

Measured quantum groupoids associated to proper dynamical quantum groups

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Abstract. Dynamical quantum groups were introduced by Etingof and Varchenko in connection with the dynamical quantum Yang–Baxter equation, and measured quantum groupoids were introduced by Enock, Lesieur and Vallin in their study of inclusions of type II_1 factors. In this article, we associate to suitable dynamical quantum groups, which are purely algebraic objects, Hopf C^* -bimodules and measured quantum groupoids on the level of von Neumann algebras. Assuming invariant integrals on the dynamical quantum group, we first construct a fundamental unitary which yields Hopf bimodules on the level of C^* -algebras and von Neumann algebras. Next, we assume properness of the dynamical quantum group and lift the integrals to the operator algebras. In a subsequent article, this construction shall be applied to the dynamical $\text{SU}_q(2)$ studied by Koelink and Rosengren.

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1. Introduction

Dynamical quantum groups were introduced by Etingof and Varchenko as an algebraic framework for the study of the dynamical quantum Yang–Baxter equation [6, 7, 8], a variant of the Yang–Baxter equation arising in statistical mechanics. Every (rigid) solution of this equation has a naturally associated tensor category of representations which turns out to be equivalent to the category of representations of some dynamical quantum group. In the case of the basic rational or basic trigonometric solution, this dynamical quantum group can be regarded as a quantization of the function algebra on some Poisson-Lie-groupoid. In general, it can be regarded as a quantum groupoid and fits into the theory of Hopf algebroids developed by Böhm and others [1].

Measured quantum groupoids were introduced by Enock, Lesieur and Vallin [2, 13] to capture generalized Galois symmetries of certain inclusions of type II_1 factors [3, 4, 15]. Apart from this fundamental example in von Neumann algebra theory,

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which was also considered in the algebraic setting [9, 19], and from the finite case, only few measured quantum groupoids have been constructed and investigated yet [13, 29].

Up to now, connections between algebraic and operator-algebraic approaches to quantum groupoids have only been explored in the finite case [14, 17, 28] and in the form of a few examples and constructions that exist on both levels. The situation is very different in the area of quantum groups, where Woronowicz's theory of compact quantum groups [35] and van Daele's theory of multiplier Hopf algebras with integrals [12, 31] form a bridge between the algebraic and operator-algebraic approaches, combining the computational convenience of the former with the power and richness of the latter.

Another approach to quantum groupoids which is equivalent to the algebraic and operator algebraic one, at least in finite dimensions, is via fusion categories [5, 19].

In this article, we associate to suitable dynamical quantum groups, which are purely algebraic objects, Hopf C^* -bimodules and measured quantum groupoids on the level of von Neumann algebras. The main example of a dynamical group we have in mind for application is the dynamical $SU_q(2)$ studied by Koelink and Rosengren [10], and in a subsequent article, we want to study the construction for this example in detail.

On the dynamical quantum groups, we have to impose several assumptions.

First, we need a left- and a right-invariant integral, which correspond to fiber-wise integration on a groupoid, and a weight on the basis that is suitably quasi-invariant, such that the resulting total integrals are faithful, positive, and coincide. In the case of the dynamical $SU_q(2)$, the left- and right-invariant integrals can be obtained from a Peter–Weyl decomposition due to Koelink and Rosengren [10], while the quasi-invariant weight on the basis can be chosen quite freely.

Second, we assume the dynamical quantum group to be proper, which is the natural analogue of compactness and unitality for quantum groupoids, and to possess a specific approximate unit in the base algebra. The dynamical $SU_q(2)$ mentioned above even is compact and thus satisfies this second assumption.

In particular, the dynamical quantum group need not be a Hopf algebroid, but only a multiplier Hopf algebroid in the sense of [25]. The latter are closely related to the weak multiplier Hopf algebras that were recently introduced by Van Daele and Wang [33, 34].

Third, we assume that the quasi-invariant weight on the basis admits a bounded GNS-construction. Like the first condition, this one is very natural. In the case of the dynamical $SU_q(2)$, the base algebra is formed by all meromorphic functions on the plane and does not admit any non-trivial bounded representations. To apply our construction, one therefore has to change the base and check that the Peter–Weyl decomposition persists.

Given these assumptions, the measured quantum groupoid is constructed as follows.

The algebraic GNS-construction, applied to the total integral on the dynamical quantum group, yields a Hilbert space of square-integrable functions on the dynamical quantum group together with a natural representation by densely defined multiplication operators. To obtain a C^* -algebra or von Neumann algebra, one has to show that these multiplication operators are bounded. To prove this and to lift the comultiplication to the resulting C^* -algebra and von Neumann algebra, we proceed as in the case of quantum groups [23] and construct a fundamental unitary which is pseudo-multiplicative on the level of von Neumann algebras and C^* -algebras in the sense of [27] and [24], respectively. The general theory of these unitaries then yields completions of the dynamical quantum group in the form a Hopf C^* -bimodule and a Hopf von-Neumann bimodule, and simultaneously a Pontrjagin dual in the same form. Finally, we extend the invariant integrals to the level of operator algebras, using properness of the dynamical quantum group and standard von Neumann algebra techniques.

This article is organized as follows.

Section 2 provides the algebraic basics on dynamical quantum groups and integration that are needed for the construction in Section 3. We first generalize the definition of a dynamical quantum group or \mathfrak{h} -Hopf algebroid, allowing the base to be non-unital, then consider left- and right-invariant integrals on the total algebra and quasi-invariant weights on the basis, and finally construct a $*$ -algebra related to the Pontrjagin dual. The main result of this section is the existence of a modular automorphism for the total integral, which follows from a strong invariance property similarly as in the setting of multiplier Hopf algebras [31].

Section 3 presents the construction of the measured quantum groupoid outlined above. It uses Connes spatial theory, in particular the relative tensor product of Hilbert modules, and the C^* -algebraic analogue of that construction [22], and introduces the necessary concepts along the way when they are needed.

We use standard notation and adopt the following conventions. All algebras will be over the ground field \mathbb{C} and we do not assume the existence of a unit element. Given a vector space V with a subset $X \subseteq V$, we denote by $\langle X \rangle \subseteq V$ the linear span and, if V is normed, by $[X] \subseteq V$ the closed linear span of X . Inner products on Hilbert spaces will be linear in the second and anti-linear in the first variable.

2. Dynamical quantum groups with integrals on the algebraic level

This section summarizes and develops the basics on dynamical quantum groups and integration used in this article. Before turning to details, let us outline the main concepts.

A dynamical quantum group is a special quantum groupoid and as such consists of an algebra B called the basis, an algebra A , an embedding $r: B \rightarrow A$ and an anti-homomorphic embedding $s: B \rightarrow A$ whose images commute, and a

comultiplication, antipode and counit. What makes it special is that the basis B is commutative, that $r(B)$ and $s(B)$ are central in A up to a twist which is controlled by an action of a group Γ on B and a bigrading of A by Γ , and that the target of the comultiplication is a well-behaved monoidal product $A \tilde{\otimes} A$.

Integration on a quantum groupoid involves several ingredients. The analogue of the left- or right-invariance property of Haar measures on groups, Haar systems on groupoids, and Haar weights on quantum groups can be formulated for conditional expectations from A to $r(B)$ or $s(B)$, respectively. To obtain a total integration on A , such a partial integral has to be composed with a suitable functional on B that is quasi-invariant with respect to the action of Γ .

Let us now turn to details. We proceed as follows.

From the beginning, we assume all our algebras to possess an involution but not necessarily a unit. We first recall terminology concerning non-unital algebras (§2.1), then describe the monoidal product $A \tilde{\otimes} A$ (§2.2), and define dynamical quantum groups or, more precisely, multiplier (B, Γ) -Hopf $*$ -algebroids (§2.3). Afterwards, we introduce and study integrals (§2.4–§2.6) and prove the existence of a modular automorphism that controls the deviation of the total integral from being a trace. Using integration, we finally construct the dual $*$ -algebra of a multiplier (B, Γ) -Hopf $*$ -algebroid (§2.7).

2.1. Preliminaries on non-unital algebras. To handle non-unital algebras, we use extra non-degeneracy assumptions and multiplier algebras [30, appendix] which are recalled below.

Let R be an algebra, not necessarily unital. Given a left R -module M , we say that R has *local units for M* if for each finite subset $F \subseteq M$, there exists some $r \in R$ such that $rm = m$ for all $m \in F$ [32]. The corresponding notion for right R -modules is defined similarly. We say that R has *local units* if it has local units for R , regarded as a left and as a right R -module.

Let R and S be algebras with local units, let N be an R - S -bimodule and assume that R and S have local units for N . A *multiplier* of N is a pair $T = (T_\rho, T_\lambda)$, where $T_\rho: R \rightarrow N$ is a left R -module map and $T_\lambda: S \rightarrow N$ a right S -module map satisfying $T_\rho(r)s = rT_\lambda(s)$ for all $r \in R, s \in S$. Given such a multiplier, we write $rT := T_\rho(r)$ and $Ts := T_\lambda(s)$ for all $r \in R, s \in S$. We denote the set of all multipliers of N by $M(N)$. Clearly, N embeds into $M(N)$ and $M(N)$ carries a natural structure of an R - S -bimodule that is compatible with this embedding.

Regarding R as an R - R -bimodule, $M(R)$ becomes an algebra via $TT' = (T'_\rho \circ T_\rho, T_\lambda \circ T'_\lambda)$, and R embeds into $M(R)$ as an essential ideal. If R is a $*$ -algebra, then so is $M(R)$, where the adjoint of a multiplier $T = (T_\rho, T_\lambda) \in M(R)$ is the pair $T^* = (T_\rho^*, T_\lambda^*)$ given by $T_\rho^*(r) = (T_\lambda(r^*))^*$ and $T_\lambda^*(r) = (T_\rho(r^*))^*$ for all $r \in R$.

The bimodule N is an $M(R)$ - $M(S)$ -bimodule via $T(rns)T' := T_\lambda(r)nT'_\rho(s)$ for all $T \in M(R), r \in R, n \in N, s \in S, T' \in M(S)$, and $M(N)$ is an $M(R)$ - $M(S)$ -bimodule via $TT'T'' := (T''_\rho \circ T'_\rho \circ T_\rho, T_\lambda \circ T'_\lambda \circ T''_\lambda)$ for all $T \in M(R), T' \in M(N), T'' \in M(S)$.

A homomorphism $\pi: R \rightarrow M(S)$ is *non-degenerate* if $\langle \pi(R)S \rangle = S = \langle S\pi(R) \rangle$; in that case, it extends uniquely to a homomorphism $M(R) \rightarrow M(S)$ which is again denoted by π (see [30]).

2.2. The category of $(B, \Gamma)^{\text{ev}}$ -algebras. Let B be a commutative $*$ -algebra with local units, let Γ be a group that acts on B on the left, and let $e \in \Gamma$ be the unit.

A (B, Γ) -module is a Γ -graded B -bimodule $V = \bigoplus_{\gamma \in \Gamma} V_\gamma$ for which B has local units, where each V_γ is a B -bimodule and $vb = \gamma(b)v$ for all $v \in V_\gamma$, $b \in B$, $\gamma \in \Gamma$. A *morphism* of (B, Γ) -modules V and W is a morphism of Γ -graded B -bimodules.

A (B, Γ) -algebra is a Γ -graded $*$ -algebra $A = \bigoplus_{\gamma \in \Gamma} A_\gamma$ which has local units in A_e and is equipped with a $*$ -homomorphism $B \rightarrow M(A)$ that turns A into a (B, Γ) -module. Such a (B, Γ) -algebra is *proper* if B maps into A .

Given a (B, Γ) -algebra A and $\gamma \in \Gamma$, we denote by $M(A)_\gamma \subseteq M(A)$ the space of all multipliers $T \in M(A)$ satisfying $TA_{\gamma'} \subseteq A_{\gamma\gamma'}$ and $A_{\gamma'}T \subseteq A_{\gamma'\gamma}$ for all $\gamma' \in \Gamma$.

A *morphism* of (B, Γ) -algebras A and C is a non-degenerate, B -linear $*$ -homomorphism $\pi: A \rightarrow M(C)$ satisfying $\pi(A_\gamma) \subseteq M(C)_\gamma$ for all $\gamma \in \Gamma$. Such a morphism is *proper* if it maps A into C .

Using the extension of non-degenerate homomorphisms to multipliers, one defines the composition of morphisms and checks that (B, Γ) -algebras form a category.

The tensor product $B \otimes B$ is a $*$ -algebra with local units and a natural action of $\Gamma \times \Gamma$. Replacing (B, Γ) by $(B, \Gamma)^{\text{ev}} := (B \otimes B, \Gamma \times \Gamma)$ in the definition above, we obtain the category of all $(B, \Gamma)^{\text{ev}}$ -algebras.

Let A be a $(B, \Gamma)^{\text{ev}}$ -algebra. We call an element $x \in A$ *homogeneous* and write $\partial_x = \gamma$, $\bar{\partial}_x = \gamma'$ if $x \in A_{\gamma, \gamma'}$ for some $\gamma, \gamma' \in \Gamma$. Thus, $\partial_x \partial_y = \partial_{xy}$, $\bar{\partial}_x \bar{\partial}_y = \bar{\partial}_{xy}$ and $\partial_x^* = \partial_x^{-1}$, $\bar{\partial}_x^* = \bar{\partial}_x^{-1}$ for all homogeneous $x, y \in A$. Define $r = r_A: B \rightarrow M(A)$ and $s = s_A: B \rightarrow M(A)$ by $r(b)a = (b \otimes 1)a$ and $s(b)a = (1 \otimes b)a$ for all $a \in A$, $b \in B$. We write ${}_r A$, A_r , ${}_s A$, A_s if we consider A as a B -module via left or right multiplication via r or s , respectively.

Clearly, B is a (B, Γ) -algebra and $B \otimes B$ is a $(B, \Gamma)^{\text{ev}}$ -algebra with respect to the trivial gradings. Every (B, Γ) -algebra A can be regarded as a $(B, \Gamma)^{\text{ev}}$ -algebra, where $A_{(\gamma, \gamma)} = A_\gamma$ and $A_{(\gamma, \gamma')} = 0$ whenever $\gamma \neq \gamma'$, and $(b \otimes b')a = bb'a$ for all $b, b' \in B$, $a \in A$. Conversely, every $(B, \Gamma)^{\text{ev}}$ -algebra A can be considered as a (B, Γ) -algebra via $r: B \rightarrow M(A)$ and the grading given by $A_\gamma := \bigoplus_{\gamma'} A_{\gamma, \gamma'}$, or via $s: B \rightarrow M(A)$ and the grading given by $A_{\gamma'} := \bigoplus_{\gamma} A_{\gamma, \gamma'}$. We write (A, r) and (A, s) , respectively, to denote the resulting (B, Γ) -algebras.

Denote by $B \rtimes \Gamma$ the crossed product for the action of Γ on B , that is, the universal algebra containing B and Γ such that $e = 1_B$ and $b\gamma \cdot b'\gamma' = b\gamma(b')\gamma\gamma'$ for all $b, b' \in B$, $\gamma, \gamma' \in \Gamma$. This is a (B, Γ) -algebra with respect to the natural inclusion $B \rightarrow B \rtimes \Gamma$ and the involution and grading given by $(b\gamma)^* = \gamma^{-1}b^*$ and $(B \rtimes \Gamma)_\gamma = B\gamma$ for all $b \in B$, $\gamma \in \Gamma$.

The *fiber product* of $(B, \Gamma)^{\text{ev}}$ -algebras A and C is defined as follows. The subalgebra

$$A \overset{\Gamma}{\otimes} C := \sum_{\gamma, \gamma', \gamma'' \in \Gamma} A_{\gamma, \gamma'} \otimes C_{\gamma', \gamma''} \subset A \otimes C$$

is a $(B, \Gamma)^{\text{ev}}$ -algebra, where $(A \overset{\Gamma}{\otimes} C)_{\gamma, \gamma''} = \sum_{\gamma'} A_{\gamma, \gamma'} \otimes C_{\gamma', \gamma''}$ for all $\gamma, \gamma'' \in \Gamma$ and $(r \times s)(b \otimes b') = r_A(b) \otimes s_C(b')$ for all $b, b' \in B$. Let $I \subseteq M(A \overset{\Gamma}{\otimes} C)$ be the ideal generated by $\{s_A(b) \otimes 1 - 1 \otimes r_C(b) : b \in B\}$. Then the quotient

$$A \tilde{\otimes} C := A \overset{\Gamma}{\otimes} C / (I(A \overset{\Gamma}{\otimes} C))$$

is a $(B, \Gamma)^{\text{ev}}$ -algebra again, called the fiber product of A and C . Write $a \tilde{\otimes} c$ for the image of an element $a \otimes c$ in $A \tilde{\otimes} C$.

The assignment $(A, C) \mapsto A \tilde{\otimes} C$ is functorial, associative and unital. Indeed, for all morphisms of $(B, \Gamma)^{\text{ev}}$ -algebras $\pi^1: A^1 \rightarrow C^1$, $\pi^2: A^2 \rightarrow C^2$, there exists a morphism

$$\pi^1 \tilde{\otimes} \pi^2: A^1 \tilde{\otimes} A^2 \rightarrow C^1 \tilde{\otimes} C^2, \quad a_1 \tilde{\otimes} a_2 \mapsto \pi^1(a_1) \tilde{\otimes} \pi^2(a_2); \quad (2.1)$$

for all $(B, \Gamma)^{\text{ev}}$ -algebras A, C, D , there exists an isomorphism

$$(A \tilde{\otimes} C) \tilde{\otimes} D \rightarrow A \tilde{\otimes} (C \tilde{\otimes} D), \quad (a \tilde{\otimes} c) \tilde{\otimes} d \mapsto a \tilde{\otimes} (c \tilde{\otimes} d), \quad (2.2)$$

and for each $(B, \Gamma)^{\text{ev}}$ -algebra A , there exist isomorphisms

$$(B \rtimes \Gamma) \tilde{\otimes} A \rightarrow A, \quad b\gamma \tilde{\otimes} a \mapsto r(b)a, \quad A \tilde{\otimes} (B \rtimes \Gamma) \rightarrow A, \quad a \tilde{\otimes} b\gamma \mapsto s(b)a. \quad (2.3)$$

These isomorphisms are compatible in a natural sense and endow the category of $(B, \Gamma)^{\text{ev}}$ -algebras with a monoidal structure. From now on, we shall use them without further notice.

The category of $(B, \Gamma)^{\text{ev}}$ -algebras carries automorphisms $(-)^{\text{op}}$ and $(-)^{\text{co}}$ such that for each (B, Γ) -algebra A and each morphism $\phi: A \rightarrow C$, we have $A^{\text{co}} = A$ as an algebra, A^{op} is the opposite $*$ -algebra of A , that is, the same vector space with the same involution and reversed multiplication, and

$$(A^{\text{op}})_{\gamma, \gamma'} = A_{\gamma^{-1}, \gamma'^{-1}} \text{ for all } \gamma, \gamma' \in \Gamma, \quad r_{A^{\text{op}}} = r_A, \quad s_{A^{\text{op}}} = s_A, \quad \phi^{\text{op}} = \phi, \quad (2.4)$$

$$(A^{\text{co}})_{\gamma, \gamma'} = A_{\gamma', \gamma} \text{ for all } \gamma, \gamma' \in \Gamma, \quad r_{A^{\text{co}}} = s_A, \quad s_{A^{\text{co}}} = r_A, \quad \phi^{\text{co}} = \phi. \quad (2.5)$$

These automorphisms are involutive and commute, that is,

$$(-)^{\text{op}} \circ (-)^{\text{op}} = \text{id}, \quad (-)^{\text{co}} \circ (-)^{\text{co}} = \text{id}, \quad (-)^{\text{op}} \circ (-)^{\text{co}} = (-)^{\text{co}} \circ (-)^{\text{op}}.$$

Furthermore, they are compatible with the monoidal structure as follows. Given (B, Γ) -algebras A, C , there exist isomorphisms $(A \tilde{\otimes} C)^{\text{op}} \rightarrow A^{\text{op}} \tilde{\otimes} C^{\text{op}}$ and $(A \tilde{\otimes} C)^{\text{co}} \rightarrow C^{\text{co}} \tilde{\otimes} A^{\text{co}}$ given by $a \tilde{\otimes} c \mapsto a \tilde{\otimes} c$ and $a \tilde{\otimes} c \mapsto c \tilde{\otimes} a$, respectively. Moreover, $(B \rtimes \Gamma)^{\text{co}} = B \rtimes \Gamma$, there exists an isomorphism $S^{B \rtimes \Gamma}: B \rtimes \Gamma \rightarrow (B \rtimes \Gamma)^{\text{op}}$, $b\gamma \mapsto \gamma^{-1}b$, and all of these isomorphisms and the isomorphisms in (2.2) and (2.3) are compatible in a natural sense.

2.3. Multiplier (B, Γ) -Hopf $*$ -algebroids. We shall work with variants of the \mathfrak{h} -Hopf algebroids and (B, Γ) -Hopf $*$ -algebroids considered in [7, 10] and [21], respectively, where the basis need no longer be unital. These variants consist of a $(B, \Gamma)^{\text{ev}}$ -algebra and a comultiplication, counit and antipode, which will be introduced one after the other. To quickly proceed to the main part of this article, we postulate all the usual properties of these maps as axioms and leave a study of the axiomatics for later.

Given a $(B, \Gamma)^{\text{ev}}$ -algebra A , we denote by $\tilde{M}(A \tilde{\otimes} A) \subseteq M(A \tilde{\otimes} A)$ the set of all $T \in M(A \tilde{\otimes} A)$ for which all products of the form

$$T(x \tilde{\otimes} 1_{M(A)}), \quad (x \tilde{\otimes} 1_{M(A)})T, \quad T(1_{M(A)} \tilde{\otimes} y), \quad (1_{M(A)} \tilde{\otimes} y)T$$

where $x \in A_{\gamma, e}, y \in A_{e, \gamma}, \gamma \in \Gamma$, lie in $A \tilde{\otimes} A$. Evidently, $\tilde{M}(A \tilde{\otimes} A)$ is a $*$ -subalgebra of $M(A \tilde{\otimes} A)$.

Definition 2.3.1. A *comultiplication* on a $(B, \Gamma)^{\text{ev}}$ -algebra A is a morphism Δ from A to $A \tilde{\otimes} A$ satisfying $\Delta(A) \subseteq \tilde{M}(A \tilde{\otimes} A)$ and $(\Delta \tilde{\otimes} \text{id}) \circ \Delta = (\text{id} \tilde{\otimes} \Delta) \circ \Delta$. A (proper) *multiplier (B, Γ) - $*$ -bialgebroid* is a (proper) $(B, \Gamma)^{\text{ev}}$ -algebra with a comultiplication. A *morphism of multiplier (B, Γ) - $*$ -bialgebroids* $(A, \Delta_A), (B, \Delta_B)$ is a morphism ϕ from A to B satisfying $\Delta_B \circ \phi = (\phi \tilde{\otimes} \phi) \circ \Delta_A$.

Let (A, Δ) be a multiplier (B, Γ) - $*$ -bialgebroid.

We shall need to form products of the form $\Delta(x)(1 \otimes y)$ or $(y \otimes 1)\Delta(x)$ when $\partial_y \neq e$ or $\bar{\partial}_y \neq e$, respectively, which are defined as follows. Let $x, y \in A$. The multiplication on $A \otimes A$ induces a canonical $A \tilde{\otimes} A$ - $A \otimes A$ -bimodule structure on ${}_s A \otimes_B {}_r A$ and a canonical $A \otimes A$ - $A \tilde{\otimes} A$ -bimodule structure on $A_s \otimes_B A_r$. Using the natural maps ${}_s M(A) \otimes_B {}_r M(A) \rightarrow M({}_s A \otimes_B {}_r A)$ and $M(A)_s \otimes_B M(A)_r \rightarrow M(A_s \otimes_B A_r)$, we define multipliers $1 \otimes y, x \otimes 1 \in M({}_s A \otimes_B {}_r A)$ and $x \otimes 1, 1 \otimes y \in M(A_s \otimes_B A_r)$. Regarding $M({}_s A \otimes_B {}_r A)$ as an $M(A \tilde{\otimes} A)$ - $M(A \otimes A)$ -bimodule and $M(A_s \otimes_B A_r)$ as an $M(A \otimes A)$ - $M(A \tilde{\otimes} A)$ -bimodule (see §2.1), we can then multiply these multipliers with $\Delta(x)$ or $\Delta(y)$, respectively.

Lemma 2.3.2. *The following linear maps are well defined:*

$$\begin{aligned} T_1: A_s \otimes_B {}_s A &\rightarrow {}_s A \otimes_B {}_r A, & x \otimes_B y &\mapsto \Delta(x)(1 \otimes_B y), \\ T_2: A_r \otimes_B {}_r A &\rightarrow A_s \otimes_B A_r, & x \otimes_B y &\mapsto (x \otimes_B 1)\Delta(y), \\ T_3: {}_s A \otimes_B A_s &\rightarrow A_s \otimes_B A_r, & x \otimes_B y &\mapsto (1 \otimes_B y)\Delta(x), \\ T_4: {}_r A \otimes_B A_r &\rightarrow {}_s A \otimes_B {}_r A, & x \otimes_B y &\mapsto \Delta(y)(x \otimes_B 1). \end{aligned}$$

Proof. We only prove the assertion concerning T_1 , the cases of T_2, \dots, T_4 being similar. Using the explanations above, we obtain a linear map $A \otimes A \rightarrow M({}_s A \otimes_B {}_r A)$, $x \otimes y \mapsto \Delta(x)(1 \otimes y)$. This map factorizes through the quotient map $A \otimes A \rightarrow A_s \otimes_B {}_s A$ because $\Delta(xs(b)) = \Delta(x)(1 \otimes s(b))$ for all $x \in A, b \in B$, and takes values in ${}_s A \otimes_B {}_r A$ because $\Delta(A)$ is contained in $\tilde{M}(A \otimes A)$. \square

We adopt the Sweedler notation and write $\Delta(x) = \sum x_{(1)} \tilde{\otimes} x_{(2)}$ for $x \in A$. This notation requires extra care because $\Delta(x)$ need not lie in $A \tilde{\otimes} A$ but only in $\tilde{M}(A \tilde{\otimes} A)$, so that $x_{(1)}$ and $x_{(2)}$ do not simply represent elements of A . In this notation, the maps introduced above take the form

$$\begin{aligned} T_1: x \otimes_B y &\mapsto \sum x_{(1)} \otimes_B x_{(2)} y, & T_2: x \otimes_B y &\mapsto \sum x y_{(1)} \otimes_B y_{(2)}, \\ T_3: x \otimes_B y &\mapsto \sum x_{(1)} \otimes_B y x_{(2)}, & T_4: x \otimes_B y &\mapsto \sum y_{(1)} x \otimes_B y_{(2)}. \end{aligned}$$

We shall almost exclusively use the Sweedler notation for products as above. A detailed explanation of this notation in the context of multiplier Hopf algebras is given in [30, 32]. Apart from the fact that we use tensor products of B -modules instead of tensor products of vector spaces, this explanation carries over easily. As in the theory of (multiplier) Hopf algebras, we extend the Sweedler notation to iterated applications of Δ , writing

$$(\Delta \tilde{\otimes} \text{id})(\Delta(x)) = \sum x_{(1)} \tilde{\otimes} x_{(2)} \tilde{\otimes} x_{(3)} = (\text{id} \tilde{\otimes} \Delta)(\Delta(x))$$

for $x \in A$, and to iterated applications of the maps T_1, \dots, T_4 , writing, for example,

$$\begin{aligned} (T_2 \otimes \text{id})((\text{id} \otimes T_1)(x \otimes_B y \otimes_B z)) &= \sum x y_{(1)} \otimes_B y_{(2)} \otimes_B y_{(3)} z \\ &= (\text{id} \otimes T_1)((T_2 \otimes \text{id})(x \otimes_B y \otimes_B z)) \end{aligned}$$

for all $x, y, z \in A$.

Definition 2.3.3. A *counit* for a multiplier (B, Γ) -*-bialgebroid (A, Δ) is a proper morphism of $(B, \Gamma)^{\text{ev}}$ -algebras $\epsilon: A \rightarrow B \rtimes \Gamma$ satisfying $(\epsilon \tilde{\otimes} \text{id}) \circ \Delta = \text{id}_A = (\text{id} \tilde{\otimes} \epsilon) \circ \Delta$.

Let (A, Δ) be a multiplier (B, Γ) - $*$ -bialgebroid with counit ϵ . Using the linear maps

$$\sharp: B \rtimes \Gamma \rightarrow B, \quad \sum_{\gamma} b_{\gamma} \gamma \mapsto \sum_{\gamma} b_{\gamma}, \quad \flat: B \rtimes \Gamma \rightarrow B, \quad \sum_{\gamma} \gamma b_{\gamma} \mapsto \sum_{\gamma} b_{\gamma},$$

we define $\epsilon^{\sharp}, \epsilon^{\flat}: A \rightarrow B$ by $\epsilon^{\sharp} := \sharp \circ \epsilon$ and $\epsilon^{\flat} := \flat \circ \epsilon$. Define $m_r: A_r \otimes_B rA \rightarrow A$ and $m_s: A_s \otimes_B sA \rightarrow A$ by $\sum_i x_i \otimes_B y_i \mapsto \sum_i x_i y_i$.

Remarks 2.3.4.

- i) Clearly, $\epsilon(A_{\gamma, \gamma'}) \subseteq (B \rtimes \Gamma)_{\gamma, \gamma'} = 0$ whenever $\gamma, \gamma' \in \Gamma$ and $\gamma \neq \gamma'$.
- ii) If ϵ' is a counit as well, then $\epsilon = \epsilon \circ (\text{id} \tilde{\otimes} \epsilon') \circ \Delta = \epsilon' \circ (\epsilon \tilde{\otimes} \text{id}) \circ \Delta = \epsilon'$.
- iii) The condition $(\epsilon \tilde{\otimes} \text{id}) \circ \Delta = \text{id}_A = (\text{id} \tilde{\otimes} \epsilon) \circ \Delta$ is equivalent to the relations

$$\sum r(\epsilon^{\sharp}(x_{(1)}))x_{(2)}y = xy = \sum xy_{(1)}s(\epsilon^{\flat}(y_{(2)})) \quad \text{for all } x, y \in A,$$

and hence to commutativity of the diagrams

$$\begin{array}{ccc} & & sA \otimes_B rA \\ & \nearrow T_1 & \searrow \epsilon^{\sharp} \otimes \text{id}_B \\ A_s \otimes_B sA & \xrightarrow{m_s} & A \end{array}, \quad \begin{array}{ccc} & & A_s \otimes_B A_r \\ & \nearrow T_2 & \searrow \text{id} \otimes \epsilon^{\flat} \\ A_r \otimes_B rA & \xrightarrow{m_r} & A \end{array}$$

Furthermore, this condition is equivalent to the relations

$$\sum xy_{(2)}r(\epsilon^{\flat}(y_{(1)})) = xy = \sum s(\epsilon^{\sharp}(x_{(2)}))x_{(1)}y \quad \text{for all } x, y \in A.$$

The definition of the antipode involves the isomorphism

$$\sigma_{A,A}: (A \tilde{\otimes} A)^{\text{co,op}} \rightarrow A^{\text{co,op}} \tilde{\otimes} A^{\text{co,op}}, \quad x \tilde{\otimes} y \mapsto y \tilde{\otimes} x.$$

Definition 2.3.5. An *antipode* for a multiplier (B, Γ) - $*$ -bialgebroid (A, Δ) with counit ϵ is an isomorphism $S: A \rightarrow A^{\text{co,op}}$ of $(B, \Gamma)^{\text{ev}}$ -algebras that makes the following diagrams commute:

$$\begin{array}{ccc} A_s \otimes_B sA & \xrightarrow{T_1} & sA \otimes_B rA, & A_r \otimes_B rA & \xrightarrow{T_2} & A_s \otimes_B A_r, \\ \epsilon^{\flat} \otimes \text{id}_B \downarrow & & \downarrow S \otimes \text{id}_B & \text{id} \otimes \epsilon^{\sharp} \downarrow & & \downarrow \text{id} \otimes S \\ A & \xleftarrow{m_r} & A_r \otimes_B rA & A & \xleftarrow{m_s} & A_s \otimes_B sA \end{array}$$

$$\begin{array}{ccc} A & \xrightarrow{S} & A^{\text{co,op}} \\ \Delta \downarrow & & \downarrow \Delta^{\text{co,op}} \\ A \tilde{\otimes} A & \xrightarrow{S \tilde{\otimes} S} & A^{\text{co,op}} \tilde{\otimes} A^{\text{co,op}} \xleftarrow{\sigma_{A,A}} (A \tilde{\otimes} A)^{\text{co,op}} \end{array}$$

A *multiplier (B, Γ) -Hopf $*$ -algebroid* is a multiplier (B, Γ) - $*$ -bialgebroid with counit and antipode.

Examples 2.3.6.

- i) The tensor product $B \otimes B$ is a multiplier (B, Γ) -Hopf $*$ -algebroid, where $\Delta(b \otimes b') = (b \otimes 1) \tilde{\otimes} (1 \otimes b')$, $\epsilon(b \otimes b') = bb'$, $S(b \otimes b') = b' \otimes b$ for all $b, b' \in B$.
- ii) The crossed product $B \rtimes \Gamma$ is a multiplier (B, Γ) -Hopf $*$ -algebroid, where $\Delta(b\gamma) = b\gamma \tilde{\otimes} \gamma = \gamma \tilde{\otimes} b\gamma$, $\epsilon = \text{id}$ and $S(\gamma b) = b\gamma^{-1}$ for all $b \in B, \gamma \in \Gamma$.

Given an antipode S on a multiplier (B, Γ) - $*$ -bialgebroid (A, Δ) and an element $a \in A$, we shall henceforth always regard $S(a)$ as an element of A and not of $A^{\text{co,op}}$.

Remarks 2.3.7. Let (A, Δ, ϵ, S) be a multiplier (B, Γ) -Hopf $*$ -algebroid.

- i) In Sweedler notation, commutativity of the diagrams in Definition 2.3.5 amount to

$$\sum S(x_{(1)})x_{(2)}y = s(\epsilon^{\flat}(x))y, \quad \sum xy_{(1)}S(y_{(2)}) = xr(\epsilon^{\sharp}(y)) \quad (2.6)$$

for all $x, y \in A$,

$$\sum S(x_{(1)})\tilde{\otimes}S(x_{(2)}) = \sum S(x)_{(2)}\tilde{\otimes}S(x)_{(1)} \quad \text{for all } x \in A. \quad (2.7)$$

- ii) If S' is an antipode as well, then $S' = S$ because for all $x, y, z \in A$,

$$\begin{aligned} xS(y)z &= S(yS^{-1}(x))z = \sum S(s(\epsilon^{\sharp}(y_{(2)}))y_{(1)}S^{-1}(x))z \\ &= \sum S(y_{(2)}S^{-1}(x))r(\epsilon^{\sharp}(y_{(2)}))z \\ &= \sum S(y_{(1)}S^{-1}(x))y_{(2)}S'(S'^{-1}(z)y_{(3)}) \\ &= xS'(y)z. \end{aligned}$$

For every multiplier (B, Γ) -Hopf $*$ -algebroid, the maps T_1, \dots, T_4 defined above are bijections.

Proposition 2.3.8. *Let (A, Δ) be a multiplier (B, Γ) - $*$ -bialgebroid. If (A, Δ) has a counit ϵ and an antipode S , then the maps T_1, \dots, T_4 are bijective and for all $x, y \in A$,*

$$\begin{aligned} T_1^{-1}(x \otimes_B y) &= \sum x_{(1)} \otimes_B S(S^{-1}(y)x_{(2)}), \\ T_2^{-1}(x \otimes_B y) &= \sum S(y_{(1)}S^{-1}(x)) \otimes_B y_{(2)}, \\ T_3^{-1}(x \otimes_B y) &= \sum x_{(1)} \otimes_B S^{-1}(x_{(2)}S(y)), \\ T_4^{-1}(x \otimes_B y) &= \sum S^{-1}(S(x)y_{(1)}) \otimes_B y_{(2)}. \end{aligned}$$

Proof. We only prove the assertion concerning T_1 . One first checks that the formula given for T_1^{-1} yields a well-defined map $T_1': {}_s A \otimes_B {}_r A \rightarrow A_s \otimes_B {}_s A$, and then that for all $x, y \in A$ and $u, v \in A_{e,e}$,

$$\begin{aligned}
(u \otimes v) \cdot (T_1 \circ T_1')(x \otimes_B y) &= \sum u x_{(1)} \otimes_B v x_{(2)} S(S^{-1}(y)x_{(3)}) \\
&= \sum u x_{(1)} \otimes_B v x_{(2)} S(x_{(3)})y \\
&= \sum u x_{(1)} \otimes_B v r(\epsilon^\#(x_{(2)}))y \\
&= \sum u s(\epsilon^\#(x_{(2)}))x_{(1)} \otimes_B v y = u x \otimes_B v y, \\
(u \otimes v) \cdot (T_1' \circ T_1)(x \otimes_B y) &= \sum u x_{(1)} \otimes_B v S(S^{-1}(x_{(3)}y)x_{(2)}) \\
&= \sum u x_{(1)} \otimes_B v S(x_{(2)})x_{(3)}y \\
&= \sum u x_{(1)} \otimes_B v s(\epsilon^b(x_{(2)}))y \\
&= \sum u x_{(1)} s(\epsilon^b(x_{(2)})) \otimes_B v y = u x \otimes_B v y. \quad \square
\end{aligned}$$

2.4. Bi-measured multiplier (B, Γ) -*-bialgebroids. We now introduce the main objects of this article — multiplier (B, Γ) -Hopf *-algebroids equipped with certain integrals. In §3, we shall construct completions of such objects in the form of measured quantum groupoids.

As on a groupoid, integration on a multiplier (B, Γ) -*-bialgebroid (A, Δ) proceeds in stages. First, one needs partial integrals $\phi, \psi: A \rightarrow B$ with suitable left or right invariance properties, and second a suitable weight $\mu: B \rightarrow \mathbb{C}$ that is compatible with the action of Γ . The results in [10] suggest that dynamical quantum groups that are compact in a suitable sense even possess a bi-invariant integral $h: A \rightarrow B \otimes B$ that can be obtained from a Peter–Weyl decomposition of A .

We first focus on the weight μ and the bi-integral h , and discuss left and right integrals in the next subsection.

Let us briefly recall some terminology. Let C be a *-algebra with local units. A linear map $\mu: C \rightarrow \mathbb{C}$ is *faithful* if $\mu(Cc) = 0$ implies $c = 0$, and *positive* if $\mu(c^*c) \geq 0$ for all $c \in C$. Assume that μ is positive. Then μ is *-linear, because positivity of $\mu((b+c)^*(b+c))$ and $\mu((b+ic)^*(b+ic))$ implies $\mu(b^*c) = \overline{\mu(c^*b)}$ for all $b, c \in C$, and faithful as soon as $\mu(c^*c) \neq 0$ whenever $c \neq 0$.

Definition 2.4.1. A *weight* for (B, Γ) is a faithful, positive linear map $\mu: B \rightarrow \mathbb{C}$ that is *quasi-invariant with respect to Γ* in the sense that for each $\gamma \in \Gamma$, there exists some $D_\gamma \in M(B)$ such that $\mu(\gamma(bD_\gamma)) = \mu(b)$ for all $b \in B$.

Remark 2.4.2. Let μ be a weight for (B, Γ) . Then

- i) each D_γ is uniquely determined and self-adjoint,
- ii) $D_{\gamma\gamma'} = \gamma'^{-1}(D_\gamma)D_{\gamma'}$ and $1 = \gamma^{-1}(D_{\gamma^{-1}})D_\gamma$ for all $\gamma, \gamma' \in \Gamma$,
- iii) $\mu(\gamma^{-1}(b)c) = \mu(b\gamma(c)D_{\gamma^{-1}}^{-1})) = \mu(b\gamma(c)D_\gamma)$ for all $b, c \in B, \gamma \in \Gamma$.

Indeed, i) and ii) follow easily from the fact that μ is faithful and the relations $\mu(\gamma(bD_\gamma^*)) = \mu(\gamma(D_\gamma b^*)) = \mu(b^*) = \mu(b)$ and $\mu(\gamma(\gamma'(bD_{\gamma\gamma'}))) = \mu(b) = \mu(\gamma'(bD_{\gamma'})) = \mu(\gamma(\gamma'(bD_{\gamma'})D_\gamma))$.

We henceforth call the family $(D_\gamma)_{\gamma \in \Gamma}$ the *Radon–Nikodym cocycle* of μ .

The following definition is inspired by the notion of a Haar functional introduced in [10].

Definition 2.4.3. A *bi-integral* on (A, Δ) is a morphism of $(B, \Gamma)^{\text{ev}}$ -modules $h: A \rightarrow B \otimes B$ satisfying $\Delta(\ker h)(1 \otimes A_{e,e}) \subseteq \ker h \otimes A$ and $\Delta(\ker h)(A_{e,e} \otimes 1) \subseteq A \otimes \ker h$. If (A, Δ) is proper and $h(r(b)s(b')) = b \otimes b'$ for all $b, b' \in B$, we call such a bi-integral *normalized*.

Lemma 2.4.4. Let (A, Δ) be proper and let h be a normalized bi-integral on (A, Δ) .

- i) $(\text{id} \otimes m_B \circ h) \circ \Delta = h = (m_B \circ h \otimes \text{id}) \circ \Delta$, where $m_B: B \otimes B \rightarrow B$ denotes the multiplication.
- ii) If h' is a normalized bi-integral on (A, Δ) , then $h' = h$.
- iii) If (A, Δ, ϵ, S) is a proper multiplier (B, Γ) -Hopf $*$ -algebroid, then $h \circ S = \sigma_B \circ h$, where $\sigma_B: B \otimes B \rightarrow B \otimes B$ denotes the flip $b \otimes c \mapsto c \otimes b$.

Proof. i) We only prove the first equation. Let $\omega: (A, r) \rightarrow B$ be a morphism of (B, Γ) -modules sending $I := \ker h$ to 0. Then

$$(\text{id} \otimes \omega)(\Delta(I))A_{e,e} = (\text{id} \otimes \omega)(\Delta(I)(A_{e,e} \otimes 1)) \subseteq (\text{id} \otimes \omega)(A \otimes I) = 0$$

and hence $(\text{id} \otimes \omega)(\Delta(I)) = 0$. Moreover, if $b, b', b'' \in B$ and $u \in A_{e,e}$, then

$$\begin{aligned} (\text{id} \otimes \omega)(\Delta(r(b)s(b'))s(b''))u &= (\text{id} \otimes \omega)(r(b)s(b''))u \otimes s(b') \\ &= r(b)s(\omega(s(b')r(b''))u). \end{aligned}$$

For $\omega = m_B \circ h$, these calculations imply for all $a \in I$ and $b, b' \in B$

$$\begin{aligned} (\text{id} \otimes m_B \circ h)(\Delta(a)) &= 0 = h(a), \\ (\text{id} \otimes m_B \circ h)(\Delta(r(b)s(b'))) &= r(b)s(b') = h(r(b)s(b')). \end{aligned}$$

Since $A = I + r(B)s(B)$, we can conclude $(\text{id} \otimes m_B \circ h) \circ \Delta = h$.

ii) Let $x \in \ker h$ and choose $u, u' \in B \otimes B$ such that $u(1 \otimes m_B(u'))h'(x) = h'(x)$. Then

$$h'(x) = h(uh'(x)s(m_B(u'))) = \sum h(ux_{(1)}s(m_B(h'(x_{(2)}u')))) = 0$$

because $\sum ux_{(1)} \otimes x_{(2)}u' \in u(\ker h) \otimes_B A$. Thus, $\ker h \subseteq \ker h'$. Since h and h' are normalized and $\ker h + B \otimes B = A$, we can conclude $h = h'$.

iii) One easily verifies that $\sigma_B \circ h \circ S$ is a normalized bi-integral. By ii), it equals h . \square

Definition 2.4.5. A proper multiplier (B, Γ) -*-bialgebroid (A, Δ) is *bi-measured* if it is equipped with a normalized bi-integral $h: A \rightarrow B \otimes B$ and a weight μ for (B, Γ) such that $\nu := (\mu \otimes \mu) \circ h$ is faithful and positive.

Remark 2.4.6. Given a bi-measured proper multiplier (B, Γ) -Hopf *-algebroid as above, h is evidently faithful, and also *-linear. To see this, note that $(\mu \otimes \mu)(h(a^*)(b \otimes c)) = \nu(a^*r(b)s(c)) = \overline{\nu(s(c^*)r(b^*)a)} = \overline{(\mu \otimes \mu)((b \otimes c)^*h(a))} = (\mu \otimes \mu)(h(a)^*(b \otimes c))$ for all $a \in A, b, c \in B$.

2.5. Left and right integrals. For large parts of this article, the multiplier (B, Γ) -Hopf *-algebroids under consideration need not be equipped with a bi-integral, but only with left and right integrals ϕ, ψ . The definition of these integrals involves slice maps of the following form.

Let (A, Δ) be a multiplier (B, Γ) -*-bialgebroid and let $\phi: (A, r) \rightarrow B$ be a morphism of (B, Γ) -modules. Then there exists a unique linear map $\text{id} \tilde{\otimes} \phi: \tilde{M}(A \tilde{\otimes} A) \rightarrow M(A)$ such that

$$((\text{id} \tilde{\otimes} \phi)(T))a = (\text{id} \otimes \phi)_B(T(a \otimes 1)), \quad a((\text{id} \tilde{\otimes} \phi)(T)) = (\text{id} \otimes \phi)_B((a \otimes 1)T)$$

for all $T \in \tilde{M}(A \tilde{\otimes} A)$ and $a \in A$, where we regard $T(a \otimes 1)$ and $(a \otimes 1)T$ as elements of ${}_s A \otimes_B {}_r A$ and $A_s \otimes_B A_r$, respectively. In the case $T = \Delta(x)$ for some $x \in A$,

$$(\text{id} \tilde{\otimes} \phi)(\Delta(x))a = \sum s(\phi(x_{(2)}))x_{(1)}a, \quad a(\text{id} \tilde{\otimes} \phi)(\Delta(x)) = \sum ax_{(1)}s(\phi(x_{(2)})). \quad (2.8)$$

Likewise, every morphism $\psi: (A, s) \rightarrow B$ yields a slice map $\psi \tilde{\otimes} \text{id}: \tilde{M}(A \tilde{\otimes} A) \rightarrow M(A)$.

Definition 2.5.1. A *left integral* on (A, Δ) is a morphism $\phi: (A, r) \rightarrow B$ satisfying $(\text{id} \tilde{\otimes} \phi) \circ \Delta = r \circ \phi$. A *right integral* on (A, Δ) is a morphism $\psi: (A, s) \rightarrow B$ satisfying $(\psi \tilde{\otimes} \text{id}) \circ \Delta = s \circ \psi$.

Remarks 2.5.2.

i) In Sweedler notation, the invariance conditions on ϕ and ψ become

$$\sum s(\phi(x_{(2)}))x_{(1)}a = r(\phi(x))a, \quad \sum ax_{(2)}r(\psi(x_{(1)})) = as(\psi(x))$$

for all $a, x \in A$.

- ii) If (A, Δ, ϵ, S) is a (B, Γ) -Hopf $*$ -algebroid, then the map $\phi \mapsto \phi \circ S$ gives a bijection between left and right integrals on (A, Δ) . This follows easily from (2.7).
- iii) If ϕ is a left integral, then $\phi(-s(b))$ is also a left integral for each $b \in B$. Likewise, if ψ is a right integral, then also $\psi(-r(b))$ is a right integral for each $b \in B$.

We shall frequently use the following strong invariance relations:

Proposition 2.5.3. *Assume that (A, Δ, ϵ, S) is a (B, Γ) -Hopf $*$ -algebroid. Then*

- i) $(\text{id} \otimes \phi)((1 \tilde{\otimes} z)\Delta(x)) = S((\text{id} \otimes \phi)(\Delta(z)(1 \tilde{\otimes} x)))$ for every left integral ϕ and all $x, z \in A$;
- ii) $(\psi \otimes \text{id})(\Delta(x)(z \tilde{\otimes} 1)) = S((\psi \otimes \text{id})(x \tilde{\otimes} 1)\Delta(z))$ for every right integral ψ and all $x, z \in A$.

Proof. Using Sweedler notation, we calculate

$$\begin{aligned} \sum x_{(1)}s(\phi(zx_{(2)})) &= \sum x_{(1)}s(\phi(z_{(2)}r(\epsilon^b(z_{(1)}))x_{(2)})) \\ &= \sum s(\epsilon^b(z_{(1)}))x_{(1)}s(\phi(z_{(2)}x_{(2)})) \\ &= \sum S(z_{(1)})z_{(2)}x_{(1)}s(\phi(z_{(3)}x_{(2)})) = \sum S(z_{(1)})r(\phi(z_{(2)}x)) \end{aligned}$$

and

$$\begin{aligned} \sum r(\psi(x_{(1)}z))x_{(2)} &= \sum r(\psi(x_{(1)}s(\epsilon^\sharp(z_{(2)}))z_{(1)}))x_{(2)} \\ &= \sum r(\psi(x_{(1)}z_{(1)}))x_{(2)}r(\epsilon^\sharp(z_{(2)})) \\ &= \sum r(\psi(x_{(1)}z_{(1)}))x_{(2)}z_{(2)}S(z_{(3)}) = s(\psi(xz_{(1)}))S(z_{(2)}). \quad \square \end{aligned}$$

Normalized bi-integrals yield left and right integrals as follows:

Lemma 2.5.4. *Assume that (A, Δ) is proper, h is a normalized bi-integral on (A, Δ) , and $\mu: B \rightarrow \mathbb{C}$ is linear. Then $\phi := (\text{id} \otimes \mu) \circ h$ and $\psi := (\mu \otimes \text{id}) \circ h$ are a left and a right integral, respectively, and $\phi \circ S^{\pm 1} = \psi$.*

Proof. Repeating the proof of Lemma 2.4.4 i) with $\omega := \phi = (\text{id} \otimes \mu) \circ h$, we find

$$\begin{aligned} (\text{id} \tilde{\otimes} \phi)(\Delta(a)) &= 0 = r(\phi(a)), \\ (\text{id} \tilde{\otimes} \phi)(\Delta(r(b)s(b'))) &= r(b\mu(b')) = \phi(r(b)s(b')) \end{aligned}$$

for all $a \in \ker h$ and $b, b' \in B$. Since $A = (\ker h) + r(B)s(B)$, we can conclude $(\text{id} \tilde{\otimes} \phi) \circ \Delta = r \circ \phi$. The assertion on ψ follows similarly, and the last equation follows from Lemma 2.4.4 iii). \square

2.6. Measured multiplier (B, Γ) -*-bialgebroids. Much of the ensuing material applies not only to bi-measured proper multiplier (B, Γ) -Hopf *-algebroids but also to the following class of objects.

Definition 2.6.1. A multiplier (B, Γ) -*-bialgebroid (A, Δ) is *measured* if it is equipped with a left integral ϕ , a right integral ψ , and a weight μ for (B, Γ) such that $\nu := \mu \circ \phi$ and $\nu^{-1} := \mu \circ \psi$ are faithful, positive, and coincide, and $\psi(A) = B = \phi(A)$.

Remarks 2.6.2.

- i) Given a measured multiplier (B, Γ) -Hopf *-algebroid as above, the maps ϕ and ψ are *-linear. This can be seen from a similar argument as in Remark 2.4.6.
- ii) If $(A, \Delta, \epsilon, S, h, \mu)$ is a bi-measured proper multiplier (B, Γ) -Hopf *-algebroid and $\phi = (\text{id} \otimes \mu) \circ h$ and $\psi = (\mu \otimes \text{id}) \circ h$, then $(A, \Delta, \epsilon, S, \phi, \psi, \mu)$ is a measured multiplier (B, Γ) -Hopf *-algebroid by Lemma 2.5.4. In that case, $\phi \circ S^{\pm 1} = \psi$ and $\nu \circ S = \nu$ by Lemma 2.4.4 iii).

Till the end of this subsection, let $(A, \Delta, \epsilon, S, \phi, \psi, \mu)$ be a measured multiplier (B, Γ) -Hopf *-algebroid and let $(D_\gamma)_\gamma$ be the Radon–Nikodym cocycle for μ . Define $D, \bar{D}: A \rightarrow A$ by

$$D(a) = r(D_{\partial_a^{-1}})a = ar(D_{\partial_a}^{-1}), \quad \bar{D}(a) = s(D_{\bar{\partial}_a^{-1}})a = as(D_{\bar{\partial}_a}^{-1}) \quad (2.9)$$

for all homogeneous $a \in A$.

Lemma 2.6.3. D and \bar{D} both are algebra and $(B, \Gamma)^{\text{ev}}$ -module automorphisms of A , and satisfy

$$\begin{aligned} (D \tilde{\otimes} \text{id}) \circ \Delta &= \Delta \circ D, & (\text{id} \tilde{\otimes} \bar{D}) \circ \Delta &= \Delta \circ \bar{D}, & (\bar{D} \tilde{\otimes} \text{id}) \circ \Delta &= (\text{id} \tilde{\otimes} D) \circ \Delta, \\ D \circ \bar{D} &= \bar{D} \circ D, & S \circ D &= \bar{D}^{-1} \circ S, & S \circ \bar{D} &= D^{-1} \circ S, \\ * \circ D &= D^{-1} \circ *, & * \circ \bar{D} &= \bar{D}^{-1} \circ *. \end{aligned}$$

Proof. The maps D and \bar{D} are bijective because D_γ is invertible for each $\gamma \in \Gamma$. The remaining assertions follow from straightforward calculations, for example,

$$\begin{aligned} D(xy) &= r(D_{\partial_{xy}^{-1}})xy = r(D_{\partial_x^{-1}} \partial_x (D_{\partial_y^{-1}}))xy = r(D_{\partial_x^{-1}})xr(D_{\partial_y^{-1}})y = D(x)D(y), \\ S(D(x)) &= S(r(D_{\partial_x}^{-1})x) = S(x)s(D_{\bar{\partial}_{S(x)}}) = \bar{D}^{-1}(S(x)), \\ D(x)^* &= x^*r(D_{\partial_x}^*) = x^*r(D_{\partial_{x^*}}) = D^{-1}(x^*) \quad \text{for all } x, y \in A. \quad \square \end{aligned}$$

Lemma 2.6.4. Let $\omega \in \{\phi, \psi, \nu\}$.

- i) $\omega(A_{\gamma, \gamma'}) = 0$ whenever $(\gamma, \gamma') \neq (e, e)$.
- ii) $\omega(r(b)s(b')a) = \omega(ar(b)s(b'))$ for all $a \in A, b, b' \in B$.

iii) $\omega(D(a)a') = \omega(aD^{-1}(a'))$ and $\omega(\bar{D}(a)a') = \omega(a\bar{D}^{-1}(a'))$ for all $a, a' \in A$.

Proof. i) For $\omega = \nu$, the assertion follows from the relation $\ker \phi + \ker \psi \subseteq \ker \nu$. To obtain the assertion for $\omega = \phi, \psi$, use the fact that μ is faithful.

ii) Let $a \in A$ and $b, b' \in B$. Then $\nu(r(b)a) = \mu(b\phi(a)) = \mu(\phi(a)b) = \nu(ar(b))$ and similarly $\nu(s(b')a) = \nu(as(b'))$. To obtain the assertion for $\omega = \phi, \psi$, use the fact that μ is faithful again.

iii) This follows immediately from equation (2.9) and i). \square

We shall now show that $\nu = \mu \circ \phi$ has a modular automorphism and thus satisfies an algebraic variant of the KMS-condition. Let us briefly recall this concept.

Let C be a $*$ -algebra with local units and a faithful, positive, linear map $\omega: C \rightarrow \mathbb{C}$. A *modular automorphism* for ω is a bijection $\theta_\omega: C \rightarrow C$ satisfying $\omega(cc') = \omega(c'\theta_\omega(c))$ for all $c, c' \in C$. If it exists, a modular automorphism θ_ω for ω is uniquely determined, an algebra automorphism, and satisfies $\omega \circ \theta_\omega = \omega$ and $\theta_\omega \circ * \circ \theta_\omega \circ * = \text{id}_C$. This follows easily from the relations

$$\begin{aligned} \omega(z\theta_\omega(xy)) &= \omega(xyz) = \omega(yz\theta_\omega(x)) = \omega(z\theta_\omega(x)\theta_\omega(y)), \\ \omega(yx) &= \overline{\omega(x^*y^*)} = \overline{\omega(y^*\theta_\omega(x^*))} = \omega(\theta_\omega(x^*)^*y) = \omega(y\theta_\omega(\theta_\omega(x^*)^*)), \end{aligned}$$

where $x, y, z \in C$.

As before, let $(A, \Delta, \epsilon, S, \phi, \psi, \mu)$ be a measured multiplier (B, Γ) -Hopf $*$ -algebroid.

Theorem 2.6.5. *Let $(A, \Delta, \epsilon, S, \phi, \psi, \mu)$ be a measured multiplier (B, Γ) -Hopf $*$ -algebroid and let $\nu = \mu \circ \phi = \mu \circ \psi$.*

i) *There exists a modular automorphism θ for ν .*

ii) *θ is a $(B, \Gamma)^{\text{ev}}$ -module automorphism of A .*

iii) *If $\nu \circ S = \nu$, then $\theta \circ S = S \circ \theta^{-1}$.*

Proof. i) The proof repeatedly uses strong invariance of ϕ and ψ , and closely follows [31], where the corresponding result was obtained for multiplier Hopf algebras. We proceed in three steps.

Step 1. Repeatedly using Remark 2.4.2 iii), we find that for all homogeneous $x, x', y, y' \in A$,

$$\begin{aligned} \bar{\partial}_{x'} = \bar{\partial}_{y'}^{-1} &\Rightarrow v^{-1}(ys(\psi(xx'))y') = \mu(\psi(yy')\bar{\partial}_{y'}^{-1}(\psi(xx'))) \\ &= \mu(\psi(xx')\bar{\partial}_{y'}(\psi(yy')D_{\bar{\partial}_{y'}})) \\ &= v^{-1}(xs(\psi(yy'))\bar{D}(x')), \end{aligned} \quad (2.10)$$

$$\begin{aligned} \bar{\partial}_x = \partial_{y'} &\Rightarrow v(yr(\psi(xx'))y') = \mu(\phi(yy')\partial_{y'}^{-1}(\psi(xx'))) \\ &= \mu(\partial_{y'}(\phi(yy')D_{\partial_{y'}})\psi(xx')) \\ &= v^{-1}(xs(\phi(D(y)y'))x'), \end{aligned} \quad (2.11)$$

$$\begin{aligned} \partial_x = \partial_y^{-1} &\Rightarrow v(yr(\phi(xx'))y') = \mu(\partial_y(\phi(xx'))\phi(yy')) \\ &= \mu(\partial_y^{-1}(\phi(yy')D_{\partial_y^{-1}})\phi(xx')) \\ &= v(xr(\phi(D(y)y'))x'). \end{aligned} \quad (2.12)$$

Step 2. Let $c, d \in A$ be homogeneous and let

$$a = \sum \bar{D}(s(\psi(dS(c_{(2)})))c_{(1)}) \in A, \quad a' = \sum d_{(2)}r(\phi(D(S(d_{(1)}))\bar{D}(c))) \in A. \quad (2.13)$$

Then the equations above and Proposition 2.5.3 imply

$$\begin{aligned} v(za) &= \sum v(z\bar{D}(s(\psi(dS(c_{(2)})))c_{(1)})) \\ &= \sum v(ds(\psi(zc_{(1)}))S(c_{(2)})) && \text{(Equation (2.10))} \\ &= \sum v(dr(\psi(z_{(1)}c)z_{(2)}) && \text{(Proposition 2.5.3)} \\ &= \sum v(z_{(1)}s(\phi(D(d)z_{(2)}))c) && \text{(Equation (2.11))} \\ &= \sum v(S(D(d_{(1)}))r(\phi(d_{(2)}z))c) && \text{(Proposition 2.5.3)} \\ &= \sum v(S(d_{(1)})r(\phi(d_{(2)}z))\bar{D}(c)) && \text{(use } S \circ D = \bar{D}^{-1} \circ S \\ & && \text{and 2.6.4 iii)} \\ &= \sum v(d_{(2)}r(\phi(D(S(d_{(1)}))\bar{D}(c)))z) = v(a'z). \end{aligned} \quad \text{(Equation 2.12)}$$

Step 3. Using bijectivity of the maps \bar{D}, S, T_1 and the relation $\langle s(\psi(A))A \rangle = A$, one finds that all elements of the form like a in (2.13) span A . A similar argument shows that the same is true for elements of the form like a' . Hence, there exists a bijection $\theta: A \rightarrow A$ such that $v(az) = v(z\theta(a))$ for all $a \in A$, and uniqueness of such a bijection follows from faithfulness of v .

ii) We first show that θ respects the grading. Let $c, d \in A$ be homogeneous. Then the element a in (2.13) is homogeneous as well, with grading given by $\partial_a = \partial_c$ and

$\bar{\partial}_a = \bar{\partial}_d$ because $\psi(dS(c_{(2)})) = 0$ unless $\bar{\partial}_d = \partial_{c_{(2)}} = \bar{\partial}_{c_{(1)}}$, and similarly a' in (2.13) is homogeneous with the same degree like a . To see that θ is $B \otimes B$ -linear, use the relation $v(y\theta(r(b)s(b')x)) = v(r(b)s(b')xy) = v(xyr(b)s(b')) = v(yr(b)s(b')\theta(x))$, where $x, y \in A$ and $b, b' \in B$, and faithfulness of v .

iii) If $v \circ S = v$, then we have $v(y\theta(S(x))) = v(S(x)y) = v(S^{-1}(y)x) = v(\theta^{-1}(x)S^{-1}(y)) = v(yS(\theta^{-1}(x)))$ for all $x, y \in A$. \square

Define $\theta_D, \theta_{\bar{D}}, \theta_{D, \bar{D}}: A \rightarrow A$ by

$$\begin{aligned}\theta_D &:= \theta \circ D^{-1} = D^{-1} \circ \theta, \\ \theta_{\bar{D}} &:= \theta \circ \bar{D}^{-1} = \bar{D}^{-1} \circ \theta, \\ \theta_{D, \bar{D}} &:= \theta \circ D^{-1} \circ \bar{D}^{-1}.\end{aligned}$$

Proposition 2.6.6. *Let $x, y \in A$ be homogeneous. Then*

- i) $\phi \circ \theta = \phi$ and $\phi(xy) = \partial_x(\phi(y\theta_D(x)))$;
- ii) $\psi \circ \theta = \psi$ and $\psi(xy) = \bar{\partial}_x(\psi(y\theta_{\bar{D}}(x)))$;
- iii) $h \circ \theta = h$ and $h(xy) = (\partial_x \otimes \bar{\partial}_x)(h(y\theta_{D, \bar{D}}(x)))$ if h is a bi-invariant integral and $v = (\mu \otimes \mu) \circ h$.

Proof. Assertion i) follows from the fact that μ is faithful and that for all homogeneous $x, y \in A$ and all $b \in B$,

$$\begin{aligned}\mu(b\phi(\theta(x))) &= v(r(b)\theta(x)) = v(\theta(r(b)x)) = v(r(b)x) = \mu(b\phi(x)), \\ \mu(b\phi(y\theta(x))) &= v(r(b)y\theta(x)) = v(xr(b)y) \\ &= v(r(\partial_x(bD_{\partial_x}))xr(D_{\partial_x}^{-1})y) \\ &= \mu(\partial_x(bD_{\partial_x})\phi(D(x)y)) = \mu(b\partial_x^{-1}(\phi(D(x)y))).\end{aligned}$$

Assertions ii) and iii) follow similarly. \square

Recall that a B -module N is called *flat* if the functor $N \otimes_B -$ on the category of B -modules is exact or, equivalently, preserves injectivity of morphisms.

Proposition 2.6.7. *Assume that A_S is a flat B -module. Then $\Delta \circ \theta_D = (S^2 \tilde{\otimes} \theta_D) \circ \Delta$.*

Proof. Let $x, y \in A$ be homogeneous. Using Sweedler notation, we calculate

$$\begin{aligned}
\sum \theta_D(x)_{(1)} s(\phi(y\theta_D(x)_{(2)})) &= \sum S(s(\phi(y_{(2)}\theta_D(x)))y_{(1)}) && \text{(Prop. 2.5.3)} \\
&= \sum S(s(\partial_x^{-1}(\phi(xy_{(2)})))y_{(1)}) && \text{(Prop. 2.6.6)} \\
&= \sum S(y_{(1)}s(\phi(xy_{(2)}))) \\
&= \sum S^2(s(\phi(x_{(2)}y))x_{(1)}) && \text{(Prop. 2.5.3)} \\
&= \sum S^2(s(\partial_{x_{(2)}}(\phi(y\theta_D(x_{(2)})))x_{(1)}) && \text{(Prop. 2.6.6)} \\
&= \sum S^2(x_{(1)}s(\phi(y\theta_D(x_{(2)}))).
\end{aligned}$$

Since A_s is a flat B -module and maps of the form $a \mapsto \phi(ya)$, where $y \in A$ is homogeneous, separate the points of A , we can conclude

$$\sum \theta_D(x)_{(1)} \tilde{\otimes} \theta_D(x)_{(2)} = \sum S^2(x_{(1)}) \tilde{\otimes} \theta_D(x_{(2)}). \quad \square$$

2.7. The dual $*$ -algebra. Let $(A, \Delta, \epsilon, S, \phi, \psi, \mu)$ be a measured multiplier (B, Γ) -Hopf $*$ -algebroid. Denote by $M(A)'$ the dual vector space of $M(A)$ and let

$$\hat{A} := \{v(x-) : x \in A\} \subseteq M(A)'$$

Then $\hat{A} = \{v(-x) : x \in A\}$ by Theorem 2.6.5 and for each $\omega \in \hat{A}$, there exist unique B -module maps ${}_r\omega : {}_rM(A) \rightarrow B$, $\omega_r : M(A)_r \rightarrow B$, ${}_s\omega : {}_sM(A) \rightarrow B$, $\omega_s : M(A)_s \rightarrow B$ whose compositions with μ are equal to ω , because $v = \mu \circ \phi = \mu \circ \psi$ and μ is faithful. Using either of these B -module maps, one can equip \hat{A} with the structure of a $*$ -algebra. We shall choose an approach that fits well with the duality on the operator-algebraic level in the next section.

First, we define an abstract Fourier transform

$$A \rightarrow \hat{A}, \quad x \mapsto \hat{x} := v(S(x)-),$$

which is a linear bijection because v is faithful. Evidently, $\hat{x}_s = \psi(S(x)-)$ and $\hat{x}_r = \phi(S(x)-)$, and by Proposition 2.6.6, ${}_s\hat{x} = \psi(-\theta(S(x)))$ and ${}_r\hat{x} = \phi(-\theta(S(x)))$. For all $x, a \in A$, we define a right convolution

$$a * \hat{x} := \sum a_{(2)} r(\hat{x}_s(a_{(1)})) = \sum a_{(2)} r(\psi(S(x)a_{(1)})) \in A. \quad (2.14)$$

Remark 2.7.1. One could also work with the transform $A \rightarrow \hat{A}$, $x \mapsto \check{x} := v(-S(x))$, and the left convolution defined by

$$\check{x} * a := \sum s({}_r\check{x}(a_{(2)}))a_{(1)} = \sum s(\phi(a_{(2)}S(x)))a_{(1)} \in A \quad \text{for all } x, a \in A. \quad (2.15)$$

If $\phi \circ S = \psi$, for example, if we are in the bi-measured case (see Remark 2.6.2 ii), then

$$\widehat{S(x)} * S(a) = \sum s(\phi(S(a)_{(2)} S^2(x))) S(a)_{(1)} = S(a)_{(2)} r(\psi(S(x) a_{(1)})) = S(a * \hat{x})$$

for all $a, x \in A$.

We collect a few useful formulas. First, for all $a, x \in A$,

$$a * \hat{x} = \sum r(\psi(a_{(1)} \theta_{\widehat{D}}(x))) a_{(2)}, \quad (\text{Proposition 2.6.6}) \quad (2.16)$$

$$\begin{aligned} a * \hat{x} &= \sum S^{-1}(r(\psi(S(x)_{(1)} a)) S(x)_{(2)}) \\ &= \sum x_{(1)} s(\psi(S(x)_{(2)} a)) \quad (\text{Proposition 2.5.3}) \end{aligned} \quad (2.17)$$

Next, for all $a, x, y \in A, b \in B, \gamma, \gamma', \delta, \delta' \in \Gamma$,

$$\begin{aligned} r(b) a * \hat{x} &= a * \widehat{s(b)x}, & ar(b) * \hat{x} &= a * \widehat{xs(b)}, \\ s(b) a * \hat{x} &= s(b)(a * \hat{x}), & as(b) * \hat{x} &= (a * \hat{x}) s(b), \end{aligned} \quad (2.18)$$

$$\begin{aligned} (a * \hat{x}) * \hat{y} &= \sum a_{(3)} r(\psi(S(y) a_{(2)} r(\psi(S(x) a_{(1)})))) \\ &= \sum a_{(2)} r(\psi(S(y) x_{(1)} s(\psi(S(x)_{(2)} a_{(1)})))) \end{aligned} \quad (2.19)$$

$$= \sum a_{(2)} r(\psi(S(x)_{(2)} r(\psi(S(y) x_{(1)}))) a_{(1)}) = a * \widehat{(x * \hat{y})},$$

$$A_{\gamma, \gamma'} * \widehat{A_{\delta, \delta'}} \subseteq \sum_{\gamma''} s(\psi(A_{\delta'-1, \delta-1} A_{\gamma, \gamma''})) A_{\gamma'', \gamma'} \subseteq \delta_{\gamma, \delta'} A_{\delta, \gamma'}, \quad (2.20)$$

where we used Lemma 2.6.4 in the last line. Finally, note that the surjectivity of T_2 (Proposition 2.3.8) and of ψ imply

$$\langle A * \hat{A} \rangle = (\psi \otimes \text{id})(T_2(A_r \otimes_B rA)) = \langle Ar(\psi(A)) \rangle = \langle Ar(B) \rangle = A. \quad (2.21)$$

The $(B, \Gamma)^{\text{ev}}$ -algebra structure on A induces the following structure on \hat{A} :

Proposition 2.7.2.

i) \hat{A} carries the structure of a non-degenerate $*$ -algebra, where $\widehat{\hat{y}\hat{x}} = \widehat{x * \hat{y}}$ and $\widehat{\hat{x}^*} = \widehat{S(x)^*}$ for all $x, y \in A$.

ii) There exist non-degenerate $*$ -homomorphisms $\hat{r}, \hat{s}: B \rightarrow M(\hat{A})$ such that

$$\hat{r}(b)\hat{x} = \widehat{xr(b)}, \quad \hat{x}\hat{r}(b) = \widehat{xs(b)}, \quad \hat{s}(b)\hat{x} = \widehat{r(b)x}, \quad \hat{x}\hat{s}(b) = \widehat{s(b)x}$$

for all $x \in A, b \in B$. The images of \hat{r} and \hat{s} commute.

iii) Let $\hat{A}^{\gamma, \gamma'} = \widehat{(A_{\gamma, \gamma'})}$ for all $\gamma, \gamma' \in \Gamma$. Then $\hat{A} = \bigoplus_{\gamma, \gamma' \in \Gamma} \hat{A}^{\gamma, \gamma'}$ as a vector space and

$$\hat{A}^{\gamma, \gamma'} \hat{A}^{\delta, \delta'} \subseteq \delta_{\gamma', \delta} \hat{A}^{\gamma, \delta'}, \quad (\hat{A}^{\gamma, \gamma'})^* = \hat{A}^{\gamma', \gamma}, \quad \hat{r}(B)\hat{s}(B)\hat{A}^{\gamma, \gamma'} \subseteq \hat{A}^{\gamma, \gamma'}$$

for all $\gamma, \gamma', \delta, \delta' \in \Gamma$.

Furthermore, for all $\gamma, \gamma', \delta, \delta' \in \Gamma$, $\hat{a} \in \hat{A}^{\gamma, \gamma'}$, $b, b' \in B$,

$$\hat{r}(b)\hat{s}(b')\hat{a} = \hat{r}(\gamma^{-1}(b'))\hat{s}(\gamma(b))\hat{a} \quad \text{and} \quad \hat{a}\hat{r}(b)\hat{s}(b') = \hat{a}\hat{r}(\gamma'^{-1}(b'))\hat{s}(\gamma'(b)).$$

Proof. i) The multiplication is associative and turns \hat{A} into a non-degenerate algebra by (2.19) and (2.21). The $*$ -operation is involutive because $* \circ S$ is involutive, and anti-multiplicative because

$$\begin{aligned} S(y * \hat{x})^* &= \sum S(y_{(2)}r(\psi(S(x)y_{(1)})))^* \\ &= \sum S(y_{(2)})^*s(\psi(y_{(1)}^*S(x)^*)) \\ &= \sum S(y)_{(1)}^*s(\psi(S(S(y)_{(2)}^*)S(x)^*)) = S(x)^* * \widehat{S(y)}^*. \end{aligned}$$

ii) For each $b \in B$, the formulas above define multipliers $\hat{r}(b), \hat{s}(b) \in M(\hat{A})$ because

$$\hat{y}(\hat{r}(b)\hat{x}) = (xr(b) * \hat{y})^\wedge = (x * \widehat{ys(b)})^\wedge = (\hat{y}\hat{r}(b))\hat{x}$$

and similarly $\hat{y}(\hat{s}(b)\hat{x}) = (\hat{y}\hat{s}(b))\hat{x}$ for all $x, y \in A$ by (2.18). The maps $\hat{r}, \hat{s}: B \rightarrow M(\hat{A})$ are non-degenerate homomorphisms because $r, s: B \rightarrow M(A)$ have the same properties, their images evidently commute, and they are involutive because

$$(\hat{x}\hat{r}(b))^* = (\widehat{xs(b)})^* = (S(xs(b))^*)^\wedge = (S(x)^*r(b^*))^\wedge = \hat{r}(b^*)\hat{x}^*$$

and similarly $(\hat{x}\hat{s}(b))^* = \hat{s}(b^*)\hat{x}^*$ for all $x \in A, b \in B$.

iii) All of these assertions follow easily from the definitions and relation (2.20), for example, $\hat{r}(b)\hat{x} = \widehat{xr(b)} = (r(\gamma(b))x)^\wedge = \hat{s}(\gamma(b))\hat{x}$ for all $\gamma, \gamma' \in \Gamma$, $x \in A_{\gamma, \gamma'}, b \in B$. \square

3. Construction of associated measured quantum groupoids

In this section, we fix a measured multiplier (B, Γ) -Hopf $*$ -algebroid $(A, \Delta, \epsilon, S, \mu, \phi, \psi)$ and construct operator-algebraic completions of this algebraic object in the form of a Hopf C^* -bimodule, Hopf–von Neumann bimodule and a measured quantum groupoid. Along the way, we shall impose further assumptions on B, Γ, μ, A which were mentioned already in the introduction, most notably properness.

The basic idea is to use the GNS-representations for the weight μ on the basis B and the functional ν on the total algebra A , respectively. Naturally, some restrictions have to be made on B, Γ, μ . To show that ν admits a bounded GNS-representation and to lift the comultiplication to the level of operator algebras, we use a fundamental unitary. To take full advantage of this unitary, we describe its domain and range as relative tensor products, and show that it is a pseudo-multiplicative unitary in the

sense of [24] and [27]. The necessary modules are introduced in §3.2, and the unitary itself is constructed in §3.3. This part uses Connes' spatial theory [20], and the relative tensor product of Hilbert spaces over C^* -algebras which was introduced in [22]. The fundamental unitary then gives rise to completions of A and \hat{A} in the form of Hopf C^* -bimodules and two Hopf–von Neumann bimodules; see §3.4–§3.6. To obtain the full structure of a measured quantum groupoid, we finally extend the integrals ϕ, ψ to the level of von Neumann algebras and show that these extensions are left- or right-invariant again in §3.7.

Before we turn to details, let us briefly sketch the construction of the fundamental unitary, which we denote by W . Its domain and range can be described as separated completions of the relative tensor products ${}_s A \otimes_B {}_r A$ and ${}_r A \otimes_B A_r$ with respect to the sesquilinear forms given by

$$\begin{aligned} \langle x \otimes_B y | x' \otimes_B y' \rangle_{({}_s A \otimes_B {}_r A)} &= \nu(x^* s(\partial_y(\phi(y^* y'))))x', \\ \langle x \otimes_B y | x' \otimes_B y' \rangle_{({}_r A \otimes_B A_r)} &= \nu(x^* r(\phi(y^* y'))x') \end{aligned} \quad (3.1)$$

for all $x, y \in A$, where y is assumed to be homogeneous in the upper line. Note that positivity of these forms is not evident because ϕ is not assumed to be completely positive in any sense. Given that positivity, the map

$$T_4: {}_r A \otimes_B A_r \rightarrow {}_s A \otimes_B {}_r A, \quad x \otimes_B y \mapsto \Delta(y)(x \otimes_B 1) = \sum y_{(1)}x \otimes_B y_{(2)},$$

extends to a unitary on the respective completions because it is surjective by Proposition 2.3.8 and isometric as shown by the calculation

$$\begin{aligned} \sum \langle y_{(1)}x \otimes_B y_{(2)} | y'_{(1)}x' \otimes_B y'_{(2)} \rangle_{({}_s A \otimes_B {}_r A)} &= \sum \nu(x^* y_{(1)}^* s(\partial_{y_{(2)}}(\phi(y_{(2)}^* y'_{(2)})))y'_{(1)}x') \\ &= \sum \nu(x^* s(\phi(y_{(2)}^* y'_{(2)}))y_{(1)}y'_{(1)}x') \\ &= \nu(x^* r(\phi(y^* y'))x') \\ &= \langle x \otimes_B y | x' \otimes_B y' \rangle_{({}_r A \otimes_B A_r)}, \end{aligned} \quad (3.2)$$

where $y_{(2)}$ is assumed to be homogeneous without loss of generality. The adjoint of this extension is the fundamental unitary W .

Similarly, one can construct and employ another unitary V which is an extension of the map $T_1: A_s \otimes_B {}_s A \rightarrow {}_s A \otimes_B {}_r A$, $x \otimes_B y \mapsto \Delta(x)(1 \otimes_B y)$. We shall focus on W because this unitary is given preference in the theory of locally compact quantum groups and measured quantum groupoids.

3.1. Preparations concerning the base. We define an inner product on B by $\langle b | b' \rangle := \mu(b^* b')$ for all $b, b' \in B$, and denote by K the Hilbert space obtained by completion, and by $\Lambda_\mu: B \rightarrow K$ the canonical inclusion.

From now on, we assume:

(A1) The weight μ admits a GNS-representation via bounded operators in the sense that the following equivalent conditions hold:

- i) for each $b \in B$, there is a $\lambda > 0$ such that $\mu(c^*b^*bc) \leq \lambda\mu(c^*c)$ for all $c \in B$;
- ii) the formula $\pi_\mu(b)\Lambda_\mu(c) = \Lambda_\mu(bc)$ defines a $*$ -homomorphism $\pi_\mu: B \rightarrow \mathcal{L}(K)$.

Remarks 3.1.1.

- i) Assumption (A1) holds if B is a pre- C^* -algebra since then $\mu(c^*b^*bc) \leq \mu(c^*\|b^*b\|c) = \|b^*b\|\mu(c^*c)$ for all $b, c \in B$. Conversely, if (A1) holds, then B can be regarded as a pre- C^* -algebra with respect to the norm given by $b \mapsto \|\pi_\mu(b)\|$.
- ii) To apply the constructions below, it may be useful to first perform a base change, similarly as described in [21, §2], to replace B by a pre- C^* -algebra of the form $C_c(\Omega)$, where Ω is a locally compact space with an action of Γ . For example, one can take Ω to be the set of all $*$ -homomorphisms $\chi: B \rightarrow \mathbb{C}$, equipped with the weakest topology that makes the function $\Omega \rightarrow \mathbb{C}$, $\chi \mapsto \chi(b)$, continuous for each $b \in B$, and perform a base change along the canonical map $B \rightarrow M(C_c(\Omega))$. Note, however, that such a base change can not simply be applied to left and right integrals, but only to bi-integrals.

Recall that a *Hilbert algebra* is a $*$ -algebra with an inner product such that left multiplication by each element is bounded, the resulting $*$ -representation is non-degenerate, and the involution is pre-closed with respect to the norm induced by the inner product. Since B is commutative, the map $\Lambda_\mu(B) \rightarrow \Lambda_\mu(B)$ given by $\Lambda_\mu(b) \mapsto \Lambda_\mu(b^*)$ extends to an anti-unitary operator J_μ on K , and hence $\Lambda_\mu(B) \subseteq K$ together with the $*$ -algebra structure inherited from B is a Hilbert algebra. We thus obtain

- a von Neumann algebra $N := \pi_\mu(B)'' \subseteq \mathcal{L}(K)$,
- a n.s.f. weight $\tilde{\mu}$ on N such that $\tilde{\mu}(\pi_\mu(b^*b)) = \langle \Lambda_\mu(b) | \Lambda_\mu(b) \rangle = \mu(b^*b)$ for all $b \in B$,
- a left ideal $\mathfrak{N}_{\tilde{\mu}} := \{x \in N : \tilde{\mu}(x^*x) < \infty\} \subseteq N$ of square-integrable elements,
- a closed map $\Lambda_{\tilde{\mu}}: \mathfrak{N}_{\tilde{\mu}} \rightarrow K$ such that $(K, \Lambda_{\tilde{\mu}}, \text{id}_N)$ is a GNS-representation for $\tilde{\mu}$; this is the closure of the map $\pi_\mu(B) \rightarrow K$ given by $\pi_\mu(b) \rightarrow \Lambda_\mu(b)$.

3.2. Various module structures. We define an inner product on A by $\langle a|a' \rangle := v(a^*a')$ for all $a, a' \in A$ and denote by H the Hilbert space obtained by completion. We call the canonical inclusion of A into H the *GNS-map* for v and denote it by Λ_v .

Lemma 3.2.1. *There exist maps $\Lambda_\phi, \Lambda_\psi, \Lambda_\phi^\dagger, \Lambda_\psi^\dagger: A \rightarrow \mathcal{L}(K, H)$ such that for all $x, y \in A, b \in B$,*

$$\begin{aligned} \Lambda_\phi(x)\Lambda_\mu(b) &= \Lambda_v(xr(b)), \\ \Lambda_\phi(x)^*\Lambda_v(y) &= \Lambda_\mu(\phi(x^*y)), \quad \Lambda_\phi(x)^*\Lambda_\phi(y) = \pi_\mu(\phi(x^*y)), \\ \Lambda_\psi(x)\Lambda_\mu(b) &= \Lambda_v(xs(b)), \\ \Lambda_\psi(x)^*\Lambda_v(y) &= \Lambda_\mu(\psi(x^*y)), \quad \Lambda_\psi(x)^*\Lambda_\psi(y) = \pi_\mu(\psi(x^*y)), \\ \Lambda_\phi^\dagger(x)\Lambda_\mu(b) &= \Lambda_v(r(b)x), \\ \Lambda_\phi^\dagger(x)^*\Lambda_v(y) &= \Lambda_\mu(\phi(y\theta(x^*))), \quad \Lambda_\phi^\dagger(x)^*\Lambda_\phi^\dagger(y) = \pi_\mu(\phi(y\theta(x^*))), \\ \Lambda_\psi^\dagger(x)\Lambda_\mu(b) &= \Lambda_v(s(b)x), \\ \Lambda_\psi^\dagger(x)^*\Lambda_v(y) &= \Lambda_\mu(\psi(y\theta(x^*))), \quad \Lambda_\psi^\dagger(x)^*\Lambda_\psi^\dagger(y) = \pi_\mu(\psi(y\theta(x^*))). \end{aligned}$$

Proof. We only prove the assertions concerning Λ_ϕ and Λ_ϕ^\dagger . They follow from the relations

$$\begin{aligned} \|\Lambda_v(xr(b))\|^2 &= v(r(b)^*x^*xr(b)) = \mu(b^*\phi(x^*x)b) \leq \|\pi_\mu(\phi(x^*x))\|\|\Lambda_\mu(b)\|^2, \\ \langle \Lambda_v(y)|\Lambda_v(xr(b)) \rangle &= v(y^*xr(b)) = \mu(\phi(y^*x)b) = \langle \Lambda_\mu(\phi(x^*y))|\Lambda_\mu(b) \rangle \end{aligned}$$

and

$$\begin{aligned} \|\Lambda_v(r(b)x)\|^2 &= v(x^*r(b^*b)x) = v(\theta^{-1}(x)x^*r(b^*b)) \\ &= \mu(\phi(\theta^{-1}(x)x^*)b^*b) \\ &\leq \|\Lambda_\mu(b)\|^2\|\pi_\mu(\theta^{-1}(x)x^*)\|, \\ \langle \Lambda_v(y)|\Lambda_v(r(b)x) \rangle &= v(y^*r(b)x) = v(\theta^{-1}(x)y^*r(b)) \\ &= \mu(\phi(\theta^{-1}(x)y^*)b) \\ &= \langle \Lambda_\mu(\phi(y\theta(x^*)))|\Lambda_\mu(b) \rangle, \end{aligned}$$

which hold for all $x, y \in A$ and $b \in B$. □

The maps introduced above yield various module structures on H as follows. Let

$$E_\phi := [\Lambda_\phi(A)], \quad E_\psi := [\Lambda_\psi(A)], \quad E_\phi^\dagger := [\Lambda_\phi^\dagger(A)], \quad E_\psi^\dagger := [\Lambda_\psi^\dagger(A)]. \quad (3.3)$$

We shall use the following concepts introduced in [22, 24]. A C^* - \mathfrak{b} -module, where $\mathfrak{b} = (K, [\pi_\mu(B)], [\pi_\mu(B)])$, consists of a Hilbert space L and a closed subset $E \subseteq \mathcal{L}(K, L)$ such that $[EK] = L$, $[E\pi_\mu(B)] = E$, $[E^*E] = [\pi_\mu(B)]$. Each

such C^* - \mathfrak{b} -module gives rise to a normal, faithful, non-degenerate representation $\rho_E: N = \pi_\mu(B)'' \rightarrow \mathcal{L}(L)$ such that $\rho_E(x)\xi = \xi x$ for all $x \in N, \xi \in E$. A C^* - $(\mathfrak{b}, \mathfrak{b})$ -module is a triple (L, E, F) such that (L, E) and (L, F) are C^* - \mathfrak{b} -modules and $[\rho_E(\pi_\mu(B))F] = F$ and $[\rho_F(\pi_\mu(B))E] = [E]$.

Lemma 3.2.2. *The Hilbert space H is a C^* - $(\mathfrak{b}, \mathfrak{b})$ -module with respect to either two of the spaces $E_\phi, E_\psi, E_\phi^\dagger, E_\psi^\dagger$. The representations $\alpha := \rho_{E_\phi^\dagger}, \beta := \rho_{E_\psi^\dagger}, \widehat{\alpha} := \rho_{E_\psi}, \widehat{\beta} := \rho_{E_\phi}$ of N on H are given by*

$$\begin{aligned} \alpha(\pi_\mu(b))\Lambda_v(a) &= \Lambda_v(r(b)a), & \beta(\pi_\mu(b))\Lambda_v(a) &= \Lambda_v(s(b)a), \\ \widehat{\beta}(\pi_\mu(b))\Lambda_v(a) &= \Lambda_v(ar(b)), & \widehat{\alpha}(\pi_\mu(b))\Lambda_v(a) &= \Lambda_v(as(b)) \end{aligned} \quad (3.4)$$

for all $b \in B, a \in A$.

Proof. Let E, F be any two of the spaces listed above. Then $[EH] = H$ and $[E\pi_\mu(B)] = E$ because $\langle r(B)s(B)Ar(B)s(B) \rangle = A$, and $[E^*E] = [\pi_\mu(B)]$ because $\phi(A) = B = \psi(A)$. Thus, (H, E) is a C^* - \mathfrak{b} -module. The formulas for the associated representations are easily verified. Using these formulas and the relation $\langle r(B)s(B)Ar(B)s(B) \rangle = A$, one easily checks that $[\rho_E(\pi_\mu(B))F] = F$ and $[\rho_F(\pi_\mu(B))E] = E$. \square

Recall that a vector ζ in a Hilbert space L is *bounded* with respect to a normal, non-degenerate representation $\rho: N \rightarrow \mathcal{L}(L)$ and the weight $\tilde{\mu}$ if the following equivalent conditions hold:

- i) there exists a $K \geq 0$ such that $\|\rho(x)\zeta\| \leq K\tilde{\mu}(x^*x)$ for all $x \in \mathfrak{N}_{\tilde{\mu}}$;
- ii) there exists an operator $R_\zeta^{\rho, \tilde{\mu}} \in \mathcal{L}(K, L)$ such that $R_\zeta^{\rho, \tilde{\mu}}\Lambda_\mu(x) = \rho(x)\zeta$ for all $x \in \mathfrak{N}_{\tilde{\mu}}$.

The set of all such bounded vectors is denoted by $D(L_\rho, \tilde{\mu})$. This spaces carries an N -valued inner product $\langle - | - \rangle_{\rho, \tilde{\mu}}$, given by $\langle \zeta | \zeta' \rangle_{\rho, \tilde{\mu}} = (R_\zeta^{\rho, \tilde{\mu}})^* R_{\zeta'}^{\rho, \tilde{\mu}}$ for all $\zeta, \zeta' \in D(L_\rho, \tilde{\mu})$, and $\rho(N)'D(L_\rho, \tilde{\mu}) = D(L_\rho, \tilde{\mu})$ and

$$\Lambda_{\tilde{\mu}}(\langle \zeta | \zeta' \rangle_{\rho, \tilde{\mu}}) = (R_\zeta^{\rho, \tilde{\mu}})^* \zeta', \quad R_{T\zeta}^{\rho, \tilde{\mu}} = TR_\zeta^{\rho, \tilde{\mu}} \quad \text{for all } T \in \rho(N)', \zeta, \zeta' \in D(L_\rho, \tilde{\mu}). \quad (3.5)$$

Lemma 3.2.3. $\Lambda_v(A) \subseteq D(H_\alpha, \tilde{\mu}) \cap D(H_\beta, \tilde{\mu}) \cap D(H_{\widehat{\alpha}}, \tilde{\mu}) \cap D(H_{\widehat{\beta}}, \tilde{\mu})$ and for all $x, y \in A$,

$$R_{\Lambda_v(x)}^{\alpha, \tilde{\mu}} = \Lambda_\phi^\dagger(x), \quad R_{\Lambda_v(x)}^{\beta, \tilde{\mu}} = \Lambda_\psi^\dagger(x), \quad R_{\Lambda_v(x)}^{\widehat{\alpha}, \tilde{\mu}} = \Lambda_\psi(x), \quad R_{\Lambda_v(x)}^{\widehat{\beta}, \tilde{\mu}} = \Lambda_\phi(x).$$

Proof. We shall only prove the assertion concerning α . Let $a \in A$. Then $\Lambda_\phi^\dagger(a)\Lambda_{\tilde{\mu}}(\pi_\mu(b)) = \Lambda_v(r(b)a) = \alpha(\pi_\mu(b))\Lambda_v(a)$ for all $b \in B$, and since $\pi_\mu(B)$ is a core for $\Lambda_{\tilde{\mu}}$, we can conclude $\Lambda_\phi^\dagger(a)\Lambda_{\tilde{\mu}}(x) = \alpha(x)\Lambda_v(a)$ for all $x \in \mathfrak{N}_{\tilde{\mu}}$. \square

The preceding result and Lemma 3.2.1 imply that for all $x, y \in A$,

$$\begin{aligned} \langle \Lambda_\nu(x) | \Lambda_\nu(y) \rangle_{\alpha, \tilde{\mu}} &= \pi_\mu(\phi(y\theta(x^*))), & \langle \Lambda_\nu(x) | \Lambda_\nu(y) \rangle_{\beta, \tilde{\mu}} &= \pi_\mu(\psi(y\theta(x^*))), \\ \langle \Lambda_\nu(x) | \Lambda_\nu(y) \rangle_{\hat{\alpha}, \tilde{\mu}} &= \pi_\mu(\psi(x^*y)), & \langle \Lambda_\nu(x) | \Lambda_\nu(y) \rangle_{\hat{\beta}, \tilde{\mu}} &= \pi_\mu(\phi(x^*y)). \end{aligned} \quad (3.6)$$

3.3. The fundamental unitary. To define the domain and the range of the fundamental unitary, we use Connes' relative tensor product of Hilbert modules and the module structures introduced above. Connes' original manuscript on the construction remained unpublished; we therefore refer to [20] and [23] for details.

The *relative tensor product* $H_\beta \otimes_{\alpha, \tilde{\mu}} H$ is the separated completion of the algebraic tensor product $D(H_\beta, \tilde{\mu}) \otimes K \otimes D(H_\alpha, \tilde{\mu})$ with respect to the sesquilinear form given by

$$\langle \xi \otimes \zeta \otimes \eta | \xi' \otimes \zeta' \otimes \eta' \rangle = \langle \zeta | \langle \xi | \xi' \rangle_{\beta, \tilde{\mu}} \langle \eta | \eta' \rangle_{\alpha, \tilde{\mu}} \zeta' \rangle. \quad (3.7)$$

This Hilbert space can naturally be identified with the separated completions of the algebraic tensor products $D(H_\beta, \tilde{\mu}) \otimes H$ and $H \otimes D(H_\alpha, \tilde{\mu})$ with respect to the sesquilinear forms given by

$$\langle \xi \otimes \eta | \xi' \otimes \eta' \rangle = \langle \eta | \alpha(\langle \xi | \xi' \rangle_{\beta, \tilde{\mu}}) \eta' \rangle \quad \text{and} \quad \langle \xi \otimes \eta | \xi' \otimes \eta' \rangle = \langle \xi | \beta(\langle \eta | \eta' \rangle_{\alpha, \tilde{\mu}}) \xi' \rangle, \quad (3.8)$$

respectively, via

$$\xi \otimes R_\xi^{\alpha, \tilde{\mu}} \zeta \equiv \xi \otimes \zeta \otimes \eta \equiv R_\xi^{\beta, \tilde{\mu}} \zeta \otimes \eta, \quad (3.9)$$

and we shall use these identifications without further notice. Replacing the representations β, α by $\hat{\beta}$ or $\hat{\alpha}, \beta$, respectively, one obtains the relative tensor products $H_\alpha \otimes_{\tilde{\mu}, \hat{\beta}} H$ and $H_{\hat{\alpha}} \otimes_{\tilde{\mu}} \beta H$.

To proceed, we impose the following simplifying assumption:

(A2) The Radon–Nikodym cocycle $(D_\gamma)_\gamma$ of μ has a *positive square root* in $M(B)$ in the sense that there exists a family $(D_\gamma^{\frac{1}{2}})_{\gamma \in \Gamma}$ in $M(B)$ such that for all $\gamma, \gamma' \in \Gamma, c \in B$,

$$\begin{aligned} D_e^{\frac{1}{2}} &= 1, & (D_\gamma^{\frac{1}{2}})^* &= D_\gamma^{\frac{1}{2}}, & (D_\gamma^{\frac{1}{2}})^2 &= D_\gamma, \\ D_{\gamma\gamma'}^{\frac{1}{2}} &= \gamma'^{-1} (D_\gamma^{\frac{1}{2}}) D_{\gamma'}^{\frac{1}{2}}, & \mu(c^* D_\gamma^{\frac{1}{2}} c) &\geq 0. \end{aligned}$$

Clearly, this condition implies the existence of a unitary representation $U: \Gamma \rightarrow \mathcal{L}(K)$ such that

$$U_\gamma \Lambda_\mu(c) = \Lambda_\mu(\gamma(c D_\gamma^{\frac{1}{2}})), \quad U_\gamma \pi_\mu(b) U_\gamma^* = \pi_\mu(\gamma(b)) \quad \text{for all } b, c \in B, \gamma \in \Gamma. \quad (3.10)$$

Similarly as in (2.9), we define linear maps $D^{\frac{1}{2}}, \bar{D}^{\frac{1}{2}}: A \rightarrow A$ by

$$D^{\frac{1}{2}}(a) = r(D_{\partial_a^{-1}}^{\frac{1}{2}})a = ar(D_{\partial_a}^{-\frac{1}{2}}), \quad \bar{D}^{\frac{1}{2}}(a) = s(D_{\partial_a^{-1}}^{\frac{1}{2}})a = as(D_{\partial_a}^{-\frac{1}{2}})$$

for all homogeneous $a \in A$. These maps share all the properties of the maps D, \bar{D} listed in Lemma 2.6.3. Short calculations show that for all homogeneous $x, y \in A$,

$$\Lambda_\phi(x)U_{\partial_x^{-1}} = \Lambda_\phi^\dagger(D^{\frac{1}{2}}(x)), \quad \langle \Lambda_\nu(D^{\frac{1}{2}}(x)) | \Lambda_\nu(D^{\frac{1}{2}}(y)) \rangle_{\alpha, \bar{\mu}} = \pi_\mu(\partial_x(\phi(x^*y))), \quad (3.11)$$

$$\Lambda_\psi(x)U_{\partial_x^{-1}} = \Lambda_\psi^\dagger(\bar{D}^{\frac{1}{2}}(x)), \quad \langle \Lambda_\nu(\bar{D}^{\frac{1}{2}}(x)) | \Lambda_\nu(\bar{D}^{\frac{1}{2}}(y)) \rangle_{\beta, \bar{\mu}} = \pi_\mu(\bar{\partial}_x(\psi(x^*y))). \quad (3.12)$$

Indeed, for all homogeneous $x, y \in A$ and $b \in B$,

$$\begin{aligned} \Lambda_\phi(x)U_{\partial_x^{-1}}\Lambda_\mu(b) &= \Lambda_\nu(xr(\partial_x^{-1}(bD_{\partial_x}^{\frac{1}{2}}))) \\ &= \Lambda_\nu(r(bD_{\partial_x}^{\frac{1}{2}})x) = \Lambda_\phi^\dagger(D^{\frac{1}{2}}(x))\Lambda_\mu(b), \\ \Lambda_\phi^\dagger(D^{\frac{1}{2}}(x))^*\Lambda_\phi^\dagger(D^{\frac{1}{2}}(y)) &= U_{\partial_x^{-1}}^*\Lambda_\phi(x)^*\Lambda_\phi(y)U_{\partial_y^{-1}} \\ &= U_{\partial_x}\pi_\mu(\phi(x^*y))U_{\partial_y^{-1}} = \pi_\mu(\partial_x(\phi(x^*y))). \end{aligned}$$

Lemma 3.3.1. *The sesquilinear forms on ${}_sA \otimes_B {}_rA$ and ${}_rA \otimes_B A_r$ defined in (3.1) are positive. Denote by $\overline{{}_sA \otimes_B {}_rA}$ and $\overline{{}_rA \otimes_B A_r}$ the respective separated completions. Then there exist isomorphisms*

$$\begin{aligned} \Lambda: \overline{{}_rA \otimes_B A_r} &\rightarrow H_{\alpha} \otimes_{\bar{\mu}} \widehat{\beta} H, \quad x \otimes y \mapsto \Lambda_\nu(x) \otimes_{\bar{\mu}} \Lambda_\nu(y), \\ \Lambda': \overline{{}_sA \otimes_B {}_rA} &\rightarrow H_{\beta} \otimes_{\bar{\mu}} \alpha H, \quad x \otimes y \mapsto \Lambda_\nu(x) \otimes_{\bar{\mu}} \Lambda_\nu(D^{\frac{1}{2}}(y)). \end{aligned}$$

Proof. The maps Λ, Λ' are surjective because $\Lambda_\nu(A) \subseteq H$ is dense, and they are well-defined and isometric because (3.8), (3.6) and (3.11) imply for all homogeneous $x, y \in A$

$$\begin{aligned} \langle \Lambda(x \otimes y) | \Lambda(x' \otimes y') \rangle &= \nu(x^*s(\phi(y^*y'))x'), \\ \langle \Lambda'(x \otimes y) | \Lambda'(x' \otimes y') \rangle &= \nu(x^*r(\partial_y(\phi(y^*y'))x')). \end{aligned} \quad \square$$

Proposition 3.3.2. *There exists a unitary $W: H_{\beta} \otimes_{\alpha} H \rightarrow H_{\alpha} \otimes_{\beta} H$ such that $W^* \circ \Lambda = \Lambda' \circ T_4$ as maps from ${}_r A \otimes_B A_r$ to $H_{\beta} \otimes_{\alpha} H$, that is, for all homogeneous $x, y \in A$,*

$$\begin{aligned} W^*(\Lambda_v(x) \otimes_{\tilde{\mu}} \Lambda_v(y)) &= \sum \Lambda_v(\bar{D}^{\frac{1}{2}}(y_{(1)})x) \otimes_{\tilde{\mu}} \Lambda_v(y_{(2)}) \\ &= \sum \Lambda_v(y_{(1)}x) \otimes_{\tilde{\mu}} \Lambda_v(D^{\frac{1}{2}}(y_{(2)})), \\ W(\Lambda_v(x) \otimes_{\tilde{\mu}} \Lambda_v(y)) &= \sum \Lambda_v(S^{-1}(D^{-\frac{1}{2}}(y_{(1)}))x) \otimes_{\tilde{\mu}} \Lambda_v(y_{(2)}) \\ &= \sum \Lambda_v(\bar{D}^{\frac{1}{2}}(S^{-1}(y_{(1)}))x) \otimes_{\tilde{\mu}} \Lambda_v(y_{(2)}). \end{aligned}$$

Proof. Calculation (3.2) and Lemma 3.3.1 imply that the map $\Lambda_v(x) \otimes_{\tilde{\mu}} \Lambda_v(y) \mapsto \sum \Lambda_v(y_{(1)}x) \otimes_{\tilde{\mu}} \Lambda_v(D^{\frac{1}{2}}(y_{(2)}))$ extends to an isometry $H_{\alpha} \otimes_{\beta} H \rightarrow H_{\beta} \otimes_{\alpha} H$. Bijectivity of this isometry and the formula for W follow from Proposition 2.3.8 and Lemma 2.6.3. \square

Similarly, the map T_1 yields a second fundamental unitary:

Proposition 3.3.3. *There exists a unitary $V: H_{\alpha} \otimes_{\beta} H \rightarrow H_{\beta} \otimes_{\alpha} H$ such that for all homogeneous $x, y \in A$,*

$$\begin{aligned} V(\Lambda_v(x) \otimes_{\tilde{\mu}} \Lambda_v(y)) &= \sum \Lambda_v(\bar{D}^{\frac{1}{2}}(x_{(1)})) \otimes_{\tilde{\mu}} \Lambda_v(x_{(2)}y) \\ &= \sum \Lambda_v(x_{(1)}) \otimes_{\tilde{\mu}} \Lambda_v(D^{\frac{1}{2}}(x_{(2)})y), \\ V^*(\Lambda_v(x) \otimes_{\tilde{\mu}} \Lambda_v(y)) &= \sum \Lambda_v(x_{(1)}) \otimes_{\tilde{\mu}} \Lambda_v(S(\bar{D}^{-\frac{1}{2}}(x_{(2)}))y) \\ &= \sum \Lambda_v(x_{(1)}) \otimes_{\tilde{\mu}} \Lambda_v(D^{\frac{1}{2}}(S(x_{(2)}))y). \end{aligned}$$

Proof. The formula above defines an isometry V . Indeed, (3.8), (3.6) and (3.12) imply

$$\begin{aligned} \sum \langle \Lambda_v(\bar{D}^{\frac{1}{2}}(x_{(1)})) \otimes_{\tilde{\mu}} \Lambda_v(x_{(2)}y) | \Lambda_v(\bar{D}^{\frac{1}{2}}(x'_{(1)})) \otimes_{\tilde{\mu}} \Lambda_v(x'_{(2)}y') \rangle_{(H_{\beta} \otimes_{\alpha} H)} \\ &= \sum v(y^* x'_{(2)} r(\bar{\partial}_{x_{(1)}}(\psi(x^*_{(1)} x'_{(1)}))) x'_{(2)} y') \\ &= \sum v(y^* x'_{(2)} x'_{(2)} r(\psi(x^*_{(1)} x'_{(1)})) y'), \\ \langle \Lambda_v(x) \otimes_{\tilde{\mu}} \Lambda_v(y) | \Lambda_v(x') \otimes_{\tilde{\mu}} \Lambda_v(y') \rangle_{(H_{\alpha} \otimes_{\beta} H)} &= v(y^* s(\psi(x^* x')) y') \end{aligned}$$

for all homogeneous $x, x', y, y' \in A$, where $x_{(1)}$ is assumed to be homogeneous without loss of generality, and by right-invariance of ψ (see Remark 2.5.2 i)), the expressions above coincide. Bijectivity of V and the inversion formula follow from Proposition 2.3.8. \square

3.4. Boundedness of the canonical representations. The first application of the fundamental unitary W is to show that left multiplication on A and right convolution by \hat{A} extend to representations on the Hilbert space H .

The proof of Theorem 3.4.2 involves operators and slice maps of the following form. For each $\xi \in D(H_\beta, \tilde{\mu})$ and $\eta \in D(H_\alpha, \tilde{\mu})$, there exist bounded linear operators

$$\lambda_\xi^{\beta, \alpha}: H \rightarrow H_\beta \otimes_{\tilde{\mu}} H, \quad \eta' \mapsto \xi \otimes_{\tilde{\mu}} \eta', \quad \rho_\eta^{\beta, \alpha}: H \rightarrow H_\beta \otimes_{\tilde{\mu}} H, \quad \xi' \mapsto \xi' \otimes_{\tilde{\mu}} \eta, \quad (3.13)$$

whose adjoints are given by

$$(\lambda_\xi^{\beta, \alpha})^*(\xi' \otimes \eta') = \alpha(\langle \xi | \xi' \rangle_{\beta, \tilde{\mu}}) \eta', \quad (\rho_\eta^{\beta, \alpha})^*(\xi' \otimes \eta') = \beta(\langle \eta | \eta' \rangle_{\alpha, \tilde{\mu}}) \xi'; \quad (3.14)$$

see also [4]. Likewise, there exist operators $\lambda_\xi^{\alpha, \beta}, \rho_\eta^{\alpha, \beta}: H \rightarrow H_\alpha \otimes_{\tilde{\mu}} H$ for all $\xi \in D(H_\alpha, \tilde{\mu})$ and $\eta \in D(H_\beta, \tilde{\mu})$ which are defined similarly. Using these operators, one defines slice maps

$$\begin{aligned} \omega_{\xi, \xi'} * \text{id}: \mathcal{L}(H_\alpha \otimes_{\tilde{\mu}} H, H_\beta \otimes_{\tilde{\mu}} H) &\rightarrow \mathcal{L}(H), \quad T \mapsto (\lambda_\xi^{\beta, \alpha})^* T \lambda_{\xi'}^{\alpha, \beta}, \\ \text{id} * \omega_{\eta, \eta'}: \mathcal{L}(H_\alpha \otimes_{\tilde{\mu}} H, H_\beta \otimes_{\tilde{\mu}} H) &\rightarrow \mathcal{L}(H), \quad T \mapsto (\rho_\eta^{\beta, \alpha})^* T \rho_{\eta'}^{\alpha, \beta} \end{aligned}$$

for all $\xi \in D(H_\beta, \tilde{\mu}), \xi' \in D(H_\alpha, \tilde{\mu}), \eta \in D(H_\alpha, \tilde{\mu}), \eta' \in D(H_\beta, \tilde{\mu})$; see [3].

Lemma 3.4.1. *Let $x, x', y, y' \in A$. Then*

$$\begin{aligned} (\text{id} * \omega_{\Lambda_v(y), \Lambda_v(y')})(W^*) \Lambda_v(x) &= \Lambda_v(ax), \quad \text{where } a = \sum \bar{D}^{-\frac{1}{2}}(y'_{(1)} s(\phi(y^* y'_{(2)}))), \\ (\omega_{\Lambda_v(x), \Lambda_v(x')} * \text{id})(W^*) \Lambda_v(y) &= \Lambda_v(y * \hat{c}), \quad \text{where } c = S^{-1}(\bar{D}^{\frac{1}{2}}(\theta^{-1}(x') x^*)). \end{aligned}$$

Proof. Without loss of generality, we may assume y to be homogeneous. Then

$$\begin{aligned}
& (\rho_{\Lambda_v(y)}^{\beta,\alpha})^* W^* \widehat{\rho}_{\Lambda_v(y')}^{\alpha,\beta} \Lambda_v(x) \\
&= \sum (\rho_{\Lambda_v(y)}^{\beta,\alpha})^* (\Lambda_v(y'_1)x \underset{\tilde{\mu}}{\otimes} \Lambda_v(D^{\frac{1}{2}}(y'_2))) \\
&= \sum \beta(\langle \Lambda_v(y) | \Lambda_v(D^{\frac{1}{2}}(y'_2)) \rangle_{\alpha, \tilde{\mu}}) \Lambda_v(y'_1)x \quad (\text{Eqn. (3.14)}) \\
&= \sum \Lambda_v(s(\partial_y(\phi(D^{-\frac{1}{2}}(y)^* y'_2)))y'_1)x) \quad (\text{Eqn. (3.11)}) \\
&= \sum \Lambda_v(y'_1 s(\phi(y^* D^{-\frac{1}{2}}(y'_2))))x) \quad (\text{Lem. 2.6.3}) \\
&= \sum \Lambda_v(\bar{D}^{-\frac{1}{2}}(y'_1) s(\phi(y^* y'_2)))x, \quad (\text{Lem. 2.6.3}),
\end{aligned}$$

$$\begin{aligned}
& (\lambda_{\Lambda_v(x)}^{\beta,\alpha})^* W^* \widehat{\lambda}_{\Lambda_v(x')}^{\alpha,\beta} \Lambda_v(y) \\
&= \sum (\lambda_{\Lambda_v(x)}^{\beta,\alpha})^* (\Lambda_v(\bar{D}^{\frac{1}{2}}(y_1)x' \underset{\tilde{\mu}}{\otimes} \Lambda_v(y_2)) \\
&= \sum \alpha(\langle \Lambda_v(x) | \Lambda_v(\bar{D}^{\frac{1}{2}}(y_1)x') \rangle_{\beta, \tilde{\mu}}) \Lambda_v(y_2) \quad (\text{Eqn. (3.14)}) \\
&= \sum \Lambda_v(r(\psi(\bar{D}^{\frac{1}{2}}(y_1)x' \theta(x^*)))y_2) \quad (\text{Eqn. (3.6)}) \\
&= \sum \Lambda_v(r(\psi(y_1) \bar{D}^{-\frac{1}{2}}(x' \theta(x^*))))y_2) \quad (\text{Lem. 2.6.3}) \\
&= \sum \Lambda_v(y_2 r(\psi(\bar{D}^{\frac{1}{2}}(\theta^{-1}(x')x^*)y_1))). \quad (\text{Eqn. (2.16)}) \square
\end{aligned}$$

Theorem 3.4.2. *Let $(A, \Delta, \epsilon, S, \mu, \phi, \psi)$ be a measured multiplier (B, Γ) -Hopf $*$ -algebroid such that μ admits a GNS-representation via bounded operators $(K, \Lambda_\mu, \pi_\mu)$ and its Radon–Nikodym cocycle has a positive square root in $M(B)$. Denote by $\Lambda_v: A \rightarrow \mathcal{L}(H)$ the GNS-map for $\nu = \mu \circ \phi = \mu \circ \psi$. Then there exist non-degenerate $*$ -homomorphisms $\pi_\nu: A \rightarrow \mathcal{L}(H)$ and $\rho: \hat{A} \rightarrow \mathcal{L}(H)$ such that*

$$\pi_\nu(x)\Lambda_\nu(y) = \Lambda_\nu(xy) \text{ and } \rho(\omega)\Lambda_\nu(y) = \Lambda_\nu(y * \omega) \quad (3.15)$$

for all $x, y \in A, \omega \in \hat{A}$.

Proof of Theorem 3.4.2. For elements a and c of the form in Lemma 3.4.1, the maps $\Lambda_\nu(y) \mapsto \Lambda_\nu(ay)$ and $\Lambda_\nu(x) \mapsto \Lambda_\nu(x * \hat{c})$ coincide with compositions of bounded operators and therefore are bounded. Since elements of this form span A , we obtain maps $\pi_\nu: A \rightarrow \mathcal{L}(H)$ and $\rho: \hat{A} \rightarrow \mathcal{L}(H)$ as in (3.15). Evidently, π_ν is a $*$ -homomorphism. The map ρ is multiplicative by (2.19) and Proposition 2.7.2, and

it is involutive because by (2.17) and Proposition 2.7.2,

$$\begin{aligned}
\langle \rho(\hat{x})^* \Lambda_v(z) | \Lambda_v(y) \rangle &= \langle \rho(\widehat{S(x)^*}) \Lambda_v(z) | \Lambda_v(y) \rangle \\
&= \sum \langle \Lambda_v(S(x)_{(1)}^* \psi(S(S(x)_{(2)}^*)z)) | \Lambda_v(y) \rangle \\
&= v(s(\psi(z^* x_{(1)})) S(x_{(2)}) y) \\
&= v(z^* x_{(1)} s(\psi(S(x_{(2)}) y))) = \langle \Lambda_v(z) | \rho(\hat{x}) \Lambda_v(y) \rangle.
\end{aligned}$$

Finally, π_v and ρ are non-degenerate because $\langle AA \rangle = A$ and $\langle A * \hat{A} \rangle = A$ (see (2.21)). \square

Remark 3.4.3. Lemma 3.4.1, Theorem 3.4.2 and self-adjointness of $\pi_v(A)$ and $\rho(\hat{A})$ imply

$$\begin{aligned}
\pi_v(A) &= \text{span}\{(\text{id} * \omega_{\Lambda_v(y), \Lambda_v(y')})(W^*) | y, y' \in A\} \\
&= \text{span}\{(\text{id} * \omega_{\Lambda_v(y), \Lambda_v(y')})(W) | y, y' \in A\}, \\
\rho(\hat{A}) &= \text{span}\{(\omega_{\Lambda_v(x), \Lambda_v(x')} * \text{id})(W^*) | x, x' \in A\} \\
&= \text{span}\{(\omega_{\Lambda_v(x), \Lambda_v(x')} * \text{id})(W) | x, x' \in A\}.
\end{aligned}$$

For later use, we calculate the slices of V , which are defined similarly as those of W^* .

Lemma 3.4.4. *Let $x, x', y, y' \in A$. Then*

$$\begin{aligned}
(\omega_{\Lambda_v(x), \Lambda_v(x')} * \text{id})(V) \Lambda_v(y) &= \Lambda_v(ay), \text{ where } a = \sum D^{-\frac{1}{2}}(x'_{(2)} r(\psi(x^* x'_{(1)}))), \\
(\text{id} * \omega_{\Lambda_v(y), \Lambda_v(y')})(V) \Lambda_v(x) &= \Lambda_v(\check{c} * x), \text{ where } c = S^{-1}(D^{-\frac{1}{2}}(y' \theta(y^*))).
\end{aligned}$$

Proof. Without loss of generality, we assume x to be homogeneous. Proceeding as in the proof of Lemma 3.4.1, we then find

$$\begin{aligned}
&(\lambda_{\Lambda_v(x)}^{\beta, \alpha})^* V \lambda_{\Lambda_v(x')}^{\alpha, \beta} \Lambda_v(y) \\
&= \sum (\lambda_{\Lambda_v(x)}^{\beta, \alpha})^* (\Lambda_v(\bar{D}^{\frac{1}{2}}(x'_{(1)})) \otimes_{\bar{\mu}} \Lambda_v(x'_{(2)} y)) \quad (\text{Def. of } V) \\
&= \sum \alpha(\langle \Lambda_v(x) | \Lambda_v(\bar{D}^{\frac{1}{2}}(x'_{(1)})) \rangle_{\beta, \bar{\mu}}) \Lambda_v(x'_{(2)} y) \\
&= \sum \Lambda_v(r(\bar{\partial}_x(\psi(\bar{D}^{-\frac{1}{2}}(x)^* x'_{(1)}))) x'_{(2)} y) \quad (\text{Eqn. (3.12)}) \\
&= \sum \Lambda_v(D^{-\frac{1}{2}}(x'_{(2)} r(\psi(x^* x'_{(1)}))) y),
\end{aligned}$$

$$\begin{aligned}
& (\rho_{\Lambda_v(y)}^{\beta,\alpha})^* V \widehat{\rho}_{\Lambda_v(y')}^{\alpha,\beta} \Lambda_v(x) \\
&= \sum (\rho_{\Lambda_v(y)}^{\beta,\alpha})^* (\Lambda_v(x_{(1)}) \underset{\bar{\mu}}{\otimes} \Lambda_v(D^{\frac{1}{2}}(x_{(2)})y')) \quad (\text{Def. of } V) \\
&= \sum \beta((\Lambda_v(y)|\Lambda_v(D^{\frac{1}{2}}(x_{(2)})y'))_{\alpha,\bar{\mu}}) \Lambda_v(x_{(1)}) \\
&= \sum \Lambda_v(s(\phi(D^{\frac{1}{2}}(x_{(2)})y'\theta(y^*)))x_{(1)}) \quad (\text{Eqn. (3.6)}) \\
&= \sum \Lambda_v(s(\phi(x_{(2)}D^{-\frac{1}{2}}(y'\theta(y^*))))x_{(1)}). \quad \square
\end{aligned}$$

3.5. The Hopf–von Neumann bimodules. We next show that the fundamental unitary W is pseudo-multiplicative in the sense of [27] and therefore yields two Hopf–von Neumann bimodules, which are completions of A and \hat{A} , respectively. First, we need further preliminaries.

The relative tensor product is functorial so that there exist bounded linear operators $S \underset{\bar{\mu}}{\otimes} T \in \mathcal{L}(H_{\beta} \underset{\bar{\mu}}{\otimes} H)$ for all $S \in \beta(N)', T \in \alpha(N)'$, as well as $S \underset{\bar{\mu}}{\otimes} T \in \mathcal{L}(H_{\alpha} \underset{\bar{\mu}}{\otimes} \widehat{H})$ for all $S \in \alpha(N)', T \in \widehat{\beta}(N)'$, both times given by $\xi \underset{\bar{\mu}}{\otimes} \eta \mapsto S \xi \underset{\bar{\mu}}{\otimes} T \eta$.

In particular, the commuting representations $\alpha, \beta, \widehat{\alpha}, \widehat{\beta}$ yield six representations $\alpha \underset{\bar{\mu}}{\otimes} \text{id}, \widehat{\alpha} \underset{\bar{\mu}}{\otimes} \text{id}, \widehat{\beta} \underset{\bar{\mu}}{\otimes} \text{id}, \text{id} \underset{\bar{\mu}}{\otimes} \beta, \text{id} \underset{\bar{\mu}}{\otimes} \widehat{\alpha}, \text{id} \underset{\bar{\mu}}{\otimes} \widehat{\beta}$ of N on $H_{\beta} \underset{\bar{\mu}}{\otimes} H$, and further six representations of N on $H_{\alpha} \underset{\bar{\mu}}{\otimes} \widehat{H}$.

Lemma 3.5.1. *The following relations hold for all $x \in N$:*

$$\begin{aligned}
W(\text{id} \underset{\bar{\mu}}{\otimes} \widehat{\beta}(x)) &= (\beta(x) \underset{\bar{\mu}}{\otimes} \text{id})W, & W(\widehat{\alpha}(x) \underset{\bar{\mu}}{\otimes} \text{id}) &= (\widehat{\alpha}(x) \underset{\bar{\mu}}{\otimes} \text{id})W, \\
W(\widehat{\beta}(x) \underset{\bar{\mu}}{\otimes} \text{id}) &= (\widehat{\beta}(x) \underset{\bar{\mu}}{\otimes} \text{id})W, & W(\alpha(x) \underset{\bar{\mu}}{\otimes} \text{id}) &= (\text{id} \underset{\bar{\mu}}{\otimes} \alpha(x))W, \\
W(\text{id} \underset{\bar{\mu}}{\otimes} \beta(x)) &= (\text{id} \underset{\bar{\mu}}{\otimes} \beta(x))W, & W(\text{id} \underset{\bar{\mu}}{\otimes} \widehat{\alpha}(x)) &= (\text{id} \underset{\bar{\mu}}{\otimes} \widehat{\alpha}(x))W.
\end{aligned}$$

Proof. This follows immediately from the fact that $\pi_{\mu}(B) \subseteq N$ is weakly dense, the definition of W , and the formulas for $\alpha, \beta, \widehat{\alpha}, \widehat{\beta}$ given in Lemma 3.2.2. \square

The relative tensor product is associative in a natural sense. The intertwining relations for W obtained above imply that all operators in the diagram below are well defined,

$$\begin{array}{ccc}
H_{\beta} \otimes_{\tilde{\mu}}^{\alpha} H_{\beta} \otimes_{\tilde{\mu}}^{\alpha} H & \xrightarrow{W_{12}} & H_{\alpha} \otimes_{\tilde{\mu}}^{\beta} H_{\beta} \otimes_{\tilde{\mu}}^{\alpha} H & \xrightarrow{W_{23}} & H_{\alpha} \otimes_{\tilde{\mu}}^{\beta} H_{\alpha} \otimes_{\tilde{\mu}}^{\beta} H, \\
\downarrow W_{23} & & & & \uparrow W_{12} \\
H_{\beta} \otimes_{\tilde{\mu}}^{\alpha} (\text{id} \otimes_{\tilde{\mu}}^{\alpha}) (H_{\alpha} \otimes_{\tilde{\mu}}^{\beta} H) & \xrightarrow{W_{13}} & (H_{\beta} \otimes_{\tilde{\mu}}^{\alpha} H)_{(\alpha \otimes \text{id})} \otimes_{\tilde{\mu}}^{\beta} H & &
\end{array} \tag{3.16}$$

where $W_{12} = W \otimes_{\tilde{\mu}} \text{id}$, $W_{23} = \text{id} \otimes_{\tilde{\mu}} W$, and W_{13} acts on the first and third tensor factor; see [27] for details.

Lemma 3.5.2. *Diagram (3.16) commutes, that is, $W_{23}W_{12} = W_{12}W_{13}W_{23}$.*

Proof. A short calculation shows that the adjoints of both compositions are given by

$$\begin{aligned}
& \Lambda_{\nu}(x) \otimes_{\tilde{\mu}} \Lambda_{\nu}(y) \otimes_{\tilde{\mu}} \Lambda_{\nu}(z) \\
& \mapsto \sum \Lambda_{\nu}(z_{(1)}y_{(1)}x) \otimes_{\tilde{\mu}} \Lambda_{\nu}(D^{\frac{1}{2}}(z_{(2)}y_{(2)})) \otimes_{\tilde{\mu}} \Lambda_{\nu}(D^{\frac{1}{2}}(z_{(3)})). \quad \square
\end{aligned}$$

Theorem 3.5.3. *Let $(A, \Delta, \epsilon, S, \mu, \phi, \psi)$ be a measured multiplier (B, Γ) -Hopf $*$ -algebroid such that μ admits a GNS-representation via bounded operators $(K, \Lambda_{\mu}, \pi_{\mu})$ and its Radon–Nikodym cocycle has a positive square root in $M(B)$. Let $\tilde{\mu}$ be the weight on $N = \pi_{\mu}(B)''$ associated to the Hilbert algebra $\Lambda_{\mu}(B)$, let $\Lambda_{\nu}: A \rightarrow \mathcal{L}(H)$ be the GNS-map for $\nu = \mu \circ \phi = \mu \circ \psi$, and define $\alpha, \beta, \hat{\beta}: N \rightarrow \mathcal{L}(H)$ as in (3.4). Then the unitaries $W: H_{\beta} \otimes_{\tilde{\mu}}^{\alpha} H \rightarrow H_{\alpha} \otimes_{\tilde{\mu}}^{\beta} H$ and $V: H_{\alpha} \otimes_{\tilde{\mu}}^{\beta} H \rightarrow H_{\beta} \otimes_{\tilde{\mu}}^{\alpha} H$ defined in Proposition 3.3.2 and 3.3.3 are pseudo-multiplicative in the sense of [27].*

Proof. The assertion on W is just Lemma 3.5.1 and Lemma 3.5.2. For V , the proof is similar. \square

Definition 3.5.4. Let $(A, \Delta, \epsilon, S, \mu, \phi, \psi)$, W and V as in Theorem 3.5.3. Then we call W and V the *left* and the *right pseudo-multiplicative unitary* of $(A, \Delta, \epsilon, S, \mu, \phi, \psi)$, respectively.

Recall from [26] that a *Hopf–von Neumann bimodule* over $(N, \tilde{\mu})$ is a von Neumann algebra M acting on a Hilbert space L together with faithful, non-degenerate, commuting normal representations $\gamma, \delta: N \rightarrow M$ and a non-degenerate, normal $*$ -homomorphism $\Delta_M: M \rightarrow M_{\delta} *_{\tilde{\mu}} M$ such that $\Delta_M \circ \gamma = \gamma \otimes_{\tilde{\mu}} \text{id}$, $\Delta_M \circ \delta = \text{id} \otimes_{\tilde{\mu}} \delta$ and $(\Delta_M * \text{id}) \circ \Delta_M = (\text{id} * \Delta_M)$, where $M_{\delta} *_{\tilde{\mu}} M = (M' \otimes_{\tilde{\mu}} M)' \subseteq \mathcal{L}(L_{\delta} \otimes_{\gamma} L)$, and $\Delta_M * \text{id}$ and $\text{id} * \Delta_M$ are suitably defined [16].

Lemma 3.5.5. *The following relations hold:*

$$\begin{aligned}\alpha(N) \cup \beta(N) &\subseteq \pi_\nu(A)'' \subseteq \widehat{\beta}(N)' \cap \widehat{\alpha}(N)', \\ \widehat{\beta}(N) \cup \alpha(N) &\subseteq \rho(\widehat{A})'' \subseteq \beta(N)' \cap \widehat{\alpha}(N)'\end{aligned}\quad (3.17)$$

and

$$\begin{aligned}\pi_\nu(A)' &= \{S \in \beta(N)' \cap \alpha(N)' \mid (S_\alpha \otimes_{\widehat{\beta}} 1)W = W(S_\beta \otimes_\alpha 1)\}, \\ \rho(\widehat{A})' &= \{T \in \alpha(N)' \cap \widehat{\beta}(N)' \mid (1_\alpha \otimes_{\widehat{\beta}} T)W = W(1_\beta \otimes_\alpha T)\}.\end{aligned}\quad (3.18)$$

Proof. The inclusions in (3.17) follow from Lemma 3.2.2, non-degeneracy of $\pi_\nu(A)$ and $\rho(\widehat{A})$ and equation (2.18). The equations (3.18) follow from (3.17) and Remark 3.4.3. \square

Using (3.17) and (3.18) and slightly abusing notation, we define faithful, normal, non-degenerate $*$ -homomorphisms

$$\begin{aligned}\Delta: \pi_\nu(A)'' &\rightarrow \mathcal{L}(H_\beta \otimes_{\widehat{\beta}} H), \quad x \mapsto W^*(\text{id} \otimes x)W, \\ \widehat{\Delta}: \rho(\widehat{A})'' &\rightarrow \mathcal{L}(H_\alpha \otimes_{\widehat{\beta}} H), \quad y \mapsto \Sigma W(y \otimes \text{id})W^* \Sigma.\end{aligned}\quad (3.19)$$

Theorem 3.5.6. *Let $\mathcal{A} = (A, \Delta, \epsilon, S, \mu, \phi, \psi)$ be a measured multiplier (B, Γ) -Hopf $*$ -algebroid such that μ admits a GNS-representation via bounded operators $(K, \Lambda_\mu, \pi_\mu)$ and the Radon–Nikodym cocycle of μ has a positive square root in $M(B)$. Let $\tilde{\mu}$ be the n.s.f. weight on $N = \pi_\mu(B)''$ associated to the Hilbert algebra $\Lambda_\mu(B)$ and let $\Lambda_\nu: A \rightarrow \mathcal{L}(H)$ be the GNS-map for $\nu = \mu \circ \phi = \mu \circ \psi$. Define $\pi_\nu: A \rightarrow \mathcal{L}(H)$, $\rho: \widehat{A} \rightarrow \mathcal{L}(H)$ as in (3.15), $\alpha, \beta, \widehat{\beta}: N \rightarrow \mathcal{L}(H)$ as in (3.4) and $\Delta, \widehat{\Delta}$ as in (3.19), where $W: H_\beta \otimes_{\widehat{\beta}} H \rightarrow H_\alpha \otimes_{\widehat{\beta}} H$ is the left pseudo-multiplicative unitary of \mathcal{A} . Then $(\pi_\nu(A)'', \alpha, \beta, \Delta)$ and $(\rho(\widehat{A})'', \widehat{\beta}, \alpha, \widehat{\Delta})$ are Hopf–von Neumann bimodules over $(N, \tilde{\mu})$.*

Proof. The tuples $(\pi_\nu(A)'', \alpha, \beta, \Delta)$ and $(\rho(\widehat{A})'', \widehat{\beta}, \alpha, \widehat{\Delta})$ are the Hopf–von Neumann bimodules associated to the pseudo-multiplicative unitary W . More precisely, the assertion follows from Proposition 10.3.10 and Theorem 10.3.11 in [23] and equation (3.18). \square

Definition 3.5.7. Let $(A, \Delta, \epsilon, S, \mu, \phi, \psi)$, $(\pi_\nu(A)'', \alpha, \beta, \Delta)$ and $(\rho(\widehat{A})'', \widehat{\beta}, \alpha, \widehat{\Delta})$ be as in Theorem 3.5.6. Then we call $(\pi_\nu(A)'', \alpha, \beta, \Delta)$ the *Hopf–von Neumann bimodule* of $(A, \Delta, \epsilon, S, \mu, \phi, \psi)$ and $(\rho(\widehat{A})'', \widehat{\beta}, \alpha, \widehat{\Delta})$ the *dual Hopf–von Neumann bimodule* of $(A, \Delta, \epsilon, S, \mu, \phi, \psi)$.

Theorem 3.5.6 above can also be deduced from the following explicit formulas for Δ and $\hat{\Delta}$:

Lemma 3.5.8. *For all $a, c, x, y \in A$,*

$$\begin{aligned}\Delta(\pi_v(a))(\Lambda_v(x) \otimes_{\tilde{\mu}} \Lambda_v(y)) &= \sum \Lambda_v(a_{(1)}x) \otimes_{\tilde{\mu}} \Lambda_v(D^{\frac{1}{2}}(a_{(2)})y), \\ \hat{\Delta}(\rho(\hat{c}))(\Lambda_v(x) \otimes_{\tilde{\mu}} \Lambda_v(y)) &= \sum \Lambda_v(x_{(2)}r(\psi(S(c)x_{(1)}y_{(1)}))) \otimes_{\tilde{\mu}} \Lambda_v(y_{(2)}),\end{aligned}$$

Proof. We calculate

$$\begin{aligned}\Delta(\pi_v(a)) \sum \Lambda_v(y_{(1)}x) \otimes_{\tilde{\mu}} \Lambda_v(D^{\frac{1}{2}}(y_{(2)})) \\ &= W^*(\text{id} \otimes_{\tilde{\mu}} \pi_v(a)) W W^*(\Lambda_v(x) \otimes_{\tilde{\mu}} \Lambda_v(y)) \\ &= W^*(\Lambda_v(x) \otimes_{\tilde{\mu}} \Lambda_v(ay)) \\ &= \sum \Lambda_v(a_{(1)}y_{(1)}x) \otimes_{\tilde{\mu}} \Lambda_v(D^{\frac{1}{2}}(a_{(2)}y_{(2)})),\end{aligned}$$

$$\begin{aligned}W^* \hat{\Delta}(\rho(\hat{c}))(\Lambda_v(x) \otimes_{\tilde{\mu}} \Lambda_v(y)) \\ &= (\rho(\hat{c}) \otimes_{\tilde{\mu}} \text{id}) W^*(\Lambda_v(x) \otimes_{\tilde{\mu}} \Lambda_v(y)) \\ &= \sum \rho(\hat{c}) \Lambda_v(y_{(1)}x) \otimes_{\tilde{\mu}} \Lambda_v(D^{\frac{1}{2}}(y_{(2)})) \\ &= \sum \Lambda_v(y_{(2)}x_{(2)}r(\psi(S(c)y_{(1)}x_{(1)}))) \otimes_{\tilde{\mu}} \Lambda_v(D^{\frac{1}{2}}(y_{(3)})) \\ &= W^* \sum \Lambda_v(x_{(2)}r(\psi(S(c)y_{(1)}x_{(1)}))) \otimes_{\tilde{\mu}} \Lambda_v(y_{(2)}). \quad \square\end{aligned}$$

Remark 3.5.9. Under the identification (3.9), for all homogeneous $a, x, y \in A$ and $\zeta \in K$

$$\Delta(\pi_v(a))(\Lambda_v(x) \otimes \zeta \otimes \Lambda_v(y)) = \sum \Lambda_v(a_{(1)}x) \otimes U_{\hat{\delta}_{a_{(1)}}} \zeta \otimes \Lambda_v(a_{(2)}y),$$

where $a_{(1)}$ is assumed to be homogeneous without loss of generality.

3.6. The Hopf C^* -bimodules. The fundamental unitary W is a regular C^* -pseudo-multiplicative unitary in the sense of [24], and therefore yields Hopf C^* -bimodules which are completions of A and \hat{A} . To prove this, we again need some preliminaries concerning the relative tensor product in the setting of C^* -algebras; for details, see [22] and [24]. The construction is parallel to the von Neumann-algebraic setting and differs mainly in notation.

As before, let $\mathfrak{b} = (K, [\pi_\mu(B)], [\pi_\mu(B)])$. The relative tensor product $H_{E_\psi^\dagger \otimes_{\mathfrak{b}} E_\phi^\dagger} H$ of the C^* - \mathfrak{b} -modules (H, E_ψ^\dagger) and (H, E_ϕ^\dagger) is the separated completion of the algebraic tensor product $E_\psi^\dagger \otimes K \otimes E_\phi^\dagger$ with respect to the sesquilinear form given by

$$\langle \xi \otimes \zeta \otimes \eta | \xi' \otimes \zeta' \otimes \eta' \rangle = \langle \zeta | (\xi^* \xi') (\eta^* \eta') \zeta' \rangle. \quad (3.20)$$

It can be regarded as a twofold internal tensor product of Hilbert C^* -modules and identified with certain separated completions $E_\psi^\dagger \otimes_\alpha H$ and $H_\beta \otimes E_\phi^\dagger$ of the algebraic tensor products $E_\psi^\dagger \otimes H$ and $H \otimes E_\phi^\dagger$, respectively, such that

$$E_\psi^\dagger \otimes_\alpha H \cong H_{E_\psi^\dagger \otimes_{\mathfrak{b}} E_\phi^\dagger} H \cong H_\beta \otimes E_\phi^\dagger, \quad \xi \otimes \eta \zeta \equiv \xi \otimes \zeta \otimes \eta \equiv \xi \zeta \otimes \eta. \quad (3.21)$$

Comparing the sesquilinear forms (3.7) with (3.20) and using (3.6), one finds that there exists an isomorphism

$$H_\beta \otimes_{\tilde{\mu}} H \cong H_{E_\psi^\dagger \otimes_{\mathfrak{b}} E_\phi^\dagger} H, \quad \Lambda_\nu(x) \otimes \zeta \otimes \Lambda_\nu(y) \equiv \Lambda_\psi^\dagger(x) \otimes \zeta \otimes \Lambda_\phi^\dagger(y). \quad (3.22)$$

For each $\xi \in E_\psi^\dagger$ and $\eta \in E_\phi^\dagger$, there exist bounded linear operators

$$|\xi\rangle_1: H \rightarrow H_{E_\psi^\dagger \otimes_{\mathfrak{b}} E_\phi^\dagger} H, \quad \eta' \mapsto \xi \otimes \eta', \quad |\eta\rangle_2: H \rightarrow H_{E_\psi^\dagger \otimes_{\mathfrak{b}} E_\phi^\dagger} H, \quad \xi' \mapsto \xi' \otimes \eta.$$

We denote their adjoints by $\langle \xi|_1$ and $\langle \eta|_2$, respectively, and write $|E_\psi^\dagger\rangle_1 = \{|\xi\rangle_1 : \xi \in E_\psi^\dagger\}$, $|E_\phi^\dagger\rangle_2 = \{|\eta\rangle_2 : \eta \in E_\phi^\dagger\}$ et cetera. Comparing with (3.13), we see that under the identification (3.22), $\lambda_{\Lambda_\nu(x)}^{\alpha, \beta} \equiv |\Lambda_\psi^\dagger(x)\rangle_1$ and $\rho_{\Lambda_\nu(y)}^{\alpha, \beta} \equiv |\Lambda_\phi^\dagger(y)\rangle_2$ for all $x, y \in A$.

Replacing E_ψ^\dagger and E_ϕ^\dagger by E_ϕ^\dagger and E_ψ^\dagger , respectively, one similarly defines the relative tensor product $H_{E_\phi^\dagger \otimes_{\mathfrak{b}} E_\psi^\dagger} H$ with a canonical isomorphism $H_{E_\phi^\dagger \otimes_{\mathfrak{b}} E_\psi^\dagger} H \cong H_\alpha \otimes_{\tilde{\beta}} H$, and operators $|\xi\rangle_1, |\eta\rangle_2: H \rightarrow H_{E_\phi^\dagger \otimes_{\mathfrak{b}} E_\psi^\dagger} H$ for all $\xi \in E_\phi^\dagger$ and $\eta \in E_\psi^\dagger$.

Thus, W can be regarded as a unitary $H_{E_\psi^\dagger \otimes_{\mathfrak{b}} E_\phi^\dagger} H \rightarrow H_{E_\phi^\dagger \otimes_{\mathfrak{b}} E_\psi^\dagger} H$.

Lemma 3.6.1. For all $x, x', y, y' \in A$ and $\gamma \in \{\alpha, \beta, \widehat{\beta}\}$, $\gamma' \in \{\alpha, \widehat{\alpha}, \beta\}$,

$$\begin{aligned} \Lambda_v(x) \otimes_{\tilde{\mu}} \Lambda_v(y) &\in D((H_\beta \otimes_{\tilde{\mu}} H)_{\text{id} \otimes \gamma, \tilde{\mu}}), \\ R_{\Lambda_v(x) \otimes_{\tilde{\mu}} \Lambda_v(y)}^{\text{id} \otimes \gamma, \tilde{\mu}} &= \lambda_{\Lambda_v(x)}^{\beta, \alpha} R_{\Lambda_v(y)}^{\gamma, \tilde{\mu}} = |\Lambda_\psi^\dagger(x)\rangle_1 R_{\Lambda_v(y)}^{\gamma, \tilde{\mu}}, \\ \Lambda_v(x') \otimes_{\tilde{\mu}} \Lambda_v(y') &\in D((H_\alpha \otimes_{\tilde{\mu}} \widehat{H})_{\gamma' \otimes \text{id}, \tilde{\mu}}), \\ R_{\Lambda_v(x') \otimes_{\tilde{\mu}} \Lambda_v(y')}^{\gamma' \otimes \text{id}, \tilde{\mu}} &= \rho_{\Lambda_v(y')}^{\alpha, \widehat{\beta}} R_{\Lambda_v(x')}^{\gamma', \tilde{\mu}} = |\Lambda_\phi(y')\rangle_2 R_{\Lambda_v(x')}^{\gamma', \tilde{\mu}}. \end{aligned}$$

Proposition 3.6.2. The following equations for subspaces of $\mathcal{L}(H, H_{E_\psi^\dagger \otimes_{\mathfrak{b}} E_\phi^\dagger} H)$ hold:

$$\begin{aligned} W[|E_\psi^\dagger\rangle_1 E_\phi] &= [|E_\phi\rangle_2 E_\psi^\dagger], & W[|E_\phi^\dagger\rangle_2 E_\psi] &= [|E_\phi\rangle_2 E_\psi], \\ W[|E_\phi^\dagger\rangle_2 E_\phi] &= [|E_\phi\rangle_2 E_\phi], & W[|E_\phi^\dagger\rangle_2 E_\phi^\dagger] &= [|E_\phi^\dagger\rangle_1 E_\phi^\dagger], \\ W[|E_\psi^\dagger\rangle_1 E_\psi] &= [|E_\phi^\dagger\rangle_1 E_\psi^\dagger], & W[|E_\psi^\dagger\rangle_1 E_\psi] &= [|E_\phi^\dagger\rangle_1 E_\psi]. \end{aligned}$$

Proof. We only prove the first equation; the others follow similarly:

$$\begin{aligned} W[|E_\psi^\dagger\rangle_1 E_\phi] &= [\{WR_\omega^{\text{id} \otimes \widehat{\beta}, \tilde{\mu}} : \omega \in \Lambda_v(A) \otimes_{\tilde{\mu}} \Lambda_v(A)\}] \quad (\text{Lem. 3.6.1 and (3.2.3)}) \\ &= [\{R_{W\omega}^{\beta \otimes \text{id}, \tilde{\mu}} : \omega \in \Lambda_v(A) \otimes_{\tilde{\mu}} \Lambda_v(A)\}] \quad (\text{Lem. 3.5.1}) \\ &= [\{R_{\omega'}^{\beta \otimes \text{id}, \tilde{\mu}} : \omega' \in \Lambda_v(A) \otimes_{\tilde{\mu}} \Lambda_v(A)\}] \quad (\text{Def. of } W) \\ &= [|E_\phi\rangle_2 E_\psi^\dagger]. \quad (\text{Lem. 3.6.1 and 3.2.3}) \quad \square \end{aligned}$$

Theorem 3.6.3. Let $\mathcal{A} = (A, \Delta, \epsilon, S, \mu, \phi, \psi)$ be a measured multiplier (B, Γ) -Hopf $*$ -algebroid such that μ admits a GNS-representation via bounded operators $(K, \Lambda_\mu, \pi_\mu)$ and the Radon–Nikodym cocycle of μ has a positive square root in $M(B)$. Let $\mathfrak{b} = (K, [\pi_\mu(B)], [\pi_\mu(B)])$, let $\Lambda_v: A \rightarrow \mathcal{L}(H)$ be the GNS-map of $v = \mu \circ \phi = \mu \circ \psi$ and define $E_\phi, E_\phi^\dagger, E_\psi, E_\psi^\dagger \subseteq \mathcal{L}(K, H)$ as in (3.3). Then the left and the right pseudo-multiplicative unitary W and V of \mathcal{A} , regarded as operators $H_{E_\psi^\dagger \otimes_{\mathfrak{b}} E_\phi^\dagger} H \rightarrow H_{E_\phi^\dagger \otimes_{\mathfrak{b}} E_\psi^\dagger} H$ and $H_{E_\psi \otimes_{\mathfrak{b}} E_\psi^\dagger} H \rightarrow H_{E_\psi^\dagger \otimes_{\mathfrak{b}} E_\phi^\dagger} H$ as above, are C^* -pseudo-multiplicative unitaries in the sense of [24].

Proof. The assertion on W is Proposition 3.6.2 and Lemma 3.5.2. For V , the proof is similar. \square

Proposition 3.6.4. *W and V are regular in the sense that $[(E_\phi^\dagger|_1 W|E_\phi^\dagger)_2] = [E_\phi^\dagger(E_\phi^\dagger)^*] \subseteq \mathcal{L}(H)$ and $[(E_\psi^\dagger|_1 V|E_\psi^\dagger)_2] = [E_\psi^\dagger(E_\psi^\dagger)^*] \subseteq \mathcal{L}(H)$.*

Proof. Let $x, x', y \in A$. Then $\Lambda_\phi^\dagger(y)\Lambda_\phi^\dagger(x)^*\Lambda_\nu(y') = \Lambda_\nu(r(\phi(y'\theta(x^*)))y)$ by Lemma 3.2.1 and

$$\begin{aligned}
& \langle \Lambda_\phi^\dagger(y)|_2 W^* | \Lambda_\phi^\dagger(x) \rangle_1 \Lambda_\nu(y') \\
&= (\rho_{\Lambda_\nu(y)}^{\beta, \alpha})^* W^* (\Lambda_\nu(x) \otimes_{\bar{\mu}} \Lambda_\nu(y')) \\
&= \sum \beta(\langle \Lambda_\nu(y) | \Lambda_\nu(D^{\frac{1}{2}}(y'_{(2)})) \rangle_{\alpha, \bar{\mu}}) \Lambda_\nu(y'_{(1)} x) \\
&= \sum \Lambda_\nu(s(\phi(D^{\frac{1}{2}}(y'_{(2)})\theta(y^*)))y'_{(1)} x) && \text{(Eqn. (3.6))} \\
&= \sum \Lambda_\nu(s(\phi(y'_{(2)} z))y'_{(1)} x) && \text{with } z := D^{-\frac{1}{2}}(\theta(y^*)) \\
&= \sum \Lambda_\nu(r(\phi(y' z_{(2)}))S^{-1}(z_{(1)} x)). && \text{(Prop. 2.5.3)}
\end{aligned}$$

Since the maps θ , $D^{-\frac{1}{2}}$, S and T_3 are bijections, we can conclude

$$[\{\Lambda_\phi^\dagger(y)\Lambda_\phi^\dagger(x)^* : x, y \in A\}] = [\{\Lambda_\phi^\dagger(x)|_2 W^* | \Lambda_\phi^\dagger(y) \rangle_1 : x, y \in A\}].$$

The assertion on V follows from a similar calculation. \square

Recall from [24] that a *Hopf C^* -bimodule* over \mathfrak{b} consists of a C^* - $(\mathfrak{b}, \mathfrak{b})$ -module (L, E, F) , a non-degenerate C^* -algebra $C \subseteq \mathcal{L}(L)$ satisfying $\rho_E(\pi_\mu(B)) \subseteq M(C)$ and $\rho_F(\pi_\mu(B)) \subseteq M(C)$, and a non-degenerate $*$ -homomorphism $\Delta_C: C \rightarrow C_F *_E C$ that is co-associative and compatible with E and F in a suitable sense, where

$$\begin{aligned}
C_F *_E C &= \{T \in \mathcal{L}(L_F \otimes_E L) : T|F)_1 + T^*|F)_1 \subseteq [(F)_1 C] \\
&\quad \text{and } T|E)_2 + T^*|E)_2 \subseteq [(E)_2 C]\}
\end{aligned}$$

is the *fiber product* of C with itself relative to F and E .

Theorem 3.6.5. *Let $\mathcal{A} = (A, \Delta, \epsilon, S, \mu, \phi, \psi)$ be a measured multiplier (B, Γ) -Hopf $*$ -algebroid such that μ admits a GNS-representation via bounded operators $(K, \Lambda_\mu, \pi_\mu)$ and its Radon–Nikodym cocycle has a positive square root in $M(B)$. Let $\mathfrak{b} = (K, [\pi_\mu(B)], [\pi_\mu(B)])$, let $\Lambda_\nu: A \rightarrow \mathcal{L}(H)$ be the GNS-map for $\nu = \mu \circ \phi = \mu \circ \psi$, let $(\pi_\nu(A)'', \alpha, \beta, \Delta)$ be the Hopf–von Neumann bimodule of \mathcal{A} , let $(\rho(\hat{A}''), \hat{\beta}, \alpha, \hat{\Delta})$ be the dual Hopf–von Neumann bimodule of \mathcal{A} , define $E_\phi, E_\phi^\dagger, E_\psi, E_\psi^\dagger \subseteq \mathcal{L}(K, H)$ as in (3.3) and regard Δ and $\hat{\Delta}$ as maps $\pi_\nu(A)'' \rightarrow \mathcal{L}(H_{E_\psi^\dagger \otimes_{\mathfrak{b}} E_\phi^\dagger} H)$ and $\rho(\hat{A})'' \rightarrow \mathcal{L}(H_{E_\phi^\dagger \otimes_{\mathfrak{b}} E_\psi^\dagger} H)$ as above. Then $((H, E_\phi^\dagger, E_\psi^\dagger), [\pi_\nu(A)], \Delta|_{[\pi_\nu(A)]})$ and $((H, E_\psi, E_\phi), [\rho(\hat{A})], \hat{\Delta}|_{[\rho(\hat{A})]})$ are Hopf C^* -bimodules over \mathfrak{b} .*

Proof. By Theorem 3.6.3 and Proposition 3.6.4, the left pseudo-multiplicative unitary W of \mathcal{A} , regarded as a unitary $H_{E_\psi} \otimes_b E_\phi H \rightarrow H_{E_\phi} \otimes_b E_\psi H$, is a regular C^* -pseudo-multiplicative unitary. By [24], the latter gives rise to two Hopf C^* -bimodules $((H, E_\phi, E_\psi), [\langle E_\phi |_2 W | E_\phi \rangle_2], \Delta)$ and $((H, E_\psi, E_\phi), [\langle E_\psi |_1 W | E_\psi \rangle_1], \hat{\Delta})$. Finally, by Lemma 3.4.1, $[\langle E_\phi |_1 W | E_\psi \rangle_1] = [\rho(\hat{A})]$ and $[\langle E_\phi |_2 W | E_\phi \rangle_2] = [\lambda(\hat{A})]$. \square

3.7. The measured quantum groupoid. To obtain a measured quantum groupoid, we finally extend ν, ϕ, ψ to normal, semi-finite, faithful weights on the level of von Neumann algebras. We impose the following simplifying assumptions:

(A3) the bimodule ${}_r A_s$ is *proper* in the sense that $r(B)s(B) \subseteq A$.

(A4) There exists a net $(u_i)_i$ in B that is *truncating for μ* in the sense that $(\pi_\mu(u_i))_i$ is a net of positive elements in the unit ball of $\pi_\mu(B)$ that converges in $M([\pi_\mu(B)])$ strictly to 1 and such that $(\pi_\mu(u_i^2))_i$ is increasing.

Note that a net $(u_i)_i$ as in (A4) exists always if we drop the condition that $(\pi_\mu(u_i^2))_i$ should be increasing.

Let us also note that in the bi-measured case where ϕ, ψ and ν arise from a bi-integral h on (A, Δ) , the extensions of ϕ, ψ, ν and the invariance of these extensions can be proved quite easily, see Remark 3.7.5 and 3.7.10.

For the extension of ν , we do not need the assumptions (A3) and (A4), but use the modular automorphism θ for ν obtained in Theorem 2.6.5, the theory of Hilbert algebras [20], and results of Kustermans and van Daele [11].

Lemma 3.7.1. $\Lambda_\nu(A) \subseteq H$ is a Hilbert algebra with respect to the $*$ -algebra structure inherited from A .

Proof. The multiplication $\Lambda_\nu(y) \mapsto \Lambda_\nu(xy)$ is bounded for each $x \in A$ by Theorem 3.4.2, and the involution $\Lambda_\nu(x) \mapsto \Lambda_\nu(x^*)$ is pre-closed because

$$\langle \Lambda_\nu(x) | \Lambda_\nu(y^*) \rangle = \nu(x^* y^*) = \nu(y^* \theta(x^*)) = \langle \Lambda_\nu(y) | \Lambda_\nu(\theta(x^*)) \rangle$$

for all $x, y \in A$. \square

The general theory of Hilbert algebras [20] now yields

- $M = \pi_\nu(A)'' \subseteq \mathcal{L}(H)$ as the associated von Neumann algebra,
- a n.s.f. weight $\tilde{\nu}$ on M such that $\tilde{\nu}(\pi_\nu(a^*a)) = \langle \Lambda_\nu(a) | \Lambda_\nu(a) \rangle = \nu(a^*a)$ for all $a \in A$,
- a left ideal $\mathfrak{N}_{\tilde{\nu}} := \{x \in M : \tilde{\nu}(x^*x) < \infty\} \subseteq M$ of square-integrable elements,
- a closed map $\Lambda_{\tilde{\nu}}: \mathfrak{N}_{\tilde{\nu}} \rightarrow H$ such that $(H, \Lambda_{\tilde{\nu}}, \text{id}_M)$ is a GNS-representation for $\tilde{\nu}$; this is the closure of the map $\pi_\nu(A) \rightarrow H$ given by $\pi_\nu(a) \rightarrow \Lambda_\nu(a)$;
- the usual objects $J_{\tilde{\nu}}, \Delta_{\tilde{\nu}}, \sigma_{\tilde{\nu}}, \mathcal{T}_{\tilde{\nu}}, \dots$ of Tomita-Takesaki theory.

The modular automorphism θ is related to the modular automorphism group $\sigma^{\tilde{\nu}}$ as follows:

Proposition 3.7.2. $\pi_\nu(A) \subseteq \mathcal{T}_{\tilde{\nu}}$ and $\sigma_{ni}^{\tilde{\nu}}(\pi_\nu(a)) = \pi_\nu(\theta^{-n}(a))$ for all $a \in A$, $n \in \mathbb{Z}$.

Proof. Use the arguments in [12, §3], in particular from Lemma 3.16 till Proposition 3.22. \square

Let $A^\theta := \{a \in A : \theta(a) = a\} \subseteq A$. Note that this space is a $*$ -subalgebra and, by (A3), contains $r(B)s(B)$.

Lemma 3.7.3.

- i) $\sigma^{\tilde{\nu}}$ acts trivially on $\pi_\nu(A^\theta)''$, in particular on $\alpha(N)$ and $\beta(N)$.
- ii) $J_{\tilde{\nu}}\alpha(x)^*J_{\tilde{\nu}} = \widehat{\beta}(x)$ and $J_{\tilde{\nu}}\beta(x)^*J_{\tilde{\nu}} = \widehat{\alpha}(x)$ for all $x \in N$.

Proof. i) The first assertion follows from the fact that $\sigma_t^{\tilde{\nu}}(x) = \Delta_{\tilde{\nu}}^{it}x\Delta_{\tilde{\nu}}^{-it}$ and $\Delta_{\tilde{\nu}}^{-1}x\Delta_{\tilde{\nu}} = x$ for each $x \in \pi_{\tilde{\nu}}(A^\theta)$ by Proposition 3.7.2, and the second assertion follows from the fact that $\sigma_t^{\tilde{\nu}}$ is normal for all $t \in \mathbb{R}$ and acts trivially on $\pi_\nu(r(B)s(B))$.

ii) Combine i) and Lemma 3.2.2. \square

Proposition 3.7.4. *There exist unique n.s.f. weights T_L from M to $\alpha(N)$ and T_R from M to $\beta(N)$ such that $\tilde{\mu} \circ \alpha^{-1} \circ T_L = \tilde{\nu} = \tilde{\mu} \circ \beta^{-1} \circ T_R$.*

Proof. This follows from Lemma 3.7.3 i) and [18, 10.1] or [20, IX Theorem 4.18]. \square

We thus obtain extensions $\tilde{\phi} := \alpha^{-1} \circ T_L$ and $\tilde{\psi} := \beta^{-1} \circ T_R$ of ϕ and ψ .

Remark 3.7.5. Assume that $\phi = (\text{id} \otimes \mu) \circ h$ and $\psi = (\mu \otimes \text{id}) \circ h$ for a normalized bi-integral h on (A, Δ) . Then the map $\Lambda_\mu(B) \otimes \Lambda_\mu(B) \rightarrow \Lambda_\nu(A)$ given by $\Lambda_\mu(b) \otimes \Lambda_\mu(b') \mapsto \Lambda_\nu(r(b)s(b'))$ extends to an isometry $\iota: K \otimes K \rightarrow H$, and a short calculation shows that $\iota^*\pi_\nu(a)\iota = (\pi_\mu \otimes \pi_\mu)(h(a))$ for all $a \in A$. We therefore get a positive, normal, linear extension $\tilde{h}: M \rightarrow N$, $x \mapsto \iota^*x\iota$, of h , and thereby the desired extensions $\tilde{\phi} = (\text{id} \otimes \tilde{\mu}) \circ \tilde{h}$, $\tilde{\psi} = (\tilde{\mu} \otimes \text{id}) \circ \tilde{h}$ and $\tilde{\nu} = (\tilde{\mu} \otimes \tilde{\mu}) \circ \tilde{h}$.

Recall that an element $\xi \in H$ is *right-bounded* with respect to the Hilbert algebra $\Lambda_\nu(A)$ if there exists an operator $R_\xi \in \mathcal{L}(H)$ such that $\pi_\nu(a)\xi = R_\xi\Lambda_\nu(a)$ for all $a \in A$. Note that then $R_\xi \in M'$.

Lemma 3.7.6.

- i) If $x \in A^\theta$, then $\Lambda_\nu(x) \in H$ is right-bounded, $R_{\Lambda_\nu(x)} = J_\nu\pi_\nu(x)^*J_\nu$ and $\|R_{\Lambda_\nu(x)}\| = \|\pi_\nu(x)\|$.
- ii) If $x \in A^\theta \cap r(B)'$, then $\pi_\nu(a)\Lambda_\phi(x) = R_{\Lambda_\nu(x)}\Lambda_\phi(a)$ for all $a \in A$.
- iii) If $a \in A$ and $\xi \in K$ is right-bounded with respect to $\Lambda_\mu(B)$, then $\Lambda_\phi(a)\xi = \widehat{\beta}(R_\xi)\Lambda_\nu(a)$.

Proof. i) For all $x \in A^\theta, a \in A$, we have $\pi_\nu(a)\Lambda_\nu(x) = \Lambda_\nu(ax) = J_\nu\pi_\nu(x)^*J_\nu\Lambda_\nu(a)$.

ii) For all $x \in A^\theta \cap r(B)', a \in A, b \in B$,

$$\begin{aligned} \pi_\nu(a)\Lambda_\phi(x)\Lambda_\mu(b) &= \Lambda_\nu(axr(b)) = \Lambda_\nu(ar(b)x) \\ &= \pi_\nu(ar(b))\Lambda_\nu(x) \\ &= R_{\Lambda_\nu(x)}\Lambda_\nu(ar(b)) = R_{\Lambda_\nu(x)}\Lambda_\phi(a)\Lambda_\mu(b). \end{aligned}$$

iii) If $a \in A$ and $\xi = \Lambda_\mu(b)$ for some $b \in B$, then $R_\xi = \pi_\mu(b)$ and $\Lambda_\phi(a)\xi = \Lambda_\nu(ar(b)) = \widehat{\beta}(\pi_\mu(b))\Lambda_\nu(a)$. Now, the assertion follows for all right-bounded ξ because $\Lambda_\mu(B)$ is a core for $\Lambda_{\tilde{\mu}}$ and the right-bounded elements coincide with $\Lambda_{\tilde{\mu}}(\mathfrak{N}_{\tilde{\mu}})$. \square

To prove Theorem 3.7.9, we construct increasing approximations of the weights $\tilde{\mu}, \tilde{\nu}, \tilde{\phi}, \tilde{\psi}$ by bounded positive maps, using an approximate unit $(u_i)_i$ in B with the properties assumed in (A4). Let $u_{i,j} := r(u_i)s(u_j) \in A$, and define for all i, j bounded, normal, positive, linear maps

$$\begin{aligned} \mu_i: N &\rightarrow \mathbb{C}, & x &\mapsto \langle \Lambda_\mu(u_i) | x \Lambda_\mu(u_i) \rangle, \\ \nu_{i,j}: M &\rightarrow \mathbb{C}, & x &\mapsto \langle \Lambda_\nu(u_{i,j}) | x \Lambda_\nu(u_{i,j}) \rangle, \\ \phi_{i,j}: M &\rightarrow N, & x &\mapsto \Lambda_\phi(u_{i,j})^* x \Lambda_\phi(u_{i,j}), \\ \psi_{i,j}: M &\rightarrow N, & x &\mapsto \Lambda_\psi(u_{i,j})^* x \Lambda_\psi(u_{i,j}). \end{aligned}$$

Given a net $(\lambda_\kappa)_\kappa$ of real numbers, we write $(\lambda_\kappa)_\kappa \nearrow \lambda$ if it is increasing and converges to λ . Likewise, given a von Neumann algebra C with a net $(\omega_\kappa)_\kappa$ in C_*^+ and a n.s.f. weight ω , we write $(\omega_\kappa)_\kappa \nearrow \omega$ if $\omega_\kappa(x^*x) \nearrow \omega(x^*x)$ for all $x \in C$.

Proposition 3.7.7.

- i) $(\mu_i)_i \nearrow \tilde{\mu}$ and $(\nu_{i,j})_{i,j} \nearrow \tilde{\nu}$;
- ii) $(\nu \circ \phi_{i,j})_{i,j} \nearrow \nu \circ \tilde{\phi}$ and $(\nu \circ \psi_{i,j})_{i,j} \nearrow \nu \circ \tilde{\psi}$ for all $\nu \in N_*^+$.

Proof. i) We only prove the assertion concerning \tilde{v} .

Let $\xi_{i,j} := \Lambda_v(u_{i,j})$ and $R_{i,j} := R_{\xi_{i,j}} = J_v \pi_v(u_{i,j}) J_v$ for all i, j .

The net $(v_{i,j})_{i,j}$ in M_*^+ is increasing because $(R_{i,j}^* R_{i,j})_{i,j}$ is increasing by assumption on $(u_i)_i$, $v_{i,j}(\pi_v(a^* a)) = \|R_{i,j} \Lambda_v(a)\|^2$ for all $a \in A$ and $\pi_v(A) \subseteq M$ is weakly dense.

Call $\xi \in H$ *right-contractive* if ξ is right-bounded and $\|R_\xi\| \leq 1$. Let $x \in M$. Then

$$\tilde{v}(x^* x) = \sup \{ \|x\xi\|^2 \mid \xi \in H \text{ is right-contractive} \}.$$

Each $\xi_{i,j}$ is right-contractive by Lemma 3.7.6 and hence $v_{i,j}(x^* x) = \|x \Lambda_v(u_{i,j})\|^2 \leq \tilde{v}(x^* x)$ for all i, j . Conversely, for each right-contractive $\xi \in H$,

$$\begin{aligned} \|x\xi\|^2 &= \lim_{i,j} \|x \pi_v(u_{i,j}) \xi\|^2 = \lim_{i,j} \|x R_\xi \Lambda_v(u_{i,j})\|^2 \\ &\leq \lim_{i,j} \|x \Lambda_v(u_{i,j})\|^2 = \lim_{i,j} v_{i,j}(x^* x) \end{aligned}$$

because $R_\xi \in M'$ and $R_\xi^* R_\xi \leq 1$. Therefore, $\tilde{v}(x^* x) \leq \lim_{i,j} v_{i,j}(x^* x)$.

ii) We only prove the assertion concerning $\tilde{\phi}$. A similar argument as above and Lemma 3.7.6 ii) show that for each $v \in N_*^+$, the net $(v \circ \phi_{i,j})_{i,j}$ is increasing. Taking pointwise limits, we obtain a normal semi-finite weight ω from M to N such that for each $y \in M$, the element $\omega(y^* y)$ in the extended positive part \hat{N}_+ is defined by $v(\omega(y)) = \sup_{i,j} v(\phi_{i,j}(y^* y))$ for all $v \in N_*^+$. Then for all $y \in M$,

$$\begin{aligned} \tilde{\mu}(\omega(y^* y)) &\stackrel{i,j,k}{\nearrow} \|y \Lambda_\phi(u_{i,j}) \Lambda_\mu(u_k)\|^2 = \|y \hat{\beta}(\pi_\mu(u_k)) \xi_{i,j}\|^2 \xrightarrow{k \rightarrow \infty} \|y \xi_{i,j}\|^2 \\ &= v_{i,j}(y^* y) \stackrel{i,j}{\nearrow} \tilde{v}(y^* y) \end{aligned}$$

and hence $\tilde{\mu} \circ \omega = \tilde{v}$. By [20, Theorem 4.18], $\omega = \tilde{\phi}$. \square

Lemma 3.7.8. $W^* \rho_{\Lambda_v(r(b)s(b'))}^{\alpha, \hat{\beta}} \beta(\pi_\mu(b'')) = \rho_{\Lambda_v(r(b'')s(b'))}^{\beta, \alpha} \alpha(\pi_\mu(b))$ for all $b, b', b'' \in B$.

Proof. Applying both sides to $\Lambda_v(a)$, where $a \in A$ is arbitrary, we obtain $W^*(\Lambda_v(s(b'')a) \otimes_{\tilde{\mu}} \Lambda_v(r(b)s(b')))$ and $\Lambda_v(r(b)a) \otimes_{\tilde{\mu}} \Lambda_v(r(b'')s(b'))$, respectively, which coincide. \square

As usual, let $\mathfrak{N}_{T_L} := \{x \in M : T_L(x^* x) \in N\}$ and similarly define \mathfrak{N}_{T_R} .

Theorem 3.7.9. Let $\mathcal{A} = (A, \Delta, \epsilon, S, \mu, \phi, \psi)$ be a measured multiplier (B, Γ) -Hopf $*$ -algebroid satisfying the following conditions:

(A1) μ admits a GNS-representation via bounded operators $(K, \Lambda_\mu, \pi_\mu)$,

(A2) the Radon–Nikodym cocycle of μ has a positive square root in $M(B)$,

(A3) the bimodule ${}_r A_s$ is proper,

(A4) there exists a truncating net for μ .

Let $\tilde{\mu}$ be the weight on $N = \pi_\mu(B)''$ associated to the Hilbert algebra $\Lambda_\mu(B)$, let $\Lambda_\nu: A \rightarrow \mathcal{L}(H)$ be the GNS-map for $\nu = \mu \circ \phi = \mu \circ \psi$, let $(\pi_\nu(A)'', \alpha, \beta, \Delta)$ be the Hopf–von Neumann bimodule of \mathcal{A} (Definition 3.5.7), let $\tilde{\nu}$ be the weight on $M = \pi_\nu(A)''$ associated to the Hilbert algebra $\Lambda_\nu(A)$, and let T_L and T_R be the n.s.f. weights from M to $\alpha(N)$ and $\beta(N)$ given by $\tilde{\mu} \circ \alpha^{-1} \circ T_L = \tilde{\nu} = \tilde{\mu} \circ \beta^{-1} \circ T_R$ (see Proposition 3.7.4). Then $(N, \tilde{\mu}, M, \alpha, \beta, \Delta, T_L, T_R, \tilde{\nu})$ is a measured quantum groupoid in the sense of [2]. In particular, T_L and T_R are left- and right-invariant with respect to Δ in the sense that

$$\begin{aligned} T_L((\lambda_\xi^{\beta, \alpha})^* \Delta(x^* x) \lambda_\xi^{\beta, \alpha}) &= \alpha((R_\xi^{\beta, \tilde{\mu}})^* T_L(x^* x) R_\xi^{\beta, \tilde{\mu}}) \\ &\quad \text{for all } x \in \mathfrak{N}_{T_L}, \xi \in D(H_\beta, \tilde{\mu}), \\ T_R((\rho_\eta^{\beta, \alpha})^* \Delta(x^* x) \rho_\eta^{\beta, \alpha}) &= \beta((R_\eta^{\alpha, \tilde{\mu}})^* T_R(x^* x) R_\eta^{\alpha, \tilde{\mu}}) \\ &\quad \text{for all } x \in \mathfrak{N}_{T_R}, \eta \in D(H_\alpha, \tilde{\mu}). \end{aligned}$$

Proof. We use the same notation as before. To prove the assertion concerning $\tilde{\phi}$ and T_L , we show that

$$\langle \tilde{\phi} | \tilde{\phi}((\lambda_\xi^{\beta, \alpha})^* \Delta(x^* x) \lambda_\xi^{\beta, \alpha}) \zeta \rangle = \|\alpha(\tilde{\phi}(x^* x))^{\frac{1}{2}} R_\xi^{\beta, \tilde{\mu}} \zeta\|^2 \quad (3.23)$$

for all $x \in \mathfrak{N}_{T_L}$, $\xi \in D(H_\beta, \tilde{\mu})$ and $\zeta \in K$. Given such x, ξ, ζ , let

$$\xi_k := \alpha(\pi_\mu(u_k)) \xi$$

and

$$c_{i,j,k} := \langle \zeta | \phi_{i,j}((\lambda_{\xi_k}^{\beta, \alpha})^* \Delta(x^* x) \lambda_{\xi_k}^{\beta, \alpha}) \zeta \rangle \quad \text{for all } i, j, k.$$

Then $R_{\xi_k}^{\beta, \tilde{\mu}} = \alpha(\pi_\mu(u_k)) R_\xi^{\beta, \tilde{\mu}}$, $\lambda_{\xi_k}^{\beta, \alpha} = (\alpha(\pi_\mu(u_k)) \otimes \text{id}) \lambda_\xi^{\beta, \alpha}$, and by Proposition 3.7.7,

$$c_{i,j,k} \xrightarrow{k \rightarrow \infty} \langle \zeta | \phi_{i,j}((\lambda_\xi^{\beta, \alpha})^* \Delta(x^* x) \lambda_\xi^{\beta, \alpha}) \zeta \rangle \nearrow_{i,j} \langle \tilde{\phi} | \tilde{\phi}((\lambda_\xi^{\beta, \alpha})^* \Delta(x^* x) \lambda_\xi^{\beta, \alpha}) \zeta \rangle.$$

On the other hand, using the relation $\Lambda_\phi(u_{i,j}) = \Lambda_\phi^\dagger(u_{i,j})$, we find

$$\begin{aligned} c_{i,j,k} &= \|(1 \otimes_\mu x) W \lambda_{\xi_k}^{\beta, \alpha} \Lambda_\phi(u_{i,j}) \zeta\|^2 && \text{(Def. of } \Delta_W \text{ and } \phi_{i,j}) \\ &= \|(1 \otimes_\mu x) W \rho_{\Lambda_\nu(u_{i,j})}^{\beta, \alpha} \alpha(\pi_\mu(u_k)) R_\xi^{\beta, \tilde{\mu}} \zeta\|^2 && \text{(Def. of } H_\beta \otimes_{\tilde{\mu}} H) \\ &= \|(1 \otimes_\mu x) \rho_{\Lambda_\nu(u_{k,j})}^{\widehat{\alpha}, \beta} \beta(\pi_\mu(u_i)) R_\eta^{\beta, \tilde{\mu}} \zeta\|^2 && \text{(Lem. 3.7.8)} \\ &= \|\widehat{\alpha}(\phi_{k,j}(x^* x))^{\frac{1}{2}} \beta(\pi_\mu(u_i)) R_\xi^{\beta, \tilde{\mu}} \zeta\|^2 \nearrow_{i,j,k} \|\widehat{\alpha}(\tilde{\phi}(x^* x))^{\frac{1}{2}} R_\xi^{\beta, \tilde{\mu}} \zeta\|^2. && \text{(Prop. 3.7.7)} \end{aligned}$$

Thus, (3.23) follows. The assertion concerning $\tilde{\psi}$ and T_R can be proven similarly, where W has to be replaced by the unitary V . \square

Remark 3.7.10. Assume that $\phi = (\text{id} \otimes \mu) \circ h$ for a normalized bi-integral h on (A, Δ) . Then for each $b \in B$, the map $\Lambda_\mu(B) \rightarrow \Lambda_\nu(A)$ given by $\Lambda_\mu(c) \mapsto \Lambda_\nu(s(b)r(c))$ is bounded with norm less than or equal to $\mu(b^*b)^{\frac{1}{2}}$, and therefore extends to an operator $\Lambda_\phi(s(b)) \in \mathcal{L}(K, H)$. One can then approximate $\tilde{\phi}$ monotonously by the maps $\phi_i: M \rightarrow N$, $x \mapsto \Lambda_\phi(s(u_i))^* x \Lambda_\phi(s(u_i))$, and a similar calculation as in Lemma 3.7.8 shows that each ϕ_i is right-invariant.

Associated to the measured quantum groupoid $(N, \tilde{\mu}, M, \alpha, \beta, \Delta, T_L, T_R, \tilde{\nu})$ are two fundamental unitaries $U'_H: H_{\tilde{\mu}} \hat{\otimes}_{\tilde{\mu}} H \rightarrow H_{\tilde{\mu}} \otimes_{\alpha} H$ and $U_H: H_{\tilde{\mu}} \hat{\otimes}_{\beta} H \rightarrow H_{\tilde{\mu}} \otimes_{\alpha} H$, characterized by

$$(\lambda_{w'}^{\beta, \alpha})^* U_H(v \otimes_{\tilde{\mu}} \Lambda_{\tilde{\nu}}(a)) = \Lambda_{\tilde{\nu}}((\omega_{w, v} * \text{id})(\Delta(a)))$$

$$\text{for all } v, w \in D(H_{\beta}, \tilde{\mu}), a \in \mathfrak{N}_{\tilde{\nu}} \cap \mathfrak{N}_{T_L},$$

$$(\rho_{w'}^{\beta, \alpha})^* U'_H(\Lambda_{\tilde{\nu}}(a') \otimes_{\tilde{\mu}} v') = \Lambda_{\tilde{\nu}}((\text{id} * \omega_{w', v'}) (\Delta(a'))),$$

$$\text{for all } v', w' \in D(H_{\alpha}, \tilde{\mu}), a' \in \mathfrak{N}_{\tilde{\nu}} \cap \mathfrak{N}_{T_R};$$

see [13, Proposition 3.17].

Proposition 3.7.11. $W^* = U_H$ and $V = U'_H$.

Proof. Let $x, y, y', z \in A$ and choose $v_i, w_i \in A$ such that $\sum \bar{D}^{\frac{1}{2}}(y_{(1)})x' \otimes_B y_{(2)} = \sum v_i \otimes_B w_i$ in ${}_s A \otimes_B {}_r A$. Then

$$\begin{aligned} (\omega_{\Lambda_\nu(x), \Lambda_\nu(x')} * \text{id})(W^*)\Lambda_\nu(y) &= \sum_i (\lambda_{\Lambda_\nu(x)}^{\beta, \alpha})^* (\Lambda_\nu(v_i) \otimes_{\tilde{\mu}} \Lambda_\nu(w_i)) \\ &= \sum_i \Lambda_\nu(r(\psi(v_i \theta(x^*)))w_i), \\ (\omega_{\Lambda_\nu(x), \Lambda_\nu(x')} * \text{id})(\Delta(y))\Lambda_\nu(z) &= \sum_i (\lambda_{\Lambda_\nu(x)}^{\beta, \alpha})^* (\Lambda_\nu(v_i) \otimes_{\tilde{\mu}} \Lambda_\nu(w_i z)) \\ &= \sum_i \pi_\nu(r(\psi(v_i \theta(x^*))))\Lambda_\nu(w_i z), \end{aligned}$$

and hence $(\omega_{\Lambda_\nu(x), \Lambda_\nu(x')} * \text{id})(W^*)\Lambda_\nu(y) = \Lambda_{\tilde{\nu}}((\omega_{\Lambda_\nu(x), \Lambda_\nu(x')} * \text{id})(\Delta(y)))$.

Likewise, with $v'_i, w'_i \in A$ such that $\sum \bar{D}^{\frac{1}{2}}(x_{(1)}) \otimes_B x_{(2)} y' = \sum v'_i \otimes_B w'_i \in {}_s A \otimes_B {}_r A$, we find

$$\begin{aligned} (\text{id} * \omega_{\Lambda_v(y), \Lambda_v(y')})(V) \Lambda_v(x) &= \sum_i (\rho_{\Lambda_v(y)}^{\beta, \alpha})^* (\Lambda_v(v'_i) \otimes_{\tilde{\mu}} \Lambda_v(w'_i)) \\ &= \sum_i \Lambda_v(s(\phi(w'_i \theta(y^*))) v'_i), \\ (\text{id} * \omega_{\Lambda_v(y'), \Lambda_v(y)})(\Delta(\pi_v(x))) \Lambda_v(z) &= \sum_i (\rho_{\Lambda_v(y)}^{\beta, \alpha})^* (\Lambda_v(v'_i z) \otimes_{\tilde{\mu}} \Lambda_v(w'_i)) \\ &= \sum_i \pi_v(s(w'_i \theta(y^*))) \Lambda_v(v'_i z) \end{aligned}$$

and hence $(\text{id} * \omega_{\Lambda_v(y), \Lambda_v(y')})(V) \Lambda_v(x) = \Lambda_{\tilde{v}}((\text{id} * \omega_{\Lambda_v(y'), \Lambda_v(y)})(\Delta(\pi_v(x))))$. \square

The adapted measured quantum groupoid $(N, \tilde{\mu}, M, \alpha, \beta, \Delta, T_L, T_R, \tilde{v})$ has an antipode \tilde{S} which is characterized by the following properties:

- i) $\text{span}\{(\omega_{w,v} * \text{id})(V) : w, v \in \mathcal{T}_{\tilde{v}, T_R}\}$ is a core for \tilde{S} ,
- ii) $\tilde{S}((\omega_{w,v} * \text{id})(V)) = (\omega_{w,v} * \text{id})(V^*)$ for all $w, v \in \mathcal{T}_{\tilde{v}, T_R}$,

where $\mathcal{T}_{\tilde{v}, T_R}$ is the set of all $x \in M$ that are analytic with respect to $\sigma^{\tilde{v}}$ and satisfy $\sigma_z^{\tilde{v}} \in \mathfrak{N}_{\tilde{v}} \cap \mathfrak{N}_{\tilde{v}}^* \cap \mathfrak{N}_{T_R} \cap \mathfrak{N}_{T_R}^*$ for all $z \in \mathbb{C}$. Likewise, one defines $\mathcal{T}_{\tilde{v}, T_L}$.

Lemma 3.7.12. $\pi_v(A) \subseteq \mathcal{T}_{\tilde{v}, T_R} \cap \mathcal{T}_{\tilde{v}, T_L}$.

Proof. Recall that $\pi_v(A) \subseteq \mathcal{T}_{\tilde{v}}$ by Proposition 3.7.2. Using Lemma 3.7.3 i), we find

$$\sigma_z^{\tilde{v}}(\pi_v(A)) = \sigma_z^{\tilde{v}}(\pi_v(As(B))) = \sigma_z^{\tilde{v}}(\pi_v(A))\beta(\pi_{\mu}(B)) \subseteq \mathfrak{N}_{\tilde{v}}\beta(\mathfrak{N}_{\tilde{\mu}}) \subseteq \mathfrak{N}_{T_R}$$

for all $z \in \mathbb{C}$. Consequently, $\pi_v(A) \subseteq \mathcal{T}_{\tilde{v}, T_R}$. A similar argument shows that $\pi_v(A) \subseteq \mathcal{T}_{\tilde{v}, T_L}$. \square

Proposition 3.7.13. $\pi_v(A) \subseteq \text{Dom}(\tilde{S})$ and $\tilde{S}(\pi_v(a)) = \pi_v(D^{\frac{1}{2}}SD^{\frac{1}{2}}(a))$ for all $a \in A$.

Proof. Let $x, x' \in A$ and $a = \sum D^{-\frac{1}{2}}(x'_{(2)}r(\psi(x^*x'_{(1)})))$. Then

$$\begin{aligned}
(\omega_{\Lambda_\nu(x), \Lambda_\nu(x')} * \text{id})(V) &= \pi_\nu(a), & (\text{Lem. 3.4.4}) \\
(\omega_{\Lambda_\nu(x), \Lambda_\nu(x')} * \text{id})(V^*) &= \left((\lambda_{\Lambda_\nu(x')}^{\beta, \alpha})^* V \widehat{\lambda}_{\Lambda_\nu(x)}^{\alpha, \beta} \right)^* \\
&= \sum \pi_\nu(D^{-\frac{1}{2}}(x_{(2)}r(\psi(x'^*x_{(1)}))))^* & (\text{Lem. 3.4.4}) \\
&= \sum \pi_\nu(D^{\frac{1}{2}}(r(\psi(x^*_{(1)}x'))x^*_{(2)})) \\
&= \sum \pi_\nu(D^{\frac{1}{2}}(S(x'_{(2)}r(\psi(x^*x'_{(1)})))))) & (\text{Prop. 2.5.3}) \\
&= \pi_\nu(D^{\frac{1}{2}}SD^{\frac{1}{2}}(a)). & \square
\end{aligned}$$

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