

Finite part of operator K-theory for groups with rapid decay

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Abstract. In this paper we study the part of the K -theory of the reduced C^* -algebra arising from torsion elements of the group, and in particular we study the pairing of K -theory with traces and when traces can detect certain K -theory elements. In the case of groups with Property RD, we give a condition on the growth of conjugacy classes that determines whether they can be detected. Moreover, in the case that they can be detected, we show that nonzero elements in the part of the K -theory generated by torsion elements are not in the image of the assembly map $K_0^G(EG) \rightarrow K_0(C^*G)$. One application of this result is a lower bound for the structure groups of certain manifolds.

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1. Introduction

Let G be a group. An element g is said to have order d for positive integer d if $g^d = 1$ and $g^k \neq 1$ for any $0 < k < d$. If no such d exists, we say that g has order ∞ . We say an element is torsion if it has finite order.

For a torsion element g of order d , we define a corresponding idempotent p as

$$p = \frac{1 + g + g^2 + \cdots + g^{d-1}}{d} \in \mathbb{C}G.$$

It is easy to see that this is an idempotent, so we can view it in $K_0(\mathbb{C}G)$. Let C_{\max}^*G denote the maximal group C^* -algebra and C_{red}^*G denote the reduced group C^* -algebra. Then one can also view the above idempotent in $K_0(C_{\text{red}}^*G)$ or $K_0(C_{\max}^*G)$. The question we would like to study is whether for idempotents p_1, \dots, p_k corresponding to torsion elements g_1, \dots, g_k of distinct orders $d_1, \dots, d_k > 1$, we have that $1, p_1, \dots, p_k$ are linearly independent in C_{red}^*G and C_{\max}^*G .

The study of idempotents arising from torsion elements was introduced in [8]. In that paper, Weinberger and Yu showed that if the group is finitely embeddable in Hilbert spaces, then the p_i are linearly independent in $K_0(C_{\max}^*(G))$. They

conjectured that the same holds for all groups. They also showed that the rank of the part of $K_0(C_{\max}^*(G))$ generated by these torsion elements bounds the non-rigidity for M . More precisely, it gives a lower bound for the rank of the structure group $S(M)$, that is, the abelian group of equivalence classes of pairs (f, M') , where M' is a compact, oriented manifold, and $f : M' \rightarrow M$ is an orientation preserving homotopy equivalence.

Our approach to proving the linear independence of p_i is to find $k + 1$ traces on the algebra such that the $(k + 1) \times (k + 1)$ matrix that arises from evaluating the traces on $1, p_1, \dots, p_k$ has non-zero determinant. If there are such traces, then we say that $1, p_1, \dots, p_k$ can be distinguished by traces. It is easy to see that being distinguished by traces implies linear independence in K_0 . For groups with rapid decay property (property RD), we prove a necessary and sufficient condition for the $1, p_1, \dots, p_k \in K_0(C_{\text{red}}^*G)$ to be distinguished by traces, namely that they can be distinguished by traces if and only if the relevant conjugacy classes grow polynomially. We discuss some examples of both our positive and our negative results. The negative results are particularly striking because they demonstrate a phenomenon in reduced group C^* -algebras that is very different from the maximal C^* -algebra case.

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2. Traces, idempotents, and rapid decay property

In this section, we explain the conjecture of Weinberger and Yu in [8], introduce the method of using traces to detect idempotents, and define the rapid decay property.

Consider a group G with torsion elements g_1, g_2, \dots, g_k of orders d_1, \dots, d_k respectively, where the d_i are distinct positive integers. For each g_i , we can consider the element

$$p_i = \frac{1 + g_i + g_i^2 + \dots + g_i^{d_i-1}}{d_i} \in \mathbb{C}G.$$

Then p_i is an idempotent, so we may consider it in the K_0 groups, $K_0(C_{\max}^*(G))$ and $K_0(C_{\text{red}}^*(G))$. We would like to explore whether

$$p_0 = 1, p_1, \dots, p_k$$

are linearly independent in these K_0 theories.

We call the subgroup of $K_0(C_{\max}^*G)$ (resp. $K_0(C_{\text{red}}^*G)$) generated by the p_i the finite part of $K_0(C_{\max}^*G)$ (resp. $K_0(C_{\text{red}}^*G)$).

It was conjectured by Weinberger and Yu in [8] that the p_i are linearly independent in $K_0(C_{\max}^*G)$. More precisely:

Conjecture 2.1. *For a group G with torsion elements g_1, g_2, \dots, g_k of orders d_1, \dots, d_k respectively, where the d_i are distinct positive integers. Let p_i denote*

the idempotent corresponding to g_i as above. Then the $1, p_1, \dots, p_k$ are linearly independent in $K_0(C_{\max}^*G)$. Moreover, any nonzero element in the subgroup generated by p_1, \dots, p_k is not in the image of the assembly map $K_0^G(EG) \rightarrow K_0(C_{\max}^*(G))$, where EG is the universal space for proper and free G -action.

The assembly map is the map $K_0^G(EG) \rightarrow K_0(C_{\max}^*(G))$ studied in [1] and [2].

To get at this question, we consider whether there are traces $\tau_0, \tau_1, \dots, \tau_k : C^*G \rightarrow \mathbb{C}$ such that the matrix $A_{ij} = \tau_i(p_j)$ has non-zero determinant. If this were true, then we know that the p_i are linearly independent, and we say that they are distinguishable by traces. If there are no such τ_i , then the p_i may still be linearly independent, but there is no way to prove this by traces, so we say that they are not distinguishable by traces.

Note that if idempotents can be separated by traces on C_{red}^*G , then they can also be separated by traces on C_{\max}^*G , because there is a natural map $C_{\max}^*G \rightarrow C_{\text{red}}^*G$, so the traces on C_{red}^*G induce traces on C_{\max}^*G in a way that does not affect the values on the idempotents. However, as we shall see, the converse is not true—indeed we shall give an example in which the idempotents can be separated by traces in C_{\max}^* , but not in C_{red}^* .

One other notion we will need is that of property RD. Recall that a length function on a group G is a function $l : G \rightarrow \mathbb{Z}^{\geq 0}$, such that $l(fg) \leq l(f) + l(g)$, $l(g^{-1}) = l(g)$, and $l(e) = 0$, such that for $S \subset \mathbb{Z}^{\geq 0}$ finite $l^{-1}(S)$ is also finite. Given such a length function on a group, there is a norm $\|\cdot\|_{H^s}$ on $\mathbb{C}G$ given by

$$\|\sum c_g g\|_{H^s}^2 = \sum |c_g|^2 (1 + l(g))^{2s}.$$

Let $H^s(G)$ denote the completion of $\mathbb{C}G$ with respect to this norm. Then we say G has rapid decay property, or property RD if for some s , there is a length function on it and a constant C such that

$$\|\sum c_g g\|_{\text{red}} \leq C \|\sum c_g g\|_{H^s}.$$

In this case, there is a natural map $H^s G \rightarrow C_{\text{red}}^*G$ obtained by extending the identity $\mathbb{C}G \rightarrow \mathbb{C}G$.

Note that for $t > s$, $\|x\|_{H^t} > \|x\|_{H^s}$ and we have natural map $H^t(G) \rightarrow H^s(G)$. So if we have property RD for s , then we also have the inequality for any $t > s$.

Jolissaint introduced this property in [6]. He also showed in [7] that for such groups the map $H^s G \rightarrow C_{\text{red}}^*G$ induces an isomorphism of K -theory $K_0(H^s G) \rightarrow K_0(C_{\text{red}}^*G)$.

3. Results

In this section, we show our main result, that in groups with property RD, the idempotents corresponding to torsion elements can be separated by traces if and

only if the conjugacy classes of the torsion elements grow polynomially.

In Subsection 3.1 we show that idempotents can be separated by traces if the conjugacy classes grow polynomially, and in Section 3.2 we show the converse.

3.1. Positive Results. Let G be a group with Property RD. For an element $g \in G$, let $C(g)$ denote its conjugacy class. Suppose that $g_1, \dots, g_n \in G$ are torsion elements such that $C(g_i)$ has polynomial growth, that is, for $C(g_i)_l = \{g \mid g \in C(g_i), l(g) = l\}$ and $n_{i,l} = |C(g_i)_l|$, we have $n_{i,l} < P(l)$ for some polynomial P . Then the p_i can be separated by traces on $H^s(G)$ for some s . More precisely, we have:

Lemma 3.1. *Let G be a group with Property RD and suppose $C(g_1)$ has polynomial growth. Then the trace $\tau : \mathbb{C}G \rightarrow G$ given by $\tau(\sum c_g g) = \sum_{g \in C(g_1)} c_g$ extends to $H^s(G)$ for sufficiently large s .*

Proof. Notice that

$$\begin{aligned} \left| \sum_{g \in C(g_1)} c_g \right| &\leq \sum_l \left| \sum_{g \in C(g_1)_l} c_g \right| \leq \sum_l \sqrt{n_{1,l}} \sqrt{\sum_{g \in C(g_1)_l} |c_g|^2} \\ &\leq C \sqrt{\sum_l n_{1,l} \left(\sum_{g \in C(g_1)_l} |c_g|^2 \right) l^2} \end{aligned} \tag{3.1}$$

for some constant C . Here, the second inequality follows from the Power Mean Inequality, which says $\frac{\sum_{i=1}^n a_i}{n} \leq \sqrt{\frac{\sum a_i^2}{n}}$, and the third inequality follows from the Cauchy Schwarz inequality, which implies that for a sequence a_l ,

$$\sum a_l \leq \sqrt{\left(\sum a_l^2 l^2\right) \left(\sum \frac{1}{l^2}\right)} \leq \frac{\pi}{\sqrt{6}} \sqrt{\left(\sum a_l^2 l^2\right)}.$$

But because $n_{1,l}$ is polynomial growth, the right hand side of (3.1) is bounded by the H^s norm for some s , so the trace can be extended, as desired. \square

Theorem 3.2. *Let G be a group with Property RD. Let g_1, \dots, g_n be torsion elements such that conjugacy classes $C(g_i)$ all have polynomial growth. Then for some s , the corresponding idempotents p_i can be separated by traces on H^s . Thus, they are linearly independent in $K_0(H^s G)$. Moreover, they are linearly independent in $K_0(C_{red}^* G)$, and in $K_0(C_{red}^* G)$, nonzero elements in the subgroup generated by p_i cannot lie in the image of the assembly map. Thus, groups with property RD where all the conjugacy classes of torsion elements grow polynomially satisfy Conjecture 2.1.*

Proof. Consider $\tau_0 : H^s(G) \rightarrow \mathbb{C}$ given by $\tau_0(\sum c_g g) = c_e$. This is clearly a bounded map. For $i = 1, \dots, k$ consider $\tau_i : H^s(G) \rightarrow \mathbb{C}$ given as above by

$$\tau_i(\sum c_g g) = \sum_{g \sim g_i} c_g.$$

Then $\tau_0(p_i) = \frac{1}{d_i}$. Also note that for $d_i > d_j$, for any t , we have $\deg(g_j^t) \leq \deg(g_j) < \deg(g_i)$, so g_j^t and g_i cannot be similar for any j . Thus for $d_i > d_j$, we have $\tau_i(p_j) = 0$.

Also note that $\tau_i(p_i) \geq \frac{1}{d_i}$, because $\tau_i(g_i^s)$ is either 0 or 1, and it is 1 for $s = 1$.

So if we suppose that $d_1 < d_2, \dots$, we get that matrix $\tau_i(p_j)$ is upper triangular with all diagonal elements non-zero, so its determinant is nonzero, as desired.

Jolissaint showed in [7] that for property RD groups, the map $H^s G \rightarrow C_{\text{red}}^* G$ is an isomorphism on the level of K-theory. That is, property RD implies $K_0(H^s(G)) \simeq K_0(C_{\text{red}}^*(G))$. Thus, the p_i are also linearly independent in $C_{\text{red}}^*(G)$.

Let us show that the non-zero elements in the subgroup generated by p_i are not in the image of the assembly map $K_0^G(EG) \rightarrow K_0(C_{\text{red}}^* G)$. For a Hilbert space H , let \mathcal{S} denote the ring of Schatten class operators, that is operators $T : H \rightarrow H$ such that there exists $p \geq 1$ such that $\text{tr}(T^* T)^{p/2} < \infty$, where the trace tr is the sum of the singular values. Let SG denote the group algebra of G with coefficients in \mathcal{S} . As in [8] and [9], let $H_0^{\text{or}G}(EG, \mathbb{K}(\mathcal{S})^{-\infty})$ be the generalized G -invariant homology theory associated to the non-connective algebraic K-theory spectrum $\mathbb{K}(\mathcal{S})^{-\infty}$. Then there is commutative diagram

$$\begin{array}{ccc} H_0^{\text{or}G}(EG, \mathbb{K}(\mathcal{S})^{-\infty}) & \xrightarrow{A} & K_0(SG) \\ \downarrow \simeq & \nearrow & \downarrow \\ K_0^G(EG) & \xrightarrow{\mu} & K_0(C_{\text{red}}^* G) \end{array}$$

There is a homomorphism $j : \mathbb{C}G \rightarrow SG$ given by $j(a) = P_0 a$ where P_0 is a rank 1 projection. We may consider the finite part of $K_0(SG)$ to be the subgroup generated by $j_*(p_i)$ where the p_i are defined as above. To show that the finite part of $K_0(C_{\text{red}}^* G)$ is not in the image of μ it suffices to show that the finite part of $K_0(SG)$ is not in the image of $H_0^{\text{or}G}(EG, \mathbb{K}(\mathcal{S})^{-\infty})$.

For an algebra \mathcal{A} , we say that $\tau_n : \mathcal{A}^{\times n+1} \rightarrow \mathbb{C}$ is a multitrace if it satisfies

$$\tau_n(a_0, a_1, \dots, a_n) = (-1)^n \tau_n(a_1, a_2, \dots, a_n, a_0)$$

and

$$\sum_{i=0}^n (-1)^i \tau_n(a_0, \dots, a_i a_{i+1}, \dots, a_{n+1}) + (-1)^{n+1} \tau_n(a_{n+1} a_0, a_1, \dots, a_n) = 0.$$

For g of finite order, consider the trace tr_g on $\mathcal{S}_1 G$ given by $\text{tr}_g(\sum_{\gamma \in C(g)} s_\gamma \gamma) = \sum_{\gamma \in C(g)} \text{tr}(s_\gamma)$, where tr is the standard trace on \mathcal{S}_1 . Then for $n = 2k$, consider $\tau_{n,g}$ on $\mathcal{S}_n(G)$ given by

$$\tau_{n,g}(a_0, \dots, a_n) = \text{tr}_g(a_0 a_1 \cdots a_n).$$

It was shown in [8] (see the proof of Lemma 3.2 of [8]) that for element $p \in K_0(\mathcal{S}G)$ in the image of A , we have that such $\tau_{n,g}, \tau_{n,g}(p, p, \dots, p) = 0$.

For groups with property RD, we have commutative diagram

$$\begin{array}{ccc} K_0^G(EG) & & \\ \downarrow & \searrow & \\ K_0(H^\infty G) & \xrightarrow{\cong} & K_0(C_{\text{red}}^* G) \end{array}$$

where $H^\infty G$ is the intersection of the $H^s G$, where the vertical maps are assembly maps.

Again as in [8], according to Connes' cyclic cohomology

$$j^* \tau_{n,g} = S^k \tau_g,$$

where $\tau_g : \mathbb{C}G \rightarrow \mathbb{C}$ is defined by $\tau_g(\sum_{\gamma \in C(g)} c_\gamma \gamma) = \sum_{\gamma \in C(g)} c_\gamma$. In [8] it was shown that for any element x in the image of the assembly map $A : H_0^{orG}(EG, \mathbb{K}(\mathcal{S})^{-\infty}) \rightarrow K_0(\mathcal{S}G)$, we have $\langle \tau_{n,g}, x \rangle = 0$ for the $\tau_{n,g}$ as defined above, but in this case, we know that there is a trace τ_{g_i} (where $\{g_i\}$ is as in Conjecture 2.1) of this form that does not vanish on x , so x cannot be in the image.

Now we show the conjecture, which is the same two statements, but for $C_{\text{max}}^* G$ instead of $C_{\text{red}}^* G$. For the first, note that there is a natural map $C_{\text{max}}^*(G) \rightarrow C_{\text{red}}^*(G)$, so linear independence in $C_{\text{red}}^* G$ implies linear independence in $C_{\text{max}}^*(G)$. For the second, note that the assembly map $K_0^G(EG) \rightarrow K_0(C_{\text{red}}^* G)$ factors through $K_0(C_{\text{max}}^* G)$, so if the elements considered are in the image of $K_0^G(EG)$ in $K_0(C_{\text{max}}^* G)$, then they are also in the image in $K_0(C_{\text{red}}^* G)$.

Thus, we have shown Conjecture 2.1 for Property RD groups whose conjugacy classes of torsion elements have polynomial growth. \square

Example 3.3. Let us discuss some examples of groups which satisfy our conditions above. One example is groups of polynomial growth, which are known to have Property RD.

Another example is a polynomial-distortion extension of a torsion free word hyperbolic group by a polynomial growth group. That is we consider group G with normal subgroup N of polynomial growth such that G/N is a torsion free word hyperbolic group and the length function of G upon restriction to N is polynomially distorted from the length function of N .

It was shown in [6] that polynomial growth groups have property RD and in [3] that hyperbolic groups have property RD. Thus, by the following result of [5], G has property RD.

Theorem 3.4. (*Garncarek*) *Let $1 \rightarrow N \rightarrow G \rightarrow H \rightarrow 1$ be an extension with polynomial distortion. If N and H have property RD, then so does G .*

All torsion elements in G are contained in N , because G/N is torsion-free. Thus, because the extension has polynomial distortion, the conjugacy classes of torsion elements have polynomial growth, and we may apply the above theorem.

Note that we have also established the conjecture for groups that are residually contained in the above class:

Definition 3.5. Let \mathcal{R} be the class of groups G with property RD such that for any torsion element g , $C(g)$ has polynomial growth. We say that a group G is residually in \mathcal{R} if for any finite subset $F \subset G$ there is a group $H \in \mathcal{R}$ and a homomorphism $\phi : G \rightarrow H$ such that ϕ is injective on F .

Theorem 3.6. *Any group that is residually in \mathcal{R} satisfies Conjecture 2.1.*

Proof. Let g_1, \dots, g_k be the torsion elements of order d_1, \dots, d_k in question, and let p_1, \dots, p_k be the corresponding idempotents. Let $F = \{g_i^{t_i} \mid 0 \leq t_i < d_i\}$. Then note that the $\phi(g_i)$ are torsion elements of order d_i in H , and their corresponding idempotents in $\mathbb{C}H$ are $\phi(p_i)$, where ϕ denotes the induced map $\phi : \mathbb{C}G \rightarrow \mathbb{C}H$. But note that any unitary representation of H is also a unitary representation of G , so ϕ also induces a map $C_{\max}^*G \rightarrow C_{\max}^*H$. By the above corollary, H satisfies the conjecture, so for $p_i \in K_0(C_{\max}^*G)$, we have $\phi(p_i) \in K_0(C_{\max}^*H)$ are linearly independent. But this means that the p_i are linearly independent in $K_0(C_{\max}^*G)$ as well.

The fact that nonzero elements in the subgroup generated by p_i are not in the image of the assembly map follows from the functoriality of the assembly map. \square

This theorem has consequences related to structure groups of manifolds. In particular we have

Corollary 3.7. *Let M be a compact manifold of dimension $4k - 1$ with $k > 1$ an integer and $\pi_1(M) = G$, and G is in the class \mathcal{R} described above. Then the rank of the structure group $S(M)$ is bounded below by the number of distinct orders of torsion elements of G .*

Proof. This follows from Theorem 1.5 in [8], which states that this corollary holds for any G satisfying Conjecture 2.1 \square

It is reasonable to conjecture that a similar statement to Conjecture 2.1 holds:

Conjecture 3.8. For a group G with torsion elements g_1, g_2, \dots, g_k of orders d_1, \dots, d_k respectively, where the d_i are distinct positive integers. Let p_i denote the idempotent corresponding to g_i . Then the $1, p_1, \dots, p_k$ are linearly independent in $K_0(C_{red}^*G)$. Moreover, any nonzero element in the subgroup generated by p_1, \dots, p_k is not in the image of the assembly map $K_0^G(EG) \rightarrow K_0(C_{max}^*(G))$, where EG is the universal space for proper and free G -action.

3.2. Negative Results. Now let us investigate the case where the group has Property RD, but some of the conjugacy classes do not have polynomial growth. We will show that all traces must be zero on these conjugacy classes.

Lemma 3.9. Let G be a group of Property RD and Let $h \in G$ be an element (which does not necessarily have to have finite order). Let $C(g)$ be its conjugacy class and let $C(g)_l \subset C(g)$ be those elements of length l , and let $n_l = |C(g)_l|$. Assume that n_l does not grow polynomially, that is, for any polynomial P , there are an infinite number of l such that $n_l > P(l)$. Then any trace $\tau : C_{red}^*G \rightarrow \mathbb{C}$ must have $\tau(h) = 0$.

Proof. The proof is in the same spirit as the proof in [6] that an amenable group that is not of polynomial growth does not have property RD. We proceed by contradiction. Suppose we have τ a trace such that $\tau(h) \neq 0$. By scaling it, we can assume $\tau(h) = 1$. Then because τ is a trace, it is constant on conjugacy classes, so for any $h \in C(g)$, we have $\tau(h) = 1$.

Since n_l is super-polynomial, we may consider an increasing sequence l_1, l_2, \dots such that $n_{l_i} > (1 + l_i)^{4i}$. That is, $n_{l_1} > (1 + l_1)^4, n_{l_2} > (1 + l_2)^8$, etc.

Now consider the element $x = \sum x_h h$ with

$$x_h = \begin{cases} n_{l(h)}^{-5/8} & l(h) \in \{l_1, l_2, \dots\} \text{ and } h \in C(g) \\ 0 & \text{else} \end{cases}.$$

Then note that for the Sobolev norm $\|\sum c_h h\|_{H^s}^2 = \sum |c_h|^2 (1 + l(h))^{2s}$, we have that the above element is in H^s with

$$\|\sum x_h h\|_{H^s} = \sum_i n_{l_i} \cdot n_{l_i}^{-10/8} (1 + l_i)^{2s} < \sum_i (1 + l_i)^{-i} (1 + l_i)^{2s}$$

which clearly converges.

In fact, $x_N = \sum_{l(h) < N} x_h h$ converges to x in the H^s norm. Therefore, this is also true in the C_r^*G , because property RD.

However, I claim that for any τ with $\tau(h) = 1$ on $C(g)$, τ does not converge on x_N , therefore τ is not a bounded linear functional on H^s . Note that the value would be

$$\sum_{i \text{ with } l_i < N} n_{l_i} \cdot n_{l_i}^{-5/8} = \sum_{i \text{ with } l_i < N} n_{l_i}^{3/8}$$

but $n_{l_i} > 1$ for all i , so clearly this sum diverges as $N \rightarrow \infty$.

Thus, τ cannot be a bounded linear functional. \square

Applying this lemma we have the following theorem:

Theorem 3.10. *Let G be a group with Property RD and let $g_1 \in G$ be a torsion element such that the conjugacy classes of g_1^t do not grow polynomially for any t . Then in $C_{red}^*(G)$ we cannot distinguish 1 , p_1 by traces.*

Proof. Applying the above lemma to each g_i^t for $0 < t < d$, we get that $\tau(g_i^t) = 0$ for all t , which means that $\tau(p_i) = \tau(\frac{1}{d_i})$, so for any trace τ , we have $\tau([p_i]) = \frac{1}{d_i} \tau([1])$, so we cannot show that 1 and p_1 are linearly dependent using traces. \square

Note that a similar result does not hold for C_{max}^*G . In particular, for torsion element $g \in G$ with order d , and p the corresponding idempotent, we can always distinguish $[1]$ and $[p]$ using the traces $\sum c_g g \mapsto c_e$ and $\sum c_g g \mapsto \sum c_g$, where the latter is a bounded map because it is the trace of the trivial action of G on \mathbb{C} .

One example satisfying the above conditions is $G = \langle x, y \rangle / x^3$ with $g_1 = x$. Then G is a hyperbolic group, so it has Property RD. Also, x and x^2 both have exponential growth conjugacy classes. In particular, the conjugacy classes contain $g x g^{-1}$ for

$$g = yx^{a_1}yx^{3-a_1}yx^{a_2}yx^{3-a_2}y \cdots yx^{a_l}yx^{3-a_l}y,$$

where $a_i \in \{1, 2\}$. Then g has length $5l + 1$ and $g x g^{-1}$ has length $10l + 3$, but there are 2^l choices of g . Thus, applying the above corollary, we see that $[1]$ and $[p_1]$ are not distinguishable by traces.

We also have another corollary to the above lemma:

Theorem 3.11. *Suppose that G is a Property RD group such that none of its conjugacy classes grow polynomially. Then the only trace on C_{red}^*G is $\sum c_g g \mapsto c_e$.*

Remark 3.12. Note that this gives another proof of de la Harpe's results in [4] for unique trace on torsion-free non-elementary word hyperbolic groups, that is, that such groups have a unique trace, because for such groups all conjugacy classes grow exponentially.

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