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Easy quantum groups and quantum subgroups of a semi-direct product quantum group

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Abstract. We consider homogeneous compact matrix quantum groups whose fundamental corepresentation matrix has entries which are partial symmetries with central support. We show that such quantum groups have a simple presentation as semi-direct product quantum groups of a group dual quantum group by an action of a permutation group. This general result allows us to completely classify easy quantum groups with the above property by certain reflection groups. We give four applications of our result. First, there are uncountably many easy quantum groups. Second, there are non-easy homogeneous hyperoctahedral quantum groups. Third, we study operator algebraic properties of the hyperoctahedral series. Finally, we prove a generalised de Finetti theorem for those easy quantum groups in the scope of this article.

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1. Introduction

Easy quantum groups were introduced by Banica and Speicher in [8], in the context of Wang's universal quantum groups [44]. Via Speicher's partitions, this class of quantum groups has a natural link to free probability theory. Together with [37], the present paper completes the classification of easy quantum groups, which was started in [5, 8, 46].

Compact quantum groups were introduced by Woronowicz in [47, 50]. As they have a natural set of axioms and satisfy a version of Tannaka–Krein duality [48], they are by now an established generalisation of compact groups to a non-commutative setting. There are mainly three sources of examples of compact quantum groups. First, there are q-deformations of compact simple Lie groups described in [17, 23, 34, 48]. Second, motivated by a question of Connes, Wang described the quantum symmetry group of n points in [45]. This later led Goswami to define quantum isometry groups in [18]. Third, Banica and Speicher defined a combinatorial class of quantum groups, which they called easy quantum groups [8]. This article is concerned with the latter class of quantum groups.

In 1995 Wang introduced the universal orthogonal and the universal unitary quantum groups O_n^+ and U_n^+ [44]. Later in 1998 he introduced the quantum permutation groups S_n^+ [45]. After work of Banica and others [1–3] introducing the free hyperoctahedral quantum group H_n^+, it became clear that all these quantum groups should be considered as a result of a "liberation process" of classical groups, which was formalised in [8]. It resembles the passage from classical probability theory to free probability theory via Speicher's partitions. The combinatorial point of view taken in [8] naturally gave rise to a new class of quantum groups which are described by Speicher's partitions. Banica and Speicher called them *easy quantum groups*. By their very definition easy quantum groups are related to free probability theory. This intuition was confirmed by several de Finetti type theorems, identifying easy quantum groups as the correct class of symmetries of certain distributions [6,27]. In [8,46] the classification of two subclasses of easy quantum groups, namely easy groups and free quantum groups (those easy quantum groups corresponding to non-crossing partitions), was settled. Moreover, in [5,46] further classification results for easy quantum groups were obtained, showing that there are exactly 13 easy quantum groups outside a family called *hyperoctahedral easy quantum groups*. At the same time, it was shown that there is at least a countable number of hyperoctahedral easy quantum groups, leaving their complete classification as an open problem.

We complete the classification of easy quantum groups by classifying hyperoctahedral easy quantum groups in this paper and in [37].

We identify a dividing line between easy quantum groups which are quantum subgroups of the semi-direct product quantum groups $\mathrm{C}^*(\mathbb{Z}_2^{*n}) \bowtie \mathrm{C}(\mathrm{S}_n)$ and those which are not. While the structure of the former class of easy quantum groups is governed by algebraic considerations, the study of the latter class remains of combinatorial nature. Indeed, in [37] we classify by combinatorial arguments the remaining hyperoctahedral easy quantum groups which are not contained in $\mathrm{C}^*(\mathbb{Z}_2^{*n}) \bowtie \mathrm{C}(\mathrm{S}_n)$. It turns out that there are only countably many of them. In contrast, in the present paper we completely classify homogeneous quantum subgroups of $\mathrm{C}^*(\mathbb{Z}_2^{*n}) \bowtie \mathrm{C}(\mathrm{S}_n)$ and only afterwards we identify easy quantum groups among them. A quantum group is called homogeneous, if it contains the permutation group S_n as a quantum subgroup (see Section 2.4.1 for a precise definition). The homogeneity assumption is natural for any classification result for compact quantum groups, since every Lie group admits many embeddings into O_n – these subgroups have to be excluded to expect reasonable classification results. The algebraic approach of this paper allows us to understand the easy quantum groups in question more profoundly, as is demonstrated by several applications. Moreover, it is a complete classification result for homogeneous quantum subgroups of a naturally defined quantum group.

Already in the preprint [36], we obtained some of the results presented in this paper by means of combinatorial methods. The present article however achieves a complete description of the quantum groups treated there, making use of algebraic methods. We refer to [38] for a concise presentation of the combinatorial approach.

Let us describe the results of this paper in more detail. We first characterise homogeneous compact matrix quantum groups (A, u) such that all u_{ij}^2 are central projections in A. A compact matrix quantum group is called homogeneous, if it admits a morphism $A \to \mathrm{C}(\mathrm{S}_n)$. The compact matrix quantum groups arising in the next theorem are called semi-direct product quantum groups (see Section 2.5). We denote them by $\mathrm{C}^*(\Gamma) \bowtie \mathrm{C}(\mathrm{S}_n)$. They generalise the compact groups $\widehat{H} \rtimes \mathrm{S}_n$, where H is a discrete abelian group carrying an action of S_n by group automorphisms.

Theorem A (See Theorem 3.1). *Let (A, u) be a homogeneous compact matrix quantum group such that $u_{ij}^* = u_{ij}$ and u_{ij}^2 is a central projection in A for all $i, j \in \{1, \dots, n\}$. Then there is a quotient $\mathbb{Z}_2^{*n} \twoheadrightarrow \Gamma$ whose kernel is invariant under the natural action of S_n such that A is a version of $C^*(\Gamma) \bowtie \mathrm{C}(\mathrm{S}_n)$. In particular, we have $A \cong C^*(\Gamma) \bowtie \mathrm{C}(\mathrm{S}_n)$ if A is in its maximal version.*

Applying the previous classification to hyperoctahedral easy quantum groups, we obtain the following structural result. A strongly symmetric reflection group is a quotient $\mathbb{Z}_2^{*n} \to \Gamma$ whose kernel is invariant under any *identification of letters* (See Definition 4.3).

Theorem B (See Theorem 4.5). *Let (A, u) be an easy quantum group associated with the category of partitions $\mathcal{C}$. Assume that $\sqcup\!\!\!/\!\!\!\sqcap \in \mathcal{C}$. Then $A \cong C^*(\Gamma) \bowtie \mathrm{C}(\mathrm{S}_n)$ for some strongly symmetric reflection group Γ on n generators. Denoting the generators of Γ by $g_1, \dots, g_n$ and the fundamental corepresentation of $\mathrm{C}(\mathrm{S}_n)$ by (p_{ij}), the fundamental corepresentation of A is identified with $(u_{g_i} p_{ij})$.*

Moreover, every strongly symmetric reflection group arises this way.

Motivated by this result, we call categories of partitions that contain $\sqcup\!\!\!/\!\!\!\sqcap$ *group-theoretical* categories of partitions. The associated easy quantum groups are called *group-theoretical* easy quantum groups.

Our results allow us to answer some open questions on easy quantum groups. First of all we show that there are uncountably many easy quantum groups.

Theorem C (See Theorem 5.6). *There are uncountably many pairwise non-isomorphic easy quantum groups.*

Our next result deals with intermediate quantum groups of free quantum groups and their classical counterparts. The half-liberated orthogonal group O_n^* is the unique easy quantum group intermediate to $\mathrm{O}_n^+ \supset O_n$. Further, in [4], it is shown that there is no intermediate quantum group $\mathrm{O}_n^* \supset G \supset \mathrm{O}_n$. However, it is an open problem, whether there are other quantum groups intermediate to $\mathrm{O}_n^+ \supset \mathrm{O}_n$. Likewise it is not known whether there is any intermediate quantum group to $\mathrm{S}_n^+ \supset \mathrm{S}_n$. In the spirit of these questions, we show that besides the abundance of easy hyperoctahedral quantum groups, there are examples of homogeneous hyperoctahedral quantum groups that are not easy.

Theorem D (See Theorem 5.8). *For every $n \geq 3$, there is an example of a homogeneous hyperoctahedral quantum group $H_n^+ \supset G$ that is not easy.*

In [5], two series of easy quantum groups were introduced. $H_n^{(s)}$ and $H_n^{[s]}$, $s \in \mathbb{N} \cup \{\infty\}$ are called the hyperoctahedral series and the higher hyperoctahedral series, respectively. These quantum groups fit into the framework of this article, which allows us to identify them explicitly. In the light of present interest in operator algebraic properties of free quantum groups, we study the hyperoctahedral series from this point of view.

As a last application of our classification result in Theorem B, we prove a uniform de Finetti theorem for all group-theoretical hyperocatahedral quantum groups in Section 5.4. It recovers the de Finetti theorem for H_n^* of Banica, Curran and Speicher [6].

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2. Preliminaries

2.1. Compact matrix quantum groups. In [47, 49] Woronowicz defined *compact matrix quantum groups* (CMQG), which are the correct analogue of compact Lie groups in the setting of his *compact quantum groups* [50]. A compact matrix quantum group is a unital C*-algebra A with an element $u \in \mathrm{M}_n(A)$ such that

- A is generated by the entries of u,
- there is a *-homomorphism $\Delta : A \to A \otimes A$ called *comultiplication* which satisfies $\Delta(u_{ij}) = \sum_k u_{ik} \otimes u_{kj}$ for all $1 \leq i, j \leq n$ and
- u is unitary and its transpose u^t is invertible.

The matrix u is called the *fundamental corepresentation* of (A, u). A morphism between CMQGs A and B is a morphism $\phi : A \to B$ of the underlying C*-algebras such that $(\phi \otimes \mathrm{id})(u_A)$ is conjugate with u_B by some element in $\mathrm{GL}(n, \mathbb{C})$. Two CMQGs are called *similar* if they are isomorphic as C*-algebras via a morphism of CMQGs. They are *isomorphic* if they are isomorphic as C*-algebras via a morphism preserving the fundamental corepresentation.

An important feature of CMQG is the existence of a *Haar state* proved by Woronowicz.

Theorem 2.1 (See [47]). *Let (A, u) be a CMQG with comultiplication Δ. Then there is a unique state, the Haar state, $\phi \in A^*$ such that*

$$(\mathrm{id} \otimes \phi) \circ \Delta(x) = \phi(x)1 = (\phi \otimes \mathrm{id}) \circ \Delta(x)$$

for all $x \in A$.

A CMQG group (A, u) is called *orthogonal*, if its fundamental corepresentation has self-adjoint entries. We write $\overline{u} = (u_{ij}^*)$ for the entrywise adjoint of u. One can show that the Haar state ϕ of an orthogonal quantum group (A, u) is *tracial*, that is $\phi(xy) = \phi(yx)$ for all $x, y \in A$. All compact matrix quantum groups considered in this article are orthogonal.

2.2. Different versions of quantum groups. Every CMQG (A, u) contains the *algebra of polynomial functions* $\mathrm{Pol}(A) = * - \mathrm{alg}(u_{ij}, i, j \in \{1, \dots, n\})$. Also the universal enveloping C*-algebra of $\mathrm{Pol}(A)$ is a CMQG, which is called the *maximal version* of A. We say that A is in its maximal version if it is isomorphic to its maximal version. We write $\mathrm{C}(G)$ for the maximal version of a CMQG, thinking of G as the quantum group. In this case we also write $\mathrm{Pol}(G) = \mathrm{Pol}(\mathrm{C}(G))$. Any CMQG whose maximal version is isomorphic with $\mathrm{C}(G)$, is called a version of $\mathrm{C}(G)$.

The Haar state ϕ of A is faithful on $\mathrm{Pol}(G)$.

For von Neumann algebraic notions we refer the reader to Section 2.7.1. Taking the weak closure of $\mathrm{Pol}(G)$ in the GNS-representation associated with ϕ, we obtain the *von Neumann algebraic version* of A denoted by $\mathrm{L}^\infty(G)$. If A is an orthogonal CMQG, then ϕ extends to a faithful normal tracial state on $\mathrm{L}^\infty(G)$, showing that it is a finite von Neumann algebra.

Let us mention that with every compact matrix quantum group G, one can associate a *dual quantum group* $\widehat{G}$, which is a discrete quantum group. We will only use the notation $\widehat{G}$ and refer to [40] for more explanation on duality for quantum groups.

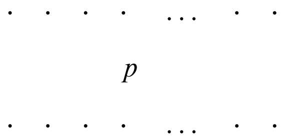
2.3. Tannaka–Krein duality for compact matrix quantum groups. Let (A, u) be a CMQG with comultiplication Δ. A *unitary corepresentation matrix* of A is a unitary element $v \in \mathrm{M}_m(A)$ such that $\Delta_A(v_{ij}) = \sum_k v_{ik} \otimes v_{kj}$ for all $i, j \in \{1, \dots, m\}$. A morphism between unitary corepresentation matrices $v \in \mathrm{M}_k(A)$ and $w \in \mathrm{M}_l(A)$ is a matrix $T \in \mathrm{M}_{l,k}(\mathbb{C})$ such that $Tv = wT$. It is also called an *intertwiner*. The space of intertwiners between two unitary corepresentation matrices v, w is denoted by $\mathrm{Hom}(v, w)$.

The *tensor product* of two corepresentation matrices $v \in \mathrm{M}_k(A)$ and $w \in \mathrm{M}_l(A)$ is defined as the Kronecker tensor product $(v \otimes w)_{(i,i')(j,j')} = v_{ij} w_{i'j'}$. Denote by $\mathrm{Corep}(A, u)$ the category whose elements are tensor powers $u^{\otimes k}$, $k \in \mathbb{N}$ and whose morphisms are intertwiners. Then $\mathrm{Corep}(A, u)$ is a *concrete compact tensor C*-category* in the sense of Woronowicz (see [47,48] or [40, Chapter 5]).

We need the following special version of Woronowicz's Tannaka–Krein duality.

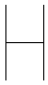
Theorem 2.2 (See [48]). *Any concrete tensor C^*-category whose objects are tensor powers of a specified object x is equivalent to the category* $\operatorname{Corep}(A, u)$ *for some orthogonal compact matrix quantum group (A, u) such that u corresponds to x under this equivalence of categories. Two orthogonal compact matrix quantum groups (A, u) and (B, v) are similar if and only if the categories* $\operatorname{Corep}(A, u)$ *and* $\operatorname{Corep}(B, v)$ *are equivalent over* FdHilb *via an equivalence that sends the isomorphism class of u to the isomorphism class of v.*

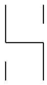
2.4. Easy quantum groups. In order to describe corepresentation categories of compact quantum matrix groups combinatorially, Banica and Speicher introduced the notions of a *category of partitions* and of *easy quantum groups* [8]. Alternatively, we speak about *partition quantum groups*. A *partition* p is given by k upper points and l lower points which may be connected by lines. This way, the set of $k + l$ points is partitioned into several *blocks*. We write a partition as a diagram in the following way:



k upper points and
l lower points

Two examples of such partitions are the following diagrams.





In the first example, all four points are connected and the partition consists only of one block. In the second example, the left upper point and the right lower point are connected, whereas neither of the two remaining points is connected to any other point.

The set of partitions on k upper and l lower points is denoted by $\mathcal{P}(k, l)$, and the set of all partitions is denoted by $\mathcal{P}$. We also write $\mathcal{P}(k)$ for the set of all partitions which have only k upper points and no lower points. We will also use *labelled partitions*, i.e. partitions whose points are labelled by natural numbers. Vice versa, we can associate with an index $i = (i_1, \dots, i_k) \in \{1, \dots, n\}^k$ the partition $\ker(i) \in \mathcal{P}(k)$ whose blocks are exactly $\{j \mid i_j = i_0\}$, where i_0 runs through $\{1, \dots, n\}$.

There are the natural operations *tensor product* $(p \otimes q)$, *composition* (pq), *involution* (p^*) and *rotation* on partitions (see [8, Definition 1.8] or [46, Definition 1.4]). If $p \in \mathcal{P}(k, l)$, $q \in \mathcal{P}(k', l')$, then $p \otimes q \in \mathcal{P}(k + k', l + l')$ is the partition obtained by writing p and q next to each other. If $p \in \mathcal{P}(k, l)$, $q \in \mathcal{P}(l, m)$, then qp is the partition obtained by writing p above q, connecting them along the l intermediate points and deleting all closed strings which are not connected to any of the $k+m$ remaining points. The involution $p^* \in \mathcal{P}(l,k)$ of $p \in \mathcal{P}(k,l)$ is obtained by turning p upside down. A basic rotation of $p \in \mathcal{P}(k,l)$ is the partition in $\mathcal{P}(k-1,l+1)$ or in $\mathcal{P}(k+1,l-1)$ arising after one turns the first leg of the upper row and puts it in front of the first leg of the lower row – or vice versa. Now, a rotation of p is a partition which arises after applying multiple basic rotations to p.

A collection $\mathcal{C}$ of subsets $\mathcal{C}(k,l) \subset \mathcal{P}(k,l)$, $k,l \in \mathbb{N}$ is called a *category of partitions* if it is closed under these operations and if it contains the pair partition ⊓ (see [8, Definition 6.1] or [46, Definition 1.4]).

Given a partition $p \in \mathcal{P}(k,l)$ and two multi-indices $(i_1,\dots,i_k)$, $(j_1,\dots,j_l)$, we can label the diagram of p with these numbers. The upper and the lower row both are labelled from left to right and we define

$$\delta_p(i,j) = \begin{cases} 1 & \text{if } p \text{ connects only equal indexes,} \\ 0 & \text{if there is a block of } p \text{ connecting unequal indexes}. \end{cases}$$

For every $n \in \mathbb{N}$, one associates a map $T_p : (\mathbb{C}^n)^{\otimes k} \to (\mathbb{C}^n)^{\otimes l}$ with p which is defined by

$$T_p(e_{i_1} \otimes \cdots \otimes e_{i_k}) = \sum_{j_1,\dots,j_l \in \{1,\dots,n\}} \delta_p(i,j) \cdot e_{j_1} \otimes \cdots \otimes e_{j_l}\,.$$

Definition 2.3 (Definition 6.1 of [8] or Definition 2.1 of [5]). An orthogonal compact matrix quantum group (A,u) is called *easy*, if there is a category of partitions $\mathcal{C}$ given by sets $\mathcal{C}(k,l) \subset \mathcal{P}(k,l)$, for all $k,l \in \mathbb{N}$ such that

$$\mathrm{Hom}(u^{\otimes k}, u^{\otimes l}) = \mathrm{span}\{T_p \mid p \in \mathcal{C}(k,l)\}\,.$$

We can apply Theorem 2.2, in order to obtain the following one-to-one correspondence between categories of partitions and easy quantum groups. It is the basis of combinatorial investigations of easy quantum groups.

Theorem 2.4 (See [8]). *There is a bijection between*

- *categories of partitions $\mathcal{C}$ and*
- *families of easy quantum groups $G_{\mathcal{C}}(n)$, $n \in \mathbb{N}$, up to similarity.*

2.4.1. Homogeneous quantum groups.

The permutation groups S_n arise as easy quantum groups associated with the category of all partitions. In the framework of compact matrix quantum groups they can be presented as

$$\mathrm{C}(\mathrm{S}_n) \cong \mathrm{C}^*(p_{ij}, 1 \le i,j \le n \mid p \text{ is unitary, } p_{ij} \text{ are commuting projections})\,,$$

where $p = (p_{ij})$ is the fundamental corepresentation.

A compact matrix quantum group (A,u) is called *homogeneous* if there is a morphism of compact matrix quantum groups $A \to \mathrm{C}(\mathrm{S}_n)$. Put differently, S_n is a quantum subgroup of the compact quantum group described by A.

Note that all easy quantum groups are homogeneous, since every category of partitions is contained in the category of all partitions.

2.4.2. Hyperoctahedral quantum groups. Properties of an easy quantum group are, in principle, completely described by their category of partitions. Let us recall some elementary instances of this fact.

A category of partitions $\mathcal{C}$ is called *hyperoctahedral* if the four block $\sqcap\!\!\sqcap\!\!\sqcap$ is in $\mathcal{C}$, but the double singleton $\uparrow \otimes \uparrow$ is not in $\mathcal{C}$.

Proposition 2.5. *Let (A, u) be an easy quantum group whose category of partitions is $\mathcal{C}$.*

(i) The entries u_{ij} of u are partial isometries if and only if $\sqcap\!\!\sqcap\!\!\sqcap \in \mathcal{C}$.

(ii) The elements u_{ij}^2 are central projections in A if and only if $\sqcup\!\!\!/\!\!\sqcap \in \mathcal{C}$.

(iii) The corepresentation matrix u is irreducible if and only if $\uparrow \otimes \uparrow \notin \mathcal{C}$.

Proof. By Tannaka–Krein duality, A is the universal C*-algebra generated by the entries u_{ij} of its fundamental corepresentation which satisfy $T_p u^{\otimes k} = u^{\otimes l} T_p$ for all $p \in \mathcal{C}(k, l)$. Now item (i) follows by comparing coefficients of T_p and $u^{\otimes 4} T_p$ for $p = \sqcap\!\!\sqcap\!\!\sqcap$. Similarly item (ii) follows by comparing coefficients of $T_p u^{\otimes 3}$ and $u^{\otimes 3} T_p$ for $p = \sqcup\!\!\!/\!\!\sqcap$.

In order to see (iii), note that u is irreducible if and only if $\dim \operatorname{Hom}(1, u \otimes u) = 1$. Since $T_{\sqcap} \in \operatorname{Hom}(1, u \otimes u)$ and $|\mathcal{P}(2)| = 2$, it follows that u is irreducible, if and only if $T_{\uparrow \otimes \uparrow} \notin \operatorname{Hom}(1, u \otimes u)$. The latter is equivalent to $\uparrow \otimes \uparrow \notin \mathcal{C}$. ☐

Definition 2.6. An orthogonal compact matrix quantum group (A, u) is called *hyperoctahedral* if the entries u_{ij} are self-adjoint partial isometries and the corepresentation matrix u is irreducible.

Item (iii) of the previous proposition can be reformulated to a characterisation of hyperoctaherdal quantum groups.

Proposition 2.7. *Let (A, u) be a homogeneous orthogonal quantum group. Then (A, u) is hyperoctahedral if and only if the elements u_{ij} are self-adjoint partial isometries and there are i, i' such that $\sum_j u_{ij} \neq \sum_j u_{i'j}$.*

Proof. Let (A, u) with $u \in \mathrm{M}_n(A)$ be a homogeneous orthogonal quantum group. Since (A, u) is orthogonal, its fundamental corepresentation u is irreducible if and only if $\dim \operatorname{Hom}(1, u \otimes u) = 1$. Since (A, u) is homogeneous, it satisfies $\operatorname{Hom}(1, u \otimes u) \subset \operatorname{span}\{T_{\sqcap}, T_{\uparrow \otimes \uparrow}\}$. Moreover, $T_{\sqcap} \in \operatorname{Hom}(1, u \otimes u)$. Assuming that all entries u_{ij} of the fundamental corepresentation are self-adjoint partial isometries, we hence have to prove that $T_{\uparrow \otimes \uparrow} \in \operatorname{Hom}(1, u \otimes u)$ if and only if $\sum_j u_{ij}$ is independent of the choice of $i \in \{1, \ldots, n\}$. This follows by comparing the coefficients of $T_{\uparrow \otimes \uparrow}$ and $u^{\otimes 2} T_{\uparrow \otimes \uparrow}$. ☐

2.5. Semi-direct product quantum groups. Let $\Gamma = \langle g_1, \ldots, g_n \rangle$ be a finitely generated group and assume that S_n acts on Γ by permuting $g_1, \ldots, g_n$. Then S_n also acts on the maximal group C*-algebra $\mathrm{C}^*(\Gamma)$. We describe a semi-direct product construction implementing this action. It is a very simple special case of the bicrossed-product constructions described in [26, 28, 29, 43].

Recall that the semi-direct product of two groups G and H, where H acts by group automorphisms $(\alpha_h)_{h \in H}$ on G is $G \times H$ as a set whose multiplication is given by $(g_1, h_1)(g_2, h_2) = (g_1 \alpha_{h_1}(g_2), h_1 h_2)$. This picture should be kept in mind in order to understand the semi-direct product construction which is described in the following proposition.

Denote by (p_{ij}) the fundamental corepresentation of $\mathrm{C}(\mathrm{S}_n)$. The CMQG described in the next proposition is called the semi-direct product of $\mathrm{C}^*(\Gamma)$ and $\mathrm{C}(\mathrm{S}_n)$ and it is denoted by $\mathrm{C}^*(\Gamma) \bowtie \mathrm{C}(\mathrm{S}_n)$.

Proposition 2.8. *The C*-algebra $\mathrm{C}^*(\Gamma) \otimes \mathrm{C}(\mathrm{S}_n)$ is a CMQG with fundamental corepresentation $(u_{g_i} p_{ij})$. Its comultiplication is given by $\Delta(u_{g_i} p_{ij}) = \sum_k u_{g_i} p_{ik} \otimes u_{g_k} p_{kj}$ for all $i, j \in \{1, \ldots, n\}$.*

Proof. We first show that $(u_{g_i} p_{ij})$ and $(u_{g_i}^* p_{ij})$ are unitaries in $\mathrm{M}_n(\mathrm{C}^*(\Gamma) \otimes \mathrm{C}(G))$. We can use the relation $\sum_k p_{ki} = 1$ so as to obtain

$$\sum_k p_{ki} u_{g_k}^* u_{g_k} p_{kj} = \delta_{ij} \sum_k p_{ki} = \delta_{ij} 1 \,.$$

So $(u_{g_i} p_{ij})$ is an isometry. Similarly it follows that $(u_{g_i}^* p_{ij})$ is an isometry. As $\mathrm{C}^*(\Gamma) \otimes \mathrm{C}(\mathrm{S}_n)$ has a faithful tracial state, this implies that both matrices are unitary.

Since $\sum_j u_{g_i} p_{ij} = u_{g_i}$ and $(u_{g_i} p_{ij})^2 = p_{ij}$, it follows that the entries of $(u_{g_i} p_{ij})$ generate $\mathrm{C}^*(\Gamma) \otimes \mathrm{C}(\mathrm{S}_n)$. Finally let us show that there is a comultiplication on $\mathrm{C}^*(\Gamma) \otimes \mathrm{C}(\mathrm{S}_n)$ which admits $(u_{g_i} p_{ij})$ as a fundamental corepresentation. The right S_n-action $\sigma(g_i) = g_{\sigma^{-1}(i)}$ gives rise to a right S_n-action on $\mathrm{C}^*(\Gamma)$. Denote by $\delta : \mathrm{C}^*(\Gamma) \to \mathrm{C}(\mathrm{S}_n) \otimes \mathrm{C}^*(\Gamma)$ the corresponding coaction satisfying $\delta(u_{g_i}) = \sum_k p_{ik} \otimes u_{g_k}$. Denote by Δ_{S_n} and Δ_Γ the comultiplication of $\mathrm{C}(\mathrm{S}_n)$ and $\mathrm{C}^*(\Gamma)$, respectively. Then $\Delta = ((\mathrm{id} \otimes \delta \otimes \mathrm{id} \otimes \mathrm{id}) \circ (\Delta_\Gamma \otimes \Delta_{\mathrm{S}_n}))_{12324}$ from $\mathrm{C}^*(\Gamma) \otimes \mathrm{C}(\mathrm{S}_n)$ to $\mathrm{C}^*(\Gamma) \otimes \mathrm{C}(\mathrm{S}_n) \otimes \mathrm{C}^*(\Gamma) \otimes \mathrm{C}(\mathrm{S}_n)$ satisfies

$$\begin{aligned}
\Delta(u_{g_i} p_{ij}) &= \left(\left(\mathrm{id} \otimes \delta \otimes \mathrm{id} \otimes \mathrm{id} \right) \left(\sum_k u_{g_i} \otimes u_{g_i} \otimes p_{ik} \otimes p_{kj} \right) \right)_{12324} \\
&= \left(\sum_{k,l} u_{g_i} \otimes p_{il} \otimes u_{g_l} \otimes p_{ik} \otimes p_{kj} \right)_{12324} \\
&= \sum_k u_{g_i} p_{ik} \otimes u_{g_k} p_{kj} \,,
\end{aligned}$$

for all $i, j \in \{1, \ldots, n\}$. This finishes the proof. □

Example 2.9. Consider the permutation action of S_n on the natural generators of $\mathbb{Z}^{\oplus n}$. Note that $\mathrm{C}^*(\mathbb{Z}^{\oplus n}) \cong \mathrm{C}(\mathbb{T}^n)$ as compact quantum groups by Pontryagin duality. We have $\mathrm{C}^*(\mathbb{Z}^{\oplus n}) \bowtie \mathrm{C}(\mathrm{S}_n) \cong \mathrm{C}(\mathbb{T}^n \rtimes \mathrm{S}_n)$ as compact matrix quantum groups. Similarly, we have $\mathrm{C}^*(\mathbb{Z}_2^{\oplus n}) \bowtie \mathrm{C}(\mathrm{S}_n) \cong \mathrm{C}(\mathbb{Z}_2^{\oplus n} \rtimes \mathrm{S}_n)$.

2.6. Diagonal subgroups. If (A, u) is a CMQG whose fundamental corepresentation is a diagonal matrix, then its diagonal entries u_{ii}, $i \in \{1, \dots, n\}$ are unitary. Let Γ be the group generated by these diagonal entries. Then A is a C^*-algebra completion of the group algebra $\mathbb{C}\Gamma$. Using this fact, one can associate with any CMQG a canonical discrete group with a fixed set of generators.

Definition 2.10. Let (A, u) be a CMQG. Denote by $\pi : A \to B$ the quotient of A by the relations $u_{ij} = 0$ for all $i \neq j$. Let Γ be the group generated by the elements $g_i = \pi(u_{ii})$. Then Γ together with the generators $g_1, \dots, g_n$ is called the *diagonal subgroup* of (A, u). We denote it by $\operatorname{diag}(A, u)$.

We will use the following proposition, guaranteeing that a CMQG in its maximal version gives rise to the maximal group C^*-algebra of its diagonal subgroup.

Proposition 2.11. *Let (A, u) be a CMQG in its maximal version and let Γ be the diagonal subgroup of (A, u). Then the C^*-algebra $A/(u_{ij} = 0$ for all $i \neq j) \cong C^*(\Gamma)$ is in its maximal version.*

Proof. Since (A, u) is in its maximal version, it is the universal enveloping C^*-algebra of its $*$-subalgebra $*\text{-alg}(u_{ij} \mid i, j \in \{1, \dots, n\})$. Denote by $\pi : A \to B$ the quotient of A by the relations $u_{ij} = 0$ for all $i \neq j$. Then B is the universal enveloping C^*-algebra of the $*$-algebra $*\text{-alg}(\pi(u_{ii}) \mid i\{1, \dots, n\}) \cong \mathbb{C}\Gamma$. So $B \cong \mathrm{C}^*(\Gamma)$. ☐

2.7. Operator algebraic properties of quantum groups. In Section 5.3 we will describe certain operator algebraic properties of easy quantum groups. Let us briefly describe von Neumann algebras and some of their properties. We refer the reader to the book [7] for more details on approximation properties for operator algebras and to [16] for an introduction to von Neumann algebras.

2.7.1. Von Neumann algebras. A separable von Neumann algebra is a strongly closed, unital $*$-subalgebra $M \subset \mathcal{B}(H)$ for some (complex) separable Hilbert space H. All von Neumann algebras in this article are separable. We say that M is finite, if there is a faithful normal tracial state on M, i.e. a σ-strongly continuous functional $\tau : M \to \mathbb{C}$ such that

- $\tau(x^*x) \geq 0$ for all $x \in M$,
- if $\tau(x^*x) = 0$ then $x = 0$ and
- $\tau(xy) = \tau(yx)$ for all $x, y \in M$.

If M has a normal faithful tracial state τ, then $(x, y) \mapsto \tau(y^*x)$ defines an inner product on M. The Hilbert space completion of M with respect to this inner product is denoted by $\mathrm{L}^2(M, \tau)$. The action of M on itself by left multiplication can be extended to a representation on $\mathrm{L}^2(M, \tau)$ called the *GNS-representation* of M associated with τ. We say that an inclusion of finite von Neumann algebras $N \subset M$ with faithful normal tracial state τ has *finite index* if the commutant $N' = \{x \in \mathcal{B}(\mathrm{L}^2(M, \tau)) \mid xy = yx \text{ for all } y \in N\}$ is a finite von Neumann algebra. If M is finitely generated as a left N module, then $N \subset M$ has finite index for all traces on M.

2.7.2. Amenability. The notion of amenability for discrete groups goes back to the work of von Neumann in [42]. All abelian groups are amenable, while the basic example of a non-amenable group is a free group $\mathbb{F}_n$. Also free products $\mathbb{Z}_s^{*(n-1)}$ for $s, n \geq 2$ and $sn \geq 6$ are non-amenable.

In [35,41], Ruan and Tomatsu describe amenability of quantum groups. We need the following very special case of their work, which already appeared in the article of Ruan.

Theorem 2.12 (See [35, Theorem 4.5]). *Let G be a compact quantum subgroup of O_n^+. Then the discrete dual $\widehat{G}$ is an amenable quantum group, if and only $\mathrm{L}^\infty(G)$ is an amenable von Neumann algebra.*

An important notion in von Neumann algebra theory is *strong solidity* introduced in [33]. A von Neumann algebra M is called diffuse, if there are no minimal projections in M. We call M strongly solid if for all amenable, diffuse von Neumann subalgebras $A \subset M$, the normaliser $\mathcal{N}_M(A)'' = \mathrm{vN}(u \in \mathcal{U}(M) \mid uAu^* = A)$ is also amenable.

2.7.3. Haagerup property. The Haagerup property for groups goes back to [19], where Haagerup proved that free groups have the Haagerup property. More generally, this property is preserved under free products.

Theorem 2.13 (See [9]). *Let G_1, G_2 be groups with the Haagerup property. Then also $G_1 * G_2$ has the Haagerup property.*

Based on a lecture by Connes the Haagerup property was defined in the setting of finite von Neumann algebra by Choda [10]. In particular, she proves that a group has the Haagerup property if and only if its group von Neumann algebra has this property.

Theorem 2.14 (See [10]). *Let G be a discrete group. Then G has the Haagerup property if and only if* $\mathrm{L}(G)$ *has the Haagerup property.*

The article [24] describes basic properties of the Haagerup property for finite von Neumann algebras.

Theorem 2.15 (See [24, Theorem 1.1]). *Let $N \subset M$ be a finite index inclusion of von Neumann algebras. Then N has the Haagerup property if and only if M has the Haagerup property.*

Recently, in [15] the notion was systematically investigated in the framework of quantum groups. We cite a special case of the main result of this article.

Theorem 2.16 (See [15]). *Let G be a compact quantum subgroup of O_n^+. Then $\widehat{G}$ has the Haagerup property if and only if $\mathrm{L}^\infty(G)$ has the Haagerup property as a von Neumann algebra.*

2.7.4. The complete metric approximation property.

The complete metric approximation property (CMAP) for groups goes back to the work of Haagerup in [19] and de Cannière-Haagerup [14]. In [39] it is proved that the free product of groups having CMAP still has CMAP.

Theorem 2.17 (See [39, Theorem 4.13]). *If G_1 and G_1 are groups with the CMAP, then also $G_1 * G_2$ hast the CMAP.*

The von Neumann algebraic analogue of CMAP is called W*-completely contractive approximation property (W*-CCAP). We state a special case of a result in [20] (see also [21]).

Theorem 2.18 (See [20]). *Let G be a discrete group. Then G has CMAP if and only if $\mathrm{L}(G)$ has the W*-CCAP.*

The following stability result for the W*-CCAP is well known. It can be proved using [7, Theorem 12.3.13].

Theorem 2.19. *Let $N \subset M$ be a finite index inclusion of von Neumann algebras. Then N has the W*-CCAP if and only if M has the W*-CCAP.*

In the context of discrete quantum groups, CMAP was studied as well.

Theorem 2.20 (See [25]). *Let G be compact quantum subgroup of O_n^+. Then $\widehat{G}$ has the CMAP if and only if $\mathrm{L}^\infty(G)$ has the W*-CCAP.*

3. Homogeneous quantum subgroups of $\mathrm{C}^*(\mathbb{Z}_2^{*n}) \bowtie \mathrm{C}(\mathrm{S}_n)$

Recall from Section 2.5 that $\mathrm{C}^*(\mathbb{Z}_2^{*n}) \bowtie \mathrm{C}(\mathrm{S}_n)$ denotes the CMQG whose C*-algebra is isomorphic with $\mathrm{C}^*(\mathbb{Z}_2^{*n}) \otimes \mathrm{C}(\mathrm{S}_n)$ and whose fundamental corepresentation is $(u_{a_i} p_{ij})$. Here $a_1, \dots, a_n$ denote the natural generators of $\mathbb{Z}_2^{*n}$ and (p_{ij}) is the fundamental corepresentation of $\mathrm{C}(\mathrm{S}_n)$. Note that $(u_{a_i} p_{ij})^2 = p_{ij}$ is a central projection in $\mathrm{C}^*(\mathbb{Z}_2^{*n}) \otimes \mathrm{C}(\mathrm{S}_n)$ for all $i, j \in \{1, \dots, n\}$. The next theorem tells us in particular that $\mathrm{C}^*(\mathbb{Z}_2^{*n}) \bowtie \mathrm{C}(\mathrm{S}_n)$ is the universal homogeneous quantum group with this property.

Theorem 3.1. *Let (A,u) be a homogeneous orthogonal compact matrix quantum group such that u_{ij}^2 is a central projection in A for all $i,j \in \{1,\dots,n\}$. Then there is a quotient $\mathbb{Z}_2^{*n} \twoheadrightarrow \Gamma$ whose kernel is invariant under the natural action of S_n, such that A is a version of $C^*(\Gamma) \bowtie \mathrm{C}(\mathrm{S}_n)$. In particular, we have $A \cong C^*(\Gamma) \bowtie \mathrm{C}(\mathrm{S}_n)$ if A is in its maximal version.*

Proof. First note that if (A,u) is a compact matrix quantum group such that u_{ij}^2 is a central projection in A for all $i,j \in \{1,\dots,n\}$, then the elements u_{ij}^2 are central projections in $\mathrm{Pol}(A)$ and hence the same is true in the maximal version of (A,u). We may hence assume that A is in its maximal version.

An embedding $\mathrm{C}(S_n) \to A$. Since u_{ij} is a self-adjoint partial isometry for all $i,j \in \{1,\dots,n\}$ and u is unitary, it follows that $\sum_k u_{ik}^2 = 1 = \sum_k u_{kj}^2$ for all $i,j \in \{1,\dots,n\}$. Since $q_{ij} = u_{ij}^2 \in A$, $i,j \in \{1,\dots,n\}$ is also a commuting family of projections, it satisfies the relations of the fundamental corepresentation of $\mathrm{C}(\mathrm{S}_n)$. So there is a morphism $\mathrm{C}(\mathrm{S}_n) \to A$ of compact quantum groups defined by $p_{ij} \mapsto q_{ij}$. Here $p = (p_{ij})$ denotes the fundamental corepresentation of $\mathrm{C}(\mathrm{S}_n)$. Since A is homogeneous, there is a morphism $A \to \mathrm{C}(\mathrm{S}_n)$ of compact matrix quantum groups, satisfying $u_{ij} \mapsto p_{ij}$. This shows that $\mathrm{C}(\mathrm{S}_n) \to A$ is an embedding.

Construction of Γ. Let now $v_i = \sum_j u_{ij}$ for $i \in \{1,\dots,n\}$. Then all v_i are self-adjoint unitaries, since they are obviously self-adjoint and

$$v_i^2 = \sum_{j,k} u_{ij}u_{ik} = \sum_j q_{ij} = 1\,.$$

Denote by Γ the diagonal subgroup of (A,u). By Proposition 2.11 the quotient $A/(u_{ij} = 0 \text{ for all } i \neq j)$ appears in its maximal version $\mathrm{C}^*(\Gamma)$, as A is in its maximal version. Denote by $\pi : A \to \mathrm{C}^*(\Gamma)$ the natural quotient map. Since $\pi(v_i) = \pi(u_{ii})$ and $v_i^2 = 1$, there is a quotient map $\mathbb{Z}_2^{*n} \to \Gamma$ mapping the i-th generator of $\mathbb{Z}_2^{*n}$ to $\pi(v_i)$.

A $*$-homomorphism $\mathrm{C}^*(\Gamma) \to A$. Denote by g_i the natural generators of Γ, which satisfy $u_{g_i} = \pi(u_{ii}) = \pi(v_i)$ for all $i \in \{1,\dots,n\}$. We show that there is a map $\mathrm{C}^*(\Gamma) \to A$ which maps u_{g_i} to v_i. By universality of $\mathrm{C}^*(\Gamma)$, it suffices to show that the unitaries v_i, $i \in \{1,\dots,n\}$ satisfy the relations of g_i, $i \in \{1,\dots,n\}$. So assume that $g_{i_1} \cdots g_{i_l} = e$. Then $\pi(v_{i_1} \cdots v_{i_l}) = u_{g_{i_1}} \cdots u_{g_{i_l}} = 1$. Let $i_{l+1},\dots,i_{l'}$ be an enumeration of $\{1,\dots,n\} \setminus \{i_1,\dots,i_l\}$. Note that $v_i^2 = 1$ implies $g_i^2 = e$. So

$$\pi(u_{i_1i_1} \cdots u_{i_li_l} q_{i_{l+1}i_{l+1}} \cdots q_{i_{l'}i_{l'}}) = 1$$

holds and implies

$$u_{i_1i_1} \cdots u_{i_li_l} q_{i_{l+1}i_{l+1}} \cdots q_{i_{l'}i_{l'}} \in q_{i_1i_1} \cdots q_{i_li_l} q_{i_{l+1}i_{l+1}} \cdots q_{i_{l'}i_{l'}} + \langle q_{ij}, i \neq j \rangle\,, \tag{3.1}$$

where the last expression denotes the ideal in A which is generated by all q_{ij}, $i \neq j$.

Note that indeed $\langle q_{ij}, i \neq j\rangle = \langle u_{ij}, i \neq j\rangle$, since $v_i q_{ij} = u_{ij}$. Next note that $\{i_1, \dots, i_{l'}\} = \{1, \dots, n\}$ implies

$$\langle q_{ij}, i \neq j\rangle \cdot q_{i_1 i_1} \cdots q_{i_l i_l} q_{i_{l+1} i_{l+1}} \cdots q_{i_{l'} i_{l'}} = 0\,, \tag{3.2}$$

because $q_{ij} q_{ij'} = 0 = q_{ij} q_{i'j}$ for all $i \neq i'$ and $j \neq j'$. Moreover, since q_{ij} is central in A, we have

$$\begin{aligned}(u_{i_1 i_1} \cdots u_{i_l i_l} q_{i_{l+1} i_{l+1}} \cdots q_{i_{l'} i_{l'}}) \cdot (q_{i_1 i_1} \cdots q_{i_l i_l} q_{i_{l+1} i_{l+1}} \cdots q_{i_{l'} i_{l'}})& \\ = u_{i_1 i_1} \cdots u_{i_l i_l} q_{i_{l+1} i_{l+1}} \cdots q_{i_{l'} i_{l'}}\,.&\end{aligned} \tag{3.3}$$

Multiplying (3.1) with $q_{i_1 i_1} \cdots q_{i_l i_l} q_{i_{l+1} i_{l+1}} \cdots q_{i_{l'} i_{l'}}$, and applying (3.2) and (3.3), we see that

$$u_{i_1 i_1} \cdots u_{i_l i_l} q_{i_{l+1} i_{l+1}} \cdots q_{i_{l'} i_{l'}} = q_{i_1 i_1} \cdots q_{i_l i_l} q_{i_{l+1} i_{l+1}} \cdots q_{i_{l'} i_{l'}}\,. \tag{3.4}$$

Applying Δ to (3.4), we obtain

$$\begin{aligned}&\sum_{k_1, \dots, k_{l'}} u_{i_1 k_1} \cdots u_{i_l k_l} q_{i_{l+1} k_{l+1}} \cdots q_{i_{l'} k_{l'}} \otimes u_{k_1 i_1} \cdots u_{k_l i_l} q_{k_{l+1} i_{l+1}} \cdots q_{k_{l'} i_{l'}} \\ &\quad = \sum_{k_1, \dots, k_{l'}} q_{i_1 k_1} \cdots q_{i_l k_l} q_{i_{l+1} k_{l+1}} \cdots q_{i_{l'} k_{l'}} \otimes q_{k_1 i_1} \cdots q_{k_l i_l} q_{k_{l+1} i_{l+1}} \cdots q_{k_{l'} i_{l'}}\end{aligned}$$

Multiplying this equation on both sides with $q_{i_1 k_1} \cdots q_{i_l k_l} q_{i_{l+1} k_{l+1}} \cdots q_{i_{l'} k_{l'}} \otimes q_{k_1 i_1} \cdots q_{k_l i_l} q_{k_{l+1} i_{l+1}} \cdots q_{k_{l'} i_{l'}}$ for a fixed index $k_1, \dots, k_{l'}$, we obtain that for all such indices

$$\begin{aligned}&u_{i_1 k_1} \cdots u_{i_l k_l} q_{i_{l+1} k_{l+1}} \cdots q_{i_{l'} k_{l'}} \otimes u_{k_1 i_1} \cdots u_{k_l i_l} q_{k_{l+1} i_{l+1}} \cdots q_{k_{l'} i_{l'}} \\ &\quad = q_{i_1 k_1} \cdots q_{i_l k_l} q_{i_{l+1} k_{l+1}} \cdots q_{i_{l'} k_{l'}} \otimes q_{k_1 i_1} \cdots q_{k_l i_l} q_{k_{l+1} i_{l+1}} \cdots q_{k_{l'} i_{l'}}\end{aligned} \tag{3.5}$$

holds. This implies that

$$u_{i_1 k_1} \cdots u_{i_l k_l} q_{i_{l+1} k_{l+1}} \cdots q_{i_{l'} k_{l'}} = q_{i_1 k_1} \cdots q_{i_l k_l} q_{i_{l+1} k_{l+1}} \cdots q_{i_{l'} k_{l'}}$$

for all $k_1, \dots, k_{l'} \in \{1, \dots, n\}$. Summing over all these indices and using $\sum_k q_{ik} = 1$ for all i, we obtain $v_{i_1} \cdots v_{i_l} = 1$. It follows that there is a $*$-homomorphism $\mathrm{C}^*(\Gamma) \to A$ sending u_{g_i} to v_i for all $\{1, \dots, n\}$.

The S_n-action on Γ. Since $g_i^2 = e$, there is a quotient map $\mathbb{Z}_2^{*n} \to \Gamma$ mapping the i-th natural generator of $\mathbb{Z}_2^{*n}$ to g_i. Let us show that the kernel of this map is S_n-invariant. Take any permutation $\sigma \in \mathrm{S}_n$ and denote by $\chi_\sigma : \mathrm{C}(\mathrm{S}_n) \to \mathbb{C}$ the associated evaluation map. We have

$$\Delta(v_i) = \Delta\Big(\sum_j u_{ij}\Big) = \sum_{k,j} u_{ik} \otimes u_{kj} = \sum_k u_{ik} \otimes v_k\,.$$

So using the quotient map $\psi : A \to \mathrm{C}(\mathrm{S}_n) : u_{ij} \to p_{ij}$, we find

$$(\chi_\sigma \circ \psi \otimes \mathrm{id})(\Delta(v_i)) = \sum_k \chi_\sigma(p_{ik}) \otimes v_k = v_{\sigma^{-1}(i)} \,.$$

Assume that $g_{i_1} \cdots g_{i_l} = e$. Then $v_{i_1} \cdots v_{i_l} = 1$ and hence $\Delta(v_{i_1} \cdots v_{i_l}) = 1 \otimes 1$. We obtain

$$1 = (\chi_{\sigma^{-1}} \circ \psi \otimes \mathrm{id})(\Delta(v_{i_1} \cdots v_{i_l})) = v_{\sigma(i_1)} \cdots v_{\sigma(i_l)} \,.$$

This implies that $g_{\sigma(i_1)} \cdots g_{\sigma(i_l)} = e$. We have shown that the kernel of $\mathbb{Z}_2^{*n} \to \Gamma$ is invariant under the natural action of S_n. So S_n acts on Γ by permuting its generators.

End of the proof. Since S_n acts on Γ, we can hence consider $\mathrm{C}^*(\Gamma) \otimes \mathrm{C}(\mathrm{S}_n)$ with the fundamental corepresentation matrix $w = (u_{g_i} p_{ij}) \in \mathrm{M}_n(\mathrm{C}^*(\Gamma) \otimes \mathrm{C}(\mathrm{S}_n))$ as described in Section 2.5. Now consider the map $\rho : \mathrm{C}^*(\Gamma) \otimes \mathrm{C}(\mathrm{S}_n) \to A$ defined by $\rho(u_{g_i}) = v_i$ and $\rho(p_{ij}) = q_{ij}$. Then $\rho(u_{g_i} p_{ij}) = u_{ij}$ saying that ρ is a morphism of CMQGs. We prove that $(\pi \otimes \psi) \circ \Delta : A \to \mathrm{C}^*(\Gamma) \otimes \mathrm{C}(\mathrm{S}_n)$ is the inverse of ρ. Indeed, it suffices to note that

$$(\pi \otimes \psi) \circ \Delta(u_{ij}) = (\pi \otimes \psi)\left(\sum_k u_{ik} \otimes u_{kj} \right) = \sum_k u_{g_i} \delta_{i,k} \otimes p_{kj} = u_{g_i} \otimes p_{ij} \,.$$

We proved that $A \cong \mathrm{C}^*(\Gamma) \bowtie \mathrm{C}(\mathrm{S}_n)$ as CMQGs, which concludes the proof. □

Before we end this section, let us mention the following proposition showing the particular relevance of diagonal subgroups for CMQGs (A, u) for which u_{ij}^2 are central projections in A. This makes the assumptions of Theorem 3.1 very natural.

Proposition 3.2. *Let (A, u) be an orthogonal CMQG and let (B, v) be the quotient of A by the relations*

$$u_{ij}^2 \textit{ is a central projection for all } i, j$$

Then $\mathrm{diag}(A, u) = \mathrm{diag}(B, v)$.

Proof. Denote by Γ_A the diagonal subgroup of A and by g_i, $i \in \{1, \ldots, n\}$ its generators. Then the diagonal subgroup of (B, u) is described by its C*-algebra via $\mathrm{C}^*(\Gamma_B) = \mathrm{C}^*(\Gamma_A)/\{u_{g_i}^2 \text{ is a central projection}\}$. Since (A, u) is orthogonal, the generators of its diagonal subgroup satisfy $g_i^2 = e$ for all $i \in \{1, \ldots, n\}$. It follows that $u_{g_i}^2 = 1$ and hence $\mathrm{C}^*(\Gamma_B) = \mathrm{C}^*(\Gamma_A)$, which finishes the proof. □

4. Easy quantum subgroups of $\mathbf{C}^*(\mathbb{Z}_2^{*n}) \bowtie \mathbf{C}(\mathbf{S}_n)$

Recall from the preliminaries that the easy quantum group $A = \mathrm{C}(H_n^{[\infty]})$ associated with the category $\langle \sqcup\!\!\!/\!\sqcap \rangle$ is the universal C*-algebra generated by the entries u_{ij},

$i, j \in \{1, \dots, n\}$ of its fundamental corepresentation subject to the relations that u_{ij}^2 are central projections in A for all $i, j \in \{1, \dots, n\}$. Since all easy quantum groups are homogeneous, Theorem 3.1 implies that $\mathrm{C}(H_n^{[\infty]}) \cong \mathrm{C}^*(\mathbb{Z}_2^{*n}) \rtimes \mathrm{C}(\mathrm{S}_n)$. In this section we achieve a complete classification of easy quantum subgroups of $\mathrm{C}^*(\mathbb{Z}_2^{*n}) \rtimes \mathrm{C}(\mathrm{S}_n)$. In [38] we present a completely combinatorial proof for the classification of easy quantum subgroups of $\mathrm{C}(H_n^{[\infty]})$, but it fails to give a description of the quantum groups. The basic ideas behind [38] and the present section are similar, but we make use of Theorem 3.1 in order to simplify combinatorial considerations.

Theorem 3.1 tells us that we need to investigate diagonal subgroups of easy quantum groups in order to describe them completely.

4.1. Diagonal subgroups of easy quantum groups. Recall that the diagonal subgroup of $\mathrm{C}(O_n^+)$ is $\mathbb{Z}_2^{*n}$ together with its natural generators. Hence, all diagonal subgroups of quantum subgroups $\mathrm{C}(O_n^+) \twoheadrightarrow (A, u)$ are quotients of $\mathbb{Z}_2^{*n}$. We want to describe these quotients for easy quantum groups.

Let $a_1, a_2, \dots$ be the natural generators of $\mathbb{Z}_2^{*\infty}$. Given a partition $p \in \mathcal{P}(k)$ we say that a labelling $(i_1, \dots, i_k)$ of p is compatible, if every block of p is labelled by exactly one index — equivalently $\delta_p(i) = 1$. Take $p \in \mathcal{P}(k)$ and a compatible labelling $(i_1, \dots, i_k)$ of p by indices in $\mathbb{N}$, we denote by $w(p, i)$ the word in $\mathbb{Z}_2^{*\infty}$ which arises by labelling p with the letters $a_{i_1}, a_{i_2}, \dots, a_{i_k}$ from left to right. We write $\underline{n} = \{1, \dots, n\}$ for $n \in \mathbb{N}^\times$ and $\underline{n} = \mathbb{N}^\times$ for $n = \infty$. If $\mathcal{C}$ is a category of partitions we write $\mathcal{C}(k) = \mathcal{C}(k, 0)$ for the partitions of length k without lower points. For $n \in \mathbb{N} \cup \{\infty\}$, we write

$$F_n(\mathcal{C}) = \{w(p, i) \,|\, k \in \mathbb{N}, p \in \mathcal{C}(k), i \in \underline{n}^k \text{ compatible}\}$$

for the set of all possible words in $\mathbb{Z}_2^{*n}$ arising from $\mathcal{C}$. The next lemma shows that $F_n(\mathcal{C})$ is always a normal subgroup of $\mathbb{Z}_2^{*n}$.

Lemma 4.1. *Let $\mathcal{C}$ be a category of partitions and $n \in \mathbb{N} \cup \{\infty\}$. Then $F_n(\mathcal{C})$ is a normal subgroup of $\mathbb{Z}_2^{*n}$.*

Proof. We first show that $F_n(\mathcal{C})$ is closed under products. Let $p \in \mathcal{C}(k)$, $p' \in \mathcal{C}(k')$ and let i,i' be compatible labellings with indices from $\underline{n}$ of length k and k', respectively. Then

$$w(p, i) w(p', i') = w(p \otimes p', (i_1, \dots, i_k, i'_1, \dots, i'_{k'}))\,.$$

Next, observe that for $p \in \mathcal{C}(k)$ with compatible labelling i, the inverse of $w(p, i)$ is given by

$$w(p, i)^{-1} = w(p^*, (i_k, i_{k-1}, \dots, i_1))\,.$$

It remains to show that $F_n(\mathcal{C})$ is normal in $\mathbb{Z}_2^{*n}$. Take p and i as before. Choose some $i_0 \in \{1, \dots, n\}$. Then

$$\mathrm{Ad}(a_{i_0})(w(p, i)) = w(p', (i_0, i_1, \dots, i_k, i_0))\,,$$

where p' is the partition arising from p by rotating the uttermost right leg of $p \otimes \sqcup$ to the left. This finishes the proof. ☐

We can now give a description of the diagonal subgroup of an arbitrary easy quantum group.

Lemma 4.2. *Let $\mathcal{C}$ be a category of partitions and let $G_{\mathcal{C}}(n)$ be the easy quantum group associated with $\mathcal{C}$ whose fundamental corepresentation matrix has size $n \times n$. Let Γ be the diagonal subgroup of $G_{\mathcal{C}}(n)$, whose natural generators we denote by $g_1, \dots, g_n$. Then $\Gamma = (\mathbb{Z}_2^{*n})/F_n(\mathcal{C})$, where the natural generators of $\mathbb{Z}_2^{*n}$ map to the generators $g_1, \dots, g_n$.*

Proof. First note that by construction $\mathrm{C}(G_{\mathcal{C}}(n))$ is the universal C^*-algebra generated by the entries of a matrix $(u_{ij})_{1 \leq i,j \leq n}$ satisfying $T_p u^{\otimes k} = u^{\otimes l} T_p$ for all $p \in \mathcal{C}(k,l)$. Since rotating partitions is implemented by repeated composition with $| \otimes \cdots \otimes | \otimes \sqcup$, one can equivalently describe the relations of (u_{ij}) by $T_p u^{\otimes k} = T_p$ for all $p \in \mathcal{C}(k)$. Writing this out, we obtain the relations

$$\sum_{i_1,\dots,i_k} \delta_p(i) u_{i_1 j_1} \cdots u_{i_k j_k} = \delta_p(j)\,,$$

for all partitions $p \in \mathcal{C}(k)$ and all indices $j = (j_1, \dots, j_k)$. Passing to the diagonal subgroup of Γ of $\mathrm{C}(G_{\mathcal{C}}(n))$ replaces u_{ij} by $\delta_{ij} u_{g_i}$ in the previous relations, where $g_1, \dots, g_n$ denote the natural generators of G. Using Proposition 2.11, we see that $\mathrm{C}^*(\Gamma)$ is the universal C^*-algebra whose generating elements $u_{g_1}, \dots, u_{g_n}$ satisfy

$$\delta_p(i) u_{g_{i_1}} \cdots u_{g_{i_k}} = \delta_p(i)\,,$$

for all $p \in \mathcal{C}(k)$ and all indices $i_1, \dots, i_k$. So Γ is the universal group generated by elements $g_1, \dots, g_n$ which satisfy $e = g_{i_1} \cdots g_{i_k}$ for all partitions $p \in \mathcal{C}(k)$ and all compatible labellings $i = (i_1, \dots, i_k)$. Note that in particular $g_i^2 = e$ for all $i \in \{1, \dots, n\}$, since $\sqcup \in \mathcal{C}$. Put differently, Γ is a quotient of $\mathbb{Z}_2^{*n}$ whose relations are exactly given by elements of $F_n(\mathcal{C})$. This finishes the proof. ☐

Next we precisely describe the possible diagonal subgroups of easy quantum groups. The key notion for the subsequent classification is strong symmetry of subgroups of $\mathbb{Z}_2^{*n}$.

Definition 4.3. Let $n \in \mathbb{N} \cup \{\infty\}$. We define the strong symmetric semigroup sS_n as the semigroup of $\mathrm{End}(\mathbb{Z}_2^{*n})$ consisting of identifications of letters. It contains precisely the endomorphisms $\sigma_\phi : a_k \mapsto a_{\phi(k)}$, $k \in \underline{n}$, where $\phi : \underline{n} \to \underline{n}$ is any map.

A *strongly symmetric reflection group* Γ is the quotient $\mathbb{Z}_2^{*n} \to \Gamma$ by an sS_n-invariant, normal subgroup together with its natural generators, which are the images of a_i in Γ.

The next theorem contains all necessary combinatorial considerations, in order to deduce our main results about easy quantum groups.

Theorem 4.4. *For all categories of partitions and all* $n \in \mathbb{N} \cup \{\infty\}$*, the subgroup* $F_n(\mathcal{C}) \leq \mathbb{Z}_2^{*n}$ *is* sS_n*-invariant. Vice versa, for every* $n \in \mathbb{N} \cup \{\infty\}$ *and every* sS_n*-invariant, normal subgroup* $N \leq \mathbb{Z}_2^{*n}$ *there is a category of partitions* $\mathcal{C}$ *such that* $F_n(\mathcal{C}) = N$.

Proof. Let $\mathcal{C}$ be a category of partitions and $n \in \mathbb{N} \cup \{\infty\}$. We prove that $F_n(\mathcal{C})$ is sS_n-invariant. So let $p \in \mathcal{C}(k)$ and i be a compatible labelling of p by elements of $\underline{n}$. Let $w(p,i) \in F_n(\mathcal{C})$ be the associated word in $\mathbb{Z}_2^{*n}$ and $\phi : \underline{n} \to \underline{n}$ be an arbitrary map. We denote by $\phi_*(i)$ the labelling $(\phi(i_1), \dots, \phi(i_k))$. Then $\sigma_\phi(w(p,i)) = w(p, \phi_*(i))$ showing that $F_n(\mathcal{C})$ is sS_n-invariant.

Fix $n \in \mathbb{N} \cup \{\infty\}$. We show that all sS_n-invariant, normal subgroups $N \leq \mathbb{Z}_2^{*n}$ arise as $N = F_n(\mathcal{C})$ for some category of partitions $\mathcal{C}$. Let $N \leq \mathbb{Z}_2^{*n}$ be an sS_n-invariant, normal subgroup and define

$$\mathcal{C} = \{p \in \mathcal{P} \mid p \text{ is a rotation of } \ker(i) \text{ for some } k \in \mathbb{N}, a_{i_1} \cdots a_{i_k} \in N\}.$$

If $p = \ker(i)$ for some $a_{i_1} \cdots a_{i_k} \in N$ and $p' \in \mathcal{C}(k)$ is a rotation of p, then also $p' = \ker(i')$ for some $a_{i'_1} \cdots a_{i'_k} \in N$. Indeed, it suffices to check this for a rotation of the uttermost left leg of p to the uttermost right of p. If p' arises from p in this way, then $p' = \ker(i_2, \dots, i_k, i_1)$ and $a_{i_2} \cdots a_{i_k} a_{i_1} = \mathrm{Ad}(a_{i_1})(a_{i_1} \cdots a_{i_k}) \in N$ by normality of $N \leq \mathbb{Z}_2^{*n}$. In particular

$$\mathcal{C}(k) = \{p \in \mathcal{P} \mid p = \ker(i) \text{ for some } a_{i_1} \cdots a_{i_k} \in N\}. \tag{4.1}$$

We show that $\mathcal{C}$ is a category of partitions.

- We have $a_1^2 = e \in N$, so $\sqcup = \ker((1,1)) \in \mathcal{C}$.
- It is clear that $\mathcal{C}$ is closed under rotation.
- We show that $\mathcal{C}$ is closed under involution. The involution of a rotation of $p \in \mathcal{C}$ is equal to a rotation of the involution of p. So we have to check that $\ker((i_k, \dots, i_1)) \in \mathcal{C}$ for all $a_{i_1} \cdots a_{i_k} \in N$. This follows from $a_{i_k} \cdots a_{i_1} = (a_{i_1} \cdots a_{i_k})^{-1}$.
- The tensor product of partitions $p, p' \in \mathcal{C}$ (not necessarily on one row) is a rotation of the tensor product of rotations of p and p' onto one row. So we have to check that for $a_{i_1} \cdots a_{i_k} \in N$ and $a_{i'_1} \cdots a_{i'_{k'}} \in N$ we have $\ker(i) \otimes \ker(i') \in \mathcal{C}$. Since N is S_n-invariant, we may apply a permutation of letters to $a_{i'_1} \cdots a_{i'_{k'}}$ in order to assume that $\{i_1, \dots, i_k\} \cap \{i'_1, \dots, i'_{k'}\} = \emptyset$. But then $a_{i_1} \cdots a_{i_k} a_{i'_1} \cdots a_{i'_{k'}} \in N$ implies $\ker(i) \otimes \ker(i') = \ker((i_1, \dots, i_k, i'_1, \dots, i'_{k'})) \in \mathcal{C}$.
- It remains to show that if $p \in \mathcal{C}(k,l)$ and $q \in \mathcal{C}(l,m)$ then also $qp \in \mathcal{C}(k,m)$. We may rotate all lower legs of p to its right and likewise for the lower legs of q, so as to obtain partition $p' \in \mathcal{C}(k+l)$ and $q' \in \mathcal{C}(l+m)$. Then qp is

a rotation of the l-fold iterated composition of $p' \otimes q'$ with partitions of the form $| \otimes \cdots \otimes | \otimes \sqcap \otimes | \otimes \cdots \otimes |$. So we have to show that the composition of any $p \in \mathcal{C}(k)$ with a partition of the form $| \otimes \cdots \otimes | \otimes \sqcap \otimes | \otimes \cdots \otimes |$ lies in $\mathcal{C}(k-2)$. By (4.1), there is $a_{i_1} \cdots a_{i_k} \in N$ such that $p = \ker(i)$. Denote by $\sqcap_{k,l}$ the partition in $\mathcal{C}(0,k)$ of the from $| \otimes \cdots \otimes | \otimes \sqcap \otimes | \otimes \cdots \otimes |$ such that the first leg of $\sqcap$ is on the l-th position of $\sqcap_{k,l}$. Denote by $\phi : \underline{n} \to \underline{n}$ the map that satisfies $\phi(i_l) = i_{l+1}$ and that fixes all other elements of $\underline{n}$. By $\mathrm{sS_n}$-invariance of N, we have

$$a_{\phi(i_1)} \cdots a_{\phi(i_{l-1})} a_{\phi(i_{l+2})} \cdots a_{\phi(i_k)} = \phi_*(a_{i_1} \cdots a_{i_k}) \in N\,.$$

So the composition $p \circ \sqcap_{k,l} = \ker((\phi(i_1), \dots, \phi(i_{l-1}), \phi(i_{i+2}), \dots, \phi(i_k)))$ lies in $\mathcal{C}(k-2)$.

We have shown that $\mathcal{C}$ is a category of partitions. Since N is S_n-invariant, (4.1) shows that $F_n(\mathcal{C}) = N$, which finishes the proof of the theorem. □

4.2. Classification of group-theoretical easy quantum groups. We can now combine the previous theorem with the classification of quantum groups in Theorem 3.1 and the classification of diagonal subgroups of easy quantum groups given in Lemma 4.2. In view of Proposition 3.2, it is natural to consider easy quantum groups, whose categories of partitions contain $\sqcup\!\!/\!\!\sqcap$. This yields a complete description of group-theoretical easy quantum groups.

Theorem 4.5. *Let (A,u) be an easy quantum group associated with the category of partitions $\mathcal{C}$. Assume that $\sqcup\!\!/\!\!\sqcap \in \mathcal{C}$. Then $A \cong C^*(\Gamma) \bowtie \mathrm{C}(\mathrm{S}_n)$ as compact matrix quantum groups for the strongly symmetric reflection group $\Gamma = \mathbb{Z}_2^{*n}/F_n(\mathcal{C})$.*

Moreover, for every strongly symmetric reflection group Γ there is an easy quantum group which is isomorphic with $C^*(\Gamma) \bowtie \mathrm{C}(\mathrm{S}_n)$ and whose category of partitions contains $\sqcup\!\!/\!\!\sqcap$.

Proof. Since A is an easy quantum group, it is a homogeneous quantum group in its maximal version. So we can apply Theorem 3.1 showing that $A \cong \mathrm{C}^*(\Gamma) \bowtie \mathrm{C}(G)$, where Γ is the diagonal subgroup of (A,u), the group G is some subgroup of S_n and the entry u_{ij} of the fundamental corepresentation of A is identified with $u_{g_i} p_{ij}$. Here $g_1, \dots, g_n$ denote the natural generators of Γ and (p_{ij}) is the fundamental corepresentation of $\mathrm{C}(G)$ given by the natural embedding $G \hookrightarrow S_n$. We have to show that $G = \mathrm{S}_n$, that $\Gamma \cong (\mathbb{Z}_2^{*n})/F_n(\mathcal{C})$ and that the latter is a strongly symmetric reflection group. First note that there is a homomorphism of CMQGs $\pi : A \to \mathrm{C}(\mathrm{S}_n)$ given by $\pi(u_{ij}) = \pi(u_{ij}^2) = p_{ij}$. It follows that $G = \mathrm{S}_n$. Next, Theorem 4.4 shows that $\Gamma = (\mathbb{Z}_2^{*n})/F_n(\mathcal{C})$ is strongly symmetric.

Theorem 4.4 also shows that for all strongly symmetric reflection groups on n generators there is a category of partitions $\mathcal{C}$ such that $\Gamma \cong \mathbb{Z}_2^{*n}/F_n(\mathcal{C})$. We can invoke Propositions 2.5 and 3.2 so as to assume $\sqcup\!\!/\!\!\sqcap \in \mathcal{C}$. But then the first part of the proof shows that $A_{\mathcal{C}}(n) \cong \mathrm{C}^*(\Gamma) \bowtie \mathrm{C}(\mathrm{S}_n)$. This finishes the proof. □

The next theorem deduces a complete classification of group-theoretical categories of partitions.

Theorem 4.6. *There is a one-to-one correspondence between*

- *categories of partitions which contain* $\sqcup\!\!/\!\!\sqcap$ *and*
- *strongly symmetric reflection groups on countably many generators*

It is given by associating $\mathbb{Z}_2^{*\infty}/F_\infty(\mathcal{C})$ *with a category of partitions* $\mathcal{C}$.

Proof. By Theorem 4.4 every sS_∞-invariant, normal subgroup $N \leq \mathbb{Z}_2^{*\infty}$ arises as $F_\infty(\mathcal{C})$ for some category of partitions $\mathcal{C}$. Since $F_\infty(\mathcal{C}) = F_\infty(\langle \mathcal{C}, \sqcup\!\!/\!\!\sqcap \rangle)$ by Proposition 3.2, we can assume that $\mathcal{C}$ contains $\sqcup\!\!/\!\!\sqcap$. This shows that $\mathcal{C} \mapsto \mathbb{Z}_2^{*\infty}/F_\infty(\mathcal{C})$ maps surjectively from all categories of partitions which contain $\sqcup\!\!/\!\!\sqcap$ onto strongly symmetric reflection groups on countably many generators. It remains to show that this map is injective.

Let $\mathcal{C}$ be a category of partitions which contains $\sqcup\!\!/\!\!\sqcap$. We show that $\mathcal{C}$ can be recovered from $\mathbb{Z}_2^{*\infty}/F_\infty(\mathcal{C})$. Note that $\mathbb{Z}_2^{*n}/F_n(\mathcal{C})$ is the subgroup of $\mathbb{Z}_2^{*\infty}/F_\infty(\mathcal{C})$ generated by the first n generators. Moreover, $\mathrm{C}(G_{\mathcal{C}}(n)) \cong \mathrm{C}^*(\mathbb{Z}_2^{*n}/F_n(\mathcal{C})) \rtimes \mathrm{C}(S_n)$ for all $n \in \mathbb{N}$ by Theorem 4.5. So Theorem 2.4 implies that $\mathcal{C}$ can be recovered from $F_\infty(\mathcal{C})$. □

5. Applications

5.1. Uncountably many pairwise non-isomorphic easy quantum groups. In this section we show that there are uncountably many easy quantum groups. We inject the lattice of varieties of groups into the lattice of hyperoctahedral categories of partitions. The generators of $\mathbb{F}_\infty$ are denoted by $x_1, x_2, \ldots$.

Definition 5.1. Let $w = w(x_1, \ldots, x_n) \in \mathbb{F}_\infty$ be a word in the letters $x_1, \ldots, x_n$ and let Γ be a group. Then the *identical relation* w *holds in* Γ if for all $g_1, \ldots, g_n \in \Gamma$, we have $w(g_1, \ldots, g_n) = e$.

Let $W \subset \mathbb{F}_\infty$ be any subset of the free group on countably many generators $x_1, x_2, \ldots$. The *variety of groups* $\mathcal{V}(W)$ associated with W is the class of all groups Γ such that for all $w \in W$ the identical relation w holds in Γ.

Let us state a classical result in the theory of varieties of groups. A subgroup $N \leq \mathbb{F}_\infty$ is called *fully characteristic* if $\phi(N) \subset N$ for all $\phi \in \mathrm{End}(\mathbb{F}_\infty)$.

Theorem 5.2 (See [30] or Theorem 14.31 in [31]). *There is a lattice anti-isomorphism between varieties of groups and fully characteristic subgroups of* $\mathbb{F}_\infty$ *sending a variety of groups to the set of all identical relations that hold in it.*

Denote by $E \leq \mathbb{Z}_2^{*\infty}$ the group consisting of all words of even length. We identify E with a free group with basis $x_1 = a_1a_2, x_2 = a_1a_3, \ldots$. The following observation is key to this section.

Proposition 5.3. *The subgroup* $E \leq \mathbb{Z}_2^{*\infty}$ *is fully characteristic. In particular, every fully characteristic subgroup of* $\mathbb{F}_\infty \cong E$ *is fully characteristic in* $\mathbb{Z}_2^{*\infty}$.

Proof. If $w = a_{i_1}a_{i_2}\cdots a_{i_{2k}} \in E$ and $\phi \in \mathrm{End}(\mathbb{Z}_2^{*\infty})$, then $\phi(w) = \phi(a_{i_1})\cdots \phi(a_{i_{2k}}) \in E$. So E is fully characteristic in $\mathbb{Z}_2^{*\infty}$. So every endomorphism of $\mathbb{Z}_2^{*\infty}$ restricts to E. This shows that a fully characteristic subgroup $N \leq E \cong \mathbb{F}_\infty$ is also a fully characteristic subgroup in $\mathbb{Z}_2^{*\infty}$. This finishes the proof. □

Remark 5.4. We do not know whether any fully characteristic subgroup $\mathbb{Z}_2^{*\infty}$ that is contained in E and is fully characteristic, is also fully characteristic in $\mathbb{F}_\infty$. If this was the case, then hyperoctahedral easy quantum groups A whose entries are subject to the condition that u_{ij}^2 is central in A, would be classified by non-trivial varieties of groups.

Combining the previous proposition with the classification of group-theoretical hyperoctahedral categories of partitions, we obtain the following result.

Theorem 5.5. *There is a lattice injection from the lattice of non-trivial varieties of groups into the lattice of easy quantum groups.*

Proof. By Theorem 2.4 there is a lattice anti-isomorphism between categories of partitions and easy quantum groups. Moreover by Theorem 4.6, group-theoretical hyperoctahedral categories of partitions are in one-one correspondence with sS_∞-invariant, normal subgroups of $\mathbb{Z}_2^{*\infty}$ and this correspondence preserves the lattice structure given by inclusion.

By Theorem 5.2, there is a lattice anti-isomorphism between varieties of groups and fully characteristic subgroups of $\mathbb{F}_\infty$.

So it suffices to find an embedding of lattices from fully characteristic subgroups of $\mathbb{F}_\infty$ into sS_∞-invariant, normal subgroups of $\mathbb{Z}_2^{*\infty}$. By Proposition 5.3, the embedding $\mathbb{F}_\infty \cong E \leq \mathbb{Z}_2^{*\infty}$ has this property. This finishes the proof. □

The correspondence from the last theorem allows us to translate classification results for varieties of groups into results on easy quantum groups. In [5], the question was raised whether or not all easy quantum groups are either classical, free, half-liberated or form part of a multi-parameter family unifying the series of quantum groups $H_n^{(s)}$ and $H_n^{[s]}$. We can answer this question in the negative.

Theorem 5.6. *There are uncountably many pairwise non-isomorphic easy quantum groups.*

This follows directly from Theorem 5.5 and the following result by Olshanskii.

Theorem 5.7 (See [32]). *The class of varieties of groups has cardinality equal to the continuum.*

5.2. Examples of non-easy quantum groups. It is was an open problem whether easy quantum groups describe all intermediate quantum groups of free quantum groups and their classical counterparts. While this remains open for $O_n^+ \supset O_n$ and $\mathrm{S}_n^+ \supset \mathrm{S}_n$, we answer this question in the negative for hyperoctahedral quantum groups.

Theorem 5.8. *For every $n \geq 3$, there is an example of a homogeneous hyperoctahedral quantum group $H_n^+ \supset G$ that is not easy.*

Proof. Let $n \geq 3$. We start by exhibiting an example of a non-easy quantum group $H_n^{[\infty]} \supset G \supset \mathrm{S}_n$. By Theorem 3.1 it must satisfy $\mathrm{C}(G) \cong \mathrm{C}^*(\Gamma) \bowtie \mathrm{C}(\mathrm{S}_n)$ for some S_n-invariant quotient $\mathbb{Z}_2^{*n} \twoheadrightarrow \Gamma$. So by Theorem 4.5 it suffices to exhibit an S_n-invariant, normal subgroup of $\mathbb{Z}_2^{*n}$ that is not sS_n-invariant.

Let $\pi : \mathbb{Z}_2^{*n} \to \mathrm{S}_{n+1}$ be the homomorphism satisfying $\pi(a_i) = (1i)$. It is surjective, since $(ij) = (1i)(1j)(1i)$. If $\phi : \{1, \ldots, n\} \to \{1, \ldots, n\}$ is any map, then $\pi(a_{\phi(i)}) = (1\phi(i))$. So if $w \in \mathbb{Z}_2^{*n}$ maps to a product of cycles $\pi(w) = (i_1 \ldots i_{k_1}) \cdots (i_{k_{l-1}} \cdots i_{k_l})$, then $\pi(\sigma_\phi(w)) = (\phi(i_1)\phi(i_2) \ldots \phi(i_{k_1})) \cdots (\phi(i_{k_{l-1}}) \cdots \phi(i_{k_l}))$. In particular, $\ker \pi$ is S_n-invariant. We show that it is not sS_n-invariant. Indeed, take for $\phi : \{1, \ldots, n\} \to \{1, \ldots, n\}$ the map defined by $\phi(4) = 1$, $\phi(i) = i$ for all $i \neq 4$. We have $(12)(34)(12)(34) = \mathrm{id}$, but applying ϕ, we obtain

$$(12)(31)(12)(31) = (132) \neq \mathrm{id}\,.$$

So $\ker \pi$ is not sS_n-invariant.

Write now $\Gamma = \mathbb{Z}_2^{*n} / \ker(\pi)$. We check that $\mathrm{C}^*(\Gamma) \bowtie \mathrm{C}(\mathrm{S}_n)$ is a hyperoctahedral quantum group. The fundamental corepresentation of $\mathrm{C}^*(\Gamma) \bowtie \mathrm{C}(\mathrm{S}_n)$ is $(u_{ij}) = (u_{g_i} p_{ij})$ as described in Section 2.5. It is clear that its entries $u_{g_i} p_{ij}$ are self-adjoint partial isometries. Moreover $\sum_j u_{g_i} p_{ij} = u_{g_i}$ for all i. Since $\Gamma \neq \mathbb{Z}_2$, we have $g_i \neq g_{i'}$ for some i,i'. So it follows from Proposition 2.7 that $\mathrm{C}^*(\Gamma) \bowtie \mathrm{C}(\mathrm{S}_n)$ is a hyperoctahedral quantum group. This completes the proof of the theorem. ☐

5.3. The hyperoctahedral series. In [5], the hyperoctahedral series and the higher hyperoctahedral series were defined. We describe these quantum groups in the context of our classification results.

Definition 5.9. The *higher hyperoctahedral series* is the sequence of compact matrix quantum groups defined by

$$\begin{aligned}\mathrm{C}(H_n^{[s]}) = \mathrm{C}^*(u_{ij}, 1 \leq i, j \leq n \,|\, u = \overline{u} \text{ unitary, } u_{ij} \text{ partial isometries and} \\ (u_{ij} u_{kl})^s = (u_{kl} u_{ij})^s \text{ for all } i, j, k, l \in \{1, \ldots, n\})\,.\end{aligned}$$

Similarly, the *hyperoctahedral series* is defined as

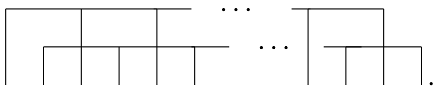
$$\begin{aligned} \mathrm{C}(H_n^{(s)}) = \mathrm{C}^*(u_{ij}, 1 \le i, j \le n \mid u = \overline{u} \text{ unitary, } u_{ij} \text{ partial isometries,}\\ (u_{ij}u_{kl})^s = (u_{kl}u_{ij})^s \text{ for all } i, j, k, l \in \{1, \dots, n\}\\ \text{and } abc = cba \text{ for all } a, b, c \in \{u_{ij} \mid i, j \in \{1, \dots, n\}\}) \,. \end{aligned}$$

All members of the hyperoctahedral series and the higher hyperoctahedral series are easy quantum groups such that u_{ij}^2 are central projections in their underlying C^*-algebra. Hence these quantum groups are in the scope of this article.

Proposition 5.10. *We have the following isomorphisms of CMQGs.*

- $\mathrm{C}(H_n^{[s]}) \cong C^*((\mathbb{Z}_2^{*n})/(a_i a_j)^s = e) \bowtie \mathrm{C}(\mathrm{S}_n)$.
- $\mathrm{C}(H_n^{(s)}) \cong C^*((\mathbb{Z}_2^{*n})/(a_i a_j)^s = e \text{ and } a_i a_j a_k = a_k a_j a_i) \bowtie \mathrm{C}(\mathrm{S}_n)$.

Proof. By [5], the higher hyperoctahedral series is associated with the category of partitions generated by ⊔⁄⊓ and

$h_s =$


The hyperoctahedral series is associated with the categories $\langle$⊓⊓, h_s, ∗$\rangle$, which also contain ⊔⁄⊓. So we can apply Lemma 4.2 and Theorem 4.5 in order to finish the proof. □

The next theorem describes some operator algebraic properties of quantum groups in the hyperoctahedral series. We refer the reader to Section 2.7 for the relevant notations.

Theorem 5.11. *The quantum groups* $\mathrm{C}(H_n^{(s)})$ *in the hyperoctahedral series are index two extensions of* $C^*(\mathbb{Z}_s^{\oplus n-1} \rtimes \mathrm{S}_n)$*, where a permutation* $\sigma \in \mathrm{S}_n$ *acts by*

$$\sigma(b_i) = \begin{cases} b_{\sigma(i)} & \sigma(n) = n \\ b_{\sigma(n)} & \sigma(i) = n \\ b_{\sigma(n)} b_{\sigma(i)} & \sigma(n) \ne n \ne \sigma(i) \end{cases}$$

on the generators $b_1, \dots, b_{n-1}$ *of* $\mathbb{Z}_s^{\oplus n-1}$.

In particular, $\widehat{H_n^{(s)}}$ *is a finite quantum group, or equivalently* $\mathrm{L}^\infty(H_n^{(s)})$ *is a finite dimensional von Neumann algebra.*

For all $s \geq 3$, $n \geq 3$ *the discrete dual quantum groups* $\widehat{H_n^{[s]}}$ *of the higher hyperoctahedral series are not amenable, but weakly amenable and they have the Haagerup property. Their von Neumann algebras* $\mathrm{L}^\infty(H_n^{[s]})$ *are strongly solid.*

Proof. Let us first consider $H_n^{(s)}$. By Proposition 5.10, the CMQG $\mathrm{C}(H_n^{(s)})$ is isomorphic with $\mathrm{C}^*((\mathbb{Z}_2^{*n})/\{(a_ia_j)^s = e \text{ and } a_ia_ja_k = a_ka_ja_i\}) \bowtie \mathrm{C}(\mathrm{S}_n)$. Consider the index two subgroup E of $\mathbb{Z}_2^{*n}/\{(a_ia_j)^s = e \text{ and } a_ia_ja_k = a_ka_ja_i\}$ consisting of words of even length. It is generated by the elements $b_i = a_na_i$, $i \in \{1, \dots, n-1\}$. We can use the relation $a_ia_ja_k = a_ka_ja_i$ in order to obtain $b_ib_j = a_na_ia_na_j = a_na_ja_na_i = b_jb_i$. Similarly, $b_ib_j = b_jb_i$ implies $a_ia_ja_ka_l = b_i^{-1}b_jb_k^{-1}b_l = b_k^{-1}b_jb_i^{-1}b_l = a_ka_ja_ia_l$ for all $i, j, k, l \in \{1, \dots, n\}$. So E is the universal group generated by $n-1$ commuting elements of order s. Hence $E \cong \mathbb{Z}_s^{\oplus n-1}$ by an isomorphism identifying b_i, $i \in \{1, \dots, n-1\}$ with the natural generators of $\mathbb{Z}_s^{\oplus n-1}$. Take $\sigma \in \mathrm{S}_n$. Then $\sigma(b_i) = \sigma(a_na_i) = a_{\sigma(n)}a_{\sigma(i)} = a_na_{\sigma(n)}a_na_{\sigma(i)}$. So σ acts on E via

$$\sigma(b_i) = \begin{cases} b_{\sigma(i)} & \sigma(n) = n \\ b_{\sigma(n)} & \sigma(i) = n \\ b_{\sigma(n)}b_{\sigma(i)} & \sigma(n) \neq n \neq \sigma(i)\,. \end{cases}$$

In particular, S_n leaves E invariant, so that $\mathrm{C}^*(E) \bowtie \mathrm{C}(\mathrm{S}_n) \subset \mathrm{C}(H_n^{(s)})$ is a compact quantum group. Note that since the Pontryagin dual of $\mathbb{Z}_s$ is isomorphic with $\mathbb{Z}_s$, we have $\mathrm{C}^*(E) \bowtie \mathrm{C}(\mathrm{S}_n) \cong \mathrm{C}(\mathbb{Z}_s^{\oplus n-1} \rtimes \mathrm{S}_n)$. Since $E \leq (\mathbb{Z}_2^{*n})/\{(a_ia_j)^s = e \text{ and } a_ia_ja_k = a_ka_ja_i\}$ has index two, $\mathrm{L}^\infty(H_n^{(s)})$ is two-dimensional as an $\mathrm{L}^\infty(\mathbb{Z}_s^{\oplus n-1} \rtimes S_n)$-module. So the inclusion $\mathrm{L}^\infty(\mathbb{Z}_s^{\oplus n-1} \rtimes \mathrm{S}_n) \subset \mathrm{L}^\infty(H_n^{(s)})$ has index two. In particular, $\mathrm{L}^\infty(H_n^{(s)})$ is a finite dimensional von Neumann algebra.

Let us now consider $\mathrm{C}(H_n^{[s]})$. By Proposition 5.10 it is isomorphic with $\mathrm{C}^*(\mathbb{Z}_2^{*n}/\{(a_ia_j)^s = e\}) \bowtie \mathrm{C}(\mathrm{S}_n)$. Denote by $E \leq \mathbb{Z}_2^{*n}/\{(a_ia_j)^s = e\}$ the subgroup of words of even length. It is generated by elements $b_i = a_na_i$, which only satisfy the relations $b_i^s = e$. Hence $E \cong \mathbb{Z}_s^{*n-1}$. Since $u_{a_i} = \sum_j u_{ij} \in \mathrm{Pol}(\mathrm{C}(H_n^{[s]}) \subset \mathrm{L}^\infty(H_n^{[s]})$, we have $\mathrm{L}(E) \subset \mathrm{L}^\infty(H_n^{[s]})$. Note that E is not amenable, because $s, n \geq 3$. It follows that $\mathrm{L}^\infty(H_n^{[s]})$ is not amenable as a von Neumann algebra. By Theorem 2.12 this implies non-amenability of $\widehat{H_n^{(s)}}$. Moreover, $\mathrm{L}(E) \subset \mathrm{L}^\infty(H_n^{[s]})$ has finite index. Since $E \cong \mathbb{Z}_s^{*n-1}$ has the CMAP by Theorem 2.17 and has the Haagerup property by Theorem 2.13, it follows that also $\mathrm{L}^\infty(H_n^{[s]})$ has the W*-CCAP and the Haagerup property. Finally note that E is a free product of hyperbolic groups, so it is hyperbolic itself. Furthermore every non-trivial conjugacy class in $E \cong \mathbb{Z}_s^{*n-1}$ is infinite, because $s, n \geq 3$. By [13], it follows that $\mathrm{L}(E)$ is a strongly solid von Neumann algebra. Since $\mathrm{L}^\infty(H_n^{[s]})$ contains $\mathrm{L}(E)$ as a finite index von Neumann subalgebra, [22, Proposition 5.2] implies that also $\mathrm{L}^\infty(H_n^{[s]})$ is strongly solid. This finishes the proof. ☐

5.4. De Finetti theorems for easy quantum subgroups of $\mathbf{C^*(\mathbb{Z}_2^{*n}) \bowtie C(S_n)}$.

After the work of Köstler and Speicher in [27], de Finetti theorems became a central piece of the theory of easy quantum groups. In this section we present a unified de Finetti theorem for all easy quantum subgroups of $\mathrm{C}^*(\mathbb{Z}_2^{*n}) \bowtie \mathrm{C}(\mathrm{S}_n)$. Unsurprisingly, it needs strong assumptions to yield the desired equivalence between invariance of the distribution of non-commutative random variables $x_1, x_2, \dots$ under the natural action of a series of easy quantum groups and independence properties of this distribution. However, the de Finetti theorem for $H_n^* = H_n^{(\infty)}$ as it is known from [6] takes its usual form, only demanding a commutation relation between the random variables $x_1, x_2, \dots$ in question. Similarly, one can formulate a simple de Finetti theorem for $H_n^{[\infty]}$. We describe these two especially interesting cases in Corollary 5.14 justifying the assumptions of Theorem 5.12.

In the next theorem we use the following notation. If $w \in \mathbb{N}^{*n}$ is any word in n letters, then we denote by $e_i(w)$ the exponent of the i-th letter in w.

Theorem 5.12. *Let Γ be a non-trivial strongly symmetric reflection group on countably many generators and $K = \ker(\mathbb{N}^{*\infty} \to \Gamma) \leq \mathbb{N}^{*\infty}$ the sS_∞-invariant subsemigroup of $\mathbb{N}^{*\infty}$ associated with it. For $n \in \mathbb{N}$ denote by Γ_n the strongly symmetric reflection groups generated by the first n generators of Γ. Let $x_1, x_2, \dots$ be a sequence of non-commutative, self-adjoint random variables in a W*-probability space (M, ϕ). Then there is a von Neumann subalgebra $B \subset M$ with ϕ-preserving expectation $\mathrm{E} : M \to B$ such that if $x_1, x_2, \dots$ satisfy*

- *$x_i^2 x_j = x_j x_i^2$ for all $i, j \in \mathbb{N}$ and*
- *$\phi(w(x_1, \dots, x_n)) = \phi(\mathrm{E}(x_1^{e_1(w)}) \cdots \mathrm{E}(x_n^{e_n(w)}))$ for all $n \in \mathbb{N}$ and all words $w \in K$ on n letters,*

then the following are equivalent.

(i) $x_1^2, x_2^2, \dots,$ are identically distributed and independent with respect to E and $\phi(w(x_1, \dots, x_n)) = 0$ for all $n \in \mathbb{N}$ and all $w \in \mathbb{N}^{*\infty} \setminus K$ in n letters.

(ii) For all n $\in \mathbb{N}$, the distribution of $x_1, \dots, x_n$ is invariant under the coaction of the easy quantum group $C^*(\Gamma_n) \bowtie \mathrm{C}(\mathrm{S}_n)$.

Proof. We may assume that $M = \mathrm{W}^*(x_1, x_2, \dots)$. Let $B = \bigcap_{n \geq 1} \mathrm{W}^*(x_n^2, x_{n+1}^2, \dots)$ and note that B lies in the centre of M. In particular, there is a ϕ-preserving normal conditional expectation $\mathrm{E} : M \to B$.

Consider the coaction α_n of $\mathrm{C}^*(\Gamma_n) \bowtie \mathrm{C}(\mathrm{S}_n)$ on $\mathbb{C}\langle X_1, X_2, \dots \rangle$ given by

$$\alpha_n(X_i) = \sum_j X_j \otimes u_{ji} = \sum_j X_j \otimes u_{g_j} p_{ji}$$

for $i \leq n$ and $\alpha(X_i) = X_i \otimes 1$ for $i > n$. Denote by ψ the state on $\mathbb{C}\langle X_1, X_2, \dots \rangle$ given by $\psi(w(X_1, \dots, X_n)) = \phi(w(x_1, \dots, x_n))$ for all $w \in \mathbb{N}^{*n}$ and all $n \in \mathbb{N}$. Then the distribution of $x_1, x_2, \dots$ is invariant under the coaction of $\mathrm{C}^*(\Gamma_n) \bowtie \mathrm{C}(\mathrm{S}_n)$ if and only if for all indices $i_1, \dots, i_l \in \{1, \dots, n\}$ we have

$$\psi(X_{i_1} \cdots X_{i_l}) = \sum_{j_1,\dots,j_l \in \{1,\dots,n\}} \psi(X_{j_1} \cdots X_{j_l}) u_{g_{j_1} \cdots g_{j_l}} p_{j_1 i_1} \cdots p_{j_l i_l}\,. \tag{5.1}$$

We are going to manipulate the right hand side of this equation. First note that we have $p_{j_1 i_1} \cdots p_{j_l i_l} \neq 0$ if and only if there is $\sigma \in \mathrm{S}_n$ such that $j_k = \sigma(i_k)$ for all $k \in \{1, \dots, l\}$. Moreover, if $\sigma|_{\{i_1,\dots,i_l\}} = \rho|_{\{i_1,\dots,i_l\}}$ for $\sigma, \rho \in \mathrm{S}_n$, then

$$\begin{aligned} &\psi(X_{\sigma(i_1)} \cdots X_{\sigma(i_l)}) u_{g_{\sigma(i_1)} \cdots g_{\sigma(i_l)}} p_{\sigma(i_1) i_1} \cdots p_{\sigma(i_l) i_l} \\ &\qquad = \psi(X_{\rho(i_1)} \cdots X_{\rho(i_l)}) u_{g_{\rho(i_1)} \cdots g_{\rho(i_l)}} p_{\rho(i_1) i_1} \cdots p_{\rho(i_l) i_l}\,. \end{aligned}$$

Let L be the cardinality of $\{i_1, \dots, i_l\}$. Then

$$\begin{aligned} &\sum_{j_1,\dots,j_l \in \{1,\dots,n\}} \psi(X_{j_1} \cdots X_{j_l}) u_{g_{j_1} \cdots g_{j_l}} p_{j_1 i_1} \cdots p_{j_l i_l} \\ &\quad = \frac{1}{(n-L)!} \sum_{\sigma \in \mathrm{S}_n} \psi(X_{\sigma(i_1)} \cdots X_{\sigma(i_l)}) u_{g_{\sigma(i_1)} \cdots g_{\sigma(i_l)}} p_{\sigma(i_1) i_1} \cdots p_{\sigma(i_l) i_l}\,. \end{aligned} \tag{5.2}$$

Let $i_{l+1}, \dots, i_{l+l'}$ be an enumeration of $\{1, \dots, n\} \setminus \{i_1, \dots, i_l\}$. Using $p_{j'i} p_{ji} = 0$ for all $i, j, j' \in \{1, \dots, n\}$, $j \neq j'$, we see that

$$\begin{aligned} &\sum_{\sigma \in \mathrm{S}_n} \psi(X_{\sigma(i_1)} \cdots X_{\sigma(i_l)}) u_{g_{\sigma(i_1)} \cdots g_{\sigma(i_l)}} p_{\sigma(i_1) i_1} \cdots p_{\sigma(i_l) i_l} \\ &\quad = \Bigg(\sum_{\sigma \in \mathrm{S}_n} \psi(X_{\sigma(i_1)} \cdots X_{\sigma(i_l)}) u_{g_{\sigma(i_1)} \cdots g_{\sigma(i_l)}} p_{\sigma(i_1) i_1} \cdots p_{\sigma(i_l) i_l} \Bigg) \\ &\qquad\qquad \cdot \Bigg(\sum_{j_{l+1},\dots,j_{l+l'} \in \{1,\dots,n\}} p_{j_{l+1} i_{l+1}} \cdots p_{j_{l+l'} i_{l+l'}} \Bigg) \qquad (5.3) \\ &\quad = (n-L)! \sum_{\sigma \in \mathrm{S}_n} \psi(X_{\sigma(i_1)} \cdots X_{\sigma(i_l)}) u_{g_{\sigma(i_1)} \cdots g_{\sigma(i_l)}} p_{\sigma(i_1) i_1} \cdots \\ &\qquad\qquad \cdots p_{\sigma(i_l) i_l} p_{\sigma(i_{l+1}) i_{l+1}} \cdots p_{\sigma(i_{l+l'}) i_{l+l'}}\,. \end{aligned}$$

We can now use $p_{ji}^2 = p_{ji}$ for all $i, j \in \{1, \dots, n\}$ with the equations (5.2) and (5.3) in order to reformulate the right hand side of (5.1) and obtain

$$\sum_{j_1,\dots,j_l \in \{1,\dots,n\}} \psi(X_{j_1} \cdots X_{j_l}) u_{g_{j_1} \cdots g_{j_l}} p_{j_1 i_1} \cdots p_{j_l i_l} \\ = \sum_{\sigma \in \mathrm{S}_n} \psi(X_{\sigma(i_1)} \cdots X_{\sigma(i_l)}) u_{g_{\sigma(i_1)} \cdots g_{\sigma(i_l)}} p_{\sigma(1)1} \cdots p_{\sigma(n)n} \, .$$

Using this equation, we see that the distribution of $x_1, x_2, \dots$ is invariant under the action of $\mathrm{C}^*(\Gamma_n) \bowtie \mathrm{C}(\mathrm{S}_n)$ if and only if

$$\psi(w(X_1, \dots, X_n)) \\ = \sum_{\sigma \in \mathrm{S}_n} \psi(w(X_{\sigma(1)}, \dots, X_{\sigma(n)})) u_{w(g_{\sigma(1)},\dots,g_{\sigma(n)}} p_{\sigma(1)1} \cdots p_{\sigma(n)n} \, , \tag{5.4}$$

for all words $w \in \mathbb{N}^{*n}$.

Assume (i) and take $n \in \mathbb{N}$. If $w \in \mathbb{N}^{*n} \setminus K$ is a word on the first n letters $X_1, \dots, X_n$, then $\psi(w(X_1, \dots, X_n)) = 0$. Furthermore, for every $\sigma \in \mathrm{S}_n$, there is $w' \in \mathbb{N}^{*n}$ such that $w(X_{\sigma(1)}, \dots, X_{\sigma(n)}) = w'(X_1, \dots, X_n)$. Since w' arises from w by permuting its letters according to σ, we see that $w' \in \mathbb{N}^{*n} \setminus K$. Hence we have $\psi(w(X_{\sigma(1)}, \dots, X_{\sigma(n)})) = 0$. Filling in this information into equation (5.4), we see that its left- and right-hand side are equal to zero for $w \in \mathbb{N}^{*n} \setminus K$.

There is a natural Γ_n-grading of $\mathbb{C}\langle X_1, X_2, \dots \rangle$ which grades $w(X_1, \dots, X_n)$ by $w(g_1, \dots, g_n)$ for all $w \in \mathbb{N}^{*n}$ and which grades X_k by e for $k \geq n + 1$. We have already proven that $\psi(w(X_1, \dots, X_n)) = 0$ for all $w \in \mathbb{N}^{*n} \setminus K$. This is equivalent to the fact that ψ respects the Γ_n-grading of $\mathbb{C}\langle X_1, X_2, \dots \rangle$. In particular, purely graded subspaces of $\mathbb{C}\langle X_1, X_2, \dots, \rangle$ are pairwise orthogonal with respect to ϕ. So the GNS-construction gives rise to a well defined Γ_n-grading of M. Every element in B is purely e-graded, because B is a subalgebra of $\mathrm{W}^*(x_1^2, x_2^2, \dots,)$. It follows that $\mathrm{E}(w(x_1, \dots, x_n)) = 0$ for all $w \in \mathbb{N}^{*\infty} \setminus K$. In particular, the fact that Γ is not trivial implies that all x_i have even distribution with respect to E. Combining this with the fact that $x_1^2, x_2^2, \dots,$ have an identical distribution with respect to E, it follows that $x_1, x_2, \dots$ are identically distributed with respect to E.

Now take $w \in \mathbb{N}^{*n} \cap K$. As before, we see that for $\sigma \in \mathrm{S}_n$, the word $w' \in \mathbb{N}^{*n}$ arising from a permutation of letters of w according σ satisfies $w(X_{\sigma(1)}, \dots, X_{\sigma(n)}) = w'(X_1, \dots, X_n)$. Since K is invariant under permutation of letters, we obtain $w' \in K$ and hence $w(g_{\sigma(1)}, \dots, g_{\sigma(n)}) = e$. This shows

$$\sum_{\sigma \in \mathrm{S}_n} \psi(w(X_{\sigma(1)}, \dots, X_{\sigma(n)})) u_{w(g_{\sigma(1)},\dots,g_{\sigma(n)})} p_{\sigma(1)1} \cdots p_{\sigma(n)n} \\ = \sum_{\sigma \in \mathrm{S}_n} \psi(w(X_{\sigma(1)}, \dots, X_{\sigma(n)})) p_{\sigma(1)1} \cdots p_{\sigma(n)n} \, . \tag{5.5}$$

For all $\sigma \in \mathrm{S}_n$, we have

$$\begin{aligned}\psi(w(X_1,\dots,X_n)) &= \phi(w(x_1,\dots,w_n))\\ &= \phi(\mathrm{E}(x_1^{e_1(w)})\cdots \mathrm{E}(x_n^{e_n(w)})) && \text{(assumption on } x_1,\dots,x_n)\\ &= \phi(\mathrm{E}(x_{\sigma(1)}^{e_1(w)})\cdots \mathrm{E}(x_{\sigma(n)}^{e_n(w)})) && (x_1,\dots,x_n \text{ identically distributed wrt E})\\ &= \psi(w(X_{\sigma(1)},\dots,X_{\sigma(n)}))\,.\end{aligned}$$

Using this formula and equation (5.5), we obtain

$$\begin{aligned}&\sum_{\sigma\in\mathrm{S}_n} \psi(w(X_{\sigma(1)},\dots,X_{\sigma(n)}))u_{w(g_{\sigma(1)},\dots,g_{\sigma(n)})}p_{\sigma(1)1}\cdots p_{\sigma(n)n}\\ &\qquad= \psi(w(X_1,\dots,X_n))\sum_{\sigma\in\mathrm{S}_n} p_{\sigma(1)1}\cdots p_{\sigma(n)n}\\ &\qquad= \psi(w(X_1,\dots,X_n))\,.\end{aligned}$$

This shows that ψ is invariant under the action of $\mathrm{C}^*(\Gamma_n)\bowtie \mathrm{C}(S_n)$.

Assume (ii). For all $n\in\mathbb{N}$ and all $w\in\mathbb{N}^{*n}$, we have

$$\begin{aligned}&\psi(w(X_1,\dots,X_n))\\ &\quad= \sum_{\sigma\in\mathrm{S}_n} \psi(w(X_{\sigma(1)},\dots,X_{\sigma(n)}))u_{w(g_{\sigma(1)},\dots,g_{\sigma(n)})}p_{\sigma(1)1}\cdots p_{\sigma(n)n}\,. \qquad (5.6)\end{aligned}$$

Let $w\in\mathbb{N}^{*n}\setminus K$. For $\sigma\in\mathrm{S}_n$, we see as before that $w(g_{\sigma(1)},\dots,g_{\sigma(n)})\neq e$. So (5.6) can only be true if $\psi(w(X_1,\dots,X_n))=0$. Since $g_k^2=e$ for all $k\in\{1,\dots,n\}$, we have $w(g_1^2,\dots,g_n^2)=e$ for all $n\in\mathbb{N}^{*n}$. Hence equation (5.6) for words of length $2n$ implies

$$\psi(w(X_1^2,\dots,X_n^2)) = \sum_{\sigma\in\mathrm{S}_n}\psi(w(X_{\sigma(1)}^2,\dots,X_{\sigma(n)}^2))p_{\sigma(1)1}\cdots p_{\sigma(n)n}\,.$$

So the same formula is true if we replace ψ by ϕ on both sides. Recall that $B=\bigcap_{n\geq 1}\mathrm{W}^*(x_n^2,x_{n+1}^2,\dots)$ admits the ϕ-preserving conditional expectation $\mathrm{E}: M\to B$. Since the $x_1^2,x_n^2,\dots$ is a commuting family of random variables, the classical de Finetti theorem implies that the distribution of $x_1^2,x_2^2,\dots$ is independent and identically distributed with respect to E. So we showed (i), which completes the proof of the theorem. ☐

Remark 5.13. We want to remark that the previous theorem confirms the special role of easy quantum groups as correct symmetries of non-commutative distributions. Indeed, it is the sS_n-invariance of the kernel $\ker(\mathbb{N}^{*n}\to\Gamma)$ for a strongly symmetric reflection group Γ, which allows for substitution of letters and hence turns the statement $\phi(w(x_1,\dots,x_n)) = \phi(\mathrm{E}(x_1)^{e_1(w)}\cdots\mathrm{E}(x_n)^{e_n(w)})$ into a reasonable condition.

We apply Theorem 5.12 to the easy quantum groups $H_n^{[\infty]}$ and H_n^* described in Section 5.3. In particular, we recover the de Finetti theorem for H_n^* first proved in [6]. We take over the notion of being "balanced" from their work. A word $w \in \mathbb{N}^{*n}$ is balanced if and only if $w \in \ker(\mathbb{N}^{*n} \to \mathbb{Z}_2^{*n}/\{a_ia_ja_k = a_ka_ja_i\})$.

Corollary 5.14. *Let $x_1, x_2, \dots$ be a sequence of non-commutative self-adjoint random variables in a W^*-probability space (M,ϕ). Then there is a von Neumann subalgebra $B \subset M$ and a ϕ-preserving conditional expectation $\mathrm{E} : M \to B$ such that the following statements hold.*

(i) If $x_i^2x_j = x_jx_i^2$ for all $i,j \in \{1,\dots,n\}$, then the following are equivalent.

- *$x_1^2, x_2^2, \dots$ is identically distributed and independent with respect to* E *and $\phi(w(x_1,\dots,x_n)) = 0$ for all words $w \in \ker(\mathbb{N}^{*n} \to \mathbb{Z}_2^{*n})$.*
- *For all $n \in \mathbb{N}$, the distribution of $x_1,\dots,x_n$ is invariant under* $\mathrm{C}(H_n^{[\infty]})$.

(ii) If $x_ix_jx_k = x_kx_jx_i$ for all $i,j,k \in \{1,\dots,n\}$, then the following are equivalent

- *$x_1^2, x_2^2, \dots$ are identically distributed and independent with respect to* E *and $\phi(w(x_1,\dots,x_n)) = 0$ for all non-balanced words $w \in \mathbb{N}^{*n}$.*
- *For all $n \in \mathbb{N}$, the distribution of $x_1,\dots,x_n$ is invariant under* $\mathrm{C}(H_n^*)$.

Proof. We may assume that $M = \mathrm{W}^*(x_1,x_2,\dots)$. Let $B = \bigcap_{n\geq 1}\mathrm{W}^*(x_n^2, x_{n+1}^2,\dots)$ and note that B lies in the centre of M. In particular, there is a ϕ-preserving conditional expectation $\mathrm{E} : M \to B$.

We are going to apply Theorem 5.12 to (i) and (ii). Before starting to consider these cases one by one, let us observe the following facts. The invariance of the E-distribution of $x_1, x_2, \dots$ by any of the quantum groups $H_n^{[\infty]}$ or H_n^*, implies that their distribution is invariant under the permutation groups S_n. In particular, as the random variables $x_1^2, x_2^2, \dots$ are pairwise commuting, they are independent and identically distributed with respect to E. Moreover, as in Theorem 5.12, it follows that the E-distribution of x_i is even for all i. So $x_1, x_2, \dots$ are identically distributed with respect to E.

Let us start to prove (i). Since $x_i^2x_j = x_jx_i^2$ for all $i,j \in \{1,\dots,n\}$ and $K = \ker(\mathbb{N}^{*n} \to \mathbb{Z}_2^{*n})$ is the smallest subsemigroup of $\mathbb{N}^{*n}$ which is invariant under $x \mapsto a_ixa_i$, we obtain inductively that $w(x_1,\dots,x_n) = x_1^{e_1(w)}\cdots x_n^{e_n(w)}$ for all $w \in K$. Furthermore, $e_1(w),\dots,e_n(w) \in 2\mathbb{N}$. Independence of $x_1^2,\dots,x_n^2$ with respect to E implies that

$$\begin{aligned}\phi(w(x_1,\dots,x_n)) = \phi(x_1^{e_1(w)}\cdots x_n^{e_n(w)}) &= (\phi\circ\mathrm{E})(x_1^{e_1(w)}\cdots x_n^{e_n(w)})\\ &= \phi(\mathrm{E}(x_1^{e_1(w)})\cdots\mathrm{E}(x_n^{e_n(w)}))\,,\end{aligned}$$

for all $w \in K$ on n letters.

So we can apply Theorem 5.12 in order to finish the proof of (i).

In order to apply Theorem 5.12 to the situation in (ii), we need to check that $\phi(w(x_1,\dots,x_n)) = \phi(\mathrm{E}(x_1^{e_1(w)})\cdots\mathrm{E}(x_n^{e_n(w)}))$ for all balanced words $w \in \mathbb{N}^{*n}$. If $w \in \mathbb{N}^{*n}$ is a balanced word, then $x_ix_jx_k = x_kx_jx_i$ for all i,j,k implies $w(x_1,\dots,x_n) = x_1^{e_1(w)}\cdots x_n^{e_n(w)}$. So

$$\begin{aligned}\phi(w(x_1,\dots,x_n)) = \phi(x_1^{e_1(w)}\cdots x_n^{e_n(w)}) &= (\phi\circ\mathrm{E})(x_1^{e_1(w)}\cdots x_n^{e_n(w)})\\ &= \phi(\mathrm{E}(x_1^{e_1(w)})\cdots\mathrm{E}(x_n^{e_n(w)})),\end{aligned}$$

where the last equality follows from the fact that $e_i(w) \in 2\mathbb{N}$ for all $i \in \{1,\dots,n\}$ and independence of $x_1^2,\dots,x_n^2$ with respect to E. So we can indeed apply Theorem 5.12. This finishes the proof.

□

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