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Spectral triples and Toeplitz operators

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Abstract. We give examples of spectral triples, in the sense of A. Connes, constructed using the algebra of Toeplitz operators on smoothly bounded strictly pseudoconvex domains in $\mathbb{C}^n$, or the star product for the Berezin–Toeplitz quantization. Our main tool is the theory of generalized Toeplitz operators on the boundary of such domains, due to Boutet de Monvel and Guillemin.

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1. Introduction

The purpose of this work is to construct spectral triples $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ using the algebra $\mathcal{A}$ of Toeplitz operators acting on Bergman or Hardy spaces $\mathcal{H}$ on a smoothly bounded strictly pseudoconvex domain $\Omega \subset \mathbb{C}^n$.

Recall that a spectral triple consists, loosely speaking, of an algebra $\mathcal{A}$ of operators, acting on a Hilbert space $\mathcal{H}$, and a certain operator $\mathcal{D}$ which has bounded commutators with elements from $\mathcal{A}$. The notion was introduced by Connes, the model example being that of $\mathcal{A}$ the algebra of C^∞ functions on a Riemannian manifold M, $\mathcal{H}$ the space of L^2-spinors and $\mathcal{D}$ the Dirac operator; and a remarkable highlight of the theory is his reconstruction theorem that, in fact, every commutative spectral triple satisfying certain conditions arises in this way (up to isomorphism) [10]. Spectral triples are thus quintessential for the "noncommutative differential geometry" program [9, 11] and there exists an extensive literature on the subject, see e.g. the books [20, 28], and the references therein.

Our main tool are the so-called generalized Toeplitz operators, or Toeplitz operators with pseudodifferential symbols, on the boundaries of such domains, whose theory was developed by Boutet de Monvel and Guillemin [8]. Upon passing from holomorphic functions on the domain to their boundary values, the generalized Toeplitz operators turn out to include also the ordinary Toeplitz operators

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on (weighted) Bergman spaces of the domain [6, 23] (see also [26] for related constructions), which have long been used in quantization on Kähler manifolds (Berezin and Berezin–Toeplitz quantizations) [3, 13, 27, 32].

The set $\Psi(M)$ of classical pseudodifferential operators on a compact Riemannian manifold is an essential tool to understand the geometry of M. For instance, if $\mathcal{A}$ is the algebra $C^\infty(M)$ and $\mathcal{D} \in \Psi(M)$ is of order one, both acting on $\mathcal{H} = L^2(M)$, then, assuming $\mathcal{D} = \mathcal{D}^*$, all commutators $[\mathcal{D}, a]$ are bounded operators for any $a \in \mathcal{A}$ and if $\mathcal{D}$ is elliptic, $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ is a spectral triple. If $\Psi^0(M)$ is the algebra of pseudodifferential operators of order less than or equal to zero, one can check that $(\Psi^0(M), L^2(M), \mathcal{D})$ is also a spectral triple. An extension to non-compact manifolds is possible by imposing for instance that $a(1 + \mathcal{D}^2)^{-1}$ is compact for any $a \in \mathcal{A}$. The same phenomenon occurs abstractly, since a regular spectral triple $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ generates a pseudodifferential calculus $\Psi(\mathcal{A})$ which is the algebra of operators P on $\mathcal{H}$ with asymptotic expansion $P \simeq a_q|\mathcal{D}|^q + a_{q-1}|\mathcal{D}|^{q-1} + \cdots$ where the a_k are in the algebra generated by the $\delta^n(a)$, $n \in \mathbb{N}$, with $\delta(a) = [|\mathcal{D}|, a]$. Then $(\Psi^0(\mathcal{A}), \mathcal{H}, |\mathcal{D}|)$ will be also a spectral triple.

Here we play around similar notions in the framework of Toeplitz operators on an open bounded domain $\Omega \subset \mathbb{C}^n$. By a result of Howe, the algebra $\Psi(\mathbb{R}^n)$ is locally isomorphic to the algebra of Toeplitz operators on the unit ball $\mathbb{B}^n$ of $\mathbb{C}^n$ [26, 35]. This result has been generalized by Boutet de Monvel and Guillemin [6, 8, 23]: the generalized Toeplitz operators on a compact manifold possessing so-called Toeplitz structure form an algebra microlocally isomorphic (modulo smoothing operators) to the algebra of pseudodifferential operators on $\mathbb{R}^n$. Indeed, this general framework can be brought back to a microlocal model for generalized Toeplitz operators, via a Fourier integral operator constructed modulo smoothing operators [5], which sets up a bijection with the algebra of pseudodifferential operators on $\mathbb{R}^n$.

Of course, Ω is not compact, but we do not fall into the technicalities related to non unital spectral triples; but the price to pay is the intricate study of the role of the boundary of Ω which is of interest in complex analysis. This analysis has a long history which we intersect here only at few points: the Heisenberg algebra with its Fock and Bergman space representations and another quantization process than the Weyl one based on this Heisenberg algebra, namely the so-called Berezin–Toeplitz quantization.

We give several spectral triples involving algebras related to Toeplitz operators on Hardy and (weighted) Bergman spaces over smoothly bounded strictly pseudoconvex domains in $\mathbb{C}^n$. Different examples of the operator $\mathcal{D}$ are investigated. The constructions of these triples are elementary from a geometrical point of view but nevertheless sufficient to get a spectral triple based on the Berezin–Toeplitz star product. We compute their spectral dimension, a point related to the appearance of the Dixmier trace. Such appearance was already detected in [7] and studied in [1, 2, 16–18, 30] (see also [24] for the bidisk).

A spectral triple on the algebra $\{T_f \mid f \in C(S^1)\} \cap \Psi^0(S^1)$ acting on $\mathcal{H} = H^2(S^1) \oplus H^2(S^1)$ with an operator $\mathcal{D}$ based on the shift and on the "number operator" has already been proposed in [12].

We expect that our spectral triples encode information about the (CR-)geometry of Ω or $\partial\Omega$ in much the same way as they encode the information about the Riemannian geometry of the manifold in the original model example mentioned above; we plan to treat this question in a subsequent work.

More concretely, we recall in Section 2 the role of the Poisson kernel and the trace map between the Sobolev spaces on $\partial\Omega$ and Sobolev spaces of harmonic functions on Ω which can be associated to a weight w on $\overline{\Omega}$, extending known results on the analysis of Toeplitz operators of Szegö type or Bergman type (see for instance [6, 14]). Section 3 is devoted to the Fock and Bergman representations of the Lie algebra of the Heisenberg group, especially for the model case of Ω the unit ball of $\mathbb{C}^n$ with the standard weights, which are used in the next section on possible Dirac-like operators on $H^2(\partial\Omega)$ or A^2_w. In Section 5, we consider spectral triples with several candidates for operators $\mathcal{D}$. The first one is defined by an elliptic generalized Toeplitz operator of order one. An example is the inverse of the Toeplitz operator associated with the defining function of Ω. But since it is positive (so with trivial K-homology class), we double the Hilbert space and construct two classes of unitary operators compatible with the notion of generalized Toeplitz operators. A second idea is to start from the usual Dirac operator on $\mathbb{R}^n$ and to construct the corresponding operator acting on $\mathcal{H}$ through isomorphisms which involve a set of representations of the Lie algebra of the Heisenberg group. In the last section, upon applying the previous results to the special case of a suitable unit disc bundle over Ω, we exhibit a spectral triple directly related to the star product arising in the Berezin–Toeplitz quantization of Ω equipped with a natural Kähler structure [3, 27, 32].

2. Bergman and Hardy spaces, and Toeplitz operators

We gather in this section the preliminary material and known results.

2.1. Notations and definitions. We consider a open strictly pseudoconvex bounded set $\Omega \subset \mathbb{C}^n \approx \mathbb{R}^{2n}$, with smooth boundary $\partial\Omega$ for n a positive integer, so $\overline{\Omega} = \Omega \cup \partial\Omega$ is compact. Let $r : \overline{\Omega} \to \mathbb{R}^+$ be a positively signed defining function for Ω, i.e.

$$r \in C^\infty(\overline{\Omega}) \text{ with } r|_\Omega > 0,\ r|_{\partial\Omega} = 0 \text{ and } (\partial_{\mathbf{n}} r)|_{\partial\Omega} \neq 0, \tag{2.1}$$

where $\partial_{\mathbf{n}}$ is the normal derivative to the boundary. The condition on the normal derivative means that r is comparable to the distance to the boundary near it.

We decompose the exterior differential $d = \partial + \bar\partial$ into holomorphic and antiholomorphic parts and denote $\partial_i := \partial_{z_i}$, $\bar\partial_i := \partial_{\bar z_i}$.

The strict pseudoconvexity guarantees that the restriction $\eta := \frac{1}{2i}(\bar\partial r - \partial r)|_{\partial\Omega}$ to the boundary of the one-form $\mathrm{Im}(-\partial r)$ is a contact form, i.e. $\nu := \eta \wedge (d\eta)^{n-1}$ is a volume element on $\partial\Omega$. Let us also consider the half-line bundle

$$\Sigma := \{(x', s\,\eta_{x'}) \in T^*(\partial\Omega) \mid x' \in \partial\Omega,\ s > 0\}, \tag{2.2}$$

and a positive weight function w on Ω which is decomposed in the following way:

$$w =: r^{m_w}\, g_w, \text{ where } m_w \in (-1, +\infty) \text{ and } 0 < g_w \in C^\infty(\overline{\Omega}). \tag{2.3}$$

As Hermitian structure, we choose the Euclidean one on $\mathbb{C}^n \simeq \mathbb{R}^{2n}$, thus

$$\|dz_j\| = \sqrt{2} \text{ and } \|\partial r\| = \sqrt{2}\,\|\eta\|.$$

In the following, all pseudodifferential operators are classical and considered in $OPS^d_{1,0}$, for some $d \in \mathbb{R}$, using Hörmander notation (see [25, Definition 7.8.1]). For any pseudodifferential operator Q, we will denote by $\sigma(Q)$ its principal symbol.

Definition 2.1. The weighted Bergman space is

$$A^2_w(\Omega) := L^2_{hol}(\Omega, w) := \{f \in L^2(\Omega, w\,d\mu),\ f \text{ is holomorphic on } \Omega\},$$

endowed with the norm derived from the weighted scalar product $\langle f, g\rangle_w := \int_\Omega f\,\bar g\,w\,d\mu$, where μ is the Lebesgue measure.

The Bergman space $A^2(\Omega)$ is the unweighted Bergman space where $w = r^0 = 1$.

When $\Omega = \mathbb{B}^n$ and the defining function r is radial, i.e. $r(z) = r(|z|)$, and for a weight $w = r^{m_w}$, $m_w \in \mathbb{N}$, we have the following orthonormal basis for $A^2_w(\mathbb{B}^n)$ (see [22, Corollary 2.5]):

$$v^w_\alpha(z) = b_\alpha\, z^\alpha := \left[\int_{\mathbb{B}^n} z^\alpha \bar z^\alpha\, w(|z|)\, d\mu(z)\right]^{-1/2} z^\alpha, \quad \alpha \in \mathbb{N}^n. \tag{2.4}$$

In particular, an orthonormal basis of the unweighted Bergman space $A^2(\mathbb{B}^n)$ is given by the family (note that $\mu(\mathbb{B}^n) = \pi^n/n!$)

$$v_\alpha(z) = b_\alpha\, z^\alpha := \left(\tfrac{(|\alpha|+n)!}{n!\,\alpha!\,\mu(\mathbb{B}^n)}\right)^{1/2} z^\alpha, \tag{2.5}$$

(see [36, Lemma 1.11] for more details) where for a multiindex $\alpha \in \mathbb{N}^n$, we set $\alpha! := \prod_{k=1}^n \alpha_k!$ and $|\alpha| := \sum_{k=1}^n \alpha_k$.

We use the standard definition of Sobolev spaces of order $s \in \mathbb{R}$ on a subset of $\mathbb{R}^n$ or its boundary, and on $\Omega \subset \mathbb{C}^n$ or $\partial\Omega$, whose construction is given in [29, Chapitre 1], and also in [21, Appendix]. We denote them by $W^s(\Omega)$ and $W^s(\partial\Omega)$, respectively. We assume that the norms in these spaces have been chosen so that $W^0(\Omega) = L^2(\Omega, d\mu)$ with μ as above, and $W^0(\partial\Omega) = L^2(\partial\Omega)$ with respect to some smooth volume element on $\partial\Omega$, absolutely continuous with respect to the surface measure (for instance, $\nu = \eta \wedge (d\eta)^{n-1}$), which we fix from now on.

Definition 2.2. For $s \in \mathbb{R}$, the holomorphic Sobolev space on Ω of order s is defined by

$$W^s_{hol}(\Omega) := \{f \in W^s(\Omega),\ f \text{ is holomorphic on } \Omega\}.$$

Thus, $W^0_{hol}(\Omega) = A^2(\Omega)$.

Harmonic Sobolev spaces $W^s_{harm}(\Omega)$ are defined analogously. The set of harmonic functions in $L^2(\Omega,\, w\, d\mu)$ is denoted by $L^2_{harm}(\Omega,\, w)$.

Definition 2.3. The Poisson operator K is the harmonic extension operator which solves the Dirichlet problem: $\Delta\, Ku = 0$ on Ω, $Ku|_{\partial\Omega} = u$, where $\Delta = \partial\overline{\partial}$ is the complex Laplacian.

Thus K maps functions on $\partial\Omega$ to harmonic functions on Ω and by elliptic regularity theory (see [29]), K extends to a continuous map from $W^s(\partial\Omega)$ onto $W^{s+1/2}_{harm}(\Omega, w)$, for all $s \in \mathbb{R}$. In particular K maps $C^\infty(\partial\Omega)$ onto $C^\infty_{harm}(\overline{\Omega})$. We denote by K_w the operator K considered as a map from $L^2(\partial\Omega)$ into $L^2(\Omega, w\, d\mu)$, and by K^*_w its Hilbert space adjoint. For an arbitrary weight w, a simple computation shows K^*_w is related to $K(= K_1)$ through $K^*_w u = K^*(wu)$. In particular, $K^*_w K_w = K^* w K$.

Note that K_w is injective since $0 = K_w u \Rightarrow K_w u|_{\partial\Omega} = 0 \Leftrightarrow u = 0$. The operator K^* acts continuously from $W^s(\Omega)$ into $W^{s+1/2}(\partial\Omega)$, for all $s \in \mathbb{R}$.

We consider the operator

$$\Lambda_w := K^*_w K_w = K^*\, w\, K. \tag{2.6}$$

Actually Λ_w is an elliptic and selfadjoint pseudodifferential operator of order $-(m_w + 1)$ on $\partial\Omega$ (hence compact) with principal symbol (see [6])

$$\sigma(\Lambda_w)(x', \xi') = 2^{-1}\Gamma(m_w + 1)\, g_w(x') \|\eta_{x'}\|^{m_w}\ \big\|\xi'\big\|^{-(m_w+1)} \qquad x' \in \partial\Omega,\ \xi' \in T^*\partial\Omega, \tag{2.7}$$

so, when $m_w \in \mathbb{N}$,

$$\sigma(\Lambda_w)(x', \xi') = 2^{-(m_w+1)}\, (\partial^{m_w}_{\mathbf{n}} w)(x')\ \big\|\xi'\big\|^{-(m_w+1)}, \qquad x' \in \partial\Omega,\ \xi' \in T^*\partial\Omega. \tag{2.8}$$

This is actually a subject of the extensive theory of calculus of boundary pseudodifferential operators due to Boutet de Monvel [4].

In particular, Λ_w acts continuously from $W^s(\partial\Omega)$ into $W^{s+m_w+1}(\partial\Omega)$, for any $s \in \mathbb{R}$. Λ_w is an injection since for $u \in \operatorname{Ker}(\Lambda_w)$ and using the injectivity of K_w, we have $0 = \langle \Lambda_w u, u\rangle = \|K_w u\|^2$. The inverse operator Λ^{-1}_w is well defined on $\operatorname{Ran}(K^*_w)$, thus we have

$$\Lambda^{-1}_w\, K^*_w K_w = \mathbb{1}_{L^2(\partial\Omega)} \text{ and } K_w\, \Lambda^{-1}_w\, K^*_w = \mathbf{\Pi}_{w,harm}\,, \tag{2.9}$$

where $\mathbf{\Pi}_{w,harm}$ is the orthogonal projection from $L^2(\Omega, w)$ onto $L^2_{harm}(\Omega, w)$: the first equality is direct. Applying K_w on both side of it, we deduce that $K_w \Lambda^{-1}_w K^*_w$ is the identity on $\overline{\operatorname{Ran}(K_w)}$ which is the closure of $W^{1/2}_{harm}(\Omega)$ in $L^2(\Omega, w)$, i.e. $L^2_{harm}(\Omega, w)$. Moreover, $K_w \Lambda_w^{-1} K_w^*$ vanishes on $\overline{\mathrm{Ran}(K_w)}^{\perp} = \mathrm{Ker}(K_w^*)$, and we get the second equality of (2.9).

As a bounded operator, K_w has the polar decomposition

$$K_w =: U_w (K_w^* K_w)^{1/2} = U_w \Lambda_w^{1/2},$$

where U_w is a unitary from $L^2(\partial\Omega)$ onto $\overline{\mathrm{Ran}(K_w)} = L^2_{harm}(\Omega, w)$: $U_w^* U_w$ maps $L^2(\partial\Omega)$ to itself since

$$U_w^* U_w = \Lambda_w^{-1/2} K_w^* K_w \Lambda_w^{-1/2} = \Lambda_w^{-1/2} \Lambda_w \Lambda_w^{-1/2} = \mathbb{1}_{L^2(\partial\Omega)},$$

while $U_w U_w^* = K_w \Lambda_w^{-1} K_w^* |_{L^2_{harm}(\Omega,w)} = \mathbb{1}_{L^2_{harm}(\Omega,w)}$ by (2.9).

Definition 2.4. The trace operator $\boldsymbol{\gamma}_w : L^2(\Omega, w) \to L^2(\partial\Omega)$ is defined by

$$\boldsymbol{\gamma}_w := \Lambda_w^{-1} K_w^* .$$

In particular, (2.9) gives $K_w \, \boldsymbol{\gamma}_w |_{L^2_{harm}(\Omega,w)} = \mathbb{1}_{L^2_{harm}(\Omega,w)}$ and $\boldsymbol{\gamma}_w \, K_w = \mathbb{1}_{L^2(\partial\Omega)}$.

The operator $\boldsymbol{\gamma}_w$ is thus a left inverse of K_w, so it takes the boundary value of any function f in $L^2_{harm}(\Omega, w)$. Again, the index w is here to recall that the operator is defined on a weighted Hilbert space. Moreover, $\boldsymbol{\gamma}_w$ extends continuously to $\boldsymbol{\gamma}_w : W^s_{harm}(\Omega) \to W^{s-1/2}(\partial\Omega)$ for $s \in \mathbb{R}$.

We now define the following spaces:

Definition 2.5. The holomorphic Sobolev space on $\partial\Omega$ of order $s \in \mathbb{R}$ is

$$W^s_{hol}(\partial\Omega) := \{u \in W^s(\partial\Omega),\ Ku \text{ is holomorphic on } \Omega\},$$

and the Hardy space (endowed with the usual norm on $L^2(\partial\Omega)$) is

$$H^2 := H^2(\partial\Omega) := W^0_{hol}(\partial\Omega).$$

Note that the Hardy space is the closure of $C^\infty_{hol}(\partial\Omega)$ in $L^2(\partial\Omega)$.

We will use two types of Toeplitz operators using bold letters to refer to operators acting on Hilbert spaces defined over the domain Ω whereas the regular roman ones concern those over its boundary $\partial\Omega$:

Definition 2.6. If $u \in C^\infty(\partial\Omega)$, then the Toeplitz operator $T_u : H^2(\partial\Omega) \to H^2(\partial\Omega)$ is

$$T_u := \Pi \, M_u,$$

where $\Pi : L^2(\partial\Omega) \to H^2(\partial\Omega)$ is the Szegö projection and M_u (or just u) is the multiplication operator by u.

For $f \in C^\infty(\overline{\Omega})$, the Toeplitz operator $\mathbf{T}_f : A^2_w(\Omega) \to A^2_w(\Omega)$ is defined as

$$\mathbf{T}_f := \boldsymbol{\Pi}_w \, \mathbf{M}_f,$$

where $\boldsymbol{\Pi}_w : L^2(\Omega, w) \to A^2_w(\Omega)$ is the orthogonal projection onto the space of holomorphic functions in $L^2(\Omega, w)$ and $\mathbf{M}_f$ is the multiplication by f.

For the Hardy (respectively, Bergman) case, we have

$$u \to T_u \text{ is linear, } T_u^* = T_{\overline{u}},\ T_1 = \mathbb{1},\ \|T_u\| \le \|u\|_\infty,\ \text{(resp., idem for } \mathbf{T}_f).$$

Remark 2.7. For any strictly positive function u in $L^\infty(\partial\Omega)$, T_u is a selfadjoint and positive definite operator on $H^2(\partial\Omega)$ since $\langle T_u v, v\rangle = \int_{\partial\Omega} u|v|^2 > 0$ for any $v \neq 0$. In particular, it is injective, so there exists an unbounded inverse T_u^{-1}, which is densely defined on $H^2(\partial\Omega)$. The same is true in the Bergman case for $\mathbf{T}_f$, $f \in L^\infty(\Omega)$.

Boutet de Monvel and Guillemin studied in [8] (see also [6]) a more general notion of Toeplitz operators acting on Hardy spaces:

Definition 2.8. For a pseudodifferential operator P on $L^2(\partial\Omega)$ of order $m \in \mathbb{R}$, let T_P be the generalized Toeplitz operator (GTO): $W^m_{hol}(\partial\Omega) \to H^2(\partial\Omega)$ defined by

$$T_P := \Pi\, P|_{W^m_{hol}(\partial\Omega)}.$$

We can alternatively extend the definition of $T_P : W^m(\partial\Omega) \to H^2(\partial\Omega)$ by taking $T_P = \Pi\, P\, \Pi$.

It often happens that $T_P = T_Q$ with $P \neq Q$. However, the restriction of the principal symbol $\sigma(P)$ of P to Σ is always determined uniquely: when $T_P = T_Q$ then either $\mathrm{ord}(P) = \mathrm{ord}(Q)$ and in that case $\sigma(P)|_\Sigma = \sigma(Q)|_\Sigma$; or, for instance, $\mathrm{ord}(P) > \mathrm{ord}(Q)$ and in that case $\sigma(P)|_\Sigma = 0$. Therefore, the following quantities are well defined:

Definition 2.9. The order and the principal symbol of a GTO T_P are respectively

$$\mathrm{ord}(T_P) := \inf\{\mathrm{ord}(Q),\ T_P = T_Q\},$$
$$\sigma(T_P) := \sigma(Q)|_\Sigma, \text{ for any } Q \text{ such that } T_Q = T_P \text{ and } \mathrm{ord}(Q) = \mathrm{ord}(T_P).$$

The order can be $-\infty$, in which case the symbol is not defined.

As shown in [8], for any GTO T_P there exists a pseudodifferential operator Q such that $T_P = T_Q$ and $[Q, \Pi] = 0$. As a consequence, the GTO's form an algebra: if P, Q are two pseudodifferential operators, there exists another pseudodifferential operator R such that $T_P\, T_Q = T_R$. We have also the usual properties

$$\mathrm{ord}(T_P\, T_Q) = \mathrm{ord}(T_P) + \mathrm{ord}(T_Q), \quad \sigma(T_P\, T_Q) = \sigma(T_P)\,\sigma(T_Q).$$

Moreover, a GTO T_P of order m maps continuously holomorphic Sobolev spaces, namely

$$T_P : W^{s+m}_{hol}(\partial\Omega) \to W^s_{hol}(\partial\Omega), \text{ for any } s \in \mathbb{R},$$

because Π is (or rather extends to) a continuous map from $W^s(\partial\Omega)$ onto $W^s_{hol}(\partial\Omega)$ for any real number s.

A GTO is said to be elliptic if its principal symbol does not vanish. Like classical pseudodifferential operators, an elliptic GTO T_P of order m admits a parametrix T_Q which is a GTO of order $-m$, verifying

$$\sigma(T_Q) = \sigma(T_P)^{-1}, \quad T_P\,T_Q \sim \mathbb{1} \ \text{ and } \ T_Q\,T_P \sim \mathbb{1}.$$

Here and below $A \sim B$ means that $A - B$ is a smoothing operator (i.e. of order $-\infty$, or equivalently having Schwartz kernel in $C^\infty(\partial\Omega \times \partial\Omega)$).

Finally, if T_P is elliptic of order $m \neq 0$, positive and selfadjoint as an operator on $H^2(\partial\Omega)$, with $\sigma(T_P) > 0$, then the power T_P^s, $s \in \mathbb{C}$ (in the sense of the spectral theorem) is a GTO of order ms. In particular, for $s = -1$, the inverse of T_P is a GTO of order $-m$ (see [14, Proposition 16] for the details).

Let $\mathbf{P}$ be a differential operator on $\mathbb{C}^n$ of order $d \in \mathbb{N}$ of the form

$$\mathbf{P} = \sum_{|\nu|\le d} a_\nu(x)\, r(x)^j\, \partial^\nu + \sum_{|\nu'|\le d} b_{\nu'}(x)\, r(x)^{j'}\, \overline{\partial}^{\,\nu'} \tag{2.10}$$

for some $j, j' \in \mathbb{R}^+$, $\nu, \nu' \in \mathbb{N}^n$ and some functions $a_\nu, b_{\nu'} \in C^\infty(\overline{\Omega})$.

We can generalize the definition of Λ_w and construct the operator

$$\Lambda_{w\mathbf{P}} := K_w^* \mathbf{P} K_w = K^* w \mathbf{P} K \tag{2.11}$$

acting on the boundary $\partial\Omega$. By Boutet de Monvel's theory [4,21,31], $\Lambda_{w\mathbf{P}}$ is again a pseudodifferential operator on the boundary and the following proposition gives a formula for its principal symbol.

Proposition 2.10. *The operator $\Lambda_{w\mathbf{P}}$ is a pseudodifferential operator on the boundary $\partial\Omega$ and $T_{\Lambda_{w\mathbf{P}}}$ is a GTO of order $d - (m_w + 1 + j)$ with principal symbol*

$$\sigma(T_{\Lambda_{w\mathbf{P}}})(x', \xi') = \tfrac{(-1)^d\, \Gamma(m_w+1+j)}{2\|\xi'\|^{-d+m_w+1+j}}\, g_w(x')\, \|\eta_{x'}\|^{-d+m_w+j} \sum_{|\nu|=d} a_\nu(x') \prod_{k=1}^{n} (\partial_k r)^{\nu_k}(x') \tag{2.12}$$

(when (2.12) vanishes, $T_{\Lambda_{w\mathbf{P}}}$ is in fact of lower order).

Proof. For $k \in \{1, \dots, n\}$, define the tangential operators Z_k and $\overline{Z}_k$ on $\partial\Omega$ by

$$Z_k := \gamma\, \partial_k\, K, \quad \overline{Z}_k := \gamma\, \overline{\partial}_k\, K. \tag{2.13}$$

As $K^* w \mathbf{P} K = \sum_\nu K^* a_\nu r^j w \partial^\nu K + \sum_{\nu'} K^* b_{\nu'} r^{j'} w \overline{\partial}^{\nu'} K = \sum_\nu \Lambda_{a_\nu r^j w} Z^\nu + \sum_{\nu'} \Lambda_{b_{\nu'} r^{j'} w} \overline{Z}^{\nu'}$ with $Z = \gamma \partial K$ and the same for $\overline{Z}$, we see from (2.7) that indeed $K^* w \mathbf{P} K$ is a pseudodifferential operator on $\partial\Omega$ of order $d - (m_w + j + 1)$ (or less if there are some cancellations in the summation on ν). Since $T_{\Lambda_{w\mathbf{P}}} = \Pi\, \Lambda_{w\mathbf{P}}\, \Pi$, we have

$$\sigma(T_{\Lambda_{w\mathbf{P}}})(x',\xi') = \sigma(\Lambda_{w\mathbf{P}}|_{H^2})\Big(x', \tfrac{\|\xi'\|}{\|\eta_{x'}\|}\,\eta_{x'}\Big)$$
$$= \sigma\Big(\sum_{\nu} \Lambda_{a_\nu r^j w} Z^\nu\Big)\Big(x', \tfrac{\|\xi'\|}{\|\eta_{x'}\|}\,\eta_{x'}\Big)$$

By a direct computation, $\sigma(Z_k)(x',\xi') = i\langle \xi', Z_k\rangle = -\frac{\|\xi'\|}{\|\eta_{x'}\|}\partial_k r$ (see also [14, p. 1440]) and from (2.7), we have

$$\sigma(T_{\Lambda_{w\mathbf{P}}})(x',\xi')$$
$$= \tfrac{(-1)^d\,\Gamma(m_w+1+j)}{2\|\xi'\|^{m_w+1+j}}\, g_w(x')\,\|\eta_{x'}\|^{m_w+j} \sum_{|\nu|=d} a_\nu(x') \prod_k \sigma(Z_k)^{\nu_k}(x',\xi'),$$

so the result follows. □

We remark that equation (2.12) is still valid when $m_w \in \mathbb{C}$ with $\mathrm{Re}(m_w) > -1$. Also, when $m_w \in \mathbb{N}$, using $\|\partial r\| = \sqrt{2}\,\|\eta_{x'}\|$, the right-hand side can be written as

$$\frac{(-1)^d\,\Gamma(m_w+1+j)}{2^{-d+m_w/2+j+1}\Gamma(m_w+1)^{-d+j}\,\|\xi'\|^{-d+m_w+1+j}}\,\big(\partial_{\mathbf{n}}^{(m_w)}w\big)^{-d+j}(x')$$
$$\cdot \sum_{|\nu|=d} a_\nu(x') \prod_{k=1}^{n} (\partial_k r)^{\nu_k}(x').$$

2.2. Links between the spaces on Ω and $\partial\Omega$. The operator T_{Λ_w} exists as a positive, elliptic and compact GTO of order $-(m_w+1)$ on $L^2(\partial\Omega)$ and maps continuously $W^s_{hol}(\partial\Omega)$ into $W^{s+m_w+1}_{hol}(\partial\Omega)$, for any $s \in \mathbb{R}$.

Let $u \in \mathrm{Ker}(T_{\Lambda_w}) \subset W^s_{hol}(\partial\Omega)$ for a certain $s \in \mathbb{R}$, then

$$0 = \langle T_{\Lambda_w}u, u\rangle_{W^s_{hol}(\partial\Omega)} = \langle \Pi\Lambda_w u, u\rangle_{W^s_{hol}(\partial\Omega)} = \langle \Lambda_w u, \Pi u\rangle_{W^s_{hol}(\partial\Omega)}.$$

Since $\Pi u = u$ we get, using the injectivity of Λ_w

$$0 = \langle \Lambda_w u, u\rangle_{W^s_{hol}(\partial\Omega)} = \|\Lambda_w^{1/2}u\|^2 \Rightarrow u = 0.$$

Thus, for any $s \in \mathbb{R}$, the inverse operator $T^{-1}_{\Lambda_w}$ exists from $\mathrm{Ran}(T_{\Lambda_w}) = W^{s+m_w+1}_{hol}(\partial\Omega)$ onto $W^s_{hol}(\partial\Omega)$.

The following result was proved in [15, Theorem 4].

Proposition 2.11. *Let T be a positive selfadjoint operator on $H^2(\partial\Omega)$ such that $T \sim T_P$, where P is an elliptic pseudodifferential operator of order $s \in \mathbb{R}$ such that $\sigma(T_P) > 0$.*

Let $W^T_{hol}(\partial\Omega)$ be the completion of $C^\infty_{hol}(\partial\Omega)$ with respect to the norm $\|u\|^2_T := \langle Tu, u\rangle_{H^2}$. Then, we have $W^T_{hol}(\partial\Omega) = W^{\mathrm{ord}(T_P)/2}_{hol}(\partial\Omega)$.

Proposition 2.12. *The operator K_w maps bijectively the space $W_{hol}^{T_{\Lambda w}}(\partial\Omega)$ onto $A^2_w(\Omega)$.*

Proof. Let $f \in C^\infty_{hol}(\overline{\Omega}) \subset A^2_w(\Omega)$ and $u = \boldsymbol{\gamma}_w f \in C^\infty_{hol}(\overline{\Omega})$. Then

$$\|f\|^2_{A^2_w} = \langle K_w u, K_w u\rangle_{L^2(\Omega,w)} = \langle \Lambda_w u, u\rangle_{L^2(\partial\Omega)}$$
$$= \langle \Pi \Lambda_w u, u\rangle_{L^2(\partial\Omega)} = \langle T_{\Lambda_w} u, u\rangle_{L^2(\partial\Omega)}.$$

Thus K_w is an isometry of $W_{hol}^{T_{\Lambda w}}(\partial\Omega)$ onto the completion of $C^\infty_{hol}(\overline{\Omega})$ in A^2_w.

In a manner completely similar to (2.9) (see [18] for details), we get

$$\boldsymbol{\Pi}_w = K_w \, \Pi \, T^{-1}_{\Lambda_w} \, \Pi \, K^*_w. \tag{2.14}$$

Since $C^\infty_{hol}(\overline{\Omega})$ is dense in A^2_w (just note that $C^\infty(\overline{\Omega})$ is dense in $L^2(\Omega, w)$, while the weighted Bergman projection $\boldsymbol{\Pi}_w$ maps each $W^s_{hol}(\Omega)$, and, hence, $C^\infty(\overline{\Omega})$ into itself), the claim follows. □

From the fact that K_w is an isomorphism of $W^s_{hol}(\partial\Omega)$ onto $W^{s+1/2}_{hol}(\Omega)$ $\forall s \in \mathbb{R}$, we also see that $A^2_w(\Omega) = K_w W^{-(m_w+1)/2}(\partial\Omega) = W^{-m_w/2}_{hol}(\Omega)$ and $\boldsymbol{\gamma}_w$ is an isomorphism of A^2_w onto $W^{-(m_w+1)/2}(\partial\Omega)$.

As we already said, $T^{1/2}_{\Lambda_w}$ is an isomorphism of $W^s_{hol}(\partial\Omega)$ onto $W^{s+\frac{m_w+1}{2}}(\partial\Omega)$ for all $s \in \mathbb{R}$, with equivalent norms. As a consequence,

Lemma 2.13. *The operator*

$$V_w := K_w \, T^{-1/2}_{\Lambda_w} \text{ is a unitary which maps } H^2(\partial\Omega) \text{ onto } A^2_w(\Omega). \tag{2.15}$$

Proof. We have $V^*_w V_w = T^{-1/2}_{\Lambda_w} K^*_w K_w \, T^{-1/2}_{\Lambda_w} = T^{-1/2}_{\Lambda_w} T_{\Lambda_w} T^{-1/2}_{\Lambda_w} = \mathbb{1}_{H^2}$. Similarly, $V_w V^*_w = \mathbb{1}_{A^2_w(\Omega)}$, see (2.14). □

Now we identify a Toeplitz operator $\mathbf{T}_f$ on $A^2_w(\Omega)$ with generalized Toeplitz operators acting on $H^2(\partial\Omega)$ via γ_w and K_w or V_w and V^*_w:

Proposition 2.14. *For $f \in C^\infty(\overline{\Omega})$, we have*

$$\begin{aligned} \boldsymbol{\gamma}_w \, \mathbf{T}_f \, K_w &= T^{-1}_{\Lambda_w} \, T_{\Lambda_{wf}} && \text{on } W^{-(m_w+1)/2}_{hol}(\partial\Omega), \\ \mathbf{T}_f &= V_w \, T^{-1/2}_{\Lambda_w} \, T_{\Lambda_{wf}} \, T^{-1/2}_{\Lambda_w} \, V^*_w && \text{on } A^2_w(\Omega). \end{aligned} \tag{2.16}$$

Proof. For any u an v in $W^{-(m_w+1)/2}_{hol}(\partial\Omega)$, we get

$$\begin{aligned} \langle \mathbf{T}_f \, K_w u, K_w v\rangle_{A^2_w(\Omega)} &= \langle \boldsymbol{\Pi}_w \, f \, K_w u, K_w v\rangle_{A^2_w(\Omega)} = \langle f \, K_w u, \boldsymbol{\Pi}_w \, K v\rangle_{A^2_w(\Omega)} \\ &= \langle f \, K_w u, K_w v\rangle_{A^2_w(\Omega)} = \langle w f \, K u, K v\rangle_{L^2(\Omega)} \\ &= \langle (K^* \, w f \, K) u, v\rangle_{H^2(\partial\Omega)} = \langle \Lambda_{wf} u, \Pi v\rangle_{H^2(\partial\Omega)} \\ &= \langle T_{\Lambda_{wf}} u, v\rangle_{H^2(\partial\Omega)} = \langle K_w \, T^{-1}_{\Lambda_w} \, T_{\Lambda_{wf}} \, u, K_w v\rangle_{H^2(\partial\Omega)}. \end{aligned}$$

Thus $\mathbf{T}_f K_w = K_w T_{\Lambda_w}^{-1} T_{\Lambda_{wf}}$ on $W_{hol}^{-(m_w+1)/2}(\partial\Omega)$, hence $\boldsymbol{\gamma}_w \mathbf{T}_f K_w = T_{\Lambda_w}^{-1} T_{\Lambda_{wf}}$.

Finally, we get $V_w^* \mathbf{T}_f V_w = V_w^* (K_w \gamma_w) \mathbf{T}_f (K_w \gamma_w) V_w = T_{\Lambda_w}^{-1/2} T_{\Lambda_{wf}} T_{\Lambda_w}^{-1/2}$. □

From the GTO theory and the mapping properties of K_w and $\boldsymbol{\gamma}_w$, we see that the right-hand side of the first formula in (2.16) extends to a bounded operator on any $W_{hol}^s(\Omega)$, hence the left-hand side enjoys the same property.

For any differential operator $\mathbf{P}$ with coefficients as in (2.10), we can define the following "Toeplitz operator with symbol $\mathbf{P}$":

$$\mathbf{T}_\mathbf{P} := \boldsymbol{\Pi}_w \mathbf{P} \quad \text{acting on } A_w^2(\Omega). \tag{2.17}$$

Lemma 2.15. *For* $\mathbf{P}$ *as above, we have*

$$\begin{aligned} \gamma_w \mathbf{T}_\mathbf{P} K_w &= T_{\Lambda_w}^{-1} T_{\Lambda_{w\mathbf{P}}} && \text{on } W_{hol}^{-(m_w+1)/2}(\partial\Omega), \\ \mathbf{T}_\mathbf{P} &= V_w T_{\Lambda_w}^{-1/2} T_{\Lambda_{w\mathbf{P}}} T_{\Lambda_w}^{-1/2} V_w^* && \text{on } A_w^2(\Omega). \end{aligned} \tag{2.18}$$

Moreover, $\mathbf{T}_\mathbf{P}$ *is selfadjoint on* $A_w^2(\Omega)$ *when* $\mathbf{P}$ *has a selfadjoint extension on* $L^2(\Omega, w)$.

Proof. Similar calculation as in the proof of (2.16) shows (2.18).

We only need to prove that $(T_{\Lambda_{w\mathbf{P}}})$ is selfadjoint, which follows from (2.11): for $u,\, v \in H^2(\partial\Omega)$,

$$\begin{aligned} \langle\, (T_{\Lambda_{w\mathbf{P}}})^* u,\, v\rangle_{H^2(\partial\Omega)} &= \langle u, K^* w\mathbf{P}K\, v\rangle_{H^2(\partial\Omega)} = \langle Ku,\, \mathbf{P}Kv\rangle_{L^2(\Omega,w)} \\ &= \langle \mathbf{P}Ku,\, Kv\rangle_{L^2(\Omega,w)} = \langle w\mathbf{P}Ku,\, Kv\rangle_{L^2(\Omega)} \\ &= \langle K^* w\mathbf{P}K\, u, v\rangle_{H^2(\partial\Omega)} = \langle T_{\Lambda_{w\mathbf{P}}} u,\, v\rangle_{H^2(\partial\Omega)}. \end{aligned}$$

□

Remark 2.16. An interesting example of selfadjoint operator $\mathbf{T}_\mathbf{P}$, where $\mathbf{P}$ is not selfadjoint on $L^2(\Omega, w)$ is given by the "weighted normal derivative" operator

$$\mathbf{P}_w := \sum_{j=1}^n \frac{\overline{\partial_j (rw)}}{w}\, \partial_j\,;$$

note that $\partial_j(rw)/w$ is smooth up to the boundary. Using Stokes' formula, for f, g in $A_w^2(\Omega)$,

$$\int_\Omega d\mu\, \partial_j (rw\, f\overline{g}) = -\int_{\partial\Omega} d\sigma\, rw\, f\,\overline{g}\, \tfrac{\partial_j r}{2\|\partial r\|} = 0$$

since $rw = 0$ on $\partial\Omega$ (here $d\sigma$ is the ordinary surface measure on $\partial\Omega$). Applying Leibniz rule to the left-hand side gives

$$\int_\Omega w d\mu\, \tfrac{\partial_j (rw)}{w}\, f\overline{g} = -\int_\Omega w d\mu\, (r\partial_j) f\, \overline{g} \quad \text{hence } \mathbf{T}_{\partial_j(rw)/w} + \mathbf{T}_{r\partial_j} = 0$$

where the Toeplitz operators act on $A_w^2(\Omega)$.

Since $\mathbf{T}_h^* = \mathbf{T}_{\overline{h}}$ and $\mathbf{T}_{h\partial_j} = \mathbf{T}_h \mathbf{T}_{\partial_j}$, we get

$$\mathbf{T}_{\mathbf{P}_w} = \sum_{j=1}^{n} \big(\mathbf{T}_{\partial_j(rw)/w}\big)^* \,\mathbf{T}_{\partial_j} = -\sum_{j=1}^{n} \big(\mathbf{T}_{r\partial_j}\big)^* \,\mathbf{T}_{\partial_j} = -\sum_{j=1}^{n} \mathbf{T}_{\partial_j}^* \,\mathbf{T}_r \,\mathbf{T}_{\partial_j}.$$

Thus $\mathbf{T}_{\mathbf{P}_w}$ is not only selfadjoint but even negative on $A_w^2(\Omega)$.

Now in the context of the unweighted Bergman space, we can even compute the symbol of the operator $\gamma\, \mathbf{T}_{\mathbf{P}_{w=1}}\, K$. Indeed, using Stokes' formula, we get for $f, g \in A^2(\Omega)$,

$$\begin{aligned}\langle \mathbf{T}_{\partial_j} f, g\rangle_{A^2(\Omega)} = \langle \partial_j f, g\rangle_{A^2(\Omega)} &= \int_\Omega d\mu\, (\partial_j f)\, \bar g \\ &= -\int_{\partial\Omega} d\sigma\, f\, \overline{g}\, \tfrac{\partial_j r}{2\|\partial r\|} - \int_\Omega d\mu\, f\, \partial_j(\overline{g}) \\ &= -\int_{\partial\Omega} d\sigma\, f\, \overline{g}\, \tfrac{\partial_j r}{2\|\partial r\|}\,.\end{aligned}$$

It follows that, on $W_{hol}^{-1/2}(\partial\Omega)$, $K^*\mathbf{T}_{\partial_j} K = -T_{\partial_j r/2\|\partial r\|}$ and

$$\begin{aligned}\gamma\, \mathbf{T}_{\partial_j} K &= -\Lambda^{-1}\, T_{\partial_j r/2\|\partial r\|}\,,\\ \mathbf{T}_{\partial_j} &= -V\, T_\Lambda^{-1/2} T_{\partial_j r/2\|\partial r\|} T_\Lambda^{-1/2}\, V^*.\end{aligned} \tag{2.19}$$

The use of (2.7), (2.12) and (2.18) gives

$$\sigma(\gamma \mathbf{T}_{\mathbf{P}_{w=1}} K)(x',\xi') = -2\, \|\eta_{x'}\|\, \big\|\xi'\big\|, \quad (x',\xi') \in \Sigma. \tag{2.20}$$

In particular $-V^*\, \mathbf{T}_{\mathbf{P}_{w=1}} V$ is a positive elliptic GTO.

Remark 2.17. The hypothesis $\mathbf{P} = \mathbf{P}^*$ in Lemma 2.15 is quite strong because the equality $\mathbf{T}_{\mathbf{P}}^* = \mathbf{T}_{\mathbf{P}^*}$ is not necessarily true: when $w = 1$, one deduces from (2.19) that

$$\mathbf{T}_{\partial_j}^* = -V\, T_\Lambda^{-1/2} T_{\overline{\partial_j r}/2\|\partial r\|} T_\Lambda^{-1/2}\, V^* \neq 0 = \mathbf{T}_{-\bar\partial_j}$$

while $-\overline{\partial}_j$ is the formal adjoint of ∂_j on the domain of smooth compactly supported functions on Ω. Of course, any selfadjoint extension of a differential operator $\mathbf{P}$ needs to take care of the boundary conditions on $\partial\Omega$.

We now establish a result on unitaries $\mathbf{U}$ on $A_w^2(\Omega)$ such that $V_w^*\, \mathbf{U}\, V_w$ is a GTO, which will be used later on in Section 5.

Lemma 2.18. *Let $\varphi \in S^0(\mathbb{R})$, i.e.*

$$\forall j \in \mathbb{N} \text{ there exists } c_j(\varphi) > 0 \text{ with } |\partial_\xi^j \varphi(\xi)| \le c_j(\varphi)(1+\|\xi\|)^{-j},\ \forall \xi \in \mathbb{R},$$

and let A be an elliptic selfadjoint pseudodifferential operator of order 1 on a compact manifold M. Then the operator $\exp(i\varphi(A))$ *is a pseudodifferential operator of order 0 on M.*

Proof. The Faà di Bruno formula states that for any smooth function f on $\mathbb{R}$,

$$\partial^n(f\circ\varphi)(\xi) = \sum_{(k_1,\ldots,k_n)\in K_n} \tfrac{n!}{k_1!\ldots k_n!}\,(\partial^{\sum_j k_j} f)\big(\varphi(\xi)\big)\,\prod_{j=1}^{n}(\tfrac{1}{j!}\,\partial^j\varphi)^{k_j}(\xi)\,,\qquad \forall\, n\in\mathbb{N},$$

where $K_n = \{(k_1,\ldots,k_n)\in\mathbb{N}^n, k_1+2k_2+\cdots+nk_n = n\}$.

Thus, with $f:\xi\in\mathbb{R}\mapsto \exp(i\,\xi)$, we get

$$|\partial^n(f\circ\varphi)(\xi)| \le \sum_{(k_1,\ldots,k_n)\in K_n} \tfrac{n!}{k_1!\ldots k_n!}\left[\prod_{j=1}^{n}(\tfrac{1}{j!}c_j(\varphi))^{k_j}\right](1+\|\xi\|)^{-n}.$$

So that there are constants $c_n(f\circ\varphi)>0$ such that

$$|\partial^n(f\circ\varphi)(\xi)| \le c_n(f\circ\varphi)(1+\|\xi\|)^{-n},$$

and we have shown that the function $\xi\in\mathbb{R}\mapsto\exp\big(i\varphi(\xi)\big)$ is in $S^0(\mathbb{R})$.

Applying [33, Theorem 1] (or [34, Theorem 1.2]), we conclude that $\exp(i\varphi(A))$ is a pseudodifferential operator of order 0 on M. □

Corollary 2.19. *Let φ be any function in $S^0(\mathbb{R})$ (for instance, $\varphi(\xi) := (1+\xi)^2\cdot(1+\xi^2)^{-1}$) and choose a pseudodifferential operator A on $\partial\Omega$ of order 1 such that A commutes with Π and $T_A = (T_{\Lambda_w})^{-1/(m_w+1)}$. Denote $V_\varphi := \exp\big(i\varphi(A)\big)$. Then T_{V_φ} is a unitary generalized Toeplitz operator on $H^2(\partial\Omega)$.*

Proof. By previous lemma, $V_\varphi := \exp\big(i\varphi(A)\big)$ is a unitary classical pseudodifferential operator on $M=\partial\Omega$ which commutes with Π, thus $T_{V_\varphi} = \exp\big[i\varphi\big(T_{\Lambda_w}^{-1/(m_w+1)}\big)\big]$ is a unitary generalized Toeplitz operator. □

Remark 2.20. In Section 5.2, we will need to find unitary operators on $A^2_w(\Omega)$ of the form $V_w T V_w^*$, with T a GTO, to deduce non-positive Dirac-like operators from positive ones. Thus we can take $\mathbf{U} := V_w T_{V_\varphi} V_w^*$, with T_{V_φ} defined in Corollary 2.19.

Another class of unitary operators on $A^2_w(\Omega)$ can be obtained as follows. Take any GTO T_P which is invertible and not a constant multiple of a positive operator. (For instance, $T_P = T_f$ with f a nonconstant zero-free holomorphic function: the zero-free condition ensures $T_f = M_f$ is invertible, while $T_f = cA$ with $A>0$ would mean that multiplication by the nonconstant holomorphic function f/c is a positive operator, which is quickly seen to lead to contradiction.) We know there exists another pseudodifferential operator Q such that $\Pi\, Q = Q\,\Pi$ and $T_P = T_Q$. Hence also $T_P^* T_P = T_Q^* T_Q = T_{Q^*Q} = T_{|Q|}^2$, implying that $U := T_P\, T_{|Q|}^{-1}$ is a unitary generalized Toeplitz operator. From Proposition 2.14, the operator $\mathbf{U} := V_w\, U\, V_w^*$ is unitary from $A^2_w(\Omega)$ onto $A^2_w(\Omega)$. Furthermore, $\mathbf{U}$ is not a multiple of the identity; for, if it were, then so would be U, hence $T_P = U\,T_{|Q|}$ would be a constant multiple of the positive operator $T_{|Q|}$, contrary to the hypothesis.

3. Fock space and Heisenberg algebra

3.1. Fock space and Segal–Bargmann transform. If $(x, y) \in \mathbb{R}^{2n}$, we denote $xy := \sum_{k=1}^{n} x_k y_k$. For $(z, z') \in \mathbb{C}^{2n}$, we use the usual scalar product $\langle z, z' \rangle := \sum_{k=1}^{n} z_k \bar{z}'_k$. Also, for $(x, z) \in \mathbb{R}^n \times \mathbb{C}^n$, we denote xz the complex number $\sum_{k=1}^{n} x_k z_k$ (so $x^2 = \sum_{k=1}^{n} x_k^2$ and $z^2 = \sum_{k=1}^{n} z_k^2$), and $|z|^2 := \sum_{k=1}^{n} |z_k|^2$.

We recall now some known results about the Fock space:

Definition 3.1. For $t > 0$, the Fock space is

$$\mathscr{F}_t := L^2_{hol}(\mathbb{C}^n, dm_t(z)), \text{ where } dm_t(z) := (\pi t)^{-n}\, e^{-|z|^2/t}\, d\mu(z).$$

This Fock space is denoted $\mathscr{F}$ in [26, Chap. 1.7] when $t = \pi^{-1}$. Here, the measure dm_t has been chosen such that $m_t(1) = 1$. The family of functions $\{u_\alpha\}_{\alpha \in \mathbb{N}^n}$, where

$$u_\alpha(z) = a_\alpha\, z^\alpha := (t^{|\alpha|}\alpha!)^{-1/2}\, z^\alpha, \tag{3.1}$$

forms an orthonormal basis of $\mathscr{F}_t$.

Definition 3.2. The Segal–Bargmann transform $W_t : L^2(\mathbb{R}^n, dx) \to \mathscr{F}_t$ is

$$(W_t\, f)(z) := (\pi t)^{-n/4} \int_{x \in \mathbb{R}^n} e^{(-z^2 + 2\sqrt{2}\, xz - x^2)/2t}\, f(x)\, dx.$$

Proposition 3.3 (see [19]). *The Segal–Bargmann transform W_t is a unitary map from $L^2(\mathbb{R}^n)$ to $\mathscr{F}_t$. Moreover, for any $f \in \mathscr{F}_t$,*

$$(W_t^{-1} f)(x) = (\pi t)^{-5n/4} \int_{z \in \mathbb{C}^n} e^{(-\bar{z}^2 + 2\sqrt{2}\, x\bar{z} - x^2)/2t}\, f(z)\, e^{-|z|^2/t}\, d\mu(z).$$

The map W_t plays an important role since it links unitarily functions of real variables to holomorphic functions of complex variables.

3.2. The Heisenberg group and its Lie algebra.

Definition 3.4. The Heisenberg group $\mathbb{H}^n$ is the set $\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}$ endowed with product:

$$(q, p, s)\, (q', p', s') = \big(q + q', p + p', s + s' + \tfrac{1}{2}(qp' - pq')\big),$$

where $(q, p) \in \mathbb{R}^{2n}$ and $s \in \mathbb{R}$.

The unit element is $(0, 0, 0)$ and $(q, p, s)^{-1} = (-q, -p, -s)$.

Let us define for $j \in \{1, \cdots, n\}$ the generators Q_j, P_j and T of $\mathbb{H}^n$ as:

$$\begin{aligned} \exp(Q_j) &:= (1_j, (0, \ldots, 0), 0), \\ \exp(P_j) &:= \big((0, \ldots, 0), 1_j, 0\big), \\ \exp(T) &:= \big((0, \ldots, 0), (0, \ldots, 0), 1\big), \end{aligned}$$

where 1_k denotes the multiindex being zero everywhere and 1 at the k^{th} position.

These generators form a basis of the Lie algebra $\mathfrak{h}^n$ of $\mathbb{H}^n$. The only non null commutation relations are $[Q_j, P_k] = \delta_{j,k} T$. We also define the two elements of $\mathfrak{h}^n$

$$a_j := \tfrac{1}{\sqrt{2}}(Q_j + iP_j) \text{ and } a_j^+ := \tfrac{1}{\sqrt{2}}(Q_j - iP_j),$$

which verify $[a_j, a_k^+] = -\frac{i}{2}([Q_j, P_k] + [Q_k, P_j]) = -i\delta_{j,k} T$.

We will also use the element N of the universal enveloping algebra $\mathscr{U}(\mathfrak{h}^n)$ of $\mathfrak{h}^n$:

$$N := \tfrac{1}{2}\sum_{j=1}^{n} a_j^+ a_j + a_j a_j^+.$$

3.3. Representations of $\mathfrak{h}^n$. Different representations of $\mathfrak{h}^n$ on the function spaces that we consider here have been studied in [26] and later in [35] with different approaches.

3.3.1. Schrödinger representation.

Definition 3.5. The Schrödinger representation ρ_t of $\mathfrak{h}^n$ on $L^2(\mathbb{R}^n, dx)$ is defined by

$$\rho_{t'}(Q_j) f(x) := x_j f(x), \quad \rho_{t'}(P_j) f(x) := -it'\,\partial_{x_j} f(x),$$
$$\rho_{t'}(T) f(x) := it' f(x),$$

where t' is a strictly positive parameter.

With this representation, the canonical commutation relations from the quantum mechanics can be recovered by setting $t' = \hbar$, the Planck constant, $\widehat{x_j} = \rho(Q_j)$ and $\widehat{p_j} = \rho(P_j)$, respectively the position and momentum operators:

$$[\widehat{x_j}, \widehat{p_j}] = -i\hbar\,\partial_{x_j} + i\hbar\,\mathbb{1} + i\hbar\,\partial_{x_j} = i\hbar\,\mathbb{1} = \rho([Q_j, P_j]).$$

We have also

$$\rho_{t'}(a_j) = \tfrac{1}{\sqrt{2}}(x_j + t'\,\partial_{x_j}), \quad \rho_{t'}(a_j^+) = \tfrac{1}{\sqrt{2}}(x_j - t'\,\partial_{x_j}),$$
$$\rho_{t'}(N) = \tfrac{1}{4}\sum_{j=1}^{n}(x_j^2 - t'^2\,\partial_{x_j}^2).$$

3.3.2. Fock representation. From the Schrödinger representation $\rho_{t'}$, we use the unitary map W_t to get a unitary representation of $\mathfrak{h}^n$ on the Fock space $\mathscr{F}_t$ by choosing $t' = t$.

Definition 3.6. The Fock representation ν_t of an element $h \in \mathfrak{h}^n$ on $\mathscr{F}_t$ is

$$\nu_t(h) := W_t\, \rho_t(h)\, W_t^{-1}.$$

Proposition 3.7. *The explicit actions on the basis vectors of $\mathscr{F}_t$ are given by*

$$\begin{aligned}
\nu_t(Q_j)\, u_\alpha &= (\tfrac{t}{2})^{1/2}\,(\sqrt{\alpha_j}\, u_{\alpha-1_j} + \sqrt{\alpha_j+1}\, u_{\alpha+1_j}),\\
\nu_t(P_j)\, u_\alpha &= -i(\tfrac{t}{2})^{1/2}\,(\sqrt{\alpha_j}\, u_{\alpha-1_j} - \sqrt{\alpha_j+1}\, u_{\alpha+1_j}),\\
\nu_t(T)\, u_\alpha &= it\, u_\alpha,\\
\nu_t(a_j)\, u_\alpha &= t^{1/2}\, \sqrt{\alpha_j}\, u_{\alpha-1_j}, \quad \nu_t(a_j^+)\, u_\alpha = t^{1/2}\, \sqrt{\alpha_j+1}\, u_{\alpha+1_j},\\
\nu_t(N)\, u_\alpha &= t\,(|\alpha| + \tfrac{n}{2})\, u_\alpha.
\end{aligned}$$

Proof. Differentiating in Definition 3.2, we get

$$(\partial_{z_j} W_t f)(z) = \big(W_t(-\tfrac{z_j}{t} + \tfrac{\sqrt{2}x_j}{t}) f\big)(z),$$

so $$\tfrac{z_j + t\partial_{z_j}}{t}\, W_t = W_t\, \tfrac{\sqrt{2}x_j}{t}, \quad \text{or} \quad W_t\, \tfrac{\sqrt{2}x_j}{t}\, W_t^{-1} = \tfrac{z_j}{t} + \partial_{z_j}.$$

Similarly, integrating by parts in Definition 3.2 we get

$$\big(W_t(\partial_{x_j} f)\big)(z) = \big(W_t(\tfrac{x_j}{t} - \tfrac{\sqrt{2}z_j}{t}) f\big)(z),$$

so $$W_t\Big(\tfrac{x_j}{t} - \partial_{x_j}\Big) = \tfrac{\sqrt{2}z_j}{t} W_t, \quad \text{or} \quad W_t\,\Big(\tfrac{x_j}{t} - \partial_{x_j}\Big)\, W_t^{-1} = \tfrac{\sqrt{2}}{t}\, z_j.$$

Consequently,

$$W_t\, x_j\, W_t^{-1} = \tfrac{1}{\sqrt{2}}(z_j + t\,\partial_{z_j}), \qquad W_t\, \partial_{x_j}\, W_t^{-1} = \tfrac{1}{\sqrt{2}}(-z_j + t\,\partial_{z_j}).$$

Using the formulas in §3.3.1, we get the result. □

3.3.3. Bergman representation. The Fock and Bergman spaces are linked through a simple change of basis via the unitary operator $\mathscr{V}_t^w : \mathscr{F}_t \to A_w^2(\Omega)$:

$$\mathscr{V}_t^w\,(u_\alpha) := v_\alpha^w\,, \tag{3.2}$$

where u_α is as in (3.1) and v_α^w is any orthonormal basis of $A_w^2(\Omega)$.

We will also need the map

$$U_t^w := \mathscr{V}_t^w\, W_t, \tag{3.3}$$

which is, by construction, a unitary operator from $L^2(\mathbb{R}^n, dx)$ to $A_w^2(\Omega)$.

For the peculiar case $\Omega = \mathbb{B}^n$, $\partial\Omega = \mathbb{S}^{2n-1}$ and $w = r^0 = 1$, we can get explicit formulas. Using the basis (2.5) of $A^2(\mathbb{B}^n)$, we denote the above change of basis by $\mathscr{V}_t : \mathscr{F}_t \to A^2(\mathbb{B}^n)$:

$$\mathscr{V}_t\,(u_\alpha) := v_\alpha\,. \tag{3.4}$$

Here again, we fix the constant t' from the Schrödinger representation ρ'_t with $t' = t$ to define the Bergman representation.

Definition 3.8. The Bergman representation τ_t^w of an element $h \in \mathfrak{h}^n$ on $A^2_w(\Omega)$ is given by

$$\tau_t^w(h) := \mathscr{V}_t^w\,\nu_t(h)\,(\mathscr{V}_t^w)^{-1} = U_t^w\,\rho_t(h)\,(U_t^w)^{-1}.$$

We denote by τ_t this representation in the case $\Omega = \mathbb{B}^n$, $\partial\Omega = \mathbb{S}^{2n-1}$ and $w = r^0 = 1$.

We get directly the following results, since the Bergman representation differs from the Fock representation by a simple change of basis:

Proposition 3.9. *The representation τ_t^w has the following properties:*

$$\begin{aligned}
\tau_t^w(Q_j)\,v_\alpha^w &= (\tfrac{t}{2})^{1/2}\big(\sqrt{\alpha_j}\,v^w_{\alpha-1_j} + \sqrt{\alpha_j+1}\,v^w_{\alpha+1_j}\big),\\
\tau_t^w(P_j)\,v_\alpha^w &= -i(\tfrac{t}{2})^{1/2}\big(\sqrt{\alpha_j}\,v^w_{\alpha-1_j} - \sqrt{\alpha_j+1}\,v^w_{\alpha+1_j}\big),\\
\tau_t^w(T)\,v_\alpha^w &= it\,v_\alpha^w,\\
\tau_t^w(a_j)\,v_\alpha^w &= t^{1/2}\sqrt{\alpha_j}\,v^w_{\alpha-1_j}, \quad \tau_t^w(a_j^+)\,v_\alpha^w = t^{1/2}\sqrt{\alpha_j+1}\,v^w_{\alpha+1_j},\\
\tau_t^w(N)\,v_\alpha^w &= t(|\alpha| + \tfrac{n}{2})\,v_\alpha^w.
\end{aligned}$$

3.4. A diagram as a summary. The figure below shows a diagram with all the representations and maps involved here: the dotted, dashed, simple and thick arrows refer to injections, surjections, isomorphisms and isometries, respectively. Double arrows indicate the action of the Lie algebra $\mathfrak{h}^n$ on the Hilbert spaces.

We must highlight the fact that only compositions of maps which do not involve dotted or dashed lines (projectors) are commutative. Indeed, for instance $U_w\,\mathbf{\Pi}_w \neq \Pi\,V_w$.

Note that K_w maps $L^2(\partial\Omega)$ into $L^2_{harm}(\Omega, w)$ with dense range and that $\gamma\,\mathbf{\Pi}_w\,K_w = T^{-1}_{\Lambda_w}\,\Pi\,\Lambda_w$ on $L^2(\partial\Omega)$ (see [15, Proposition 8]), so $\gamma\,\mathbf{\Pi}_w\,K_w\,\Pi = \Pi$.

For the case $\Omega = \mathbb{B}^n$, $w = r^0 = 1$, replace $W_{hol}^{-(m_w+1)/2}(\partial\Omega)$, $\mathscr{V}_t^w$, U_t^w and V_w of (2.15) respectively by $W_{hol}^{-1/2}(\partial\Omega)$, $\mathscr{V}_t$ as in (3.4), U_t as in (3.3) and V.

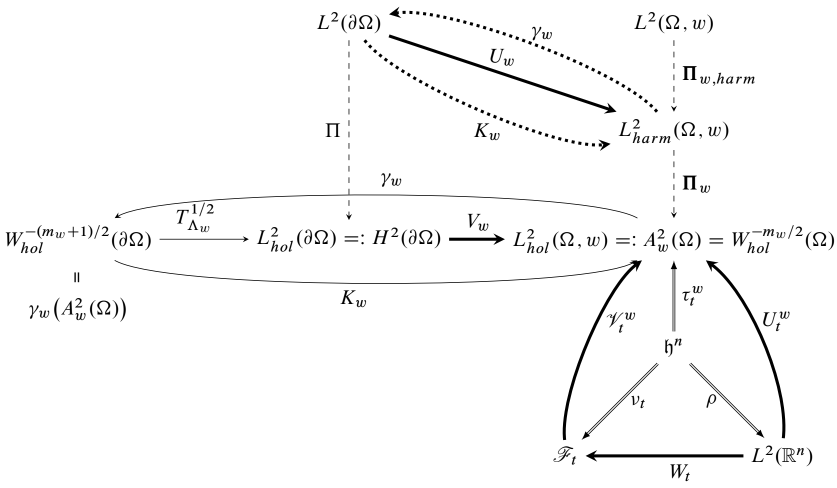

Figure 1.

4. Pseudodifferential and Toeplitz operators

The relation between pseudodifferential operators and Toeplitz operators has been studied in [26], [8] and [23]. The authors show that these operators enjoy a similar symbolic calculus. Moreover, there exists an isomorphism between pseudodifferential operators of order k on $L^2(\mathbb{R}^n)$ and Toeplitz operators of order $k/2$ on the Bergman space $A^2(\mathbb{B}^n)$. This isomorphism is nothing else than the conjugation by the unitary U_t defined above (see [35, Appendix B] for a detailed proof).

4.1. Toeplitz operators on $A^2(\mathbb{B}^n)$ as representation of elements in $\mathscr{U}(\mathfrak{h}^n)$. We express here the Toeplitz operators $\mathbf{T}_{z^\alpha}$ and $\mathbf{T}_{\partial^\alpha}$ acting on the Bergman space $A^2(\mathbb{B}^n)$ as representations of elements in the enveloping algebra $\mathscr{U}(\mathfrak{h}^n)$. We follow [26, Chap. 4.2] for the first result.

Proposition 4.1. *For $\alpha \in \mathbb{N}^n$, let*

$$a^\alpha := \prod_{j=1}^n a_j^{\alpha_j}, \quad (a^+)^\alpha := \prod_{j=1}^n (a_j^+)^{\alpha_j}$$

(the order is not important since $[a_j, a_k] = [a_j^+, a_k^+] = 0$), and

$$g_\alpha^\pm := \prod_{k=1}^{|\alpha|} \big(N - i(\tfrac{n}{2} \pm k)T\big)$$

be elements of $\mathscr{U}(\mathfrak{h}^n)$.

The Toeplitz operators $\mathbf{T}_{z^\alpha}$ *and* $\mathbf{T}_{\partial^\alpha}$ *from* $A^2(\mathbb{B}^n)$ *to* $A^2(\mathbb{B}^n)$ *can be written as*

$$\mathbf{T}_{z^\alpha} = \tau_t\big((a^+)^\alpha\big)\,\big(\tau_t(g_\alpha^+)\big)^{-1/2} \quad \textit{and} \quad \mathbf{T}_{\partial^\alpha} = \tau_t\big(a^\alpha\big)\,\big(\tau_t(g_\alpha^-)\big)^{-1/2}.$$

Note that the t-dependences in the respective right-hand sides compensate, so these sides do not depend on t.

Proof. Since $z \mapsto z^\alpha$ is holomorphic, $\mathbf{T}_{z^\alpha} = \mathbf{M}_{z^\alpha}$, and we have also $\mathbf{T}_{\partial^\alpha} = \mathbf{\Pi}_w \partial^\alpha = \partial^\alpha$ on $A^2(\mathbb{B}^n)$. Applied on the basis, they yield

$$\begin{aligned} \mathbf{T}_{z^\alpha}\, v_\beta &= z^\alpha\, v_\beta = \tfrac{b_\alpha}{b_{\beta+\alpha}}\, v_{\beta+\alpha} = \big(\tfrac{(\beta+\alpha)!}{\beta!}\,\tfrac{(|\beta|+n)!}{(|\beta|+|\alpha|+n)!}\big)^{1/2}\, v_{\beta+\alpha}\,, && \forall\alpha,\beta\,,\\ \mathbf{T}_{\partial^\alpha}\, v_\beta &= \partial^\alpha\, v_\beta = \tfrac{\beta!}{(\beta-\alpha)!}\,\tfrac{b_\alpha}{b_{\beta-\alpha}}\, v_{\beta-\alpha} = \big(\tfrac{\beta!}{(\beta-\alpha)!}\,\tfrac{(|\beta|+n)!}{(|\beta|-|\alpha|+n)!}\big)^{1/2}\, v_{\beta-\alpha}\,, && \beta\geq\alpha\,, \end{aligned} \tag{4.1}$$

where for the second equality, the case $\alpha > \beta$ corresponds to the null operator. So that for the rest, we only consider $\mathbf{T}_{\partial^\alpha}$ on the domain $\mathrm{Span}\{v_\beta\}_{\beta\geq\alpha}$. From Proposition 3.9, we deduce

$$\begin{aligned} \tau_t\big((a^+)^\alpha\big)\, v_\beta &= t^{|\alpha|/2}\big(\tfrac{(\beta+\alpha)!}{\beta!}\big)^{1/2}\, v_{\beta+\alpha}, && \forall\alpha,\beta,\\ \tau_t(a^\alpha)\, v_\beta &= t^{|\alpha|/2}\big(\tfrac{\beta!}{(\beta-\alpha)!}\big)^{1/2}\, v_{\beta-\alpha}, && \beta\geq\alpha. \end{aligned}$$

Moreover, the representations of the elements $g_\alpha^\pm$ act on $A^2(\mathbb{B}^n)$ as

$$\begin{aligned} \tau_t(g_\alpha^\pm)\, v_\beta = \prod_{k=1}^{|\alpha|}\big(\tau_t(N) - i(\tfrac{n}{2}\pm k)\tau_t(T)\big)\, v_\beta &= t^{|\alpha|}\prod_{k=1}^{|\alpha|}(|\beta|+n\pm k)\, v_\beta\\ &= t^{|\alpha|}\,\tfrac{(|\beta|\pm|\alpha|+n)!}{(|\beta|+n)!}\, v_\beta\,. \end{aligned}$$

Thus the operators $\tau_t(g_\alpha^\pm)$ are invertible on $A^2(\mathbb{B}^n)$ and we get the claimed formulae. □

Lemma 4.2. *The operator*

$$\mathbf{R} := \sum_{j=1}^n \mathbf{T}_{z_j}\mathbf{T}_{\partial_j} = \sum_{j=1}^n \mathbf{T}_{z_j\partial_j} \quad \textit{on } A^2(\mathbb{B}^n)$$

is positive and

$$\tau_t(P_j) = -i(\tfrac{t}{2})^{1/2}\big[\mathbf{T}_{\partial_j}\,(\mathbf{R}+n)^{-1/2} - (\mathbf{T}_{\partial_j}\,(\mathbf{R}+n)^{-1/2})^*\big] \tag{4.2}$$

(showing again that $\tau_t(P_j)$ is selfadjoint).

The operator $\gamma\tau_t(P_j)K$ is a GTO of order $\frac{1}{2}$ with principal symbol

$$\sigma\big(\gamma\tau_t(P_j)K\big)(x',\xi') = 2^{3/4}(\tfrac{t}{2})^{1/2}\big(\tfrac{\|\eta_{x'}\|}{-R(r)(x')}\big)^{1/2}\,\tfrac{\xi'_j}{\|\xi'\|^{1/2}}\,, \quad (x',\xi')\in\Sigma. \tag{4.3}$$

Proof. By (4.1), $\mathbf{R}\, v_\beta = |\beta|\, v_\beta$ for any multiindex β, so $\mathbf{R} \geq 0$. Since

$$\mathbf{T}_{\partial_j}\,(\mathbf{R}+n)^{-1/2}\,v_\alpha = \sqrt{\alpha_j}\,v_{\alpha-1_j}\,, \quad (\mathbf{R}+n)^{1/2}\mathbf{T}_{z_j}\,v_\alpha = (\alpha_j+1)^{1/2}\,v_{\alpha+1_j}$$
$$(\mathbf{T}_{z_j})^*\,v_\alpha = \big(\tfrac{\alpha_j}{|\alpha|+n}\big)^{1/2}\,v_{\alpha-1_j}\,,$$

we get

$$\mathbf{T}^*_{z_j} = (\mathbf{R}+n+1)^{-1}\mathbf{T}_{\partial_j} = \mathbf{T}_{\partial_j}\,(\mathbf{R}+n)^{-1}. \tag{4.4}$$

(These relations could also be deduced from $(T^{\mathscr{F}_t}_{z_j})^* = t\,T^{\mathscr{F}_t}_{\partial_j}$ where $T^{\mathscr{F}_t}$ refers to the Toeplitz operator on the Fock space, and for instance,

$$\mathbf{T}_{\partial_j} = \sqrt{t}\,\mathscr{V}_t\,T^{\mathscr{F}_t}_{\partial_j}\,\mathscr{V}_t^{-1}\,(\mathbf{R}+n)^{1/2}.)$$

By Proposition 3.9, this yields

$$\tau_t(P_j)\,v_\alpha = -i\,(\tfrac{t}{2})^{1/2}\Big[\tfrac{1}{\sqrt{|\alpha|+n}}\,\mathbf{T}_{\partial_j}v_\alpha - \sqrt{|\alpha|+n+1}\,\mathbf{T}_{z_j}v_\alpha\Big],$$

and $\tau_t(P_j) = -i\,(\frac{t}{2})^{1/2}\big[\mathbf{T}_{\partial_j}\,(\mathbf{R}+n)^{-1/2} - \mathbf{T}_{z_j}(\mathbf{R}+n+1)^{1/2}\big]$. Thus, (4.4) implies (4.2).

As in the proof of Remark 2.16, we get for $f, g \in A^2(\Omega)$ (omitting the sum over j):

$$\langle\,(\mathbf{R}+n)f, g\rangle = \int_\Omega d\mu\,\partial_j(z_j f)\,\overline{g} = \int_\Omega d\mu\,\partial_j(z_j f\overline{g}) = -\int_{\partial\Omega} d\sigma\; f\,\overline{g}\,\tfrac{z_j\partial_j r}{2\|\partial r\|}\,. \tag{4.5}$$

Since $\mathbf{R}+n \geq 0$, $-\mathbf{R}(r)$ is a positive function on $\partial\Omega$ (for instance, when $r(z) = 1-\|z\|^2$, $\mathbf{R}(r)(x') = -\|x'\|^2$). Thus, on $W^{-1/2}_{hol}(\partial\Omega)$, $K^*(\mathbf{R}+n)\,K = T_{-\mathbf{R}(r)/2\|\partial r\|} =: X$ with $X \geq 0$, implying $\mathbf{R}+n = V\,T_\Lambda^{-1/2} X T_\Lambda^{-1/2}\,V^*$ and $(\mathbf{R}+n)^{-1/2} = V\,[T_\Lambda^{-1/2} X T_\Lambda^{-1/2}]^{-1/2}\,V^*$.

Finally $\gamma\,(\mathbf{R}+n)^{-1/2}\,K = T_\Lambda^{-1/2}\,[T_\Lambda^{-1/2} X T_\Lambda^{-1/2}]^{-1/2}\,T_\Lambda^{1/2}$. Thus

$$\sigma(\gamma\,(\mathbf{R}+n)^{-1/2}\,K) = \sigma(\Lambda)^{1/2}\,\sigma(X)^{-1/2}$$

(so this symbol is positive as it has to be).

Using (4.2) with $A = \gamma\,\mathbf{T}_{\partial_j}\,K\,\gamma\,(\mathbf{R}+n)^{-1/2}\,K$, we obtain

$$\begin{aligned}\sigma\big(\gamma\tau_t(P_j)K\big) &= -i\,(\tfrac{t}{2})^{1/2}[\sigma(A)-\sigma(A^*)] = 2(\tfrac{t}{2})^{1/2}\,\mathrm{Im}(\sigma(A))\\ &= 2(\tfrac{t}{2})^{1/2}\,\sigma(\gamma\,(\mathbf{R}+n)^{-1/2}\,K)\,\mathrm{Im}(\sigma\big(\gamma\,\mathbf{T}_{\partial_j}\,K)\big).\end{aligned}$$

Since $\gamma\,\mathbf{T}_{\partial_j}\,K = \Lambda^{-1}\,T_{-\partial_j r/2\|\partial r\|}$, we get $\sigma(\gamma\,\mathbf{T}_{\partial_j}\,K) = \sigma(\Lambda)^{-1}\,(-\partial_j r/2\,\|\partial r\|)$. Finally,

$$\begin{aligned}\sigma\big(\gamma\tau_t(P_j)K\big)(x',\xi') &= -2(\tfrac{t}{2})^{1/2}\Big(\sigma(\Lambda)^{-1/2}\,\sigma(X)^{-1/2}\,(2\,\|\partial r\|)^{-1}\,\mathrm{Im}(\partial_j r)\Big)(x',\xi')\\ &= -2(\tfrac{t}{2})^{1/2}\,(2\,\|\xi'\|)^{1/2}\,\big(\tfrac{-\mathbf{R}(r)(x')}{2\|\partial r\|}\big)^{-1/2}\,(2\,\|\partial r\|)^{-1}\,\mathrm{Im}(\partial_j r)(x')\\ &= -2(\tfrac{t}{2})^{1/2}\,\big(\tfrac{\|\xi'\|}{\|\partial r\|(-R(r)(x'))}\big)^{1/2}\,\mathrm{Im}(\partial_j r)(x')\\ &= -2(\tfrac{t}{2})^{1/2}\,\big(\tfrac{\|\xi'\|}{\sqrt{2}\|\eta_{x'}\|(-R(r)(x'))}\big)^{1/2}\,\mathrm{Im}(\partial_j r)(x').\end{aligned}$$

Moreover, $-\mathrm{Im}(\partial_j r)(x') = (\eta_{x'})_j = \frac{\|\eta'_x\|}{\|\xi'\|}\,\xi'_j$, which yields the result. □

4.2. Dirac-like operators on $A^2_w(\Omega)$ and $H^2(\partial\Omega)$. Our goal is to construct spectral triples $(\mathcal{A},\mathcal{H},\mathcal{D})$ (see Definition 5.1 below) using the algebra $\mathcal{A}$ of Toeplitz operators acting on Hilbert spaces $\mathcal{H}$ of functions on bounded domains such as Bergman and Hardy spaces. The natural candidates $\mathcal{D}_{\partial\Omega}$ and $\mathcal{D}_\Omega$ defined below are the images of the usual Dirac operator $\not{D} := -i\sum_{j=1}^n \Gamma_j\,\partial_{x_j}$ on $\mathbb{R}^n$ through the maps involved in the diagram of Section 1. Here the Γ_j are the usual selfadjoint gamma matrices which represent the n-dimensional Clifford algebra $\mathcal{C}_n \cong \mathbb{C}^{2^{[n/2]}}$. The resulting operators act on $H^2(\partial\Omega)\otimes\mathcal{C}_n$ or $A^2_w(\Omega)\otimes\mathcal{C}_n$ respectively, and are even generalized Toeplitz operators of order 1/2, according to our previous result.

Since the representation of $P_j \in \mathfrak{h}^n$ on $L^2(\mathbb{R}^n)$ is $\rho_t(P_j) = -it\,\partial_{x_j}$, we define:

$$\mathcal{D}_{\partial\Omega} := (\tfrac{t}{2})^{-1/2}\sum_{j=1}^n \Gamma_j\,V_w^*\,\tau_t^w(P_j)\,V_w\,, \tag{4.6}$$

$$\mathcal{D}_\Omega := V_w\,\mathcal{D}_{\partial\Omega}\,V_w^* = (\tfrac{t}{2})^{-1/2}\sum_{j=1}^n \Gamma_j\,U_t^w\,\rho_t(P_j)\,U_t^{w\,*}, \tag{4.7}$$

acting respectively on $H^2(\partial\Omega)\otimes\mathcal{C}_n$ and $A^2_w(\Omega)\otimes\mathcal{C}_n$. They are selfadjoint and again, do not depend on t.

Another way to construct operators $\mathcal{D}$ on $A^2_w(\Omega)$ is to consider an operator $\mathbf{T}_\mathbf{P}$ of the form (2.17), with $\mathbf{P}$ a selfadjoint differential operator on $L^2(\Omega,w)$ as in Proposition 2.10.

5. Spectral triples

Definition 5.1. A (unital) spectral triple is defined by the data $(\mathcal{A},\mathcal{H},\mathcal{D})$ with

- an involutive unital algebra $\mathcal{A}$,
- a faithful representation π of $\mathcal{A}$ on a Hilbert space $\mathcal{H}$,

– a selfadjoint operator $\mathcal{D}$ acting on $\mathcal{H}$ with compact resolvent such that for any $a \in \mathcal{A}$, the extended operator of $[\mathcal{D}, \pi(a)]$ is bounded.

The spectral dimension of the triple is $d := \inf\{d' > 0 \text{ such that } \operatorname{Tr}|\mathcal{D}|^{-d'} < \infty\}$. The spectral triple is called regular if the spaces $\mathcal{A}$ and $[\mathcal{D}, \mathcal{A}]$ are contained in the domain of δ^k, for all $k \in \mathbb{N}$, where $\delta(a) := [|\mathcal{D}|, a]$, $a \in \mathcal{A}$.

5.1. Spectral triples for Hardy space on Ω.

Proposition 5.2. *For a bounded domain Ω as in Section 2.1, let $\mathcal{A}_H$ be the algebra of all GTO's of order ≤ 0, with the identity representation π on $\mathcal{H} := H^2(\partial\Omega)$, and $\mathcal{D}$ be a selfadjoint elliptic generalized Toeplitz operator of order* 1 *on $\mathcal{H}$. Then $(\mathcal{A}_H, \mathcal{H}, \mathcal{D})$ is a regular spectral triple of dimension $n = dim_{\mathbb{C}}\,\Omega$.*

Proof. Clearly $\mathcal{A}_H$ is an algebra with unit $T_1 = \mathbb{1}$, and involution $T_P^* = T_{P^*}$, where P^* is the adjoint of P in $L^2(\partial\Omega)$, and trivially π is faithful. Since $\mathcal{D}$ is elliptic of order 1, it has a parametrix of order -1, hence compact, so $\mathcal{D}$ has compact resolvent. Moreover, for any $T_P \in \mathcal{A}_H$, the commutator $[\mathcal{D}, T_P]$ is bounded since, as commutator of GTO's, we get the inequalities

$$\operatorname{ord}([\mathcal{D}, T_P]) \leq \operatorname{ord}(\mathcal{D}) + \operatorname{ord}(T_P) - 1 \leq 1 + 0 - 1 = 0.$$

Thus $(\mathcal{A}_H, \mathcal{H}, \mathcal{D})$ is a spectral triple.

Since $|\mathcal{D}|$ is of order 1 (see for instance [14, Proposition 16]), one can check recursively that for all $k \in \mathbb{N}$ and $T_P \in \mathcal{A}_H$, $\delta^k(T_P) = [|\mathcal{D}|, T_k]$, where T_k is a GTO of order 0, so the commutator is bounded. The same is true for elements of the form $T_P = [\mathcal{D}, T_Q]$, $T_Q \in \mathcal{A}_H$, so the regularity follows.

For the dimension computation, we follow [18, Theorem 3]. We sort the points λ_j of the spectrum of $|\mathcal{D}| := (\mathcal{D}^*\mathcal{D})^{1/2}$ counting multiplicities as $0 < \lambda_1 \leq \lambda_2 \leq \cdots$. Let $M(\lambda)$ be the number of λ_j's less than λ. We can apply [8, Theorem 13.1] to $|\mathcal{D}|$ which is of order 1:

$$M(\lambda) \underset{\lambda\to\infty}{=} \tfrac{\operatorname{vol}(\Sigma_{\mathcal{D}})}{(2\pi)^n}\lambda^n + \mathcal{O}(\lambda^{n-1}),$$

where, using (2.2), $\Sigma_{\mathcal{D}} := \{(x,\xi) \in \Sigma : \sigma(\mathcal{D})(x,\xi) \leq 1\}$. Thus, if $c := (2\pi)^{-n}\operatorname{vol}(\Sigma_D)$, we get for large λ:

$$\lambda^n = \tfrac{M(\lambda)}{c} + \mathcal{O}(\lambda^{n-1}) = \tfrac{M(\lambda)}{c} + \mathcal{O}\big(\lambda^{-1}M(\lambda)\big).$$

Since $M(\lambda)^{-1/n} \sim \mathcal{O}(\lambda^{-1})$, we have

$$\lambda^n = \tfrac{M(\lambda)}{c} + \mathcal{O}(M(\lambda)^{1-1/n}) = \tfrac{M(\lambda)}{c}[1 + O(M(\lambda)^{-1/n})]$$

as $\lambda \to \infty$, so given $d \in \mathbb{R}$,

$$\lambda^{-d} = \tfrac{c^{d/n}[1+O(M(\lambda)^{-1/n})]}{M(\lambda)^{d/n}} = \tfrac{c^{d/n}}{M(\lambda)^{d/n}} + \mathcal{O}\Big(\tfrac{1}{M(\lambda)^{(d+1)/n}}\Big).$$

Thus

$$\begin{aligned}\operatorname{Tr}|\mathcal{D}|^{-d} &= \sum_{j=1}^{\infty} \lambda_j^{-d} = \int_{\lambda_1}^{\infty} \lambda^{-d}\, dM(\lambda) \\ &= \int_{\lambda_1}^{\infty} \big(\tfrac{c^{d/n}}{M(\lambda)^{d/n}} + \mathcal{O}(\tfrac{1}{M(\lambda)^{(d+1)/n}})\big)\, dM(\lambda) \\ &= \int_{1}^{\infty} \big(\tfrac{c^{d/n}}{M^{d/n}} + \mathcal{O}(\tfrac{1}{M^{(d+1)/n}})\big)\, dM\end{aligned}$$

is finite if and only if $d > n$. □

Remark 5.3. If we assume in above proposition that $\mathcal{D}$ is of order $s < 1$, then the commutators with T_P will be GTOs of order $s - 1$, hence not only bounded but even compact.

5.2. Spectral triples for Bergman space on Ω. In the Bergman case, we have a similar result as Proposition 5.2:

Proposition 5.4. *For a bounded domain Ω as in Section 2.1, let $\mathcal{A}_B$ be the algebra generated by the Toeplitz operators $\mathbf{T}_f$, with $f \in C^\infty(\overline{\Omega})$, with the identity representation π on $\mathcal{H} := A^2_w(\Omega)$, and $\mathcal{D} := V_w\, T\, V_w^*$, where T is a selfadjoint elliptic GTO of order 1 and V_w as in (2.15). Then $(\mathcal{A}_B, \mathcal{H}, \mathcal{D})$ is a regular spectral triple of dimension $n = \dim_{\mathbb{C}} \Omega$.*

Proof. As in the Hardy case, clearly $\mathcal{A}_B$ is a unital involutive algebra with a faithful representation on $\mathcal{H}$. Since T has a parametrix of order -1, hence compact, $\mathcal{D}$ has compact resolvent by unitary equivalence.

To see that $[\mathcal{D}, \mathbf{T}_f]$ is bounded for all $\mathbf{T}_f$ in $\mathcal{A}_B$, we use (2.18) and remark that

$$[\mathcal{D}, \mathbf{T}_f] = V_w\, [T,\, T_{\Lambda_w}^{-1/2} T_{\Lambda_w f} T_{\Lambda_w}^{-1/2}]\, V_w^*. \tag{5.1}$$

Since the orders of the GTOs T and $T_{\Lambda_w}^{-1/2} T_{\Lambda_w f} T_{\Lambda_w}^{-1/2}$ are respectively 1 and less than or equal to 0, the commutator on the right hand side has order less than or equal to 0, hence is in particular bounded on $H^2(\partial\Omega)$.

Since $|\mathcal{D}| = V_w\, |T|\, V_w^*$ and $|\mathcal{D}|^{-s} = V_w\, |T|^{-s}\, V_w^*$, for $s \in \mathbb{R}$, the regularity and dimension computation are shown by using the same arguments as in Proposition 5.2. □

Let $\mathcal{T}^{-\infty}$ denote the ideal in $\mathcal{A}_H$ of GTO's of order $-\infty$, i.e. of (smoothing) generalized Toeplitz operators with Schwartz kernel in $C^\infty(\partial\Omega \times \partial\Omega)$.

Proposition 5.5. *The map $\psi : \mathcal{A}_B \ni a \to V_w^*\, a\, V_w \in \mathcal{A}_H$ is a $*-$isomorphism of $\mathcal{A}_B$ onto the subalgebra $\psi(\mathcal{A}_B)$ of $\mathcal{A}_H$. Moreover, $\mathcal{A}_H = \psi(\mathcal{A}_B) + \mathcal{T}^{-\infty}$.*

Proof. We first show that $\psi(\mathcal{A}_B) \subset \mathcal{A}_H$: thanks to (2.16), $\psi(\mathbf{T}_f) = V_w^* \mathbf{T}_f V_w = T_{\Lambda_w}^{-1/2} T_{\Lambda_w f} T_{\Lambda_w}^{-1/2}$, so $\psi(\mathbf{T}_f) \in \mathcal{A}_H$. Since the algebra $\mathcal{A}_B$ is generated by the $\mathbf{T}_f$, the map ψ defines an isomorphism from $\mathcal{A}_B$ into $\psi(\mathcal{A}_B)$ which preserves the adjoint.

We now prove the equality: let $T \in \mathcal{A}_H$, let $-s \leq 0$ be the order of T and $u_0(x')\|\xi'\|^{-s}$, $u_0 \in C^\infty(\partial\Omega)$, its principal symbol. If

$$f_0(x) := \tfrac{\Gamma(m_w+1)}{\Gamma(m_w+s+1)} K(\|\eta\|^s u_0)(x),$$

then $f_0 \in C^\infty(\overline{\Omega})$ and by Proposition 2.14, (2.7) and (2.12), the operator $\psi(\mathbf{T}_{r^s f_0})$ is a GTO also of order $-s$ and with the same principal symbol as T. Thus $T_1 := T - \psi(\mathbf{T}_{r^s f_0})$ is a GTO of order $-s-1$. Applying the same reasoning to T_1 in the place of T yields $f_1 \in C^\infty(\overline{\Omega})$ such that $\psi(\mathbf{T}_{r^{s+1} f_1})$ has the same order and principal symbol as T_1, hence $T_2 := T - \psi(\mathbf{T}_{r^s f_0 + r^{s+1} f_1})$ is a GTO of order $-s-2$. By iteration, this yields a sequence $f_2, f_3, \dots$. Finally, let $f \in C^\infty(\overline{\Omega})$ be a function which has the same boundary jet as the formal sum $\sum_{j=0}^\infty r^j f_j$; that is, such that

$$f - \sum_{j=0}^k r^j f_j = \mathcal{O}(r^{k+1})$$

vanishes to order $k+1$ at the boundary, for any $k = 0, 1, 2, \dots$. (Such an f can be obtained in a completely standard manner along the lines of the classical Borel theorem.) Set $g := r^s f$. Then by Proposition 2.14 and (2.7) again, for any $k \in \mathbb{N}$, the difference

$$\begin{aligned} R := T - \psi(\mathbf{T}_g) &= T - \psi(\mathbf{T}_{\sum_{j=0}^k r^{s+j} f_j}) - \psi(\mathbf{T}_{r^s(f - \sum_{j=0}^k r^j f_j)}) \\ &= T_{k+1} - \psi(\mathbf{T}_{\mathcal{O}(r^{k+s+1})}) \end{aligned}$$

is a GTO of order (at most) $-s-k-1$. Since k is arbitrary, R is a GTO of order $-\infty$, i.e. $R \in \mathcal{T}^{-\infty}$, and the proof is complete. □

For a function f in $C^\infty(\overline{\Omega})$ vanishing to order $j \in \mathbb{N}$ on the boundary, the order of $\psi(\mathbf{T}_f)$ is $-j$, since, from (2.8), the expression for the principal symbol is

$$\sigma(\psi(\mathbf{T}_f))(x', \xi') = \tfrac{2^{-j}\,\Gamma(m_w+1+j)}{\Gamma(m_w+1) j!}\, \partial_{\mathbf{n}}^j f(x')\, \|\xi'\|^{-j}.$$

Hence all functions f that vanish to infinite order on the boundary (so they are not analytic on the boundary) are such that $\psi(\mathbf{T}_f) \in \mathcal{T}^{-\infty}$.

Question 5.6. *Is the inclusion $\psi(\mathcal{A}_B) \subset \mathcal{A}_H$ strict? (Or, is $\mathcal{T}^{-\infty}$ not contained in $\psi(\mathcal{A}_B)$?)*

We now give two examples of the operator $\mathcal{D}$ for these Bergman triples:

Using Proposition 2.10, we may take $\mathcal{D} = \mathbf{T_P}$, with any differential operator $\mathbf{P}$ of order 1 on $\overline{\Omega}$ such that $\mathbf{T_P} = \mathbf{T_P^*}$ and (2.12) is nonzero on Σ. Thus a first example is given in Remark 2.16.

As a second example, we deduce from Remark 2.7 that $(\mathbf{T}_r)^{-1}$ exists on $\mathrm{Ran}(\mathbf{T}_r)$ which is dense in $A^2_w(\Omega)$. Thus we can take $\mathcal{D} = (\mathbf{T}_r)^{-1}$ as an example of Dirac operator on $A^2_w(\Omega)$ and construct the spectral triple $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ with the same $\mathcal{A}$ and $\mathcal{H}$ as in Proposition 5.4. However, the positivity of $\mathcal{D} = (\mathbf{T}_r)^{-1}$ induces a trivial K-homology class for the spectral triple.

We now get around this triviality:

Proposition 5.7. *Let $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ be the spectral triple of Proposition 5.4 with $\mathcal{D} = (\mathbf{T}_r)^{-1}$. Define $\widetilde{\mathcal{A}}$ as the algebra of all $\mathbf{T}_f$'s acting diagonally on $\widetilde{\mathcal{H}} := \mathcal{H} \oplus \mathcal{H}$ and let $\widetilde{\mathcal{D}}$ be the operator*

$$\widetilde{\mathcal{D}} := \begin{pmatrix} 0 & \mathbf{U}\,(\mathbf{T}_r)^{-1} \\ (\mathbf{T}_r)^{-1}\,\mathbf{U}^* & 0 \end{pmatrix}$$

where $\mathbf{U}$ is a unitary operator on $A^2_w(\Omega)$. If $\mathbf{U}$ is such that

$$V_w^*\,\mathbf{U}\,V_w \text{ is a unitary GTO,} \tag{5.2}$$

then $(\widetilde{\mathcal{A}}, \widetilde{\mathcal{H}}, \widetilde{\mathcal{D}})$ is a regular spectral triple. The triples $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ and $(\widetilde{\mathcal{A}}, \widetilde{\mathcal{H}}, \widetilde{\mathcal{D}})$ have the same dimension.

Proof. We first check the boundedness of $[\widetilde{\mathcal{D}}, \widetilde{\mathbf{T}}_f]$. For any $\widetilde{\mathbf{T}}_f \in \widetilde{\mathcal{A}}$, we have $[\widetilde{\mathcal{D}}, \widetilde{\mathbf{T}}_f] = \begin{pmatrix} 0 & D_1 \\ D_2 & 0 \end{pmatrix}$ with $D_1 := [\mathbf{U}\,\mathbf{T}_r^{-1}, \mathbf{T}_f]$ and $D_2 := [\mathbf{T}_r^{-1}\mathbf{U}^*, \mathbf{T}_f]$. From Proposition 2.14, we have the relations $\mathbf{T}_f = V_w\, T_{\Lambda_w}^{-1/2} T_{\Lambda_{wf}} T_{\Lambda_w}^{-1/2}\, V_w^*$, and $(\mathbf{T}_r)^{-1} = V_w\, T_{\Lambda_w}^{1/2} T_{\Lambda_{wr}}^{-1} T_{\Lambda_w}^{1/2}\, V_w^*$. We get

$$\begin{aligned}
D_1 &= \mathbf{U}\mathbf{T}_r^{-1}\,\mathbf{T}_f - \mathbf{T}_f\,\mathbf{U}\mathbf{T}_r^{-1} \\
&= \mathbf{U}\,(V_w T_{\Lambda_w}^{1/2} T_{\Lambda_{wr}}^{-1} T_{\Lambda_w}^{1/2} V_w^*)\,(V_w T_{\Lambda_w}^{-1/2} T_{\Lambda_{wf}} T_{\Lambda_w}^{-1/2} V_w^*) \\
&\qquad - (V_w T_{\Lambda_w}^{-1/2} T_{\Lambda_{wf}} T_{\Lambda_w}^{-1/2} V_w^*)\,\mathbf{U}\,(V_w T_{\Lambda_w}^{1/2} T_{\Lambda_{wr}}^{-1} T_{\Lambda_w}^{1/2} V_w^*) \\
&= \mathbf{U}\,(V_w T_{\Lambda_w}^{1/2} T_{\Lambda_{wr}}^{-1} T_{\Lambda_{wf}} T_{\Lambda_w}^{-1/2} V_w^*) \\
&\qquad - V_w T_{\Lambda_w}^{-1/2} T_{\Lambda_{wf}} T_{\Lambda_w}^{-1/2}\,(V_w^*\mathbf{U}V_w)\,T_{\Lambda_w}^{1/2} T_{\Lambda_{wr}}^{-1} T_{\Lambda_w}^{1/2} V_w^* \\
&= (V_w V_w^*)\,\mathbf{U}\,V_w T_{\Lambda_w}^{1/2} T_{\Lambda_{wr}}^{-1}\,(T_{\Lambda_w}^{1/2} T_{\Lambda_w}^{-1/2})\,T_{\Lambda_{wf}} T_{\Lambda_w}^{-1/2} V_w^* \\
&\qquad - V_w T_{\Lambda_w}^{-1/2} T_{\Lambda_{wf}} T_{\Lambda_w}^{-1/2}\,(V_w^*\mathbf{U}V_w)\,T_{\Lambda_w}^{1/2} T_{\Lambda_{wr}}^{-1} T_{\Lambda_w}^{1/2} V_w^* \\
&= V_w\,[(V_w^*\mathbf{U}V_w)\,T_{\Lambda_w}^{1/2} T_{\Lambda_{wr}}^{-1} T_{\Lambda_w}^{1/2},\, T_{\Lambda_w}^{-1/2} T_{\Lambda_{wf}} T_{\Lambda_w}^{-1/2}]\,V_w^*\,.
\end{aligned}$$

From the hypothesis, $V_w \mathbf{U} V_w^*$ is a bounded GTO, $T_{\Lambda_{wr}}^{-1} T_{\Lambda_w}$ is a GTO of order 1 and $T_{\Lambda_w}^{-1} T_{\Lambda_{wf}}$ is a GTO of order less than or equal to 0, so the commutator is a GTO of order less than or equal to 0, thus is a bounded operator on $A_w^2(\Omega)$. Similar arguments show that

$$D_2 = V_w\,[T_{\Lambda_w}^{1/2} T_{\Lambda_{wr}}^{-1} T_{\Lambda_w}^{1/2}\,(V_w^* \mathbf{U}^* V_w),\, T_{\Lambda_w}^{-1/2} T_{\Lambda_{wf}} T_{\Lambda_w}^{-1/2}]\, V_w^*$$

is also bounded on $A_w^2(\Omega)$, which makes $[\widetilde{\mathcal{D}}, \widetilde{\mathbf{T}}_f]$ bounded on the direct sum $\widetilde{\mathcal{H}}$.

We remark that the expressions for D_1 and D_2 differ from (5.1) by the term $V_w^* \mathbf{U} V_w$ which is a GTO of order 0. So that the regularity of the spectral triple is proven as in Proposition 5.4.

Finally $\widetilde{\mathcal{D}}$ has compact resolvent since $\widetilde{\mathcal{D}}^{-1} = \begin{pmatrix} 0 & \mathbf{U}\mathbf{T}_r \\ \mathbf{T}_r\mathbf{U}^* & 0 \end{pmatrix}$ is compact because the operators $\mathbf{U}\mathbf{T}_r$ and $\mathbf{T}_r\mathbf{U}^*$ are compact.

Since $\widetilde{\mathcal{D}}^2 = \begin{pmatrix} \mathbf{U}\mathbf{T}_r^{-2}\mathbf{U}^* & 0 \\ 0 & \mathbf{T}_r^{-2} \end{pmatrix}$, we deduce that the unitary $\mathbf{U}$ does not influence the calculation of eigenvalues. □

Remark 5.8. The two classes of unitaries $\mathbf{U}$ defined in Remark 2.20 satisfy (5.2), so provide examples of spectral triples $(\widetilde{\mathcal{A}}, \widetilde{\mathcal{H}}, \widetilde{\mathcal{D}})$ on (the sum of two copies of) the Bergman space with non-positive $\widetilde{\mathcal{D}}$ when $\mathcal{D} = (\mathbf{T}_r)^{-1}$.

For the case of the unit ball with a radial weight, Proposition 5.4 can be made much more explicit. Indeed, if f is a radial function in $C^\infty(\overline{\mathbb{B}^n})$ and the weight w is as in (2.3), the family $\{v_\alpha\}_{\alpha\in\mathbb{N}^n}$ defined in (2.4) diagonalizes $\mathbf{T}_f : A_w^2(\mathbb{B}^n) \to A_w^2(\mathbb{B}^n)$ and the eigenvalues only depend on $|\alpha|$. Namely,

$$\langle \mathbf{T}_f\, v_\alpha, v_\beta \rangle_{A_w^2} = \frac{\delta_{\alpha\beta}}{\int_0^1 t^{2n+2|\alpha|-1}\, w(t)\, dt} \int_0^1 t^{2n+2|\alpha|-1}\, f(t)\, w(t)\, dt,$$

as is easily seen by passing to the polar coordinates.

Thus, assume that $w = r^{m_w}$, $m_w \in \mathbb{N}$, where the function r of the form (2.1) depends only on the variable $|x|$, for x in $\mathbb{B}^n$. For convenience, we temporarily denote here $\partial := \partial_{\mathbf{n}}$. Then, since the first $m_w - 1$ derivatives of w vanish on $\partial\mathbb{B}^n = \mathbb{S}^{2n-1}$, and $\partial^k(w\,r)$ is non-zero only for $k > m_w$, we have for $\alpha \in \mathbb{N}^n$:

$$\begin{aligned}
\int_0^1 w(t)\, t^{2n+2|\alpha|-1}\, dt &= \frac{(-1)^{m_w}\, \partial^{m_w} w(1)}{\prod_{k=0}^{m_w}(2n+2|\alpha|+k)} \\
&\quad - \frac{(-1)^{m_w}}{\prod_{k=0}^{m_w}(2n+2|\alpha|+k)} \int_0^1 \partial^{m_w+1} w(t)\, t^{2n+2|\alpha|+m_w}\, dt, \\
\int_0^1 w(t)\, r(t)\, t^{2n+2|\alpha|-1}\, dt &= \frac{(-1)^{m_w+1}\, \partial^{m_w+1}(w\,r)(1)}{\prod_{k=0}^{m_w+1}(2n+2|\alpha|+k)} \\
&\quad - \frac{(-1)^{m_w+1}}{\prod_{k=0}^{m_w+1}(2n+2|\alpha|+k)} \int_0^1 \partial^{m_w+2}(w\,r)(t)\, t^{2n+2|\alpha|+m_w+1}\, dt.
\end{aligned}$$

Since $\partial^{m_w} w(1) \neq 0$ by hypothesis, we deduce, applying the Leibniz formula for $\partial^{m_w+1}(w\,f)(1)$,

$$\int_0^1 w(t)\, t^{2n+2|\alpha|-1}\, dt \underset{|\alpha|\to\infty}{\sim} \frac{(-1)^{m_w}\, \partial^{m_w} w(1)}{\prod_{k=0}^{m_w}(2n+2|\alpha|+k)},$$

$$\int_0^1 w(t)\, r(t)\, t^{2n+2|\alpha|-1}\, dt \underset{|\alpha|\to\infty}{\sim} \frac{(-1)^{m_w+1}\, \partial^{m_w} w(1)\, \partial r(1)}{\prod_{k=0}^{m_w+1}(2n+2|\alpha|+k)}.$$

We choose now $\mathcal{D} = \mathbf{T}_r$ and from the previous result, $\mathcal{D}$ is diagonal in the basis $\{v_\alpha\}_{\alpha\in\mathbb{N}^n}$ and we obtain the asymptotic behavior of the eigenvalues:

$$\langle \mathcal{D}\, v_\alpha, v_\alpha\rangle_{A^2_w} \underset{|\alpha|\to\infty}{\sim} -\frac{1}{2n+2|\alpha|+m_w+1}\, \partial r(1) \underset{|\alpha|\to\infty}{\sim} -\frac{1}{2|\alpha|}\, \partial r(1).$$

Since $\operatorname{Tr}|\mathcal{D}|^{-d'} = \sum_{k=0}^{\infty} \binom{n-1+k}{n-1} \frac{\partial r(1)}{(2k)^{d'}}$ and $\binom{n-1+k}{n-1} \underset{k\to\infty}{\sim} \frac{k^{n-1}}{(n-1)!}$, we have $\operatorname{Tr}|\mathcal{D}|^{-d'} < \infty$ if and only if $\frac{1}{(n-1)!}\sum_{k=0}^{\infty} \frac{k^{n-1}}{k^{d'}} < \infty$, so for each $d' > d = n$.

To give an example, we choose the function $r : z \in \Omega \mapsto 1 - |z|^2$, and the weight $w = r^0 = 1$. A direct calculation shows that $\{v_\alpha\}_{\alpha\in\mathbb{N}^n}$ defined in (2.5) diagonalizes the operator $\mathbf{T}_{1-|z|^2}$, acting on $A^2(\mathbb{B}^n)$ with eigenvalues $\lambda_\alpha := \frac{1}{n+|\alpha|+1}$ and multiplicity $\binom{n-1+|\alpha|}{n-1}$. Indeed:

$$\begin{aligned}
\mathbf{T}_{1-|z|^2}\, v_\alpha &= v_\alpha - \sum_{\beta\in\mathbb{N}^n}\sum_{j=1}^{n} \langle z_j \bar{z}_j\, v_\alpha, v_\beta\rangle\, v_\beta \\
&= v_\alpha - \sum_{\beta\in\mathbb{N}^n}\sum_{j=1}^{n} \langle b_\alpha z^{\alpha+1_j}, b_\beta z^{\beta+1_j}\rangle\, v_\beta \\
&= (1 - \sum_{j=1}^{n} b_\alpha^2 \|z^{\alpha+1_i}\|^2)\, v_\alpha = (1 - \sum_{j=1}^{n} \frac{b_\alpha^2}{b_{\alpha+1_j}^2})\, v_\alpha.
\end{aligned}$$

Since $b_{\alpha+1_j}^2 = \frac{(n+|\alpha|+1)!}{n!\alpha!(\alpha_j+1)\,\mu(\mathbb{B}^n)}$, we get $\lambda_\alpha = 1 - \sum_{j=1}^{n} \frac{(n+|\alpha|)!}{n!\alpha!} \frac{n!\alpha!(\alpha_j+1)}{(n+|\alpha|+1)!} = 1 - \frac{n+|\alpha|}{n+|\alpha|+1} = \frac{1}{n+|\alpha|+1}$, and the multiplicity follows because λ_α depends only on $|\alpha|$.

5.3. Example of spectral triples on the unit ball of $\mathbb{C}^n$ without weight. We consider in this section the model case $\Omega = \mathbb{B}^n$ with $w = 1$.

Proposition 5.9. *Let $\mathcal{A} = \{T_u,\ u \in C^\infty(\mathbb{S}^{2n-1})\}$ be the algebra generated by Toeplitz operators acting on $\mathcal{H} = H^2(\mathbb{S}^{2n-1}) \otimes \mathcal{C}_n$ via the representation $\pi(T_u) := T_u \otimes I$, and $\mathcal{D}_{\mathbb{S}^{2n-1}}$ the operator from* (4.6)*. Then $(\mathcal{A}, \mathcal{H}, \mathcal{D}_{\mathbb{S}^{2n-1}})$ is a regular spectral triple of dimension $2n$.*

Let $\mathcal{A} = \{\mathbf{T}_f,\ f \in C^\infty(\overline{\mathbb{B}^n})\}$ be the algebra generated by Toeplitz operators on $\mathcal{H} = A^2(\mathbb{B}^n) \otimes \mathcal{C}_n$ (via the previous representation) and $\mathcal{D}_{\mathbb{B}^n}$ the operator from (4.7)*. Then $(\mathcal{A}, \mathcal{H}, \mathcal{D}_{\mathbb{B}^n})$ is a regular spectral triple of dimension $2n$.*

Proof. The requirement of compact resolvent is fulfilled automatically, since it is fulfilled for the standard Dirac operator on $\mathbb{R}^n$, from which $\mathcal{D}_{\mathbb{S}^{2n-1}}$ and $\mathcal{D}_{\mathbb{B}^n}$ were obtained by transferring via various $*$−isomorphisms, which also shows they are selfadjoint.

From Lemma 4.2, $\mathcal{D}_{\mathbb{S}^{2n-1}} = (t/2)^{1/2} \sum_j \Gamma_j\, T_\Lambda^{1/2} \boldsymbol{\gamma} \tau_t(P_j) K T_\Lambda^{-1/2}$, is a GTO of order $1/2$ and we use the same argument as in the proof of Proposition 5.2. The result for the Bergman case follows from the identity $\mathcal{D}_{\mathbb{B}^n} = V \mathcal{D}_{\mathbb{S}^{2n-1}} V^*$ and using a similar reasoning as in the proof of Proposition 5.4. □

5.4. Dixmier traces. In all the examples of spectral triples above, one can also give a formula for the Dixmier traces $\mathrm{Tr}_\omega(a|\mathcal{D}|^{-d})$ where $a \in \mathcal{A}$ and d is the spectral dimension.

First, recall that by [18, Theorem 3], if T_P is a GTO on $\partial\Omega$ of order $-n$, then T_P is in the Dixmier class, is measurable, and

$$\mathrm{Tr}_\omega(T_P) = \tfrac{1}{n!(2\pi)^n} \int_{\partial\Omega} \sigma(T_P)(x', \eta_{x'})\, \nu_{x'}\,. \tag{5.3}$$

This formula is independent of the choice of the defining function r (see [18, Remark 4]).

In the context of the Hardy space spectral triple from Section 5.1, we thus have for any $u \in C^\infty(\partial\Omega)$ and $\mathcal{D}$ as in Proposition 5.2

$$\mathrm{Tr}_\omega(T_u|\mathcal{D}|^{-n}) = \tfrac{1}{n!(2\pi)^n} \int_{\partial\Omega} u(x')\, |\sigma(\mathcal{D})(x', \eta_{x'})|^{-n}\, \nu_{x'}\,, \tag{5.4}$$

and similarly for T_u replaced by any GTO T_Q of order 0: the $u(x')$ in the integrand is then replaced by $\sigma(T_Q)(x', \eta_{x'})$.

For the Bergman case, the Dirac operator in Proposition 5.4 is of the form $\mathcal{D} = V_w T V_w^*$, where T is a selfadjoint elliptic GTO of order 1. Thus we have for any $f \in C^\infty(\overline{\Omega})$

$$\begin{aligned}\mathrm{Tr}_\omega(\mathbf{T}_f|\mathcal{D}|^{-n}) &= \mathrm{Tr}_\omega(V_w\, T_{\Lambda_w}^{-1/2}\, T_{\Lambda_w f}\, T_{\Lambda_w}^{-1/2}\, V_w^*\, V_w\, |T|^{-n}\, V_w^*)\\ &= \mathrm{Tr}_\omega(T_{\Lambda_w}^{-1/2}\, T_{\Lambda_w f}\, T_{\Lambda_w}^{-1/2}\, |T|^{-n}),\end{aligned}$$

which is treated as above.

For $\mathcal{D} = \mathbf{T}_{\mathbf{P_n}}, \mathbf{P_n} = \sum_j \overline{\partial_j r} \partial_j$ as in Remark 2.16, we use a similar trick to compute

$$\begin{aligned}\mathrm{Tr}_\omega(\mathbf{T}_f|\mathcal{D}|^{-n}) &= \mathrm{Tr}_\omega(V\, T_\Lambda^{-1/2}\, T_{\Lambda_f}\, T_\Lambda^{-1/2}\, V^*\, V\, T_\Lambda^{1/2}\, (\gamma \mathbf{T}_{\mathbf{P_n}} K)\, T_\Lambda^{-1/2}\, V^*)\\ &= \mathrm{Tr}_\omega(T_\Lambda^{-1/2}\, T_{\Lambda_w f}\, (\gamma \mathbf{T}_{\mathbf{P_n}} K)\, T_\Lambda^{-1/2}),\end{aligned}$$

and we get the result from (2.20).

For the triple $(\widetilde{\mathcal{A}}, \widetilde{\mathcal{H}}, \widetilde{\mathcal{D}})$ from Proposition 5.7, the Dixmier traces get multiplied by 2 due to the appearance of 2×2 block matrices (see the last paragraph of the proof of the proposition).

Similarly, a factor n appears in the computation of Dixmier traces in the context of Section 5.3 since the Dirac operators $\mathcal{D}_{\mathbb{S}^{2n-1}}$ and $\mathcal{D}_{\mathbb{B}^n}$ involved in Proposition 5.9 contain gamma matrices. Using $V^* \tau(P_j) V = T_\Lambda^{1/2} (\gamma \tau(P_j) K) T_\Lambda^{-1/2}$ and Lemma 4.2, we get $\mathcal{D}_{\mathbb{S}^{2n-1}} = (t/2)^{-1/2} \sum_j \Gamma_j T_{Q_j}$, where the T_{Q_j} are GTOs of order 1/2 whose symbols are known. Hence $\operatorname{Tr}_\omega(|\mathcal{D}_{\mathbb{S}^{2n-1}}|^{-2n}) = \operatorname{Tr}_\omega(|\mathcal{D}_{\mathbb{B}^n}|^{-2n})$ is finite. We use (5.4) again to compute $\operatorname{Tr}_\omega(T_u |\mathcal{D}_{\mathbb{S}^{2n-1}}|^{-2n})$ and the Bergman case follows as above.

Remark 5.10. Note that pseudodifferential operators of order k on $\mathbb{R}^n$ are transformed in GTOs of order $k/2$ on the boundary of Ω (in the beginning of Section 4), which might seem to be at odds with the fact that Dixmier-trace operators correspond to both pseudodifferential operators and GTOs of order $-n$, respectively on a compact real manifold of dimension n and on the boundary of a complex domain of dimension n. The point is that on a compact manifold of dimension n this is true, but on $\mathbb{R}^n$ this fails: for instance the operator $(\mathbb{1} - \Delta)^{-n/2}$ on $\mathbb{R}^n$ is not even compact, much less in the Dixmier class. What one needs is first of all not to use Hörmander but Shubin (also known as Grossman–Loupias–Stein) classes of pseudodifferential operators (i.e. with prescribed decay of symbols not only as ξ goes to infinity, but as (x, ξ) goes to infinity), and secondly, the order needed for the Dixmier-class is then not $-n$ but $-2n$ (see for instance [1, Theorem 4.1] where the result is stated for pseudodifferential operators on $\mathbb{R}^n$ of Weyl type, but it is the same for the Kohn–Nirenberg type). Since $(-2n)/2 = -n$, this yields precisely the correct order for GTOs, and the contradiction disappears.

Actually, this can be recast the following way: for a non unital spectral triple $(\mathcal{A}, \mathcal{H}, \mathcal{D})$, the axiom "$\mathcal{D}$ has a compact resolvent" is replaced by "$\pi(a)(\mathbb{1} + \mathcal{D}^2)^{-1}$ is a compact operator for any $a \in \mathcal{A}$". For instance, in a triple on $\mathbb{R}^n$ like $\big(\text{Functions}(\mathbb{R}^n), \mathcal{H} = L^2(\mathbb{R}^n) \otimes \mathcal{C}_n, \mathcal{D} = -i \sum_j \Gamma_j \partial_{x_j}\big)$, where Functions$(\mathbb{R}^n)$ is a subalgebra of $C^\infty(\mathbb{R}^n)$, one can choose for $\mathcal{A}$ the Schwartz space on $\mathbb{R}^n$ to secure this property. While $(\mathbb{1} + \mathcal{D}^2)^{-n/2} = \big((\mathbb{1} - \Delta) \otimes \mathbb{1}_{\mathcal{C}_n}\big)^{-n/2}$ is not Dixmier-traceable, $\pi(f)(\mathbb{1} + \mathcal{D}^2)^{-n/2}$ is, so the dimension n appears twice: one in the power of $|\mathcal{D}|$ and the other through the algebra $\mathcal{A}$ (via the n variables of f).

6. Berezin–Toeplitz star products

One may, in a sense, glue the spectral triples from §5.2 with different weights w into a single "composed" spectral triple, much as Toeplitz operators on weighted Bergman spaces are "glued" together in the Berezin–Toeplitz quantization [3] [32]; this actually yields a spectral triple directly related to the standard Berezin–Toeplitz star product on Ω. Let us give the details.

Assume that $\log 1/r$ is strictly plurisubharmonic on Ω (defining functions r with this property exist in abundance due to the strict pseudoconvexity of Ω), so that $g_{j\bar{k}}(z) := \partial_j \bar{\partial}_k \log \frac{1}{r(z)}$ defines a Kähler metric on Ω; and let $g := r^{n+1} \det[g_{j\bar{k}}]$.

By matrix manipulations, one can check that $g \in C^\infty(\overline{\Omega})$ and (thanks to strict pseudoconvexity) $g > 0$ on $\partial\Omega$ (in fact g coincides with the Monge–Ampère determinant $g = -\det\begin{bmatrix} r & \partial r \\ \bar{\partial} r & \partial\bar{\partial} r \end{bmatrix}$).

Consider the weighted Bergman spaces $A^2(\Omega, w_m)$ with $w_m := r^m g$, $m \in \mathbb{N}$, which we will now denote by A^2_m for brevity. Let

$$\mathbf{H} := \bigoplus_{m=0}^{\infty} A^2_m$$

be their orthogonal direct sum, and let π_m stand for the orthogonal projection in $\mathbf{H}$ onto the m-th summand. Denote by $\mathbf{N}$ the "number operator" $\mathbf{N} := \bigoplus_m (m + n + 1)\, \pi_m$.

For $f \in C^\infty(\overline{\Omega})$, we then have the orthogonal sums

$$\mathbf{T}^\oplus_f := \bigoplus_m (\mathbf{T}_f \text{ on } A^2_m)$$

of the Toeplitz operators $\mathbf{T}_f$ from Section 2, acting on $\mathbf{H}$. Clearly each $\mathbf{T}^\oplus_f$ is again bounded with $\|\mathbf{T}^\oplus_f\| \le \|f\|_\infty$, $(\mathbf{T}^\oplus_f)^* = \mathbf{T}^\oplus_{\bar{f}}$, and $[\mathbf{T}^\oplus_f, \pi_m] = 0$ for all m.

Let $\mathcal{B}$ denote the subset of all bounded linear operators M on $\mathbf{H}$ for which $[M, \pi_m] = 0$ for all $m \in \mathbb{N}$ and which possess an asymptotic expansion

$$M \approx \sum_{j=0}^{\infty} \mathbf{N}^{-j}\, \mathbf{T}^\oplus_{f_j} \tag{6.1}$$

with some $f_j \in C^\infty(\overline{\Omega})$ (depending on M). Here "$\approx$" means that

$$\Big\| \pi_m \big(M - \sum_{j=0}^{k-1} \mathbf{N}^{-j}\, \mathbf{T}^\oplus_{f_j} \big)\, \pi_m \Big\| = O(m^{-k}) \quad \text{as } m \to +\infty \text{ for any } k = 0, 1, 2, \dots.$$

It is the main result of the Berezin–Toeplitz quantization on Ω that finite products of $\mathbf{T}^\oplus_f$ belong to $\mathcal{B}$. More specifically, one has [3]

$$\mathbf{T}^\oplus_f\, \mathbf{T}^\oplus_g \approx \sum_{j=0}^{\infty} \mathbf{N}^{-j}\, \mathbf{T}^\oplus_{C_j(f,g)}$$

where

$$\sum_{j=0}^{\infty} h^j\, C_j(f,g) =: f \star g\,,$$

defines a star product on $(\Omega, g_{j\bar{k}})$.

This is the so-called Berezin–Toeplitz star product. Symbolically (making the identification $h := (m+n+1)^{-1}$), we can write

$$\mathbf{T}^{\oplus}_{f\star g} = \mathbf{T}^{\oplus}_f \mathbf{T}^{\oplus}_g .$$

Another result is, incidentally, that

$$\|\pi_m \mathbf{T}^{\oplus}_f \pi_m\| \to \|f\|_\infty \quad \text{as } m \to +\infty, \tag{6.2}$$

implying, in particular, that for a given $M \in \mathcal{B}$ the sequence $\{f_m\}_{m\in\mathbb{N}}$ in (6.1) is determined uniquely.

There is a neat representation for this whole situation, as follows (see e.g. [13, p. 235], for details in this setting; the idea however goes back to Forelli and Rudin). Consider the "unit disc bundle" over Ω:

$$\widetilde{\Omega} := \{(z,t) \in \Omega \times \mathbb{C} : |t|^2 < r(z)\}.$$

The fact that r is a defining function for Ω implies that $\widetilde{\Omega}$ is smoothly bounded, and the facts that Ω is strictly pseudoconvex and $\log 1/r$ is strictly plurisubharmonic imply that $\widetilde{\Omega}$ is strictly pseudoconvex. Thus we have the Hardy space $\widetilde{H} := H^2(\widetilde{\Omega})$ of $\widetilde{\Omega}$ and the GTOs $\widetilde{T}_P$ there, whose symbols P are now pseudodifferential operators on $\partial\widetilde{\Omega}$. A function in $\widetilde{H}$ has the Taylor expansion in the fiber variable

$$f(z,t) = \sum_{m=0}^{\infty} f_m(z)\, t^m .$$

Denote by $\widetilde{H}_m$, $m \in \mathbb{N}$, the subspace in $\widetilde{H}$ of functions of the form $f_m(z)\, t^m$ (i.e. for which all the Taylor coefficients vanish except for the m-th); alternatively, $\widetilde{H}_m$ is the subspace of functions in $\widetilde{H}$ satisfying

$$f(z, e^{i\theta} t) = e^{mi\theta} f(z,t), \quad \forall \theta \in \mathbb{R}.$$

Then the correspondence

$$f_m(z)\, t^m \longleftrightarrow f_m(z)$$

is an isometric (up to a constant factor) isomorphism of $\widetilde{H}_m$ onto A^2_{m-n-1}. Thus (note that $A^2_{m-n-1} = \{0\}$ for $m \le n$)

$$\widetilde{H} = \bigoplus_{m=0}^{\infty} \widetilde{H}_{m+n+1} \cong \bigoplus_{m=0}^{\infty} A^2_m = \mathbf{H}.$$

Furthermore, viewing a function $f \in C^\infty(\overline{\Omega})$ also as the function $f(z,t) := f(z)$ on $\partial\widetilde{\Omega}$ (i.e. identifying f with its pullback via the projection map), one has, under the above isomorphism,

$$\widetilde{T}_f \cong \bigoplus_{m=0}^{\infty} (\mathbf{T}_f \text{ on } A^2_m) = \mathbf{T}^{\oplus}_f .$$

Finally, let $\widetilde{K}$ be the Poisson operator for $\widetilde{\Omega}$, and set as before $\widetilde{\Lambda} := \widetilde{K}^*\widetilde{K}$. Thus $\widetilde{\Lambda}$ is a pseudodifferential operator on $\partial\widetilde{\Omega}$ of order -1, and a positive selfadjoint compact operator on $\widetilde{H}$. Since the fiber rotations $(z,t) \mapsto (z, e^{i\theta}t)$, $\theta \in \mathbb{R}$, preserve holomorphy and harmonicity of functions, both $\widetilde{K}$, $\widetilde{\Lambda}$ and the Szegö projection $L^2(\partial\widetilde{\Omega}) \to \widetilde{H}$ must commute with them. The GTO $\widetilde{T}_{\widetilde{\Lambda}}$ on $\widetilde{H}$ therefore likewise commutes with these rotations, and hence commutes also with the projections in $\widetilde{H}$ onto $\widetilde{H}_m$, i.e. is diagonalized by the decomposition $\widetilde{H} = \bigoplus_m \widetilde{H}_m$.

Let $L := \bigoplus_m L_m$ be the operator corresponding to $\widetilde{T}_{\widetilde{\Lambda}}$ under the isomorphism $\widetilde{H} \cong \mathbf{H} = \bigoplus_m A^2_m$.

Proposition 6.1. *Let $\mathcal{A}$ be the algebra (no closures taken) generated by $\mathbf{T}^{\oplus}_f$, $f \in C^\infty(\overline{\Omega})$ acting (via identity representation) on $\mathcal{H} := \mathbf{H}$ and $\mathcal{D} := L^{-1}$. Then $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ is a regular spectral triple of dimension $n+1$.*

Proof. Using the above isomorphisms, we can actually switch from $\mathbf{H}$ to the space $\widetilde{H}$, from $\mathcal{A}$ to the algebra generated by $\widetilde{T}_f$, $f \in C^\infty(\overline{\Omega})$ (identified via pullback with functions on $\partial\widetilde{\Omega}$), and from $\mathcal{D}$ to $\widetilde{\Lambda}^{-1}$. Everything then follows in exactly the same way as in Section 5.1 noting that $\dim_{\mathbb{C}} \widetilde{\Omega} = n+1$ (in fact, it is even the special case of the result from that section for functions f on $\partial\widetilde{\Omega}$ that are pullbacks of functions on $\overline{\Omega}$). □

In the spirit of deformation quantization, we now build a spectral triple whose algebra is a certain subalgebra of formal power series generated by $f \in C^\infty(\overline{\Omega})$ with the product $\star$, taking for π the representation $f \mapsto \mathbf{T}^{\oplus}_f$. The relation (6.1) induces a linear map κ from $\mathcal{B}$ into the ring of formal power series

$$\mathcal{N} := C^\infty(\overline{\Omega})[[h]]$$

(equipped with the usual involution $\big(\sum_{j\in\mathbb{N}} h^j f_j(z)\big)^* := \sum_{j\in\mathbb{N}} h^j \overline{f_j(z)}$) given by

$$\kappa : M \longmapsto \sum_{j=0}^{\infty} h^j f_j(z)$$

for M as in (6.1). As noted previously, κ is well defined owing to (6.2) (although it is not injective), and, extending as usual $\star$ from functions to all of $\mathcal{N}$ by $\mathbb{C}[[h]]$-linearity,

$$\kappa(MN) = \kappa(M) \star \kappa(N), \quad \kappa(M^*) = \kappa(M)^*,$$

i.e. $\kappa : (\mathcal{B}, \circ) \to (\mathcal{N}, \star)$ is a $*-$algebra homomorphism. Then we have the following:

Theorem 6.2. *Let $\mathcal{A}$ be the subalgebra over $\mathbb{C}[[h]]$ (no closures taken) of $(\mathcal{N}, \star)$ generated by $\kappa(\mathbf{T}^{\oplus}_f)$, $f \in C^\infty(\overline{\Omega})$ endowed with the representation π on $\mathcal{H} := \mathbf{H}$ be determined by*

$$\pi(h^j f) := \mathbf{N}^{-j}\, \mathbf{T}^{\oplus}_f, \quad f \in C^\infty(\overline{\Omega}),\ j \in \mathbb{N}, \tag{6.3}$$

which is well-defined from $\mathcal{A}$ into $\mathcal{B}$, and $\mathcal{D} := \bigoplus_m L_m^{-1}$ on $\mathbf{H}$. Then $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ is a regular spectral triple of dimension $n+1$.

Proof. In view of the preceding result, the only thing we need to check is that π is well-defined and faithful. The former is immediate from (6.3) and the fact that $\kappa : (\mathcal{B}, \circ) \to (\mathcal{N}, \star)$ is a $*$−algebra homomorphism. For the faithfulness, note that $\kappa \circ \pi = \text{id}$ on $\mathcal{A}$; thus $\pi(a) = 0$ implies $a = \kappa\big(\pi(a)\big) = 0$. □

Again, proceeding as in Proposition 5.7, one can adjoin to the last construction an appropriate unitary GTOs on $\partial\widetilde{\Omega}$ to obtain also non-positive operators $\widetilde{\mathcal{D}}$ (cf. Remark 5.8).

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