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$\mathcal{Z}$ is universal

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Abstract. We use order zero maps to express the Jiang–Su algebra $\mathcal{Z}$ as a universal C*-algebra on countably many generators and relations, and we show that a natural deformation of these relations yields the stably projectionless algebra $\mathcal{W}$ studied by Kishimoto, Kumjian and others. Our presentation is entirely explicit and involves only *-polynomial and order relations.

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1. Introduction

In Elliott's programme to classify simple, nuclear C*-algebras using K-theoretic invariants, the Jiang–Su algebra $\mathcal{Z}$ plays a particularly prominent role (see [18]). While there are various ways of characterizing $\mathcal{Z}$ (see for example [4] and [14]), its most concise description (due to the second named author, in [20]) is as the unique initial object in the category of strongly self-absorbing C*-algebras. Here, a separable, unital C*-algebra $\mathcal{D} \neq \mathbb{C}$ is *strongly self-absorbing* if there is an isomorphism $\varphi : \mathcal{D} \to \mathcal{D} \otimes \mathcal{D}$ that is approximately unitarily equivalent to the first factor embedding, cf. [16]. The statement that $\mathcal{Z}$ is an initial object in this category is equivalent to saying that every strongly self-absorbing C*-algebra absorbs $\mathcal{Z}$ tensorially (i.e. is '$\mathcal{Z}$-stable').

Apart from $\mathcal{Z}$, the known strongly self-absorbing algebras are: the Cuntz algebras $\mathcal{O}_2$ and $\mathcal{O}_\infty$, UHF algebras of infinite type, and such UHF algebras tensored with $\mathcal{O}_\infty$. These all admit presentations as universal C*-algebras (see Section 5 for a discussion), and Theorem 3.1 of this article provides such a description for $\mathcal{Z}$ which, although complicated, is explicit and algebraic in the sense that it involves only *-polynomial and order relations. The proof relies on the 'order zero' presentations of prime dimension drop algebras described in [14] (see Section 2), and gives a

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construction of $\mathcal{Z}$ as an inductive limit of such algebras with connecting maps defined in terms of generators and relations.

The Jiang–Su algebra may be thought of as a stably finite analogue of $\mathcal{O}_\infty$, and the C*-algebra $\mathcal{W}$ constructed in [3] (and studied in another form in [5]) has been similarly proposed as a stably finite analogue of $\mathcal{O}_2$. The conjecture that $\mathcal{W} \otimes \mathcal{W} \cong \mathcal{W}$, while still open, is known to have interesting consequences. For example, it is shown in [3] that among the C*-algebras classified in [11], those that are simple and have trivial K-theory would absorb $\mathcal{W}$ tensorially. On the other hand, L. Robert proves in [12] that the Cuntz semigroup of a $\mathcal{W}$-stable C*-algebra is determined by the cone of its lower semicontinuous 2-quasitraces. These results indicate that $\mathcal{W}$ may play an important role in the classification of nuclear, stably projectionless C*-algebras. In this article, we examine the structure of $\mathcal{W}$ rather than its role in classification, by showing in Theorem 4.3 how to obtain $\mathcal{W}$ as a nonunital deformation of $\mathcal{Z}$.

The paper is organized as follows. In Section 2 we establish notation and recall various basic facts about order zero maps and dimension drop algebras. Section 3 contains the presentation of $\mathcal{Z}$ as a universal C*-algebra (Theorems 3.1 and 3.3), and Section 4 contains the corresponding description of $\mathcal{W}$ (Theorem 4.3). We conclude with some open questions in Section 5.

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2. Preliminaries

In this section, we collect some well-known facts about order zero maps and dimension drop algebras that are used throughout the article. (Detailed exposition of order zero maps can be found in [21] and [22].) We denote by e_{ij} (or $e_{ij}^{(n)}$) the canonical (i,j)-th matrix unit in $M_n = M_n(\mathbb{C})$.

Recall that a completely positive (c.p.) map $\varphi : B \to A$ has *order zero* if it preserves orthogonality. Every completely positive and contractive (c.p.c.) order zero map $\varphi : B \to A$ (for B unital) is of the form $\varphi(\cdot) = \pi_\varphi(\cdot)\varphi(1_B) = \varphi(1_B)\pi_\varphi(\cdot)$ for a *-homomorphism $\pi_\varphi : B \to A^{**}$ called the *supporting *-homomorphism of φ*. We frequently use the notion of positive functional calculus provided by this decomposition: if $f \in C_0(0,1]$ is positive with $\|f\| \le 1$ then the map $f(\varphi) : B \to A$ given by $f(\varphi)(\cdot) := \pi_\varphi(\cdot) f(\varphi(1_B))$ is a well-defined c.p.c. order zero map. It is easy to see that if $p \in B$ is a projection, then $f(\varphi)(p) = f(\varphi(p))$. On the other hand, if $\varphi(1_B)$ is a projection, then φ is in fact a *-homomorphism.

Finally, c.p.c. order zero maps $B \to A$ correspond bijectively to *-homomorphisms $C_0((0,1],B) \to A$. For $B = M_n$, one way of interpreting this fact is to say that the cone $C_0((0,1],M_n)$ is *the universal* C**-algebra generated by a c.p.c. order zero* *map on* M_n. Equivalently, it is easy to check that $C_0((0,1], M_n)$ is the universal C*-algebra on generators $x_1, \dots, x_n$ subject to the relations $\mathcal{R}_n^{(0)}$ given by

$$\|x_i\| \le 1, \quad x_1 \ge 0, \quad x_i x_i^* = x_1^2, \quad x_j^* x_j \perp x_i^* x_i \quad \text{for} \quad 1 \le i \ne j \le n \tag{2.1}$$

(for example by mapping x_j to $t^{1/2} \otimes e_{1j}$, so that $t \otimes e_{ij}$ corresponds to $x_i^* x_j$). One can therefore view the statement

$$C_0((0,1], M_n) = \mathrm{C}^*(\varphi \mid \varphi \text{ c.p.c. order zero on } M_n) \tag{2.2}$$

as an abbreviation for these relations.

Remark 2.1. In the case $n = 2$, $C_0((0,1], M_2)$ is the universal C*-algebra $\mathrm{C}^*(x \mid \|x\| \le 1, x^2 = 0)$. Therefore, if A is a C*-algebra and $v \in A$ is a contraction with $v^2 = 0$, then there is a unique c.p.c. order zero map $\psi : M_2 \to A$ with $\psi^{1/2}(e_{12}) = v$ (so that $\psi(e_{11}) = vv^*$ and $\psi(e_{22}) = v^*v$).

By a *prime dimension drop algebra*, we mean a C*-algebra of the form

$$Z_{p,q} := \{ f \in C([0,1], M_p \otimes M_q) \mid f(0) \in M_p \otimes 1_q, f(1) \in 1_p \otimes M_q \}, \tag{2.3}$$

where p and q are coprime natural numbers. The Jiang–Su algebra $\mathcal{Z}$ is the unique inductive limit of prime dimension drop algebras which is simple and has a unique tracial state (see [4]).

The order zero notation (2.2) essentially appears in [14, Proposition 2.5], where the presentation of prime dimension drop algebras described in [4, Proposition 7.3] is reinterpreted in terms of order zero maps. Specifically, the prime dimension drop algebra $Z_{p,q}$ is the universal unital C*-algebra

$$\mathrm{C}^*(\alpha, \beta \mid \alpha \text{ c.p.c. order zero on } M_p, \beta \text{ c.p.c. order zero on } M_q,$$
$$\alpha(1_p) + \beta(1_q) = 1, [\alpha(M_p), \beta(M_q)] = 0),$$

with generators corresponding to the obvious embeddings of $C_0([0,1), M_p)$ and $C_0((0,1], M_q)$ into $Z_{p,q}$.

When $q = p+1$, there is another presentation of $Z_{p,p+1}$ in terms of order zero maps that does not involve a commutation relation. The following is essentially contained in [14, Proposition 5.1], and we note that these relations have already proved highly useful, for example in [19], [21], [15] and [8].

Proposition 2.2. *Let* $Z^{(n)}$ *denote the universal unital* C*-*algebra* $\mathrm{C}^*(\varphi, \psi \mid \mathcal{R}_n)$*, where* $\mathcal{R}_n$ *is the set of relations:*

(i) φ *and* ψ *are c.p.c. order zero maps on* M_n *and* M_2 *respectively;*

(ii) $\psi(e_{11}) = 1 - \varphi(1_n)$*;*

(iii) $\psi(e_{22})\varphi(e_{11}) = \psi(e_{22})$.

Then $Z^{(n)} \cong Z_{n,n+1}$.

In Section 3, we use Proposition 2.2 to write $\mathcal{Z}$ as a limit of dimension drop algebras in a universal way. We make analogous use of Proposition 4.1, a nonunital version of Proposition 2.2, to present $\mathcal{W}$.

3. Generators and relations for the Jiang–Su algebra

In this section, we will construct an inductive system $(Z^{(q(k))}, \alpha_k)$, where $q(k) = p^{3^k}$ for some fixed $p \geq 2$ ($p = 2$ will do) and $Z^{(q(k))} = \mathrm{C}^*(\varphi_k, \psi_k \mid \mathcal{R}_{q(k)}) \cong Z_{q(k),q(k)+1}$ (as in Proposition 2.2), and we will check that the inductive limit is simple with a unique tracial state. It will then follow from the classification theorem of [4] that $\mathcal{Z} \cong \varinjlim(Z^{(q(k))}, \alpha_k)$.

If this procedure is to provide an explicit presentation of $\mathcal{Z}$ as a universal C*-algebra, we need to be able to describe the connecting maps α_k in terms of generators and relations. (This is perhaps the key difference between our construction and the original construction of $\mathcal{Z}$ as an inductive limit in [4].) In other words, for every $k \in \mathbb{N}$ we will find c.p.c. order zero maps $\widehat{\varphi}_k : M_{q(k)} \to Z^{(q(k+1))}$ and $\widehat{\psi}_k : M_2 \to Z^{(q(k+1))}$ that satisfy the relations $\mathcal{R}_{q(k)}$ of Proposition 2.2. By universality, we will then have unital connecting maps $\alpha_k : Z^{(q(k))} \to Z^{(q(k+1))}$ with $\alpha_k \circ \varphi_k = \widehat{\varphi}_k$ and $\alpha_k \circ \psi_k = \widehat{\psi}_k$.

Before giving the connecting maps, it is instructive to note that there are obvious choices for $\widehat{\varphi}_k$ and $\widehat{\psi}_k$. Since $q(k+1) = q(k)^3$, we can identify $M_{q(k+1)}$ with $M_{q(k)} \otimes M_{q(k)} \otimes M_{q(k)}$ (and $e_{11}^{(q(k+1))}$ with $e_{11}^{(q(k))} \otimes e_{11}^{(q(k))} \otimes e_{11}^{(q(k))}$). We could then set $\widehat{\varphi}_k = \varphi_{k+1} \circ (\mathrm{id}_{M_{q(k)}} \otimes 1_{q(k)} \otimes 1_{q(k)})$ and $\widehat{\psi}_k = \psi_{k+1}$; it is easy to see that these maps satisfy the relations $\mathcal{R}_{q(k)}$, but the corresponding inductive limit certainly would not be simple. The idea is therefore to define $\widehat{\varphi}_k$ in such a way as to ensure that $[0, 1]$ is chopped up into suitably small pieces under the induced *-homomorphism α_k; $\widehat{\psi}_k^{1/2}(e_{12})$ will then be some partial-isometry-like element that facilitates the relations $\mathcal{R}_{q(k)}$.

One way of doing this is as follows. Define $\rho_k : M_{q(k)} \to M_{q(k+1)}$ by

$$\rho_k = (\mathrm{id}_{M_{q(k)}} \otimes 1_{q(k)-1} \otimes 1_{q(k)}) \oplus \left(\bigoplus_{i=1}^{q(k)} \frac{i}{q(k)} \left(\mathrm{id}_{M_{q(k)}} \otimes e_{q(k),q(k)} \otimes e_{ii} \right) \right). \tag{3.1}$$

Note that ρ_k is c.p.c. order zero, with supporting *-homomorphism $\pi_{\rho_k} = \mathrm{id}_{M_{q(k)}} \otimes 1_{q(k)} \otimes 1_{q(k)}$. We may then define $\widehat{\varphi}_k := \varphi_{k+1} \circ \rho_k$. For this to work, we need to be able to transport the defect $1 - \varphi_{k+1}(\rho_k(1_{q(k)})) = (1 - \varphi_{k+1}(1_{q(k+1)})) + \varphi_{k+1}(1_{q(k+1)} - \rho_k(1_{q(k)}))$ underneath $\varphi_{k+1}(\rho_k(e_{11}^{(q(k))}))$, and the basic idea is to do this in two steps.

Step 1. Use $\psi_{k+1}(e_{12})$ to transport the corner

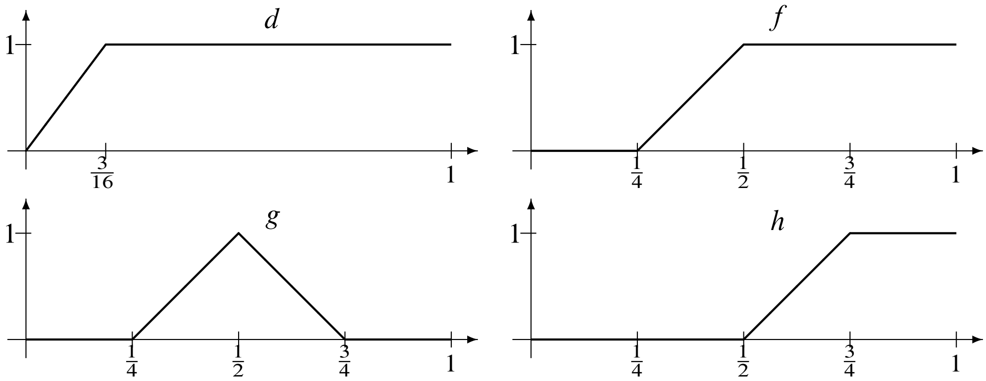
$$\pi_{\psi_{k+1}}(e_{11})(1 - \varphi_{k+1}(\rho_k(1_{q(k)})))\pi_{\psi_{k+1}}(e_{11})$$

$$\text{underneath } \pi_{\psi_{k+1}}(e_{22})\varphi_{k+1}(e_{11}^{(q(k+1))})\pi_{\psi_{k+1}}(e_{22}) \leq \varphi_{k+1}(e_{11}^{(q(k+1))})$$

$$\leq \varphi_{k+1}(\rho_k(e_{11}^{(q(k))})).$$

Step 2. Use a partial isometry $v_{k+1} \in M_{q(k+1)}$ to transport (a projection bigger than) $1_{q(k+1)} - \rho_k(1_{q(k)})$ underneath (a projection smaller than) $\rho_k(e_{11}^{(q(k))}) - e_{11}^{(q(k+1))}$, so that $\varphi_{k+1}(v_{k+1})$ transports the rest of $1 - \varphi_{k+1}(\rho_k(1_{q(k)}))$ underneath $\varphi_{k+1}(\rho_k(e_{11}^{(q(k))})) - \varphi_{k+1}(e_{11}^{(q(k+1))})$.

Although this is essentially the right idea, it needs fine-tuning in the guise of functional calculus. We achieve this in Theorem 3.3 by adjusting the relations for $Z^{(q(k))}$, while for Theorem 3.1, we modify $\widehat{\varphi}_k$ and $\widehat{\psi}_k$ using the following piecewise linear functions:



These are chosen so that, writing $\overline{d}(t) = d(1-t)$, we have

$$g = f - h, \quad hf = h, \quad (1-f)\overline{d} = 1 - f \quad \text{and} \quad g\overline{d} = g. \tag{3.2}$$

For use in Section 4, we also note that if $\widehat{d}$ is the function $\widehat{d}(t) = d(t(1-t))$ then we have

$$(f - f^2)\widehat{d} = f - f^2 \quad \text{and} \quad g\widehat{d} = g. \tag{3.3}$$

Finally, to accomplish Step 2, we choose a partial isometry

$$v_{k+1} \in M_{q(k+1)} \tag{3.4}$$

such that

$$v_{k+1}v_{k+1}^* = 1_{q(k)} \otimes e_{q(k),q(k)} \otimes 1_{q(k)-1}$$

and

$$v_{k+1}^* v_{k+1} = (e_{11} \otimes 1_{q(k)-1} \otimes 1_{q(k)}) + (e_{11} \otimes e_{q(k),q(k)} \otimes e_{q(k),q(k)}) - (e_{11} \otimes e_{11} \otimes e_{11}).$$

This is possible since both of these projections have rank $q(k)^2 - q(k)$; since they are orthogonal, we moreover have $v_{k+1}^2 = 0$. This v_{k+1} then satisfies:

(i) $v_{k+1}^* v_{k+1} \perp e_{11} \otimes e_{11} \otimes e_{11} = e_{11}^{(q(k+1))}$ (in fact, $v_{k+1}v_{k+1}^*$ is orthogonal to $e_{11}^{(q(k+1))}$, too);

(ii) $v_{k+1}^* v_{k+1}$ is dominated by $\rho_k(e_{11}^{(q(k))})$ (and therefore by $\rho_k(e_{11}^{(q(k))}) - e_{11}^{(q(k+1))}$); and

(iii) $v_{k+1} v_{k+1}^*$ acts like a unit on

$$1_{q(k+1)} - \rho_k(1_{q(k)}) = \bigoplus_{i=1}^{q(k)} \left(1 - \frac{i}{q(k)}\right)(1_{q(k)} \otimes e_{q(k),q(k)} \otimes e_{ii}). \tag{3.5}$$

Theorem 3.1. *Let the functions $d, f, g, h \in C_0(0,1]$, the partial isometries $v_{k+1} \in M_{q(k+1)}$, and the c.p.c. order zero maps $\rho_k : M_{q(k)} \to M_{q(k+1)}$ be as above for each $k \in \mathbb{N}$. Define $\mathcal{Z}_U$ to be the universal unital* C**-algebra generated by c.p.c. order zero maps φ_k on $M_{q(k)}$ ($k \in \mathbb{N}$) and ψ_k on M_2 ($k \in \mathbb{N}$) such that for each k, these maps satisfy the relations $\mathcal{R}_{q(k)}$, i.e.*

$$\psi_k(e_{11}) = 1 - \varphi_k(1_{q(k)}) \tag{3.6}$$

and

$$\psi_k(e_{22})\varphi_k(e_{11}) = \psi_k(e_{22}), \tag{3.7}$$

together with the additional relations $\mathcal{S}_{q(k)}$ given by

$$\varphi_k = f(\varphi_{k+1}) \circ \rho_k, \tag{3.8}$$

$$\psi_k^{1/2}(e_{12}) = \big(1 - f(\varphi_{k+1})(1_{q(k+1)}) + g(\varphi_{k+1})(1_{q(k+1)} - \rho_k(1_{q(k)}))\big)^{1/2} d(\psi_{k+1})(e_{12}) + h(\varphi_{k+1})(1_{q(k+1)} - \rho_k(1_{q(k)}))^{1/2} f(\varphi_{k+1})(v_{k+1}). \tag{3.9}$$

Then $\mathcal{Z}_U \cong \mathcal{Z}$.

Proof. For each k, define $\widehat{\varphi}_k : M_{q(k)} \to Z^{(q(k+1))} = \mathrm{C}^*(\varphi_{k+1}, \psi_{k+1} \mid \mathcal{R}_{q(k+1)})$ and $\widehat{\psi}_k : M_2 \to Z^{(q(k+1))}$ by

$$\widehat{\varphi}_k = f(\varphi_{k+1}) \circ \rho_k \tag{3.10}$$

and

$$\widehat{\psi}_k^{1/2}(e_{12}) = \gamma_k + \delta_k, \tag{3.11}$$

where

$$\gamma_k := \big(1 - f(\varphi_{k+1})(1_{q(k+1)}) + g(\varphi_{k+1})(1_{q(k+1)} - \rho_k(1_{q(k)}))\big)^{1/2} d(\psi_{k+1})(e_{12}) \tag{3.12}$$

and

$$\delta_k := h(\varphi_{k+1})\big(1_{q(k+1)} - \rho_k(1_{q(k)})\big)^{1/2} f(\varphi_{k+1})(v_{k+1}). \tag{3.13}$$

We need to check that $\widehat{\varphi}_k$ and $\widehat{\psi}_k$ satisfy the relations $\mathcal{R}_{q(k)}$. First, it is obvious that $\widehat{\varphi}_k$ is c.p.c. order zero since φ_{k+1} and ρ_k are, and f is contractive. Next, to show that (3.11) genuinely defines a c.p.c. order zero map $\widehat{\psi}_k$, it suffices to check that $\gamma_k + \delta_k$ is a contraction that squares to zero (see Remark 2.1). In fact, this would follow automatically from the relations (3.6) and (3.7) for $\widehat{\varphi}_k$ and $\widehat{\psi}_k$ (where, for the moment, we interpret $\widehat{\psi}_k(e_{11})$ and $\widehat{\psi}_k(e_{22})$ as *notation* for $\widehat{\psi}_k^{1/2}(e_{12})\widehat{\psi}_k^{1/2}(e_{12})^*$ and $\widehat{\psi}_k^{1/2}(e_{12})^*\widehat{\psi}_k^{1/2}(e_{12})$ respectively). Indeed, $1-\widehat{\varphi}_k(1_{q(k)})$ is certainly a contraction, and (3.6) and (3.7) would imply that

$$\begin{aligned}\widehat{\psi}_k(e_{22})\widehat{\psi}_k(e_{11}) &= \widehat{\psi}_k(e_{22})(1-\widehat{\varphi}_k(1_{q(k)}))\\ &= \widehat{\psi}_k(e_{22}) - \sum_{i=1}^{n}\widehat{\psi}_k(e_{22})\widehat{\varphi}_k(e_{11})\widehat{\varphi}_k(e_{ii}) = 0,\end{aligned} \tag{3.14}$$

and hence that $\left(\widehat{\psi}_k^{1/2}(e_{12})\right)^2 = 0$. Let us now check that $\widehat{\varphi}_k$ and $\widehat{\psi}_k$ really do satisfy these relations.

Claim 1. $\widehat{\psi}_k(e_{11}) = 1 - \widehat{\varphi}_k(1_{q(k)})$.

Proof of Claim 1. First note that, using (3.7) and property (i) of the partial isometry v_{k+1}, we have

$$\begin{aligned}&d(\psi_{k+1})(e_{12})f(\varphi_{k+1})(v_{k+1}^*)\\ &\quad = d^{1/2}(\psi_{k+1})(e_{12})d^{1/2}(\psi_{k+1})(e_{22})\varphi_{k+1}(e_{11})f(\varphi_{k+1})(v_{k+1}^*v_{k+1}v_{k+1}^*) = 0.\end{aligned}$$

Therefore, the cross terms $\gamma_k\delta_k^*$ and $\delta_k\gamma_k^*$ in the expansion of

$$\widehat{\psi}_k(e_{11}) = \widehat{\psi}_k^{1/2}(e_{12})\widehat{\psi}_k^{1/2}(e_{12})^*$$

vanish.

Using the fact that $fh = h$, and property (iii) of v_{k+1}, we have

$$\begin{aligned}&h(\varphi_{k+1})(1_{q(k+1)} - \rho_k(1_{q(k)}))f(\varphi_{k+1})(v_{k+1})f(\varphi_{k+1})(v_{k+1}^*)\\ &\quad = h(\varphi_{k+1})((1_{q(k+1)}-\rho_k(1_{q(k)})v_{k+1}v_{k+1}^*) = h(\varphi_{k+1})(1_{q(k+1)}-\rho_k(1_{q(k)})).\end{aligned}$$

Thus, $\delta_k\delta_k^* = h(\varphi_{k+1})(1_{q(k+1)} - \rho_k(1_{q(k)}))$. From (3.6) we have

$$d(\psi_{k+1})(e_{11}) = d(\psi_{k+1}(e_{11})) = d(1-\varphi_{k+1}(1_{q(k+1)})) = \overline{d}(\varphi_{k+1}(1_{q(k+1)})),$$

where $\overline{d}(t) = d(1-t)$ as in (3.2), whence we also obtain

$$\begin{aligned}(1-f(\varphi_{k+1})(1_{q(k+1)}))d(\psi_{k+1})(e_{11}) &= (1-f)(\varphi_{k+1}(1_{q(k+1)}))\overline{d}(\varphi_{k+1}(1_{q(k+1)}))\\ &= (1-f)(\varphi_{k+1}(1_{q(k+1)}))\\ &= 1-f(\varphi_{k+1})(1_{q(k+1)}).\end{aligned}$$

Similarly, we have $g(\varphi_{k+1})(1_{q(k+1)})d(\psi_{k+1})(e_{11}) = g(\varphi_{k+1})(1_{q(k+1)})$, hence

$$g(\varphi_{k+1})(1_{q(k+1)} - \rho_k(1_{q(k)}))d(\psi_{k+1})(e_{11}) = g(\varphi_{k+1})(1_{q(k+1)} - \rho_k(1_{q(k)})).$$

We therefore have $\gamma_k\gamma_k^* = 1 - f(\varphi_{k+1})(1_{q(k+1)}) + g(\varphi_{k+1})(1_{q(k+1)} - \rho_k(1_{q(k)}))$. Since $g + h = f$, it follows that

$$\begin{aligned}\widehat{\psi}_k(e_{11}) &= \gamma_k\gamma_k^* + \delta_k\delta_k^* \\ &= 1 - f(\varphi_{k+1})(1_{q(k+1)}) + g(\varphi_{k+1})(1_{q(k+1)} - \rho_k(1_{q(k)})) \\ &\qquad + h(\varphi_{k+1})(1_{q(k+1)} - \rho_k(1_{q(k)})) \\ &= 1 - f(\varphi_{k+1})(\rho_k(1_{q(k)})) \\ &= 1 - \widehat{\varphi}_k(1_{q(k)}).\end{aligned}$$

Claim 2. $\widehat{\psi}_k(e_{22})\widehat{\varphi}_k(e_{11}) = \widehat{\psi}_k(e_{22})$.

Proof of Claim 2. Since $fh = h$ and v_{k+1} is a partial isometry with property (ii), we have

$$\begin{aligned}h(\varphi_{k+1})(1_{q(k+1)} - \rho_k(1_{q(k)}))f(\varphi_{k+1})(v_{k+1})f(\varphi_{k+1})(\rho_k(e_{11})) \\ = h(\varphi_{k+1})((1_{q(k+1)} - \rho_k(1_{q(k)}))v_{k+1}) \\ = h(\varphi_{k+1})(1_{q(k+1)} - \rho_k(1_{q(k)}))f(\varphi_{k+1})(v_{k+1}).\end{aligned}$$

Thus, $\delta_k\widehat{\varphi}_k(e_{11}) = \delta_k$. Next, it follows from (3.7), upon approximating $d^{1/2}$ and f uniformly by polynomials, that

$$d^{1/2}(\psi_{k+1})(e_{22})f(\varphi_{k+1})(e_{11}) = f(1)d^{1/2}(\psi_{k+1})(e_{22}) = d^{1/2}(\psi_{k+1})(e_{22}).$$

Since $e_{11}^{(q(k+1))} \perp (\rho_k(e_{11}^{(q(k))}) - e_{11}^{(q(k+1))})$ and $f(\varphi_{k+1})$ is order zero, we therefore have $d^{1/2}(\psi_{k+1})(e_{22})f(\varphi_{k+1})(\rho_k(e_{11})) = d^{1/2}(\psi_{k+1})(e_{22})$, hence $d(\psi_{k+1})(e_{12})f(\varphi_{k+1})(\rho_k(e_{11})) = d(\psi_{k+1})(e_{12})$. Therefore, $\gamma_k\widehat{\varphi}_k(e_{11}) = \gamma_k$, and so $\widehat{\psi}_k(e_{22})\widehat{\varphi}_k(e_{11}) = (\gamma_k^* + \delta_k^*)(\gamma_k + \delta_k)\widehat{\varphi}_k(e_{11}) = \widehat{\psi}_k(e_{22})$.

We have now shown that $\widehat{\varphi}_k$ and $\widehat{\psi}_k$ satisfy the relations $\mathcal{R}_{q(k)}$. This means that, for any $k \in \mathbb{N}$, (3.8) and (3.9) do not introduce any new relations on φ_{k+1} and ψ_{k+1}; thus, the sub-C*-algebra generated by φ_{k+1} and ψ_{k+1} within $\mathcal{Z}_U$ is isomorphic to the universal C*-algebra on relations $\mathcal{R}_{q(k+1)}$ (that is, to $Z^{(q(k+1))}$), and moreover contains the sub-C*-algebra generated by φ_k and ψ_k. Therefore, by Proposition 2.2, $\mathcal{Z}_U$ is isomorphic to an inductive limit of prime dimension drop algebras.

The strategy for the remainder of the proof is to pass from the abstract picture of $\mathcal{Z}_U$ as a universal C*-algebra, to a concrete description as an inductive limit $\varinjlim(Z^{(q(k))}, \alpha_k)$, where the (unital) connecting maps $\alpha_k : Z^{(q(k))} \to Z^{(q(k+1))}$

are determined by (3.8) and (3.9) (i.e. $\alpha_k \circ \varphi_k = \widehat{\varphi}_k$ and $\alpha_k \circ \psi_k = \widehat{\psi}_k$). We will obtain explicit descriptions of the maps α_k, and use these to show that $\mathcal{Z}_U$ is simple and has a unique tracial state.

For each $k \in \mathbb{N}$, let us fix an identification of $Z^{(q(k))}$ with $Z_{q(k),q(k)+1}$ via the order zero map $M_{q(k)} \to Z_{q(k),q(k)+1}$ (which, abusing notation, we also call φ_k) defined by:

$$\varphi_k(a)(t) = u_k(t)(a \otimes 1_{q(k)})u_k(t)^* \oplus (1-t)(a \otimes e_{q(k)+1,q(k)+1}) \tag{3.15}$$

for $a \in M_{q(k)}$ and $t \in [0,1]$. (Here, u_k is a unitary in the algebra $C([0,1], M_{q(k)} \otimes M_{q(k)})$, included nonunitally in the top left corner of $C([0,1], M_{q(k)} \otimes M_{q(k)+1})$, with $u_k(0) = 1$ and $u_k(1)$ implementing the flip in $M_{q(k)} \otimes M_{q(k)}$.) It is easy to write down a suitable ψ_k, but for the purpose of computing the connecting map $Z_{q(k),q(k)+1} \to Z_{q(k+1),q(k+1)+1}$ (also called α_k), this is not necessary.

For each $t \in [0,1]$, let us write α_k^t for the map $\mathrm{ev}_t \circ \alpha_k : Z_{q(k),q(k)+1} \to M_{q(k+1)} \otimes M_{q(k+1)+1}$, where ev_t denotes evaluation at t. Then α_k^t is a finite-dimensional representation of $Z_{q(k),q(k)+1}$, so is a direct sum of finitely many irreducible representations $\pi_1^t, \dots, \pi_{m(t)}^t$ of $Z_{q(k),q(k)+1}$ (corresponding up to unitary equivalence and, at the endpoints, up to multiplicity, to point evaluations). Since $\mathrm{C}^*(\varphi_k(1_{q(k)})) \subset Z_{q(k),q(k)+1}$ separates the points of $[0,1]$, it is easy to see that the unitary equivalence classes of $\pi_1^t, \dots, \pi_{m(t)}^t$ can be determined by computing $\alpha_k^t(\varphi_k(1_{q(k)}))$. To do this, note that

$$f(\varphi_{k+1})(b)(t) = u_{k+1}(t)(b \otimes 1_{q(k+1)})u_{k+1}(t)^* \oplus f(1-t)(b \otimes e_{q(k+1)+1,q(k+1)+1}) \tag{3.16}$$

for $b \in M_{q(k+1)}$, and recall the definition (3.1) of ρ_k. We then have, for $a \in M_{q(k)}$ and $t \in [0,1]$,

$$\begin{aligned}
\alpha_k^t(\varphi_k(a)) &= f(\varphi_{k+1})(\rho_k(a))(t) \\
&= u_{k+1}(t)(a \otimes 1_{q(k)-1} \otimes 1_{q(k)} \otimes 1_{q(k+1)})u_{k+1}(t)^* \\
&\oplus u_{k+1}(t)\left(\bigoplus_{i=1}^{q(k)} \frac{i}{q(k)}\left(a \otimes e_{q(k),q(k)} \otimes e_{ii} \otimes 1_{q(k+1)}\right)\right)u_{k+1}(t)^* \\
&\oplus f(1-t)(a \otimes 1_{q(k)-1} \otimes 1_{q(k)} \otimes e_{q(k+1)+1,q(k+1)+1}) \\
&\oplus f(1-t)\left(\bigoplus_{i=1}^{q(k)} \frac{i}{q(k)}\left(a \otimes e_{q(k),q(k)} \otimes e_{ii} \otimes e_{q(k+1)+1,q(k+1)+1}\right)\right) \\
&\sim_{\mathrm{u}} \left(\bigoplus_{m=1}^{q(k+1)} \bigoplus_{i=1}^{q(k)-1} \varphi_k(a)\left(1 - \frac{i}{q(k)}\right)\right)
\end{aligned}$$

$$\oplus \left(\bigoplus_{m=1}^{q(k)(q(k)-1)} \varphi_k(a)(1-f(1-t)) \right)$$
$$\oplus \left(\bigoplus_{i=1}^{q(k)} \varphi_k(a) \left(1 - \frac{if(1-t)}{q(k)}\right) \right),$$

where $\sim_{\mathrm{u}}$ denotes unitary equivalence. Write $h_i(t) = 1 - \frac{if(1-t)}{q(k)}$ (so that, in fact, $h_{q(k)} = 1 - f(1-t) = h(t)$). By our earlier reasoning it then follows that, for every $t \in [0,1]$, there is a unitary $w_k^t \in M_{q(k+1)} \otimes M_{q(k+1)+1}$ such that

$$\alpha_k^t = w_k^t \left(\left(\bigoplus_{m=1}^{q(k+1)} \bigoplus_{i=1}^{q(k)-1} \mathrm{ev}_{\frac{i}{q(k)}} \right) \oplus \left(\bigoplus_{m=1}^{q(k)(q(k)-1)} \mathrm{ev}_{h(t)} \right) \oplus \left(\bigoplus_{i=1}^{q(k)} \mathrm{ev}_{h_i(t)} \right) \right) {w_k^t}^*. \tag{3.17}$$

It could be that $t \mapsto w_k^t$ is not continuous, but this does not matter. (Moreover, it is not difficult to show that, up to approximate unitary equivalence, continuity may be assumed anyway.)

We can also give a description of the connecting map $\alpha_{k,k+n} = \alpha_{k+n-1} \circ \cdots \circ \alpha_k$. For each $j \in \mathbb{N}$, let Λ_j be the sequence of continuous functions given by listing each constant function $i/q(j)$ (for $1 \le i \le q(j)-1$) with multiplicity $q(j+1)$, then h with multiplicity $q(j)(q(j)-1)$ and then each h_i for $1 \le i \le q(j)$. Then $\alpha_{k,k+n}$ is fibrewise unitarily equivalent to the direct sum of all maps of the form $\mathrm{ev}_{F_1 \circ \cdots \circ F_n}$ with $F_j \in \Lambda_{k+j-1}$ for $1 \le j \le n$.

Let us write $T(A)$ for the space of tracial states on a C*-algebra A. Recall that every tracial state on $Z_{q(j),q(j)+1}$ is of the form $\int \mathrm{tr} \circ \mathrm{ev}_t(\cdot) d\mu(t)$ for some Borel probability measure μ on $[0,1]$, where tr is the unique tracial state on $M_{q(j)} \otimes M_{q(j)+1}$. In particular, every such trace extends to a trace on $C([0,1], M_{q(j)} \otimes M_{q(j)+1})$, and is invariant under fibrewise unitary equivalence.

Since $\mathcal{Z}_U \cong \varinjlim Z_{q(k),q(k)+1}$ with unital connecting maps α_k, we have $T(\mathcal{Z}_U) \cong \varprojlim T(Z_{q(k),q(k)+1})$. Thus $T(\mathcal{Z}_U)$ is an inverse limit of nonempty compact Hausdorff spaces, so is nonempty. That is, $\mathcal{Z}_U$ has at least one tracial state. For uniqueness, we need to show that for every $k \in \mathbb{N}$, every $\epsilon > 0$, and every $b \in Z_{q(k),q(k)+1}$ we have

$$|\tau_1(\alpha_{k,k+n}(b)) - \tau_2(\alpha_{k,k+n}(b))| < \epsilon \tag{3.18}$$

for all sufficiently large n and every $\tau_1, \tau_2 \in T(Z_{q(k+n),q(k+n)+1})$. The key observation for this is that for each j, most of the elements in the sequence Λ_j

defined above are constant functions. In fact, the proportion of functions in Λ_j that are *not* constant is

$$\begin{aligned}\frac{q(j)(q(j)-1)+q(j)}{q(j+1)(q(j)-1)+q(j)(q(j)-1)+q(j)} &= \frac{q(j)^2}{q(j)^4-q(j)^3+q(j)^2}\\ &= \frac{1}{q(j)^2-q(j)+1}. \qquad (3.19)\end{aligned}$$

Since $F_1 \circ \cdots \circ F_n$ is constant if any of the F_i are constant, it follows that for fixed $b \in Z_{q(k),q(k)+1}$, $\alpha_{k,k+n}(b)$ is fibrewise unitarily equivalent to a direct sum of continuous $M_{q(k)} \otimes M_{q(k)+1}$-valued functions, most of which are constant except for a small corner. But any two tracial states agree on the constant pieces, and the small corner has trace at most $\|b\| \prod_{j=k}^{k+n-1} \frac{1}{q(j)^2-q(j)+1}$, which of course converges to 0 as $n \to \infty$. Thus (3.18) holds, and so $\mathcal{Z}_U$ has a unique tracial state.

It is well known that, to establish simplicity, it suffices to show the following (see for example [14, Theorem 3.4]): if b is a nonzero element of $Z_{q(k),q(k)+1}$, then $\alpha_{k,r}(b)$ generates $Z_{q(r),q(r)+1}$ as a (closed, two-sided) ideal for every sufficiently large r (which is the case if and only if $\alpha_{k,r}^t(b)$ is nonzero for every $t \in [0,1]$). Suppose that b is such an element, so that there is an interval in $(0,1)$ of width $\epsilon > 0$ on which b is nonzero. For each $n \in \mathbb{N}$ and $t \in [0,1]$, $\alpha_{k,k+n+1}^t(b)$ contains summands unitarily equivalent to $b\left(h^{(n)}\left(\frac{i}{q(k+n)}\right)\right)$ for $1 \le i \le q(k+n)-1$, where $h^{(n)} := \overbrace{h \circ \cdots \circ h}^{n}$. Moreover, $h^{(n)}$ is of the form

$$h^{(n)}(t) = \begin{cases} 0, & 0 \le t \le l_n/4^n \\ 4^n t - l_n, & l_n/4^n \le t \le (1+l_n)/4^n \\ 1, & (1+l_n)/4^n \le t \le 1 \end{cases}$$

for some l_n, and so it suffices to show that for large n we have $\frac{1}{q(k+n)} < \frac{\epsilon}{4^n}$. But this is true for all large n since $\frac{4^n}{q(k+n)} = \frac{4^n}{p^{3^{k+n}}} \longrightarrow 0$ as $n \to \infty$. Thus $\mathcal{Z}_U$ is simple.

It now follows from the classification theorem [4, Theorem 6.2] that $\mathcal{Z}_U \cong \mathcal{Z}$. □

Remark 3.2. One point that should be emphasized is that, despite the use of functional calculus, the relations of Theorem 3.1 really are *algebraic*, or at least C*-algebraic in the sense that they involve only *-polynomial and order relations. This can be made explicit by encoding the relations (3.2) satisfied by the functions d, f, g and h into the relations for the building blocks $Z^{(q(k))}$.

More specifically, it is not difficult to derive from Proposition 2.2 that the dimension drop algebra $Z_{n,n+1}$ is isomorphic to the universal C*-algebra on generators φ, ψ and h with relations:

(i) φ, ψ and h are c.p.c. order zero maps on M_n, M_2 and $\mathbb{C}$ respectively (in particular, h is just a positive contraction);

(ii) $[\psi(e_{11}), \varphi(M_n)] = [h, \varphi(M_n)] = 0$;

(iii) $\psi(e_{11})h = h$;

(iv) $h(1 - \varphi(1_n)) = 1 - \varphi(1_n)$;

(v) $\psi(e_{22})\varphi(e_{11}) = \psi(e_{22})$.

(It is a straightforward exercise in functional calculus to write down inverse isomorphisms between the universal C*-algebra determined by these relations and $Z^{(n)} \cong Z_{n,n+1}$.) The following is then proved in exactly the same way as Theorem 3.1.

Theorem 3.3. *The Jiang–Su algebra $\mathcal{Z}$ is isomorphic to the universal unital* C*-*algebra generated by c.p.c. order zero maps φ_k on $M_{q(k)}$ ($k \in \mathbb{N}$) and ψ_k on M_2 ($k \in \mathbb{N}$), and positive contractions h_k ($k \in \mathbb{N}$), together with (for each $k \in \mathbb{N}$) the relations:*

$$\begin{aligned}
[\psi_k(e_{11}), \varphi_k(M_{q(k)})] &= [h_k, \varphi_k(M_{q(k)})] = 0,\\
\psi_k(e_{11})h_k &= h_k,\\
h_k(1 - \varphi_k(1_{q(k)})) &= 1 - \varphi_k(1_{q(k)}),\\
\psi_k(e_{22})\varphi_k(e_{11}) &= \psi_k(e_{22}),\\
\varphi_k &= \varphi_{k+1} \circ \rho_k,\\
\frac{1}{\sqrt{2}}(1 + h_k)^{1/2}\psi_k^{1/2}(e_{12}) &= (h_{k+1} + (1 - h_{k+1})\varphi_{k+1}(v_{k+1}v_{k+1}^*))^{1/2}\psi_{k+1}^{1/2}(e_{12})\\
&\quad + (1 - \psi_{k+1}(e_{11}))^{1/2}\varphi_{k+1}^{1/2}(v_{k+1}),
\end{aligned}$$

where the c.p.c. order zero maps $\rho_k : M_{q(k)} \to M_{q(k+1)}$ and the partial isometries $v_k \in M_{q(k)}$ are as in (3.1) and (3.4) respectively. □

4. $\mathcal{W}$ as a universal C*-algebra

The article [10] (or, in a much more general setting, [11]) contains a classification by tracial data of simple inductive limits of building blocks

$$W_{n,m} := \{f \in C([0,1], M_n \otimes M_m) \mid f(0) = a \otimes 1_m, f(1) = a \otimes 1_{m-1}, a \in M_n\}, \tag{4.1}$$

where $n, m \in \mathbb{N}, m > 1$.

Such building blocks are easily seen to be stably projectionless, and it can moreover be shown that they have trivial K-theory (this is why the classifying invariant is purely tracial). The classification is also complete in the sense that every permissible value of the invariant is attained—see [17] or [3, Proposition 5.3]. Then, $\mathcal{W}$ may be defined as the unique C*-algebra in this class which has a unique tracial state (and no unbounded traces).

An explicit construction of $\mathcal{W}$ is given in [3], and in this section we obtain another one by adapting the previous section's universal characterization of $\mathcal{Z}$. To begin with, notice that $W_{n,n+1}$ is isomorphic to a subalgebra of the dimension drop algebra $Z_{n,n+1}$; the following indicates that it in fact may be thought of as its nonuntial analogue (compare with Proposition 2.2).

Proposition 4.1. *Let $W^{(n)}$ denote the universal* C*-*algebra* $\mathrm{C}^*(\varphi, \psi \mid \widehat{\mathcal{R}}_n)$, *where $\widehat{\mathcal{R}}_n$ is the set of relations:*

(i) *φ and ψ are c.p.c. order zero maps on M_n and M_2 respectively;*

(ii) *$\psi(e_{11}) = \varphi(1_n)(1 - \varphi(1_n))$;*

(iii) *$\psi(e_{22})\varphi(e_{11}) = \psi(e_{22})$.*

Then $W^{(n)} \cong W_{n,n+1}$.

Proof. The proof is almost identical to that of Proposition 2.2, but we include it here for completeness. Define $\varphi : M_n \to W_{n,n+1}$ by

$$\varphi(a)(t) = (a \otimes 1_n) \oplus (1 - t)(a \otimes e_{n+1,n+1})$$

for $a \in M_n$ and $t \in [0, 1]$. Then φ is clearly a c.p.c. order zero map. Equivalently, if we write

$$x_i(t) = (e_{1i} \otimes 1_n) \oplus (1 - t)^{1/2}(e_{1i} \otimes e_{n+1,n+1}) = \varphi^{1/2}(e_{1i})(t)$$

for $1 \le i \le n$, then the x_i satisfy the order zero relations $\mathcal{R}_n^{(0)}$ and $\varphi(e_{ij}) = x_i^* x_j$. Next, define

$$v(t) = t^{1/2}(1 - t)^{1/2} \sum_{j=1}^{n} e_{j1} \otimes e_{n+1,j}.$$

Then $vv^* = \varphi(1_n)(1 - \varphi(1_n))$ and $vx_1 = v$, and (so) $\|v\| \le 1$ and $v^2 = 0$. In particular, there is a unique c.p.c. order zero map $\psi : M_2 \to W_{n,n+1}$ with $\psi^{1/2}(e_{12}) = v$, i.e.

$$\psi(e_{12})(t) = t(1 - t) \sum_{j=1}^{n} e_{j1} \otimes e_{n+1,j},$$

so that $\psi(e_{11}) = vv^*$, $\psi(e_{22}) = v^*v$ and φ and ψ satisfy all of the relations $\widehat{\mathcal{R}}_n$.

Next, we check that v and the x_i generate $W_{n,n+1}$ as a C*-algebra. Write $A := \mathrm{C}^*(\{v, x_1, \ldots, x_n\})$. We have

$$v^* x_i(t) = t^{1/2}(1 - t)(e_{1i} \otimes e_{1,n+1})$$

and

$$v^*x_ivx_j(t) = t(1-t)^{3/2}(e_{1j}\otimes e_{1i})$$

for $1 \le i, j \le n$. Thus, for $t \in (0,1)$, the elements $v^*x_i(t)$ and $v^*x_ivx_j(t)$ give all matrix units $\{e_{1k}\otimes e_{1l}\}_{1\le k\le n, 1\le l\le n+1}$, so generate all of $M_n \otimes M_{n+1}$, and so the irreducible representation $\mathrm{ev}_t : W_{n,n+1} \to M_n \otimes M_{n+1}$ restricts to an irreducible representation of A. For $t \in \{0,1\}$, the x_i generate all the matrix units of M_n in the endpoint irreducible representation $\mathrm{ev}_\infty : W_{n,n+1} \to M_n$. Thus every irreducible representation of $W_{n,n+1}$ restricts to an irreducible representation of A. Also, since $x_1(s)$ is not unitarily equivalent to $x_1(t)$ for distinct $s, t \in (0,1)$, it follows that inequivalent irreducible representations of $W_{n,n+1}$ restrict to inequivalent irreducible representations of A. Therefore, by Stone-Weierstrass (i.e. [2, Proposition 11.1.6]), we do indeed have $\mathrm{C}^*(\{v, x_1, \ldots, x_n\}) = W_{n,n+1}$.

It remains to show that these generators of $W_{n,n+1}$ enjoy the appropriate universal property: for every representation $\{\widehat{\varphi}, \widehat{\psi}\}$ of the given relations, we need to show that there is a *-homomorphism $W_{n,n+1} \to \mathrm{C}^*(\widehat{\varphi}, \widehat{\psi})$ sending φ to $\widehat{\varphi}$ and ψ to $\widehat{\psi}$. By [7, Lemma 3.2.2], it suffices to consider the case where $\{\widehat{\varphi}, \widehat{\psi}\}$ is an *irreducible* representation on some Hilbert space H (i.e. has trivial commutant in $\mathfrak{B}(H)$). Note that the irreducible representations of $W_{n,n+1}$ are (up to unitary equivalence), the evaluation maps $\mathrm{ev}_t : W_{n,n+1} \to M_{n(n+1)}$ for $t \in (0,1)$ together with the endpoint representation $\mathrm{ev}_\infty : W_{n,n+1} \to M_n$. We will therefore show that (again, up to unitary equivalence) $\widehat{\varphi} = \mathrm{ev}_t \circ \varphi$ and $\widehat{\psi} = \mathrm{ev}_t \circ \psi$ for some $t \in (0,1) \cup \{\infty\}$.

For each $i \in \{1, \ldots, n\}$, let $\widehat{\psi}_i : M_2 \to \mathrm{C}^*(\widehat{\varphi}, \widehat{\psi})$ be the c.p.c. order zero map defined by $\widehat{\psi}_i^{1/2}(e_{12}) = \widehat{\psi}^{1/2}(e_{12})\widehat{\varphi}^{1/2}(e_{1i})$, so that $\widehat{\psi}_i(e_{11}) = \widehat{\varphi}(1_n)(1-\widehat{\varphi}(1_n))$ and $\widehat{\psi}_i(e_{22})\widehat{\varphi}(e_{ii}) = \widehat{\psi}_i(e_{22})$. Define

$$z := \widehat{\psi}(e_{11}) + \sum_{i=1}^n \widehat{\psi}_i(e_{22}) \in \mathrm{C}^*(\widehat{\varphi}, \widehat{\psi}).$$

Then

$$[z, \widehat{\varphi}(e_{1j})] = \widehat{\psi}_1(e_{22})\widehat{\varphi}(e_{1j}) - \widehat{\varphi}(e_{1j})\widehat{\psi}_j(e_{22}) = 0,$$

and

$$\begin{aligned}[z, \widehat{\psi}(e_{12})] &= \widehat{\psi}^2(e_{12}) + \sum_{i=1}^n \widehat{\psi}_i(e_{22})\widehat{\psi}^{1/2}(e_{11})\widehat{\psi}^{1/2}(e_{12}) \\ &\qquad - \sum_{i=1}^n \widehat{\psi}(e_{12})\widehat{\varphi}(e_{11})\widehat{\varphi}(e_{ii})\widehat{\psi}_i(e_{22}) \\ &= \widehat{\psi}^2(e_{12}) + 0 - \widehat{\psi}^2(e_{12}) \\ &= 0,\end{aligned}$$

so z is central in $C^*(\widehat{\varphi}, \widehat{\psi})$, and is therefore $\zeta 1$ for some scalar ζ. Moreover, z is positive with $\|z\| = \|\widehat{\psi}(e_{11})\| = \|\widehat{\varphi}(1_n)(1 - \widehat{\varphi}(1_n))\| \leq 1/4$, so $0 \leq \zeta \leq 1/4$.

If $\zeta = 0$ then $\widehat{\psi} = 0$ and $\widehat{\varphi}(1_n)$ is therefore a projection. It follows that $\widehat{\varphi}$ is a *-homomorphism giving an irreducible representation of M_n on H. Thus (up to unitary equivalence) $H = \mathbb{C}^n$ and $\widehat{\varphi} = \mathrm{ev}_\infty \circ \varphi$.

Suppose that $\zeta > 0$. Then $\zeta \widehat{\psi}(e_{11}) = z\widehat{\psi}(e_{11}) = (\widehat{\psi}(e_{11}))^2$, so $p := \zeta^{-1}\widehat{\psi}(e_{11})$ and $q_i := \zeta^{-1}\widehat{\psi}_i(e_{22})$ are equivalent orthogonal projections with $p + q_1 + \cdots + q_n = 1$. Since p commutes with $\widehat{\varphi}(M_n)$, the maps $p\widehat{\varphi}(\cdot)p$ and $(1-p)\widehat{\varphi}(\cdot)(1-p)$ are c.p.c. order zero. In fact,

$$\zeta\widehat{\varphi}(1_n)(1-p) = \widehat{\varphi}(1_n)(z - \widehat{\psi}(e_{11})) = z - \widehat{\psi}(e_{11}) = \zeta(1-p),$$

i.e. $(1-p)\widehat{\varphi}(1_n)(1-p) = 1-p$. Thus, $(1-p)\widehat{\varphi}(\cdot)(1-p)$ is a *unital* c.p.c. order zero map into the corner $(1-p)\mathfrak{B}(H)(1-p) \cong \mathfrak{B}((1-p)H)$, so is a *-homomorphism into this corner. Also, $p\widehat{\varphi}(1_n)p$ commutes with (the WOT-closure of) the corner $pC^*(\widehat{\varphi}, \widehat{\psi})p = pC^*(\widehat{\varphi})p$ (which, by irreducibility, is all of $p\mathfrak{B}(H)p \cong \mathfrak{B}(pH)$) so $p\widehat{\varphi}(1_n)p = tp$ for some $t \in [0, 1]$. So $t^{-1}p\widehat{\varphi}(\cdot)p$ is also a *-homomorphism, and is in fact an *irreducible* representation of M_n on pH. In particular, up to unitary equivalence, $pH = \mathbb{C}^n$ and $p\widehat{\varphi}(\cdot)p = t \cdot \mathrm{id}_{M_n}$.

Moreover, since every q_i is equivalent to p, they all have trace n $(= \mathrm{tr}(p))$. Thus (again up to unitary equivalence) $(1-p)H = \mathbb{C}^{n^2}$ (so $H = \mathbb{C}^{n(n+1)}$) and $(1-p)\widehat{\varphi}(\cdot)(1-p) : M_n \to M_{n^2}$ is just $a \mapsto \mathrm{diag}(a, \ldots, a)$. Finally, since

$$\begin{aligned} t(1-t)p = tp(p - tp) &= \widehat{\varphi}(1_n)p(p - \widehat{\varphi}(1_n)p) \\ &= p\widehat{\varphi}(1_n)(1 - \widehat{\varphi}(1_n)) = p\widehat{\psi}(e_{11}) = \zeta p, \end{aligned}$$

we have $t(1-t) = \zeta$. Therefore, $\widehat{\varphi} = p\widehat{\varphi}(\cdot)p + (1-p)\widehat{\varphi}(\cdot)(1-p) = \mathrm{ev}_{1-t} \circ \varphi$ and, since $\zeta^{-1/2}\widehat{\psi}^{1/2}(e_{12})$ is a partial isometry implementing an equivalence between q_1 and p, $\widehat{\psi} = \mathrm{ev}_{1-t} \circ \psi$ (up to conjugation by a unitary). Thus $W_{n,n+1}$ has the required universal property. □

Remark 4.2. It should also be possible to detect *-homomorphisms from $W_{n,n+1}$ to a stable rank one C*-algebra A at the level of the Cuntz semigroup $W(A)$ (just as for $Z_{n,n+1}$ in [14, Proposition 5.1]). The existence of $\langle x \rangle \in W(A)$ and a positive contraction $y \in A$ with $n\langle x \rangle = \langle y \rangle$ and $\langle y - y^2 \rangle \ll \langle x \rangle$ (where $\ll$ denotes the relation of compact containment) is probably necessary and sufficient, but perhaps this is not the most useful characterization.

Finally, we present $\mathcal{W}$ as a nonunital deformation of $\mathcal{Z}$.

Theorem 4.3. *Choose positive functions* $d, f, g, h \in C_0(0, 1]$, *partial isometries* $v_{k+1} \in M_{q(k+1)}$, *and c.p.c. order zero maps* $\rho_k : M_{q(k)} \to M_{q(k+1)}$ *as in Theorem 3.1. Define* $\mathcal{W}_U$ *to be the universal* C**-algebra generated by c.p.c. order* *zero maps φ_k on $M_{q(k)}$ ($k \in \mathbb{N}$) and ψ_k on M_2 ($k \in \mathbb{N}$) such that for each k, these maps satisfy the relations $\widehat{\mathcal{R}}_{q(k)}$, i.e.*

$$\psi_k(e_{11}) = \varphi_k(1_{q(k)})(1 - \varphi_k(1_{q(k)})) \tag{4.2}$$

and

$$\psi_k(e_{22})\varphi_k(e_{11}) = \psi_k(e_{22}), \tag{4.3}$$

together with the additional relations $\widehat{\mathcal{S}}_{q(k)}$ given by

$$\varphi_k = f(\varphi_{k+1}) \circ \rho_k, \tag{4.4}$$

$$\begin{aligned} &\psi_k^{1/2}(e_{12}) \\ &\quad = f(\varphi_{k+1})(\rho_k(1_{q(k)}))^{1/2}\Big(h(\varphi_{k+1})(1_{q(k+1)} - \rho_k(1_{q(k)}))^{1/2} f(\varphi_{k+1})(v_{k+1}) \\ &\quad + \big(1 - f(\varphi_{k+1})(1_{q(k+1)}) + g(\varphi_{k+1})(1_{q(k+1)} - \rho_k(1_{q(k)}))\big)^{1/2} d(\psi_{k+1})(e_{12})\Big). \end{aligned} \tag{4.5}$$

Then $\mathcal{W}_U \cong \mathcal{W}$.

Proof. The proof is essentially the same as that of Theorem 3.1, so we omit most of the details. As before, let us write $\widehat{\varphi}_k = f(\varphi_{k+1}) \circ \rho_k$ and $\widehat{\psi}_k^{1/2}(e_{12}) = \gamma_k + \delta_k$, where this time

$$\gamma_k := f(\varphi_{k+1})(\rho_k(1_{q(k)}))^{1/2}\lambda_k d(\psi_{k+1})(e_{12})$$

and

$$\delta_k := f(\varphi_{k+1})(\rho_k(1_{q(k)}))^{1/2}\mu_k f(\varphi_{k+1})(v_{k+1}),$$

with

$$\lambda_k := (1 - f(\varphi_{k+1})(1_{q(k+1)}) + g(\varphi_{k+1})(1_{q(k+1)} - \rho_k(1_{q(k)})))^{1/2}$$

and

$$\mu_k := h(\varphi_{k+1})(1_{q(k+1)} - \rho_k(1_{q(k)}))^{1/2}.$$

To show that $\widehat{\psi}_k(e_{11}) = \widehat{\varphi}_k(1_{q(k)})(1 - \widehat{\varphi}_k(1_{q(k)}))$, we proceed exactly as in the proof of Claim 1. The only difference is that we now have

$$\begin{aligned} d(\psi_{k+1})(e_{11}) = d(\psi_{k+1}(e_{11})) &= d(\varphi_{k+1}(1_{q(k+1)})(1 - \varphi_{k+1}(1_{q(k+1)}))) \\ &= \widehat{d}(\varphi_{k+1}(1_{q(k+1)})), \end{aligned}$$

where $\widehat{d}(t) = d(t(1-t))$ as in (3.3). We also have

$$\begin{aligned} &f(\varphi_{k+1})(\rho_k(1_{q(k)}))(1 - f(\varphi_{k+1})(1_{q(k+1)})) \\ &\qquad = \pi_{\varphi_{k+1}}(\rho_k(1_{q(k+1)}))(f - f^2)(\varphi_{k+1}(1_{q(k+1)})). \end{aligned}$$

Since $\widehat{d}(f - f^2) = f - f^2$, this therefore gives

$$f(\varphi_{k+1})(\rho_k(1_{q(k)}))(1 - f(\varphi_{k+1})(1_{q(k+1)}))d(\psi_{k+1})(e_{11}) \\ = f(\varphi_{k+1})(\rho_k(1_{q(k)}))(1 - f(\varphi_{k+1})(1_{q(k+1)})),$$

and the rest of the argument carries over mutatis mutandis. (Note in particular that λ_k and μ_k both commute with $f(\varphi_{k+1})(\rho_k(1_{q(k)}))^{1/2}$.) The proof that $\widehat{\psi}_k(e_{22})\widehat{\varphi}_k(e_{11}) = \widehat{\psi}_k(e_{22})$ is literally the same as the proof of Claim 2.

We now know that $\mathcal{W}_U$ is isomorphic to an inductive limit $\varinjlim(W_{q(k),q(k+1)}, \beta_k)$. Moreover, arguing exactly as before, we see that the connecting maps β_k are (fibrewise) unitarily equivalent to the connecting maps α_k obtained earlier. That is, there are unitaries $z_k^t \in M_{q(k+1)} \otimes M_{q(k+1)+1}$ such that

$$\beta_k^t = z_k^t \left(\left(\bigoplus_{m=1}^{q(k+1)} \bigoplus_{i=1}^{q(k)-1} \mathrm{ev}_{\frac{i}{q(k)}} \right) \oplus \left(\bigoplus_{m=1}^{q(k)(q(k)-1)} \mathrm{ev}_{h(t)} \right) \oplus \left(\bigoplus_{i=1}^{q(k)} \mathrm{ev}_{h_i(t)} \right) \right) z_k^{t\,*} \tag{4.6}$$

for every $t \in [0, 1]$.

The same arguments as with $\mathcal{Z}_U$ show that $\mathcal{W}_U$ is simple and has a unique tracial state. (One has to perhaps be slightly careful about the *existence* of a trace, since the space of tracial states of a nonunital C*-algebra need not be compact. But this is not an issue.) The only minor technicality is that, since the building blocks $W_{q(k),q(k+1)}$ are nonunital and the connecting maps β_k are degenerate, $\mathcal{W}_U$ may have unbounded traces. However, one can easily show, using (3.19), that this is not the case. It therefore follows from the classification theorem of [10] (or indeed from the more general result proved in [11]) that $\mathcal{W}_U \cong \mathcal{W}$. □

Corollary 4.4. *There exists a trace-preserving embedding of $\mathcal{W}$ into $\mathcal{Z}$. Such an embedding is canonical at the level of the Cuntz semigroup, and is unique up to approximate unitary equivalence.*

Proof. This follows immediately from Theorem 4.3 and Theorem 3.1. The result can already be deduced from the main theorem of [11], which also gives the uniqueness statement. □

5. Outlook

5.1. It might be interesting to characterize other C*-algebras as we have done for $\mathcal{Z}$ and $\mathcal{W}$. It should in particular be possible, for any $n \geq 2$, to obtain a universal construction of a simple, monotracial, stably projectionless C*-algebra $\mathcal{W}_n$ with $(K_0(\mathcal{W}_n), K_1(\mathcal{W}_n)) = (0, \mathbb{Z}/(n-1)\mathbb{Z})$. Candidate building blocks could be of the form

$$\{f \in C([0,1], M_m \otimes M_{(n-1)(m+1)}) : f(0) = a \otimes 1_{(n-1)(m+1)},\\ f(1) = a \otimes 1_{(n-1)m},\ a \in M_m\},$$

which at least have the right K-theory. Of course, $\mathcal{W}_2$ is just $\mathcal{W}$, obtained as in Theorem 4.3.

It was proved in [11] that $\mathcal{W} \otimes \mathcal{K} \cong \mathcal{O}_2 \rtimes \mathbb{R}$ for certain 'quasi-free' actions of $\mathbb{R}$ on the Cuntz algebra $\mathcal{O}_2$ (see for example [5] and [1]). More generally, one would expect (i.e. the Elliott conjecture predicts) that $\mathcal{W}_n \otimes \mathcal{K} \cong \mathcal{O}_n \rtimes \mathbb{R}$, and in this sense $\mathcal{W}_n$ might be thought of as a stably projectionless analogue of $\mathcal{O}_n$. (Similar speculation is made in the article [9].)

It is unclear what interpretation the corresponding universal *unital* algebras might have. Note for example that the Jiang–Su algebra is not stably isomorphic to a crossed product of a Kirchberg algebra by $\mathbb{R}$ (when simple, such a crossed product is either traceless or stably projectionless—see [6, Proposition 4]).

5.2. One of our motivations for presenting $\mathcal{Z}$ as a universal C*-algebra was to find a direct proof of its strong self-absorption (i.e. one that does not rely on classification). To put this problem into context, consider the other strongly self-absorbing C*-algebras. On the one hand, UHF algebras of infinite type can also be described in terms of order zero generators and relations, for example:

$$M_{2^\infty} \cong \mathrm{C}^*((\varphi_k)_{k=1}^\infty \mid \varphi_k \text{ order zero on } M_{q(k)},\\ \varphi_k(1_{q(k)}) = \varphi_k(1_{q(k)})^2, \varphi_k = \varphi_{k+1} \circ \mathrm{id}_{q(k)} \otimes 1_{q(k)} \otimes 1_{q(k)})$$

(where $q(k)$ is still 2^{3^k}), and the proof of strong self-absorption in this case amounts to linear algebra. On the other hand, while $\mathcal{O}_2$ and $\mathcal{O}_\infty$ are presented simply as $\mathrm{C}^*(s_1, s_2 \mid s_i^* s_i = 1 = s_1 s_1^* + s_2 s_2^*)$ and $\mathrm{C}^*((s_i)_{i=1}^\infty \mid s_i^* s_j = \delta_{ij})$ respectively, the proofs that $\mathcal{O}_2 \otimes \mathcal{O}_2 \cong \mathcal{O}_2$ and $\mathcal{O}_\infty \otimes \mathcal{O}_\infty \cong \mathcal{O}_\infty$ require some difficult analysis (see for example [13]). It is conceivable that our presentation of $\mathcal{Z}$ lies somewhere in the middle of this spectrum.

That being said, it is at least possible to show from our relations, in connection with [11], that the C*-algebra $\mathcal{Z}_U^{\otimes\infty}$ is strongly self-absorbing. (One first shows that $\mathcal{Z}_U^{\otimes\infty}$ has stable rank one and strict comparison, and then use the main theorem of [11] to show that any two of the canonical embeddings of $\mathcal{Z}_U$ into $\mathcal{Z}_U^{\otimes\infty}$ are approximately unitarily equivalent; this then yields strong self-absorption of $\mathcal{Z}_U^{\otimes\infty}$. Details will be given in the second named author's forthcoming CBMS monograph.)

Meanwhile, it remains an open problem to prove that $\mathcal{W} \otimes \mathcal{W} \cong \mathcal{W}$.

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