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Almost normal operators mod Hilbert–Schmidt and the K-theory of the algebras $E\Lambda(\Omega)$

Dan-Virgil Voiculescu*

Abstract. Is there a mod Hilbert–Schmidt analogue of the BDF-theorem, with the Pincus g-function playing the role of the index? We show that part of the question is about the K-theory of certain Banach algebras. These Banach algebras, related to Lipschitz functions and Dirichlet algebras have nice Banach-space duality properties. Moreover their corona algebras are C^*-algebras.

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1. Introduction

The BDF-theorem [7] classifies, up to unitary equivalence, the normal elements of the Calkin algebra, by the spectrum and the index of the resolvent. If the ideal of compact operators is replaced by the trace-class, for operators with trace-class self-commutator, the Pincus g-function ([8], [9]) is an L^1-function on $\mathbb{C}$ which extends the index of the essential resolvent. The g-function has been related to algebraic K-theory by L. G. Brown ([5], [6]) and in another direction, after work of J. W. Helton and R. Howe ([18]), the distribution to which the g-function gives rise, has been interpreted in terms of cyclic cohomology by A. Connes ([13]).

These developments around the g-function, were however not accompanied by a corresponding BDF-type result. In ([28], [26], [27]) we formulated conjectures about operators with trace-class self-commutator, an affirmative answer to which would fill this gap. Besides the initial evidence in favor of these conjectures, there was no further progress. The situation is roughly that the g-function viewed in the cyclic cohomology framework covers the index part and our work on Hilbert–Schmidt perturbations of normal operators ([25]) covers the part about trivial extensions, while the rest is wide open. The absence on the technical side of a normal dilation result which would correspond to the existence of inverses in Ext and which in

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the BDF context can be derived from the Choi–Effros completely positive lifting theorem, is a noted difficulty.

Our aim here is to decouple the normal dilation from the rest by introducing the algebras $E\Lambda(\Omega)$. In this way we are also able to bring K-theory to the study of this problem since we are led to the K_0-group of such an algebra.

The Banach $*$-algebras $E\Lambda(\Omega)$ are the natural framework to study operators with trace-class self-commutator which are obtained from compressions of normal operators to mod Hilbert–Schmidt reducing projections. Roughly $E\Lambda(\Omega)$, where Ω is a Borel subset of $\mathbb{C}$ is an algebra of operators in $L^2(\Omega, \lambda)$ with Hilbert–Schmidt commutators with the multiplication operators by Lipschitz functions, a construction reminiscent of Paschke-duality ([22]).

The algebras $E\Lambda(\Omega)$ have nice properties as Banach algebras. They resemble the Lipschitz algebras of [30], up to the use of a Hilbert–Schmidt norm instead of a uniform norm, which is a feature of the Dirichlet algebras of non-commutative potential theory ([1], [10], [11]). Actually the ideal $\mathcal{K}\Lambda(\Omega)$ of compact operators in $E\Lambda(\Omega)$ is a Dirichlet algebra and we show that $E\Lambda(\Omega)$ can be viewed both as the algebra of multipliers or as the bidual of $\mathcal{K}\Lambda(\Omega)$, when Ω is bounded. Since all this has the flavor of Banach algebra analogues of basic C^*-algebras, it is perhaps unexpected that the corona $E\Lambda(\Omega)/\mathcal{K}\Lambda(\Omega)$ which is the analogue of the Calkin algebra is really a C^*-algebra. Note, however, that while the Dirichlet algebra $\mathcal{K}\Lambda(\Omega)$ has the same simple K-theory as the algebra $\mathcal{K}(\mathcal{H})$ of compact operators, the K-theory of $E\Lambda(\Omega)$ and hence of $E\Lambda(\Omega)/\mathcal{K}\Lambda(\Omega)$, which interests us in connection with operators with trace-class self-commutator, is certainly richer.

On the technical side an essential ingredient is the existence of a bounded approximate unit consisting of projections for $\mathcal{K}\Lambda(\Omega)$, which is a consequence of our work on norm-ideal perturbations of Hilbert-space operators ([25], [29]).

Concerning the relation of the operator theory problems to the K-theory of the algebras $E\Lambda(\Omega)$, we should point out that while the K-theory problem is so to speak the operator theory problem minus the dilation problem, actually certain outcomes of the K-theory problem could provide a negative answer to the dilation problem. If the K-theory of $E\Lambda(\Omega)$ exhibits some integrality property making K_0 less rich this would answer in the negative the dilation problem.

In addition to the first section, which is the introduction, the paper has five more sections.

Section 2 contains background material about the conjectures about almost normal operators modulo Hilbert–Schmidt. Details of certain connections between these problems, left out previously, are included for the reader's convenience.

Section 3 introduces the algebras $E\Lambda(\Omega)$ and some of their basic properties. We also consider the ideal of compact operators $\mathcal{K}\Lambda(\Omega)$ of $E\Lambda(\Omega)$ and the Banach algebra $E\Lambda(\mathbb{C})_0$ which is the inductive limit of the $E\Lambda(\Omega)$ for bounded sets Ω.

In section 4 we look at the K-theory of the Banach algebras considered. We show that the problem about a mod Hilbert–Schmidt BDF-type theorem for almost normal operators is equivalent to the normal dilation problem plus the problem whether the K_0-group of $E\Lambda(\mathbb{C})_0$ is isomorphic via the Pincus g-function to the group $L^1_{\mathrm{re}}(\mathbb{C},\lambda)$ of real-valued L^1-functions with bounded support.

Section 5 returns to the algebras $\mathcal{K}\Lambda(\Omega)$, $E\Lambda(\Omega)$ and $(E/\mathcal{K})\Lambda(\Omega)$ and gives results about duality, multipliers and the relation to C^*-algebras.

Section 6 contains concluding remarks in several directions: the action of bi-Lipschitz homeomorphisms on the algebras, the center of $(E/\mathcal{K})\Lambda(\Omega)$, the relation to Dirichlet algebras and non-commutative potential theory, the possibility of similar constructions with other Schatten–von Neumann classes $\mathcal{C}_p$ replacing the Hilbert–Schmidt class.

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2. Background

2.1. If $\mathcal{H}$ is a separable infinite-dimensional Hilbert space over $\mathbb{C}$, then $\mathcal{B}(\mathcal{H})$ will denote the bounded operators on $\mathcal{H}$ and $\mathcal{C}_p(\mathcal{H})$ the *Schatten–von Neumann p-class*. The p-norm $|\cdot|_p$ is $|T|_p = \mathrm{Tr}(T^*T)^{p/2}$. In particular, $\mathcal{C}_1(\mathcal{H})$ is the *trace-class* and $\mathcal{C}_2(\mathcal{H})$ is the *Hilbert–Schmidt class*.

2.2. An operator $T \in B(\mathcal{H})$ is *almost normal* if its *self-commutator* $[T^*,T]$ is in $\mathcal{C}_1(\mathcal{H})$. Equivalently, if $T = A + iB$ with $A = A^*$, $B = B^*$ then $[A,B] \in \mathcal{C}_1$ since $2i[A,B] = [T^*,T]$. We shall denote by $\mathcal{AN}(\mathcal{H})$ the set of almost operators. Background material and references to the literature for many facts about operators with trace-class self-commutator can be found in the books [12], [21].

2.3. If $T = A + iB \in \mathcal{AN}(\mathcal{H})$ and if $Q,R \in \mathbb{C}[X,Y]$ are polynomials in two commuting indeterminates, then since $\tilde{A},\tilde{B}$ the class of A,B in $\mathcal{B}(\mathcal{H})/\mathcal{C}_1(\mathcal{H})$ commute, we shall also write $Q(A,B)$, $R(A,B)$ for elements in $\mathcal{B}(\mathcal{H})$ so that $\widetilde{Q(A,B)} = Q(\tilde{A},\tilde{B})$, $\widetilde{R(A,B)} = R(\tilde{A},\tilde{B})$. Clearly these are only defined up to a $\mathcal{C}_1$ perturbation. The *Helton–Howe measure* P_T of $T = A + iB \in \mathcal{AN}(\mathcal{H})$ ([18]) is a compactly supported measure on $\mathbb{R}^2$ so that

$$\mathrm{Tr}[Q(A,B),R(A,B)] = (2\pi i)^{-1}\int \mathcal{J}(Q,R)\,dP_T$$

where

$$\mathcal{J}(Q,R) = \frac{\partial Q}{\partial X}\frac{\partial R}{\partial Y} - \frac{\partial Q}{\partial Y}\frac{\partial R}{\partial X}\,.$$

Then $\mathrm{supp}\, P_T \subset \sigma(T)$ and P_T is absolutely continuous w.r.t. Lebesgue measure λ and the Radon–Nikodym derivative $\frac{dP_T}{d\lambda} = g_T \in L^1(\mathbb{R}^2)$ is the *Pincus principal function of T* (also called *Pincus g-function*).

2.4. Let $R_1^+(\mathcal{H}) = \{X \in B(\mathcal{H}) : X \text{ finite rank}, 0 \le X \le 1\}$, which is a directed ordered set. Then the obstruction to the existence of quasicentral approximate units relative to the Hilbert–Schmidt class ([25]) is

$$k_2(T_1, \ldots, T_n) = \liminf_{X \in R_1^+(\mathcal{H})} \max_{1 \le j \le n} |[T_j, X]|_2.$$

In [28] we showed that: *if* $T_1, T_2 \in \mathcal{AN}(\mathcal{H})$, $k_2(T_1) = 0$ *and* $T_1 - T_2 \in \mathcal{C}_2$, *then* $P_{T_1} = P_{T_2}$ *(or equivalently* $g_{T_1} = g_{T_2}$ *a.e.).*

2.5. We recall two of the conjectures about almost normal operators ([28] conjectures 3 and 4). Note that the second of these is a consequence of the first.

Conjecture 3 in [28]. *If* $T_1, T_2 \in \mathcal{AN}(\mathcal{H})$ *are so that* $P_{T_1} = P_{T_2}$ *then there is a normal operator* $N \in B(\mathcal{H})$ *and a unitary operator* $U \in B(\mathcal{H} \oplus \mathcal{H})$ *so that* $T_1 \oplus N - U(T_2 \oplus N)U^* \in \mathcal{C}_2$.

If true, this statement would represent a kind of BDF-theorem with $\mathcal{AN}(\mathcal{H})$ and the Helton–Howe measure replacing the operators with compact self-commutator and respectively the index-data. Note also that the unitary equivalence is mod $\mathcal{C}_2$ (not $\mathcal{C}_1$).

Conjecture 4 in [28]. *If* $T \in \mathcal{AN}(\mathcal{H})$ *then there is* $S \in \mathcal{AN}(\mathcal{H})$ *and a normal operator* $M \in B(\mathcal{H} \oplus \mathcal{H})$ *so that* $T \oplus S - M \in \mathcal{C}_2$.

This conjecture is an analogue of the existence of inverses in Ext in the analogue of the "Ext is a group" part of the BDF theorem. Note that the analogue of the results for trivial extensions (i.e., Weyl–von Neumann theorem part) is covered by our results in [25]. For the derivation of Conjecture 4 from Conjecture 3 one also uses the result of R. V. Carey and J. D. Pincus that every L^1-function is the g-function of some $T \in \mathcal{AN}(\mathcal{H})$ (see 4.9).

2.6. We would like to remark that Conjectures 3 and 4 in [28] don't bring the essential spectrum of the almost normal operators into the discussion. With consideration of the essential spectrum $\sigma_e(T)$, one might ask if $P_{T_1} = P_{T_2}$ and $\sigma_e(T_1) = \sigma_e(T_2)$ would imply $T_1 - UT_2U^* \in \mathcal{C}_2$, form some unitary U, when $T_j \in \mathcal{AN}(\mathcal{H})$, $j = 1, 2$.

We didn't discuss the possibility of such a strengthening, because it seems to have to do also with phenomena of another kind involving perturbations of isolated points in $\sigma(T) \backslash \sigma_e(T)$.

2.7. A consequence of Conjecture 4 and hence also of Conjecture 3 is the following conjecture.

Conjecture 1 in [28]. *If* $T \in \mathcal{AN}(\mathcal{H})$ *then* $k_2(T) = 0$.

The proof which was omitted in [28], involves using *a result of* [25], *that* $k_2(N) = 0$ *for every normal operator* N. Indeed, if Conjecture 4 holds for T, then $T \in \mathcal{AN}(\mathcal{H})$ is unitarily equivalent mod $\mathcal{C}_2$ to a compression $PN \mid P\mathcal{H}$

where $P = P^* = P^2$ is a projection, N is normal and $[P, N] \in \mathcal{C}_2$. We infer that $k_2(T) = k_2(PN \mid P\mathcal{H})$. On the other hand $k_2(N) = 0$ implies there are $X_n \in R_1^+(\mathcal{H})$, $X_n \uparrow I$ as $n \to \infty$, so that $\lim_{n\to\infty} |[X_n, N]|_2 = 0$. If $Y_n = PX_nP$ then $Y_n \in R_1^+$ and we have $Y_n \uparrow P$ as $n \to \infty$. We have

$$|[Y_n, PNP]|_2 = |P[X_n, PNP]P|_2 \le |P[X_n, NP]P|_2 + |[I - X_n, [P, N]P]|_2.$$

Since $[P, N]P \in \mathcal{C}_2$ and $I - X_n \downarrow 0$ we have $|[I - X_n, [P, N]P]|_2 \to 0$ as $n \to \infty$. On the other hand

$$|P[X_n, NP]P|_2 \le |P[I - X_n, N]P|_2 + |P[N, P](I - X_n)P|_2$$

which converges to 0 as $n \to \infty$. Thus, Conjecture 1 holds for T, i.e., $k_2(T) = 0$.

2.8. We will also need to recall some of the results for normal operators which follow from [25]. Since $k_2(N) = 0$ for every normal operator N, we can use the kind of non-commutative Weyl–von Neumann results in [25] to infer that: if N_1 and N_2 are normal operators on $\mathcal{H}$ and $\sigma(N_1) = \sigma(N_2) = \sigma_e(N_1) = \sigma_e(N_2)$ then there is a unitary operator U so that $UN_1U^* - N_2 \in \mathcal{C}_2$ and $|UN_1U^* - N_2|_2 < \varepsilon$ for a given $\varepsilon > 0$.

Also, if $T \in \mathcal{AN}(\mathcal{H})$ and N is a normal operator with $\sigma(N) = \sigma_e(T)$ then there is a unitary operator $U : \mathcal{H} \to \mathcal{H} \oplus \mathcal{H}$ so that $(T \oplus N)U - UT$ is Hilbert–Schmidt and $|(T \oplus N)U - UT|_2 < \varepsilon$ for a given $\varepsilon > 0$,

3. The Banach Algebras $E\Lambda(\Omega)$

3.1. We shall define here the algebras $E\Lambda(\Omega)$ and give a few of their basic properties.

If $\Omega \subset \mathbb{C}$ is a Borel set and $f \in L^\infty(\Omega, \lambda)$, with λ denoting Lebesgue measure, let M_f be the multiplication operator by f on $L^2(\Omega, \lambda)$ and Df be the difference quotient

$$Df(s, t) = \frac{f(s) - f(t)}{s - t}(s \ne t)$$

which is the class up to null-sets of a Lebesgue-measurable function on $\Omega \times \Omega$. Let further

$$\Lambda(\Omega) = \{f \in L^\infty(\Omega, \lambda) \mid Df \in L^\infty(\Omega \times \Omega, \lambda \otimes \lambda)$$

be the subalgebra of essentially Lipschitz functions. If $T \in \mathcal{B}(L^2(\Omega, \lambda))$ let $L(T)$ be given by

$$L(T) = \sup\{|[M_f, T]|_2 \mid f \in \Lambda(\Omega), \|Df\|_\infty \le 1\}.$$

We define $E\Lambda(\Omega)$ to be the subalgebra of $\mathcal{B}(L^2(\Omega))$

$$E\Lambda(\Omega) = \{T \in \mathcal{B}(L^2(\Omega, \lambda)) \mid L(T) < \infty\}.$$

It is easily seen that $E\Lambda(\Omega)$ is a $*$-subalgebra of $\mathcal{B}(L^2(\Omega,\Omega))$. Even more, $E\Lambda(\Omega)$ is an involutive Banach algebra with respect to the norm $|||T||| = \|T\| + L(T)$ and the involution is isometric $|||T||| = |||T^*|||$. The proof is along standard lines and will be left to the reader.

3.2. If Ω is specified and $w \in \mathbb{C}$, let $(e(w))(z) = \exp(i\,\mathrm{Re}(z\overline{w}))$ and let $U(w) = M_{e(w)}$, which is a unitary operator on $L^2(\Omega,\lambda)$. Also, if Ω is bounded, the multiplication operators by the functions which at $x+iy$ equal $x+iy$, x, y will be denoted by Z, X, Y.

Proposition 3.3. *If $T \in \mathcal{B}(L^2(\Omega,\lambda))$ and*

$$L_1(T) = \sup\{|w|^{-1}|[T,U(w)]|_2 \mid w \in \mathbb{C}\backslash\{0\}\}$$

then we have $L_1(T) \leq L(T) \leq 2L_1(T)$ and $|||T|||_1 = \|T\| + L_1(T)$ is an equivalent Banach algebra norm on $E\Lambda(\Omega)$.

If Ω is bounded then we have

$$L(T) = |[T,Z]|_2.$$

Proof. We first establish the assertions of the proposition in case $T \in \mathcal{C}_2$. Then T is given by a kernel $K \in L^2(\Omega\times\Omega,\lambda\otimes\lambda)$ and the kernel of $[M_f,T]$ is $(f(s)-f(t))K(s,t)$. The supremum of $\mathcal{C}_2$-norms of $[M_f,T]$ over all f with $\|Df\|_\infty \leq 1$ will then equal the L^2-norm of $(s-t)K(s,t)$, which for bounded Ω is the kernel of $[Z,T]$. On the other hand, if $f = e(w)|w|^{-1}$ we have $\|Df\|_\infty \leq 1$, so that $L_1(T) \leq L(T)$. Further, taking $w = \varepsilon w_0$, for some w_0 with $|w_0| = 1$ and letting $\varepsilon \downarrow 0$, the supremum of L^2-norms of the corresponding $(f(s)-f(t))K(s,t)$ will be the L^2-norm of $\mathrm{Re}((s-t)\overline{w}_0)K(s,t)$. The bound $L(T) \leq 2L_1(T)$ is then obtained taking for instance $w_0 = 1$ and $w_0 = i$.

To deal with general T, we first take up the assertion that $L(T) = |[Z,T]|_2$ when Ω is bounded. Clearly it suffices to show that $L(T) \leq |[Z,T]|_2$ the opposite inequality being obvious. By our results in [25], since Z is a normal operator, there are finite rank projections $P_n \uparrow I$ so that $|[P_n,Z]|_2 \to 0$ as $n\to\infty$. Then if f is such that $\|Df\|_\infty \leq 1$, using the result for the Hilbert–Schmidt case, we have

$$\begin{aligned}
|[M_f,T]|_2 &\leq \limsup_{n\to\infty} |[M_f,P_nTP_n]|_2\\
&\leq \limsup_{n\to\infty} |[Z,P_nTP_n]|_2\\
&\leq \limsup_{n\to\infty} (2|[Z,P_n]|_2\|T\| + |[Z,T]|_2)\\
&= |[Z,T]|_2.
\end{aligned}$$

To prove the assertion about $L_1(T)$ for unbounded Ω and general T, we proceed along similar lines, after showing that there exist finite rank projections $P_n \uparrow 1$ so that

$$\lim_{n\to\infty}\left(\sup_{w\in\mathbb{C}\setminus\{0\}} |[w^{-1}U(w), P_n]|_2\right) = 0.$$

Let $\Omega_m = \{z \in \Omega \mid m-1 \le |z| < m\}$ so that Ω is the disjoint union of the Ω_m, $m \in \mathbb{N}$. On $L^2(\Omega_m, \lambda)$ we can find, by our result from [25], finite rank projections P_{km} so that $P_{km} \uparrow I$ as $k \to \infty$ and $|[P_{km}, Z]|_2 \le (k^2 m)^{-1}$. Observe that by the result about $|[Z, T]|_2$ we proved, this gives $L(P_{km}) \le (k^2 m)^{-1}$. We then define the projection P_m acting on $L^2(\Omega, \lambda) = L^2(\Omega_1, \lambda) \oplus L^2(\Omega_2, \lambda) \oplus \ \dots$ to be $P_{m1} \oplus P_{m2} \oplus \cdots \oplus P_{mm} \oplus 0 \oplus 0 \oplus \dots$ so that $P_m \uparrow I$ and $L(P_m) \le L(P_{m1}) + \cdots + L(P_{mm}) \le Cm^{-1}$. Since $\|Dw^{-1}e(w)\|_\infty \le 1$ we have $|[w^{-1}U(w), P_m]|_2 \le Cm^{-1}$ which clearly converges to zero as $m \to \infty$ uniformly for $w \in \mathbb{C}\setminus\{0\}$. We then have for $f \in \Lambda(\Omega)$ with $\|Df\|_\infty \le 1$ and $T \in \mathcal{B}(L^2(\Omega, \lambda))$

$$\begin{aligned}
|[M_f, T]|_2 &\le \limsup_{n\to\infty} |[M_f, P_n T P_n]|_2 \\
&\le \limsup_{n\to\infty} 2L_1(P_n T P_n) \\
&\le \limsup_{n\to\infty} (4L_1(P_n)\|T\| + 2L_1(T)) \\
&= 2L_1(T).
\end{aligned}$$

□

3.4. If $\Omega = \mathbb{C}$ the proposition provides a characterization of the algebra $E\Lambda(\Omega)$ which translates well after Fourier transform. Let $\mathcal{F} : L^2(\mathbb{C}, \lambda) \to L^2(\mathbb{C}, \lambda)$ be the unitary Fourier transform

$$(\mathcal{F}f)(w) = c\int_{\mathbb{C}} f(z)(e(-w))(z)\,d\lambda(z).$$

Then $\mathcal{F}U(w_0) = V(w_0)\mathcal{F}$ where $(V(w_0)g)(w) = g(w - w_0)$ and we have the following corollary.

Corollary 3.5. *If*

$$S, T \in \mathcal{B}(L^2(\mathbb{C}, \lambda))$$

and

$$M(S) = \sup\{|w_0|^{-1}|S - V(w_0)SV(w_0)^*|_2 \mid w_0 \in \mathbb{C}\setminus\{0\}\}$$

then we have

$$M(\mathcal{F}T\mathcal{F}^{-1}) = L_1(T) \text{ and } \mathcal{F}E\Lambda(\mathbb{C})\mathcal{F}^{-1} = \{S \in \mathcal{B}(L^2(\mathbb{C}, \lambda)) | M(S) < \infty\}.$$

3.6. If $\Omega_1 \subset \Omega_2$ let

$$i(\Omega_2, \Omega_1) : \mathcal{B}(L^2(\Omega_1, \lambda)) \to \mathcal{B}(L^2(\Omega_2, \lambda))$$

be the inclusion homomorphism defined by $i(\Omega_2, \Omega_1)(T) = T \oplus 0$ with respect to the decomposition $L^2(\Omega_2, \lambda) = L^2(\Omega_1, \lambda) \oplus L^2(\Omega_2 \backslash \Omega_1, \lambda)$. There is also a conditional expectation $\varepsilon(\Omega_1, \Omega_2) : \mathcal{B}(L^2(\Omega_2, \lambda)) \to \mathcal{B}(L^2(\Omega_1, \lambda))$, $\varepsilon(\Omega_1, \Omega_2)(S) = M_{\mathcal{X}_{\Omega_1}} S M_{\mathcal{X}_{\Omega_1}} \mid L^2(\Omega_1, \lambda)$ where $\mathcal{X}_{\Omega_1}$ is the indicator function of the subset Ω_1 of Ω_2. It is easily checked that the Banach algebras $E\Lambda(\Omega)$ behave well with respect to the $i(\Omega_2, \Omega_1)$ and $\varepsilon(\Omega_2, \Omega_1)$.

Proposition 3.7. *If $\Omega_1 \subset \Omega_2$ then we have*

$$i(\Omega_2, \Omega_1)(E\Lambda(\Omega_1)) \subset E\Lambda(\Omega_2)$$

and the inclusion is isometric with respect to the $|||\cdot|||$-norms and also with respect to the $|||\cdot|||_1$-norms and $L(\cdot)$ and $L_1(\cdot)$ are preserved. We also have $\varepsilon(\Omega_1, \Omega_2)(E\Lambda(\Omega_2)) = E\Lambda(\Omega_1)$ and $\varepsilon(\Omega_1, \Omega_2)$ is contractive both in the $|||\cdot|||$-norms and in the $|||\cdot|||_1$-norms and we have $\varepsilon(\Omega_1, \Omega_2) i(\Omega_2, \Omega_1)(T) = T$.

3.8. We define the Banach subalgebra $E\Lambda(\Omega)_0 \subset E\Lambda(\Omega)$ to be the closure in $E\Lambda(\Omega)$ of $\bigcup\{i(\Omega, \Omega_1) E\Lambda(\Omega_1) \mid \Omega_1 \subset \Omega, \Omega_1 \text{bounded Borel set}\}$. Equivalently $E\Lambda(\Omega)_0$ is the closure in $E\Lambda(\Omega)$ of $\bigcup_{r>0} i(\Omega, \Omega \cap r\mathbb{D}) E\Lambda(\Omega \cap r\mathbb{D})$ where $\mathbb{D}$ is the unit disk.

Proposition 3.9. *$E\Lambda(\Omega)_0$ is an ideal in $E\Lambda(\Omega)$. If $\mathcal{X}_{\Omega \cap n\mathbb{D}}$ is the indicator function of $n\mathbb{D} \cap \Omega$ as a subset of Ω and $M_n = M_{\mathcal{X}_{\Omega \cap n\mathbb{D}}}$, then $(M_n)_{n \geq 1}$ is an approximate unit of $E\Lambda(\Omega)_0$.*

Proof. Since $|||M_n||| = \|M_n\| = 1$ and $M_n x = x M_n = x$ for any $x \in \bigcup\{i(\Omega, \Omega_1) E\Lambda(\Omega_1) \mid \Omega_1 \subset \Omega, \Omega_1 \text{bounded Borel}\}$ as soon as n is large enough, we clearly have that $(M_n)_{n \geq 1}$ is an approximate unit of $E\Lambda(\Omega)_0$. To prove that $E\Lambda(\Omega)_0$ is a two-sided ideal in $E\Lambda(\Omega)$ it will suffice now to show that $TM_n \in E\Lambda(\Omega)_0$ and $M_n T \in E\Lambda(\Omega)_0$. Actually since we deal with involutive algebras it will suffice to show that $TM_n \in E\Lambda(\Omega)_0$ and this in turn reduces to checking that $|||(I - M_m) T M_n||| \to 0$ as $m \to +\infty$. It is easily seen that $L(T) < \infty$ implies $(I - M_{n+1}) T M_n \in \mathcal{C}_2$ and hence $\|(I - M_m) T M_n\| \leq |(I - M_m)(I - M_{n+1}) T M_n|_2 \to 0$ as $m \to +\infty$. Also if $K(z_1, z_2)$ is the kernel of $(I - M_{n+1}) T M_n$ then $L((I - M_{n+1}) T M_n) < \infty$ means $(z_1 - z_2) K(z_1, z_2)$ is in $L^2(\Omega \times \Omega, \lambda \otimes \lambda)$. Then if $m > n + 1$, $L((I - M_m) T M_n)$ is the L^2-norm of the kernel

$$(1 - \mathcal{X}_{\Omega \cap m\mathbb{D}}(z_1))(z_1 - z_2) K(z_1, z_2)$$

which converges to zero as $m \to +\infty$. □

3.10. If Ω is bounded $\mathcal{C}_2(L^2(\Omega,\lambda)) \subset E\Lambda(\Omega)$ and $|||X||| \le (1+d)|X|_2$ where d is the diameter of Ω when $X \in \mathcal{C}_2(L^2(\Omega,\lambda))$. If Ω is unbounded the $\mathcal{C}_2\Lambda(\Omega) = \mathcal{C}_2(L^2(\Omega,\lambda)) \cap E\Lambda(\Omega)$ is only a subset of $\mathcal{C}_2(L^2(\Omega,\lambda))$. Similarly $R\Lambda(\Omega)$ will denote $\mathcal{R}(L^2(\Omega,\lambda)) \cap E\Lambda(\Omega)$ where $\mathcal{R}(\mathcal{H})$ stands for the finite rank operator on $\mathcal{H}$. Remark also that if $L^2\Lambda(\Omega)$ denotes functions $f \in L^2(\Omega,\lambda)$ so that $f(z)(1+|z|) \in L^2$ then the linear span of $\langle \cdot, f\rangle g$ is in $\mathcal{R}\Lambda(\Omega)$ when $f, g \in L^2\Lambda(\Omega)$. Note also that if $f \in L^\infty(\Omega,\lambda)$ then $|||M_f||| = |||Mf|||_1 = \|f\|_\infty = \|M_f\|$ since $L(M_f) = 0$ and $ML^\infty(\Omega) = \{M_f \mid f \in L^\infty(\Omega,\lambda)\} \subset E\Lambda(\Omega)$.

The following lemma records a consequence of the diagonalizability mod $\mathcal{C}_2$ of normal operators, which appeared in the last part of the proof of Proposition 3.3.

Lemma 3.11. *In $E\Lambda(\Omega)$ there are finite rank projections P_n, so that $P_n \uparrow I$ and*

$$\lim_{n\to\infty} L(P_n) = 0.$$

Moreover we have $P_n \in i(\Omega, \Omega\cap n\mathbb{D})E\Lambda(\Omega\cap n\mathbb{D})$ and $[P_n, M_{\mathcal{X}_{\Omega\cap m\mathbb{D}}}] = 0$ for all $m \in \mathbb{N}$.

We will also find it useful to have the following technical lemma when Ω is unbounded.

Lemma 3.12. *Let $M_n = M_{\mathcal{X}_n} \in ML^\infty(\Omega,\lambda)$ where $\mathcal{X}_n$ is the indicator function of $\Omega \cap n\mathbb{D}$ as a subset of Ω and let $T \in E\Lambda(\Omega)$. Then we have $L(T - M_nTM_n) \to 0$ as $n \to \infty$.*

Proof. If $\Omega_n = \Omega \cap n\mathbb{D}$, then we have $M_nTM_n = i(\Omega,\Omega_n)\varepsilon(\Omega_n,\Omega)(T)$. With T_n denoting $\varepsilon(\Omega_n,\Omega)(T)$ and X_n denoting $i(\Omega,\Omega_n)([Z,T_n])$ we have the following martingale properties. If $m \ge n$ then $M_nX_mM_n = X_n$ and $|X_n|_2 = L(T_n) \le L(T)$. Hence, if X is a weak limit of some subsequence of the X_m's as $m \to \infty$ we will have $|X|_2 < \infty$ and $X_n = M_nXM_n$. Thus if $m \ge n$

$$\begin{aligned} L(M_mTM_m - M_nTM_n) &= L(\varepsilon(\Omega_m,\Omega)(M_mTM_m - M_nTM_n)) \\ &= |[Z, \varepsilon(\Omega_m,\Omega)(M_mTM_m - M_nTM_n)]|_2 \\ &= |X_m - X_n|_2. \end{aligned}$$

Since M_mTM_m converges weakly to T and X_m converges in 2-norm to X as $m \to \infty$, we infer

$$L(T - M_nTM_n) \le \sup_{m\ge n} L(M_mTM_m - M_nTM_n) = \sup_{m\ge n} |X_m - X_n|_2 = |X_m - X_n|_2.$$

The assertion of the lemma follows from

$$|X - X_n|_2 = |X - M_nXM_n|_2 \to 0$$

as $n \to \infty$. □

3.13. We define $\mathcal{K}\Lambda(\Omega) = \{T \in E\Lambda(\Omega) \mid T \text{compact}\}$. Clearly, $\mathcal{K}\Lambda(\Omega)$ is a closed ideal in $E\Lambda(\Omega)$.

Proposition 3.14. *The ideal $\mathcal{K}\Lambda(\Omega)$ of $E\Lambda(\Omega)$ has an approximate unit $(P_n)_{n\geq 1}$ where P_n's are self-adjoint projections with the properties outlined in Lemma 3.11. In particular $\bigcup_{n\geq 1} P_n\mathcal{B}(L^2(\Omega,\lambda))P_n$ is a dense subalgebra in $\mathcal{K}\Lambda(\Omega)$ in $|||\cdot|||$-norm.*

Proof. If $T \in \mathcal{K}\Lambda(\Omega)$ then with the notation in Lemma 3.12 we actually have $|||T - M_nTM_n||| \to 0$ as $n \to \infty$ in view of the lemma and of the compactness of T which gives $\|T - M_nTM_n\| \to 0$. In view of the involution, the proof reduces to showing that $|||T - P_mT||| \to 0$ as $m \to \infty$ where P_m are the projections in Lemma 3.11 and $T \in \mathcal{K}\Lambda(\Omega)$ satisfies $T = M_nTM_n$ for some fixed n.

Clearly T being compact we have $\|T - P_mT\| \to 0$ as $m \to \infty$.

On the other hand if $m \geq n$, $T - P_mT = i(\Omega, \Omega\cap n\mathbb{D})(T' - P'_mT')$ where $T' = \varepsilon(\Omega\cap n\mathbb{D},\Omega)(T)$ satisfies $i(\Omega,\Omega\cap n\mathbb{D})(T') = T$ and $P'_m = \varepsilon(\Omega\cap n\mathbb{D},\Omega)(P_m) = \varepsilon(\Omega\cap n\mathbb{D},\Omega)(P_mM_n)$ is a projection. We have

$$\begin{aligned} L(T - P_mT) &= L(T' - P'_mT') \\ &= |[Z,(I - P'_m)T']|_2 \\ &\leq L(I - P'_m)\|T\| + |(I - P'_m)[Z,T']|_2 \to 0 \end{aligned}$$

since $L(P'_m) \leq L(P_m) \to 0$ and $[Z,T'] \in \mathcal{C}_2$, $P'_m \uparrow I$.

The remaining assertion follows from the fact that P_n is an approximate unit once we remark that $P_n\mathcal{B}(L^2(\Omega,\lambda))P_n = P_nE\Lambda(\Omega)P_n = P_n\mathcal{K}\Lambda(\Omega)P_n$ because $P_n = M_nP_nM_n$. □

Proposition 3.15. *The unit ball of $E\Lambda(\Omega)$ in $|||\cdot|||$-norm or $|||\cdot|||_1$-norm is closed in the weak operator topology and hence is weakly compact. Moreover, $E\Lambda(\Omega)$ is inverse-closed as a subalgebra of $\mathcal{B}(L^2(\Omega))$ and also closed under C^∞-functional calculus for normal elements. In particular if $T \in E\Lambda(\Omega)$ has bounded inverse and $T = V|T|$ is its polar decomposition, then $V, |T|$ are in $E\Lambda(\Omega)$.*

The proof is an exercise along standard lines and will be omitted.

3.16. We shall denote by $(E/\mathcal{K})\Lambda(\Omega)$ the quotient-Banach algebra $E\Lambda(\Omega)/\mathcal{K}\Lambda(\Omega)$ and by $p : E\Lambda(\Omega) \to (E/\mathcal{K})\Lambda(\Omega)$ the canonical surjection.

Remark also that we have $\mathcal{K}\Lambda(\Omega) \subset E\Lambda(\Omega)_0$ since the dense subalgebra of $\mathcal{K}\Lambda(\Omega)$ appearing in Proposition 3.14 is in $E\Lambda(\Omega)_0$. The quotient $E\Lambda(\Omega)_0/\mathcal{K}\Lambda(\Omega)$ will also be denoted $(E_0/\mathcal{K})\Lambda(\Omega)$.

Proposition 3.17. *Given $n \in \mathbb{N}$ there are $U_k \in E\Lambda(\Omega)$, $1 \leq k \leq n$, such that $U : L^2(\Omega,\lambda) \to L^2(\Omega,\lambda) \otimes \mathbb{C}^n$ defined by $Uh = \sum_k U_kh \otimes e_k$ is a unitary operator. In particular we have $UE\Lambda(\Omega)U^* = E\Lambda(\Omega) \otimes M_n$ and $T \to UTU^*$*

is a spatial isomorphism of $E\Lambda(\Omega)$ *and* $E\Lambda(\Omega) \otimes \mathcal{M}_n$. *Additionally we also have that*

$$UE\Lambda(\mathbb{C})_0 U^* = E\Lambda(\mathbb{C})_0 \otimes \mathcal{M}_n.$$

Proof. The existence of U is a consequence of our results on normal operator mod $\mathcal{C}_2$ ([25], 2.8). There will be some additional technicalities due to the fact that Ω may be unbounded. From ([25], 2.8) we get the existence of unitary operators $V_m : L^2(\Omega_m, \lambda) \to L^2(\Omega_m, \lambda) \otimes \mathbb{C}^n$, so that $|V_m Z - (Z \otimes I_n) V_m|_2 < 2^{-m-1}$, where $\Omega_m = \Omega \cap ((m+1)\mathbb{D} \backslash m\mathbb{D})$. If $V_m h = \sum_k V_{mk} h \otimes e_k$, we have $|[V_{mk}, Z]|_2 < 2^{-m-1}$ and hence $L(U_k) < 1$ where $U_k = \bigoplus_{m \geq 0} V_{mk}$, so that if $Uh = \sum_k U_k h \otimes e_k$ we will have that U is unitary and $U_k \in E\Lambda(\Omega)$, $1 \leq k \leq n$. It follows that $UE\Lambda(\Omega)U^* \subset E\Lambda(\Omega) \otimes \mathcal{M}_n$ and $U^*(E\Lambda(\Omega) \otimes \mathcal{M}_n)U \subset E\Lambda(\Omega)$, which implies that $UE\Lambda(\Omega)U^* = E\Lambda(\Omega) \otimes \mathcal{M}_n$ and that $T \to UTU^*$ is a spatial isomorphism of $E\Lambda(\Omega)$ and $E\Lambda(\Omega) \otimes \mathcal{M}_n$.

For the last assertion to be proved, note that the operator U which we constructed, satisfies

$$U(i(\Omega, \Omega \cap n\mathbb{D}))E\Lambda(\Omega \cap n\mathbb{D}))U^* = (i(\Omega, \Omega \cap n\mathbb{D})E\Lambda(\Omega \cap n\mathbb{D})) \otimes \mathcal{M}_n.$$

The assertion then follows from the density of $\bigcup_{n \geq 1} i(\Omega, \Omega \cap n\mathbb{D})E\Lambda(\Omega \cap n\mathbb{D})$ in $E\Lambda(\Omega)_0$. □

3.18. Along similar lines with 3.17 one can show that $E\Lambda(\Omega)$ is a huge algebra. For instance, since Z and $Z \otimes I_{\mathcal{H}}$ are unitarily equivalent mod $\mathcal{C}_2$ and since $I \otimes \mathcal{B}(\mathcal{H})$ is in the commutant of $Z \otimes I_{\mathcal{H}}$, ($\mathcal{H}$ a separable Hilbert space), one infers that $E\Lambda(\Omega)$ contains a subalgebra spatially isomorphic to $I \otimes \mathcal{B}(\mathcal{H})$.

In the remainder of this section we exhibit a few special operators which are in $E\Lambda(\Omega)$.

Proposition 3.19. *Let* Ω *be a bounded open set and let* $A^2(\Omega)$ *be the Bergman space of square-integrable analytic functions. Assume moreover that the rational functions with poles in* $\mathbb{C}\backslash\overline{\Omega}$ *are dense in* $A^2(\Omega)$. *Then we have* $P_\Omega \in E\Lambda(\Omega)$, *where* P_Ω *is the orthogonal projection of* $L^2(\Omega, \lambda)$ *onto the subspace* $A^2(\Omega)$.

Proof. This is a consequence of the Berger–Shaw inequality (see for instance [21, p. 128, Theorem 1.3]). Indeed $T = Z \mid A^2(\Omega)$ is a subnormal operator and the constant function 1 is a rationally cyclic vector for T. The Berger–Shaw inequality then gives $\mathrm{Tr}[T^*, T] < \infty$. With the simplified notation $P = P_\Omega$, we have

$$\mathrm{Tr}[PZ^*P, PZP] < \infty.$$

Since $(I-P)ZP = 0$ and $[Z^*, Z] = 0$ this gives

$$[PZ^*P, PZP] = PZ(I-P)Z^*P$$

and hence

$$[P, Z] = PZ(I-P) \in \mathcal{C}_2.$$

□

3.20. The Hilbert-transform singular integral operator on $\mathbb{C}$ ([20],[23])

$$Hf(\zeta) = \lim_{\varepsilon\downarrow 0} \int_{|z-\zeta|>\varepsilon} \frac{f(z)}{(\zeta - z)^2} d\lambda(z)$$

is a bounded operator on $L^2(\mathcal{C}, \lambda)$ and hence also its compression H_Ω to $L^2(\Omega, d\lambda)$, where Ω is bounded, is a bounded operator. Then also $T_\Omega = [Z, H_\Omega]$ is a bounded operator and

$$T_\Omega f(z) = \lim_{\varepsilon\downarrow 0} \int_{|z-\zeta|>\varepsilon} \frac{f(z)}{\zeta - z} d\lambda(z).$$

We have $[Z, T_\Omega] = \langle \cdot, 1\rangle 1$ where 1 denotes the constant function equal to 1. Since $[Z, T_\Omega]$ is rank one, we have $T_\Omega \in E\Lambda(\Omega)$. Since z^{-1} is not in $L^2(\mathbb{D}, \lambda)$, T_Ω is not in $\mathcal{C}_2$. It can be shown that $T_\Omega \in \mathcal{C}_2^+$ (the ideal of compact operators with singular numbers $s_n = O(n^{-1/2})$). Also clearly the linear span of operators of the form $M_f T_\Omega M_g$ gives operators K in $E\Lambda(\Omega)$ which are in $\mathcal{C}_2^+$ and the commutators of which $[Z, K]$ are dense in $\mathcal{C}_2(L^2(\Omega, \lambda))$.

4. About the K-theory of $E\Lambda(\Omega)$

4.1. Passing via almost normal operators, the Pincus g-function gives a homomorphism of the K_0-group of $E\Lambda(\Omega)$ to L^1-functions. We shall prove that the Conjecture 3 about almost normal operators (see 2.5) implies that this homomorphism completely determines the group $K_0(E\Lambda(\mathbb{C})_0)$. Conversely, assuming Conjecture 4, we will show that such a result about the K-theory of $E\Lambda(\mathbb{C})_0$ implies Conjecture 3.

We begin with some technical facts.

Lemma 4.2. *If $F = F^2 \in E\Lambda(\Omega)$ and $P = P^* = P^2 \in \mathcal{B}(L^2(\Omega, \lambda))$ is the orthogonal projection onto $F(L^2(\Omega, \lambda))$ then $P \in E\Lambda(\Omega)$ and P and F have the same class in K_0.*

Proof. The orthogonal projection P is equal to $\psi(FF^*)$ for some C^∞-function ψ. Hence $P \in E\Lambda(\Omega)$ is a consequence of Proposition 3.15 and $tP + (1-t)F$, $t \in [0, 1]$ is a continuous path of projections, so $[P]_0 = [F]_0$. □

Lemma 4.3. *Let $P \in E\Lambda(\Omega)$ be a self-adjoint projection, which is not finite rank and assume Ω is bounded. Then we have*

$$PZP \in \mathcal{AN}(L^2(\Omega,\lambda).$$

Proof. We have

$$[PZ^*P, PZP] = PZ(I-P)Z^*P - PZ^*(I-P)ZP \in \mathcal{C}_1$$

since $(I-P)ZP = (I-P)[Z,P] \in \mathcal{C}_2$ and $PZ(I-P) = [P,Z](I-P) \in \mathcal{C}_2$. □

Proposition 4.4. *Assume Ω is bounded. For every $\alpha \in K_0(E\Lambda(\Omega))$ there is a self-adjoint projection $P \in E\Lambda(\Omega)$, not of finite rank, so that $[P]_0 = \alpha$. The Pincus g-function g_{PZP} depends only on α (i.e., not on the choice of P). Moreover, the map $K_0 \to L^1(\mathbb{C},\lambda)$ which associates to a class α the L^1-function g_{PZP} is a homomorphism.*

Proof. The existence of unitary "Cuntz n-tuples" $U_1,\ldots,U_n$ in $E\Lambda(\Omega)$, which was shown in Proposition 3.17, implies that $[I]_0 = 0$ and that for a projection $Q \in \mathcal{M}_n(E\Lambda(\Omega))$ there is a projection $P \in E\Lambda(\Omega)$ with $[P]_0 = [Q]_0$ so that $-[Q]_0 = [I-P]_0$. Hence $K_0(E\Lambda(\Omega))$ consists of classes of idempotents in $E\Lambda(\Omega)$ and these can be chosen to be self-adjoint by Lemma 4.2.

Again using Proposition 3.15 and Proposition 3.17 the fact that the map $\alpha \to g_{PZP}$ is a well-defined homomorphism is a consequence of the following two facts: a) if $P \in E\Lambda(\Omega)$ is a self-adjoint projection and $W \in E\Lambda(\Omega)$ is unitary, then $g_{(WPW^*)Z(WPW^*)} = g_{PZP}$ and b) if $P_1, P_2 \in E\Lambda(\Omega)$ are self-adjoint projections and $P_1P_2 = 0$, then $g_{P_1ZP_1} + g_{P_2ZP_2} = g_{(P_1+P_2)Z(P_1+P_2)}$.

To show that a) holds, remark that $g_{WPW^*ZWPW^*} = g_{PW*ZWP}$ by unitary equivalence and $PW^*ZWP - PZP \in \mathcal{C}_2$. Moreover, in view of the argument in 2.7 we have $k_2(PZP) = 0$, $k_2(PW^*ZWP) = 0$ and we can then use 2.4 to get that $g_{PZP} = g_{PW^*ZWP}$.

Assertion b) is proved by the same kind of combination of facts. By the argument of 2.7, we have

$$k_2(P_1ZP_1) = k_2(P_2ZP_2) = k_2((P_1+P_2)Z(P_1+P_2)) = 0.$$

We then remark that

$$P_1ZP_1 + P_2ZP_2 - (P_1+P_2)Z(P_1+P_2) \in \mathcal{C}_2$$

and we can then use 2.4 to get

$$g_{(P_1+P_2)Z(P_1+P_2)} = g_{P_1ZP_1+P_2ZP_2} = g_{P_1ZP_1} + g_{P_2ZP_2},$$

where we used the fact that

$$k_2(P_1ZP_1 + P_2ZP_2) = k_2(P_1ZP_1 \oplus P_2ZP_2) = 0.$$

□

4.5. The homomorphism $K_0(E\Lambda(\Omega)) \to L^1_{rc}(\mathbb{C}, \lambda)$, constructed in Proposition 4.4, will be denoted by $\Gamma(\Omega)$ or simply Γ, when the bounded set Ω is not in doubt ($L^1_{rc}(\mathbb{C}, \lambda)$ being the L^1-space of real-valued functions with compact support).

We shall also denote by $\mathcal{AND}(\mathcal{H})$ the almost normal operators for which Conjecture 4 (see 2.5) holds. We shall call such almost-normal operators dilatable. It is easily seen that this is equivalent to the fact that the almost-normal operator is a Hilbert–Schmidt perturbation of an almost-normal operator which is a compression PNP of a normal operator N by a projection P so that $[P, N] \in \mathcal{C}_2$.

In 2.7 we showed that if $T \in \mathcal{AND}(\mathcal{H})$ then $k_2(T) = 0$. Next we will give a few simple facts about K-theory for some of the algebras related to $E\Lambda(\Omega)$ and get some variants of the homomorphism Γ.

4.6. If $\Omega_1 \subset \Omega_2$ are bounded Borel sets, then it is immediate from the construction of Γ that

$$\Gamma(\Omega_2) \circ (i(\Omega_2, \Omega_1))_* = \Gamma(\Omega_1).$$

In view of 3.8, $E\Lambda(\mathbb{C})_0$ is the inductive limit of the $E\Lambda(\Omega)$ with Ω bounded (the inclusion will be denoted $i_0(\mathbb{C}, \Omega)$). Then $K_0(E\Lambda(\mathbb{C})_0)$ is the inductive limit of the $K_0(E\Lambda(\Omega))$, with bounded Ω, and there is a homomorphism

$$\Gamma_\infty : K_0(E\Lambda(\mathbb{C})_0) \to L^1_{rc}(\mathbb{C}, \lambda)$$

so that

$$\Gamma_\infty \circ (i_0(\mathbb{C}, \Omega))_* = \Gamma(\Omega).$$

Lemma 4.7. *We have* $K_0(\mathcal{K}\Lambda(\Omega)) \cong \mathbb{Z}$, $K_1(\mathcal{K}\Lambda(\Omega)) = 0$, *for any* Ω *(not of measure* 0*), the isomorphism for* K_0 *being given by the trace on* $B(L^2(\Omega, \lambda))$. *Moreover we have isomorphisms*

$$K_0(E\Lambda(\Omega)) \xrightarrow{p_*} K_0((E/\mathcal{K})\Lambda(\Omega))$$
$$K_0(E\Lambda(\mathbb{C})_0) \xrightarrow{p_*} K_0((E_0/\mathcal{K})\Lambda(\mathbb{C})).$$

Proof. The assertions about the K-theory of $\mathcal{K}\Lambda(\Omega)$ are a consequence of the last assertion in Proposition 3.14.

To get the isomorphisms between K_0-groups of $E\Lambda(\Omega)$ and $(E/\mathcal{K})\Lambda(\Omega)$ and respectively $E\Lambda(\mathbb{C})_0$ and $(E_0/\mathcal{K})\Lambda(\mathbb{C})$ we use the 6-term K-theory exact sequences associated with

$$0 \to \mathcal{K}\Lambda(\Omega) \to E\Lambda(\Omega) \to (E/\mathcal{K})\Lambda(\Omega) \to 0$$

and

$$0 \to \mathcal{K}\Lambda(\mathbb{C}) \to E\Lambda(\mathbb{C})_0 \to (E_0/\mathcal{K})\Lambda(\mathbb{C}) \to 0.$$

Since $K_1(\mathcal{K}\Lambda(\Omega)) = 0$ we have that the homomorphisms p_* are surjective. The injectivity of the p_* means to show the connecting homomorphisms $K^1 \to K^0$

are surjective. This is easily seen to be the case if we can prove $E\Lambda(\Omega)$ and $E\Lambda(\mathbb{C})_0 + \mathbb{C}I$ contain a Fredholm operator of index 1. If Ω is bounded, there is a Fredholm operator of index 1, $T \in \mathcal{B}(L^2(\Omega, \lambda))$ so that $[T, Z] \in \mathcal{C}_2$. This in turn follows from the easily seen fact that Z is unitarily equivalent to $Z \oplus \mu I_{\mathcal{H}} + K$, where $\mathcal{H}$ is some infinite-dimensional Hilbert space, $\mu \in \sigma(Z)$ and $K \in \mathcal{C}_2$. For $E\Lambda(\mathbb{C})_0 + \mathbb{C}I$ we can use the Fredholm operator $T \in E\Lambda(\Omega)$ and consider $T \oplus I_{L^2(\mathbb{C}\setminus\Omega,\lambda)} \in E\Lambda(\mathbb{C})_0 + \mathbb{C}I$. □

4.8. In view of Lemma 4.7 we infer for bounded Ω the existence of homomorphisms

$$\tilde{\Gamma}(\Omega) : K_0((E/\mathcal{K})\Lambda(\Omega)) \to L^1_{rc}(\Omega, \lambda)$$

and

$$\tilde{\Gamma}_\infty : K_0((E_0/\mathcal{K})\Lambda(\mathbb{C})) \to L^1_{rc}(\mathbb{C}, \lambda)$$

so that

$$\tilde{\Gamma}(\Omega) \circ p_* = \Gamma(\Omega) \text{ and}$$
$$\tilde{\Gamma}_\infty \circ p_* = \Gamma_\infty.$$

Fact 4.9. *The following assertions are equivalent.*

(i) *Conjecture 3 is true.*

(ii) *Conjecture 4 is true and* Γ_∞ *is an isomorphism.*

(iii) *Conjecture 4 is true and* Γ_∞ *is injective.*

Proof. Since (ii) ⇒ (iii) it will be sufficient to show that (i) ⇒ (ii) and (iii) ⇒ (i).

(i) ⇒ (ii). Remark first that Conjecture 3 implies Conjecture 4. Indeed, if $T \in \mathcal{AN}(\mathcal{H})$ we can find $S_1 \in \mathcal{AN}(\mathcal{H})$ so that $g_{S_1} = -g_T$ (see [21] for instance). Then Conjecture 3 implies that there is a normal operator N_1 so that $T \oplus S_1 \oplus N_1 - N \in \mathcal{C}_2$ where N is a normal operator. Thus we can take $S = S_1 \oplus N_1$ and then $S \in \mathcal{AN}$ and $T \oplus S$ is equal $N \bmod \mathcal{C}_2$, which is the assertion of Conjecture 4 for T.

To show Γ_∞ is surjective consider $g \in L^1_{rc}(\mathbb{C}, \lambda)$. By the work of Carey–Pincus there is $T \in \mathcal{AN}(\mathcal{H}_1)$ so that $g_T = g$. By Conjecture 4 and the fact that it implies Conjecture 1 we see that T can be chosen to be $QN \mid Q\mathcal{H}$ where N is a normal operator and Q an orthogonal projection, so that $[Q, N] \in \mathcal{C}_2$. We may also assume $\sigma(N) = n\overline{\mathbb{D}}$ for some $n \in \mathbb{N}$. Then by our results on normal operators mod $\mathcal{C}_2$, there is a unitary operator $U : \mathcal{H} \to L^2(n\mathbb{D}, \lambda)$ so that $ZU - UN \in \mathcal{C}_2$. Then taking $P = UQU^*$, we have $PZP - UQNQU^* \in \mathcal{C}_2$ and hence $g_{PZP} = g_{QNQ} = g$ so that $\Gamma(n\mathbb{D})[P]_0 = g$. Clearly then $\Gamma_\infty([i_0(\mathbb{C}, n\mathbb{D})(P)]_0) = g$.

To prove that assuming Conjecture 3 holds, Γ_∞ is injective, let $\alpha \in K_0(E\Lambda(\mathbb{C})_0)$ be so that $\Gamma_\infty(\alpha) = 0$. Using 4.6 and Proposition 4.4 there is a self-adjoint projection $P \in E\Lambda(n\mathbb{D})$ for some $n \in \mathbb{N}$, so that $(i_0(\mathbb{C}, n\mathbb{D}))_*[P]_0 = \alpha$ and $\Gamma(n\mathbb{D})([P]_0) = \Gamma_\infty(\alpha) = 0$. Hence $g_{PZP} = 0$. Then Conjecture 3 gives that there is $m \geq n$ and there are normal operators N and N_1 with $\sigma(N) = \sigma(N_1) = m\overline{\mathbb{D}}$ so that

$$N - PZ \mid PL^2(n\mathbb{D}, \lambda) \oplus N_1 \in \mathcal{C}_2.$$

Since we will use the operators Z in $E\Lambda(n\mathbb{D})$ and $E\Lambda(m\mathbb{D})$ simultaneously, we shall denote them here by Z_n and Z_m. Clearly, we may use a unitary equivalence and a $\mathcal{C}_2$-perturbation to choose N_1. Similarly N can be chosen unitarily equivalent to Z_m. Thus, we get a unitary operator

$$U : PL^2(n\mathbb{D}, \lambda) \oplus L^2(m\mathbb{D}, \lambda) \to L^2(m\mathbb{D}, \lambda)$$

so that $Z_m U - U(PZ_n \mid PL^2(n\mathbb{D}, \lambda) \oplus Z_m) \in \mathcal{C}_2$. This means that U gives rise to a partial isometry $W \in \mathcal{B}(L^2(m\mathbb{D}, \lambda) \oplus L^2(m\mathbb{D}, \lambda))$ so that $W^*W = i(m\mathbb{D}, n\mathbb{D})(P) \oplus I$ and $WW^* = 0 \oplus I$ with the property that $[W, Z_m \oplus Z_m] \in \mathcal{C}_2$. Then we have $W \in \mathcal{M}_2(E\Lambda(m\mathbb{D}))$. This gives $i(m\mathbb{D}, n\mathbb{D})_*[P]_0 + [I]_0 = [I]_0$ in $K_0(E\Lambda(m\mathbb{D}, \lambda))$, so that $[i(m\mathbb{D}, n\mathbb{D})(P)]_0 = 0$. But then we must have $\alpha = [i_0(\mathbb{C}, n\mathbb{D})(P)]_0 = i_0(\mathbb{C}, m\mathbb{D})_*[i(m\mathbb{D}, n\mathbb{D})(P)]_0 = 0$.

(iii) $\Rightarrow$ (i). Assume (iii) holds and let $T_1, T_2 \in \mathcal{AN}(\mathcal{H})$ with $g_{T_1} = g_{T_2}$. Since Conjecture 4 is part of the assumption (iii) we have $T_1, T_2 \in \mathcal{AND}(\mathcal{H})$. This implies there are self-adjoint projection $P_1, P_2 \in E\Lambda(n\mathbb{D})$ for some $n \in \mathbb{N}$, so that T_j is unitarily equivalent to a $\mathcal{C}_2$-perturbation of $P_j Z \mid P_j L^2(n\mathbb{D}, \lambda)$, $j = 1, 2$. Moreover, we have $\Gamma(n\mathbb{D})[P_1]_0 = \Gamma(n\mathbb{D})[P_2]_0$ because $g_{T_1} = g_{T_2}$. It follows that $\Gamma_\infty([i_0(\mathbb{C}, n\mathbb{D})(P_1)]_0) = \Gamma_\infty([i_0(\mathbb{C}, n\mathbb{D})(P_2)]_0)$ so that by (iii) we have $i_0(\mathbb{C}, n\mathbb{D})_*[P_1]_0 = i_0(\mathbb{C}, n\mathbb{D})[P_2]_0$. Since $E\Lambda(\mathbb{C})_0$ is the inductive limit of the $E\Lambda(m\mathbb{D})$ we infer that $[i(m\mathbb{D}, n\mathbb{D})(P_1)]_0 = [i(m\mathbb{D}, n\mathbb{D})(P_2)]_0$ for some $m \geq n$. Hence there is a unitary equivalence in $\mathcal{M}_{p+q+1}(E\Lambda(m\mathbb{D}))$ between the $Q_j = i(m\mathbb{D}, n\mathbb{D})P_j \oplus I \oplus \cdots \oplus I \oplus 0 \oplus \cdots \oplus 0$, $j = 1, 2$ (there are p summands I and q summands 0). Indeed the equality of K_0-classes implies there is an invertible element intertwining Q_1, Q_2 and using Proposition 3.17 and Proposition 3.15 we can pass to the unitary in the polar decomposition of this invertible element of $\mathcal{M}_{p+q+1}(E\Lambda(n\mathbb{D}))$. This unitary will then commute with $Z \oplus \cdots \oplus Z$ modulo $\mathcal{C}_2$ and hence will intertwine mod $\mathcal{C}_2$ the compressions $Q_j(Z \oplus \cdots \oplus Z)Q_j$, $j = 1, 2$. These compressions are unitarily equivalent to

$$P_j Z \mid P_j L^2(n\mathbb{D}, \lambda) \oplus N_j$$

for some normal operators N_j, $j = 1, 2$. Thus $T_j \oplus N_j$, being unitarily equivalent mod $\mathcal{C}_2$ to these compressions, will also be unitarily equivalent mod $\mathcal{C}_2$, which proves (i) under the assumption (iii). $\square$

4.10. In view of Lemma 4.7 and of 4.8 we have that Fact 4.9 also holds with Γ_∞ replaced by $\tilde{\Gamma}_\infty$.

5. Multipliers, Corona and Bidual of $\mathcal{K}\Lambda(\Omega)$

5.1. We shall consider bounded multipliers $\mathcal{M}(\mathcal{K}\Lambda(\Omega))$, that is double centralizer pairs (T', T'') of bounded linear maps $\mathcal{K}\Lambda(\Omega) \to \mathcal{K}\Lambda(\Omega)$ so that $T'(x)y = xT''(y)$.

Proposition 5.2. *We have* $\mathcal{M}(\mathcal{K}\Lambda(\Omega)) = E\Lambda(\Omega)$*, that is, if* $(T', T'') \in \mathcal{M}(\mathcal{K}\Lambda(\Omega))$*, then there is* $T \in E\Lambda(\Omega)$ *so that* $T'(x) = xT$ *and* $T''(x) = Tx$*.*

Proof. Let $(P_n)_{n\geq 1}$ be the approximate unit provided by Proposition 3.14 and define $K_n = T'(P_n)P_n = P_nT''(P_n)$. Clearly, the norms $|||K_n|||$ will be bounded by some constant C and if $m > n$ we have

$$\begin{aligned} P_nK_mP_n &= P_nT'(P_m)P_mP_n \\ &= P_nT'(P_m)P_n = P_nP_mT''(P_n) \\ &= P_nT''(P_n) = K_n. \end{aligned}$$

Hence the weak limit T of the K_n's exists, and we shall have $P_nTP_n = K_n$. Also $L(T) \leq \sup_n(L(K_n) + 2\|T\|L(P_n)) < \infty$, so that $T \in E\Lambda(\Omega)$. Moreover, we have

$$\begin{aligned} T'(P_n) &= w - \lim_{m\to\infty} T'(P_n)P_m \\ &= w - \lim_{m\to\infty} P_nT''(P_m) \\ &= w - \lim_{m\to\infty} P_nP_mT''(P_m) = P_nT \end{aligned}$$

and similarly $T''(P_n) = TP_n$. This gives $P_nT''(x) = T'(P_n)x = P_nTx$ if $x \in \mathcal{K}\Lambda(\Omega)$ and hence

$$T''(x) = \lim_{n\to\infty} P_nT''(x) = \lim_{n\to\infty} P_nTx = Tx.$$

Similarly $T'(x)P_n = xTP_m$ and $T(x) = \lim_{n\to\infty} T'(x)P_n = xT$. □

Proposition 5.3. *The involutive Banach algebra* $(E/\mathcal{K})\Lambda(\Omega)$ *is a* C^**-algebra. Actually if* $x \in E\Lambda(\Omega)$ *the norm of* $p(x)$ *in* $(E/\mathcal{K})\Lambda(\Omega)$ *is equal to the norm of* $x + \mathcal{K}$ *in the Calkin algebra* $\mathcal{B}/\mathcal{K}$*. In particular* $(E/\mathcal{K})\Lambda(\Omega)$ *is isometrically isomorphic to a* C^**-subalgebra of* $\mathcal{B}/\mathcal{K}$*.*

Proof. It is easily seen that all assertions follow from the equality of the norm of $p(x)$ with the norm of $x + \mathcal{K}$ in the Calkin algebra. This in turn will follow from the fact that with $(P_n)_{n\geq 1}$ denoting the approximate unit of $\mathcal{K}\Lambda(\Omega)$ in Proposition 3.14

$$\lim_{n\to\infty} \|(I - P_n)x(I - P_n)\|$$

equals the Calkin norm of $x+\mathcal{K}$, if we will also show that

$$\lim_{n\to\infty} L((I-P_n)x(I-P_n))=0.$$

In case Ω is bounded we indeed have

$$L((I-P_n)x(I-P_n))\le \lim_{n\to\infty}(2\|x\|\,|[I-P_n,z]|_2+|(I-P_n)[Z,x](I-P_n)|_2=0.$$

In case Ω is unbounded we use Lemma 3.12 and write $x = x_0 + x_1$ where $x_0 = M_m x M_m$ with m chosen so that $L(x_1) < \varepsilon$. We have

$$\limsup_{n\to\infty} L((I-P_n)x_1(I-P_n))\le L(x_1)<\varepsilon$$

and since $\varepsilon > 0$ can be chosen arbitrarily small it will suffice to show that

$$\limsup_{n\to\infty} L((I-P_n)x_0(I-P_n))=0.$$

This in turn can be seen as follows. Let Z_k be the multiplication operator by $z(1\wedge k|z|^{-1})$. Then for any $y\in E\Lambda(\Omega)$ we have

$$L(y)=\limsup_{k\to\infty}|[Z_k,y]|_2.$$

Moreover if $k\ge m$, $[Z_k,x_0]=[Z_m,x_0]$. Hence

$$L((I-P_n)x_0(I-P_n))\le 2\|x_0\|L(I-P_n)+|(I-P_n)[Z_m,x_0](I-P_n)|_2\to 0$$

as $n\to\infty$. □

Remark 5.4. The C^*-algebra

$$\{p(M_f)\in (E/\mathcal{K})\Lambda(\Omega)\mid f\in C\Lambda(\Omega)\},$$

where $C\Lambda(\Omega)$ denotes the norm closure of $\Lambda(\Omega)$ in $L^\infty(\Omega,\lambda)$, is in the center of $(E/\mathcal{K})\Lambda(\Omega)$.

Indeed, if $f\in\Lambda(\Omega)$ then $[M_f,x]\in\mathcal{C}_2\Lambda\subset\mathcal{K}\Lambda(\Omega)$ if $x\in E\Lambda(\Omega)$ so that $p(M_f)$ is in the center of $(E/\mathcal{K})\Lambda(\Omega)$. Since $|||M_f|||=\|M_f\|=\|f\|_\infty$ if $f\in L^\infty(\Omega)$ and the center is clearly norm-closed in $(E/\mathcal{K})\Lambda(\Omega)$, the assertion follows.

5.5. We pass to describing the dual of $\mathcal{K}\Lambda(\Omega)$ for bounded Ω. Throughout $\mathcal{C}_1$ and $\mathcal{C}_2$ will stand for $\mathcal{C}_1(L^2(\Omega,\lambda))$ and respectively $\mathcal{C}_2(L^2(\Omega,\lambda))$.

Proposition 5.6. *Assuming Ω is bounded, the dual of $\mathcal{K}\Lambda(\Omega)$ can be identified isometrically with $(\mathcal{C}_1\times\mathcal{C}_2)/\mathcal{N}$ where*

$$\mathcal{N}=\{([Z,H],H)\in\mathcal{C}_1\times\mathcal{C}_2\mid H\in\mathcal{C}_2 \text{ with } [Z,H]\in\mathcal{C}_1\}$$

and the duality map $\mathcal{K}\Lambda(\Omega)\times(\mathcal{C}_1\times\mathcal{C}_2)\to\mathbb{C}$ is $(T,(x,y))=\mathrm{Tr}(Tx+[Z,T]y)$.

Proof. Since $T \to T \oplus [Z, T]$ identifies $\mathcal{K}\Lambda(\Omega)$ isometrically with a closed subspace of $\mathcal{K} \oplus \mathcal{C}_2$ endowed with the norm $\|K \oplus H\| = \|K\| + |H|_2$, the dual of which is $\mathcal{C}_1 \times \mathcal{C}_2$, the proof will boil down to showing that $\mathcal{N}$ is the annihilator of

$$\{T \oplus [Z, T] \in \mathcal{K} \oplus \mathcal{C}_2 \mid T \in \mathcal{K}\Lambda(\Omega)\}.$$

Since the set $\mathcal{R}$ of finite rank operators is dense in $\mathcal{K}\Lambda(\Omega)$, it will be sufficient to show that $\mathcal{N}$ is the annihilator of

$$\{R \oplus [Z, R] \in \mathcal{K} \oplus \mathcal{C}_2 \mid R \in \mathcal{R}\}.$$

If $R \in \mathcal{R}$ and $(x, y) \in \mathcal{N}$ we have

$$\operatorname{Tr}(Rx + [Z, R]y) = \operatorname{Tr}(R[Z, y] + [Z, R]y) = \operatorname{Tr}([Z, Ry]) = 0.$$

Conversely if $(x, y) \in \mathcal{C}_1 \times \mathcal{C}_2$ is such that

$$\operatorname{Tr}(Rx + [Z, R]y) = 0 \text{ for all } R \in \mathcal{R},$$

then

$$\operatorname{Tr}(R(x - [Z, y])) = 0 \text{ for all} R \in \mathcal{R}$$

and hence $x = [Z, y]$, that is $(x, y) \in \mathcal{N}$. □

Lemma 5.7. *Under the same assumptions and notations like in 5.6,*

$$\{([Z, R], R) \in \mathcal{C}_1 \times \mathcal{C}_2 \mid R \in \mathcal{R}\}$$

is dense in $\mathcal{N}$.

Proof. Let $(x, y) \in \mathcal{N}$, that is $y \in \mathcal{C}_2$ is such that $x = [Z, y] \in \mathcal{C}_1$. Let $(P_n)_{n \geq 1}$ be self-adjoint projections of finite rank so that $P_n \uparrow I$ and $|[P_n, Z]|_2 \to 0$. Then we have $|yP_n - y|_2 \to 0$ and also

$$\begin{aligned} |[Z, yP_n] - [Z, y]|_1 &= |[Z, y]P_n + y[Z, P_n] - [Z, y]|_1 \\ &\leq |y|_2|[Z, P_n]|_2 + |[Z, y](I - P_n)|_1 \to 0 \end{aligned}$$

as $n \to \infty$. □

Proposition 5.8. *If Ω is bounded, with the same notations as in Proposition 5.6, the dual of $(\mathcal{C}_1 \times \mathcal{C}_2)/\mathcal{N}$ identifies with $E\Lambda(\Omega)$ via the duality map*

$$(T, (x, y)) \to \operatorname{Tr}(Tx + [Z, T]y).$$

In particular $E\Lambda(\Omega)$ identifies with the bidual of $\mathcal{K}\Lambda(\Omega)$.

Proof. The dual of $(\mathcal{C}_1 \times \mathcal{C}_2)/\mathcal{N}$ is the orthogonal of $\mathcal{N}$ in $\mathcal{B} \oplus \mathcal{C}_2 = (\mathcal{C}_1 \times \mathcal{C}_2)^d$ (the usual duality based on the trace). Since Lemma 5.7 provides a dense subset of $\mathcal{N}$, it suffices to show that $\{T \oplus [Z, T] \in \mathcal{B} \oplus \mathcal{C}_2 \mid T \in E\Lambda(\Omega)\}$ is the orthogonal in $\mathcal{B} \oplus \mathcal{C}_2$ of $\{([Z, R], R) \in \mathcal{C}_1 \times \mathcal{C}_2 \mid R \in \mathcal{R}\}$. Indeed, if $T \oplus H \in \mathcal{B} \oplus \mathcal{C}_2$ is such that $\mathrm{Tr}(T[Z, R] + HR) = 0$ for all $R \in \mathcal{R}$, then $\mathrm{Tr}((-[Z, T] + H)R) = 0$ for all $R \in \mathcal{R}$ and hence $H = [Z, T]$, which also implies $T \in E\Lambda(\Omega)$. Clearly, also if $T \in E\Lambda(\Omega)$ and $R \in \mathcal{R}$ we have

$$\mathrm{Tr}(T[Z, R] + [Z, T]R) = \mathrm{Tr}([Z, TR]) = 0.$$

□

6. Concluding Remarks

6.1. Isomorphisms induced by bi-Lipschitz map. Let Ω_1 and Ω_2 be Borel subsets of $\mathbb{C}$ and let $F : \Omega_1 \to \Omega_2$ be a map which is Lipschitz and has an inverse which is also Lipschitz (i.e., F is bi-Lipschitz). Then if λ_j is the restriction of Lebesgue measure to Ω_j, the measures $F_*\lambda_1$ are λ_2 are mutually absolutely continuous with bounded Radon–Nikodym derivatives and the same holds for $(F^{-1})_*\lambda_2$ and λ_1 ([17]). This gives rise to a unitary operator

$$U(\Omega_2, \Omega_1)L^2(\Omega_1, \lambda_1) \to L^2(\Omega_2, \lambda_1)$$

which maps $f \in L^2(\Omega_1, \lambda_1)$ to $(f \circ F^{-1}) \cdot (dF_*\lambda_1/d\lambda_2)^{1/2}$. If $g \in L^\infty(\Omega_2, \lambda_2)$ then

$$U(\Omega_2, \Omega_1)^{-1} M_g U(\Omega_2, \Omega_1) = M_{g \circ F}.$$

The map $g \to g \circ F$ gives isomorphisms of $L^\infty(\Omega_2, \lambda_2)$ with $L^\infty(\Omega_1, \lambda_1)$ and of $\Lambda(\Omega_2)$ with $\Lambda(\Omega_1)$. Further $T \to U(\Omega_2, \Omega_1)^{-1} T U(\Omega_2, \Omega_1)$ is an isomorphism of $E\Lambda(\Omega_2)$ and $E\Lambda(\Omega_1)$. This is an isomorphism of Banach algebras with involution, which however is not isometric, since its norm depends on the Lipschitz constants of F and F^{-1}. These isomorphisms preserve finite-rank operators and hence $\mathcal{K}\Lambda(\Omega_2)$ is mapped onto $\mathcal{K}\Lambda(\Omega)$. This in turn implies there is an induced C^*-algebra isomorphism of $(E/\mathcal{K})\Lambda(\Omega_2)$ with $(E/\mathcal{K})\Lambda(\Omega_1)$.

In particular the group of bi-Lipschitz homeomorphisms of a Borel set Ω has automorphic actions on $E\Lambda(\Omega)$ and $(E/\mathcal{K})\Lambda(\Omega)$.

6.2. In view of 5.4 it is a natural question to ask, *what is the center of* $(E/\mathcal{K})\Lambda(\Omega)$? This question which appeared in the preprint version of this paper has now been answered in [4]. In the case of bounded Ω the center is the C^*-algebra generated by $p(Z)$. Note that the answer to the Calkin-algebra analogue of this question, that is the determination of the center of the commutant of a separable commutative C^*-subalgebra of the Calkin algebra, is a particular case of our Calkin algebra bicommutant theorem ([24]).

6.3. $\mathcal{K}\Lambda(\Omega)$ **as a Dirichlet algebra.** The algebras $\mathcal{K}\Lambda(\Omega)$ are examples of Dirichlet algebras in the sense of non-commutative potential theory ([1], [10], [11]). The Dirichlet form can be described for instance via the construction of Dirichlet forms from derivations (Theorem 4.5 in [10] or Theorem 8.3 in [11]). This corresponds to working with the C^*-algebra of compact operators $\mathcal{K} = \mathcal{K}(L^2(\Omega, \lambda))$ and its trace Tr, which is densely defined, faithful, semifinite and lower semicontinuous. The Hilbert space $\mathcal{H} = \mathcal{C}_2 \oplus \mathcal{C}_2$, where $\mathcal{C}_2 = \mathcal{C}_2(L^2(\Omega, \lambda))$ is a $\mathcal{K} - \mathcal{K}$-bimodule and $\mathcal{J}(x \oplus y) = x^* \oplus y^*$ is an isometric antilinear involution of $\mathcal{H}$ exchanging the right and left actions of $\mathcal{K}$ on $\mathcal{H}$. Clearly $\mathcal{C}_2$ identifies with $L^2(\mathcal{K}, \mathrm{Tr})$ and there is an L^2-closable derivation ∂ of $\mathcal{K}\Lambda(\Omega) \cap \mathcal{C}_2 \to \mathcal{C}_2$. The definition in case Ω is bounded, is $\partial a = [X, a] \oplus [Y, a]$. In general the definition can be given in terms of he kernel $K(z_1, z_2)$ of an element $a \in \mathcal{K}\Lambda(\Omega) \cap \mathcal{C}_2$. Then the components of ∂a have kernels $(x_1 - x_2)K(z_1, z_2)$ and respectively $(y_1 - y_2)K(z_1, z_2)$, which are square integrable since $a \in \mathcal{K}\Lambda(\Omega)$. Also clearly viewed the domain of definition of ∂ as part of $L^2(\mathcal{K}, \mathrm{Tr})$, the map ∂ is L^2-closed. Moreover ∂ satisfies the symmetry condition $\mathcal{J}\partial a = \partial a^*$. Then the Dirichlet form $\mathcal{E}$ which is obtained as the closure $\mathcal{E}[a] = \|\partial a\|^2_{\mathcal{H}}$ is easily seen to be precisely square of the L^2-norm of $(z_1 - z_2)K(z_1, z_2)$ which is the same as $(L(a))^2$ defined for $a \in \mathcal{K}\Lambda(\Omega)$. The Markovian semigroup T_t will then act on elements $a \in \mathcal{K}\Lambda(\Omega) \cap \mathcal{C}_2$ which have kernels $K(z_1, z_2)$ as a multiplier which produces the element with kernel $e^{-t|z_1 - z_2|^2} K(z_1, z_2)$. In view of the Markovianity it is easy to see that T_t extends to a semigroup of completely positive contraction on $\mathcal{K}\Lambda(\Omega)$, $E\Lambda(\Omega)$ and also on $\mathcal{K}$ and $\mathcal{B}$. Moreover T_t also induces a semigroup of completely positive contractions on $(E/\mathcal{K})\Lambda(\Omega)$.

6.4. Replacing $\mathcal{C}_2$ by some other $\mathcal{C}_p$. One may wonder about the consequences of replacing the Hilbert–Schmidt class $\mathcal{C}_2$ by some other $\mathcal{C}_p$-class in the definition of $E\Lambda(\Omega)$. This would mean to consider operators T so that $[T, M_f] \in \mathcal{C}_p$ for all $f \in \Lambda(\Omega)$ with $\|Df\|_\infty \le 1$. The questions about $\mathcal{C}_p$-perturbations of normal operators are still covered by our results ([25], [29]), however the passage of multiplication operators by Lipschitz functions would require the use of more difficult results on commutators and functional calculus, like those in [2].

6.5. Perhaps the study of the K-theory of the $E\Lambda(\Omega)$ may benefit from more recent developments of bivariant K-theory beyond C^*-algebras (see [15]).

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D.-V. Voiculescu, Department of Mathematics, University of California at Berkeley, Berkeley, CA 94720-3840

E-mail: dvv@math.berkeley.edu