

Bivariant cyclic cohomology and Connes' bilinear pairings in noncommutative motives

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Abstract. In this article we further the study of noncommutative motives. We prove that the bivariant cohomology and the bivariant Chern character of any additive invariant E become representable in the category of noncommutative motives. This applies in particular to bivariant cyclic cohomology and its variants. When E is moreover symmetric monoidal we prove that the associated Chern character is multiplicative and characterize it by a precise universal property. In the particular case of bivariant cyclic cohomology the associated Chern character becomes the universal lift of the Dennis trace map. Then, we prove that under the above representability result, the composition operation in the category of noncommutative motives identifies with Connes' bilinear pairings. As an application, we obtain a simple model, given by Karoubi's infinite matrices, for the (de)suspension of these bivariant cohomology theories.

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1. Introduction and statement of results

1.1. Noncommutative motives. A *differential graded (=dg) category*, over a commutative base ring k , is a category enriched over complexes of k -modules (morphisms sets are complexes) in such a way that composition fulfills the Leibniz rule: $d(f \circ g) = d(f) \circ g + (-1)^{\deg(f)} f \circ d(g)$. Dg categories enhance and solve many of the technical problems inherent to triangulated categories; see Keller's ICM address [25]. In *noncommutative algebraic geometry* in the sense of Bondal, Drinfeld, Kaledin, Kapranov, Kontsevich, Van den Bergh, and others, dg categories are considered as dg-enhancements of bounded derived categories of (quasi-)coherent sheaves on a hypothetical noncommutative space; see [4, 5, 11, 12, 21, 27, 28, 29].

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All the classical invariants such as cyclic homology (and its variants), algebraic K -theory, and even topological cyclic homology, extend naturally from k -algebras to dg categories. In order to study all these invariants simultaneously the author introduced in [36, §15] the notion of *additive invariant*. This notion makes use of the language of Grothendieck derivators (see §2.4), a formalism which allows us to state and prove precise universal properties. Let $E : \text{HO}(\text{dgcats}) \rightarrow \mathbb{D}$ be a morphism of derivators, from the derivator associated to the derived Morita model structure on dg categories (see §2.2), to a triangulated derivator. We say that E is an additive invariant if it preserves filtered homotopy colimits and sends split exact sequences of dg categories (see [36, §13]) to direct sums in the base category $\mathbb{D}(e)$

$$0 \longrightarrow \mathcal{A} \overset{\leftarrow}{\rightleftarrows} \mathcal{B} \overset{\leftarrow}{\rightleftarrows} \mathcal{C} \longrightarrow 0 \quad \mapsto \quad E(\mathcal{A}) \oplus E(\mathcal{C}) \simeq E(\mathcal{B}). \tag{1.1}$$

Thanks to the work of Keller, Thomason–Trobough, Blumberg–Mandell, and others, (see [3, 26, 41, 42]) all the mentioned invariants give rise to additive invariants. In [36, §15] the *universal additive invariant* $U : \text{HO}(\text{dgcats}) \rightarrow \mathbb{M}$ was constructed. Given any triangulated derivator \mathbb{D} we have an induced equivalence of categories

$$U^* : \underline{\text{Hom}}_1(\mathbb{M}, \mathbb{D}) \xrightarrow{\sim} \underline{\text{Hom}}_{\text{add}}(\text{HO}(\text{dgcats}), \mathbb{D}), \tag{1.2}$$

where the left-hand side denotes the category of homotopy colimit preserving morphisms of derivators, and the right-hand side denotes the category of additive invariants. Because of this universality property, which is reminiscent of motives, \mathbb{M} is called the *additive motivator*, and its base triangulated category $\mathbb{M}(e)$ the category of *noncommutative motives*. By construction, the additive motivator admits a stable Quillen model \mathcal{M} (see [34]) and the universal additive invariant is induced by a functor $\mathcal{U} : \text{dgcats} \rightarrow \mathcal{M}$. The same holds for all the above additive invariants. Among many important applications, the category of noncommutative motives has allowed a streamlined construction of the Chern characters, a unified and conceptual proof of the fundamental theorem, and even a description of the fundamental isomorphism conjecture in terms of the classical Farrell–Jones isomorphism conjecture; see [2, 38, 40].

A fundamental problem in the theory of noncommutative motives is the computation of morphisms in $\mathbb{M}(e)$ and the description of its composition operation. In [36] an important step towards the solution of this problem was taken: let \mathcal{A} be a *finite dg cell* (the dg categorical analogue of a finite CW-complex); see §2.3. Then, for any dg category \mathcal{B} we have natural isomorphisms

$$\text{Hom}_{\mathbb{M}(e)}(U(\mathcal{A}), U(\mathcal{B})[-n]) \simeq K_n \text{rep}(\mathcal{A}, \mathcal{B}) \quad n \in \mathbb{Z}, \tag{1.3}$$

where K denotes algebraic K -theory and $\text{rep}(-, -)$ the internal Hom-functor in the homotopy category of dg categories (see §2.2). The composition operation is induced by the tensor product of bimodules. In particular, when \mathcal{A} is the dg

category \underline{k} associated to the base ring k (with one object and k as the dg algebra of endomorphisms) we obtain

$$\mathrm{Hom}_{\mathbb{M}(e)}(U(\underline{k}), U(\mathcal{B})[-n]) \simeq K_n(\mathcal{B}) \quad n \in \mathbb{Z}. \tag{1.4}$$

At this point it is natural to ask the following motivational questions :

Question A: Which (further) invariants of dg categories can be expressed in terms of morphism sets in the category of noncommutative motives ?

Question B: How to explicitly describe the composition operation in these cases ?

Roughly speaking, our answer is “Bivariant cohomology, with the composition operation given by a bilinear pairing with algebraic K -theory”.

1.2. Bivariant (cyclic) cohomology. Let \mathcal{N} be a Quillen model category and $E : \mathrm{dgc}at \rightarrow \mathcal{N}$ a functor. We say that E is an *additive functor* if it sends derived Morita equivalences (see §2.2) to weak equivalences, preserves filtered homotopy colimits, and sends split exact sequences of dg categories to direct sums in the homotopy category $\mathrm{Ho}(\mathcal{N})$. Such a functor gives rise to an additive invariant $E : \mathrm{HO}(\mathrm{dgc}at) \rightarrow \mathrm{HO}(\mathcal{N})$ and hence by (1.2) to a homotopy colimit preserving morphism $E_{\mathrm{add}} : \mathbb{M} \rightarrow \mathrm{HO}(\mathcal{N})$ such that $E = E_{\mathrm{add}} \circ U$. As a consequence, one obtains a well-defined triangulated functor

$$E_{\mathrm{add}} : \mathbb{M}(e) \longrightarrow \mathrm{Ho}(\mathcal{N}). \tag{1.5}$$

The *bivariant cohomology* $E^*(-, -)$ associated to E is by definition the bifunctor

$$\mathrm{dgc}at^{\mathrm{op}} \times \mathrm{dgc}at \longrightarrow \mathrm{Gr}_{\mathbb{Z}}(\mathrm{Ab}) \quad (\mathcal{B}, \mathcal{C}) \mapsto \mathrm{Hom}_{\mathrm{Ho}(\mathcal{N})}(E(\mathcal{B})[-*], E(\mathcal{C}))$$

with values in \mathbb{Z} -graded abelian groups. For example the bivariant cohomology $\mathcal{U}^*(-, -)$ associated to $\mathcal{U} : \mathrm{dgc}at \rightarrow \mathcal{M}$ agrees with $K_{-*}\mathrm{rep}(-, -)$ when the first entry is a finite dg cell. The triangulated functor (1.5) gives then rise to a natural transformation between bifunctors

$$\mathcal{U}^*(-, -) \Rightarrow E^*(-, -) \tag{1.6}$$

which we call the *bivariant Chern character* associated to E . Our answer to the above Question A is the following:

Theorem 1.7. *Let E be an additive functor as above. Then:*

- (i) *the associated triangulated functor (1.5) admits a right adjoint functor R^E ;*
- (ii) *the following isomorphism*

$$\mathrm{Hom}_{\mathbb{M}(e)}(U(\mathcal{B})[-*], T^E(U(\mathcal{C}))) \simeq E^*(\mathcal{B}, \mathcal{C}) \tag{1.8}$$

holds for any two dg categories \mathcal{B} and \mathcal{C} , with $T^E := R^E \circ E_{\mathrm{add}}$;

(iii) *under the isomorphism (1.8), the bivariant Chern character (1.6) is represented by the unit $\epsilon_E : \text{Id} \Rightarrow T^E$ of the adjunction (E_{add}, R^E) .*

Intuitively speaking, Theorem 1.7 shows us that the bivariant cohomology, as well as the bivariant Chern character, associated to an additive functor can be represented *inside* the triangulated category of noncommutative motives.

The tensor product extends naturally from k -algebras to dg categories, giving rise to a (derived) symmetric monoidal structure $-\otimes^{\mathbb{L}}-$ on $\text{HO}(\text{dgcats})$ with \otimes -unit \underline{k} ; see §2.2. In [7] this symmetric monoidal structure was extended to \mathbb{M} in a universal way, i.e. U becomes symmetric monoidal and (1.2) admits a \otimes -sharpening

$$U^* : \underline{\text{Hom}}_!^{\otimes}(\mathbb{M}, \mathbb{D}) \xrightarrow{\sim} \underline{\text{Hom}}_{\text{add}}^{\otimes}(\text{HO}(\text{dgcats}), \mathbb{D}). \tag{1.9}$$

Given a symmetric monoidal stable Quillen model category $(\mathcal{N}, \otimes, \mathbf{1})$ and a symmetric monoidal additive functor E , the functor (1.5) becomes then symmetric monoidal and so (1.6) restricts to a natural transformation between functors

$$ch_E : K_0(-) \simeq \mathcal{U}^0(\underline{k}, -) \Rightarrow E^0(\underline{k}, -) := \text{Hom}_{\text{Ho}(\mathcal{N})}(\mathbf{1}, E(-)) \tag{1.10}$$

which we call the *Chern character* associated to E . Note that the symmetric monoidal structures of $\mathbb{M}(e)$ and $\text{Ho}(\mathcal{N})$ give rise to bilinear pairings

$$K_0(\mathcal{A}) \otimes_{\mathbb{Z}} K_0(\mathcal{B}) \rightarrow K_0(\mathcal{A} \otimes \mathcal{B}) \quad E^0(\underline{k}, \mathcal{A}) \otimes_{\mathbb{Z}} E^0(\underline{k}, \mathcal{B}) \rightarrow E^0(\underline{k}, \mathcal{A} \otimes \mathcal{B}). \tag{1.11}$$

In particular, when $\mathcal{A} = \mathcal{B} = \underline{A}$, with A a (dg) commutative k -algebra, the abelian groups $K_0(\underline{A})$ and $E^0(\underline{k}, \underline{A})$ become endowed with a ring structure.

Theorem 1.12. *Let E be a symmetric monoidal additive functor as above. Then:*

- (i) *the associated Chern character (1.10) is multiplicative;*
- (ii) *we have a natural isomorphism of abelian groups*

$$\text{Nat}(K_0(-), E^0(\underline{k}, -)) \xrightarrow{\sim} \text{Hom}_{\mathbb{M}(e)}(U(\underline{k}), T^E(U(\underline{k}))), \tag{1.13}$$

where $\text{Nat}(-, -)$ stands for the abelian group of all natural transformations. Moreover, under the isomorphism (1.13) the Chern character (1.10) corresponds to the evaluation of the unit $\epsilon_E : \text{Id} \Rightarrow T^E$ at $U(\underline{k})$.

Informally speaking, item (i) shows us that the multiplicativity of the Chern character is a consequence of the fact that E is symmetric monoidal. Item (ii) characterizes the Chern character among all possible natural transformations as the evaluation of the unit of the adjunction at the \otimes -unit of $\mathbb{M}(e)$.

Jones and Kassel, by drawing inspiration from Kasparov’s KK -theory [23], introduced in [20] the bivariant cyclic cohomology theory of unital associative k -algebras. One of the fundamental properties of this bivariant theory is the fact that it simultaneously extends both negative cyclic homology as well as

cyclic cohomology. Bivariant cyclic cohomology $HC^*(-, -)$, as well as bivariant Hochschild cohomology $HH^*(-, -)$ [30, §5.5], extend naturally from k -algebras to dg categories. They are the bivariant cohomologies associated to the symmetric monoidal additive functors

$$HH : \text{dgc}at \longrightarrow \mathcal{C}(k) \quad C : \text{dgc}at \longrightarrow \mathcal{C}(\Lambda), \tag{1.14}$$

where C denotes the mixed complex construction and $\mathcal{C}(\Lambda)$ the category of mixed complexes; see [7, Examples 7.9 and 7.10]. By Theorem 1.7 we obtain then triangulated functors $T^{HH}, T^C : \mathbb{M}(e) \rightarrow \mathbb{M}(e)$ and natural isomorphisms

$$\text{Hom}_{\mathbb{M}(e)}(U(\mathcal{B})[-*], T^{HH}(U(\mathcal{C}))) \simeq HH^*(\mathcal{B}, \mathcal{C}) \tag{1.15}$$

$$\text{Hom}_{\mathbb{M}(e)}(U(\mathcal{B})[-*], T^C(U(\mathcal{C}))) \simeq HC^*(\mathcal{B}, \mathcal{C}). \tag{1.16}$$

Recall from [30, §5.5.1] and [20] that $HH^*(\mathcal{B}, \underline{k}) \simeq HH^*(\mathcal{B})$, $HC^*(\mathcal{B}, \underline{k}) \simeq HC^*(\mathcal{B})$, $HH^*(\underline{k}, \mathcal{C}) \simeq HH_{-*}(\mathcal{C})$, and $HC^*(\underline{k}, \mathcal{C}) \simeq HC_{-*}(\mathcal{C})$. Therefore, if in (1.15)–(1.16) we replace \mathcal{C} by \underline{k} we observe that the cohomology theories $HH^*(-)$ and $HC^*(-)$ become representable in $\mathbb{M}(e)$ by $T^{HH}(U(\underline{k}))$ and $T^C(U(\underline{k}))$; consult the proof of Theorem 1.7(iii) for the construction of these noncommutative motives. On the other hand, if we replace \mathcal{B} by \underline{k} we obtain Chern characters $ch_{HH} : K_0(-) \Rightarrow HH_0(-)$ and $ch_C : K_0(-) \Rightarrow HC_0^-(-)$, which by Theorem 1.12(ii) correspond to the morphisms $\epsilon_{HH}(U(\underline{k})) : U(\underline{k}) \rightarrow T^{HH}(U(\underline{k}))$ and $\epsilon_C(U(\underline{k})) : U(\underline{k}) \rightarrow T^C(U(\underline{k}))$. The natural map from bivariant cyclic cohomology to bivariant cyclic homology is induced by the forgetful functor $\Phi : \mathcal{C}(\Lambda) \rightarrow \mathcal{C}(k)$. Since $\Phi \circ C = HH$ we have an induced natural transformation $\Phi : T^C \Rightarrow T^{HH}$ verifying the equality $\Phi \circ \epsilon_C = \epsilon_{HH}$.

Theorem 1.17. (i) *The natural transformations ch_{HH} and ch_C agree with the Dennis trace map and the Chern character ch^- (see [30, §8]).*

(ii) *The morphism $\epsilon_C(U(\underline{k}))$ can be characterized as the unique morphism from $U(\underline{k})$ to $T^C(U(\underline{k}))$ making the following diagram commute*

$$\begin{array}{ccc}
 & & T^C(U(\underline{k})) \\
 & \nearrow \epsilon_C(U(\underline{k})) & \downarrow \Phi(U(\underline{k})) \\
 U(\underline{k}) & \xrightarrow{\epsilon_{HH}(U(\underline{k}))} & T^{HH}(U(\underline{k})).
 \end{array} \tag{1.18}$$

Note that by combining item (i) of Theorems 1.17 and 1.12 we recover the multiplicativity of the Dennis trace map and of the Chern character ch^- . On the other hand, by combining item (ii) of these theorems we obtain a conceptual characterization of the Chern character ch^- as the universal lift of the Dennis trace map; found originally by Goodwillie [14].

In what concerns bivariant periodic cyclic cohomology there exist two definitions in the literature. On one hand, Jones and Kassel [20] defined $HP^*(-, -)$ by inverting the S -operation on $HC^*(-, -)$. On the other hand, Cuntz and Quillen [9, 10] considered towers of supercomplexes (calculating periodic cyclic homology) and appropriate mapping supercomplexes between them. Cuntz–Quillen’s approach satisfies excision, while Kassel–Jones’ approach does not. Since the construction of the towers of supercomplexes (see [9, §2]) is not well-behaved with respect to filtered (homotopy) colimits, Cuntz–Quillen’s approach does not fit in the framework of Theorem 1.7. In contrast, the following result holds:

Theorem 1.19. *There exists a triangulated functor $T^{HP} : \mathbb{M}(e) \rightarrow \mathbb{M}(e)$ such that for any two dg categories \mathcal{B} and \mathcal{C} , with \mathcal{B} a finite dg cell, we have*

$$\mathrm{Hom}_{\mathbb{M}(e)}(U(\mathcal{B})[-*], T^{HP}(U(\mathcal{C}))) \simeq HP^*(\mathcal{B}, \mathcal{C}). \tag{1.20}$$

Connes’ bilinear pairings. In his foundational work on noncommutative geometry, in the early eighties, Connes [8] discovered bilinear pairings

$$\langle -, - \rangle : K_0(\mathcal{B}) \times HC^{2j}(\mathcal{B}) \longrightarrow k \quad j \geq 0 \tag{1.21}$$

relating the Grothendieck group with the even part of cyclic cohomology. These bilinear pairings, which were the main motivation behind the development of a cyclic theory, consist roughly on the evaluation of a cyclic cochain at an idempotent representing a finitely generated projective module over \mathcal{B} .

Now, the above isomorphisms (1.3)–(1.4) and (1.8) show us that both algebraic K -theory as well as the different bivariant cohomologies can be expressed in terms of morphism sets in the category of noncommutative motives. Therefore, given dg categories \mathcal{A} , \mathcal{B} and \mathcal{C} , with \mathcal{A} a finite dg cell, the composition operation in the category $\mathbb{M}(e)$, combined with these isomorphisms, furnish us bilinear pairings

$$K_n \mathrm{rep}(\mathcal{A}, \mathcal{B}) \times E^m(\mathcal{B}, \mathcal{C}) \longrightarrow E^{m-n}(\mathcal{A}, \mathcal{C}). \tag{1.22}$$

In particular, using (1.15)–(1.16) and (1.20), we obtain:

$$K_n \mathrm{rep}(\mathcal{A}, \mathcal{B}) \times HH^m(\mathcal{B}, \mathcal{C}) \longrightarrow HH^{m-n}(\mathcal{A}, \mathcal{C}) \tag{1.23}$$

$$K_n \mathrm{rep}(\mathcal{A}, \mathcal{B}) \times HC^m(\mathcal{B}, \mathcal{C}) \longrightarrow HC^{m-n}(\mathcal{A}, \mathcal{C}) \tag{1.24}$$

$$K_n \mathrm{rep}(\mathcal{A}, \mathcal{B}) \times HP^m(\mathcal{B}, \mathcal{C}) \longrightarrow HP^{m-n}(\mathcal{A}, \mathcal{C}) \tag{1.25}$$

Our answer to Question B is the above pairing (1.22) and the following result:

Theorem 1.26. *The bilinear pairing (1.24), with $n = 0$, $\mathcal{A} = \mathcal{C} = \underline{k}$ and $m = 2j$, corresponds to Connes’ original bilinear pairing (1.21).*

Theorem 1.26 supports the Grothendieckian belief that all classical constructions in (noncommutative) geometry should become conceptually clear in the correct category of (noncommutative) motives. The above pairings (1.23)–(1.25), which correspond to the composition operation in the category of noncommutative motives, can therefore be considered as an extension of Connes’ foundational work.

Localizing invariants. By replacing (1.1) with the condition that exact sequences of dg categories are mapped to distinguished triangles in the base category $\mathbb{D}(e)$

$$0 \rightarrow \mathcal{A} \rightarrow \mathcal{B} \rightarrow \mathcal{C} \rightarrow 0 \quad \mapsto \quad E(\mathcal{A}) \rightarrow E(\mathcal{B}) \rightarrow E(\mathcal{C}) \rightarrow E(\mathcal{A})[1]$$

one obtains the notion of *localizing invariant*; see [36, §10]. As in the additive case, there exists also a *universal localizing invariant* $U_l : \text{HO}(\text{dgcats}) \rightarrow \mathbb{M}_l$ inducing equivalences analogous to (1.2) and (1.9); see [7]. Isomorphism (1.3) holds also but in this case K -theory is replaced by nonconnective K -theory and \mathcal{A} is assumed to be *saturated* in the sense of Kontsevich [27, 28], i.e. its complexes of morphisms are perfect and \mathcal{A} is perfect as a bimodule over itself; see [6][7, Thm. 9.2]. In what concerns Theorems 1.7 and 1.12 the same results hold. The proof that the functor (1.5) admits a right adjoint functor R^E is different and so we have incorporated it in the proof of Theorem 1.7(i). The functors (1.14) are not only additive but moreover localizing and so isomorphisms (1.15)–(1.16) hold also. Finally, Theorems 1.17, 1.19 and 1.26 and isomorphisms (1.22)–(1.25) are similar, with \mathcal{B} assumed to be a saturated dg category in (1.20) and (1.22)–(1.25).

2. Preliminaries

Throughout the article k will denote a commutative base ring with unit 1. Given a (dg) k -algebra A , we will denote by \underline{A} the associated dg category with a single object and A as the (dg) k -algebra of endomorphisms. Adjunctions will be displayed vertically with the left (resp. right) adjoint on the left- (resp. right-) hand-side.

2.1. Quillen model categories. We will use freely the language of Quillen model categories; see [16, 17, 34]. Given a model category \mathcal{N} , we will denote by $\text{Ho}(\mathcal{N})$ its homotopy category and by $\text{Map}_{\mathcal{N}}(-, -)$ its homotopy function complex; see [16, Def. 17.4.1]. Recall from [7, Def. 4.2] that an object $X \in \mathcal{N}$ is called *homotopically finitely presented* if for any filtered system of objects $\{Y_j\}_{j \in J}$, the induced map

$$\text{hocolim}_{j \in J} \text{Map}_{\mathcal{N}}(X, Y_j) \rightarrow \text{Map}_{\mathcal{N}}(X, \text{hocolim}_{j \in J} Y_j)$$

is a weak equivalence of pointed simplicial sets.

2.2. Dg categories. Let $\mathcal{C}(k)$ be the category of (unbounded) complexes of k -modules. A *differential graded (=dg) category* is a category enriched over $\mathcal{C}(k)$ and a *dg functor* is a functor enriched over $\mathcal{C}(k)$; consult Keller's survey [25]. The category of dg categories will be denoted by dgcats .

Given dg categories \mathcal{A} and \mathcal{B} their *tensor product* $\mathcal{A} \otimes \mathcal{B}$ is defined as follows: the set of objects is the cartesian product and given objects (x, z) and (y, w) in $\mathcal{A} \otimes \mathcal{B}$, we set $(\mathcal{A} \otimes \mathcal{B})((x, z), (y, w)) := \mathcal{A}(x, y) \otimes \mathcal{B}(z, w)$.

A dg functor $F : \mathcal{A} \rightarrow \mathcal{B}$ is called a *derived Morita equivalence* if it induces an equivalence $\mathcal{D}(\mathcal{A}) \xrightarrow{\sim} \mathcal{D}(\mathcal{B})$ between the associated derived categories. Thanks to [37, Thm. 5.3] the category $\text{dgc}at$ carries a (cofibrantly generated) Quillen model structure [34] whose weak equivalences are the derived Morita equivalences. We denote by Hmo the homotopy category hence obtained.

The tensor product of dg categories can be derived into a bifunctor $-\otimes^{\mathbb{L}}-$ on Hmo . Moreover, this bifunctor admits an internal Hom-functor $\text{rep}(-, -)$. Given dg categories \mathcal{A} and \mathcal{B} , $\text{rep}(\mathcal{A}, \mathcal{B})$ is the full triangulated subcategory of $\mathcal{D}(\mathcal{A}^{\text{op}} \otimes^{\mathbb{L}} \mathcal{B})$ spanned by the \mathcal{A} - \mathcal{B} -bimodules X such that for every object x in \mathcal{A} the right \mathcal{B} -module $X(-, x)$ is a compact object (see [33, Def. 4.2.7]) in the triangulated category $\mathcal{D}(\mathcal{B})$; see [7, §2.4]. Let $\mathcal{C}(\mathcal{A}, \mathcal{B})$ be the category of \mathcal{A} - \mathcal{B} -bimodules and $\mathcal{R}(\mathcal{A}, \mathcal{B})$ the subcategory which has the same objects as $\text{rep}(\mathcal{A}, \mathcal{B})$ and whose morphisms are the quasi-isomorphisms. As explained in [25, §4], there is a canonical weak equivalence of pointed simplicial sets between $\text{Map}_{\text{dgc}at}(\mathcal{A}, \mathcal{B})$ and the nerve of $\mathcal{R}(\mathcal{A}, \mathcal{B})$.

2.3. Finite dg cells. For $n \in \mathbb{Z}$, let S^n be the complex $k[n]$ (with k concentrated in degree n) and let D^n be the mapping cone on the identity of S^{n-1} . We denote by $\mathcal{S}(n)$ the dg category with two objects 1 and 2 such that $\mathcal{S}(n)(1, 1) = k$, $\mathcal{S}(n)(2, 2) = k$, $\mathcal{S}(n)(2, 1) = 0$, $\mathcal{S}(n)(1, 2) = S^n$ and composition given by multiplication. We denote by $\mathcal{D}(n)$ the dg category with two objects 3 and 4 such that $\mathcal{D}(n)(3, 3) = k$, $\mathcal{D}(n)(4, 4) = k$, $\mathcal{D}(n)(4, 3) = 0$, $\mathcal{D}(n)(3, 4) = D^n$ and with composition given by multiplication. Finally, let $\iota(n) : \mathcal{S}(n-1) \rightarrow \mathcal{D}(n)$ be the dg functor that sends 1 to 3, 2 to 4 and S^{n-1} into D^n via the map $\text{incl} : S^{n-1} \rightarrow D^n$ which is the identity on k in degree $n-1$:

$$\begin{array}{ccc}
 \mathcal{S}(n-1) & \xrightarrow{\iota(n)} & \mathcal{D}(n) \\
 \parallel & & \parallel \\
 \begin{array}{c} k \\ \curvearrowright \\ 1 \\ \downarrow S^{n-1} \\ 2 \\ \curvearrowright \\ k \end{array} & \xrightarrow{\quad \text{incl} \quad} & \begin{array}{c} k \\ \curvearrowright \\ 3 \\ \downarrow D^n \\ 4 \\ \curvearrowright \\ k \end{array}
 \end{array}
 \quad \text{where} \quad
 \begin{array}{ccc}
 S^{n-1} & \xrightarrow{\text{incl}} & D^n \\
 \parallel & & \parallel \\
 \begin{array}{ccc} 0 & \longrightarrow & 0 \\ \downarrow & & \downarrow \\ 0 & \longrightarrow & k \\ \downarrow & \text{id} & \downarrow \text{id} \\ k & \longrightarrow & k \\ \downarrow & & \downarrow \\ 0 & \longrightarrow & 0 \end{array} & & \text{(degree } n-1)
 \end{array}$$

We denote by I the set consisting of the dg functors $\{\iota(n)\}_{n \in \mathbb{Z}}$ and the dg functor $\emptyset \rightarrow \underline{k}$ (where the empty dg category \emptyset is the initial one). A dg category \mathcal{A} is called a *finite dg cell* if it is obtained from \emptyset by a finite number of pushouts along the dg functors of the set I . The (small) category of finite dg cells will be denoted by cells .

2.4. Grothendieck derivators. Derivators allow us to state and prove precise universal properties and to dispense with many of the technical problems one faces in using Quillen model categories; consult Grothendieck's original manuscript [15]. Given a Quillen model category \mathcal{N} , we will denote by $\text{HO}(\mathcal{N})$ its associated derivator. In order to simplify the exposition, a morphism of derivators and its value at the base category e (which has one object and one morphism) will be denoted by the same symbol. It will be clear from the context to which situation we are referring.

3. Proof of Theorem 1.7

Proof. We start with item (i). Recall from [36, Def. 15.1] that the stable Quillen model \mathcal{M} of the additive motivator \mathbb{M} (denoted by $\mathcal{M}_{\text{dg}}^{\text{add}}$ in *loc. cit.*) is constructed in four steps:

- (i) First, we restrict ourselves to the category $\text{cells} \subset \text{dgcats}$ of finite dg cells.
- (ii) Then, we consider the category $\widehat{\text{cells}}$ of presheaves of pointed simplicial sets on cells . This category carries a projective model structure and comes equipped with the Yoneda functor

$$h(-) : \text{dgcats} \longrightarrow \widehat{\text{cells}} \quad \mathcal{A} \mapsto N.\mathcal{R}(-, \mathcal{A})|_{\text{cells}},$$

where $N.\mathcal{R}$ agrees with the homotopy function complex of dgcats ; see §2.2.

- (iii) We then perform a left Bousfield localization $L\widehat{\text{cells}}$ of $\widehat{\text{cells}}$ so that the functor $h(-)$ sends derived Morita equivalences to weak equivalences, preserves filtered (homotopy) colimits, and sends split exact sequences of dg categories to homotopy fiber sequences.
- (iv) Finally, we take the associated category of spectra $\text{Sp}(L\widehat{\text{cells}})$ in the sense of Hovey [18]. The associated homotopy category $\text{Ho}(\text{Sp}(L\widehat{\text{cells}}))$ is the triangulated category $\mathbb{M}(e)$ of noncommutative motives.

The functor $\mathcal{U} : \text{dgcats} \rightarrow \mathcal{M}$ is given by the composition

$$\mathcal{U} : \text{dgcats} \xrightarrow{h(-)} \widehat{\text{cells}} \xrightarrow{L(-)} L\widehat{\text{cells}} \xrightarrow{\Sigma^\infty(-)} \text{Sp}(L\widehat{\text{cells}}).$$

Given a finite dg cell \mathcal{B} and any object F of $\widehat{\text{cells}}$, i.e. any presheaf of pointed simplicial sets on cells , we have the identification $\text{Map}_{\widehat{\text{cells}}}(h(\mathcal{B}), F) \simeq F(\mathcal{B})$ of pointed simplicial sets. Since (homotopy) colimits in $\widehat{\text{cells}}$ are computed objectwise we then conclude that the set $\{h(\mathcal{B}) \mid \mathcal{B} \in \text{cells}\}$ satisfies the following two conditions:

- (a) If $\text{Map}_{\widehat{\text{cells}}}(h(\mathcal{B}), F) \simeq *$ for every finite dg cell \mathcal{B} , then F is the trivial presheaf.
- (b) The objects $h(\mathcal{B})$, with \mathcal{B} a finite dg cell, are homotopy finitely presented in $\widehat{\text{cells}}$; see §2.1.

The left Bousfield localization in step (iii) is performed with respect to a set of morphisms between homotopically finitely presented objects. Hence, by [36, Lemma 7.1] we conclude that the set $\{(L \circ h)(\mathcal{B}) \mid \mathcal{B} \in \text{cells}\}$ of objects in $L\widehat{\text{cells}}$ also satisfies the above conditions (a)–(b). Lemma 8.2 of [36] allows us then to conclude that the set of noncommutative motives

$$\{U(\mathcal{B})[-m] \mid \mathcal{B} \in \text{cells}, m \in \mathbb{Z}\} \tag{3.1}$$

is a set of compact generators (in the sense of [33, §8]) for the triangulated category $\mathbb{M}(e)$. Hence, $\mathbb{M}(e)$ is compactly generated in the sense of Neeman and so by [33, Thm. 8.3.3] it satisfies the representability theorem. The triangulated functor (1.5) is obtained via (1.2) and so it preserves arbitrary sums. Therefore, by applying [33, Thm. 8.4.4] to the functor (1.5) we obtain the searched right adjoint functor R^E .

Let us now prove the analogous result in the localizing case. Recall from [36, Def. 10.2] that the Quillen model \mathcal{M}_l of the localizing motivator \mathbb{M}_l (denoted by $\mathcal{M}_{\text{dg}}^{\text{loc}}$ in *loc. cit.*) is constructed in five steps:

- (i), (ii) These two steps are similar.
- (iii) We perform a left Bousfield localization $L_1\widehat{\text{cells}}$ of $\widehat{\text{cells}}$ so that the functor $h(-)$ preserves filtered (homotopy) colimits and sends derived Morita equivalences to weak equivalences.
- (iv) Then, we take the associated category of spectra $\text{Sp}(L_1\widehat{\text{cells}})$.
- (v) Finally, we perform a left Bousfield localization $L_2(\text{Sp}(L_1\widehat{\text{cells}}))$ so that exact sequences of dg categories are mapped to distinguished triangles.

The functor $\mathcal{U}_l : \text{dgc} \rightarrow \mathcal{M}_l$ is given by the composition

$$\mathcal{U}_l : \text{dgc} \xrightarrow{h(-)} \widehat{\text{cells}} \xrightarrow{L_1(-)} L_1\widehat{\text{cells}} \xrightarrow{\Sigma^\infty(-)} \text{Sp}(L_1\widehat{\text{cells}}) \xrightarrow{L_2(-)} L_2(\text{Sp}(L_1\widehat{\text{cells}})).$$

The left Bousfield localization in step (v) is performed with respect to a set of morphisms between objects which are *not* homotopically finitely presented. As a consequence the objects $U_l(\mathcal{B})[-m]$, with \mathcal{B} a finite dg cell, generate the triangulated category $\mathbb{M}_l(e)$ but are *not* compact. Hence, the argument used in the additive case does not apply.

Recall from [13, §2][35, §3] that a Quillen model category \mathcal{N} is called *combinatorial* (in the sense of Smith) if it is cofibrantly generated and the underlying category is locally presentable. On one hand, the notion of a cofibrantly generated Quillen model is standard and can be found in [17, Def. 2.1.17]. on the other hand, the notion of local presentability can be found in [1, §1.B][13, Def. 2.2]. Concretely, it means that \mathcal{N} is co-complete and that there exists a regular cardinal λ and a set of objects \mathcal{O} in \mathcal{N} such that:

- (LP1) every object in \mathcal{O} is small with respect to λ -filtered colimits;
- (LP2) every object in \mathcal{N} can be expressed as a λ -filtered colimit of elements of \mathcal{O} .

Standard examples of locally presented categories include simplicial sets, spectra, and categories of presheaves on them (over a small category); see [35, Example 3.6][32, A.2.8.2]. By construction the Quillen model category $\mathcal{M}_l = L_2\text{Sp}(L_1\widehat{\text{cells}})$ is cofibrantly generated. Since the underlying category identifies with the category of presheaves of spectra on cells, we conclude that \mathcal{M}_l is in fact combinatorial. By [35, Prop. 6.10] the associated homotopy category $\mathbb{M}_l(e)$ is then well-generated in the sense of Neeman. Hence, by [33, Prop. 8.4.2] it satisfies the representability theorem. Therefore, by applying [33, Thm. 8.4.4] to the functor (1.5) (which preserves arbitrary sums) we obtain then the searched right adjoint functor R^E .

Item (ii) follows from the equality $E = E_{\text{add}} \circ U$ and from the following adjunction isomorphism

$$\text{Hom}_{\mathbb{M}(e)}(U(\mathcal{B})[-*], T^E(U(\mathcal{C}))) \simeq \text{Hom}_{\text{Ho}(\mathcal{N})}(E(\mathcal{B})[-*], E(\mathcal{C})) = E^*(\mathcal{B}, \mathcal{C}).$$

In what concerns item (iii), the adjunction (E_{add}, R^E) combined with the equality $E = E_{\text{add}} \circ U$ show us that the Chern character $\mathcal{U}^*(\mathcal{B}, \mathcal{C}) \rightarrow E^*(\mathcal{B}, \mathcal{C})$ identifies (under (1.8)) with the morphism

$$\text{Hom}_{\mathbb{M}(e)}(U(\mathcal{B})[-*], U(\mathcal{C})) \longrightarrow \text{Hom}_{\mathbb{M}(e)}(U(\mathcal{B})[-*], T^E(U(\mathcal{C})))$$

induced by $\epsilon_E(U(\mathcal{C}))$. Hence, we conclude that the bivariant Chern character (1.10) is represented by the unit $\epsilon_E : \text{Id} \Rightarrow T^E$ of the adjunction.

Given a dg category \mathcal{C} , let us now describe (following [33, §8.2]) the noncommutative motive $T^E(U(\mathcal{C}))$ as well as the unit map $U(\mathcal{C}) \rightarrow T^E(U(\mathcal{C}))$. The construction of $T^E(U(\mathcal{C}))$ is inductive.

Starting step: let U_0 be the indexing set $\bigcup_{(\mathcal{B}, m)} E^m(\mathcal{B}, \mathcal{C})$, with $\mathcal{B} \in \text{cells}$ and $m \in \mathbb{Z}$.

Note that an element of U_0 consists of a triple (\mathcal{B}, m, α) , with $\alpha \in E^m(\mathcal{B}, \mathcal{C})$. Under this notation, let $T_0^E(U(\mathcal{C}))$ be the noncommutative motive $\bigoplus_{(\mathcal{B}, m, \alpha)} U(\mathcal{B})[-m]$. Note

that we have a canonical map $U(\mathcal{C}) \rightarrow T_0^E(U(\mathcal{C}))$ corresponding to the factor with $\mathcal{B} = \mathcal{C}$, $n = 0$ and $\alpha = \text{id}$. Now, consider the composed functor

$$\Theta : \mathbb{M}(e) \xrightarrow{E_{\text{add}}} \text{Ho}(\mathcal{N}) \xrightarrow{\text{Hom}_{\text{Ho}(\mathcal{N})}(-, E(\mathcal{C}))} \text{Ab}.$$

Since E_{add} preserves arbitrary sums, Θ sends sums to products. Hence, $\Theta(T_0^E(U(\mathcal{C})))$ identifies with $\prod_{(\mathcal{B}, m, \alpha)} E^m(\mathcal{B}, \mathcal{C})$. As a consequence, $\Theta(T_0^E(U(\mathcal{C})))$ carries a canonical element (given by α in the factor corresponding to (\mathcal{B}, m, α)) and so we obtain a well-defined natural transformation $\text{Hom}_{\mathbb{M}(e)}(-, T_0^E(U(\mathcal{C}))) \Rightarrow \Theta$.

Inductive step: suppose now that we have constructed a noncommutative motive $T_i^E(U(\mathcal{C}))$ and a natural transformation

$$\mathrm{Hom}_{\mathbb{M}(e)}(-, T_i^E(U(\mathcal{C}))) \Rightarrow \Theta. \tag{3.2}$$

Let

$$U_{i+1} := \bigcup_{(\mathcal{B}, m)} \mathrm{Ker}\{\mathrm{Hom}_{\mathbb{M}(e)}(U(\mathcal{B})[-m], T_i^E(U(\mathcal{C}))) \rightarrow E^m(\mathcal{B}, \mathcal{C})\}.$$

An element of U_{i+1} consists of a triple (\mathcal{B}, m, f) , where f is a map from $U(\mathcal{B})[-m]$ to $T_i^E(U(\mathcal{C}))$. Let K_{i+1} be the noncommutative motive $\bigoplus_{(\mathcal{B}, m, f)} U(\mathcal{B})[-m]$ and

$K_{i+1} \rightarrow T_i^E(U(\mathcal{C}))$ the map which is f on the factor corresponding to (\mathcal{B}, m, f) . This data allows us to construct the distinguished triangle

$$K_{i+1} \longrightarrow T_i^E(U(\mathcal{C})) \longrightarrow T_{i+1}^E(U(\mathcal{C})) \longrightarrow K_{i+1}[1].$$

Since Θ sends triangles to long exact sequences the natural transformation (3.2) can be extended to a natural transformation $\mathrm{Hom}_{\mathbb{M}(e)}(-, T_{i+1}^E(U(\mathcal{C}))) \Rightarrow \Theta$. As a consequence, we obtain a well-defined sequence of maps

$$U(\mathcal{C}) \longrightarrow T_0^E(U(\mathcal{C})) \longrightarrow \dots \longrightarrow T_i^E(U(\mathcal{C})) \longrightarrow T_{i+1}^E(U(\mathcal{C})) \longrightarrow \dots \tag{3.3}$$

The searched noncommutative motive $T^E(U(\mathcal{C}))$ is then the homotopy colimit of (3.3) (see [33, Def. 1.6.4]) and the unit map $U(\mathcal{C}) \rightarrow T^E(U(\mathcal{C}))$ is the (transfinite) composition (3.3). \square

4. Proof of Theorem 1.12

Proof. In what concerns item (i), one needs to show that the diagram

$$\begin{array}{ccc} K_0(\mathcal{A}) \otimes_{\mathbb{Z}} K_0(\mathcal{B}) & \longrightarrow & K_0(\mathcal{A} \otimes \mathcal{B}) \\ \mathrm{ch}_E \otimes \mathrm{ch}_E \downarrow & & \downarrow \mathrm{ch}_E \\ E^0(\underline{k}, \mathcal{A}) \otimes_{\mathbb{Z}} E^0(\underline{k}, \mathcal{B}) & \longrightarrow & E^0(\underline{k}, \mathcal{A} \otimes \mathcal{B}) \end{array} \tag{4.1}$$

is commutative, where the horizontal maps are the bilinear pairings (1.11). Recall that we have the following commutative diagram

$$\begin{array}{ccc} \mathrm{dgc}at & \xrightarrow{E} & \mathcal{N} \longrightarrow \mathrm{Ho}(\mathcal{N}) \\ u \downarrow & & \nearrow \\ \mathcal{M} & & \\ \downarrow & & E_{\mathrm{add}} \\ \mathbb{M}(e) & & . \end{array}$$

Since by hypothesis E is symmetric monoidal, one concludes from the equivalence (1.9) that E_{add} is also symmetric monoidal. Therefore, for any two dg categories \mathcal{A} and \mathcal{B} the following diagram commutes (we have omitted the subscripts $\mathbb{M}(e)$ and $\text{Ho}(\mathcal{N})$ in order to simplify the exposition)

$$\begin{CD} \text{Hom}(U(\underline{k}), U(\mathcal{A})) \otimes_{\mathbb{Z}} \text{Hom}(U(\underline{k}), U(\mathcal{B})) @>>> \text{Hom}(U(\underline{k}), U(\mathcal{A}) \otimes U(\mathcal{B})) \\ @V E_{\text{add}} \otimes E_{\text{add}} VV @VV E_{\text{add}} V \\ \text{Hom}(\mathbf{1}, E(\mathcal{A})) \otimes_{\mathbb{Z}} \text{Hom}(\mathbf{1}, E(\mathcal{B})) @>>> \text{Hom}(\mathbf{1}, E(\mathcal{A}) \otimes E(\mathcal{B})), \end{CD} \tag{4.2}$$

where the horizontal maps are induced by the symmetric monoidal structures of $\mathbb{M}(e)$ and $\text{Ho}(\mathcal{N})$. Since the functors U and E are symmetric monoidal we conclude that (4.1) agrees with (4.2) and so the proof is finished.

Let us now prove item (ii). Since by hypothesis E is symmetric monoidal, \underline{k} is mapped to $\mathbf{1}$ and so the functor $E^0(\underline{k}, -)$ is given by the composition

$$\text{dgc}at \xrightarrow{E} \mathcal{N} \longrightarrow \text{Ho}(\mathcal{N}) \xrightarrow{\text{Hom}_{\text{Ho}(\mathcal{N})}(\mathbf{1}, -)} \text{Ab}.$$

Hence, since E is additive, $E^0(\underline{k}, -)$ sends derived Morita equivalences to isomorphisms and split exact sequences of dg categories to direct sums of abelian groups. On the other hand, by (1.4) the functor $K_0(-)$ is given by the composition

$$\text{dgc}at \xrightarrow{\mathcal{U}} \mathcal{M} \longrightarrow \mathbb{M}(e) \xrightarrow{\text{Hom}_{\mathbb{M}(e)}(U(\underline{k}), -)} \text{Ab}.$$

Therefore, since \mathcal{U} is an additive functor, $K_0(-)$ sends also derived Morita equivalence to isomorphisms and split exact sequences of dg categories to direct sums of abelian groups. By [38, Prop. 4.1] we obtain then the natural isomorphism

$$\text{Nat}(K_0(-), E^0(\underline{k}, -)) \xrightarrow{\sim} E^0(\underline{k}, \underline{k}) \quad \eta \mapsto \eta(\underline{k})([\underline{k}]), \tag{4.3}$$

where $[k]$ stands for the class of k (as a module over itself) in the Grothendieck group $K_0(\underline{k}) = K_0(k)$. The searched isomorphism (1.13) is then obtained by combining (4.3) with the adjunction isomorphism $E^0(\underline{k}, \underline{k}) \simeq \text{Hom}_{\mathbb{M}(e)}(U(\underline{k}), T^E(U(\underline{k})))$. Recall that the evaluation of the Chern character (1.10) at \underline{k} is given by the homomorphism

$$K_0(k) \simeq \text{Hom}_{\mathbb{M}(e)}(U(\underline{k}), U(\underline{k})) \longrightarrow \text{Hom}_{\text{Ho}(\mathcal{N})}(E(\underline{k}), E(\underline{k}))$$

induced by the functor (1.5). Under the isomorphism $K_0(\underline{k}) \simeq \text{Hom}_{\mathbb{M}(e)}(U(\underline{k}), U(\underline{k}))$ the class $[k]$ of k corresponds to the identity morphism $\text{id}_{U(\underline{k})} \in \text{Hom}_{\mathbb{M}(e)}(U(\underline{k}), U(\underline{k}))$. As a consequence, the image of the Chern character (1.10) under (4.3) is the identity morphism $\text{id}_{E(\underline{k})} \in \text{Hom}_{\text{Ho}(\mathcal{N})}(E(\underline{k}), E(\underline{k})) = E^0(\underline{k}, \underline{k})$. By the adjunction (E_{add}, R^E) , this identity morphism $\text{id}_{E(\underline{k})}$ corresponds to the unit morphism $\epsilon_E(U(\underline{k})) : U(\underline{k}) \rightarrow T^E(U(\underline{k}))$ and so the proof is finished. \square

5. Proof of Theorem 1.17

Proof. We start with item (i). By applying the above natural isomorphism (4.3) to $E = HH$ and $E = C$ we obtain

$$\text{Nat}(K_0(-), HH_0(-)) \xrightarrow{\sim} HH^0(\underline{k}, \underline{k}) \quad \text{Nat}(K_0(-), HC_0^-(-)) \xrightarrow{\sim} HC^0(\underline{k}, \underline{k}).$$

These isomorphisms combined with

$$HH^0(\underline{k}, \underline{k}) \simeq HH_0(\underline{k}) \xrightarrow{\psi_0} k \quad HC^0(\underline{k}, \underline{k}) \simeq HC_0^-(\underline{k}) \xrightarrow{\psi^-} k, \quad (5.1)$$

where the isomorphisms ψ_0 and ψ^- are described in [38, §3], give rise to

$$\text{Nat}(K_0(-), HH_0(-)) \simeq k \quad \text{Nat}(K_0(-), HC_0^-(-)) \simeq k. \quad (5.2)$$

Since the identity morphisms $\text{id}_{HH(k)} \in \text{Hom}_{\mathcal{D}(k)}(HH(k), HH(k))$ and $\text{id}_{HC(k)} \in \text{Hom}_{\mathcal{D}(\Lambda)}(HC(k), HC(k))$ are mapped under (5.1) to $1 \in k$, we conclude that the Chern character ch_{HH} and ch_C are mapped under (5.2) to $1 \in k$. By construction, the isomorphisms (5.2) agree with the isomorphism (1.4)–(1.5) of [38, Thm. 1.3]. In *loc. cit.* the Dennis trace map and the Chern character ch^- were characterized as the unique natural transformations which are mapped to $1 \in k$ under (5.2). Therefore, we conclude that ch_{HH} and with ch_C agree respectively with the Dennis trace map and the Chern character ch^- .

Let us now prove item (ii). Thanks to the adjunction (C_{add}, R^C) we have the identification $\text{Hom}_{\mathbb{M}(e)}(U(\underline{k}), T^C(U(\underline{k}))) \simeq \text{Hom}_{\mathcal{D}(\Lambda)}(C(\underline{k}), C(\underline{k}))$ with $\epsilon_C(U(\underline{k}))$ corresponding to $\text{id}_{C(\underline{k})}$. Similarly, the adjunction $(HH_{\text{add}}, R^{HH})$ gives rise to the identification $\text{Hom}_{\mathbb{M}(e)}(U(\underline{k}), T^{HH}(U(\underline{k}))) \simeq \text{Hom}_{\mathcal{D}(k)}(HH(\underline{k}), HH(\underline{k}))$ with $\epsilon_{HH}(U(\underline{k}))$ corresponding to $\text{id}_{HH(\underline{k})}$. Moreover, the commutativity of the diagram (1.18) is equivalent to the fact that the morphism

$$\text{Hom}_{\mathcal{D}(\Lambda)}(C(\underline{k}), C(\underline{k})) \longrightarrow \text{Hom}_{\mathcal{D}(k)}(HH(\underline{k}), HH(\underline{k})), \quad (5.3)$$

induced by the forgetful functor $\Phi : \mathcal{D}(\Lambda) \rightarrow \mathcal{D}(k)$, sends $\text{id}_{C(\underline{k})}$ to $\text{id}_{HH(\underline{k})}$. Recall from the proof of item (i) that, under the isomorphisms

$$\text{Hom}_{\mathcal{D}(\Lambda)}(C(\underline{k}), C(\underline{k})) \simeq k \quad \text{Hom}_{\mathcal{D}(k)}(HH(\underline{k}), HH(\underline{k})) \simeq k,$$

$\text{id}_{C(\underline{k})}$ and $\text{id}_{HH(\underline{k})}$ correspond to the unit $1 \in k$. As a consequence we conclude that (5.3) is an isomorphism. Therefore, there exists a unique element in $\text{Hom}_{\mathcal{D}(\Lambda)}(C(\underline{k}), C(\underline{k}))$ which is mapped by (5.3) to $\text{id}_{HH(\underline{k})} \in \text{Hom}_{\mathcal{D}(k)}(HH(\underline{k}), HH(\underline{k}))$. Equivalently, there exists a unique morphism η from $U(\underline{k})$ to $T^C(U(\underline{k}))$ making the following diagram commutative

$$\begin{array}{ccc} & & T^C(U(\underline{k})) \\ & \nearrow \eta & \downarrow \Phi(U(\underline{k})) \\ U(\underline{k}) & \xrightarrow{\epsilon_{HH}(U(\underline{k}))} & T^{HH}(U(\underline{k})). \end{array}$$

Since $\Phi \circ \epsilon_C = \epsilon_{HH}$ this morphism η is necessarily the evaluation $\epsilon_C(U(\underline{k}))$ of the unit at $U(\underline{k})$ and so the proof is finished. \square

6. Proof of Theorem 1.19

Proof. As explained in [20, §1], we have a periodicity map S in $\text{Ext}_\Lambda^2(M, M) \simeq \text{Hom}_{\mathcal{D}(\Lambda)}(M, M[2])$ for every M in $\mathcal{D}(\Lambda)$. As a consequence we obtain an induced natural transformation of triangulated functors

$$S : T^C = (R^C \circ C_{\text{add}}) \Rightarrow (R^C \circ (-[2]) \circ C_{\text{add}}) =: T^C[2]$$

and hence the following diagram of natural transformations of triangulated functors

$$T^C \xrightarrow{S} T^C[2] \xrightarrow{S} \dots \xrightarrow{S} T^C[2r] \xrightarrow{S} \dots \tag{6.1}$$

Let us denote by T^{HP} the homotopy colimit (see [33, Def. 1.6.4]) of (6.1). Given dg categories \mathcal{B} and \mathcal{C} , with $\mathcal{B} \in \text{cells}$, the following isomorphisms hold :

$$\begin{aligned} \text{Hom}_{\mathbb{M}(e)}(U(\mathcal{B})[-*], T^P(U(\mathcal{C}))) &= \text{Hom}_{\mathbb{M}(e)}(U(\mathcal{B})[-*], \text{hocolim}_r T^C[2r](U(\mathcal{C}))) \\ &\simeq \text{hocolim}_r \text{Hom}_{\mathbb{M}(e)}(U(\mathcal{B})[-*], T^C[2r](U(\mathcal{C}))) \tag{6.2} \\ &\simeq \text{hocolim}_r \text{Hom}_{\mathbb{M}(e)}(U(\mathcal{B})[-*], T^C(U(\mathcal{C}))[2r]) \tag{6.3} \\ &\simeq \text{hocolim}_r \text{Hom}_{\mathbb{M}(e)}(U(\mathcal{B})[-* - 2r], T^C(U(\mathcal{C}))) \\ &\simeq \text{hocolim}_r HC^{*+2r}(\mathcal{B}, \mathcal{C}) \tag{6.4} \\ &= HP^*(\mathcal{B}, \mathcal{C}). \tag{6.5} \end{aligned}$$

Isomorphism (6.2) follows from the compactness of $U(\mathcal{B})$ in the triangulated category $\mathbb{M}(e)$; see the proof of Theorem 1.7(i). Isomorphism (6.3) follows from the natural equivalence of triangulated functors

$$T^C[2r] := (R^C \circ (-[2r]) \circ C_{\text{add}}) \simeq (R^C \circ C_{\text{add}})[2r].$$

Isomorphism (6.4) follows from the isomorphism (1.16). Finally, equality (6.5) follows from the definition of bivariant periodic cyclic cohomology; see [20, Def. 8.1]. \square

7. Proof of Theorem 1.26

Proof. Recall from [30, §8.3.10] that Connes' bilinear pairings (1.21) can be expressed as the following compositions

$$\langle -, - \rangle : K_0(\mathcal{B}) \times HC^{2j}(\mathcal{B}) \xrightarrow{ch_{2j} \times \text{id}} HC_{2j}(\mathcal{B}) \times HC^{2j}(\mathcal{B}) \xrightarrow{\text{ev}} k \quad j \geq 0, \tag{7.1}$$

where ch_{2j} is the Chern character map (see [30, §8.3]) and ev is induced by the evaluation of cyclic cochains on cyclic chains. Thanks to the adjunction

$$\begin{array}{ccc} & \mathcal{D}(\Lambda) & \\ & \uparrow \downarrow & \\ C_{\text{add}} & & R^C \\ & \mathbb{M}(e) & \end{array}$$

we obtain a commutative square (where we have omitted the subscripts $\mathbb{M}(e)$ and $\mathcal{D}(\Lambda)$ in order to simplify the exposition)

$$\begin{array}{ccc} \text{Hom}(U(\underline{k}), U(\mathcal{B})) \times \text{Hom}(U(\mathcal{B})[-2j], T^C(U(\underline{k}))) & \xrightarrow{\text{comp}} & \text{Hom}_{\mathbb{M}(e)}(U(\underline{k})[-2j], T^C(U(\underline{k}))) \\ C_{\text{loc}} \times \varphi \downarrow & & \sim \downarrow \varphi \\ \text{Hom}(C(\underline{k}), C(\mathcal{B})) \times \text{Hom}(C(\mathcal{B})[-2j], C(\underline{k})) & \xrightarrow{\text{comp}} & \text{Hom}_{\mathcal{D}(\Lambda)}(C(\underline{k})[-2j], C(\underline{k})), \end{array}$$

where φ is the natural isomorphism given by the adjunction and the horizontal maps are the composition operations in $\mathbb{M}(e)$ and $\mathcal{D}(\Lambda)$. Theorem 1.17(i) combined with isomorphisms (1.3) and (1.16) show us that the above commutative square corresponds to the following diagram

$$\begin{array}{ccc} K_0(\mathcal{B}) \times HC^{2j}(\mathcal{B}) & \longrightarrow & HC^{2j}(\underline{k}) \simeq k \\ ch^-(\mathcal{B}) \times \text{id} \downarrow & & \parallel \\ HC_0^-(\mathcal{B}) \times HC^{2j}(\mathcal{B}) & \xrightarrow{\text{comp}} & HC^{2j}(\underline{k}) \simeq k, \end{array} \tag{7.2}$$

where the upper horizontal map is the pairing (1.24) (with $n = 0$, $\mathcal{A} = \mathcal{C} = \underline{k}$ and $m = 2j$). Now, recall from [30, §5.1.8] that there exist natural maps

$$U_j : HC_0^-(\mathcal{B}) \longrightarrow HC_{2j}(\mathcal{B}) \quad j \geq 0$$

such that $U_j \circ ch^-(\mathcal{B}) = ch_{2j}(\mathcal{B})$. Thanks to the description of the composition operation in $\mathcal{D}(\Lambda)$ given in [20, Thm. 5.1] we have the following diagram

$$\begin{array}{ccc} HC_0^-(\mathcal{B}) \times HC^{2j}(\mathcal{B}) & \xrightarrow{\text{comp}} & k \\ U_j \times \text{id} \downarrow & & \parallel \\ HC_{2j}(\mathcal{B}) \times HC^{2j}(\mathcal{B}) & \xrightarrow{\text{ev}} & k. \end{array} \tag{7.3}$$

Finally, by combining diagram (7.2) with diagram (7.3) we conclude that the pairing (1.24) (with $n = 0$, $\mathcal{A} = \mathcal{C} = \underline{k}$ and $m = 2j$) identifies with the above composition (7.1), and so with Connes' original bilinear pairing (1.21). This achieves the proof. \square

8. An application : (de)suspension of bivariant cohomology theories

In [39] it was shown that Karoubi's infinite matrices [22] provide a simple model for the suspension in the triangulated category of noncommutative motives. Consider the k -algebra Γ of $\mathbb{N} \times \mathbb{N}$ -matrices A which satisfy the following two conditions : the set $\{A_{i,j} \mid i, j \in \mathbb{N}\}$ is finite and there exists a natural number n_A such that each row and each column has at most n_A non-zero entries. Let Σ be the quotient of Γ by the two-sided ideal consisting of those matrices with finitely many non-zero entries; see [39, §3]. Alternatively, consider the (left) localization of Γ with respect to the matrices $\overline{I}_n, n \geq 0$, with entries $(\overline{I}_n)_{i,j} = \mathbf{1}$ for $i = j > n$ and 0 otherwise. Then, for any dg category \mathcal{A} we have a canonical isomorphism

$$U(\Sigma(\mathcal{A})) \xrightarrow{\sim} U(\mathcal{A})[1] \tag{8.1}$$

in $\mathbb{M}(e)$, where $\Sigma(\mathcal{A}) = \mathcal{A} \otimes \underline{\Sigma}$. By combining isomorphism (8.1) with isomorphisms (1.8), (1.15)–(1.16) and (1.20) we obtain the following result.

Theorem 8.2. *Given dg categories \mathcal{B} and \mathcal{C} , we have the following isomorphisms*

$$\begin{aligned}
 E^{*+1}(\Sigma(\mathcal{B}), \mathcal{C}) &\simeq E^*(\mathcal{B}, \mathcal{C}) & E^{*-1}(\mathcal{B}, \Sigma(\mathcal{C})) &\simeq E^*(\mathcal{B}, \mathcal{C}) \\
 HH^{*+1}(\Sigma(\mathcal{B}), \mathcal{C}) &\simeq HH^*(\mathcal{B}, \mathcal{C}) & HH^{*-1}(\mathcal{B}, \Sigma(\mathcal{C})) &\simeq HH^*(\mathcal{B}, \mathcal{C})
 \end{aligned} \tag{8.3}$$

$$HC^{*+1}(\Sigma(\mathcal{B}), \mathcal{C}) \simeq HC^*(\mathcal{B}, \mathcal{C}) \quad HC^{*-1}(\mathcal{B}, \Sigma(\mathcal{C})) \simeq HC^*(\mathcal{B}, \mathcal{C}) \tag{8.4}$$

$$HP^{*+1}(\Sigma(\mathcal{B}), \mathcal{C}) \simeq HP^*(\mathcal{B}, \mathcal{C}) \quad HP^{*-1}(\mathcal{B}, \Sigma(\mathcal{C})) \simeq HP^*(\mathcal{B}, \mathcal{C}), \tag{8.5}$$

where E is an additive functor and (8.5) holds under the assumption that \mathcal{B} is a finite dg cell.

Theorem 8.2 extends Kassel's previous work [24, §III Thm. 3.1] on bivariant cyclic cohomology on ordinary algebras defined over a field to dg categories defined over a general commutative base ring. Hence, it can now be applied to schemes. Given a (quasi-compact and quasi-separated) k -scheme X , it is well-known that the category of perfect complexes of \mathcal{O}_X -modules admits a dg-enhancement $\text{perf}_{\text{dg}}(X)$; see for instance [31] or [7, Example 4.5]. Moreover, given a pair (X, Y) of k -schemes, the bivariant Hochschild, cyclic, and periodic, homology of (X, Y) can be obtained from the pair of dg categories $(\text{perf}_{\text{dg}}(X), \text{perf}_{\text{dg}}(Y))$ by applying the corresponding bivariant theory. Therefore, when $\mathcal{B} = \text{perf}_{\text{dg}}(X)$ and $\mathcal{C} = \text{perf}_{\text{dg}}(Y)$, the above isomorphisms (8.3)–(8.5) reduce to the corresponding isomorphisms associated to the schemes X and Y .

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