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A Vanishing Theorem for $H^p(X, \Omega^q(B))$ on Weakly 1-Complete Manifolds

By

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§ 1. Introduction

Let X be a connected complex manifold of (complex) dimension n. X is called weakly 1-complete if there exists an exhaustion function Φ which is C^∞ and plurisubharmonic on X. We set $X_c = \{x \in X; \Phi(x) < c\}$ for any real number c. Since S. Nakano established a vanishing theorem for positive bundles (cf. [8], [9]), there have been a lot of activities concerning analytic cohomology groups of weakly 1-complete manifolds (cf. [3], [6], [10], [11], [12], [13], [15], [16]). The aim of these works is to treat the cohomology groups from differential geometric viewpoint based on the curvature conditions on vector bundles rather than the strong pseudoconvexity of the base manifold X. So they are regarded as natural generalizations of the results obtained for compact manifolds. Lately, generalizing J. Girbau's work [4], O. Abdelkader [1] proved the following

Theorem 1. *Let X be a weakly 1-complete Kähler manifold and let B be a semi-positive line bundle over X whose curvature form has everywhere at least $n-k+1$ positive eigenvalues. Then*

$$H^p(X_c, \Omega^q(B)) = 0 \qquad \text{for } p+q \geqq n+k \text{ and any real } c\,.$$

In [16], the first author generalized Theorem 1 as follows:

Theorem 2. *Let X be a weakly 1-complete Kähler manifold and B a semipositive line bundle whose curvature form has at least $n-k+1$ positive eigenvalues outside a proper compact subset $K \subset X$. Then*

$$H^p(X, \Omega^n(B)) = 0 \qquad \text{for any } p \geqq k\,.$$

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The present note is a continuation of the above works. We shall prove the following

Theorem 3. *Let X and B be as in Theorem* 1, *then*

$$H^p(X, \Omega^q(B))=0 \qquad \text{for } p+q\geqq n+k.$$

The key tool here is an approximation theorem of Runge type which was introduced by the second author in [12].

During the preparation of this work (and the submission of [16]), H. Skoda's note [14] appeared, proving Theorem 3 for $p\geqq k$ and $q=n$. But his method does not seem to be applicable to prove Theorem 3 in general (cf. [14] Remark 2).

§2. Notations and Basic Formulae

Let $B\xrightarrow{\pi}X$ be a holomorphic line bundle over a complex manifold X and let $\{b_{ij}\}$ be a system of transition functions with respect to a coordinate cover $\{U_i\}_{i\in I}$ with holomorphic coordinates $(z_i^1,\dots, z_i^n)$. We fix a hermitian metric $\{a_i\}_{i\in I}$ along the fibers of B with respect to $\{U_i\}_{i\in I}$ and assume that X is provided with a Kähler metric ds^2. We set

$$ds^2=\sum_{\alpha,\beta=1}^{n} g_{i,\alpha\bar\beta}dz_i^\alpha d\bar z_i^\beta .$$

Let $C^{p,q}(X, B)$ be the space of B-valued differential forms on X, of class C^∞ and of type (p, q), and let $C_0^{p,q}(X, B)$ be the space of the forms in $C^{p,q}(X, B)$ with compact supports. We express $\varphi=\{\varphi_i\}_{i\in I}\in C^{p,q}(X, B)$ as

$$\varphi_i=\frac{1}{p!\cdot q!}\sum_{\substack{\alpha_1,\dots,\alpha_p\\ \beta_1,\dots,\beta_q}}\varphi_{i,\alpha_1\dots\alpha_p,\bar\beta_1\dots\bar\beta_q}dz_i^{\alpha_1}\wedge\cdots\wedge dz_i^{\alpha_p}\wedge d\bar z_i^{\beta_1}\wedge\cdots\wedge d\bar z_i^{\beta_q},$$

following the notation of [7]. For $\varphi\in C^{p,q}(X, B)$, we set

$$\begin{aligned}&\varphi_i^{\bar\alpha_1,\dots,\bar\alpha_p,\beta_1,\dots,\beta_q}\\ &=\sum_{\substack{c_1,\dots,c_p\\ d_1,\dots,d_q}} g_i^{\bar\alpha_1 c_1}\cdots g_i^{\bar\alpha_p c_p}g_i^{\bar d_1\beta_1}\cdots g_i^{\bar d_q\beta_q}\varphi_{i,c_1\cdots c_p\bar d_1\cdots\bar d_q}\,.\end{aligned}$$

For simplicity we write

$$\varphi_i^{\bar A_p B_q}=\sum g_i^{\bar A_p C_p}g_i^{\bar D_q B_q}\varphi_{i,C_p\bar D_q},$$

where $A_p=(\alpha_1,\dots, \alpha_p)$, $B_q=(\beta_1,\dots, \beta_q)$, and so on. With respect to $\{a_i\}_{i\in I}$ and ds^2, we set

$$(2.1) \qquad \langle\varphi, \psi\rangle = a_i \sum_{A_p, B_q} \varphi_{iA_p\bar{B}_q} \overline{\psi_i^{\bar{A}_p B_q}},$$

where $A_p=(\alpha_1,\dots,\alpha_p)$ and $B_q=(\beta_1,\dots,\beta_q)$ with $1\leqq\alpha_1<\alpha_2<\cdots<\alpha_p\leqq n$ and $1\leqq\beta_1<\cdots<\beta_q\leqq n$. Letting $*$ be the star operator and dV the volume element, we have

$$a_i\varphi_i \wedge *\bar{\psi}_i = \langle\varphi, \psi\rangle dV.$$

For any real valued C^∞-function Φ on X, we put

$$(2.2) \qquad \begin{cases} \text{i)} & \langle\varphi, \psi\rangle_\Phi = \langle\varphi, \psi\rangle e^{-\Phi} \\ \text{ii)} & (\varphi, \psi)_\Phi = \int_X \langle\varphi, \psi\rangle_\Phi dV, \qquad \text{for} \quad \varphi, \psi \in C_0^{p,q}(X, B). \end{cases}$$

In particular, we set

$$(\varphi, \psi) = (\varphi, \psi)_0,$$

$$\|\varphi\|^2 = (\varphi, \varphi),$$

$$\|\varphi\|_\Phi^2 = (\varphi, \varphi)_\Phi.$$

We denote by ϑ_Φ (resp. ϑ) the formal adjoint operator of $\bar\partial$: $C_0^{p,q}(X, B)\to C_0^{p,q+1}(X, B)$ with respect to the inner product $(\varphi, \psi)_\Phi$ (resp. (φ, ψ)). We define the Laplace-Beltrami operator $\square_\Phi$ (resp. $\square$) by

$$\square_\Phi = \bar\partial\vartheta_\Phi + \vartheta_\Phi\bar\partial \qquad (\text{resp. } \square = \bar\partial\vartheta + \vartheta\bar\partial).$$

We denote by $L^{p,q}(X, B, \Phi)$ the space of the square integrable B-valued (p, q) forms with respect to $\| \ \|_\Phi$. We denote by $\bar\partial$: $L^{p,q}(X, B, \Phi)\to L^{p,q+1}(X, B, \Phi)$ the maximal closed extension of the original $\bar\partial$. Since $\bar\partial$ is a closed densely defined operator, the adjoint operator $\bar\partial_\Phi^*$ (resp. $\bar\partial^*$) with respect to $(\varphi, \psi)_\Phi$ (resp. (φ, ψ)) can be defined. We denote the domain, the range, and the nullity of $\bar\partial$ in $L^{p,q}(X, B, \Phi)$, by $D_{\bar\partial}^{p,q}$, $R_{\bar\partial}^{p,q}$ and $N_{\bar\partial}^{p,q}$, respectively. Similarly $D_{\bar\partial_\Phi^*}^{p,q}$, $R_{\bar\partial_\Phi^*}^{p,q}$ and $N_{\bar\partial_\Phi^*}^{p,q}$ are defined. We denote by $e(\xi)$ the exterior multiplication by a differential form ξ on X. Let ω be the fundamental form of the Kähler metric ds^2 on X, and let

$$(2.3) \qquad \begin{aligned} L &= e(\omega) \\ \Lambda &= (-1)^{p+q} * L * . \end{aligned}$$

We set

$$(2.4) \qquad \chi = \sqrt{-1} \sum_{\alpha,\beta=1}^{n} \theta_{i,\alpha\bar\beta} dz_i^\alpha \wedge d\bar{z}_i^\beta,$$

where
$$\theta_{i,\alpha\bar{\beta}} = -\frac{\partial^2 \log a_i}{\partial z_i^\alpha \partial \bar{z}_i^\beta}.$$

Basic formulae:

$$e(\chi)\Lambda - \Lambda e(\chi) = \Box - *^{-1}\Box * \tag{2.5}$$

(cf. [8] p. 489 (1.16)),

$$\begin{aligned} \langle(\Lambda e(\chi) - e(\chi)\Lambda)\varphi, \varphi\rangle &= \Big(\sum_{\alpha,\beta=1}^{n} g_i^{\bar{\beta}\alpha}\theta_{i,\alpha\bar{\beta}}\Big)\langle\varphi, \varphi\rangle \\ &- \frac{1}{(p-1)!\,q!}\sum_{A_{p-1},B_q}\sum_{\alpha,\gamma}\sum_{\beta=1}^{n} g_i^{\bar{\beta}\alpha}\theta_{i,\gamma\bar{\beta}}\varphi_{i,\alpha A_{p-1}\bar{B}_q}\overline{\varphi_i^{\bar{\gamma}\bar{A}_{p-1}B_q}} \\ &- \frac{1}{p!(q-1)!}\sum_{A_p,B_{q-1}}\sum_{\beta,\gamma}\sum_{\alpha=1}^{n} g_i^{\bar{\beta}\alpha}\theta_{i,\alpha\bar{\gamma}}\varphi_{iA_p\bar{\beta}B_{q-1}}\overline{\varphi_i^{\bar{A}_p\gamma\beta_{q-1}}} \end{aligned} \tag{2.6}$$

(cf. [7] p. 132~133).

§3. Approximation Theorem

In accordance with the definition of q-pseudoconvexity of complex spaces (cf. [2]), we adopt the following definitions:

Definition 1. X is said to be *weakly* 1-*complete* if there exists an exhausting plurisubharmonic function Φ of class C^∞. Φ is called an *exhaustion function.*

Definition 2. A holomorphic line bundle $B \xrightarrow{\pi} X$ is said to be *k-semipositive* if there exists a trivializing coordinate cover $\{U_i\}_{i\in I}$ with holomorphic coordinates $(z_i^1,\dots, z_i^n)$ and a metric $\{a_i\}$ along the fibers of B such that the hermitian matrix $(-\partial^2\log a_i/\partial z_i^\alpha\partial\bar{z}_i^\beta)$ is positive semi-definite and has everywhere at least $n-k+1$ positive eigenvalues.

From now on, let X be a weakly 1-complete Kähler manifold with an exhaustion function Φ and let $B \xrightarrow{\pi} X$ be a k-semi-positive line bundle. We set $X_c = \{x \in X;\ \Phi(x) < c\}$ for any real number c.

Let (c, d) be a pair of real numbers such that d is a non-critical value of Φ and $c > d > 0$. We put $\lambda(t) = -2n\log(d-t)$. λ satisfies

$$\int_0^d \sqrt{\lambda''(t)}\,dt = +\infty. \tag{3.1}$$

Let $\{\lambda_\mu\}_{\mu\geqq 1}$ be a sequence of C^∞-strictly convex increasing functions on $(-\infty, c)$ such that

$$
(3.2)\quad \begin{cases} \text{i)} \displaystyle\int_0^c \sqrt{\lambda_\mu''(t)}\,dt = +\infty, \quad \text{for any} \quad \mu \geqq 1, \\ \text{ii)} \text{ for every } d' < d \text{ and every non-negative integer } \nu, \\ \qquad \displaystyle\lim_{\mu\to+\infty} \sup_{t\in(-\infty,d')} |\lambda_\mu^{(\nu)}(t) - \lambda^{(\nu)}(t)| = 0, \end{cases}
$$

where $\lambda_\mu^{(\nu)}$ (resp. $\lambda^{(\nu)}$) denotes the ν-th derivative of λ_μ (resp. λ).

Let $\{a_i\}$ be a fiber metric of B which corresponds to the assumption and set

$$
(3.3)\qquad \Theta_i = (\theta_{i,\alpha\bar\beta})_{1\leqq\alpha,\beta\leqq n}, \quad \text{where} \quad \theta_{i,\alpha\bar\beta} = -\frac{\partial^2 \log a_i}{\partial z_i^\alpha \partial \bar z_i^\beta}.
$$

We set

$$
(3.4)\qquad G_i = (g_{i,\alpha\bar\beta})_{1\leqq\alpha,\beta\leqq n},
$$

where $ds^2 = \sum_{\alpha,\beta=1}^n g_{i,\alpha\bar\beta} dz_i^\alpha d\bar z_i^\beta$ is the given Kähler metric on X. For some positive constant κ, which is determined later, we put

$$
(3.5)\quad \begin{cases} \text{i)} \ ds_\lambda^2 = \displaystyle\sum_{\alpha,\beta=1}^n (\kappa g_{i,\alpha\bar\beta} + \theta_{i,\alpha\bar\beta} + \partial_\alpha\partial_{\bar\beta}\lambda(\Phi)) dz_i^\alpha d\bar z_i^\beta, \text{ on } X_d, \\ \text{ii)} \ ds_\mu^2 = \displaystyle\sum_{\alpha,\beta=1}^n (\kappa g_{i,\alpha\bar\beta} + \theta_{i,\alpha\bar\beta} + \partial_\alpha\partial_{\bar\beta}\lambda_\mu(\Phi)) dz_i^\alpha d\bar z_i^\beta, \text{ on } X_c, \text{ for } \mu \geqq 1, \end{cases}
$$

$$
(3.6)\quad \begin{cases} \text{i)} \ G_{\lambda,i} = (\kappa g_{i,\alpha\bar\beta} + \theta_{i,\alpha\bar\beta} + \partial_\alpha\partial_{\bar\beta}\lambda(\Phi))_{1\leqq\alpha,\beta\leqq n}, \\ \text{ii)} \ G_{\mu,i} = (\kappa g_{i,\alpha\bar\beta} + \theta_{i,\alpha\bar\beta} + \partial_\alpha\partial_{\bar\beta}\lambda_\mu(\Phi))_{1\leqq\alpha,\beta\leqq n}, \text{ for } \mu \geqq 1, \\ \text{iii)} \ \Theta_{(\lambda),i} = (\theta_{i,\alpha\bar\beta} + \partial_\alpha\partial_{\bar\beta}\lambda(\Phi))_{1\leqq\alpha,\beta\leqq n}, \\ \text{iv)} \ \Theta_{\mu,i} = (\theta_{i,\alpha\bar\beta} + \partial_\alpha\partial_{\bar\beta}\lambda_\mu(\Phi))_{1\leqq\alpha,\beta\leqq n}, \text{ for } \mu \geqq 1, \end{cases}
$$

$$
(3.7)\quad \begin{cases} \text{i)} \ a_{\lambda,i} = a_i \cdot \exp(-\lambda(\Phi)), \\ \text{ii)} \ a_{\mu,i} = a_i \cdot \exp(-\lambda_\mu(\Phi)), \text{ for } \mu \geqq 1. \end{cases}
$$

Proposition 1. *The hermitian metric ds_μ^2 (resp. ds_λ^2) is a complete metric on X_c (resp. X_d) for every $\mu \geqq 1$ and $\kappa > 0$.*

Proof. From (3.1) and (3.2) i), it follows similarly as Proposition 1 in [8].

We can choose a matrix T_i which depends, together with T_i^{-1}, differentiably on $x \in U_i$, satisfying $G_i = {}^tT_i \overline{T}_i$. Then we have

$$
(3.8)\qquad {}^tT_i^{-1}\Theta_{\mu,i}\overline{T}_i^{-1} \geqq {}^tT_i^{-1}\Theta_i\overline{T}_i^{-1}, \quad \text{for} \quad \mu \geqq 1.
$$

Let $v_{\mu,1} \geqq v_{\mu,2} \geqq, \dots, \geqq v_{\mu,n} \geqq 0$ be the eigenvalues of ${}^tT_i^{-1}\Theta_{\mu,i}\overline{T}_i^{-1}$ at x_0. By (3.8), for any point $x_0 \in X_c$,

$$
(3.9)\qquad v_{\mu,n-k+1} \geqq \kappa_0, \quad \text{for} \quad \mu \geqq 1,
$$

where

$$\kappa_0=\inf_{x\in\bar{X}_c}\min_{\substack{L\lhd C^n\\ \dim L=n-k}}\max_{v\perp L}\frac{{}^t v^t T_i^{-1}\Theta_i \bar{T}_i^{-1}\bar{v}}{\sum_{\alpha=1}^{n}|v^\alpha|^2}>0.$$

Setting

$$\kappa=\frac{\kappa_0}{2n-1}, \tag{3.10}$$

we obtain the following proposition with respect to (3.5) ii) and (3.7) ii).

Proposition 2.

$$\|\varphi\|_{\lambda_\mu}^2\leqq 2\{\|\bar\partial\varphi\|_{\lambda_\mu}^2+\|\bar\partial_{\lambda_\mu}^*\varphi\|_{\lambda_\mu}^2\} \tag{3.11}$$

for any $\varphi\in D_{\bar\partial}^{p,q}\cap D_{\bar\partial_{\lambda_\mu}^*}^{p,q}\subset L^{p,q}(X_c, B, \lambda_\mu(\Phi))$ *with* $p+q\geqq n+k$ *and* $\mu\geqq 1$.

Proof. Since the base metric ds_μ^2 is complete for $\mu\geqq 1$ (cf. Proposition 1), $C_0^{p,q}(X_c, B)$ is dense in $D_{\bar\partial}^{p,q}\cap D_{\bar\partial_{\lambda_\mu}^*}^{p,q}$ with respect to the graph norm $(\|\varphi\|_{\lambda_\mu}^2+\|\bar\partial\varphi\|_{\lambda_\mu}^2+\|\bar\partial_{\lambda_\mu}^*\varphi\|_{\lambda_\mu}^2)^{1/2}$ (cf. [17] Theorem 1.1). Therefore it suffices to prove (3.11) for $\varphi\in C_0^{p,q}(X_c, B)$. According to Girbau's idea [4] our proof proceeds as follows.

We put

$$\omega_\mu=\sqrt{-1}\sum_{\alpha,\beta=1}^{n}(\kappa g_{i,\alpha\bar\beta}+\theta_{i,\alpha\bar\beta}+\partial_\alpha\partial_{\bar\beta}\lambda_\mu(\Phi))dz_i^\alpha\wedge d\bar z_i^\beta,$$

$$\chi_\mu=\sqrt{-1}\sum_{\alpha,\beta=1}^{n}(\theta_{i,\alpha\bar\beta}+\partial_\alpha\partial_{\bar\beta}\lambda_\mu(\Phi))dz_i^\alpha\wedge d\bar z_i^\beta,$$

$$L_\mu=e(\omega_\mu),\quad \Lambda_\mu=(-1)^{p+q}*_\mu L_\mu *_\mu,$$

where $*_\mu$ denotes the star operator with respect to ds_μ^2. Fix μ and let $p+q\geqq n+k$. We take a system of local coordinates $(z_i^1,\dots, z_i^n)$ around $x_0\in X_c$ so that we have

$$G_{\mu,i}=((\kappa+v_{\mu,\alpha})\delta_{\alpha\beta})_{1\leqq\alpha,\beta\leqq n}$$

$$G_{\mu,i}^{-1}\Theta_{\mu,i}=\left(\frac{v_{\mu,\alpha}}{\kappa+v_{\mu,\alpha}}\delta_{\alpha\beta}\right)_{1\leqq\alpha,\beta\leqq n}.$$

We choose an $m=m(x_0)\geqq 1$ so that $v_{\mu,n-k+m}>0$ and $v_{\mu,n-k+m+1}=0$. Combining these with (2.6), we have

$$(*)\quad \langle (\Lambda_\mu e(\omega_\mu) - e(\omega_\mu)\Lambda_\mu)\varphi, \varphi\rangle_{\lambda_\mu}(x_0)$$

$$= \frac{1}{p!q!} \sum_{A_p, B_q} \left(\sum_{\gamma=1}^{n} \frac{v_{\mu,\gamma}}{\kappa + v_{\mu,\gamma}}\right) \prod_{\sigma=1}^{p} \left(\frac{1}{\kappa + v_{\mu,\alpha_\sigma}}\right) \prod_{\tau=1}^{q} \left(\frac{1}{\kappa + v_{\mu,\beta_\tau}}\right) a_{\mu,i} |\varphi_{i,A_p\bar{B}_q}|^2$$

$$- \frac{1}{(p-1)!q!} \sum_{A_p, B_q} \left(\sum_{\sigma=1}^{p} \frac{v_{\mu,\alpha_\sigma}}{\kappa + v_{\mu,\alpha_\sigma}}\right) \prod_{\sigma=1}^{p} \left(\frac{1}{\kappa + v_{\mu,\alpha_\sigma}}\right) \prod_{\tau=1}^{q} \left(\frac{1}{\kappa + v_{\mu,\beta_\tau}}\right) \times a_{\mu,i} |\varphi_{i,A_p\bar{B}_q}|^2$$

$$- \frac{1}{p!(q-1)!} \sum_{A_p, B_q} \left(\sum_{\tau=1}^{q} \frac{v_{\mu,\beta_\tau}}{\kappa + v_{\mu,\beta_\tau}}\right) \prod_{\sigma=1}^{p} \left(\frac{1}{\kappa + v_{\mu,\alpha_\sigma}}\right) \prod_{\tau=1}^{q} \left(\frac{1}{\kappa + v_{\mu,\beta_\tau}}\right) \times a_{\mu,i} |\varphi_{i,A_p\bar{B}_q}|^2$$

$$= \sum_{A_p, B_q} \left(\sum_{\gamma=1}^{n} \frac{v_{\mu,\gamma}}{\kappa + v_{\mu,\gamma}} - \sum_{\sigma=1}^{p} \frac{v_{\mu,\alpha_\sigma}}{\kappa + v_{\mu,\alpha_\sigma}} - \sum_{\tau=1}^{q} \frac{v_{\mu,\beta_\tau}}{\kappa + v_{\mu,\beta_\tau}}\right) \prod_{\sigma=1}^{p} \left(\frac{1}{\kappa + v_{\mu,\alpha_\sigma}}\right) \times \prod_{\tau=1}^{q} \left(\frac{1}{\kappa + v_{\mu,\beta_\tau}}\right) a_{\mu,i} |\varphi_{i,A_p\bar{B}_q}|^2.$$

We put

$$\varepsilon_\gamma = 1 - \frac{v_{\mu,\gamma}}{\kappa + v_{\mu,\gamma}}.$$

From (3.10), we have $0 < \varepsilon_\gamma \leqq 1/2n$ for $\gamma \leqq n-k+1$ and $0 < \varepsilon_\gamma \leqq 1$ for $n-k+1 < \gamma \leqq n$.

Let s_1 (resp. s_2) be the number of indices with $\alpha_\sigma \leqq n-k+m$ (resp. $\beta_\tau \leqq n-k+m$), then $s_1 \geqq 1$, $s_2 \geqq 1$ and $s_1 + s_2 \geqq n-k+2m$. In fact, since $p+q \geqq n+k$, we have $p+q+n-k+1 \geqq 2n+1$, thus any blocks A_p of p indices and B_q of q indices taken from $\{1, 2, \dots, n\}$ contain one of the indices $\{1, 2, \dots, n-k+m\}$. Hence $s_1 \geqq 1$ and $s_2 \geqq 1$. On the other hand, in the indices $\{1, 2, \dots, n\} - \{\alpha_1, \dots, \alpha_p\}$ and $\{1, 2, \dots, n\} - \{\beta_1, \dots, \beta_q\}$, the sum of the number of indices contained in $\{1, 2, \dots, n-k+m\}$ is $2(n-k+m) - (s_1 + s_2)$. This number does not exceed $n-k$. Hence we have $s_1 + s_2 \geqq n-k+2m$. So we have

$$(**)\quad \sum_{\gamma=1}^{n} \frac{v_{\mu,\gamma}}{\kappa + v_{\mu,\gamma}} - \sum_{\sigma=1}^{p} \frac{v_{\mu,\alpha_\sigma}}{\kappa + v_{\mu,\alpha_\sigma}} - \sum_{\tau=1}^{q} \frac{v_{\mu,\beta_\tau}}{\kappa + v_{\mu,\beta_\tau}}$$

$$= n-k+m-(s_1+s_2) + \sum_{\alpha_\sigma \leqq n-k+m} \varepsilon_{\alpha_\sigma} + \sum_{\beta_\tau \leqq n-k+m} \varepsilon_\tau - \sum_{\gamma=1}^{n-k+m} \varepsilon_\gamma$$

$$\leqq -1 + \sum_{\alpha_\sigma \leqq n-k+1} \varepsilon_{\alpha_\sigma} + \sum_{\beta_\tau \leqq n-k+1} \varepsilon_{\beta_\tau} - \sum_{\gamma=1}^{n-k+1} \varepsilon_\gamma$$

$$-(m-1- \sum_{n-k+2 \leqq \alpha_\sigma \leqq n-k+m} \varepsilon_{\alpha_\sigma}) + (\sum_{n-k+2 \leqq \beta_\tau \leqq n-k+m} \varepsilon_{\beta_\tau} - \sum_{\gamma=n-k+2}^{n-k+m} \varepsilon_\gamma)$$

$$\leqq -1 + 2(\sum_{\gamma=1}^{n-k+1} \varepsilon_\gamma) - \sum_{\gamma=1}^{n-k+1} \varepsilon_\gamma$$

$$\leqq -1+\frac{n-k+1}{2n}$$

$$\leqq -\frac{1}{2}.$$

Therefore, from (*) and (**), we have

$$(***) \qquad \langle (\Lambda_\mu e(\omega_\mu)-e(\omega_\mu)\Lambda_\mu)\varphi,\ \varphi\rangle_{\lambda_\mu}(x_0) \leqq -\frac{1}{2}\langle \varphi,\ \varphi\rangle_{\lambda_\mu}(x_0).$$

Combining (2.5) with (***), we have

$$\|\bar\partial\varphi\|^2_{\lambda_\mu}+\|\bar\partial^*_{\lambda_\mu}\varphi\|^2_{\lambda_\mu}=(\square_{\lambda_\mu}\varphi,\ \varphi)_{\lambda_\mu}=(*^{-1}_{\lambda_\mu}\square_{\lambda_\mu}*_{\lambda_\mu}\varphi,\ \varphi)_{\lambda_\mu} +((e(\omega_\mu)\Lambda_\mu-\Lambda_\mu e(\omega_\mu)\varphi,\ \varphi)_{\lambda_\mu}\geqq\frac{1}{2}\|\varphi\|^2_{\lambda_\mu}. \qquad \text{q.e.d.}$$

Now we take the functions $\{\lambda_\mu\}_{\mu\geqq 1}$ satisfying (3.2) and the following additional condition:

(3.12) There exists a constant C such that for every μ and $\varphi\in L^{p,q}(X_c,\ B,\ \lambda_\mu(\Phi))$,

$$\|\varphi|_{X_d}\|_\lambda\leqq C\|\varphi\|_{\lambda_\mu}.$$

The existence of such functions has been proved (for any p, q) in [12], where B was assumed to be positive unnecessarily. We can easily obtain $\{\lambda_\mu\}_{\mu\geqq 1}$ with required properties without any significant modifications.

Approximation Theorem. *If $p+q\geqq n+k-1$, then for any $f\in L^{p,q}(X_d,\ B,\ \lambda(\Phi))$ with $\bar\partial f=0$ and for every $\varepsilon>0$, there exists an integer μ_0 and $\tilde f\in L^{p,q}(X_c,\ B,\ \lambda_{\mu_0}(\Phi))$ with $\bar\partial\tilde f=0$ and $\|\tilde f|_{X_d}-f\|_\lambda<\varepsilon$.*

Proof. It suffices to show that if $g\in L^{p,q}(X_d,\ B,\ \lambda(\Phi))$ and $(g,\tilde f|_{X_d})_\lambda=0$ for any $\tilde f\in\bigcup_{\mu=1}^{\infty}L^{p,q}(X_c,\ B,\ \lambda_\mu(\Phi))$ with $\bar\partial\tilde f=0$, then $(g,f)_\lambda=0$ for any $f\in L^{p,q}(X_d,\ B,\ \lambda(\Phi))$ with $\bar\partial f=0$. Now for any $u\in L^{p,q}(X_c,\ B,\ \lambda_\mu(\Phi))$ we have

$$|(g,\ u|_{X_d})_\lambda|\leqq C\|g\|_\lambda\|u\|_{\lambda_\mu},$$

where C is the constant introduced in (3.12).

This implies that $(g,\ \cdot|_{X_d})_\lambda$ is a continuous linear functional on $L^{p,q}(X_c,\ B,\ \lambda_\mu(\Phi))$, hence from the Riesz representation theorem there exists a $g_\mu\in L^{p,q}(X_c,\ B,\ \lambda_\mu(\Phi))$ such that $(g_\mu,\ u)_{\lambda_\mu}=(g,\ u|_{X_d})_\lambda$ for every $u\in L^{p,q}(X_c,\ B,\ \lambda_\mu(\Phi))$ and $\|g_\mu\|_{\lambda_\mu}\leqq C\|g\|_\lambda$. Since for every $\varphi\in C^{p,q}_0(X_c\backslash\bar X_d,\ B)$, we have $(g_\mu,\ \varphi)_{\lambda_\mu}$

$=(g, \varphi|_{X_d})_\lambda=0$, supp g_μ is contained in $\bar{X}_d$ and $\|g_\mu|_{X_d}\|_\lambda \leqq C^2\|g\|_\lambda$. By (3.2) ii) and the fact that supp $g_\mu \subset \bar{X}_d$ for $\mu \geqq 1$, we have $(g_\mu|_{X_d}, v)_\lambda \to (g, v)_\lambda$ as $\mu \to +\infty$ for every $v \in C_0^{p,q}(X_d, B)$. Therefore $\{g_\mu|_{X_d}\}$ converges weakly to g in $L^{p,q}(X_d, B, \lambda(\Phi))$. On the other hand, since g_μ is orthogonal to $N_{\bar\partial}^{p,q}$ in $L^{p,q}(X_c, B, \lambda_\mu(\Phi))$, g_μ is contained in the closure of $R_{\bar\partial^*_{\lambda_\mu}}^{p,q} \subset L^{p,q}(X_c, B, \lambda_\mu(\Phi))$. From (3.11), there exists a $w_\mu \in L^{p,q+1}(X_c, B, \lambda_\mu(\Phi))$ for any μ such that $g_\mu=\bar\partial^*_{\lambda_\mu} w_\mu$ and $\|w_\mu\|_{\lambda_\mu} \leqq 2^{1/2}\|g_\mu\|_{\lambda_\mu}$ (cf. [5] Lemma 4.1.1 and Lemma 4.1.2). We have $\|w_\mu|_{X_d}\|_\lambda \leqq C\|w_\mu\|_{\lambda_\mu} \leqq 2^{1/2}C^3\|g\|_\lambda$. Hence a subsequence $\{w_{\mu_k}|_{X_d}\}_{k\geqq 1}$ of $\{w_\mu|_{X_d}\}_{\mu \geqq 1}$ converges weakly to some $w \in L^{p,q+1}(X_d, B, \lambda(\Phi))$. For any $v \in C_0^{p,q}(X_d, B)$, we have $(g, v)_\lambda = \lim_{k\to+\infty}(g_{\mu_k}, v)_{\lambda_{\mu_k}} = \lim_{k\to+\infty}(w_{\mu_k}, \bar\partial v)_{\lambda_{\mu_k}} = \lim_{k\to+\infty}(w_{\mu_k}, \bar\partial v)_\lambda = (w, \bar\partial v)_\lambda$. Since ds_λ^2 is a complete metric on X_d, $C_0^{p,q}(X_d, B)$ is dense in $D_{\bar\partial}^{p,q} \subset L^{p,q}(X_d, B, \lambda(\Phi))$ with respect to the graph norm $(\|\varphi\|_\lambda^2+\|\bar\partial\varphi\|_\lambda^2)^{1/2}$ (cf. [17] Theorem 1.1). Thus $(g, v)_\lambda=(w, \bar\partial v)_\lambda$ for any $v \in D_{\bar\partial}^{p,q}$, whence $\bar\partial_\lambda^* w=g$ in $L^{p,q}(X_c, B, \lambda(\Phi))$. Therefore, for every $f \in L^{p,q}(X_d, B, \lambda(\Phi))$ with $\bar\partial f=0$, $(g, f)_\lambda=(\bar\partial_\lambda^* w, f)=(w, \bar\partial f)_\lambda$ $=0$. q. e. d.

§4. Proof of Theorem 3

By Sard's theorem, we can choose a sequence $\{c_\nu\}_{\nu=0,1,\dots}$ of real numbers such that

i) $c_{\nu+1}>c_\nu>0$ and $c_\nu \to +\infty$ as $\nu \to +\infty$,

ii) the boundary ∂X_{c_ν} of $\{x \in X; \Phi(x) \leqq c_\nu\}$ is smooth for any $\nu \geqq 0$.

For any $\nu \geqq 0$, we set

i) $\lambda^\nu(t)=-2n\log(c_\nu-t)$,

ii) $X_\nu=\{x \in X; \Phi(x)<c_\nu\}$.

For any pair $(c_{\nu+2}, c_\nu)$ $(\nu \geqq 0)$ and λ^ν, we choose a sequence of C^∞-strictly convex increasing functions $\{\lambda_\mu^{\nu+2}\}_{\mu \geqq 1}$ on $(-\infty, c_{\nu+2})$ satisfying the properties (3.2) and (3.12). The Approximation Theorem holds for any pair $(c_{\nu+2}, c_\nu)$ $(\nu \geqq 0)$. We denote by $L_{\mathrm{loc}}^{p,q}(X, B)$ (resp. $L_{\mathrm{loc}}^{p,q}(X_\nu, B)$) the set of the locally square integrable (p, q) forms on X (resp. X_ν) with values in B. For $p \geqq 1$, there is a natural isomorphism

(4.1) $H^p(X, \Omega^q(B))$

$$\simeq \frac{\{f \in L_{\mathrm{loc}}^{p,q}(X, B);\ \bar\partial f=0\}}{\{f \in L_{\mathrm{loc}}^{p,q}(X, B);\ f=\bar\partial g \text{ for some } g \in L_{\mathrm{loc}}^{p,q-1}(X, B)\}}$$

Therefore, in order to prove $H^p(X, \Omega^q(B))=0$ for $p+q\geqq n+k$, it suffices to show that for any $\varphi\in L^{p,q}_{\text{loc}}(X, B)$ with $\bar\partial\varphi=0$, there exists a $\psi\in L^{p,q-1}_{\text{loc}}(X, B)$ such that $\varphi=\bar\partial\psi$. We set $\varphi_\nu=\varphi\,|_{X_\nu}$ for any $\nu\geqq 0$. Then from Theorem 1 and (4.1), there exists a $\psi'_\nu\in L^{p,q-1}_{\text{loc}}(X_\nu, B)$ with $\varphi_\nu=\bar\partial\psi'_\nu$ for every $\nu\geqq 2$.

For any $\nu\geqq 1$, let $L^{p,q}(X_\nu, B)$ be the completion of $C^{p,q}_0(X_\nu, B)$ by the norm $\|\ \|_{X_\nu}$ with respect to the original Kähler metric ds^2 and the fiber metric $\{a_i\}$. Inductively, we choose a sequence $\{\psi_\nu\}_{\nu\geqq 1}$ so that

$$(4.2)\quad \begin{cases} \text{i)}\ \ \psi_\nu\in L^{p,q}(X_\nu, B), \\ \text{ii)}\ \ \bar\partial\psi_\nu=\varphi_\nu, \\ \text{iii)}\ \ \|\psi_{\nu+1}-\psi_\nu\|^2_{X_{\nu-1}}<\dfrac{1}{2^\nu}. \end{cases}$$

First we set $\psi_1=\psi'_2\,|_{X_1}$. Since $\varphi_2=\bar\partial\psi'_2$ in $L^{p,q}_{\text{loc}}(X_2, B)$, $\psi'_2\,|_{X_1}\in D^{p,q-1}_{\bar\partial}\subset L^{p,q-1}(X_1, B)$ and so $\bar\partial\psi_1=\varphi_1$ on X_1. Suppose $\psi_1,\dots,\psi_{\nu-1}$ are chosen. Then $(\psi'_{\gamma+1}-\psi_{\nu-1})\,|_{X_{\nu-1}}\in L^{p,q-1}(X_{\nu-1}, B, \lambda^{\nu-1}(\Phi))$ and $\bar\partial(\psi'_{\nu+1}-\psi_{\nu-1})\,|_{X_{\nu-1}}=0$.

By Approximation Theorem, for any $\varepsilon>0$, there exists a $g\in L^{p,q-1}(X_{\nu+1}, B, \lambda^{\nu+1}_{\mu_0}(\Phi))$ such that $\|g\,|_{X_{\nu-1}}-(\psi'_{\nu+1}-\psi_{\nu-1})\,|_{X_{\nu-1}}\|^2_{\lambda^{\nu-1}}<\varepsilon$ and $\bar\partial g=0$. Since $\|\ \|_{\lambda^{\nu-1},X_{\nu-2}}$ and $\|\ \|_{X_{\nu-2}}$ are equivalent norms on $L^{p,q-1}(X_{\nu-2}, B)$, we may assume

$$\|g\,|_{X_{\nu-2}}-(\psi'_{\nu+1}-\psi_{\nu-1})\,|_{X_{\nu-2}}\|^2_{X_{\nu-2}}<\frac{1}{2^{\nu-1}}.$$

We set $\psi_\nu=(\psi'_{\nu+1}-g)\,|_{X_\nu}$. Then we have

i) $\psi_\nu\in D^{p,q-1}_{\bar\partial}\subset L^{p,q-1}(X_\nu, B)$,

ii) $\varphi_\nu=\bar\partial\psi_\nu$,

iii) $\|\psi_\nu-\psi_{\nu-1}\|^2_{X_{\nu-2}}>\dfrac{1}{2^{\nu-1}}$.

Thus $\{\psi_\nu\}_{\nu\geqq 1}$ has been chosen. From (4.2), for any ν, $\{\psi_\mu\}_{\mu\geqq\nu+1}$ converges with respect to the norm $\|\ \|_{X_\nu}$ and clearly the limit is the same as the restriction of $\lim_{\mu\geqq\eta}\psi_\mu$ for any $\eta\geqq\nu+2$. Thus we can define an element ψ of $L^{p,q-1}_{\text{loc}}(X, B)$ by $\psi=\lim_{\nu\to+\infty}\psi_\nu$. Since $\bar\partial$ is a closed operator in $L^{p,q-1}(X_\nu, B)$ for every $\nu\geqq 1$, we have

$$\varphi_\nu=\bar\partial\psi \quad\text{in}\quad L^{p,q}(X_\nu, B) \qquad (\nu\geqq 1).$$

Hence we have $\varphi=\bar\partial\psi$ in $L^{p,q}_{\text{loc}}(X, B)$. q.e.d.

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