ(f_\phi^*)^{-1}$ on cohomology. Hence we have again on cohomology

$$
\begin{aligned}
&f_\phi^*[(-1)^{k(p-k)}(T'_{X'})^2]\rho_{h'}^1(\mathrm{id}\times(\Xi_{f,g}^X)^{-1})(g_\phi^*\otimes\mathrm{id})(T_X\otimes\mathrm{id})\\
&\quad = f_\phi^*\circ\rho_{h'}^1\circ(\mathrm{id}\times(\Xi_{f,g}^X)^{-1})(\mathrm{id}\otimes\Xi_{f,g}^X)\circ(f\circ g)_\phi^*\circ(f_\phi^*)^{-1}(T_X\otimes\mathrm{id})\\
&\quad = f_\phi^*\circ\rho_{h'}^1\circ(f\circ g)_\phi^*\circ(f_\phi^*)^{-1}(T_X\otimes\mathrm{id}).
\end{aligned}
$$

But, we know from the Poincaré Lemma for foliations that $\rho_{h'}^1\circ(f\circ g)_\phi^*$ induces the identity on cohomology. Therefore from the last line above we get the desired equality $f_\phi^* T'_{X'}(f_\phi^*)^\sharp = T_X\otimes\mathrm{id}$ on cohomology.

It thus remains to prove (4.4). We compute for $\psi_1\in\mathcal{E}_{X,E}$, $\psi_2\in\mathcal{E}_{X'}^X(f)$, $\eta_1\in\mathcal{E}_{X'E'}$, $\eta_2\in\mathcal{E}_X^{X'}(g)$ and $\eta_3\in\mathcal{E}_{X'}^X(f)$,

$$
\begin{aligned}
&\langle\psi_1\otimes\psi_2, f_\phi^*\Lambda'(\eta_1\otimes\eta_2\otimes\eta_3)\rangle\\
&\quad=\langle\psi_1\otimes\psi_2,\epsilon_f\Psi_f^\phi\Lambda'(\eta_1\otimes\eta_2\otimes\eta_3)\rangle\\
&\quad=\langle\psi_1\otimes\psi_2,\epsilon_f\Psi_f^\phi\rho_{h'}^1(\mathrm{id}\otimes(\Xi_{f,g}^X)^{-1})(\eta_1\otimes\eta_2\otimes\eta_3)\rangle.
\end{aligned}
$$

But we have $f_\phi^*\circ\rho_{h'}^1=\rho_{h'}^1\circ(f_\phi^*\otimes\mathrm{id})$. Therefore we get on cohomology

$$
\begin{aligned}
&\langle\psi_1\otimes\psi_2,\epsilon_f\Psi_f^\phi\rho_{h'}^1(\mathrm{id}\otimes(\Xi_{f,g}^X)^{-1})(\eta_1\otimes\eta_2\otimes\eta_3)\rangle\\
&\quad=\langle\psi_1\otimes\psi_2,\rho_{h'}^1(\epsilon_f\Psi_f^\phi\otimes\mathrm{id})(\mathrm{id}\otimes(\Xi_{f,g}^X)^{-1})(\eta_1\otimes\eta_2\otimes\eta_3)\rangle\\
&\quad=\langle\psi_1\otimes\psi_2,\rho_{h'}^1(\mathrm{id}\otimes(\Xi_{f,g}^X)^{-1})(\epsilon_f\Psi_f^\phi\otimes\mathrm{id})(\eta_1\otimes\eta_2\otimes\eta_3)\rangle\\
&\quad=\langle\psi_1\otimes\psi_2,\rho_{h'}^1((g\circ f)_\phi^*(g_\phi^*)^{-1})(\eta_1\otimes\eta_2\otimes\eta_3)\rangle\\
&\quad=\langle\psi_1\otimes\psi_2,((\Psi_g^\phi)^{-1}\epsilon_g^{-1}\otimes\mathrm{id})(\eta_1\otimes\eta_2\otimes\eta_3)\rangle\\
&\quad=\langle\psi_1\otimes\psi_2,\lambda_{g,\phi}^*\epsilon_g^{-1}(\eta_1\otimes\eta_2)\otimes\eta_3\rangle\\
&\quad=\langle\epsilon_g\circ(\lambda_{g,\phi}^*)^\sharp\psi_1\otimes\psi_2,\eta_1\otimes\eta_2\otimes\eta_3\rangle,
\end{aligned}
$$

where $(\lambda_{g,\phi}^*)^\sharp$ is the adjoint of the operator $\lambda_{g,\phi}^*$ defined above. To compute $(\lambda_{g,\phi}^*)^\sharp$, we recall that the inner product on $C_c^\infty(\mathcal{G}_X^{V'}(g),\pi_1^*E')$ is given by

$$
\langle\xi_1,\xi_2\rangle(\gamma)=\int_{\mathcal{G}_{r(\gamma)}^{V'}(g)}\xi_1\wedge\star'_g\xi_2
$$

for $\xi_1,\xi_2\in C_c^\infty(\mathcal{G}_X^{V'}(g),\pi_1^*E')$, where $\star'_g$ is the Hodge $\star$-operator on $\mathcal{G}_X^{V'}(g)$. Now let $\eta\in C_c^\infty(\mathcal{G}_X^{V'}(g),\pi_1^*E')$ and $\xi\in C_c^\infty(\mathcal{G}_X, r^*E)$ be closed k-forms. Then we have

$$
\begin{aligned}
\langle \lambda_{g,\phi}^* \eta, \xi \rangle(\gamma) &= \int_{\mathscr{G}_{r(\gamma)}} (\lambda_{g,\phi}^*)_{r(\gamma)} \eta \wedge \star \xi \\
&= \int_{\mathscr{G}_{r(\gamma)}} (\lambda_{g,\phi}^*)_{r(\gamma)} \eta \wedge (\lambda_{g,\phi}^*)_{r(\gamma)} \circ (\pi_{2,g,\phi}^*)_{r(\gamma)} \star \xi \\
&= \int_{\mathscr{G}_{r(\gamma)}} (\lambda_{g,\phi}^*)_{r(\gamma)} (\eta \wedge (\pi_{2,g,\phi}^*)_{r(\gamma)} \star \xi) \\
&= \int_{\mathscr{G}_{r(\gamma)}^{V'}(g)} \eta \wedge (\pi_{2,g,\phi}^*)_{r(\gamma)} \star \xi \\
&= \int_{\mathscr{G}_{r(\gamma)}^{V'}(g)} \eta \wedge \star_g' ((-1)^{k(p-k)} \star_g' (\pi_{2,g,\phi}^*)_{r(\gamma)} \star) \xi \\
&= \langle \eta, (-1)^{k(p-k)} \star_g' (\pi_{2,g,\phi}^*)_{r(\gamma)} \star) \xi \rangle (\gamma),
\end{aligned}
$$

where in the above computation we have used the fact that $\pi_{2,g,\phi}^*$ is the inverse of $\lambda_{g,\phi}^*$ on cohomology and since λ_g is a homotopy equivalence, $\lambda_{g,\phi}^*$ preserves fundamental cycles. Therefore, $(\lambda_{g,\phi}^*)_x^\sharp = (-1)^{k(p-k)} \star_g' (\pi_{2,g,\phi}^*)_x \star$, and so it induces an adjoint on the Hilbert-modules given by $(\lambda_{g,\phi}^*)^\sharp = (-1)^{k(p-k)} T_g' \pi_{2,g,\phi}^* T_X = (-1)^{k(p-k)} T_g' \Psi_g^\phi T_X$. Thus we have

$$
\begin{aligned}
&\langle \psi_1 \otimes \psi_2, \epsilon_f \Psi_f^\phi \rho_{h'}^1 (\mathrm{id} \otimes (\Xi_{f,g}^X)^{-1})(\eta_1 \otimes \eta_2 \otimes \eta_3) \rangle \\
&\quad = \langle \epsilon_g \circ ((-1)^{k(p-k)} T_g' \Psi_g^\phi T_X) \psi_1 \otimes \psi_2, \eta_1 \otimes \eta_2 \otimes \eta_3 \rangle \\
&\quad = (-1)^{k(p-k)} \langle (\epsilon_g \circ T_g' \otimes \mathrm{id})(\Psi_g^\phi \otimes \mathrm{id})(T_X \psi_1 \otimes \psi_2), \eta_1 \otimes \eta_2 \otimes \eta_3 \rangle.
\end{aligned}
$$

Now a similar computation gives

$$
\begin{aligned}
&\langle (-1)^{k(p-k)} [T_{X'}' \Lambda' (g_\phi^* \otimes \mathrm{id})(T_X \otimes \mathrm{id})](\psi_1 \otimes \psi_2), \Lambda'(\eta_1 \otimes \eta_2 \otimes \eta_3) \rangle \\
&\quad = \langle [\Lambda' (T_g' \otimes \mathrm{id})(\epsilon_g \Psi_g^\phi \otimes \mathrm{id})(T_X \otimes \mathrm{id})](\psi_1 \otimes \psi_2), \Lambda'(\eta_1 \otimes \eta_2 \otimes \eta_3) \rangle \\
&\quad = (-1)^{k(p-k)} \langle (T_g' \otimes \mathrm{id})(\epsilon_g \Psi_g^\phi \otimes \mathrm{id})(T_X \otimes \mathrm{id})(\psi_1 \otimes \psi_2), \eta_1 \otimes \eta_2 \otimes \eta_3 \rangle \\
&\quad = (-1)^{k(p-k)} \langle (\epsilon_g T_g' \otimes \mathrm{id})(\Psi_g^\phi \otimes \mathrm{id})(T_X \otimes \mathrm{id})(\psi_1 \otimes \psi_2), \eta_1 \otimes \eta_2 \otimes \eta_3 \rangle,
\end{aligned}
$$

where in the above computations we have used the facts that Λ' is an isometric isomorphism and ϵ_g intertwines the Poincaré duality operators. The above computations thus yield the equality on closed k-forms,

$$
\begin{aligned}
&\langle \psi_1 \otimes \psi_2, f_\phi^* \Lambda'(\eta_1 \otimes \eta_2 \otimes \eta_3) \rangle_{\mathcal{E}_{X,E} \otimes \mathcal{E}_{X'}^X(f)} \\
&\quad = (-1)^{k(p-k)} \langle [T_{X'}' \Lambda' (g_\phi^* \otimes \mathrm{id})(T_X \otimes \mathrm{id})](\psi_1 \otimes \psi_2), \Lambda'(\eta_1 \otimes \eta_2 \otimes \eta_3) \rangle_{\mathcal{E}_{X',E'}}.
\end{aligned}
$$

Thus the proof of the proposition is complete. □

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Received May 19, 2011; revised November 9, 2012

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