

Equivalence of two approaches to Yang–Mills on noncommutative torus

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Abstract. There are two notions of Yang–Mills action functional in noncommutative geometry. We show that for noncommutative n -torus both these notions agree. We also prove a structure theorem on the Hermitian structure of a finitely generated projective modules over spectrally invariant subalgebras of C^* -algebras.

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1. Introduction

There are two approaches to noncommutative geometry due to Alain Connes. In the first approach [3] one begins with $(\mathcal{A}, G, \alpha, \tau)$ a C^* -dynamical system along with an invariant trace. Moreover one also assumes that the dynamics is governed by a Lie group. In this setting Connes introduces the basic notions like Hermitian vector bundles, connections, curvature etc. and eventually along with Rieffel ([6]) he introduces the notion of Yang–Mills action functional as a certain functional $YM(\nabla)$ defined on the space of compatible connections $C(\mathcal{E})$ on a finitely generated projective \mathcal{A} module \mathcal{E} with a Hermitian structure. Critical points of this functional has been studied by Rieffel in ([11]). Variations of this concept have been studied in [9]. Later ([4]) Connes gave a spectral formulation of noncommutative geometry. In this formulation a noncommutative geometric space is described by a certain triple called spectral triple. This formulation is more popular today. In this setting as well he introduced the concept of compatible connections $\tilde{C}(\mathcal{E})$ and Yang–Mills action functional. There is a general recipe to produce a “spectral triple” from a C^* -dynamical system. Here we have put spectral triple with in quotation because the general recipe does not tell you that the resulting object is a true spectral triple but they are candidates and one has to verify the relevant conditions on a case by case basis. However for noncommutative torus, the prime test case in the subject it is easy to see that indeed one obtains a genuine spectral triple. Then one encounters

the natural question whether these two notions agree. This is important because even though the spectral approach is the popular one, whenever it comes to YM on noncommutative torus ([9],[5]) one goes back to the dynamical system path. Connes addressed the issue of their equivalence for noncommutative two torus. Proposition 13, in the last chapter of [4] states that the notions of compatible connections are the same in both the approaches and the concepts of Yang–Mills action functionals also agree up to a normalizing factor. Since the applications deal with higher dimensional cases, in this paper we take up those and show that even in these cases the notions of compatible connections are same in the sense that there is an affine isomorphism between the spaces $C(\mathcal{E})$ and $\tilde{C}(\mathcal{E})$ and Yang–Mills action functionals also agree up to a normalizing factor. Along the way we also prove a structural result on finitely generated projective modules with Hermitian structure over spectrally invariant subalgebras of C^* -algebras. The result is the following. If \mathcal{A} is a spectrally invariant subalgebra of a C^* -algebra A , that is $\mathcal{A} \subseteq A$ is a $*$ -subalgebra closed under holomorphic function calculus, and \mathcal{E} is a finitely generated projective \mathcal{A} module with a Hermitian structure, then there is a self adjoint projection $p \in M_n(\mathcal{A})$ such that $\mathcal{E} \cong p\mathcal{A}^n$ and \mathcal{E} has the induced Hermitian structure. Our proof makes crucial use of the hypothesis that \mathcal{A} is closed under holomorphic function calculus. We do not have any counter example but we believe it is necessary to assume that \mathcal{A} is spectrally invariant.

Organization of the paper is as follows. In section two we recall the definition of Yang–Mills action functional in the dynamical system approach and work out the expression for the noncommutative n -tori. In section three we show that finitely generated projective modules with a Hermitian structure over a spectrally invariant subalgebra of a C^* -algebra is always isomorphic with a submodule of a free module with induced Hermitian structure. Section four is devoted to the description of Yang–Mills in the frame work of spectral triples. Finally in the fifth section we explicitly work out the Yang–Mills functional for the noncommutative torus and show that it agrees with the first approach. This result is an higher dimensional analog of the corresponding result of Connes.

2. First approach to Yang–Mills functional

We briefly recall the setting of ([6]) for Yang–Mills functional on a C^* -dynamical system with an invariant, faithful trace. Let $(\mathcal{A}, G, \alpha, \tau)$ be one such, where \mathcal{A} is a unital C^* -algebra, G is a connected Lie group, $\alpha : G \rightarrow \text{Aut}(\mathcal{A})$, a homomorphism such that for all $a \in \mathcal{A}$, the map $g \mapsto \alpha_g(a)$ is continuous and τ is a G -invariant, faithful trace on \mathcal{A} . We say that $a \in \mathcal{A}$ is smooth if the map $g \mapsto \alpha_g(a)$ from G to the normed space \mathcal{A} is smooth. The involutive algebra $\mathcal{A}^\infty = \{a \in \mathcal{A} : a \text{ is smooth}\}$ is a norm dense subalgebra of \mathcal{A} , called the smooth subalgebra. Note that this is unital as well. One crucial property enjoyed by this

algebra is that it is closed under the holomorphic function calculus inherited from the ambient C^* -algebra \mathcal{A} .

Let \mathcal{E} be a finitely generated projective module over \mathcal{A} . Unless otherwise stated we will only consider right modules. We will say f.g.p module to mean finitely generated projective module. There exists a f.g.p \mathcal{A}^∞ -module \mathcal{E}^∞ , unique upto isomorphism, such that \mathcal{E} is isomorphic to $\mathcal{E}^\infty \otimes_{\mathcal{A}^\infty} \mathcal{A}$. Conversely if \mathcal{E}^∞ is a f.g.p \mathcal{A}^∞ -module then $\mathcal{E}^\infty \otimes_{\mathcal{A}^\infty} \mathcal{A}$ becomes a f.g.p module over \mathcal{A} . Since we shall never work with \mathcal{A} and \mathcal{E} but only with \mathcal{A}^∞ and \mathcal{E}^∞ , from now on, for notational simplicity, we denote the latter by \mathcal{A} and \mathcal{E} . Define \mathcal{E}^* as the space of \mathcal{A} linear mappings from \mathcal{E} to \mathcal{A} . Clearly \mathcal{E}^* is a right \mathcal{A} module.

Definition 2.1. A *Hermitian* structure on \mathcal{E} is an \mathcal{A} -valued positive-definite sesquilinear mapping $\langle \cdot, \cdot \rangle_{\mathcal{A}}$ such that,

- (a) $\langle \xi, \xi' \rangle_{\mathcal{A}}^* = \langle \xi', \xi \rangle_{\mathcal{A}}, \forall \xi, \xi' \in \mathcal{E}$.
- (b) $\langle \xi, \xi'.a \rangle_{\mathcal{A}} = (\langle \xi, \xi' \rangle_{\mathcal{A}}).a, \forall \xi, \xi' \in \mathcal{E}, \forall a \in \mathcal{A}$.
- (c) The map $\xi \mapsto \Phi_\xi$ from \mathcal{E} to \mathcal{E}^* , given by $\Phi_\xi(\eta) = \langle \xi, \eta \rangle_{\mathcal{A}}, \forall \eta \in \mathcal{E}$, gives conjugate linear \mathcal{A} -module isomorphism between \mathcal{E} and \mathcal{E}^* . This property will be referred as the self-duality of \mathcal{E} .

Any free \mathcal{A} -module $\mathcal{E}_0 = \mathcal{A}^q$ has a *Hermitian* structure, given by $\langle \xi, \eta \rangle_{\mathcal{A}} = \sum_{j=1}^q \xi_j^* \eta_j, \forall \xi = (\xi_1, \dots, \xi_q), \eta = (\eta_1, \dots, \eta_q) \in \mathcal{E}_0$. We refer this as the canonical *Hermitian* structure on \mathcal{A}^q . The following lemma shows that every f.g.p module admits a *Hermitian* structure.

Lemma 2.2. (a) A f.g.p module of the form $p\mathcal{A}^q$, where $p \in \mathcal{A} \otimes M_q(\mathbb{C})$ a projection, has a *Hermitian* structure.

(b) Every finitely generated projective module \mathcal{E} over \mathcal{A} is isomorphic as a f.g.p module with $p\mathcal{A}^q$ where p is a self-adjoint idempotent, that is a projection. Hence \mathcal{E} has a *Hermitian* structure on it.

Proof. (a) With respect to the canonical Hermitian structure $\langle p^*\xi, \eta \rangle_{\mathcal{A}} = \langle \xi, p\eta \rangle_{\mathcal{A}}$ holds for any $p \in M_q(\mathcal{A})$. Suppose $\mathcal{E} = p\mathcal{A}^q$ be a f.g.p module with p a projection in $M_q(\mathcal{A})$. The canonical structure $\langle \xi, \eta \rangle_{\mathcal{A}} = \sum \xi_j^* \eta_j$ on \mathcal{A}^q will induce a pairing on \mathcal{E} . We have to show that $\xi \mapsto \Phi_\xi$ gives an \mathcal{A} -module isomorphism between \mathcal{E} and \mathcal{E}^* . It is enough to check only the surjectivity of this map. In order to do so let's take an element $f \in \mathcal{E}^*$. Then $\tilde{f} = f \circ \pi$ is an element of $(\mathcal{A}^q)^*$ where $\pi : \mathcal{A}^q \rightarrow p\mathcal{A}^q$ is the map $\xi \mapsto p\xi$. By definition (2.1) there exists $\eta \in \mathcal{A}^q$ s.t $\tilde{f} = \Phi_\eta$. Consider any element $p\xi \in \mathcal{E}$ with $\xi \in \mathcal{A}^q$. Then,

$$\begin{aligned} f(p\xi) &= f \circ \pi(p\xi) = \tilde{f}(p\xi) = \langle \eta, p\xi \rangle_{\mathcal{A}} \\ &= \langle p^*\eta, p\xi \rangle_{\mathcal{A}} = \langle p\eta, p\xi \rangle_{\mathcal{A}} = \Phi_{p\eta}(p\xi). \end{aligned}$$

Hence $f = \Phi_{p\eta}$ with $\eta \in \mathcal{A}^q$. So the induced pairing on \mathcal{E} gives a *Hermitian* structure on it.

(b) Let \mathcal{E} be a f.g.p module over \mathcal{A} . There exists an \mathcal{A} -module \mathcal{F} such that $\mathcal{E} \oplus \mathcal{F} \cong \mathcal{A}^q$ for some natural number q . Once we fix such an \mathcal{F} we let $p : \mathcal{A}^q \rightarrow \mathcal{A}^q$ given by $p(e + f) = e$ for $e \in \mathcal{E}$ and $f \in \mathcal{F}$. So p is an idempotent in $M_q(\mathcal{A})$ with $\mathcal{E} = p\mathcal{A}^q$. By ([13],page 101) we see that in a C^* -algebra (or $*$ -subalgebra of a C^* -algebra which is stable under holomorphic function calculus) every idempotent is similar to a selfadjoint idempotent i.e., a projection and this similarity is witnessed by the invertible element $z = ((2p^* - 1)(2p - 1) + 1)^{1/2}$. Since \mathcal{A} is closed under holomorphic function calculus the invertible element z actually belongs to $M_q(\mathcal{A})$. Hence $\tilde{p} = zpz^{-1}$ is a projection in $M_q(\mathcal{A})$ and $\tilde{\mathcal{E}} = \tilde{p}\mathcal{A}^q \cong p\mathcal{A}^q = \mathcal{E}$. Then one restricts the *Hermitian* structure on \mathcal{A}^q to $\tilde{\mathcal{E}}$ and endows \mathcal{E} with the *Hermitian* structure obtained via the isomorphism between \mathcal{E} and $\tilde{\mathcal{E}}$. \square

Remark 2.3. The concept of *Hermitian* structure can be defined for f.g.p modules over involutive algebras and part (a) of Lemma (2.2) still holds. But part (b) requires the more finer property of closure under holomorphic function calculus.

Remark 2.4 (Open Question). We do not know whether a finitely generated projective module over an involutive algebra always admits Hermitian structure.

Let $Lie(G)$ be the Lie algebra of G . Then we have a representation δ of $Lie(G)$ into the Lie algebra $Der(\mathcal{A})$ of derivations on \mathcal{A} given by

$$\delta_X(a) = \left. \frac{d}{dt} \right|_{t=0} \alpha_{exp(tX)}(a). \quad (2.1)$$

where $exp : Lie(G) \rightarrow G$ is the exponential map.

Definition 2.5. Let \mathcal{E} be a f.g.p module over \mathcal{A} with a Hermitian Structure. A connection (on \mathcal{E}) is a \mathbb{C} -linear map $\nabla : \mathcal{E} \rightarrow \mathcal{E} \otimes Lie(G)^*$ such that, for all $X \in Lie(G)$ and $\xi \in \mathcal{E}, a \in \mathcal{A}$ one has

$$\nabla_X(\xi \cdot a) = \nabla_X(\xi) \cdot a + \xi \cdot \delta_X(a). \quad (2.2)$$

We shall say that ∇ is compatible with respect to the *Hermitian* structure on \mathcal{E} iff :

$$\langle \nabla_X \xi, \xi' \rangle_{\mathcal{A}} + \langle \xi, \nabla_X \xi' \rangle_{\mathcal{A}} = \delta_X(\langle \xi, \xi' \rangle_{\mathcal{A}}), \quad \forall \xi, \xi' \in \mathcal{E}, \forall X \in Lie(G).$$

As discussed in ([3]) compatible connection always exists. We will denote the set of compatible connections on \mathcal{E} by $C(\mathcal{E})$. The algebra $End(\mathcal{E})$ has a natural involution given by ,

$$\langle T^* \xi, \eta \rangle_{\mathcal{A}} = \langle \xi, T \eta \rangle_{\mathcal{A}} \quad \forall \xi, \eta \in \mathcal{E}, T \in End(\mathcal{E}).$$

For any two compatible connections ∇, ∇' it can be easily checked that $\nabla_X - \nabla'_X$ is a skew-adjoint element of $End(\mathcal{E})$ for each $X \in Lie(G)$. The curvature Θ_{∇} of a connection ∇ is the alternating bilinear $End(\mathcal{E})$ -valued form on $Lie(G)$ defined by,

$$\Theta_{\nabla} : \wedge^2(Lie(G)) \rightarrow End(\mathcal{E})$$

$$\Theta_{\nabla}(X \wedge Y) = [\nabla_X, \nabla_Y] - \nabla_{[X, Y]}, \quad \forall X, Y \in Lie(G).$$

This measures the extent to which ∇ fails to be a Lie algebra homomorphism. A simple calculation will assure that Θ_{∇} takes value in $End(\mathcal{E})$. Infact more can be said about the range of Θ_{∇} . We define $End(\mathcal{E})_{skew} = \{T \in End(\mathcal{E}) : T^* = -T\}$, the subset of skew-adjoint elements of $End(\mathcal{E})$.

Lemma 2.6. *Range of Θ_{∇} is contained in $End(\mathcal{E})_{skew}$.*

Proof. We have to show that $\langle \Theta_{\nabla}(X \wedge Y)(\xi), \eta \rangle_{\mathcal{A}} = -\langle \xi, \Theta_{\nabla}(X \wedge Y)(\eta) \rangle_{\mathcal{A}}$ for all $\xi, \eta \in \mathcal{E}$.

$$\begin{aligned} \langle \Theta_{\nabla}(X \wedge Y)(\xi), \eta \rangle_{\mathcal{A}} &= \langle ([\nabla_X, \nabla_Y] - \nabla_{[X, Y]})(\xi), \eta \rangle_{\mathcal{A}} \\ &= \langle \nabla_X(\nabla_Y(\xi)) - \nabla_Y(\nabla_X(\xi)) - \nabla_{[X, Y]}(\xi), \eta \rangle_{\mathcal{A}} \\ &= \langle \nabla_X(\nabla_Y(\xi)), \eta \rangle_{\mathcal{A}} - \langle \nabla_Y(\nabla_X(\xi)), \eta \rangle_{\mathcal{A}} - \langle \nabla_{[X, Y]}(\xi), \eta \rangle_{\mathcal{A}} \\ &= \delta_X(\langle \nabla_Y(\xi), \eta \rangle_{\mathcal{A}}) - \langle \nabla_Y(\xi), \nabla_X(\eta) \rangle_{\mathcal{A}} - \delta_Y(\langle \nabla_X(\xi), \eta \rangle_{\mathcal{A}}) \\ &\quad + \langle \nabla_X(\xi), \nabla_Y(\eta) \rangle_{\mathcal{A}} - \delta_{[X, Y]}(\langle \xi, \eta \rangle_{\mathcal{A}}) + \langle \xi, \nabla_{[X, Y]}(\eta) \rangle_{\mathcal{A}} \\ &= \delta_X(\delta_Y(\langle \xi, \eta \rangle_{\mathcal{A}}) - \langle \xi, \nabla_Y(\eta) \rangle_{\mathcal{A}}) - \langle \nabla_Y(\xi), \nabla_X(\eta) \rangle_{\mathcal{A}} \\ &\quad - \delta_Y(\delta_X(\langle \xi, \eta \rangle_{\mathcal{A}}) - \langle \xi, \nabla_X(\eta) \rangle_{\mathcal{A}}) + \langle \nabla_X(\xi), \nabla_Y(\eta) \rangle_{\mathcal{A}} \\ &\quad - \delta_{[X, Y]}(\langle \xi, \eta \rangle_{\mathcal{A}}) + \langle \xi, \nabla_{[X, Y]}(\eta) \rangle_{\mathcal{A}} \\ &= [\delta_X, \delta_Y](\langle \xi, \eta \rangle_{\mathcal{A}}) - \delta_{[X, Y]}(\langle \xi, \eta \rangle_{\mathcal{A}}) + \langle \xi, \nabla_{[X, Y]}(\eta) \rangle_{\mathcal{A}} \\ &\quad + \langle \nabla_X(\xi), \nabla_Y(\eta) \rangle_{\mathcal{A}} + \delta_Y(\langle \xi, \nabla_X(\eta) \rangle_{\mathcal{A}}) \\ &\quad - \langle \nabla_Y(\xi), \nabla_X(\eta) \rangle_{\mathcal{A}} - \delta_X(\langle \xi, \nabla_Y(\eta) \rangle_{\mathcal{A}}) \\ &= \langle \xi, \nabla_{[X, Y]}(\eta) \rangle_{\mathcal{A}} - \langle \xi, \nabla_X \nabla_Y(\eta) \rangle_{\mathcal{A}} + \langle \xi, \nabla_Y \nabla_X(\eta) \rangle_{\mathcal{A}} \\ &= \langle \xi, \nabla_{[X, Y]}(\eta) \rangle_{\mathcal{A}} - \langle \xi, [\nabla_X, \nabla_Y](\eta) \rangle_{\mathcal{A}} \\ &= -\langle \xi, ([\nabla_X, \nabla_Y] - \nabla_{[X, Y]})(\eta) \rangle_{\mathcal{A}} \\ &= -\langle \xi, \Theta_{\nabla}(X \wedge Y)(\eta) \rangle_{\mathcal{A}}. \quad \square \end{aligned}$$

We fix an inner product on $Lie(G)$ and this will remain fixed throughout. We next choose an orthonormal basis $\{Z_1, \dots, Z_n\}$ of $Lie(G)$. The bilinear form on the space of alternating 2-forms with values in $End(\mathcal{E})$ is given by,

$$\{\Phi, \Psi\}_{\mathcal{E}} = \sum_{i < j} \Phi(Z_i \wedge Z_j) \Psi(Z_i \wedge Z_j).$$

Recall that we have a G -invariant faithful trace τ on \mathcal{A} . We can extend it to a canonical faithful trace $\tilde{\tau}$ on $End(\mathcal{E})$ with the help of the following lemma from ([6]).

Lemma 2.7. *If \mathcal{E} is f.g.p \mathcal{A} -module with a Hermitian structure, then every element of $End(\mathcal{E})$ can be written as a linear combination of elements of the form $\langle \xi, \eta \rangle_{\mathcal{E}}$ for $\xi, \eta \in \mathcal{E}$, where $\langle \xi, \eta \rangle_{\mathcal{E}}(\zeta) = \xi(\eta, \zeta)_{\mathcal{A}}, \forall \zeta \in \mathcal{E}$.*

Proof. Let $\mathcal{E} = p\mathcal{A}^q$ where $p \in M_q(\mathcal{A})$ is an idempotent and $\{e_1, \dots, e_q\}$ be the standard basis for \mathcal{A}^q . For any given $T \in End(\mathcal{E})$ one can write $T = \bigoplus_{i=1}^q T_i$, where $T_i = \pi_i \circ T$, π_i denotes the projection onto the i -th component of \mathcal{A}^q . Then $T_i(\xi) = \langle \eta_i, \xi \rangle_{\mathcal{A}}$ for some $\eta_i \in \mathcal{E}$, which follows from self duality of \mathcal{E} . Then one can show directly that $T = \sum \langle pe_i, \eta_i \rangle_{\mathcal{E}}$. \square

Now, using this lemma, we define a linear functional $\tilde{\tau}$ on $End(\mathcal{E})$ as,

$$\tilde{\tau} : End(\mathcal{E}) \longrightarrow \mathbb{C}$$

$$\tilde{\tau}(\langle \xi, \eta \rangle_{\mathcal{E}}) = \tau(\langle \eta, \xi \rangle_{\mathcal{A}}).$$

Lemma 2.8. *$\tilde{\tau}$ defined above, is a trace on $End(\mathcal{E})$.*

Proof. One can easily check that $\langle \xi_1, \eta_1 \rangle_{\mathcal{E}} \langle \xi_2, \eta_2 \rangle_{\mathcal{E}} = \langle \xi_1 \langle \eta_1, \xi_2 \rangle_{\mathcal{A}}, \eta_2 \rangle_{\mathcal{E}}$. Now use the fact that τ is a trace on \mathcal{A} . \square

Moreover it can be shown that $\tilde{\tau}$ is faithful (see [6]). Finally, the Yang–Mills functional on $C(\mathcal{E})$ is given by,

$$YM(\nabla) = -\tilde{\tau}(\{\Theta_{\nabla}, \Theta_{\nabla}\}_{\mathcal{E}})$$

Notice that by Lemma (2.6) Θ_{∇} takes value in $End(\mathcal{E})_{skew}$. Hence this minus sign will force YM to take nonnegative real values.

Now we will deal with $\mathcal{A} = \mathcal{A}_{\Theta}$, the noncommutative n -torus. We recall non-commutative n -torus \mathcal{A}_{Θ} as defined in ([10]). Let Θ be a $n \times n$ real skew-symmetric matrix. Denote by \mathcal{A}_{Θ} , the universal C^* -algebra generated by n unitaries U_1, \dots, U_n satisfying $U_k U_m = e^{2\pi i \Theta_{km}} U_m U_k$, where $k, m \in \{1, \dots, n\}$. Throughout this paper i will stand for $\sqrt{-1}$. On the noncommutative n -torus \mathcal{A}_{Θ} , $G = \mathbb{T}^n$ (connected Lie group) acts as follows:

$$\alpha_{(z_1, \dots, z_n)}(U_k) = z_k U_k, k = 1, \dots, n.$$

The smooth subalgebra of \mathcal{A}_{Θ} , is given by

$$\mathcal{A}_{\Theta}^{\infty} := \left\{ \sum a_{\mathbf{r}} U^{\mathbf{r}} : \{a_{\mathbf{r}}\} \in \mathbb{S}(\mathbb{Z}^n), \mathbf{r} = (r_1, \dots, r_n) \in \mathbb{Z}^n \right\}$$

where $\mathbb{S}(\mathbb{Z}^n)$ denotes vector space of multisequences $(a_{\mathbf{r}})$ that decay faster than the inverse of any polynomial in $\mathbf{r} = (r_1, \dots, r_n)$.

This subalgebra (also \mathcal{A}_{Θ}) is equipped with a unique G -invariant *tracial state*, given by $\tau(a) = a_{\mathbf{0}}$, where $\mathbf{0} = (0, \dots, 0)$. One can further assume that the lattice Λ_{Θ} generated by columns of Θ is such that $\Lambda_{\Theta} + \mathbb{Z}^n$ is dense in \mathbb{R}^n . The advantage of choosing such a matrix Θ is that \mathcal{A}_{Θ} (hence $\mathcal{A}_{\Theta}^{\infty}$) becomes simple (see [7], Page 537). But in our case simpleness of $\mathcal{A}_{\Theta}^{\infty}$ is not needed and hence **we do not require any assumption on Θ except skew-symmetry**. The *Hilbert space* obtained by applying the G.N.S. construction to τ can be identified with $l^2(\mathbb{Z}^n)$ ([10]).

From now on we will work with $\mathcal{A}_{\Theta}^{\infty}$ only and hence for notational brevity we denote it by \mathcal{A}_{Θ} . In this case $\mathcal{L} = Lie(G)$ is \mathbb{R}^n . Let $\{v_1, v_2, \dots, v_n\}$ be the standard basis of \mathbb{R}^n and the associated derivations $\delta_{v_1}, \dots, \delta_{v_n}$. We will denote δ_{v_j} by $\tilde{\delta}_j$.

The derivations $\{\tilde{\delta}_1, \dots, \tilde{\delta}_n\}$ on \mathcal{A}_{Θ} are given by,

$$\tilde{\delta}_j(\sum_{\mathbf{r}} a_{\mathbf{r}} U^{\mathbf{r}}) = i \sum_{\mathbf{r}} r_j a_{\mathbf{r}} U^{\mathbf{r}} \tag{2.3}$$

It can be easily checked that these derivations commute and they are $*$ -derivations of \mathcal{A}_{Θ} i.e.

$$(\tilde{\delta}_j(a))^* = \tilde{\delta}_j(a^*) \quad ; \quad \tilde{\delta}_j(ab) = \tilde{\delta}_j(a)b + a\tilde{\delta}_j(b).$$

A connection is given by n maps $\nabla_{\tilde{\delta}_j} : \mathcal{E} \rightarrow \mathcal{E}$ such that $\nabla_{\tilde{\delta}_j}(\xi.a) = \nabla_{\tilde{\delta}_j}(\xi)a + \xi\tilde{\delta}_j(a)$. So the space of compatible connections ∇ consists of n -tuples of maps $(\nabla_{\tilde{\delta}_1}, \dots, \nabla_{\tilde{\delta}_n})$ such that,

$$\nabla(\xi) = \sum_{j=1}^n \nabla_{\tilde{\delta}_j}(\xi) \otimes e_j. \tag{2.4}$$

$$\langle \nabla_{\tilde{\delta}_j}(\xi), \eta \rangle_{\mathcal{A}_{\Theta}} + \langle \xi, \nabla_{\tilde{\delta}_j}(\eta) \rangle_{\mathcal{A}_{\Theta}} = \tilde{\delta}_j(\langle \xi, \eta \rangle_{\mathcal{A}_{\Theta}}). \tag{2.5}$$

Here $\{e_1, \dots, e_n\}$ denotes the basis dual to $\{v_1, \dots, v_n\}$ of the dual of the Lie algebra \mathbb{R}^n . The *curvature* of a *connection* ∇ is given by, $\Theta_{\nabla}(\tilde{\delta}_j \wedge \tilde{\delta}_k) = [\nabla_{\tilde{\delta}_j}, \nabla_{\tilde{\delta}_k}]$ because $[\tilde{\delta}_j, \tilde{\delta}_k] = 0$ in this case. We have $[\nabla_{\tilde{\delta}_j}, \nabla_{\tilde{\delta}_k}]^* = -[\nabla_{\tilde{\delta}_j}, \nabla_{\tilde{\delta}_k}]$ by Lemma (2.6). The bilinear form on space of alternating 2-forms with values in $End(\mathcal{E})$ becomes,

$$\{\Phi, \Psi\}_{\mathcal{E}} = \sum_{j < k} \Phi(\tilde{\delta}_j \wedge \tilde{\delta}_k) \Psi(\tilde{\delta}_j \wedge \tilde{\delta}_k).$$

Finally, the Yang–Mills functional of ∇ is given by,

$$\begin{aligned} YM(\nabla) = -\tilde{\tau}(\{\Theta_{\nabla}, \Theta_{\nabla}\}_{\mathcal{E}}) &= -\tilde{\tau}\left(\sum_{j < k} [\nabla_{\delta_j}^{\sim}, \nabla_{\delta_k}^{\sim}]^2\right) \\ &= \tilde{\tau}\left(\sum_{j < k} [\nabla_{\delta_j}^{\sim}, \nabla_{\delta_k}^{\sim}]^* [\nabla_{\delta_j}^{\sim}, \nabla_{\delta_k}^{\sim}]\right) \\ &\geq 0 \quad . \end{aligned}$$

For notational simplicity we write,

$$YM(\nabla) = \sum_{j < k} \tilde{\tau}([\nabla_j, \nabla_k]^* [\nabla_j, \nabla_k]) \quad . \quad (2.6)$$

3. Finitely generated projective modules with a Hermitian structure

It is almost by definition that a finitely generated projective module over an associative algebra can be embedded in a free module as a complemented submodule. However the situation is different for finitely generated projective modules with a Hermitian structure over involutive subalgebras of C^* -algebras. Let \mathcal{A} be an involutive subalgebra of a C^* -algebra A and \mathcal{E} be a finitely generated projective module over \mathcal{A} with a Hermitian structure. Note that any free \mathcal{A} module has a canonical Hermitian structure and one may ask does there exist an embedding $i : \mathcal{E} \rightarrow \mathcal{A}^n$ such that (i) there exists a submodule \mathcal{F} of \mathcal{A}^n with the property $i(\mathcal{E}) \oplus \mathcal{F} = \mathcal{A}^n$ and (ii) the Hermitian structure of \mathcal{E} is the one induced from \mathcal{A}^n . In this section we show that this is indeed the case provided \mathcal{A} is a subalgebra of a C^* -algebra and is closed under holomorphic function calculus. In this result the emphasis is on this condition which is often overlooked. In fact we do not know whether the result is true in general.

Let \mathcal{A} be a unital subalgebra of a C^* -algebra stable under holomorphic functional calculus and represented faithfully on a Hilbert space \mathcal{H} . Let \mathcal{E} be a f.g.p (right)module over \mathcal{A} equipped with a *Hermitian* structure on it. There is a right \mathcal{A} -module \mathcal{F} such that $\mathcal{E} \oplus \mathcal{F} \cong \mathcal{A}^q$ for some q . Since \mathcal{A}^q has a topology, \mathcal{E} inherits the topology from \mathcal{A}^q . Also \mathcal{E}^* inherits topology from \mathcal{A}^q because $\mathcal{E}^* \oplus \mathcal{F}^* \cong (\mathcal{A}^q)^* \cong \mathcal{A}^q$. As because we have topology now, we can expect the isomorphism between \mathcal{E} and \mathcal{E}^* to be topological, which turns out to be true by the following lemma.

Lemma 3.1. *If two finitely generated projective \mathcal{A} -modules \mathcal{E}_1 and \mathcal{E}_2 are algebraically isomorphic then they are also isomorphic as topological vector spaces.*

Proof. Since both the modules are projective, we can find \mathcal{F}_1 and \mathcal{F}_2 such that $\mathcal{E}_1 \oplus \mathcal{F}_1 \cong \mathcal{A}^k$ and $\mathcal{E}_2 \oplus \mathcal{F}_2 \cong \mathcal{A}^l$. Then, $\mathcal{E}_1 \oplus \mathcal{F}_1 \oplus \mathcal{A}^l \cong \mathcal{A}^{k+l}$ and $\mathcal{E}_2 \oplus \mathcal{F}_2 \oplus \mathcal{A}^k \cong \mathcal{A}^{k+l}$. Hence we can write $\mathcal{E}_1 = p_1 \mathcal{A}^{k+l}$ and $\mathcal{E}_2 = p_2 \mathcal{A}^{k+l}$, where $p_1, p_2 \in M_{k+l}(\mathcal{A})$ are idempotents. Let $u_j : \mathcal{A}^{k+l} \rightarrow \mathcal{E}_j$ denote the projection maps and $v_j : \mathcal{E}_j \rightarrow \mathcal{A}^{k+l}$ denote the inclusion maps for $j = 1, 2$. If we denote the isomorphism between \mathcal{E}_1 and \mathcal{E}_2 by ϕ then considering $f = v_2 \circ \phi \circ u_1$ and $g = v_1 \circ \phi^{-1} \circ u_2$ in $Hom_{\mathcal{A}}(\mathcal{A}^{k+l}, \mathcal{A}^{k+l})$, it is easily seen that $f \circ g = p_2$ and $g \circ f = p_1$. If we choose

$$\tilde{p}_1 = \begin{pmatrix} p_1 & 0 \\ 0 & 0 \end{pmatrix}, \quad \tilde{p}_2 = \begin{pmatrix} p_2 & 0 \\ 0 & 0 \end{pmatrix}$$

$$U = \begin{pmatrix} f & 1 - f \circ g \\ 1 - g \circ f & g \end{pmatrix}$$

then we see that $\tilde{p}_2 = U \tilde{p}_1 U^{-1}$. Since f, g both are \mathcal{A} -linear maps, U is also an \mathcal{A} -linear map from \mathcal{A}^q to \mathcal{A}^q where $q = 2(k + l)$. Since \mathcal{A} is unital, $U \in M_q(\mathcal{A})$. Hence U is bounded and induces a topological isomorphism between $\mathcal{E}_1, \mathcal{E}_2$. \square

Lemma 3.2. *All Hermitian structures on a free module over \mathcal{A} are isomorphic to each other.*

Proof. The canonical Hermitian structure on \mathcal{A}^q was given by $\langle \xi, \eta \rangle_{\mathcal{A}} = \sum_{k=1}^q \xi_k^* \eta_k$. We show that any other Hermitian structure is isomorphic to this one. Let $\langle \cdot, \cdot \rangle' : \mathcal{A}^q \times \mathcal{A}^q \rightarrow \mathcal{A}$ be another Hermitian structure on \mathcal{A}^q . Let $\{e_1, \dots, e_q\}$ be standard basis of \mathcal{A}^q . Let $T = ((t_{rs}))$ be given by $t_{sr} = \langle e_r, e_s \rangle'$. Then $\langle \xi, \eta \rangle' = \sum_{r,s} \langle e_r \xi_r, e_s \eta_s \rangle' = \sum_{r,s} \xi_r^* \langle e_r, e_s \rangle' \eta_s$. That is, $\langle \xi, \eta \rangle' = \xi^* T \eta$, where

$T \in M_q(\mathcal{A})$ is positive-definite. Hence T is a positive element in the C^* -algebra $M_q(\mathcal{B}(\mathcal{H}))$. Note that for $\xi \in \mathcal{A}^q$, $\xi^* = (\xi_1^*, \dots, \xi_q^*)$ where $\xi = (\xi_1, \dots, \xi_q)$. We consider elements of \mathcal{A}^q as column vector, whereas their $*$ will denote row vector. So here ξ^* is a row vector and ξ is a column vector. We denote $\langle \cdot, \cdot \rangle'$ by $\langle \cdot, \cdot \rangle_T$. Hence, Hermitian structures on \mathcal{A}^q are parametrized by such T . We show that T is one to one. Suppose $T\xi = 0$. Then for any $\eta \in \mathcal{E}$, we get $\Phi_{\xi}(\eta) = \xi^* T \eta = (T\xi)^* \eta = 0$, showing $\Phi_{\xi} = 0$. Since $\xi \mapsto \Phi_{\xi}$ is an isomorphism, we get $\xi = 0$. Hence T is one to one. To see T is onto, we pick any ζ from \mathcal{A}^q . Then $\eta \mapsto \zeta^* \eta$ is a \mathcal{A} -linear map on \mathcal{A}^q taking value in \mathcal{A} (we are dealing with right \mathcal{A} -module). Hence there exists ξ in \mathcal{A}^q such that,

$$\Phi_{\xi}(\eta) = \zeta^* \eta = \xi^* T \eta = (T\xi)^* \eta$$

Hence $\zeta = T\xi$, showing T is onto. We define

$$\tilde{T} : \mathcal{A}^q \rightarrow \mathcal{A}^q$$

$$\tilde{T}(T\xi) = \xi$$

To show this map is continuous, let $T\xi_k \rightarrow T\xi$ in \mathcal{A}^q . Then $\xi_k^* T\eta \rightarrow \xi^* T\eta$ for any η because multiplication is continuous with respect to the topology of \mathcal{A}^q . Hence $\Phi_{\xi_k} \rightarrow \Phi_\xi$. By Lemma (3.1) $\xi \mapsto \Phi_\xi$ is a continuous isomorphism. Hence we get $\xi_k \rightarrow \xi$, which shows continuity of \widetilde{T} . Thus T has a bounded inverse \widetilde{T} implying spectrum of T is away from zero. Since T is positive, \sqrt{T} is a holomorphic function of T . Now define,

$$\begin{aligned} \Psi : \mathcal{A}^q &\longrightarrow \mathcal{A}^q \\ \Psi(\xi) &= \sqrt{T}\xi \end{aligned}$$

Then, $\langle \xi, \eta \rangle_T = \xi^* T\eta = \xi^* \sqrt{T} \sqrt{T} \eta = \langle \Psi(\xi), \Psi(\eta) \rangle$. Since \mathcal{A} is stable under holomorphic functional calculus in $\mathcal{B}(\mathcal{H})$, inverse of T i.e. \widetilde{T} lies in $M_q(\mathcal{A}) \subseteq M_q(\mathcal{B}(\mathcal{H}))$ (see [12]). Invertibility of T in $M_q(\mathcal{A})$ gives invertibility of Ψ . So Ψ gives an isomorphism between the canonical Hermitian structure $\langle \cdot, \cdot \rangle_{\mathcal{A}}$ on \mathcal{A} and Hermitian structure obtained through T . Hence we are done. \square

Using this lemma we can conclude the following fact about Hermitian structures on a f.g.p module which is also important in our calculation of Yang–Mills.

Theorem 3.3. *Let \mathcal{E} be a f.g.p \mathcal{A} -module with a Hermitian structure. Then we can have a self-adjoint idempotent $p \in M_q(\mathcal{A})$ such that $\mathcal{E} = p\mathcal{A}^q$ and \mathcal{E} has the induced Hermitian structure.*

Proof. Let \mathcal{E} be a f.g.p \mathcal{A} -module with a Hermitian structure $\langle \cdot, \cdot \rangle_{\mathcal{E}}$. Because \mathcal{E} is projective, we can have an \mathcal{A} -module \mathcal{F} such that $\mathcal{E} \oplus \mathcal{F} \cong \mathcal{A}^q$ for some natural number q . Since \mathcal{F} is also f.g.p \mathcal{A} -module, by Lemma (2.2) \mathcal{F} has a Hermitian structure say $\langle \cdot, \cdot \rangle_{\mathcal{F}}$. Then $\mathcal{E} \oplus \mathcal{F}$ possesses a Hermitian structure $\langle \cdot, \cdot \rangle$ given by,

$$\langle (e_1, f_1), (e_2, f_2) \rangle = \langle e_1, e_2 \rangle_{\mathcal{E}} + \langle f_1, f_2 \rangle_{\mathcal{F}}.$$

i.e. we get a Hermitian structure on \mathcal{A}^q coming from \mathcal{E} and \mathcal{F} . By our previous lemma, this Hermitian structure is isomorphic with the canonical one. Note that \mathcal{E} is orthogonal to \mathcal{F} with respect to this Hermitian structure. Let p be a projection from \mathcal{A}^q onto \mathcal{E} , i.e. $p(e + f) = e$. Then $\mathcal{E} = p\mathcal{A}^q$. Now,

$$\begin{aligned} \langle p(e_1 + f_1), (e_2 + f_2) \rangle &= \langle e_1, e_2 \rangle_E + \langle e_1, f_2 \rangle_F \\ &= \langle e_1, e_2 \rangle_E \\ &= \langle e_1, p(e_2 + f_2) \rangle_E \\ &= \langle e_1 + f_1, p(e_2 + f_2) \rangle. \end{aligned}$$

which shows that p is self-adjoint. Once we have a self-adjoint p , we can now restrict the Hermitian structure on \mathcal{A}^q to \mathcal{E} (recall proof of part (a) of Lemma 2.2) and hence \mathcal{E} has the induced Hermitian structure. \square

4. Second approach to Yang–Mills

We first recall the differential graded algebra from [4].

Definition 4.1. A spectral triple $(\mathcal{A}, \mathcal{H}, D)$, over an algebra \mathcal{A} with involution $*$, consists of the following things :

- (1) a $*$ -representation of \mathcal{A} on a Hilbert space \mathcal{H} .
- (2) an unbounded selfadjoint operator D .
- (3) D has compact resolvent and $[D, a]$ extends to a bounded operator on \mathcal{H} for every $a \in \mathcal{A}$.

We shall assume that \mathcal{A} is unital and the unit $1 \in \mathcal{A}$ acts as the identity on \mathcal{H} . If $|D|^{-d}$ is in the ideal of Dixmier traceable operators $\mathcal{L}^{(1, \infty)}$ then we say that the spectral triple is (d, ∞) -summable.

Let $\Omega^\bullet(\mathcal{A}) = \bigoplus_{k=0}^\infty \Omega^k(\mathcal{A})$ be the reduced universal differential graded algebra over \mathcal{A} . It is by definition equal to \mathcal{A} in degree 0 and is generated by symbols da ($a \in \mathcal{A}$) of degree 1 with the following presentation :

- (1) $d(ab) = (da)b + adb$ ($\forall a, b \in \mathcal{A}$)
- (2) $d1 = 0$.

The involution $*$ of \mathcal{A} extends uniquely to an involution on $\Omega^\bullet(\mathcal{A})$ with the rule $(da)^* = -da^*$. The differential d on $\Omega^\bullet(\mathcal{A})$ is defined unambiguously by

$$d(a_0 da_1 \dots da_k) = da_0 da_1 \dots da_k \quad \forall a_j \in \mathcal{A},$$

and it satisfies the relations

- (1) $d^2\omega = 0 \quad \forall \omega \in \Omega^\bullet(\mathcal{A})$,
- (2) $d(\omega_1 \omega_2) = (d\omega_1)\omega_2 + (-1)^{deg(\omega_1)} \omega_1 d\omega_2, \quad \forall \omega_j \in \Omega^\bullet(\mathcal{A})$.

For $\omega \in \Omega^k(\mathcal{A})$, $deg(\omega) = 0, +1$ or -1 accordingly as whether k is zero, even or odd. We get a $*$ -representation π of $\Omega^\bullet(\mathcal{A})$ on \mathcal{H} by,

$$\pi(a_0 da_1 \dots da_k) = a_0 [D, a_1] \dots [D, a_k]; \quad a_j \in \mathcal{A}.$$

Let $J_0^{(k)} = \{\omega \in \Omega^k : \pi(\omega) = 0\}$ and $J' = \bigoplus J_0^{(k)}$. Since J' fails to be a differential graded ideal, the quotient Ω^\bullet/J' is not a differential graded algebra. This problem can be overcome by letting $J^\bullet = \bigoplus J^{(k)}$ where $J^{(k)} = J_0^{(k)} + dJ_0^{(k-1)}$. Then J^\bullet becomes a differential graded two-sided ideal and the quotient $\Omega_D^\bullet = \Omega^\bullet/J^\bullet$ becomes a differential graded algebra.

The representation π gives an isomorphism,

$$\Omega_D^k \cong \pi(\Omega^k) / \pi(dJ_0^{k-1}).$$

Arbitrary element of Ω_D^k can be viewed as a class of elements

$$\rho = \sum_j a_0^j [D, a_1^j] \cdots [D, a_k^j]$$

modulo the sub-bimodule of elements of the form,

$$\sum_j [D, b_0^j][D, b_1^j] \cdots [D, b_{k-1}^j] : b_r^j \in \mathcal{A}; \quad \sum_j b_0^j [D, b_1^j] \cdots [D, b_{k-1}^j] = 0 .$$

The abstract differential d induces a differential \tilde{d} on the complex $\Omega_D^\bullet(\mathcal{A})$ so that we get a chain complex $(\Omega_D^\bullet(\mathcal{A}), \tilde{d})$ and a chain map $\pi_D : \Omega^\bullet(\mathcal{A}) \rightarrow \Omega_D^\bullet(\mathcal{A})$ such that the following diagram

$$\begin{array}{ccc} \Omega^\bullet(\mathcal{A}) & \xrightarrow{\pi_D} & \Omega_D^\bullet(\mathcal{A}) \\ d \downarrow & & \downarrow \tilde{d} \\ \Omega^{\bullet+1}(\mathcal{A}) & \xrightarrow{\pi_D} & \Omega_D^{\bullet+1}(\mathcal{A}) \end{array}$$

commutes.

Suppose we are given a unital $*$ -algebra \mathcal{A} and a (d, ∞) -summable spectral triple $(\mathcal{A}, \mathcal{H}, D)$ over \mathcal{A} . We also assume \mathcal{A} is closed under holomorphic function calculus so that theorem (3.3) applies.

Definition 4.2. Let \mathcal{E} be a Hermitian, f.g.p module over \mathcal{A} . A compatible connection on \mathcal{E} is a \mathbb{C} -linear mapping $\nabla : \mathcal{E} \rightarrow \mathcal{E} \otimes_{\mathcal{A}} \Omega_D^1$ such that,

- (a) $\nabla(\xi a) = (\nabla \xi)a + \xi \otimes da, \quad \forall \xi \in \mathcal{E}, a \in \mathcal{A};$
- (b) $\langle \xi, \nabla \eta \rangle - \langle \nabla \xi, \eta \rangle = d \langle \xi, \eta \rangle_{\mathcal{A}} \quad \forall \xi, \eta \in \mathcal{E}$ (Compatibility).

The meaning of the last equality in Ω_D^1 is, if $\nabla(\xi) = \sum \xi_j \otimes \omega_j$, with $\xi_j \in \mathcal{E}, \omega_j \in \Omega_D^1(\mathcal{A})$, then $\langle \nabla \xi, \eta \rangle = \sum \omega_j^* \langle \xi_j, \eta \rangle_{\mathcal{A}}$. Any f.g.p right module has a connection. An example of a compatible connection is the Grassmannian connection ∇_0 on $\mathcal{E} = p\mathcal{A}^q$, given by $\nabla_0(\xi) = pd\xi$, where $d\xi = (d\xi_1, \dots, d\xi_q)$. This connection is compatible with the *Hermitian* structure,

$$\langle \xi, \eta \rangle_{\mathcal{A}} = \sum_{k=1}^q \xi_k^* \eta_k, \quad \forall \xi, \eta \in p\mathcal{A}^q.$$

Also, any two compatible connections can only differ by an element of $Hom_{\mathcal{A}}(\mathcal{E}, \mathcal{E} \otimes_{\mathcal{A}} \Omega_D^1(\mathcal{A}))$. That is, the space of all compatible connections on \mathcal{E} , which we denote by $\tilde{C}(\mathcal{E})$, is an affine space with associated vector space $Hom_{\mathcal{A}}(\mathcal{E}, \mathcal{E} \otimes_{\mathcal{A}} \Omega_D^1(\mathcal{A}))$. The connection ∇ extends to a unique linear map $\tilde{\nabla}$ from $\mathcal{E} \otimes \Omega_D^1$ to $\mathcal{E} \otimes \Omega_D^2$ such that,

$$\tilde{\nabla}(\xi \otimes \omega) = (\nabla \xi)\omega + \xi \otimes \tilde{d}\omega, \quad \forall \xi \in \mathcal{E}, \omega \in \Omega_D^1.$$

It can be easily checked that $\widetilde{\nabla}$, defined above, satisfies the Leibniz rule, i.e.

$$\widetilde{\nabla}(\eta a) = \widetilde{\nabla}(\eta)a - \eta \tilde{d}a, \quad \forall a \in \mathcal{A}, \eta \in \mathcal{E} \otimes \Omega_D^1.$$

A simple calculation shows that $\Theta = \widetilde{\nabla} \circ \nabla$ is an element of $Hom_{\mathcal{A}}(\mathcal{E}, \mathcal{E} \otimes_{\mathcal{A}} \Omega_D^2)$. Our next goal is to define an inner-product on $Hom_{\mathcal{A}}(\mathcal{E}, \mathcal{E} \otimes_{\mathcal{A}} \Omega_D^2)$. In order to do so, recall that $\Omega_D^2 \cong \pi(\Omega^2)/\pi(dJ_0^{(1)})$. Let \mathcal{H}_2 be the Hilbert space completion of $\pi(\Omega^2)$ with the inner-product

$$\langle T_1, T_2 \rangle = Tr_{\omega}(T_1^* T_2 |D|^{-d}), \quad \forall T_1, T_2 \in \pi(\Omega^2). \tag{4.1}$$

Let $\widetilde{\mathcal{H}}_2$ be the Hilbert space completion of $\pi(dJ_0^{(1)})$ with the above inner-product. Clearly $\widetilde{\mathcal{H}}_2 \subseteq \mathcal{H}_2$. Let P be the orthogonal projection of \mathcal{H}_2 onto the orthogonal complement of the subspace $\pi(dJ_0^{(1)})$. Now define $\langle [T_1], [T_2] \rangle_{\Omega_D^2} = \langle P T_1, P T_2 \rangle$, for all $[T_j] \in \Omega_D^2$. This gives a well defined inner-product on Ω_D^2 . Viewing $\mathcal{E} = p\mathcal{A}^q$ we see that $Hom_{\mathcal{A}}(\mathcal{E}, \mathcal{E} \otimes_{\mathcal{A}} \Omega_D^2) = Hom_{\mathcal{A}}(p\mathcal{A}^q, p\mathcal{A}^q \otimes_{\mathcal{A}} \Omega_D^2) \cong Hom_{\mathcal{A}}(p\mathcal{A}^q, p(\Omega_D^2)^q)$, which is contained in $Hom_{\mathcal{A}}(\mathcal{A}^q, (\Omega_D^2)^q)$. Now for $\phi, \psi \in Hom_{\mathcal{A}}(\mathcal{E}, \mathcal{E} \otimes_{\mathcal{A}} \Omega_D^2)$, define $\langle \langle \phi, \psi \rangle \rangle = \sum_k \langle \phi(p\tilde{e}_k), \psi(p\tilde{e}_k) \rangle_{\Omega_D^2}$ where $\{\tilde{e}_1, \dots, \tilde{e}_q\}$ is the standard basis of \mathcal{A}^q . Finally, the Yang–Mills functional on $\widetilde{\mathcal{C}}(\mathcal{E})$ is given by,

$$YM(\nabla) = \langle \langle \Theta, \Theta \rangle \rangle. \tag{4.2}$$

Remark 4.3. To see that this does not depend on the choice of the projection used to describe \mathcal{E} , let $\mathcal{E} \cong p_1\mathcal{A}^q$ and $\mathcal{E} \cong p_2\mathcal{A}^l$ be two isomorphisms. Then there is a unitary $U \in M_{q+l}(\mathcal{A})$ such that $\widetilde{p}_2 = U\widetilde{p}_1U^*$ where

$$\widetilde{p}_1 = \begin{pmatrix} p_1 & 0 \\ 0 & 0 \end{pmatrix}, \quad \widetilde{p}_2 = \begin{pmatrix} p_2 & 0 \\ 0 & 0 \end{pmatrix}.$$

Hence w.l.o.g. we can assume $l = n = q$ and there is a unitary $U \in M_n(\mathcal{A})$ such that $p_2 = Up_1U^*$. Let $\mathcal{M} = \{A \in M_n(\Omega_D^2) : p_1A = A\}$ and $\mathcal{M}' = \{B \in M_n(\Omega_D^2) : p_2B = B\}$. We have obvious bijections $Hom_{\mathcal{A}}(p_1\mathcal{A}^n, p_1\mathcal{A}^n \otimes_{\mathcal{A}} \Omega_D^2) \rightarrow \mathcal{M}$ and $Hom_{\mathcal{A}}(p_2\mathcal{A}^n, p_2\mathcal{A}^n \otimes_{\mathcal{A}} \Omega_D^2) \rightarrow \mathcal{M}'$ such that the induced bijection $\Psi : \mathcal{M} \rightarrow \mathcal{M}'$ is given by $A \mapsto UA$. Observe that Ψ makes sense on $M_n(\pi(\Omega^2))$. Using the inner product (4.1) we can convert $M_n(\pi(\Omega^2))$ into an inner product space. Clearly Ψ is inner product preserving and respects $\pi(dJ_0^{(1)})$. Recall that $\Omega_D^2 \cong \pi(\Omega^2)/\pi(dJ_0^{(1)})$, hence Ψ induces an inner product preserving map on $M_n(\Omega_D^2)$. The equality $\langle \langle \Theta, \Theta \rangle \rangle = \langle \langle \Psi(\Theta), \Psi(\Theta) \rangle \rangle$ shows that YM does not depend on the choice of projection p in $\mathcal{E} = p\mathcal{A}^q$.

5. Comparison between the two approaches

In this section we work out the Yang–Mills action functional in the second formulation and show that this is same as the one coming from the C^* -dynamical

system formulation. Given a C^* -dynamical system (\mathcal{A}, G, α) where G is a connected Lie group with a G -invariant faithful trace τ on \mathcal{A} , we can consider the G.N.S Hilbert space $\widetilde{\mathcal{H}} = L^2(\mathcal{A}, \tau)$. If *dimension* of the Lie group G is m , letting $t = 2^{\lfloor m/2 \rfloor}$, we know that there exist m matrices in $M_t(\mathbb{C})$ denoted by $\gamma_1, \gamma_2, \dots, \gamma_m$ (called Clifford gamma matrices), such that, $\gamma_r \gamma_s + \gamma_s \gamma_r = 2\delta_{rs}$, $r, s \in \{1, \dots, m\}$, where δ_{rs} is the Kronecker delta function. In our case of non-commutative n torus \mathcal{A}_θ , the Lie group is \mathbb{T}^n and hence we get n Clifford gamma matrices $\gamma_1, \dots, \gamma_n$. We define $D := \sum_{j=1}^n \delta_j \otimes \gamma_j$ where $\delta_j = (-i)\widetilde{\delta}_j$ (recall definition of $\widetilde{\delta}_j$ from 2.3). Then D becomes self-adjoint on $\mathcal{H} = \widetilde{\mathcal{H}} \otimes \mathbb{C}^N$ with domain $\mathcal{A}_\theta \otimes \mathbb{C}^N$, $N = 2^{\lfloor n/2 \rfloor}$. Moreover $|D|^{-n}$ lies in $\mathcal{L}^{(1, \infty)}$ with $Tr_\omega(|D|^{-n}) = 2N\pi^{n/2}/(n(2\pi)^n \Gamma(n/2))$ (see [7], Page 545) and $(\mathcal{A}_\theta, \mathcal{H}, D)$ gives us a (n, ∞) -summable spectral triple. Following propositions determine the \mathcal{A}_θ -bimodules Ω_D^1 and Ω_D^2 upto bimodule isomorphisms :

Proposition 5.1. $\Omega_D^1 \cong \underbrace{\mathcal{A}_\theta \oplus \dots \oplus \mathcal{A}_\theta}_{n \text{ times}}$

Proof. We know that $\Omega_D^1 \cong \pi(\Omega^1)$. Let $\omega \in \Omega^1$, so $\omega = \sum_j a_j db_j$, $a_j, b_j \in \mathcal{A}_\theta$. Then,

$$\begin{aligned} \pi(\omega) &= \sum_j (a_j \otimes I)[D, b_j] \\ &= \sum_j \left(\sum_{l=1}^n a_j \delta_l(b_j) \otimes \gamma_l \right). \end{aligned}$$

Since $\{\gamma_1, \dots, \gamma_n\} \subseteq M_N(\mathbb{C})$ is a linearly independent set, their linear span forms a n -dimensional vector space \mathbb{C}^n where we identify γ_l with $\alpha_l = (0, \dots, 1, \dots, 0) \in \mathbb{C}^n$ with 1 in the l -th place. $\{\alpha_1, \dots, \alpha_n\}$ is the canonical basis for \mathbb{C}^n . Hence we get $\Omega_D^1 \subseteq \mathcal{A}_\theta \otimes \mathbb{C}^n$. To see surjectivity notice that for any $a \in \mathcal{A}_\theta$, we can write $a = aU_l^*U_l = aU_l^*\delta_l(U_l)$ where the element $aU_l^*\delta_l(U_l) \in \Omega^1$. \square

Remark 5.2. Henceforth throughout this article $\{\sigma_1, \dots, \sigma_n\}$ will denote the standard basis of \mathcal{A}_θ^n as free \mathcal{A}_θ -bimodule where $\sigma_k = \underbrace{(0, \dots, 1, \dots, 0)}_{n \text{ tuple}}$

the k -th place; whereas $\{\widetilde{e}_1, \dots, \widetilde{e}_q\}$ will stand for the standard basis of \mathcal{A}_θ^q where $\widetilde{e}_l = \underbrace{(0, \dots, 1, \dots, 0)}_{q \text{ tuple}}$ with 1 in the l -th place. We will reserve this notation in the

rest of this article. Under the identification in the above proposition, σ_k is identified with $U_k^*\delta_k(U_k) \otimes \gamma_k$ in Ω_D^1 for $k \in \{1, \dots, n\}$.

Proposition 5.3. $\Omega_D^2 \cong \underbrace{\mathcal{A}_\theta \oplus \dots \oplus \mathcal{A}_\theta}_{n(n-1)/2 \text{ times}}$

Proof. We know that $\Omega_D^2 \cong \pi(\Omega^2)/\pi(dJ_0^{(1)})$. Let $\omega \in \Omega^2$ and write $\omega = \sum_r a_r db_r dc_r$, where $a_r, b_r, c_r \in \mathcal{A}_\Theta$. Then,

$$\begin{aligned} \pi(\omega) &= \sum_r (a_r \otimes I)[D, b_r][D, c_r] \\ &= \sum_r (a_r \otimes I) \left(\sum_{j=1}^n \delta_j(b_r) \otimes \gamma_j \right) \left(\sum_{k=1}^n \delta_k(c_r) \otimes \gamma_k \right) \\ &= \sum_r \left(\sum_{j=1}^n a_r \delta_j(b_r) \otimes \gamma_j \right) \left(\sum_{k=1}^n \delta_k(c_r) \otimes \gamma_k \right) \\ &= \sum_r \left(\left(\sum_{j=1}^n a_r \delta_j(b_r) \delta_j(c_r) \otimes I \right) \right. \\ &\quad \left. + \sum_{p < q} (a_r \delta_p(b_r) \delta_q(c_r) - a_r \delta_q(b_r) \delta_p(c_r)) \otimes \gamma_p \gamma_q \right) \end{aligned}$$

Since we know that, $\gamma_l^2 = I$ and $\gamma_l \gamma_m = -\gamma_m \gamma_l$ for $l \neq m$, $a_r \delta_p(b_r) \delta_q(c_r) \otimes \gamma_p \gamma_q + a_r \delta_q(b_r) \delta_p(c_r) \otimes \gamma_q \gamma_p = (a_r \delta_p(b_r) \delta_q(c_r) - a_r \delta_q(b_r) \delta_p(c_r)) \otimes \gamma_p \gamma_q$. Now $\gamma_l \gamma_m$ is independent with all $\gamma_p \gamma_q$ if $l, m \notin \{p, q\}$. Hence, $\pi(\Omega^2) \subseteq$

$$\bigoplus_{l=1} \mathcal{A}_\Theta^{(l)}, \text{ where } \mathcal{A}_\Theta^{(l)} = \mathcal{A}_\Theta, \forall l \text{ because total number of the elements}$$

$(a_r \delta_p(b_r) \delta_q(c_r) \otimes \gamma_p \gamma_q - a_r \delta_q(b_r) \delta_p(c_r) \otimes \gamma_p \gamma_q)$ is $n(n-1)/2$. To show equality we take any non-zero $a \in \mathcal{A}_\Theta$ and $b = U_1, c = U_1^*$. Then $adU_1dU_1^* \in \Omega^2$ and $\pi(adU_1dU_1^*) = -a \otimes I$ is a non-zero element of $\pi(\Omega^2)$. Similarly for each p, q we consider $aU_q^*U_p^*d(U_p)d(U_q) \in \Omega^2$. Then $\pi(aU_q^*U_p^*d(U_p)d(U_q)) = a \otimes \gamma_p \gamma_q$. This shows that the above inclusion is an equality.

Now we calculate $\pi(dJ_0^{(1)})$. We have $\omega \in J_0^{(1)}$ implies $\omega = \sum_s a_s db_s$ where $a_s, b_s \in \mathcal{A}_\Theta$, such that $\sum_s (a_s \otimes I)[D, b_s] = 0$. So we get,

$$\sum_s (a_s \otimes I) \left(\sum_{j=1}^n \delta_j(b_s) \otimes \gamma_j \right) = 0,$$

that is,

$$\sum_{j=1}^n \left(\sum_s a_s \delta_j(b_s) \right) \otimes \gamma_j = 0.$$

But, $\gamma_1, \dots, \gamma_n$ being linearly independent we get,

$$\sum_s a_s \delta_j(b_s) \otimes \gamma_j = 0, \quad \forall j = 1, \dots, n. \tag{5.1}$$

Now $d\omega = \sum_s da_s db_s$. So,

$$\begin{aligned} \pi(d\omega) &= \sum_s [D, a_s][D, b_s] \\ &= \sum_s \left(\sum_{j=1}^n \delta_j(a_s) \otimes \gamma_j \right) \left(\sum_{k=1}^n \delta_k(b_s) \otimes \gamma_k \right) \\ &= \sum_s \left(\left(\sum_{j=1}^n \delta_j(a_s) \delta_j(b_s) \otimes I \right) + \dots \right. \\ &\quad \left. + (\delta_p(a_s) \delta_q(b_s) - \delta_q(a_s) \delta_p(b_s)) \otimes \gamma_p \gamma_q + \dots \right) \end{aligned}$$

Now from eqn. (5.1) we get,

$$\sum_s \delta_p(a_s) \delta_q(b_s) \otimes \gamma_p \gamma_q = - \sum_s a_s \delta_p \delta_q(b_s) \otimes \gamma_p \gamma_q$$

and,

$$\sum_s \delta_q(a_s) \delta_p(b_s) \otimes \gamma_q \gamma_p = - \sum_s a_s \delta_q \delta_p(b_s) \otimes \gamma_q \gamma_p$$

Hence,

$$\begin{aligned} (\delta_p(a_s) \delta_q(b_s) - \delta_q(a_s) \delta_p(b_s)) \otimes \gamma_p \gamma_q &= (-a_s \delta_p \delta_q(b_s) + a_s \delta_q \delta_p(b_s)) \otimes \gamma_p \gamma_q \\ &= 0 \end{aligned}$$

because, $\delta_p \delta_q = \delta_q \delta_p$, $\forall p, q \in \{1, \dots, n\}$. Hence, $\pi(dJ_0^{(1)}) \subseteq \mathcal{A}_\Theta$ and to show the equality take any $a \in \mathcal{A}_\Theta$. Consider $\omega = a(U_1^* dU_1 - 1/2 \times U_1^{-2} d(U_1^2)) \in \Omega^1$. Then we get $\pi(\omega) = 0$ but $\pi(d\omega) = a \otimes I \neq 0$ (which also shows non-triviality of ω). Hence we conclude $\pi(dJ_0^{(1)}) \cong \pi(\mathcal{A}_\Theta) \cong \mathcal{A}_\Theta$. \square

Now we want to determine the differential $\tilde{d} : \pi(\mathcal{A}_\Theta) \rightarrow \Omega_D^1$ so that, $\tilde{d}(\pi(a)) = \pi(da)$, $\forall a \in \mathcal{A}_\Theta$.

Lemma 5.4. $\tilde{d} : \pi(\mathcal{A}_\Theta) \rightarrow \Omega_D^1$ is given by, $\pi(a) \mapsto (\delta_1(a), \dots, \delta_n(a))$.

Proof. Pick any element $\pi(a) \in \pi(\mathcal{A}_\Theta)$. Then $da \in \Omega^1$ and hence $\pi(da) =$

$$[D, a] = \sum_{j=1}^n \delta_j(a) \otimes \gamma_j. \text{ This is an element in } \Omega_D^1, \text{ which is isomorphic to } \mathcal{A}_\Theta^n \text{ and}$$

under this isomorphism, $\sum_{j=1}^n \delta_j(a) \otimes \gamma_j$ goes to $(\delta_1(a), \dots, \delta_n(a))$ in \mathcal{A}_Θ^n . Hence

the above definition of \tilde{d} is justified. \square

Next we want to determine the differential $\tilde{d} : \Omega_D^1 \longrightarrow \Omega_D^2$ so that, $\tilde{d}(\pi(\omega)) = \pi(d\omega), \forall \omega \in \Omega^1$.

Lemma 5.5. $\tilde{d} : \Omega_D^1 \longrightarrow \Omega_D^2$ is given by,

$$(0, \dots, a, \dots, 0) \longmapsto ((\delta_p(aU_j^*)\delta_q(U_j) - \delta_q(aU_j^*)\delta_p(U_j)))_{1 \leq p < q \leq n}$$

for a in the j -th place.

Proof. For $(0, \dots, a, \dots, 0) \in \Omega_D^1$ with a in the j -th place, we have $aU_j^*dU_j \in \Omega^1$, such that $\pi(aU_j^*dU_j)$ is identified with $(0, \dots, a, \dots, 0)$. Now, $d(aU_j^*dU_j) = d(aU_j^*)dU_j$, an element of Ω^2 . Now,

$$\begin{aligned} \pi(d(aU_j^*)dU_j) &= [D, aU_j^*][D, U_j] \\ &= \left(\sum_{l=1}^n \delta_l(aU_j^*) \otimes \gamma_l \right) \left(\sum_{k=1}^n \delta_k(U_j) \otimes \gamma_k \right) \\ &= \sum_{l=1}^n \delta_l(aU_j^*)\delta_l(U_j) \otimes I \\ &\quad + \sum_{p < q} (\delta_p(aU_j^*)\delta_q(U_j) - \delta_q(aU_j^*)\delta_p(U_j)) \otimes \gamma_p\gamma_q. \end{aligned}$$

Under the isomorphism $\Omega_D^2 \cong \mathcal{A}_\Theta^{n(n-1)/2}, \sum_{p < q} (\delta_p(aU_j^*)\delta_q(U_j) - \delta_q(aU_j^*)\delta_p(U_j)) \otimes \gamma_p\gamma_q$ goes to the required point in $\mathcal{A}_\Theta^{n(n-1)/2}$. □

Finally the product map is recognized by the following lemma.

Lemma 5.6. The product map $\tilde{\prod} : \Omega_D^1 \times \Omega_D^1 \longrightarrow \Omega_D^2$ is given by,

$$(a_1, \dots, a_n).(b_1, \dots, b_n) := ((a_p b_q - a_q b_p))_{1 \leq p < q \leq n}$$

Proof. We have a product $\prod : \Omega^1 \times \Omega^1 \longrightarrow \Omega^2$ given by $\prod(a_1 da_2, b_1 db_2) = a_1 da_2 b_1 db_2 = a_1 d(a_2 b_1) db_2 - a_1 a_2 db_1 db_2$. Choose two elements (a_1, \dots, a_n) and (b_1, \dots, b_n) in Ω_D^1 . We have seen previously that $\pi(\sum_{m=1}^n a_m U_m^* d(U_m))$ in $\pi(\Omega^1)$ is identified with (a_1, \dots, a_n) . Similarly for b_m in place of a_m . Let

$$\omega = \sum_{m=1}^n a_m U_m^* d(U_m) \quad \text{and} \quad \omega' = \sum_{m=1}^n b_m U_m^* d(U_m).$$

Now,

$$\begin{aligned}
\Pi(\omega, \omega') &= \left(\sum_{m=1}^n a_m U_m^* d(U_m) \right) \left(\sum_{j=1}^n b_j U_j^* d(U_j) \right) \\
&= \sum_{m,j=1}^n a_m U_m^* d(U_m) b_j U_j^* d(U_j) \\
&= \sum_{m,j=1}^n (a_m U_m^* d(U_m b_j U_j^*) d(U_j) - a_m d(b_j U_j^*) d(U_j)).
\end{aligned}$$

It is an element of Ω^2 . Applying π on it we get

$$\begin{aligned}
\pi(\Pi(\omega, \omega')) &= \sum_{m,j=1}^n (a_m U_m^* [D, U_m b_j U_j^*] [D, U_j] - a_m [D, b_j U_j^*] [D, U_j]) \\
&= \sum_{m,j=1}^n (a_m U_m^* (\sum_{k=1}^n \delta_k (U_m b_j U_j^*) \otimes \gamma_k) (\sum_{l=1}^n \delta_l (U_j) \otimes \gamma_l) \\
&\quad - a_m (\sum_{r=1}^n \delta_r (b_j U_j^*) \otimes \gamma_r) (\sum_{s=1}^n \delta_s (U_j) \otimes \gamma_s)) \\
&= \sum_{p < q} \left(\sum_{m,j=1}^n a_m U_m^* \delta_p (U_m b_j U_j^*) \delta_q (U_j) \right. \\
&\quad \left. - \sum_{m,j=1}^n a_m U_m^* \delta_q (U_m b_j U_j^*) \delta_p (U_j) - \sum_{m,j=1}^n a_m \delta_p (b_j U_j^*) \delta_q (U_j) \right. \\
&\quad \left. + \sum_{m,j=1}^n a_m \delta_q (b_j U_j^*) \delta_p (U_j) \right) \otimes \gamma_p \gamma_q.
\end{aligned}$$

For each p and q ,

$$\begin{aligned}
&\sum_{p < q} \left(\sum_{m,j=1}^n (a_m U_m^* \delta_p (U_m b_j U_j^*) \delta_q (U_j) - \sum_{m,j=1}^n a_m U_m^* \delta_q (U_m b_j U_j^*) \delta_p (U_j)) \right. \\
&\quad \left. - \sum_{m,j=1}^n a_m \delta_p (b_j U_j^*) \delta_q (U_j) + \sum_{m,j=1}^n a_m \delta_q (b_j U_j^*) \delta_p (U_j) \right)
\end{aligned}$$

$$\begin{aligned}
 &= \sum_{m=1}^n (a_m U_m^* \delta_p (U_m b_q U_q^*) U_q - a_m \delta_p (b_q U_q^*) U_q \\
 &\quad + a_m \delta_q (b_p U_p^*) U_p - a_m U_m^* \delta_q (U_m b_p U_p^*) U_p) \\
 &= \sum_{m=1}^n (a_m U_m^* \delta_p (U_m b_q) - a_m \delta_p (b_q) \\
 &\quad + a_m \delta_q (b_p) - a_m U_m^* \delta_q (U_m b_p)) \\
 &= \sum_{m=1}^n (a_m U_m^* \delta_p (U_m) b_q - a_m U_m^* \delta_q (U_m) b_p) \\
 &= a_p b_q - a_q b_p.
 \end{aligned}$$

Hence for $(a_1, \dots, a_n), (b_1, \dots, b_n) \in \Omega_D^1$, we get $\widetilde{\prod}((a_1, \dots, a_n), (b_1, \dots, b_n)) = ((a_p b_q - a_q b_p))_{1 \leq p < q \leq n}$. \square

It can be easily checked that both the \widetilde{d} , defined above, are derivations. We first prove the following lemmas which will help us in the computation.

Lemma 5.7. *The canonical trace τ on \mathcal{A}_Θ equals $1/Tr_\omega(|D|^{-n}) \int$ where Tr_ω denotes Dixmier trace and $\int a := Tr_\omega((a \otimes I)|D|^{-n})$ for all $a \in \mathcal{A}_\Theta$.*

Proof. We have $\tau(a) = \tau(\alpha_{\mathbf{g}}(a)), \forall \mathbf{g} \in \mathbb{T}^n$ because τ is G -invariant on \mathcal{A}_Θ . The G.N.S Hilbert space $L^2(\mathcal{A}_\Theta, \tau)$ is identified with $l^2(\mathbb{Z}^n)$. For $\mathbf{g} \in \mathbb{T}^n, \alpha_{\mathbf{g}}(U_1^{k_1} \dots U_n^{k_n}) = \mathbf{g}^{\mathbf{k}} U_1^{k_1} \dots U_n^{k_n}$. Here $\mathbf{g} = (g_1, \dots, g_n) \in \mathbb{T}^n; \mathbf{g}^{\mathbf{k}} = g_1^{k_1} \dots g_n^{k_n}$. Define,

$$\begin{aligned}
 U_{\mathbf{g}} : L^2(\mathcal{A}_\Theta, \tau) &\longrightarrow L^2(\mathcal{A}_\Theta, \tau) \\
 a &\longmapsto \alpha_{\mathbf{g}}(a)
 \end{aligned}$$

It is easy to check this map is isometry with dense range. Hence extends as unitary on $L^2(\mathcal{A}_\Theta, \tau)$. For $e_{\mathbf{k}} \in l^2(\mathbb{Z}^n), U_{\mathbf{g}}(e_{\mathbf{k}}) = \mathbf{g}^{\mathbf{k}} e_{\mathbf{k}}$. Since $D(e_{\mathbf{k}} \otimes M) = \sum_{j=1}^n k_j e_{\mathbf{k}} \otimes \gamma_j M$ for $M \in M_N(\mathbb{C})$, it follows that $D(U_{\mathbf{g}} \otimes I) = (U_{\mathbf{g}} \otimes I)D$ on $L^2(\mathcal{A}_\Theta, \tau) \otimes \mathbb{C}^N$. But $(U_{\mathbf{g}} \otimes I)D(U_{\mathbf{g}}^* \otimes I) = D \Rightarrow (U_{\mathbf{g}} \otimes I)|D|(U_{\mathbf{g}}^* \otimes I) = |D|$ which further implies $(U_{\mathbf{g}} \otimes I)|D|^{-n}(U_{\mathbf{g}}^* \otimes I) = |D|^{-n}$. Hence,

$$\begin{aligned}
 Tr_\omega((U_{\mathbf{g}} a U_{\mathbf{g}}^* \otimes I)|D|^{-n}) &= Tr_\omega((U_{\mathbf{g}} \otimes I)(a \otimes I)|D|^{-n}(U_{\mathbf{g}}^* \otimes I)) \\
 &= Tr_\omega((a \otimes I)|D|^{-n})
 \end{aligned}$$

which shows that $1/Tr_\omega(|D|^{-n}) \int$ is also a G -invariant trace on \mathcal{A}_Θ . Now uniqueness of G -invariant trace on \mathcal{A}_Θ gives $\tau(a) = Tr_\omega((a \otimes I)|D|^{-n})/Tr_\omega(|D|^{-n})$ where $Tr_\omega(|D|^{-n})$ is a positive constant. \square

Lemma 5.8. *If $\{\gamma_1, \dots, \gamma_n\}$ are Clifford gamma matrices in $M_N(\mathbb{C})$ then they enjoys the property $Trace(\gamma_l \gamma_m) = 0$ for $l \neq m$.*

Proof. This follows immediately from the fact that Clifford gamma matrices satisfy the relation $\gamma_l \gamma_m + \gamma_m \gamma_l = 2\delta_{lm}$ for all l, m . \square

Lemma 5.9. *The positive linear functional $f : T \mapsto Tr_\omega(T|D|^{-n})/Tr_\omega(|D|^{-n})$, for $T \in M_N(\mathcal{A}_\Theta)$, equals with $\tau \otimes Trace$, where ‘Trace’ denotes the ordinary matrix trace (normalized) on $M_N(\mathbb{C})$.*

Proof. Since $D^2 = \sum \delta_j^2 \otimes I_N$, $|D|^{-n}$ commutes with $1 \otimes M_N(\mathbb{C})$ it follows that f is a trace on $M_N(\mathcal{A}_\Theta) \cong \mathcal{A}_\Theta \otimes M_N(\mathbb{C})$. Our requirement is now fulfilled because of the fact that $\tau \otimes Trace$ is the unique extention (normalized) of τ on $M_N(\mathcal{A}_\Theta)$. \square

Lemma 5.10. *If $l \neq m$ then any $a \otimes \gamma_l \gamma_m$ lies in the range of P where P was the orthogonal projection onto the orthogonal complement of $\pi(dJ_0^{(1)}) \subseteq \Omega_D^2$.*

Proof. Recall that any element of $\pi(dJ_0^{(1)})$ looks like $x \otimes I$. Now $\langle a \otimes \gamma_l \gamma_m, x \otimes I \rangle_{\pi(\Omega^2)} = Tr_\omega((a^* x \otimes \gamma_l \gamma_m)|D|^{-n}) = Tr_\omega(|D|^{-n})\tau(a^* x)Trace(\gamma_l \gamma_m) = 0$, since $Trace(\gamma_l \gamma_m) = 0$ by Lemma (5.8). \square

Now we are ready to calculate the Yang–Mills for \mathcal{A}_Θ . Since $\Omega_D^1 \cong \mathcal{A}_\Theta^n$, any compatible connection $\nabla : \mathcal{E} \rightarrow \mathcal{E} \otimes \Omega_D^1$ is given by n -tuple of maps $(\nabla_1, \dots, \nabla_n)$, where $\nabla_j : \mathcal{E} \rightarrow \mathcal{E}$ such that,

$$\nabla(\xi) = \sum_{j=1}^n \nabla_j(\xi) \otimes \sigma_j \tag{5.2}$$

$$\langle \xi, \nabla_j(\eta) \rangle - \langle \nabla_j(\xi), \eta \rangle = \delta_j(\langle \xi, \eta \rangle_{\mathcal{A}_\Theta}). \tag{5.3}$$

Here $\{\sigma_1, \dots, \sigma_n\}$ is the standard basis of \mathcal{A}_Θ^n as free \mathcal{A}_Θ -bimodule. Then $\tilde{\nabla} : \mathcal{E} \otimes \Omega_D^1 \rightarrow \mathcal{E} \otimes \Omega_D^2$ is given by, $\tilde{\nabla}(\xi \otimes \sigma_m) = \left(\sum_{j=1}^n \nabla_j(\xi) \otimes \sigma_j \right) \sigma_m + \xi \otimes \tilde{d}(\sigma_m)$ for each $m = 1, \dots, n$.

Proposition 5.11. *The curvature $\Theta = \tilde{\nabla} \circ \nabla$ is given by $\sum_{m < j} [\nabla_m, \nabla_j](\cdot) \otimes \sigma_m \sigma_j$ where $\sigma_m, \sigma_j \in \mathcal{A}_\Theta^n$ and $\sigma_m \sigma_j$ is the element in $\mathcal{A}_\Theta^{n(n-1)/2}$ produced by the product map $\tilde{\prod}$ of Lemma (5.6).*

Proof. Through direct computation we get,

$$\begin{aligned} \Theta(\xi) &= \tilde{\nabla} \circ \nabla(\xi) \\ &= \sum_{m=1}^n \tilde{\nabla}(\nabla_m(\xi) \otimes \sigma_m) \\ &= \sum_m \left(\left(\sum_j \nabla_j(\nabla_m(\xi)) \otimes \sigma_j \right) \sigma_m + \nabla_m(\xi) \otimes \tilde{d}(\sigma_m) \right) \\ &= \sum_{m,j} \nabla_j(\nabla_m(\xi)) \otimes \sigma_j \sigma_m + \nabla_m(\xi) \otimes \tilde{d}(\sigma_m) \\ &= \sum_{m < j} [\nabla_m, \nabla_j](\xi) \otimes \sigma_m \sigma_j + \sum_m \nabla_m(\xi) \otimes \tilde{d}(\sigma_m). \end{aligned}$$

But,

$$\sum_m \nabla_m(\xi) \otimes \tilde{d}(\sigma_m) = \sum_m \nabla_m(\xi) \otimes ((\delta_p(U_m^*)\delta_q(U_m) - \delta_q(U_m^*)\delta_p(U_m))_{1 \leq p < q \leq n}) = 0$$

because $\delta_j(U_m^*) = -U_m^* \delta_j(U_m) U_m^*$. Hence $\Theta = \sum_{m < j} [\nabla_m, \nabla_j] \otimes \sigma_m \sigma_j$. \square

Proposition 5.12. $YM(\nabla) = \sum_{m < j} \tau_q([\nabla_m, \nabla_j]^* [\nabla_m, \nabla_j])$ upto a positive factor where τ_q denotes the extended trace $\tau \otimes \text{Trace}$ on $M_q(\mathcal{A}_\Theta)$.

Proof. Recall that $\langle\langle \Theta, \Theta \rangle\rangle = \sum_{k=1}^q \langle \Theta(p\tilde{e}_k), \Theta(p\tilde{e}_k) \rangle_{\Omega_D^2}$ where $\{\tilde{e}_1, \dots, \tilde{e}_q\}$ denotes standard basis of \mathcal{A}_Θ^q and $\mathcal{E} = p\mathcal{A}_\Theta^q$. Let $[\nabla_m, \nabla_j](p\tilde{e}_k) = \eta^{(mjk)} = p\eta^{(mjk)} = (\eta_1^{(mjk)}, \dots, \eta_q^{(mjk)}) \in \mathcal{A}_\Theta^q$. Then from proposition (5.11) we get

$$\Theta(p\tilde{e}_k) = \sum_{m < j} (\eta_1^{(mjk)} \sigma_m \sigma_j, \dots, \eta_q^{(mjk)} \sigma_m \sigma_j),$$

an element of $(\Omega_D^2)^q$. It is easily seen that as \mathbb{C} vector spaces $Hom(\mathcal{E}, \mathcal{E} \otimes \Omega_D^2) \cong \bigoplus Hom(\mathcal{E}, \mathcal{E})$. We can view $End(\mathcal{E})$ as $pM_q(\mathcal{A}_\Theta)p \subseteq M_q(\mathcal{A}_\Theta)$. We have an inner-product on $\bigoplus M_q(\mathcal{A}_\Theta)$ given by $\langle\langle (A_1, \dots, A_t), (B_1, \dots, B_t) \rangle\rangle = \sum_{j=1}^t \tau_q(A_j^* B_j)$. Following calculation shows this inner-product becomes same with the one on $Hom(\mathcal{E}, \mathcal{E} \otimes \Omega_D^2)$.

$$\begin{aligned}
 \langle \Theta(p\tilde{e}_k), \Theta(p\tilde{e}_k) \rangle &= \sum_{m < j, l < r} \langle (\eta_1^{(mjk)} \sigma_m \sigma_j, \dots, \eta_q^{(mjk)} \sigma_m \sigma_j), \\
 &\qquad\qquad\qquad (\eta_1^{(lrk)} \sigma_l \sigma_r, \dots, \eta_q^{(lrk)} \sigma_l \sigma_r) \rangle \\
 &= \sum_{m < j, l < r} \sum_{s=1}^q \langle \eta_s^{(mjk)} \sigma_m \sigma_j, \eta_s^{(lrk)} \sigma_l \sigma_r \rangle_{\Omega_D^2} \\
 &= \sum_{m < j, l < r} \sum_{s=1}^q \langle [\eta_s^{(mjk)} \otimes \gamma_m \gamma_j], [\eta_s^{(lrk)} \otimes \gamma_l \gamma_r] \rangle_{\Omega_D^2} \\
 &= \sum_{m < j, l < r} \sum_{s=1}^q \langle P(\eta_s^{(mjk)} \otimes \gamma_m \gamma_j), P(\eta_s^{(lrk)} \otimes \gamma_l \gamma_r) \rangle_{\pi(\Omega^2)} \\
 &= \sum_{m < j, l < r} \sum_{s=1}^q Tr_{\omega} \left((\eta_s^{(mjk)})^* \eta_s^{(lrk)} \otimes \gamma_j \gamma_m \gamma_l \gamma_r |D|^{-n} \right).
 \end{aligned}$$

Last equality follows from Lemma (5.10). Now use of Lemma (5.8) and (5.9) shows the following,

$$\begin{aligned}
 \langle \langle \Theta, \Theta \rangle \rangle &= Tr_{\omega}(|D|^{-n}) \sum_{k=1}^q \sum_{m < j} \sum_{s=1}^q \tau \left(\eta_s^{(mjk)*} \eta_s^{(mjk)} \right) \\
 &= Tr_{\omega}(|D|^{-n}) \sum_{k=1}^q \sum_{m < j} \tau \left(\langle [\nabla_m, \nabla_j](p\tilde{e}_k), [\nabla_m, \nabla_j](p\tilde{e}_k) \rangle_{\mathcal{A}_{\Theta}} \right) \\
 &= Tr_{\omega}(|D|^{-n}) \sum_{k=1}^q \sum_{m < j} \tau \left(\langle p\tilde{e}_k, [\nabla_m, \nabla_j]^* [\nabla_m, \nabla_j](p\tilde{e}_k) \rangle_{\mathcal{A}_{\Theta}} \right) \\
 &= Tr_{\omega}(|D|^{-n}) \sum_{m < j} \tau_q \left([\nabla_m, \nabla_j]^* [\nabla_m, \nabla_j] \right).
 \end{aligned}$$

The last equality follows from the fact that for any $T = ((t_{rs})) \in pM_q(\mathcal{A}_{\Theta})p$ where $p \in M_q(\mathcal{A}_{\Theta})$ is a projection, $\sum_{k=1}^q \langle \tilde{e}_k, T\tilde{e}_k \rangle_{\mathcal{A}_{\Theta}} = \sum_{k=1}^q \langle p\tilde{e}_k, Tp\tilde{e}_k \rangle_{\mathcal{A}_{\Theta}} = \sum_{r=1}^q t_{rr}$. Hence follows the proposition. \square

Recall that $\{e_1, \dots, e_n\}$ denotes the standard basis chosen for \mathbb{R}^n and $\{\sigma_1, \dots, \sigma_n\}$ is the standard basis of Ω_D^1 . We have an one to one correspondence between these sets, both being finite sets of same cardinality. The following theorem points out the main result.

Theorem 5.13. *Let $C(\mathcal{E})$ and $\tilde{C}(\mathcal{E})$ denote the affine space of compatible connections for the first and second approaches respectively. Then both these are in one to one correspondence through an affine isomorphism, and the value of Yang–Mills functional on corresponding elements of these spaces are same upto a positive scalar factor. That is to say the following diagram commutes.*

$$\begin{array}{ccc}
 C(\mathcal{E}) & \xrightarrow{\Phi} & \tilde{C}(\mathcal{E}) \\
 \searrow^{cYM} & & \swarrow_{YM} \\
 & \mathbb{R}_+ \cup \{0\} &
 \end{array}$$

where $c = 2N\pi^{n/2}/(n(2\pi)^n\Gamma(n/2))$.

Proof. Recall from equation (2.4) for any $\nabla \in C(\mathcal{E})$, $\nabla(\xi) = \sum_{j=1}^n \nabla_j(\xi) \otimes e_j$ where $\nabla_j : \mathcal{E} \rightarrow \mathcal{E}$. We define $\Phi(\nabla) = \tilde{\nabla}$ where,

$$\tilde{\nabla}(\xi) = \sum_{j=1}^n (-i)\nabla_j(\xi) \otimes \sigma_j$$

It is easy to see that $\tilde{\nabla}$ defines a connection. Given compatibility of ∇ , we have to check whether $\tilde{\nabla}$ is compatible with respect to the *Hermitian* structure. This follows from a direct calculation.

$$\begin{aligned}
 \langle \xi, \tilde{\nabla}(\eta) \rangle - \langle \tilde{\nabla}(\xi), \eta \rangle &= \sum_{j=1}^n (\langle \xi, (-i)\nabla_j(\eta) \otimes \sigma_j \rangle - \langle (-i)\nabla_j(\xi) \otimes \sigma_j, \eta \rangle) \\
 &= \sum_{j=1}^n (\langle \xi, \nabla_j(\eta) \rangle_{\mathcal{A}_\Theta} (-i)\sigma_j - i\sigma_j^* \langle \nabla_j(\xi), \eta \rangle_{\mathcal{A}_\Theta}) \\
 &= \sum_{j=1}^n (\langle \xi, \nabla_j(\eta) \rangle_{\mathcal{A}_\Theta} (-i)\sigma_j - i\sigma_j \langle \nabla_j(\xi), \eta \rangle_{\mathcal{A}_\Theta}) \\
 &= (-i)(\langle \xi, \nabla_1(\eta) \rangle_{\mathcal{A}_\Theta} + \langle \nabla_1(\xi), \eta \rangle_{\mathcal{A}_\Theta}, \dots, \\
 &\qquad \qquad \qquad \langle \xi, \nabla_n(\eta) \rangle_{\mathcal{A}_\Theta} + \langle \nabla_n(\xi), \eta \rangle_{\mathcal{A}_\Theta}) \\
 &= (-i)(\tilde{\delta}_1(\langle \xi, \eta \rangle_{\mathcal{A}_\Theta}), \dots, \tilde{\delta}_n(\langle \xi, \eta \rangle_{\mathcal{A}_\Theta})) \\
 &= (\delta_1(\langle \xi, \eta \rangle_{\mathcal{A}_\Theta}), \dots, \delta_n(\langle \xi, \eta \rangle_{\mathcal{A}_\Theta})) \\
 &= \tilde{d}(\langle \xi, \eta \rangle_{\mathcal{A}_\Theta}).
 \end{aligned}$$

which shows compatibility of $\tilde{\nabla}$ with respect to the *Hermitian* structure and hence $\tilde{\nabla}$ belongs to $\tilde{C}(\mathcal{E})$. Conversely, for given $\tilde{\nabla} \in \tilde{C}(\mathcal{E})$ recall from equation (5.2) that

$\tilde{\nabla}(\xi) = \sum_{j=1}^n \tilde{\nabla}_j(\xi) \otimes \sigma_j$ where $\tilde{\nabla}_j : \mathcal{E} \rightarrow \mathcal{E}$. We define $\Phi^{-1}(\tilde{\nabla}) = \nabla$ where,

$$\nabla(\xi) = \sum_{j=1}^n i \tilde{\nabla}_j(\xi) \otimes e_j$$

An absolutely similar computation shows the compatibility of ∇ . So elements of $C(\mathcal{E})$ and $\tilde{C}(\mathcal{E})$ are in one-one correspondence. Recall from equation (2.6), for finitely generated projective \mathcal{A}_θ -module $\mathcal{E} = p\mathcal{A}_\theta^q$ we obtained for $\nabla \in C(\mathcal{E})$,

$$YM(\nabla) = \sum_{j < k} \tilde{\tau}([\nabla_j, \nabla_k]^* [\nabla_j, \nabla_k])$$

where $\tilde{\tau}$ was the trace on $End(\mathcal{E})$. For $\Phi(\nabla) = \tilde{\nabla}$ we obtain from Proposition (5.12),

$$YM(\tilde{\nabla}) = Tr_\omega(|D|^{-n}) \sum_{j < k} \tau_q([\nabla_j, \nabla_k]^* [\nabla_j, \nabla_k])$$

where τ_q was the extended trace of τ on $M_q(\mathcal{A}_\theta)$. Identifying $End(\mathcal{E})$ with $pM_q(\mathcal{A}_\theta)p \subseteq M_q(\mathcal{A}_\theta)$ we see that both $\tilde{\tau}$ and τ_q are equal with $\tau \otimes Trace$. Hence follows the equality of Yang–Mills for both the approaches except for the positive scalar factor $Tr_\omega(|D|^{-n}) = 2N\pi^{n/2}/(n(2\pi)^n \Gamma(n/2))$. \square

Remark 5.14. The question studied here makes sense whenever we have a C^* -dynamical system with an invariant trace. Now, Kang has already considered ([8]) the dynamical system approach to Yang–Mills for the quantum Heisenberg manifolds. Even in that case we have shown ([2]) that both these approaches agree.

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References

- [1] B. Blackadar : *K-theory for operator algebras*, Mathematical Science Research Institute Publications, 5. Springer-Verlag, New York, 1986. [Zbl 0597.46072](#) [MR 859867](#)
- [2] P. S. Chakraborty; S. Guin : *Yang–Mills on quantum Heisenberg manifolds*. *Comm. Math. Phys.* **330** (2014), no. 3, 1327–1337. [Zbl 1303.46063](#) [MR 3227514](#)
- [3] A. Connes : *C*-algèbres et géométrie différentielle*, *C.R. Acad. Sc. Paris Ser. A-B* **290** (1980), no. 13, A599–A604. [Zbl 0433.46057](#) [MR 572645](#)
- [4] A. Connes : *Noncommutative Geometry*, Academic Press, 1994. [Zbl 0818.46076](#) [MR 1303779](#)

- [5] A. Connes; M. R. Douglas; A. Schwarz : *Noncommutative geometry and matrix theory: compactification on tori*, J. High Energy Phys. 1998, no. 2, Paper 3, 35 pp. [Zbl 1018.81052](#) [MR 1613978](#)
- [6] A. Connes; M. A. Rieffel : *Yang–Mills for non-commutative two-tori*, Contemp. Math. **62** (1987) 237–266. [Zbl 0633.46069](#) [MR 878383](#)
- [7] J. Gracia-Bondía; J. Várilly; H. Figueroa : *Elements of Noncommutative Geometry*, Birkhäuser, 2001. [Zbl 0958.46039](#) [MR 1789831](#)
- [8] S. Kang, : *The Yang–Mills functional and Laplace’s equation on quantum Heisenberg manifolds*, J. Funct. Anal., **258**, (2010), no. 1, 307–327. [Zbl 1186.58007](#) [MR 2557965](#)
- [9] A. Konechny; A. Schwarz : *Supersymmetry algebra and BPS states of super Yang–Mills theories on noncommutative tori*, Phys. Lett. B **453** (1999), no. 1–2, 23–29. [Zbl 1058.58500](#) [MR 1690319](#)
- [10] M. A. Rieffel : *Non-commutative Tori—A case study of non-commutative differentiable manifolds*, Contemp. Math. **105** (1990) 191–211. [Zbl 04173687](#) [MR 1047281](#)
- [11] M. A. Rieffel : *Critical points of Yang–Mills for noncommutative two-tori*, J. Differential Geom. **31** (1990), no. 2, 535–546. [Zbl 0820.58007](#) [MR 1037414](#)
- [12] L. B. Schweitzer : *Spectral invariance of dense subalgebras of operator algebras*, Internat. J. Math (1993) no. 2, 289–317. [Zbl 0804.46081](#) [MR 1217384](#)
- [13] N. E. Wegge-Olsen : *K-theory and C*-algebras: A friendly approach*, Oxford, 1993. [Zbl 0780.46038](#) [MR 1222415](#)

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