J. Noncommut. Geom. 8 (2014), 1147–1169
DOI 10.4171/JNCG/182

Journal of Noncommutative Geometry

Noncommutative unfolding of hypersurface singularity

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Abstract. A version of Kontsevich formality theorem is proven for smooth DG algebras. As an application of this, it is proven that any quasiclassical datum of noncommutative unfolding of an isolated surface singularity can be quantized.

Mathematics Subject Classification (2010). 13D10, 14A22, 32S25.

Keywords. Kontsevich formality theorem, dg algebras, operads, hypersurface singularity.

1. Introduction

An isolated hypersurface singularity is a polynomial $f \in k[x_1,\dots,x_n]$ for which the Milnor number

$$\mu(f) = \dim k[x_1,\dots,x_n]/(\frac{\partial f}{\partial x_1},\dots\frac{\partial f}{\partial x_n})$$

is finite.

An unfolding of a hypersurface singularity is a family of hypersurface singularities parametrized by an affine space. From algebraic point of view, the description of unfoldings of f is nothing but the problem of deformations of the $k[y]$-algebra $k[x_1,\dots,x_n]$, with the algebra structure defined by the assignment $y = f(x_1,\dots,x_n)$.

In this paper we suggest studying non-commutative unfoldings of hypersurface singularities, that is deformations of $k[y]$-algebra $A = k[x_1,\dots,x_n]$, in the world of associative algebras. In other words, we are interested in studying the Hochschild cochain complex of A considered as $k[y]$-algebra, having in mind Kontsevich Formality theorem as a possible ideal answer.

Were A smooth as $k[y]$-algebra, one could use the version of the Formality theorem proven in [2] which would provide a weak equivalence of the Hochschild complex to the algebra of polyvector fields. Our case is only slightly more general: A is quasiisomorphic to a smooth dg algebra over $k[y]$. Fortunately, the proof of Formality theorem presented in [2] can be easily generalized to this setup. As a result, we can replace the Hochschild cochain complex with a certain algebra of polyvector fields (which in our case is also a dg algebra). This considerably simplifies the study of noncommutative unfoldings.

The paper consists of two parts. In the first part (Sections 2 and 3) we prove the following version of the Formality theorem for smooth dg algebras.

Theorem 1.1. *Let $R \supset \mathbb{Q}$ be a commutative ring and let A be a commutative smooth dg R-algebra, that is non-positively graded, semifree over A^0 which is smooth as R-algebra. Then the Hochschild cochain complex of A over R is equivalent to the dg algebra of polyvector fields as (homotopy) Gerstenhaber algebras.*

Recall that for a smooth dg algebra A the algebra of polyvector fields is defined as $S_A(T[-1])$ where $T = \mathrm{Der}_R(A, A)$ is the A-module of R-derivations of A; it is cofibrant when A is as indicated above. The proof of the theorem is an adaptation (and simplification) of the proof given in [2].

Since $S_A(T[-1])$ is a Gerstenhaber algebra, its Harrison chain complex $B_{\mathtt{Com}^\perp}(S_A(T[-1]))$ has a structure of dg Lie bialgebra. Homotopy Gerstenhaber algebra structure on the Hochschild complex $C(A)$ can be also described via a dg Lie bialgebra structure on $F^*_{\mathtt{Lie}}(C(A)[1])$, see [6, §6.2] or Subsection 2.3.1 below.

An equivalence between Gerstenhaber algebras of polyvector fields $S_A(T[-1])$ and the Hochschild complex $C(A)$ is presented on the level of these Lie bialgebra models: we present a dg Lie bialgebra $\xi(A)$ an two weak equivalences $\xi(A) \to B_{\mathtt{Com}^\perp}(S_A(T[-1]))$ and $\xi(A) \to F^*_{\mathtt{Lie}}(C(A))[1])$ of dg Lie bialgebras. The proof of the first weak equivalence is straightforward; the second weak equivalence is deduced from a dg version of Hochschild–Kostant–Rosenberg theorem; however, the setup of dg smooth algebras makes this deduction quite nontrivial; this part presented in Subsection 3.5 is our main deviation from the proof of [2].

1.2. In the second part of the paper we apply the formality theorem to studying noncommutative unfoldings of hypersurface singularities.

The famous consequence of the Kontsevich formality theorem says that any Poisson bracket on an affine space (or, more generally, on a C^∞ manifold) can be extended to a star-product. Poisson bracket appears in this picture as a representative of the first-order deformation extendable to a second-order deformation. Having this in mind, we suggest the following.

Definition 1.2.1. A quasiclassical datum of quantization of a B-algebra A is its deformation over $k[h]/(h^2)$ extendable to $k[h]/(h^3)$.

Thus, a quasiclassical datum for a quantization of the ring of smooth functions on a manifold is precisely a Poisson bracket on the manifold.

Let $f \in k[x_1, \dots, x_n]$ define an isolated singular hypersurface and let W be a vector subspace of $k[x_1, \dots, x_n]$ complement to the ideal $(\frac{\partial f}{\partial x_1}, \dots \frac{\partial f}{\partial x_n})$.

We prove (see Proposition 4.5.1) that the quasiclassical data for a NC unfolding of an isolated singularity $f \in k[x_1, \dots, x_n]$ are given by pairs (p, S) where $p \in W$ and S is a Poisson vector field satisfying the condition $[f, S] = 0$.

The main result of the second part of the paper is the following.

Theorem 1.3. *(see Corollary 4.6.2) Let $f \in k[x,y,z]$ define an isolated surface singularity. Then any quasiclassical datum of NC unfolding of f can can be quantized to a noncommutative unfolding over $k[[h]]$.*

Acknowledgements. We are grateful to Martin Markl (who did this some 8 years ago) and Michel Van den Bergh for having pointed out an error in the first version of the manuscript.

2. Preliminaries

Let A be a commutative k-algebra and T a Lie algebroid over A. Then the symmetric algebra $S_A(T[-1])$ has a natural structure of Gerstenhaber algebra (in what follows, G-algebra): the commutative multiplication is that of the symmetric algebra and the degree -1 Lie bracket is induced from the Lie bracket on T.

The G-algebra $S_A(T[-1])$ satisfies an obvious universal property: given a G-algebra X, a map $\alpha : A \to X$ of commutative algebras and a map $\beta : T \to X[1]$ of Lie algebras, so that β is also a map of modules over α and α is a map of T-modules via β, there is a unique map of G-algebras $S_A(T[-1]) \to X$.

Recall that the Hochschild cochain complex $C(A)$ has a G-algebra structure (however, in a *weak* sense, see 2.3 below).

Thus, we may try using the above universal property to construct a map $S_A(T[-1]) \longrightarrow C(A)$ [1] of G-algebras: the pair of obvious maps

$$\begin{array}{ccl} A & = & C^0(A) \longrightarrow C(A) \\ T & \longrightarrow & \mathrm{Hom}_k(A,A) = C^1(A) \longrightarrow C(A)[1] \end{array}$$

should satisfy all necessary properties. This would give an exceptionally simple proof of Kontsevich formality theorem. The main obstacle to this plan is that $C(A)$ is not a genuine G-algebra; it has only a structure of Gerstenhaber algebra up to homotopy. This obstacle can be, however, overcome, with a bit of Koszul duality and a standard homotopy theory for colored operads.

The proof presented below is a result of processing the proof by Dolgushev, Tamarkin and Tsygan [2]. The main theorem of [2] is generalized to smooth (non-positively graded) dg algebras over a commutative $\mathbb{Q}$-algebra. We have slightly streamlined the argument working with dg Lie bialgebras instead of $\mathsf{G}^{\perp}$-coalgebras. On the other hand, the usage of HKR theorem has become more painful in our generalized context of smooth dg algebras.

[1]Which has a good chance of being a quasiisomorphism by the Hochschild–Kostant–Rosenberg theorem.

2.1. Colored operads. The operads appearing in these notes have more than one[2] color. Colored operads were introduced in [1] back in the 1970s, but are in much less use than their colorless version.

A colored operad $\mathcal{O}$ has a set of colors (denoted $[\mathcal{O}]$) and a collection of operations $\mathcal{O}(c, d)$ for any finite collection of colors $c : I \to [\mathcal{O}]$ and another color $d \in [\mathcal{O}]$. There is an associative composition of operations, and unit elements in $\mathcal{O}(\{c\}, c)$ for all $c \in [\mathcal{O}]$.

The results of [5] about model category structure for operads and operad algebras in complexes extend easily to the colored setup. In particular, for $k \supset \mathbb{Q}$ and for any colored operad $\mathcal{O}$ in complexes over k, the category $\mathtt{Alg}_{\mathcal{O}}$ of $\mathcal{O}$-algebras has a model structure with quasiisomorphisms as weak equivalences and componentwise surjections as fibrations. The category of colored operads with a fixed collection of colors is itself the category of algebras over a certain colored operad, therefore a model structure on operads in characteristic zero.

A map $f : \mathcal{P} \to \mathcal{Q}$ of operads induces a forgetful functor $f^* : \mathtt{Alg}_{\mathcal{Q}} \longrightarrow \mathtt{Alg}_{\mathcal{P}}$ and its left adjoint $f_! : \mathtt{Alg}_{\mathcal{P}} \longrightarrow \mathtt{Alg}_{\mathcal{Q}}$. This is a Quillen pair; it is a Quillen equivalence if $f : \mathcal{P} \to \mathcal{Q}$ is a quasiisomorphism.

The above claims are proven for colorless operads in [5]. Their colored versions can be found in [7].

2.2. Koszul duality. The material of this subsection is standard. The details can be found in [3], [4], [6], [10], and [13].

2.2.1. Let us recall some standard notation connected to Koszul duality of operads. Let k be a field of characteristic zero. Let $\mathcal{O}$ be a (possibly colored) Koszul operad in graded vector spaces over k and $\mathcal{O}^{\perp}$ the corresponding quadratic dual cooperad.

Any $\mathcal{O}$-algebra X gives rise to a differential in the cofree $\mathcal{O}^{\perp}$ coalgebra $F^*_{\mathcal{O}^{\perp}}(X)$. Dually, any $\mathcal{O}^{\perp}$-coalgebra Y defines a differential on the free $\mathcal{O}$-algebra $F_{\mathcal{O}}(Y)$. These assignments define a pair of adjoint functors

$$\Omega_{\mathcal{O}} : \mathtt{Coalg}_{\mathcal{O}^{\perp}} \rightleftarrows \mathtt{Alg}_{\mathcal{O}} : B_{\mathcal{O}^{\perp}}. \tag{1}$$

A map of $\mathcal{O}$-algebras is called weak equivalence if it is a quasiisomorphism. A map of $\mathcal{O}^{\perp}$-coalgebras is called weak equivalence if the functor $\Omega_{\mathcal{O}}$ carries it to a quasiisomorphism.

The unit and the counit of the adjunction are weak equivalences; in particular, the map $\Omega_{\mathcal{O}} \circ B_{\mathcal{O}^{\perp}}(A) \longrightarrow A$ is a quasiisomorpism.

2.2.2. $\mathcal{O}_\infty$-algebras. Here $\mathcal{O}$ is a colored Koszul operad. By definition, an $\mathcal{O}_\infty$-algebra structure on a graded vector space X is just a differential on the graded

[2]Actually, two.

$\mathcal{O}^\perp$-coalgebra $F^*_{\mathcal{O}^\perp}(X)$ converting it into a dg $\mathcal{O}^\perp$-coalgebra. This differential is defined by a collection of maps

$$d_n : F^{*n}_{\mathcal{O}^\perp}(X) \to X[1]$$

satisfying the condition expressing the property $d^2 = 0$. The component d_1 yields a differential on X.

We use the notation $B_{\mathcal{O}^\perp}(X)$ for the differential graded $\mathcal{O}^\perp$-coalgebra $(F^*_{\mathcal{O}^\perp}(X), d)$.

Any dg $\mathcal{O}$-algebra X can be considered as a $\mathcal{O}_\infty$-algebra, so a canonical map

$$\mathcal{O}_\infty \longrightarrow \mathcal{O} \tag{2}$$

of dg operads is defined. It is a quasiisomorphism.

There are two different notions of morphism of $\mathcal{O}_\infty$-algebras. The first is just a morphism of algebras over the dg operad $\mathcal{O}_\infty$. This is a map of complexes $f : X \to Y$ preserving the $\mathcal{O}_\infty$-algebra structure. The second, more general, is called an $\mathcal{O}_\infty$-morphism and it is defined as a morphism $F : B_{\mathcal{O}^\perp}(X) \to B_{\mathcal{O}^\perp}(Y)$ of dg coalgebras. It is defined by its components $F_n : F^{*n}_{\mathcal{O}^\perp}(X) \longrightarrow Y$ satisfying some quadratic identities.

An $\mathcal{O}_\infty$-morphism F is called a weak equivalence if it is a weak equivalence of the dg $\mathcal{O}^\perp$-coalgebras. One can easily check that F is a weak equivalence iff $F_1 : X \to Y$ is a quasiisomorphism. If A is an $\mathcal{O}_\infty$-algebra, $B_{\mathcal{O}^\perp}(A)$ is a dg $\mathcal{O}^\perp$-coalgebra and one has an $\mathcal{O}_\infty$-weak equivalence

$$A \longrightarrow \Omega_{\mathcal{O}} \circ B_{\mathcal{O}^\perp}(A)$$

whose first component is a quasiisomorphism of complexes described, if one forgets the differential, as the composition of $A = F^{*1}_{\mathcal{O}^\perp}(A) \to F^*_{\mathcal{O}^\perp}(A)$ with $B_{\mathcal{O}^\perp}(A) = F^1_{\mathcal{O}}(B_{\mathcal{O}^\perp}(A)) \to F_{\mathcal{O}}(B_{\mathcal{O}^\perp}(A))$.

2.2.3. Examples. The following operads are Koszul.

- $\mathcal{O} = \mathtt{Com}, \mathtt{Ass}, \mathtt{Lie}$ with $\mathcal{O}^\perp = \mathtt{Lie}^*\{1\}, \mathtt{Ass}^*\{1\}, \mathtt{Com}^*\{1\}$.
- $\mathtt{G}$, the operad for Gerstenhaber algebras, with $\mathtt{G}^\perp = \mathtt{G}^*\{2\}$.
- $\mathcal{O} = \mathtt{CM}$, the two-color operad governing pairs (A, M) where A is a commutative algebra and M is an A-module. Similarly, $\mathtt{LM}$ is the two-color operad governing pairs (L, M) where L is a Lie algebra and M is an L-module. Both operads are Koszul with $\mathtt{CM}^\perp = \mathtt{LM}^*\{1\}$ and $\mathtt{LM}^\perp = \mathtt{CM}^*\{1\}$.
- $\mathtt{LA}$, the two-color for Lie algebroids. An $\mathtt{LA}$-algebra is a pair (A, T) consisting of a commutative algebra A and a Lie algebroid T. The operations include, apart of the commutative multiplication on A and a Lie bracket on T, an A-module structure on T and a T-module structure on A. One has $\mathtt{LA}^\perp = \mathtt{LA}^*\{1\}$. $\mathtt{LA}$ is a Koszul operad (see [13]), but we will not use this fact.

2.2.4. A slight generalization. Let R be a commutative k-algebra. Given an operad $\mathcal{O}$ over k, it makes sense to talk about $\mathcal{O}$-algebras with values in the category of complexes $\mathtt{dg}(R)$. The category of such algebras is denoted $\mathtt{Alg}_{\mathcal{O}}(\mathtt{dg}(R))$. If $\mathcal{O}$ is Koszul, one still has an adjoint pair

$$\Omega_{\mathcal{O}} : \mathtt{Coalg}_{\mathcal{O}^{\perp}}(\mathtt{dg}(R)) \rightleftarrows \mathtt{Alg}_{\mathcal{O}}(\mathtt{dg}(R)) : B_{\mathcal{O}^{\perp}}, \tag{3}$$

defined by the same formulas as for $R = k$ but using the symmetric monoidal category $\mathtt{dg}(R)$ of complexes over R instead of that over k. The canonical map

$$\Omega_{\mathcal{O}} \circ B_{\mathcal{O}^{\perp}}(A) \to A$$

is still a weak equivalence for each $A \in \mathtt{Alg}_{\mathcal{O}}(\mathtt{dg}(R))$.

The notions of $\mathcal{O}_{\infty}$-algebra and of $\mathcal{O}_{\infty}$-morphism extend without difficulty to algebras in $\mathtt{dg}(R)$.

2.3. Algebra structure on Hochschild cochain complex.

2.3.1. $\widetilde{\mathtt{B}}$-algebras. Let X be a $\mathtt{G}$-algebra. Then $X[1]$ has a Lie algebra structure, so that $B_{\mathtt{Com}^{\perp}}(X)[1]$ which is $F^{*}_{\mathtt{Lie}}(X[1])$ considered as a graded vector space, acquires a dg Lie bialgebra structure. Vice versa, any dg Lie bialgebra structure on $F^{*}_{\mathtt{Lie}}(X[1])$ gives rise to a $\mathtt{G}_{\infty}$-structure on X. This leads to definition of another dg operad $\widetilde{\mathtt{B}}$ whose action on a complex X is given by a dg Lie bialgebra structure $F^{*}_{\mathtt{Lie}}(X[1])$ extending the standard Lie coalgebra structure and the differential on X.

Since any $\mathtt{G}$-algebra has a natural structure of $\widetilde{\mathtt{B}}$-algebra, and any $\widetilde{\mathtt{B}}$-algebra structure on X extends to a $\mathtt{G}_{\infty}$-algebra, one has a decomposition

$$\mathtt{G}_{\infty} \longrightarrow \widetilde{\mathtt{B}} \longrightarrow \mathtt{G}, \tag{4}$$

of the canonical map $\mathtt{G}_{\infty} \to \mathtt{G}$.

The $\widetilde{\mathtt{B}}$-structure on X is given by the collection of the following operations:

- $\ell_{m,n} : F^{*m}_{\mathtt{Lie}}(X[1]) \otimes F^{*n}_{\mathtt{Lie}}(X[1]) \to X[1]$,
- $d_n : F^{*n}_{\mathtt{Lie}}(X[1]) \to X[2]$,

defining the Lie bracket and the differential on $F^{*}_{\mathtt{Lie}}(X[1])$, subject to certain relations which assure that $d^2 = 0$, d is a derivation of the bracket, and the cocycle condition connecting the bracket with the cobracket.

Note for book-keeping the degrees of $\ell_{m,n}$ and d_n.

- $\ell_{m,n} : \mathtt{Lie}(m)^{*} \otimes \mathtt{Lie}(n)^{*} \longrightarrow \widetilde{\mathtt{B}}(m+n)^{1-m-n},\ m,n \geq 1$,
- $d_n : \mathtt{Lie}(n)^{*} \longrightarrow \widetilde{\mathtt{B}}(n)^{2-n},\ n \geq 2$.

2.3.2. Lie bialgebras versus $\mathtt{G}$-algebras. The operad $\mathtt{G}$ is Koszul, so we have a standard Koszul duality pair of adjoint functors

$$\Omega_{\mathtt{G}} : \mathtt{Coalg}_{\mathtt{G}^{\perp}} \rightleftarrows \mathtt{Alg}_{\mathtt{G}} : B_{\mathtt{G}^{\perp}}.$$

There is another pair of adjoint functors, a sort of "relative Koszul duality", based on the fact expressed in 2.3.1: if $X \in \mathtt{Alg}_{\mathtt{G}}$, the dg Lie coalgebra $B_{\mathtt{Com}^{\perp}}(X)[1]$ has a structure of Lie bialgebra. Dually, given a dg Lie bialgebra Y, the commutative algebra $\Omega_{\mathtt{Com}}(Y[-1])$ has a structure of $\mathtt{G}$-algebra. This leads to the pair of adjoint functors

$$\Omega'_{\mathtt{Com}} : \mathtt{LBA} \rightleftarrows \mathtt{Alg}_{\mathtt{G}} : B'_{\mathtt{Com}^{\perp}}, \tag{5}$$

where $\mathtt{LBA}$ denotes the category of dg Lie bialgebras and

$$B'_{\mathtt{Com}^{\perp}}(X) = B_{\mathtt{Com}^{\perp}}(X)[1] \text{ and } \Omega'_{\mathtt{Com}}(Y) = \Omega_{\mathtt{Com}}(Y[-1]).$$

As for the conventional Koszul duality (1) an arrow in $\mathtt{LBA}$ will be called a weak equivalence iff its image under $\Omega'_{\mathtt{Com}}$ is a quasiisomorphism. We use the same notation $B'_{\mathtt{Com}^{\perp}}$ for the obvious extension of the functor to $\widetilde{\mathtt{B}}$-algebras.

2.3.3. Deligne conjecture. Deligne conjecture asserts that the cohomological Hochschild complex $C(A)$ of an associative algebra A has a structure of an algebra over an operad of (chains of) little squares. Even though Deligne conjecture is very much relevant for the Formality theorem, the version we need is extremely easy.

Define a $\mathtt{B}_\infty$-algebra structure on a graded vector space X as the structure of dg bialgebra on the free associative coalgebra $F^*_{\mathtt{Ass}}(X[1])$. Similarly to $\widetilde{\mathtt{B}}$-algebras, this leads to a dg operad $\mathtt{B}_\infty$ governing such algebras. This operad is generated by the operations

- $m_{p,q} : X^{\otimes p} \otimes X^{\otimes q} \longrightarrow X[1-p-q]$, the components of the product, and
- $m_n : X^{\otimes n} \longrightarrow X[2-n]$, the components of the differential,

defining the associative multiplication and the differential on $F^*_{\mathtt{Ass}}(X[1])$, subject to certain relations which assure that $d^2 = 0$, d is a derivation of the bracket, and the condition describing compatibility of the product with the coproduct.

The Hochschild complex $C(A)$ has a canonical $\mathtt{B}_\infty$-algebra structure defined by the formulas:

- m_2 is the cup-product.
- $m_k = 0$ for $k > 2$.
- $m_{1,l}$ are the brace operations $x_0, \ldots, x_l \mapsto x_0\{x_1, \ldots, x_l\}$ having degree $-l$.
- $m_{k,l} = 0$ for $k > 1$.

The associative cup-product together with the brace operations generate an operad $\mathtt{Br}$ called the operad of braces. Thus, the action of $\mathtt{B}_\infty$ on $C(A)$ factors through $\mathtt{Br}$ whose action on $C(A)$ is more or less tautological.

It turns out that the operad $\mathtt{Br}$ is equivalent to the operad of small squares, so the action of $\mathtt{Br}$ on $C(A)$ described above "solves" Deligne conjecture.

What is much more important for us is that there exists a canonical map of operads $\widetilde{\mathtt{B}} \longrightarrow \mathtt{B}_\infty$ (depending of a choice of associator) so that any $\mathtt{B}_\infty$-algebra is endowed with a canonical $\widetilde{\mathtt{B}}$-algebra structure. This remarkable result was proven by Tamarkin in his 1998 paper on Kontsevich formality theorem; see [6], [11], [12]. The proof is based on Etingof–Kazhdan theory of quantization (and dequantization) of Lie bialgebras.[3]

3. Equivalence of Lie bialgebra models

In this section we are working in the symmetric monoidal category of complexes over a commutative ring $R \supset \mathbb{Q}$.

A is a smooth dg algebra over R and $C = C(A)$ is the Hochschild cochain complex of R-algebra A. All operads considered will live over $\mathbb{Q}$; all our $\mathcal{O}$-algebra will be in $\mathtt{Alg}_{\mathcal{O}}(\mathtt{dg}(R))$.

3.1. According to the above, the Hochschild complex $C = C(A)$ admits a $\widetilde{\mathtt{B}}$-algebra structure expressible (in a very nontrivial way) via the cup product and the brace operations on C. The corresponding dg Lie bialgebra structure on $F^*_{\mathtt{Lie}}(C[1])$ is given by the collection of maps $\ell_{m,n}$, d_n described in (2.3.1), and together they form a dg Lie bialgebra denoted $B'_{\mathtt{Com}^\perp}(C)$.

The algebra of polyvector fields $S_A(T[-1])$ is a (strict) $\mathtt{G}$-algebra, so it leads to dg Lie bialgebra $B'_{\mathtt{Com}^\perp}(S_A(T[-1]))$. In order to prove the main theorem, we will present a pair of weak equivalences

$$B'_{\mathtt{Com}^\perp}(S_A(T[-1])) \xleftarrow{\iota} \xi(A) \xrightarrow{\varkappa} B'_{\mathtt{Com}^\perp}(C(A)) \tag{6}$$

in the category of dg Lie bialgebras.

We will proceed as follows. First of all we identify a dg Lie coalgebra $\xi(A)$ which naturally maps to $B'_{\mathtt{Com}^\perp}(S_A(T[-1]))$. We can easily check the map is a weak equivalence, it is injective, and that its image is closed with respect to Lie bracket. This endows $\xi(A)$ with a structure of Lie bialgebra.

On the other hand, we will see that the pair of obvious embeddings $\alpha : A \to C$ and $\beta : T \to C[1]$ induces a map of dg Lie bialgebras $\xi(A) \to B'_{\mathtt{Com}^\perp}(C(A))$. Finally, the fact that it is a weak equivalence follows from the Hochschild–Kostant–Rosenberg theorem.

[3]We have no doubt that the maps $\widetilde{\mathtt{B}} \to \mathtt{B}_\infty \to \mathtt{Br}$ are quasiisomorphisms; unfortunately we were unable to find a reference for this fact.

3.2. There is a pair of adjoint functors

$$F : \mathtt{Alg}_{\mathtt{CM}} \rightleftarrows \mathtt{Alg}_{\mathtt{Com}} : G \tag{7}$$

defined by the formulas

$$G(A) = (A, A[1]); \quad F(A, M) = S_A(M[-1]).^{4}$$

On the Koszul-dual side, there is a pair of adjoint functors

$$f : \mathtt{Coalg}_{\mathtt{CM}^\perp} \rightleftarrows \mathtt{Coalg}_{\mathtt{Com}^\perp} : g \tag{8}$$

defined by the formulas

$$g(C) = (C, C[1]); \quad f(C, N) = C \oplus N[-1],$$

with the cobracket on $f(C, N)$ determined by the cobracket on C and the coaction $\delta : N \to N \otimes C$ so that the value of the cobracket at $x \in M[-1] \subset f(C, N)$ is $\delta(x) - \sigma \circ \delta(x)$ where $\sigma : C \otimes N[-1] \to N[-1] \otimes C$ is the standard commutativity constraint.

The functors G and g commute with the Bar-construction, so that the compositions $B_{\mathtt{CM}^\perp} \circ G$ and $g \circ B_{\mathtt{Com}^\perp}$ are naturally isomorphic. This yields the composition

$$f \circ B_{\mathtt{CM}^\perp} \longrightarrow f \circ B_{\mathtt{CM}^\perp} \circ G \circ F = f \circ g \circ B_{\mathtt{Com}^\perp} \circ F \longrightarrow B_{\mathtt{Com}^\perp} \circ F. \tag{9}$$

To get a feeling for what is going on, let $(A, M) \in \mathtt{Alg}_{\mathtt{CM}}$. The composition $B_{\mathtt{Com}^\perp} \circ F$ applied to (A, M) gives the (shifted) dg Lie coalgebra Koszul dual to the commutative algebra $S_A(M[-1])$ which is graded by powers of M. The composition $f \circ B_{\mathtt{CM}^\perp}$ applied to (A, M) is the dg subcoalgebra of $B_{\mathtt{CM}^\perp}(S_A(M[-1]))$ consisting of the elements of degree ≤ 1. In particular, the map (9) is injective.

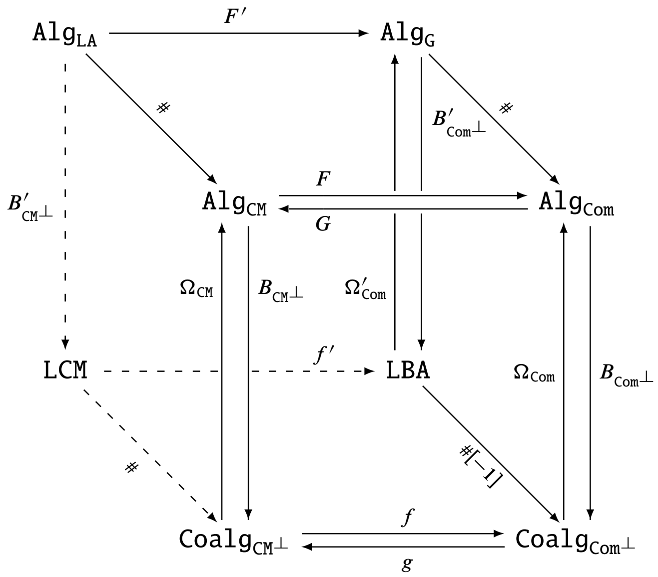
3.3. We wish to apply the map of functors (9) to a Lie algebroid (A, T). The functor F applied to a Lie algebroid, yields a $\mathtt{G}$-algebra, so we upgrade it to the functor

$$F' : \mathtt{Alg}_{\mathtt{LA}} \to \mathtt{Alg}_{\mathtt{G}}.$$

In the diagram below we draw the categories and the functors described above. The arrows denoted # are forgetful functors (#[−1] is the composition of the forgetful functor with a shift).

The diagram looks more symmetric if one adds an extra vertex which we denote $\mathtt{LCM}$. This is the category of $\mathtt{CM}^\perp$-coalgebras X together with a Lie bialgebra structure on $f(X)[1]$. One has a forgetful functor $\mathtt{LCM} \to \mathtt{Coalg}_{\mathtt{CM}^\perp}$ and an obvious functor $f' : \mathtt{LCM} \to \mathtt{LBA}$. The functor $B'_{\mathtt{CM}^\perp}$ is defined later on, see Lemma 3.3.1 and the discussion after it.

[4] The symmetric algebra $S_A(M) = \bigoplus_{n\geq 0} S^n_A(M)$ makes sense even if the commutative algebra A has no unit. This is important as our operads are non-unital.



(10)

Recall that the fuctors G and g (see the front face of the cube) commute with the Bar-constructions, which leads to a canonical morphism of functors

$$f \circ B_{\mathtt{CM}^\perp} \longrightarrow B_{\mathtt{Com}^\perp} \circ F.$$

Let now (A, T) be a Lie algebroid. We define $\xi(A, T) = f(B_{\mathtt{CM}^\perp}(A, T))[1]$. By definition, this is a dg Lie coalgebra and one has a canonical injective map

$$\iota : \xi(A, T) \longrightarrow B_{\mathtt{Com}^\perp}(F(A, T))[1] = B'_{\mathtt{Com}^\perp}(F'(A, T)). \tag{11}$$

Recall that $F'(A, T) = S_A(T[-1])$ is precisely what we need, so we have constructed a dg Lie subcoalgebra $\xi(A, T)$. It is easy to check (see Lemma 3.3.1 below) that $\xi(A, T)$ is also a Lie subalgebra, so that $\xi(A, T) \in \mathtt{LBA}$. But even before doing this, let us check that the embedding $\iota : \xi(A, T) \to B_{\mathtt{Com}^\perp}(F(A, T))[1]$ is a weak equivalence of dg Lie coalgebras.

In fact, one has a weak equivalence $\Omega_{\mathtt{CM}} \circ B_{\mathtt{CM}^\perp}(A, T) \to (A, T)$. Applying the functor F and we get a weak equivalence

$$F \circ \Omega_{\mathtt{CM}} \circ B_{\mathtt{CM}^\perp}(A, T) \to F(A, T). \tag{12}$$

Since the map $\Omega_{\mathtt{Com}} \circ B_{\mathtt{Com}^\perp}(F(A,T)) \to F(A,T)$ is also a weak equivalence, the commutative diagram

$$\begin{array}{ccc} \Omega_{\mathtt{Com}} \circ f \circ B_{\mathtt{CM}^\perp}(A,T) & \xrightarrow{\Omega_{\mathtt{Com}}(\iota)} & \Omega_{\mathtt{Com}} \circ B_{\mathtt{Com}^\perp} \circ F(A,T) \\ \| & & \downarrow \\ F \circ \Omega_{\mathtt{CM}} \circ B_{\mathtt{CM}^\perp}(A,T) & \longrightarrow & F(A,T) \end{array} \tag{13}$$

asserts that ι is a weak equivalence in $\mathtt{Coalg}_{\mathtt{Com}^\perp}$.

Lemma 3.3.1. *The image of $f(B_{\mathtt{CM}^\perp}(A,T))$ in $B'_{\mathtt{Com}^\perp}(F'(A,T))$ is a Lie subalgebra.*

Proof of the lemma. Denote $V = F'(A,T) = S_A(T[-1]) = \oplus S^n_A(T[-1])$. The Lie bialgebra $B'_{\mathtt{Com}^\perp}(V)$ as a graded space is just

$$F^*_{\mathtt{Lie}}(V[1]) = \bigoplus_{n \geq 1} (\mathtt{Lie}(n)^* \otimes V^{\otimes n}[n])^{S_n}.$$

The Lie bracket on it is extended from the Lie bracket on $V[1]$. The space V is graded, and this grading induces a grading on $F^*_{\mathtt{Lie}}(V[1])$. The image of $f(B_{\mathtt{CM}^\perp}(A,T))$ consists of elements having degree ≤ 1. The Lie gracket has degree -1 with respect to this grading; therefore, the image is closed with respect to the Lie bracket. $\square$

3.4. From now on A is a smooth dg commutative algebra over $R \supset \mathbb{Q}$ and $T = \mathrm{Der}_R(A,A)$. This means that A^0 is a smooth commutative R-algebra and the map $A^0 \to A$ is a finitely generated cofibration (that is, A is generated as a graded A^0-algebra by a finite number of free variables x_i of negative degree).

In what follows we will write $\xi(A)$ instead of $\xi(A,T)$. According to 3.1 the complex $B'_{\mathtt{Com}^\perp}(C) = (F^*_{\mathtt{Lie}}(C[1]), d)$ has a structure of dg Lie bialgebra. We will present a Lie bialgebra map $\varkappa : \xi(A) \to B'_{\mathtt{Com}^\perp}(C)$ and prove it is a weak equivalence.

Our plan is as follows. First of all we will present a map of dg Lie coalgebras $\varkappa : \xi(A) \to B'_{\mathtt{Com}^\perp}(C)$, then we will check it is a Lie algebra homomorphism, and after that we will check it is a weak equivalence of Lie bialgebras.

To present a map of dg Lie coalgebras, it suffices to have a map

$$B_{\mathtt{CM}^\perp}(A,T) \longrightarrow g(B'_{\mathtt{Com}^\perp}(C)).$$

The latter is given by a pair of maps (α, β) where

$$\alpha : A \longrightarrow C \tag{14}$$

is a map of $\mathtt{Com}_\infty$ algebras and

$$\beta : T \longrightarrow C[1] \tag{15}$$

is a map of $\mathtt{Com}_\infty$-modules over α. The maps are precisely the maps we were talking about from the very beginning, $\alpha : A \to C^0(A) = A$ and $\beta : T \to \mathrm{Hom}(A, A) = C(A)[1]^0$.

To check that α induces a map of $\mathtt{Com}_\infty$-algebras, we need to check that the map $\alpha : A \to C$ induces a map of the Bar-constructions which commutes with the differentials. This is equivalent to checking that the higher components of the differential

$$d_n : F^{*n}_{\mathtt{Com}^\perp}(C) \to C[1],\ n > 3,$$

vanish on $F^{*n}_{\mathtt{Com}^\perp}(A)$ and d_2 coincides with the multiplication in A.

Similarly, in order to check that β is a map of $\mathtt{Com}_\infty$-modules over α, one needs to verify that $d_n,\ n > 2$ also vanish on the part of $F^{*n}_{\mathtt{Com}^\perp}(C)$ having $n-1$ component C^0 and one component C^1. Both statements are independent of A; they are verified in Theorem 3 of [2]. Thus, we already know that $\varkappa : \xi(A) \to B'_{\mathtt{Com}^\perp}(C)$ is a map of dg Lie coalgebras. To check it preserves the bracket, it suffices to compose $\varkappa$ with the projection to cogenerators $C[1]$ of $B'_{\mathtt{Com}^\perp}(C)$.[5]

Since the projection itself is a Lie algebra homomorphism, we have to verify that the composition

$$\xi(A) \xrightarrow{\varkappa} B'_{\mathtt{Com}^\perp}(C) \longrightarrow C[1] \tag{16}$$

is a Lie algebra homomorphism.

We can forget about the differentials. The inclusion $\xi(A) \to F^{*}_{\mathtt{Com}^\perp}(S_A(T[-1]))$ induces a grading on $\xi(A)$; the composition $\xi(A) \to C[1]$ is zero on all components except for degree 1. Thus, it factors as

$$\xi(A) \to A \oplus T[-1] \to C[1],$$

so it remains to check that the obvious map

$$A \oplus T[-1] \to C[1] \tag{17}$$

preserves the Lie bracket. This is obvious when $C[1]$ is endowed with the Gerstenhaber bracket. The rest follows from the following proposition.

[5] Let us explain the last point. The bracket map $X \otimes X \to X$ in a Lie bialgebra is a coderivation, for an appropriate notion of coderivation from a comodule to a Lie coalgebra. A composition of a coderivation with a homomorphism of Lie coalgebras gives a coderivation. We have to compare two coderivations into a cofree Lie coalgebra. It is sufficient to compare their corestrictions on the cogenerators.

Proposition 3.4.1. *The* $\mathtt{Lie}_\infty$*-structure on C[1] defined by the* $\widetilde{\mathtt{B}}$*-structure, coincides with the (strict) Gerstenhaber bracket.*

Proof. The claim is independent of A and is precisely Theorem 2 of [2]. □

The map $\varkappa : \xi(A) \to B'_{\mathtt{Com}^\perp}(C)$ is therefore a map of Lie bialgebras.

3.5. $\varkappa$ is a weak equivalence. We have to verify that $\varkappa : \xi(A) \to B'_{\mathtt{Com}^\perp}(C)$ is a weak equivalence, that is that the map

$$\Omega_{\mathtt{Com}}(\varkappa) : F \circ \Omega_{\mathtt{CM}} \circ B_{\mathtt{CM}^\perp}(A,T) = \Omega_{\mathtt{Com}} \circ f \circ B_{\mathtt{CM}^\perp}(A,T) \to \Omega_{\mathtt{Com}} \circ B_{\mathtt{Com}^\perp}(C) \quad (18)$$

is a quasiisomorphism. We will deduce this from a dg version of Hochschild–Kostant–Rosenberg (HKR) theorem for smooth dg algebras.

Let A be as in 3.4. According to [9, §5.4.5.1], the homological HKR map $C_*(A,A) \longrightarrow S_A(\Omega_A[1])$ is a quasiisomorphism. This is a map of cofibrant A-modules, so it induces a quasiisomorphism of complexes

$$\mathtt{HKR} : S_A(T[-1]) \longrightarrow C = \mathrm{Hom}(C_*(A,A),A). \quad (19)$$

Note that the cohomology of HKR is compatible with the Gerstenhaber structures.

This immediately implies $\Omega_{\mathtt{Com}}(\varkappa)$ is a quasiisomorpism in the case A has trivial differential, for instance, when A is a (conventional) smooth algebra. In fact, the map (18) induces in cohomology a Gerstenhaber algebra homomorphism from $S_A(T[-1])$ to $H(C)$ which coincides with HKR on A and on T. Then by HKR theorem it is a quasiisomorphism.

In general, $H(C)$ needs not be generated by $H(T)$ over $H(A)$, so the above reasoning does not work. It is still true that the maps $\Omega_{\mathtt{Com}}(\varkappa)$ and HKR induce, after some identification, the same map in cohomology. This will require, however, a certain nontrivial effort.

It is convenient to replace $\Omega_{\mathtt{Com}}(\varkappa)$ with a strict map of algebras over $\mathtt{Com}_\infty$. One has a pair of adjoint functors

$$\mathtt{Alg}_{\mathtt{CM}_\infty} \underset{G_\infty}{\overset{F_\infty}{\rightleftarrows}} \mathtt{Alg}_{\mathtt{Com}_\infty}$$

defined similarly to the pair (F,G): $G_\infty(C) = (C, C[1])$, and F_∞ is left adjoint to G_∞.

The pair $(\alpha : A \to C, \beta : T \to C[1])$ that defined the map $\varkappa$ gives rise to a map $(A,T) \to G_\infty(C)$, hence a map

$$\varkappa_\infty : F_\infty(A,T) \longrightarrow C \quad (20)$$

is defined. This is a strict map of $\mathtt{Com}_\infty$-algebras.

Lemma 3.5.1. *$\varkappa$ is an equivalence iff $\varkappa_\infty$ is a quasiisomorphism.*

Proof. Three lines of the diagram

$$\begin{array}{ccc}
\mathtt{Alg}_{\mathtt{CM}_\infty} & \underset{G_\infty}{\overset{F_\infty}{\rightleftarrows}} & \mathtt{Alg}_{\mathtt{Com}_\infty} \\
\big\downarrow {\scriptstyle B_{\mathtt{CM}^\perp}} & & \big\downarrow {\scriptstyle B_{\mathtt{Com}^\perp}} \\
\mathtt{Coalg}_{\mathtt{CM}^\perp} & \underset{g}{\overset{f}{\rightleftarrows}} & \mathtt{Coalg}_{\mathtt{Com}^\perp} \\
\big\downarrow {\scriptstyle \Omega_{\mathtt{CM}}} & & \big\downarrow {\scriptstyle \Omega_{\mathtt{Com}}} \\
\mathtt{Alg}_{\mathtt{CM}} & \underset{G}{\overset{F}{\rightleftarrows}} & \mathtt{Alg}_{\mathtt{Com}}
\end{array} \tag{21}$$

convert to the same pair of adjoint functors between the homotopy categories $\mathrm{Ho}\,\mathtt{Alg}_{\mathtt{CM}}$ and $\mathrm{Ho}\,\mathtt{Alg}_{\mathtt{Com}}$. Thus the maps $\varkappa_\infty$ and $\Omega_{\mathtt{Com}}(\varkappa)$ represent isomorphic arrows in the homotopy category. □

3.5.2. It remains to verify $\varkappa_\infty$ is a quasiisomorphism.

The canonical projection $\pi : \mathtt{Com}_\infty \to \mathtt{Com}$ induces a map $\psi_\pi : F_\infty(A, T) \to F(A, T)$ which is a quasiisomorphism. We will prove that the maps $\varkappa_\infty$ and $\mathtt{HKR} \circ \psi_\pi$ induce the same map in cohomology.

First of all, we need a generalization of the functor F_∞. Let $\mathtt{P}$ be a dg operad and let $\mathtt{PM}$ be the operad governing pairs (A, M) where A is a $\mathtt{P}$-algebra and M is an A-module. The canonical map $\mathtt{PM} \to \mathtt{P}$ of operads defines a pair of adjoint functors $F_{\mathtt{P}} : \mathtt{Alg}_{\mathtt{PM}} \to \mathtt{Alg}_{\mathtt{P}}$ and $G_{\mathtt{P}} : \mathtt{Alg}_{\mathtt{P}} \to \mathtt{Alg}_{\mathtt{PM}}$ with $G_{\mathtt{P}}(A) = (A, A[1])$. We have $F_{\mathtt{Com}_\infty} = F_\infty$ and $F_{\mathtt{Com}} = F$.

Let now $\mathtt{P}$ be endowed with a pair of maps $j : \mathtt{P} \to \mathtt{Br}$ and $p : \mathtt{P} \to \mathtt{Com}$. The map $p : \mathtt{P} \to \mathtt{Com}$ defines a map $\mathtt{PM} \to \mathtt{CM}$ of colored operads, so the pair (A, T) can be considered as an $\mathtt{PM}$-algebra.

We will call the triple $(\mathtt{P}, j, p)$ *admissible* if the map $(\alpha, \beta) : (A, T) \to (C, C[1])$ defined as in beginning of Section 2, is a map of $\mathtt{PM}$-algebras. We know at least two examples of admissible triples:

- $\mathtt{P} = \mathtt{Com}_\infty$ with the obvious projection to $\mathtt{Com}$ and the map to $\mathtt{Br}$ defined as the composition $\mathtt{Com}_\infty \to \widetilde{\mathtt{B}} \to \mathtt{B}_\infty \to \mathtt{Br}$.
- $\mathtt{P} = \mathtt{Ass}$ with the obvious projection $p : \mathtt{Ass} \to \mathtt{Com}$ and the map to $\mathtt{Br}$ defined by the cup-product.

Moreover, given an admissible triple $(\mathtt{P}, j, p)$ and a map of operads $q : \mathtt{Q} \to \mathtt{P}$, the triple $(\mathtt{Q}, jq, pq)$ is also admissible.

In this case one has a natural map $\psi_q : F_{\mathtt{Q}}(A, T) \to q^*(F_{\mathtt{P}}(A, T))$.

An admissible triple $(\mathtt{P}, j, p)$ defines a map $\phi_{\mathtt{P}} : F_{\mathtt{P}}(A, T) \to p^*(C)$ of $\mathtt{P}$-algebras, so that for any $q : \mathtt{Q} \to \mathtt{P}$ one has

$$\phi_{\mathtt{Q}} = q^*(\phi_{\mathtt{P}}) \circ \psi_q. \tag{22}$$

Look at the diagram of operads

$$\begin{array}{ccc} \mathtt{Ass}_\infty & \xrightarrow{\tau} & \mathtt{Ass} \\ {\scriptstyle q}\downarrow & & \downarrow \\ \mathtt{Com}_\infty & \longrightarrow & \mathtt{Br}, \end{array} \tag{23}$$

where τ is the obvious projection, the bottom horisontal map is defined as the composition $\mathtt{Com}_\infty \to \widetilde{\mathtt{B}} \to \mathtt{B}_\infty \to \mathtt{Br}$ and the map $q : \mathtt{Ass}_\infty \to \mathtt{Com}_\infty$ is defined as follows. Recall that a $\mathtt{Com}_\infty$ structure on X is a structure of dg Lie coalgebra on $F^*_{\mathtt{Lie}}(X[1])$. Having a dg Lie coalgebra, one can apply the "enveloping coalgebra" functor to get a dg coalgebra structure on $F^*_{\mathtt{Ass}}(X[1])$ which is an $\mathtt{Ass}_\infty$-algebra structure on X. This shows that any $\mathtt{Com}_\infty$-algebra X has a canonical $\mathtt{Ass}_\infty$ structure; whence, a map $q : \mathtt{Ass}_\infty \to \mathtt{Com}_\infty$.

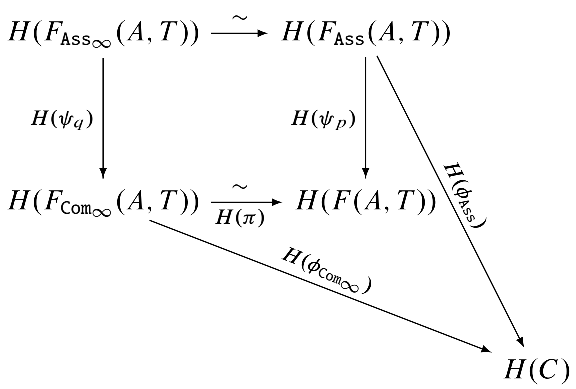
In Lemma 3.5.3 below we check that the diagram above is homotopy commutative.

The diagram of operads (23) gives rise to a diagram

$$\begin{array}{ccc} F_{\mathtt{Ass}_\infty}(A, T) & \xrightarrow{\psi_\tau} & F_{\mathtt{Ass}}(A, T) \\ {\scriptstyle \psi_q}\downarrow & & \downarrow{\scriptstyle \phi_{\mathtt{Ass}}} \\ F_{\mathtt{Com}_\infty}(A, T) & \xrightarrow{\phi_{\mathtt{Com}_\infty}} & C. \end{array} \tag{24}$$

According to Lemma 3.5.4 below it is commutative in the derived category of R-modules.

The following diagram is obtained from (24) by passing to the cohomology and adding an extra vertex $H(F(A,T))$.



(25)

Both convex quadrilaterals in (25) are commutative.

The HKR isomorphism is defined as the composition of $H(\phi_{\mathtt{Ass}})$ with the symmetrization map *sym* : $H(F(A,T)) \to H(F_{\mathtt{Ass}}(A,T))$ induced by the canonical map from the symmetric algebra of $T[-1]$ to its tensor algebra and splitting the canonical projection $H(\psi_p)$.

Thus, if we define $\theta : H(F(A,T)) \longrightarrow H(C)$ as the composition

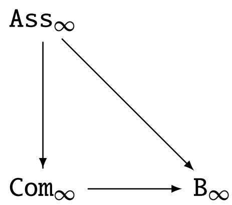
$$\theta = H(\psi_{\mathtt{Com}_\infty}) \circ H(\pi)^{-1},$$

One has $\theta \circ H(\psi_p) = H(\phi_{\mathtt{Ass}})$, which implies $\theta = \mathtt{HKR}$. □

It remains to prove Lemmas 3.5.3 and 3.5.4.

Lemma 3.5.3. *The diagram of operads (23) is homotopy commutative.*

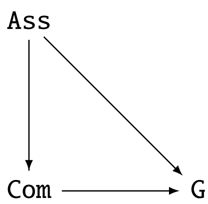
Proof. We will prove that the diagram



(26)

is homotopy commutative—this will imply the homotopy commutativity of the original diagram.

Now, the diagram (26) maps quasiisomorphically to the commutative diagram



(27)

of its cohomology. This proves the claim since $\mathtt{Ass}_\infty$ is cofibrant. □

Lemma 3.5.4. *Let* $f_0,\ f_1 : \mathtt{P} \longrightarrow \mathtt{Q}$ *be a pair of (left) homotopic maps of dg operads. Let* $(A, T) \in \mathtt{Alg}_{\mathtt{PM}}$ *and* $C \in \mathtt{Alg}_{\mathtt{Q}}$*. Assume a pair of maps* $\alpha : A \to C$ *and* $\beta : T \to C[1]$ *is compatible with both* $\mathtt{P}$*-algebra structures on* C *defined by* f_0 *and* f_1*. Then two maps*

$$F_{\mathtt{P}}(A, T) \to C$$

induced by two different $\mathtt{P}$*-module structures on* C *defined by* f_0 *and* f_1*, determine the same map in the derived category of complexes.*

Proof. Let

$$\mathtt{P} \sqcup \mathtt{P} \longrightarrow \widetilde{\mathtt{P}} \xrightarrow{\ \pi\ } \mathtt{P}$$

be a cylinder object for $\mathtt{P}$ so that a map $h : \widetilde{\mathtt{P}} \to Q$ realizes the homotopy between f and g. Replacing $\mathtt{Q}$ with $\widetilde{\mathtt{P}}$ and C with $h^*(C)$ we immediately reduce the claim to the case $\mathtt{Q} = \widetilde{\mathtt{P}}$.

Choose a surjective quasiisomorphism $C' \to C$ with cofibrant C' in $\mathtt{Alg}_{\mathtt{Q}}$. The map $C' \to \pi^*\pi_!(C')$ is a quasiisomorphism since π is a quasiisomorphism of operads.

Without loss of generality we can assume the pair (A, T) is freely generated as a $\mathtt{PM}$-algebra by a sequence of elements

$$a_i \in A,\ i \in I, \quad t_i \in T,\ i \in J,$$

such that $d(a_i)$ (resp., $d(t_i)$) is expressed via generators with smaller indices (with respect to a certain total order on $I \sqcup J$).

One can easily see that the maps $(A, T) \to G_P(f_i^*(C)),\ i = 0, 1$, defined by the pair (α, β) (this is the same map for $i = 0, 1$ if considered as a map of complexes) can be lifted to maps $j_i : (A, T) \to G_P(f_i^*(C'))$ so that the compositions

$$(A, T) \xrightarrow{\ j_i\ } G_P(f_i^*(C')) \xrightarrow{\ p\ } G_P(f_i^* \circ \pi^*(\pi_!(C'))) = G_P(\pi_!(C'))$$

coincide. In fact, since (A, T) is freely generated by $a_i,\ t_i$, two maps from (A, T) to $G_P(\pi_!(C'))$ coincide if they coincide on a_i, t_j. We will lift the map $(\alpha, \beta) : (A, T) \to G_P(f_i^*(C))$ to $G_P(f_i^*(C'))$ by lifting the images of the generators one by one. Thus, the values of two maps $(A, T) \to G_P(f_i^*(C'))$ will coincide on the generators. These will be two different maps to $(C', C'[1])$ but they will coincide at the generators; the compositions of these maps with $C' \to \pi^* \circ \pi_!(C')$ will coincide.

Passing to the adjoint maps, we get the diagram

$$\begin{array}{ccc} & f_i^*(C') & \xrightarrow{\ p\ } \pi_!(C') \\ {}^{j_i}\nearrow & \downarrow q & \\ F_P(A, T) \longrightarrow & f_i^*(C) & \end{array} \tag{28}$$

where p and q are quasiisomorphisms. Since the compositions of j_0, j_1 with p coincide in the derived category, j_0 and j_1 coincide in the derived category; therefore, their compositions with q coincide as well. □

4. Application: non-commutative unfolding

4.1. Let k be a field of characteristic zero and let f be a polynomial in $A = k[x_1 \ldots, x_n]$. We put $B = k[y]$ and we define a B-algebra structure on A via $y = f(x_1, \ldots, x_n)$.

Denote $P = B[x_1, \ldots, x_n, e]$ the semifree B-algebra generated by x_i in degree 0, e in degree -1, with the differential defined as

$$de = f(x_1, \ldots, x_n) - y.$$

The obvious projection $\pi : P \to A$ carrying x_i to x_i and e to 0, is a quasiisomorphism. It is split as a homomorphism of k-algebras by $\iota : A \to P$ defined by $\iota(x_i) = x_i$.

Lemma 4.1.1. *A is free as B-module.*

Proof. It is a standard fact that there is an automorphism of A given by the formulas

$$x_i \mapsto x_i + x_n^{N_i}, \quad x_n \mapsto x_n, \tag{29}$$

for suitable N_i, such that the image of f is a monic polynomial in x_n with coefficients in $k[x_1, \ldots, x_{n-1}]$. This allows one to assume, without loss of generality, that f is monic in x_n. In this case the sequence of elements in A

$$x_1, \ldots, x_{n-1}, f$$

is regular and A is free over $k[x_1, \ldots, x_{n-1}, f]$. This implies that A is also free as $B = k[f]$-module. □

4.2. Comparison of Hochschild complexes. Let k be a commutative ring, A and A' two dg k-algebras cofibrant as complexes over k. We are going to show that if A and A' are quasiisomorphic, then their Hochschild cochain complexes are quasiisomorphic as dg Lie algebras.

Note that according to a deep result of Keller [8] the Hochschild cochain complexes $C(A)$ and $C(A')$ should be equivalent as B_∞-algebras. We present below a much more elementary result so as not to be compelled to extend [8] to dg setup.

Let A be a dg algebra over k. The Hochschild cochain complex $C(A)$ can be defined as follows. We endow a unital cofree dg associative coalgebra $A^\vee = \oplus_{n\geq 0} A^{\otimes n}[n]$ with a differential encoding the differential and multiplication in A. The (graded) coderivations of $A^\vee$ form a dg Lie algebra which is precisely $C(A)[1]$.

Let $f : A \to A'$ be a surjective quasiisomorphism of dg algebras over k. Assume furthermore that both A and A' are cofibrant as complexes over k.

Let us show that the Hochschild complexes $C(A)$ and $C(A')$ are equivalent as dg Lie algebras.

The map f induces a map of dg coalgebras

$$f^\vee : A^\vee \longrightarrow A'^\vee.$$

This yields the pair of maps ϕ and ψ in the diagram

$$\begin{array}{ccc} X & \overset{\psi'}{\dashrightarrow} & \mathrm{Coder}(A^\vee) \\ {\scriptstyle\phi'}\downarrow & & \downarrow{\scriptstyle\phi} \\ \mathrm{Coder}(A'^\vee) & \xrightarrow{\psi} & \mathrm{Coder}^f(A^\vee, A'^\vee), \end{array} \tag{30}$$

where Coder^f is the collection of maps $\delta : A^\vee \to A'^\vee$ satisfying the condition

$$\Delta \circ \delta = (\delta \otimes f + f \otimes \delta) \circ \Delta.$$

Define X by the cartesian diagram above. Then X inherits the dg Lie algebra structure. The maps ϕ and ψ are both quasiisomorphisms and ϕ is surjective, so the maps ϕ' and ψ' are quasiisomorphism of dg Lie algebras.

Corollary 4.2.1. *Let A and A' be two dg algebras over k which are cofibrant as complexes. If A and A' are quasiisomorphic, their Hochschild complexes $C(A)[1]$ and $C(A')[1]$ are quasiisomorphic as dg Lie algebras.*

Proof. Any pair of quasiisomorphic algebras can be connected by a pair of surjective quasiisomorphisms from a cofibrant algebra which is automatically cofibrant as a complex of k-modules. □

4.3. We are now back to our unfoldings. According to the above, the dg Lie algebra governing deformations of B-algebra A, is the algebra of polyvector fields $S_P(T_P[-1])[1]$ where $T_P = \mathrm{Der}_B(P)$. In a more detail, T_P is is a P-module freely generated by the elements $\partial_i = \frac{\partial}{\partial x_i}$ of degee 0 and the element $\partial_e = \frac{\partial}{\partial e}$ of degree 1, with the differentials given by the formula

$$d(\partial_e) = 0; \quad d(\partial_i) = \frac{\partial f}{\partial x_i}\partial_e. \tag{31}$$

It is convenient to compare T_P with a dg Lie algebroid T over A generated by the same ∂_i and ∂_e over A, with the differential given by (31).

Note the following lemma.

Lemma 4.3.1. *The differential in T is inner, given by the formula*

$$d(x) = -[f\partial_e, x].$$

□

The Lie algebroid T_P can be described via T as follows.

Lemma 4.3.2. *Let P be a commutative (dg) A-algebra, T a Lie algebroid over A and let a map of Lie algebras and left A-modules $\alpha : T \to \mathrm{Der}(P)$ makes commutative the following diagram*

$$\begin{array}{ccc} T & \xrightarrow{\alpha} & \mathrm{Der}(P) \\ \downarrow & & \downarrow \\ \mathrm{Der}(A) & \longrightarrow & \mathrm{Der}(A, P), \end{array} \tag{32}$$

where the maps to $\mathrm{Der}(A, P)$ are defined via composition with the algebra map $A \to P$. Then α uniquely defines a structure of P-Lie algebroid on $P \otimes_A T$ and a map $(A, T) \to (P, P \otimes_A T)$ in $\mathtt{Alg}_{\mathtt{LA}}$.

Proof. Straightforward. □

The lemma identifies T_P with $P \otimes_A T$ and, in particular, defines a Lie algebra map $S_A(T[-1]) \longrightarrow S_P(T_P[-1])$. It is, obviously, a quasiisomorphism.

By Lemma 4.3.1 the differential in $S_A(T[-1])$ is also given by the formula $d(x) = -\mathrm{ad}_{f\partial_e}$.

4.4. Calculation. Denote $\mathfrak{g} = S_A(T[-1])[1]$. One has

$$\begin{array}{lcl} \mathfrak{g}^{-1} & = & A, \\ \mathfrak{g}^0 & = & \bigoplus_{i=1}^n A\partial_i, \\ \mathfrak{g}^1 & = & A\partial_e \oplus \bigoplus_{i,j} A\partial_i \wedge \partial_j, \\ \mathfrak{g}^2 & = & \bigoplus_i A\partial_e \wedge \partial_i \oplus \bigoplus_{i,j,k} \partial_i \wedge \partial_j \wedge \partial_k. \end{array} \tag{33}$$

Let $(R, \mathfrak{m})$ be a local artinial k-algebra with the maximal ideal $\mathfrak{m}$.

Let $w = p\partial_e + S \in \mathfrak{m} \otimes \mathfrak{g}^1$ with $S \in \bigoplus \mathfrak{m} \otimes A\partial_i \wedge \partial_j$. One has $dw + \frac{1}{2}[w, w] = dS + [p\partial_e, S] + \frac{1}{2}[S, S]$. The first two summands are divisible by ∂_e whereas the third summand is not. Thus, w satisfies Maurer–Cartan equation iff

$$\begin{cases} dS + [p\partial_e, S] & = 0 \\ [S, S] & = 0 \end{cases} \tag{34}$$

or (taking into account that $[f\partial_e, S] = 0$ iff $[f, S] = 0$)

$$\begin{cases} [f - p, S] & = 0 \\ [S, S] & = 0. \end{cases} \tag{35}$$

Let $T_A = \mathrm{Der}(A, A) = \oplus A\partial_i$.

Note that the commutative algebra $S_A(T_A[-1])$ endowed with the differential $d = \mathrm{ad}_f$ identifies with the Koszul complex of A constructed on the sequence $(\partial_1 f, \ldots, \partial_n f)$.

From now on we assume that f is an isolated singularity, that is that $\partial_i f$ form a regular sequence. This implies that $S_A(T[-1]), \mathrm{ad}_f)$ is acyclic. Moreover, for any artinian local $(R, \mathfrak{m})$ and any $p \in \mathfrak{m} \otimes A$ the complex $(R \otimes S_A(T[-1]), \mathrm{ad}_{f-p})$ is also acyclic as a deformation of acyclic complex. Therefore, $[f - p, S] = 0$ if and only if there exists a trivector field T on A such that $S = [f - p, T]$.

This proves the following result.

Proposition 4.4.1. *Let $f \in A = k[x_1, \ldots, x_n]$ define an isolated hypersurface singularity. A solution of Maurer–Cartan equation for a noncommutative unfolding is given by a pair (p, T) where $p \in \mathfrak{m} \otimes A$ and $T \in \mathfrak{m} \otimes \wedge^3_A(T_A)$ satisfying the condition*

$$[[f - p, T], [f - p, T]] = 0. \tag{36}$$

4.5. Quasiclassical data for NC unfoldings. Recall that quasiclassical datum is defined as deformations of B-algebra A over $k[h]/(h^2)$ extendable to $k[h]/(h^3)$.

Deformations over $k[h]/(h^2)$ are described by the first cohomology of $\mathfrak{g}$. Cocycles are given by pairs (ph, Sh) with $p \in A$, $S \in \oplus A\partial_i \wedge \partial_j$ satisfying the condition $[f, S] = 0$, pairs $(p_1 h, S_1 h)$ and $(p_2 h, S_2 h)$ being homologous iff $S_1 = S_2$ and $p_1 - p_2 \in (\partial_1 f, \ldots, \partial_n f)$.

Maurer–Cartan solutions over $k[h]/(h^3)$ are described by pair (p, S) where $p = p_1 h + p_2 h^2$ and $S = S_1 h + S_2 h^2$ satisfying (35). This imposes two extra conditions on (p_1, S_1):

1. $[S_1, S_1] = 0$.
2. There exists S_2 such that $[p_1, S_1] = [f, S_2]$.

Note that the second condition is equivalent to the condition $[f, [p_1, S_1]] = 0$ which is always fulfilled as $[f, S_1] = 0$ and $[f, p_1] = 0$.

Choose a vector subspace W in $k[x_1, \ldots, x_n]$ such that

$$k[x_1, \ldots, x_n] = W \oplus (\partial_1, \ldots, \partial_n).$$

We have proven

Proposition 4.5.1. *Quasiclassical data for NC unfolding of an isolated hypersurface singularity* $f \in k[x_1, \ldots, x_n]$ *are given by pairs* (p, S) *where*

1. $p \in W$.
2. S *is a Poisson bivector field satisfying* $[f, S] = 0$.

4.6. Quantization. We doubt that any quasiclassical datum can be quantized in general. This is, however, true for $n = 3$ (this case includes the classical ADE singularities) as shown in the following lemma.

Lemma 4.6.1. *Let* $A = k[x_1, x_2, x_3]$. *Any bivector field* S *satisfying* $[f, S] = 0$ *is Poisson.*

Proof. Recall that the differential in the Koszul complex $(S_A(T_A[-1]), d)$ is given by the formula $dx = [f, x]$. Let $S = [f, T] = dT$. One has

$$[S, S] = [dT, dT] = d[T, dT] = 0$$

as $[T, dT]$ is a four-vector. □

Corollary 4.6.2. *Any quasiclassical datum for NC unfolding of a surface isolated singularity can be quantized.*

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Received 27 December, 2012; revised 13 March, 2013

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