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Pimsner algebras and Gysin sequences from principal circle actions

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Abstract. A self Morita equivalence over an algebra B, given by a B-bimodule E, is thought of as a line bundle over B. The corresponding Pimsner algebra $\mathcal{O}_E$ is then the total space algebra of a noncommutative principal circle bundle over B. A natural Gysin-like sequence relates the KK-theories of $\mathcal{O}_E$ and of B. Interesting examples come from $\mathcal{O}_E$ a quantum lens space over B a quantum weighted projective line (with arbitrary weights). The KK-theory of these spaces is explicitly computed and natural generators are exhibited.

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1. Introduction

In the present paper we put in close relation two notions that seem to have touched each other only occasionally in the recent literature. These are the notion of a Pimsner (or Cuntz–Krieger–Pimsner) algebra on one hand and that of a noncommutative (in general) principal circle bundle on the other.

At the C^*-algebraic level one needs a self Morita equivalence over a C^*-algebra B, thus we look at a full Hilbert C^*-module E over B together with an isomorphism of B with the compacts on E. Through a natural universal construction this data gives rise to a C^*-algebra, the *Pimsner algebra* $\mathcal{O}_E$ generated by E. In the case where both E and its Hilbert C^*-module dual E^* are finitely generated projective over B one obtains that the $*$-subalgebra generated by the elements of E and B becomes the total space of a noncommutative principal circle bundle with base space B.

At the purely algebraic level we start from a $\mathbb{Z}$-graded $*$-algebra $\mathscr{A}$ which forms the total space of a *quantum principal circle bundle* with base space the $*$-subalgebra of invariant elements $\mathscr{A}_{(0)}$ and with a coaction of the Hopf algebra $\mathcal{O}(U(1))$ coming from the $\mathbb{Z}$-grading. Provided that $\mathscr{A}$ comes equipped with a C^*-norm, which is compatible with the circle action likewise defined by the $\mathbb{Z}$-grading, we show that the closure of $\mathscr{A}$ has the structure of a Pimsner algebra. Indeed, the first spectral subspace $\mathscr{A}_{(1)}$ is then finitely generated and projective over the algebra $\mathscr{A}_{(0)}$. The closure E of $\mathscr{A}_{(1)}$ will become a Hilbert C^*-module over B, the closure of $\mathscr{A}_{(0)}$, and the couple (E, B) will lend itself to a Pimsner algebra construction.

The commutative version of this part of our program was spelled out in [11, Prop. 5.8]. This amounts to showing that the continuous functions on the total space of a (compact) principal circle bundle can be described as a Pimsner algebra generated by a classical line bundle over the compact base space.

With a Pimsner algebra there come two natural six term exact sequences in KK-theory, which relate the KK-theories of the Pimsner algebra $\mathcal{O}_E$ with that of the C^*-algebra of (the base space) scalars B. The corresponding sequences in K-theory are noncommutative analogues of the Gysin sequence which in the commutative case relates the K-theories of the total space and of the base space. The classical cup product with the Euler class is in the noncommutative setting replaced by a Kasparov product with the identity minus the generating Hilbert C^*-module E. Predecessors of these six term exact sequences are the Pimsner-Voiculescu six term exact sequences of [19] for crossed products by the integers.

Interesting examples are quantum lens spaces over quantum weighted projective lines. The latter spaces $W_q(k,l)$ are defined as fixed points of weighted circle actions on the quantum 3-sphere S^3_q. On the other hand, quantum lens spaces $L_q(dlk;k,l)$ are fixed points for the action of a finite cyclic group on S^3_q. For general (k,l) coprime positive integers and any positive integer d, the coordinate algebra of the lens space is a quantum principal circle bundle over the corresponding coordinate algebra for the quantum weighted projective space, thus generalizing the cases studied in [5].

At the C^*-algebra level the lens spaces are given as Pimsner algebras over the C^*-algebra of the continuous functions over the weighted projective spaces (see §6). Using the associated exact sequences coming from the construction of [18], we explicitly compute in §7 the KK-theory of these spaces for general weights. A central character in this computation is played by an integer matrix whose entries are index pairings. These are in turn computed by pairing the corresponding Chern–Connes characters in cyclic theory. The computation of the KK-theory of our class of q-deformed lens spaces is, to the best of our knowledge, a novel one. Also, it is worth emphasizing that the quantum lens spaces and weighted projective spaces are in general not KK-equivalent to their commutative counterparts.

Pimsner algebras were introduced in [18]. This notion gives a unifying framework for a range of important C^*-algebras including crossed products by the integers, Cuntz-Krieger algebras [8, 9], and C^*-algebras associated to partial automorphisms [10]. Generalized crossed products, a notion which is somewhat easier to handle, were independently invented in [3]. More recently, Katsura has constructed Pimsner algebras for general C^*-correspondences [15]. In the present paper we work in a simplified setting (see Assumption 2.1 below) which is close to the one of [3].

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2. Pimsner algebras

We start by reviewing the construction of Pimsner algebras associated to Hilbert C^*-modules as given in [18]. Rather than the full fledged generality we aim at a somewhat simplified version adapted to the context of the present paper, and motivated by our geometric intuition coming from principal circle bundles.

Our reference for the theory of Hilbert C^*-modules is [16]. Throughout this section E will be a countably generated (right) Hilbert C^*-module over a separable C^*-algebra B, with B-valued (and right B-linear) inner product denoted $\langle\cdot,\cdot\rangle_B$; or simply $\langle\cdot,\cdot\rangle$ to lighten notations. Also, E is taken to be full, that is the ideal $\langle E, E\rangle := \mathrm{span}_{\mathbb{C}}\big\{\langle\xi,\eta\rangle \,|\, \xi,\eta\in E\big\}$ is dense in B.

Given two Hilbert C^*-modules E and F over the same algebra B, we denote by $\mathscr{L}(E,F)$ the space of bounded *adjointable* homomorphisms $T : E\to F$. For each of these there exists a homomorphism $T^* : F\to E$ (the adjoint) with the property that $\langle T^*\xi,\eta\rangle = \langle\xi, T\eta\rangle$ for any $\xi\in F$ and $\eta\in E$. Given any pair $\xi\in F, \eta\in E$, an adjointable operator $\theta_{\xi,\eta} : E\to F$ is defined by

$$\theta_{\xi,\eta}(\zeta) = \xi\langle\eta,\zeta\rangle\,, \quad \forall\,\zeta\in E\,.$$

The closed linear subspace of $\mathscr{L}(E,F)$ spanned by elements of the form $\theta_{\xi,\eta}$ as above is denoted $\mathscr{K}(E,F)$, the space of compact homomorphisms. When $E = F$, it results that $\mathscr{L}(E) := \mathscr{L}(E,E)$ is a C^*-algebra with $\mathscr{K}(E) := \mathscr{K}(E,E) \subseteq \mathscr{L}(E)$ the (sub) C^*-algebra of compact endomorphisms of E.

2.1. The algebras and their universal properties.

On top of the above basic conditions, the following will remain in effect as well.

Assumption 2.1. *There is a $*$-homomorphism $\phi : B\to\mathscr{L}(E)$ which induces an isomorphism $\phi : B\to\mathscr{K}(E)$.*

Next, let E^* be the dual of E (when viewed as a Hilbert C^*-module):

$$E^* := \big\{\phi\in\mathrm{Hom}_B(E,B) \,|\, \exists\,\xi\in E \text{ with } \phi(\eta) = \langle\xi,\eta\rangle \;\forall\eta\in E\big\}\,.$$

Thus, with $\xi\in E$, if $\lambda_\xi : E\to B$ is the operator defined by $\lambda_\xi(\eta) = \langle\xi,\eta\rangle$, for all $\eta\in E$, every element of E^* is of the form λ_ξ for some $\xi\in E$. By its definition, $E^* := \mathscr{K}(E,B)$. The dual E^* can be given the structure of a (right) Hilbert C^*-module over B. Firstly, the right action of B on E^* is given by

$$\lambda_\xi\, b := \lambda_\xi\circ\phi(b)\,.$$

Then, with operator $\theta_{\xi,\eta}\in\mathscr{K}(E)$ for $\xi,\eta\in E$, the inner product on E^* is given by

$$\langle\lambda_\xi,\lambda_\eta\rangle := \phi^{-1}(\theta_{\xi,\eta})\,,$$

and E^* is full as well. With the $*$-homomorphism $\phi^* : B\to\mathscr{L}(E^*)$ defined by $\phi^*(b)(\lambda_\xi) := \lambda_{\xi\cdot b^*}$, the pair (ϕ^*, E^*) satisfies the conditions in Assumption 2.1.

We need the interior tensor product $E\widehat{\otimes}_\phi E$ of E with itself over B. As a first step, one constructs the quotient of the vector space tensor product $E \otimes_{\mathrm{alg}} E$ by the ideal generated by elements of the form

$$\xi b \otimes \eta - \xi \otimes \phi(b)\eta\,, \qquad \text{for} \quad \xi, \eta \in E\,, \quad b \in B\,. \tag{2.1}$$

There is a natural structure of right module over B with the action given by

$$(\xi \otimes \eta)b = \xi \otimes (\eta b)\,, \qquad \text{for} \quad \xi, \eta \in E\,, \quad b \in B\,,$$

and a B-valued inner product given, on simple tensors, by

$$\langle \xi_1 \otimes \eta_1, \xi_2 \otimes \eta_2 \rangle = \langle \eta_1, \phi(\langle \xi_1, \xi_2 \rangle)\eta_2 \rangle \tag{2.2}$$

and extended by linearity. The inner product is well defined and has all required properties; in particular, the null space $N = \{\zeta \in E \otimes_{\mathrm{alg}} E\,;\, \langle \zeta, \zeta \rangle = 0\}$ is shown to coincide with the subspace generated by elements of the form in (2.1). One takes $E \otimes_\phi E := E \otimes_{\mathrm{alg}} E/N$ and defines $E\widehat{\otimes}_\phi E$ to be the Hilbert module obtained by completing with respect to the norm induced by (2.2). The construction can be iterated and, for $n > 0$, we denote by $E^{\widehat{\otimes}_\phi n}$, the n-fold interior tensor power of E over B. Like-wise, $(E^*)^{\widehat{\otimes}_{\phi^*} n}$ denotes the n-fold interior tensor power of E^* over B.

To lighten notation, in the following we define, for each $n \in \mathbb{Z}$, the modules

$$E^{(n)} := \begin{cases} E^{\widehat{\otimes}_\phi n} & n > 0 \\ B & n = 0 \\ (E^*)^{\widehat{\otimes}_{\phi^*}(-n)} & n < 0 \end{cases}.$$

Clearly, $E^{(1)} = E$ and $E^{(-1)} = E^*$. We define the Hilbert C^*-module over B:

$$E_\infty := \bigoplus_{n \in \mathbb{Z}} E^{(n)}\,.$$

For each $\xi \in E$ we have a bounded adjointable operator $S_\xi : E_\infty \to E_\infty$ defined component-wise by

$$\begin{aligned} S_\xi(b) &:= \xi \cdot b\,, && b \in B\,, \\ S_\xi(\xi_1 \otimes \cdots \otimes \xi_n) &:= \xi \otimes \xi_1 \otimes \cdots \otimes \xi_n\,, && n > 0\,, \\ S_\xi(\lambda_{\xi_1} \otimes \cdots \otimes \lambda_{\xi_{-n}}) &:= \lambda_{\xi_2 \cdot \phi^{-1}(\theta_{\xi_1,\xi})} \otimes \lambda_{\xi_3} \otimes \cdots \otimes \lambda_{\xi_{-n}}\,, && n < 0\,. \end{aligned}$$

In particular, $S_\xi(\lambda_{\xi_1}) = \phi^{-1}(\theta_{\xi,\xi_1}) \in B$.
The adjoint of S_ξ is easily found to be given by $S_{\lambda_\xi} := S_\xi^* : E_\infty \to E_\infty$:

$$\begin{aligned} S_{\lambda_\xi}(b) &:= \lambda_\xi \cdot b\,, && b \in B\,, \\ S_{\lambda_\xi}(\xi_1 \otimes \ldots \otimes \xi_n) &:= \phi(\langle \xi, \xi_1 \rangle)(\xi_2) \otimes \xi_3 \otimes \ldots \otimes \xi_n\,, && n > 0\,, \\ S_{\lambda_\xi}(\lambda_{\xi_1} \otimes \ldots \otimes \lambda_{\xi_{-n}}) &:= \lambda_\xi \otimes \lambda_{\xi_1} \otimes \ldots \otimes \lambda_{\xi_{-n}}\,, && n < 0\,; \end{aligned}$$

and in particular $S_{\lambda_\xi}(\xi_1) = \langle \xi, \xi_1 \rangle \in B$.

From its definition, each $E^{(n)}$ has a natural structure of Hilbert C^*-module over B and, with $\mathscr{K}$ again denoting the Hilbert C^*-module compacts, we have isomorphisms

$$\mathscr{K}(E^{(n)}, E^{(m)}) \simeq E^{(m-n)} .$$

Definition 2.2. The *Pimsner algebra* of the pair (ϕ, E) is the smallest C^*-subalgebra of $\mathscr{L}(E_\infty)$ which contains the operators $S_\xi : E_\infty \to E_\infty$ for all $\xi \in E$. The Pimsner algebra is denoted by $\mathcal{O}_E$ and comes with an inclusion $\widetilde{\phi} : \mathcal{O}_E \to \mathscr{L}(E_\infty)$.

There is an injective $*$-homomorphism $i : B \to \mathcal{O}_E$. This is induced by the injective $*$-homomorphism $\phi : B \to \mathscr{L}(E_\infty)$ defined component-wise by

$$\begin{aligned} \phi(b)(b') &:= b \cdot b' , \\ \phi(b)(\xi_1 \otimes \ldots \otimes \xi_n) &:= \phi(b)(\xi_1) \otimes \xi_2 \otimes \ldots \otimes \xi_n , \\ \phi(b)(\lambda_{\xi_1} \otimes \ldots \otimes \lambda_{\xi_n}) &:= \phi^*(b)(\lambda_{\xi_1}) \otimes \lambda_{\xi_2} \otimes \ldots \otimes \lambda_{\xi_n} \\ &= \lambda_{\xi_1 \cdot b^*} \otimes \lambda_{\xi_2} \otimes \ldots \otimes \lambda_{\xi_n} , \end{aligned}$$

and which factorizes through the Pimsner algebra $\mathcal{O}_E \subseteq \mathscr{L}(E_\infty)$. Indeed, for all $\xi, \eta \in E$ it holds that $S_\xi S_\eta^* = i(\phi^{-1}(\theta_{\xi,\eta}))$, that is the operator $S_\xi S_\eta^*$ on E_∞ is right-multiplication by the element $\phi^{-1}(\theta_{\xi,\eta}) \in B$.

A Pimsner algebra is universal in the following sense [18, Thm. 3.12]:

Theorem 2.3. *Let C be a C^*-algebra and let $\sigma : B \to C$ be a $*$-homomorphism. Suppose that there exist elements $s_\xi \in C$ for all $\xi \in E$ such that*

(1) $\alpha s_\xi + \beta s_\eta = s_{\alpha\xi+\beta\eta}$ *for all* $\alpha, \beta \in \mathbb{C}$ *and* $\xi, \eta \in E$,

(2) $s_\xi \sigma(b) = s_{\xi b}$ *and* $\sigma(b) s_\xi = s_{\phi(b)(\xi)}$ *for all* $\xi \in E$ *and* $b \in B$,

(3) $s_\xi^* s_\eta = \sigma(\langle \xi, \eta \rangle)$ *for all* $\xi, \eta \in E$,

(4) $s_\xi s_\eta^* = \sigma\big(\phi^{-1}(\theta_{\xi,\eta})\big)$ *for all* $\xi, \eta \in E$.

Then there is a unique $*$-homomorphism $\widetilde{\sigma} : \mathcal{O}_E \to C$ with $\widetilde{\sigma}(S_\xi) = s_\xi$ for all $\xi \in E$.

Also, in the context of this theorem the identity $\widetilde{\sigma} \circ i = \sigma$ follows automatically.

Remark 2.4. In the paper [18], the pair (ϕ, E) was referred to as a *Hilbert bimodule*, since the map ϕ (taken to be injective there) naturally endows the right Hilbert module E with a left module structure. As mentioned, our Assumption 2.1 simplifies the construction to a great extent (see also [3]). For the pair (ϕ, E) with a general $*$-homomorphism $\phi : B \to \mathscr{L}(E)$, (in particular, a non necessarily injective one), the name C^*-*correspondence* over B has recently emerged as a more common one, reserving the terminology Hilbert bimodule to the more restrictive case where one has both a left and a right inner product satisfying an extra compatibility relation.

2.2. Six term exact sequences. With a Pimsner algebra there come two six term exact sequences in KK-theory. Firstly, since $\phi : B \to \mathscr{L}(E)$ factorizes through the compacts $\mathscr{K}(E) \subseteq \mathscr{L}(E)$, the following class is well defined.

Definition 2.5. The class in $KK_0(B, B)$ defined by the even Kasparov module $(E, \phi, 0)$ (with trivial grading) will be denoted by $[E]$.

Next, let $P : E_\infty \to E_\infty$ denote the orthogonal projection with

$$\mathrm{Im}(P) = \big(\oplus_{n=1}^{\infty} E^{(n)} \big) \oplus B \subseteq E_\infty \, .$$

Notice that $[P, S_\xi] \in \mathscr{K}(E_\infty)$ for all $\xi \in E$ and thus $[P, S] \in \mathscr{K}(E_\infty)$ for all $S \in \mathcal{O}_E$.

Then, let $F := 2P - 1 \in \mathscr{L}(E_\infty)$ and recall that $\widetilde{\phi} : \mathcal{O}_E \to \mathscr{L}(E_\infty)$ is the inclusion.

Definition 2.6. The class in $KK_1(\mathcal{O}_E, B)$ defined by the odd Kasparov module $(E_\infty, \widetilde{\phi}, F)$ will be denoted by $[\partial]$.

For any separable C^*-algebra C we then have the group homomorphisms

$$[E] : KK_*(B, C) \to KK_*(B, C) \, , \qquad [E] : KK_*(C, B) \to KK_*(C, B)$$

and

$$[\partial] : KK_*(C, \mathcal{O}_E) \to KK_{*+1}(C, B) \, , \qquad [\partial] : KK_*(B, C) \to KK_{*+1}(\mathcal{O}_E, C) \, ,$$

which are induced by the Kasparov product.

The six term exact sequences in KK-theory given in the following theorem were constructed by Pimsner, see [18, Thm. 4.8].

Theorem 2.7. *Let $\mathcal{O}_E$ be the Pimsner algebra of the pair (ϕ, E) over the C^*-algebra B. If C is any separable C^*-algebra, there are two exact sequences:*

$$\begin{array}{ccccc}
KK_0(C,B) & \xrightarrow{1-[E]} & KK_0(C,B) & \xrightarrow{i_*} & KK_0(C,\mathcal{O}_E) \\
{\scriptstyle [\partial]}\big\uparrow & & & & \big\downarrow{\scriptstyle [\partial]} \\
KK_1(C,\mathcal{O}_E) & \xleftarrow[i_*]{} & KK_1(C,B) & \xleftarrow[1-[E]]{} & KK_1(C,B)
\end{array}$$

and

$$\begin{array}{ccccc}
KK_0(B,C) & \xleftarrow[1-[E]]{} & KK_0(B,C) & \xleftarrow[i^*]{} & KK_0(\mathcal{O}_E,C) \\
\big\downarrow{\scriptstyle [\partial]} & & & & {\scriptstyle [\partial]}\big\uparrow \\
KK_1(\mathcal{O}_E,C) & \xrightarrow{i^*} & KK_1(B,C) & \xrightarrow{1-[E]} & KK_1(B,C)
\end{array}$$

with i^*, i_* the homomorphisms in KK-theory induced by the inclusion $i : B \to \mathcal{O}_E$.

Remark 2.8. For $C = \mathbb{C}$, the first sequence above reduces to

$$\begin{array}{ccccc} K_0(B) & \xrightarrow{1-[E]} & K_0(B) & \xrightarrow{i_*} & K_0(\mathcal{O}_E) \\ {\scriptstyle [\partial]}\uparrow & & & & \downarrow{\scriptstyle [\partial]} \\ K_1(\mathcal{O}_E) & \xleftarrow[i_*]{} & K_1(B) & \xleftarrow[1-[E]]{} & K_1(B) \end{array} .$$

This could be considered as a generalization of the classical *Gysin sequence* in K-theory (see [14, IV.1.13]) for the 'line bundle' E over the 'noncommutative space' B and with the map $1-[E]$ having the role of the *Euler class* $\chi(E) := 1-[E]$ of the line bundle E. The second sequence would then be an analogue in K-homology:

$$\begin{array}{ccccc} K^0(B) & \xleftarrow[1-[E]]{} & K^0(B) & \xleftarrow[i^*]{} & K^0(\mathcal{O}_E) \\ \downarrow{\scriptstyle [\partial]} & & & & {\scriptstyle [\partial]}\uparrow \\ K^1(\mathcal{O}_E) & \xrightarrow{i^*} & K^1(B) & \xrightarrow{1-[E]} & K^1(B) \end{array} .$$

Examples of Gysin sequences in K-theory were given in [2] for line bundles over quantum projective spaces and leading to a class of quantum lens spaces. These examples will be generalized later on in the paper to a class of quantum lens spaces as circle bundles over quantum weighted projective spaces with arbitrary weights.

3. Pimsner algebras and circle actions

An interesting source of Pimsner algebras consists of C^*-algebras which are equipped with a circle action and subject to an extra completeness condition on the associated spectral subspaces. We now investigate this relationship.

Throughout this section A will be a C^*-algebra and $\{\sigma_z\}_{z\in S^1}$ will be a strongly continuous action of the circle S^1 on A.

3.1. Algebras from actions.

For each $n \in \mathbb{Z}$, define the spectral subspace

$$A_{(n)} := \big\{\xi \in A \mid \sigma_z(\xi) = z^{-n}\,\xi \quad \text{for all } z \in S^1\big\} .$$

Then the invariant subspace $A_{(0)} \subseteq A$ is a C^*-subalgebra and each $A_{(n)}$ is a (right) Hilbert C^*-module over $A_{(0)}$ with right action induced by the algebra structure on A and $A_{(0)}$-valued inner product just $\langle \xi, \eta\rangle := \xi^*\,\eta$, for all $\xi, \eta \in A_{(n)}$.

Assumption 3.1. *The data (A, σ_z) as above is taken to satisfy the conditions:*

(1) *The C^*-algebra $A_{(0)}$ is separable.*

(2) *The Hilbert C^*-modules $A_{(1)}$ and $A_{(-1)}$ are full and countably generated over the C^*-algebra $A_{(0)}$.*

Lemma 3.2. *With the $*$-homomorphism $\phi : A_{(0)} \to \mathscr{L}(A_{(1)})$ simply defined by $\phi(a)(\xi) := a\,\xi$, the pair $(\phi, A_{(1)})$ satisfies the conditions of Assumption 2.1.*

Proof. To prove that $\phi : A_{(0)} \to \mathscr{L}(A_{(1)})$ is injective, let $a \in A_{(0)}$ and suppose that $a\,\xi = 0$ for all $\xi \in A_{(1)}$. It then follows that $a\,\xi\,\eta^* = 0$ for all $\xi, \eta \in A_{(1)}$. But this implies that $a\,\langle v, w\rangle = 0$ for all $v, w \in A_{(-1)}$. Since $A_{(-1)}$ is full this shows that $a = 0$. We may thus conclude that $\phi : A_{(0)} \to \mathscr{L}(A_{(1)})$ is injective, and the image of ϕ is therefore closed.

To conclude that $\mathscr{K}(A_{(1)}) \subseteq \phi(A_{(0)})$ it is now enough to show that the operator $\theta_{\xi,\eta} \in \phi(A_{(0)})$ for all $\xi, \eta \in A_{(1)}$. But this is clear since $\theta_{\xi,\eta} = \phi(\xi\,\eta^*)$.

To prove that $\phi(A_{(0)}) \subseteq \mathscr{K}(A_{(1)})$ it suffices to check that the operator $\phi(\langle v, w\rangle) \in \mathscr{K}(A_{(1)})$ for all $v, w \in A_{(-1)}$ (again since $A_{(-1)}$ is full). But this is true being $\phi(\langle v, w\rangle) = \theta_{v^*,w^*}$. □

The condition that both $A_{(1)}$ and $A_{(-1)}$ are full over $A_{(0)}$ has the important consequence that the action $\{\sigma_z\}_{z\in S^1}$ is semi-saturated in the sense of the following.

Definition 3.3. A circle action $\{\sigma_z\}_{z\in S^1}$ on a C^*-algebra A is called *semi-saturated* if A is generated, as a C^*-algebra, by the fixed point algebra $A_{(0)}$ together with the first spectral subspace $A_{(1)}$.

Proposition 3.4. *Suppose that $A_{(1)}$ and $A_{(-1)}$ are full over $A_{(0)}$. Then the circle action $\{\sigma_z\}_{z\in S^1}$ is semi-saturated.*

Proof. With cl$(\cdot)$ refering to the norm-closure, we show that the Banach algebra

$$\mathrm{cl}\Big(\sum_{n=0}^{\infty} A_{(n)}\Big) \subseteq A$$

is generated by $A_{(1)}$ and $A_{(0)}$. A similar proof in turn shows that

$$\mathrm{cl}\Big(\sum_{n=0}^{\infty} A_{(-n)}\Big) \subseteq A$$

is generated by $A_{(-1)}$ and $A_{(0)}$. Since the span $\sum_{n\in\mathbb{Z}} A_{(n)}$ is norm-dense in A (see [10, Prop. 2.5]), this proves the proposition. We show by induction on $n \in \mathbb{N}$ that

$$(A_{(1)})^n := \mathrm{span}\big\{x_1 \cdot \ldots \cdot x_n \mid x_1, \ldots, x_n \in A_{(1)}\big\}$$

is dense in $A_{(n)}$. For $n = 1$ the statement is void.

Suppose thus that the statement holds for some $n \in \mathbb{N}$. Then, let $x \in A_{(n+1)}$ and choose a countable approximate identity $\{u_m\}_{m\in\mathbb{N}}$ for the separable C^*-algebra $A_{(0)}$. Let $\varepsilon > 0$ be given. We need to construct an element $y \in (A_{(1)})^{n+1}$ such that

$$\|x - y\| < \varepsilon\,.$$

To this end we first remark that the sequence $\{x \cdot u_m\}_{m\in\mathbb{N}}$ converges to $x \in A_{(n+1)}$. Indeed, this follows due to $x^* x \in A_{(0)}$ and since, for all $m \in \mathbb{N}$,

$$\|x \cdot u_m - x\|^2 = \|u_m x^* x u_m + x^* x - x^* x u_m - u_m x^* x\| .$$

We may thus choose an $m \in \mathbb{N}$ such that

$$\|x \cdot u_m - x\| < \varepsilon/3 .$$

Since $A_{(1)}$ is full over $A_{(0)}$, there are elements $\xi_1, \ldots, \xi_k$ and $\eta_1, \ldots, \eta_k \in A_{(1)}$ so that

$$\Big\| x \cdot u_m - \sum_{j=1}^{k} x \cdot \xi_j^* \cdot \eta_j \Big\| < \varepsilon/3 .$$

Furthermore, since $x \cdot \xi_j^* \in A_{(n)}$ we may apply the induction hypothesis to find elements $z_1, \ldots, z_k \in (A_{(1)})^n$ such that

$$\Big\| \sum_{j=1}^{k} x \cdot \xi_j^* \cdot \eta_j - \sum_{j=1}^{k} z_j \cdot \eta_j \Big\| < \varepsilon/3 .$$

Finally, it is straightforward to verify that for the element

$$y := \sum_{j=1}^{k} z_j \cdot \eta_j \in (A_{(1)})^{n+1}$$

it holds that: $\|x - y\| < \varepsilon$. This proves the present proposition. $\square$

Having a semi-saturated action one is lead to the following theorem [3, Thm. 3.1].

Theorem 3.5. *The Pimsner algebra $\mathcal{O}_{A_{(1)}}$ is isomorphic to A. The isomorphism is given by $S_\xi \mapsto \xi$ for all $\xi \in A_{(1)}$.*

3.2. $\mathbb{Z}$-graded algebras. In much of what follows, the C^*-algebras of interest with a circle action, will come from closures of dense $\mathbb{Z}$-graded $*$-algebras, with the $\mathbb{Z}$-grading defining the circle action in a natural fashion.

Let $\mathscr{A} = \oplus_{n\in\mathbb{Z}} \mathscr{A}_{(n)}$ be a $\mathbb{Z}$-graded unital $*$-algebra. The grading is compatible with the involution *, this meaning that $x^* \in \mathscr{A}_{(-n)}$ whenever $x \in \mathscr{A}_{(n)}$ for some $n \in \mathbb{Z}$. For $w \in S^1$, define the $*$-automorphism $\sigma_w : \mathscr{A} \to \mathscr{A}$ by

$$\sigma_w : x \mapsto w^{-n} x \qquad \text{for} \quad x \in \mathscr{A}_{(n)} , \; n \in \mathbb{Z} .$$

We will suppose that we have a C^*-norm $\| \cdot \| : \mathscr{A} \to [0, \infty)$ on $\mathscr{A}$ satisfying

$$\|\sigma_w(x)\| \leq \|x\| \qquad \text{for all} \quad w \in S^1 , \; x \in \mathscr{A} ,$$

thus the action has to be isometric. The completion of $\mathscr{A}$ is denoted by A.

The following standard result is here for the sake of completeness and its use below. The proof relies on the existence of a conditional expectation naturally associated to the action.

Lemma 3.6. *The collection $\{\sigma_w\}_{w \in S^1}$ extends by continuity to a strongly continuous action of S^1 on A. Each spectral subspace $A_{(n)}$ agrees with the closure of $\mathscr{A}_{(n)} \subseteq A$.*

Proof. Once $\mathscr{A}_{(n)}$ is shown to be dense in $A_{(n)}$ the rest follows from standard arguments. Thus, for $n \in \mathbb{Z}$, define the bounded operator $E_{(n)} : A \to A_{(n)}$ by

$$E_{(n)} : x \mapsto \int_{S^1} w^n \, \sigma_w(x) \, \mathrm{d}w \, ,$$

where the integration is carried out with respect to the Haar-measure on S^1. We have that $E_{(n)}(x) = x$ for all $x \in A_{(n)}$ and then that $\|E_{(n)}\| \leq 1$. This implies that $\mathscr{A}_{(n)} \subseteq A_{(n)}$ is dense. □

Let now $d \in \mathbb{N}$ and consider the unital $*$-subalgebra $\mathscr{A}^{1/d} := \oplus_{n \in \mathbb{Z}} \mathscr{A}_{(nd)} \subseteq \mathscr{A}$. Then $\mathscr{A}^{1/d}$ is a $\mathbb{Z}$-graded unital $*$-algebra as well and we denote the associated circle action by $\sigma_w^{1/d} : \mathscr{A}^{1/d} \to \mathscr{A}^{1/d}$. Let $w \in S^1$ and choose a $z \in S^1$ such that $z^d = w$. Then

$$\sigma_w^{1/d}(x_{nd}) = w^n \cdot x_{nd} = z^{nd} \cdot x_{nd} = \sigma_z(x_{nd}) \, , \qquad \text{for all } x_{nd} \in \mathscr{A}_{(nd)} \, ,$$

and it follows that $\sigma_w^{1/d}(x) = \sigma_z(x)$ for all $x \in \mathscr{A}^{1/d}$. With the C^*-norm obtained by restriction $\| \cdot \| : \mathscr{A}^{1/d} \to [0, \infty)$, it follows in particular that

$$\|\sigma_w^{1/d}(x)\| \leq \|x\|$$

by our standing assumption on the compatibility of $\{\sigma_w\}_{w \in S^1}$ with the norm on $\mathscr{A}$. The C^*-completion of $\mathscr{A}^{1/d}$ is denoted by $A^{1/d}$.

Proposition 3.7. *Suppose that $\{\sigma_w\}_{w \in S^1}$ is semi-saturated on A and let $d \in \mathbb{N}$. Then we have unitary isomorphisms of Hilbert C^*-modules*

$$(A_{(1)})^{\widehat{\otimes}_\phi d} \simeq (A^{1/d})_{(1)} \qquad \textit{and} \qquad (A_{(-1)})^{\widehat{\otimes}_\phi d} \simeq (A^{1/d})_{(-1)}$$

induced by the product $\psi : x_1 \otimes \ldots \otimes x_d \mapsto x_1 \cdot \ldots \cdot x_d$.

Proof. We only consider the case of $A_{(1)}$ since the the proof for $A_{(-1)}$ is the same.

Observe firstly that $(\mathscr{A}^{1/d})_{(1)} = \mathscr{A}_{(d)}$. Thus Lemma 3.6 yields $A_{(d)} = (A^{1/d})_{(1)}$. This implies that the product $\psi : (\mathscr{A}_{(1)})^{\otimes_{\mathscr{A}_{(0)}} d} \to (\mathscr{A}^{1/d})_{(1)}$ is a well-defined homomorphism of right modules over $\mathscr{A}_{(0)}$ (here "$\otimes_{\mathscr{A}_{(0)}}$" refers to the algebraic tensor product of bimodules over $\mathscr{A}_{(0)}$). Furthermore, since

$$\langle x_1 \otimes \ldots \otimes x_d, y_1 \otimes \ldots \otimes y_d \rangle = x_d^* \cdot \ldots \cdot x_1^* \cdot y_1 \cdot \ldots \cdot y_d \, ,$$

we get that ψ extends to a homomorphism $\psi : (A_{(1)})^{\widehat{\otimes}_\phi d} \to A^{1/d}_{(1)}$ of Hilbert C^*-modules over $A_{(0)}$ with $\langle \psi(\xi), \psi(\eta) \rangle = \langle \xi, \eta \rangle$ for all $\xi, \eta \in (A_{(1)})^{\widehat{\otimes}_\phi d}$.

It is therefore enough to show that $\mathrm{Im}(\psi) \subseteq (A^{1/d})_{(1)}$ is dense. But this is a consequence of [10, Prop. 4.8]. □

Lemma 3.8. *Suppose that $\{\sigma_w\}_{w \in S^1}$ satisfies the conditions of Assumption 3.1. Then $\{\sigma_w^{1/d}\}_{w \in S^1}$ satisfies the conditions of Assumption 3.1 for all $d \in \mathbb{N}$.*

Proof. We only need to show that the Hilbert C^*-modules $A_{(d)}$ and $A_{(-d)}$ are full and countably generated over $A_{(0)}$.

By Proposition 3.4 we have that $\{\sigma_w\}_{w \in S^1}$ is semi-saturated. It thus follows from Proposition 3.7 that

$$A_{(d)} \simeq (A_{(1)})^{\widehat{\otimes}_\phi d} \qquad \text{and} \qquad A_{(-d)} \simeq (A_{(-1)})^{\widehat{\otimes}_\phi d} . \tag{3.1}$$

Since both $A_{(1)}$ and $A_{(-1)}$ are full and countably generated by assumption these unitary isomorphisms prove the lemma. □

The following result is a stronger version of Theorem 3.5 since it incorporates all the spectral subspaces and not just the first one.

Theorem 3.9. *Suppose that the circle action $\{\sigma_w\}_{w \in S^1}$ on A satisfies the conditions in Assumption 3.1. Then the Pimsner algebra $\mathcal{O}_{A_{(d)}} \simeq \mathcal{O}_{(A_{(1)})^{\widehat{\otimes} d}}$ is isomorphic to the C^*-algebra $A^{1/d}$ for all $d \in \mathbb{N}$. The isomorphism is given by $S_\xi \mapsto \xi$ for all $\xi \in A_{(d)}$.*

Proof. This follows by combining Lemma 3.8, Proposition 3.7 and Theorem 3.5. □

We finally investigate what happens when the C^*-norm on $\mathscr{A} = \oplus_{n \in \mathbb{Z}} \mathscr{A}_{(n)}$ is changed. Thus, let $\| \cdot \|' : \mathscr{A} \to [0, \infty)$ be an alternative C^*-norm on $\mathscr{A}$ satisfying

$$\|\sigma_w(x)\|' \leq \|x\|' \qquad \text{for all } w \in S^1 \text{ and } x \in \mathscr{A} .$$

The corresponding completion A' will carry an induced circle action $\{\sigma'_w\}_{w \in S^1}$. The next theorem can be seen as a manifestation of the gauge-invariant uniqueness theorem, [15, Thm. 6.2 and Thm. 6.4]. This property was indirectly used already in [18, Thm. 3.12] for the proof of the universal properties of Pimsner algebras.

Theorem 3.10. *Suppose that $\|x\| = \|x\|'$ for all $x \in \mathscr{A}_{(0)}$. Then $\{\sigma_w\}_{w \in S^1}$ satisfies the conditions of Assumption 3.1 if and only if $\{\sigma'_w\}$ satisfies the conditions of Assumption 3.1. In this case, the identity map $\mathscr{A} \to \mathscr{A}$ induces an isomorphism $A \to A'$ of C^*-algebras. In particular, we have that $\|x\| = \|x\|'$ for all $x \in \mathscr{A}$.*

Proof. Remark first that the identity map $\mathscr{A}_{(n)} \to \mathscr{A}_{(n)}$ induces an isometric isomorphism of Hilbert C^*-modules $A_{(n)} \to A'_{(n)}$ for all $n \in \mathbb{Z}$. This is a consequence of the identity $\|x\| = \|x\|'$ for all $x \in \mathscr{A}_{(0)}$. But then we also have isomorphisms

$$(A_{(1)})^{\widehat{\otimes}_\phi n} \simeq (A'_{(1)})^{\widehat{\otimes}_\phi n} \qquad \text{and} \qquad (A_{(-1)})^{\widehat{\otimes}_\phi n} \simeq (A'_{(-1)})^{\widehat{\otimes}_\phi n}$$

for all $n \in \mathbb{N}$. These observations imply that $\{\sigma_w\}_{w\in S^1}$ satisfies the conditions of Assumption 3.1 if and only if $\{\sigma'_w\}$ satisfies the conditions of Assumption 3.1. But it then follows from Theorem 3.5 that

$$A \simeq \mathcal{O}_{A_{(1)}} \simeq \mathcal{O}_{A'_{(1)}} \simeq A' ,$$

with corresponding isomorphism $A \simeq A'$ induced by the identity map $\mathscr{A} \to \mathscr{A}$. □

4. Quantum principal bundles and $\mathbb{Z}$-graded algebras

We start by recalling the definition of a quantum principal $U(1)$-bundle.

Later on in the paper we shall exhibit a novel class of quantum lens spaces as principal $U(1)$-bundles over quantum weighted projective lines with arbitrary weights.

4.1. Quantum principal bundles.

Define the unital complex algebra

$$\mathcal{O}(U(1)) := \mathbb{C}[z, z^{-1}]/\langle 1 - zz^{-1}\rangle$$

where $\langle 1 - zz^{-1}\rangle$ denotes the ideal generated by $1 - zz^{-1}$ in the polynomial algebra $\mathbb{C}[z, z^{-1}]$ in two variables. The algebra $\mathcal{O}(U(1))$ is a Hopf algebra by defining, for all $n \in \mathbb{Z}$, coproduct $\Delta : z^n \mapsto z^n \otimes z^n$, antipode $S : z^n \mapsto z^{-n}$ and counit $\varepsilon : z^n \mapsto 1$. We simply write $\mathcal{O}(U(1)) = \big(\mathcal{O}(U(1)), \Delta, S, \varepsilon\big)$ for short.

Let $\mathscr{A}$ be a complex unital algebra and suppose in addition that it is a right comodule algebra over $\mathcal{O}(U(1))$, that is we have a homomorphism of unital algebras

$$\Delta_R : \mathscr{A} \to \mathscr{A} \otimes \mathcal{O}(U(1)) ,$$

which also provides a coaction of the Hopf algebra $\mathcal{O}(U(1))$ on $\mathscr{A}$.

Let $\mathscr{B} := \{x \in \mathscr{A} \mid \Delta_R(x) = x \otimes 1\}$ denote the unital subalgebra of $\mathscr{A}$ consisting of coinvariant elements for the coaction.

Definition 4.1. One says that the datum $\big(\mathscr{A}, \mathcal{O}(U(1)), \mathscr{B}\big)$ is a *quantum principal $U(1)$-bundle* when the *canonical map*

$$\text{can} : \mathscr{A} \otimes_{\mathscr{B}} \mathscr{A} \to \mathscr{A} \otimes \mathcal{O}(U(1)) , \qquad x \otimes y \mapsto x \cdot \Delta_R(y) ,$$

is an isomorphism.

Remark 4.2. One ought to qualify Definition 4.1 by saying that the quantum principal bundle is 'for the universal differential calculus' [6]. In fact, the definition above means that the right comodule algebra $\mathscr{A}$ is a $\mathcal{O}(U(1))$-*Galois extension*, and this is equivalent (in the present context) by [12, Prop. 1.6] to the bundle being a quantum principal bundle for the universal differential calculus.

4.2. Relation with $\mathbb{Z}$-graded algebras. We now provide a detailed analysis of the case where the quantum principal bundle structure comes from a $\mathbb{Z}$-grading of the 'total space' algebra. This will lead to an alternative characterization of quantum $U(1)$-principal bundles in this setting. While this description is not new (see for instance [21, Lemma 5.1]), it is certainly more manageable. In particular, we will apply it in §6 below for the case of quantum lens spaces as $U(1)$-principal bundles over quantum weighted projective lines.

Let $\mathscr{A} = \oplus_{n\in\mathbb{Z}}\mathscr{A}_{(n)}$ be a $\mathbb{Z}$-graded unital algebra and let $\mathcal{O}(U(1))$ be the Hopf algebra defined in the previous section. Define the unital algebra homomorphism

$$\Delta_R : \mathscr{A} \to \mathscr{A} \otimes \mathcal{O}(U(1)) \qquad x \mapsto x \otimes z^{-n}\,, \quad \text{for } x \in \mathscr{A}_{(n)}\,.$$

It is then clear that Δ_R turns $\mathscr{A}$ into a right comodule algebra over $\mathcal{O}(U(1))$. The unital subalgebra of coinvariant elements coincides with $\mathscr{A}_{(0)}$.

Theorem 4.3. *The triple* $\big(\mathscr{A}, \mathcal{O}(U(1)), \mathscr{A}_{(0)}\big)$ *is a quantum principal* $U(1)$*-bundle if and only if there exist finite sequences*

$$\{\xi_j\}_{j=1}^N\,,\ \{\beta_i\}_{i=1}^M \text{ in } \mathscr{A}_{(1)} \qquad \text{and} \qquad \{\eta_j\}_{j=1}^N\,,\ \{\alpha_i\}_{i=1}^M \text{ in } \mathscr{A}_{(-1)}$$

such that there hold identities:

$$\sum_{j=1}^N \xi_j\eta_j = 1_{\mathscr{A}} = \sum_{i=1}^M \alpha_i\beta_i\,.$$

Proof. Suppose first that $\big(\mathscr{A}, \mathcal{O}(U(1)), \mathscr{A}_{(0)}\big)$ is a quantum principal $U(1)$-bundle. Thus, that the canonical map

$$\mathrm{can} : \mathscr{A} \otimes_{\mathscr{A}_{(0)}} \mathscr{A} \to \mathscr{A} \otimes \mathcal{O}(U(1))$$

is an isomorphism. For each $n \in \mathbb{Z}$, define the idempotents

$$P_{(n)} : \mathcal{O}(U(1)) \to \mathcal{O}(U(1))\,, \quad P_{(n)} : z^m \mapsto \delta_{nm} z^m \qquad \text{and}$$

$$E_{(n)} : \mathscr{A} \to \mathscr{A}\,, \qquad E_{(n)} : x_m \mapsto \delta_{nm} x_m$$

where $x_m \in \mathscr{A}_{(m)}$ and where $\delta_{nm} \in \{0, 1\}$ denotes the Kronecker delta. Clearly,

$$\mathrm{can} \circ (1 \otimes E_{(-n)}) = (1 \otimes P_{(n)}) \circ \mathrm{can} : \mathscr{A} \otimes_{\mathscr{A}_{(0)}} \mathscr{A} \to \mathscr{A} \otimes \mathcal{O}(U(1))\,. \tag{4.1}$$

for all $n \in \mathbb{Z}$.

Let us now define the element

$$\gamma := \mathrm{can}^{-1}(1_{\mathscr{A}} \otimes z) = \sum_{j=1}^{N} \gamma_j^0 \otimes \gamma_j^1 \,.$$

It then follows from (4.1) that

$$\gamma = (1 \otimes E_{(-1)})(\gamma) = \sum_{j=1}^{N} \gamma_j^0 \otimes E_{(-1)}(\gamma_j^1)$$

To continue, we remark that

$$m(\gamma) = m \circ \mathrm{can}^{-1}(1_{\mathscr{A}} \otimes z) = (\mathrm{id} \otimes \varepsilon)(1_{\mathscr{A}} \otimes z) = 1_{\mathscr{A}}$$

where $m : \mathscr{A} \otimes_{\mathscr{A}_{(0)}} \mathscr{A} \to \mathscr{A}$ is the algebra multiplication. And this implies that

$$1_{\mathscr{A}} = \sum_{j=1} \gamma_j^0 \cdot E_{(-1)}(\gamma_j^1) = \sum_{j=1}^{N} E_{(1)}(\gamma_j^0) \cdot E_{(-1)}(\gamma_j^1) \,.$$

We therefore put,

$$\xi_j := E_{(1)}(\gamma_0^j) \qquad \text{and} \qquad \eta_j := E_{(-1)}(\gamma_1^j)\,, \qquad \text{for all } \; j = 1, \ldots, N \,.$$

Next, we define the element

$$\delta := \mathrm{can}^{-1}(1_{\mathscr{A}} \otimes z^{-1}) = \sum_{i=1}^{M} \delta_i^0 \otimes \delta_i^1 \,.$$

An argument similar to the one before then shows that $\sum_{i=1}^{M} \alpha_i \cdot \beta_i = 1_{\mathscr{A}}$, with

$$\alpha_i := E_{(-1)}(\delta_i^0) \qquad \text{and} \qquad \beta_i := E_{(1)}(\delta_i^1)\,, \qquad \text{for all } \; i = 1, \ldots, M \,.$$

This proves the first half of the theorem.

To prove the second half we suppose there exist sequences $\{\xi_j\}_{j=1}^N$, $\{\beta_i\}_{i=1}^M$ in $\mathscr{A}_{(1)}$ and $\{\eta_j\}_{j=1}^N$, $\{\alpha_i\}_{i=1}^M$in $\mathscr{A}_{(-1)}$ such that $\sum_{j=1}^N \xi_j \eta_j = 1_{\mathscr{A}} = \sum_{i=1}^M \alpha_i \beta_i$.

We then define the map $\mathrm{can}^{-1} : \mathscr{A} \otimes \mathcal{O}(U(1)) \to \mathscr{A} \otimes_{\mathscr{A}_{(0)}} \mathscr{A}$ by the formula

$$\mathrm{can}^{-1} : x \otimes z^n \mapsto \begin{cases} \sum_{J \in \{1,\ldots,N\}^n} x\, \xi_{j_1} \cdot \ldots \cdot \xi_{j_n} \otimes \eta_{j_n} \cdot \ldots \cdot \eta_{j_1}\,, & \text{for } \; n \geq 0 \\ x \otimes 1, & \text{for } \; n = 0 \,. \\ \sum_{I \in \{1,\ldots,M\}^{-n}} x\, \alpha_{i_1} \cdot \ldots \cdot \alpha_{i_{-n}} \otimes \beta_{i_{-n}} \cdot \ldots \cdot \beta_{i_1}\,, & \text{for } \; n \leq 0 \end{cases}$$

It is then straightforward to check that

$$\mathrm{can}^{-1} \circ \mathrm{can} = \mathrm{id} \qquad \text{and} \qquad \mathrm{can} \circ \mathrm{can}^{-1} = \mathrm{id}\,.$$

This ends the proof of the theorem. □

Remark 4.4. The above theorem shows that $\big(\mathscr{A}, \mathcal{O}(U(1)), \mathscr{A}_{(0)}\big)$ is a quantum principal $U(1)$-bundle if and only if $\mathscr{A}$ is *strongly* $\mathbb{Z}$-graded, see [17, Lem. I.3.2]. Our next corollary is thus a consequence of [17, Cor. I.3.3]. We present a proof here since we need the explicit form of the idempotents later on.

Corollary 4.5. *With the same conditions as in Theorem 4.3, the right-modules $\mathscr{A}_{(1)}$ and $\mathscr{A}_{(-1)}$ are finitely generated and projective over $\mathscr{A}_{(0)}$.*

Proof. With the ξ's and the η's as above, define the module homomorphisms

$$\Phi_{(1)} : \mathscr{A}_{(1)} \to (\mathscr{A}_{(0)})^N , \qquad \Phi_{(1)}(\zeta) = \begin{pmatrix} \eta_1 \zeta \\ \eta_2 \zeta \\ \vdots \\ \eta_N \zeta \end{pmatrix} \qquad \text{and}$$

$$\Psi_{(1)} : (\mathscr{A}_{(0)})^N \to \mathscr{A}_{(1)} , \qquad \Psi_{(1)} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_N \end{pmatrix} = \sum_{i=1}^N \xi_i x_i .$$

It then follows that $\Psi_{(1)}\Phi_{(1)} = \mathrm{id}_{\mathscr{A}_{(1)}}$. Thus $E_{(1)} := \Phi_{(1)}\Psi_{(1)}$ is an idempotent in $M_N(\mathscr{A}_{(0)})$ and this proves the first half of the corollary.

Similarly, with the α's and the β's as above, define the module homomorphisms

$$\Phi_{(-1)} : \mathscr{A}_{(-1)} \to (\mathscr{A}_{(0)})^N , \qquad \Phi_{(-1)}(\zeta) = \begin{pmatrix} \beta_1 \zeta \\ \beta_2 \zeta \\ \vdots \\ \beta_M \zeta \end{pmatrix} \qquad \text{and}$$

$$\Psi_{(-1)} : (\mathscr{A}_{(0)})^N \to \mathscr{A}_{(-1)} , \qquad \Psi_{(-1)} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_M \end{pmatrix} = \sum_{i=1}^M \alpha_i x_i .$$

Now one gets $\Psi_{(-1)}\Phi_{(-1)} = \mathrm{id}_{\mathscr{A}_{(-1)}}$. Thus $E_{(-1)} := \Phi_{(-1)}\Psi_{(-1)}$ is an idempotent in $M_M(\mathscr{A}_{(0)})$ as well. This finishes the proof of the corollary. □

Let $d \in \mathbb{N}$ and consider the $\mathbb{Z}$-graded unital $\mathbb{C}$-algebra $\mathscr{A}^{1/d} := \oplus_{n\in\mathbb{Z}} \mathscr{A}_{(dn)}$. As a consequence of Theorem 4.3 we obtain the following:

Proposition 4.6. *Suppose $\big(\mathscr{A}, \mathcal{O}(U(1)), \mathscr{A}_{(0)}\big)$ is a quantum principal $U(1)$-bundle. Then $\big(\mathscr{A}^{1/d}, \mathcal{O}(U(1)), \mathscr{A}_{(0)}\big)$ is a quantum principal $U(1)$-bundle for all $d \in \mathbb{N}$.*

Proof. Let the finite sequences $\{\xi_j\}_{j=1}^N$, $\{\beta_i\}_{i=1}^M$ in $\mathscr{A}_{(1)}$ and $\{\eta_j\}_{j=1}^N$, $\{\alpha_i\}_{i=1}^M$ in $\mathscr{A}_{(-1)}$ be as in Theorem 4.3. For each multi-index $J \in \{1, \ldots, N\}^d$ and each multi-index $I \in \{1, \ldots, M\}^d$ define the elements

$$\xi_J := \xi_{j_1} \cdot \ldots \cdot \xi_{j_d}\,, \qquad \beta_I := \beta_{i_d} \cdot \ldots \cdot \beta_{i_1} \in \mathscr{A}_{(d)} \qquad \text{and}$$
$$\eta_J := \eta_{j_d} \cdot \ldots \cdot \eta_{j_1}\,, \qquad \alpha_I := \alpha_{i_1} \cdot \ldots \cdot \alpha_{i_d} \in \mathscr{A}_{(-d)}\,.$$

It is then clear that

$$\sum_{J \in \{1,\ldots,N\}^d} \xi_J\, \eta_J = 1_{\mathscr{A}^{1/d}} = \sum_{I \in \{1,\ldots,M\}^d} \alpha_I\, \beta_I\,.$$

This proves the proposition by an application of Theorem 4.3. □

Note that it follows from Proposition 4.6 and Corollary 4.5 that when $\big(\mathscr{A}, \mathcal{O}(U(1)), \mathscr{A}_{(0)}\big)$ is a quantum principal bundle then the right modules $\mathscr{A}_{(d)}$ and $\mathscr{A}_{(-d)}$ are finitely generated projective over $\mathscr{A}_{(0)}$ for all $d \in \mathbb{N}$.

5. Quantum weighted projective lines

We recall the definition of the quantum weighted projective lines as fixed point algebras of circle actions on the quantum 3-sphere. These algebras play the role of the coordinate functions on the base space which parametrizes the lines generating the quantum lens spaces (as total spaces). Corresponding C^*-algebras will be the analogues of continuous functions on the base and total space respectively. The latter C^*-algebra will be given as a Pimsner algebra coming from the line bundles.

5.1. Coordinate algebras.

Let $n \in \mathbb{N}_0$ and let $q \in (0, 1)$.

Definition 5.1. The coordinate algebra $\mathcal{O}(S_q^{2n+1})$ of the *quantum sphere* S_q^{2n+1} is the universal unital $*$-algebra with generators $z_0, \ldots, z_n$ and relations

$$z_i z_j = q z_j z_i \;\text{ for } i < j\,, \qquad z_i z_j^* = q z_j^* z_i \;\text{ for } i \neq j\,,$$
$$z_i z_i^* = z_i^* z_i + (q^{-2} - 1) \sum_{m=i+1}^{n} z_m z_m^*\,, \qquad \sum_{m=0}^{n} z_m z_m^* = 1\,.$$

This algebra was introduced in [22]. Next, let $L = (l_0, \ldots, l_n) \in \mathbb{N}^{n+1}$ be fixed. We then have a circle action $\{\sigma_w^L\}_{w \in S^1}$ on $\mathcal{O}(S_q^{2n+1})$ defined on generators by

$$\sigma_w^L : z_i \mapsto w^{l_i} z_i \qquad \text{for all } i \in \{0, \ldots, n\}\,.$$

Definition 5.2. The coordinate algebra $\mathcal{O}(W_q(L))$ of the *quantum weighted projective space* $W_q(L)$ is the fixed point algebra of the circle action $\{\sigma_w^L\}_{w \in S^1}$. Thus

$$\mathcal{O}(W_q(L)) := \big\{x \in \mathcal{O}(S_q^{2n+1}) \mid \sigma_w^L(x) = x \text{ for all } w \in S^1\big\}\,.$$

From now on, we will suppose that $n = 1$ and that $k := l_0$ and $l := l_1$ are coprime. By [5, Thm. 2.1], the algebraic quantum projective line $\mathcal{O}(W_q(k,l))$ agrees with the unital $*$-subalgebra of $\mathcal{O}(S_q^3)$ generated by the elements $z_0^l(z_1^*)^k$ and $z_1 z_1^*$. Alternatively, one may identify $\mathcal{O}(W_q(k,l))$ with the universal unital $*$-algebra with generators a, b, subject to the relations

$$b^* = b \ , \qquad ba = q^{-2l}\, ab \ ,$$
$$aa^* = q^{2kl}\, b^k \prod_{m=0}^{l-1}(1-q^{2m}b) \ , \qquad a^*a = b^k \prod_{m=1}^{l}(1-q^{-2m}b) \, .$$

The identification is just $a \mapsto z_0^l(z_1^*)^k$ and $b \mapsto z_1 z_1^*$ (we have exchanged the names of generators with respect to [5]). In particular $\mathcal{O}(W_q(1,1)) = \mathcal{O}(\mathbb{C}P_q^1)$, while $\mathcal{O}(W_q(1,l))$ was named *quantum teardrop* in [5].

5.2. C^*-completions. We fix $k, l \in \mathbb{N}$ to be coprime positive integers.

Definition 5.3. The algebra of continuous functions on the *quantum weighted projective line* $W_q(k,l)$ is the universal enveloping C^*-algebra, denoted $C(W_q(k,l))$, of the coordinate algebra $\mathcal{O}(W_q(k,l))$.

Let $\mathscr{K}$ denote the C^*-algebra of compact operators on the separable Hilbert space $l^2(\mathbb{N}_0)$ of all square summable sequences indexed by $\mathbb{N}_0$, with orthonormal basis $\{e_p\}_{p\in\mathbb{N}_0}$. It was shown in [5, Prop. 5.1] that $C(W_q(k,l))$ is isomorphic to the unital C^*-algebra

$$\widetilde{\oplus_{s=1}^{l}\mathscr{K}} \subseteq \mathscr{L}\big(\oplus_{s=1}^{l}\, l^2(\mathbb{N}_0)\big) \, ,$$

where $\widetilde{\cdot}$ denotes the unitalization functor. The isomorphism is induced by the direct sum of representations $\oplus_{s=1}^{l}\pi_s : \mathcal{O}(W_q(k,l)) \to \mathscr{L}\big(\oplus_{s=1}^{l}\, l^2(\mathbb{N}_0)\big)$ where each π_s is defined on generators by

$$\pi_s(z_1 z_1^*)(e_p) := q^{2s}\, q^{2lp}\, e_p \ , \qquad \pi_s(z_0^l(z_1^*)^k)(e_0) := 0 \, ,$$
$$\pi_s(z_0^l(z_1^*)^k)(e_p) := q^{k(lp+s)} \prod_{m=1}^{l}(1-q^{2(lp+s-m)})^{1/2}\, e_{p-1} \ , \ p \geq 1 \, . \tag{5.1}$$

Note that the C^*-algebra $C(W_q(k,l))$ does not depend on k. As a consequence one has the following corollary due to Brzeziński and Fairfax, see [5, Cor. 5.3].

Corollary 5.4. *The K-groups of $C(W_q(k,l))$ are:*

$$K_0(C(W_q(k,l))) = \mathbb{Z}^{l+1} \, , \qquad K_1(C(W_q(k,l))) = 0 \, .$$

Notice that the K-theory groups of the quantum weighted projective lines do not agree with the K-theory groups of their commutative counterparts: In the commutative case, the K_0-group is given by $K_0(C(W(k,l))) = \mathbb{Z}^2$ independently of both weights k and l, see [1, Prop. 2.5].

Definition 5.5. The algebra of continuous functions on the *quantum* 3*-sphere* S^3_q is the universal enveloping C^*-algebra, $C(S^3_q)$, of the coordinate algebra $\mathcal{O}(S^3_q)$.

The (weighted) circle action $\{\sigma^{(k,l)}_w\}_{w\in S^1}$ on $\mathcal{O}(S^3_q)$ will be denoted simply by $\{\sigma_w\}_{w\in S^1}$. It induces a strongly continuous circle action on $C(S^3_q)$. We let $C(S^3_q)_{(0)}$ denote the fixed point algebra of this action.

Lemma 5.6. *The inclusion $\mathcal{O}(W_q(k,l)) \subseteq \mathcal{O}(S^3_q)$ induces an isomorphism of unital C^*-algebras,*

$$i : C(W_q(k,l)) \to C(S^3_q)_{(0)} .$$

Proof. Clearly, one has $\mathrm{Im}(i) \subseteq C(S^3_q)_{(0)}$ and $\mathrm{Im}(i)$ is dense by the argument used in the proof of Lemma 3.6.

It therefore suffices to show that $i : C(W_q(k,l)) \to C(S^3_q)$ is injective. To this end, consider the $*$-homomorphism

$$\pi := \oplus_{s=1}^{l} \pi_s : \mathcal{O}(W_q(k,l)) \to \mathscr{L}\big(\oplus_{s=1}^{l} l^2(\mathbb{N}_0)\big).$$

Then, by [5, Prop. 2.4] there exist a $*$-homomorphism $\rho : \mathcal{O}(S^3_q) \to \mathscr{L}(l^2(\mathbb{N}_0))$ and an isomorphism $\phi : \mathscr{L}\big(\oplus_{s=1}^{l} l^2(\mathbb{N}_0)\big) \to \mathscr{L}(l^2(\mathbb{N}_0))$ such that

$$\phi \circ \pi = \rho \circ i : \mathcal{O}(W_q(k,l)) \to \mathscr{L}(l^2(\mathbb{N}_0)) .$$

Let now $x \in \mathcal{O}(W_q(k,l))$. It follows from the above, that

$$\|x\| = \|\pi(x)\| = \|(\phi \circ \pi)(x)\| = \|(\rho \circ i)(x)\| \leq \|i(x)\| .$$

This proves that $i : C(W_q(k,l)) \to C(S^3_q)_{(0)}$ is an isometry and it is therefore injective. □

Let $\mathscr{L}^1$ denotes the trace class operators on the Hilbert space $l^2(\mathbb{N}_0)$.

Lemma 5.7. *The $*$-homomorphism $\pi := \oplus_{s=1}^{l} \pi_s : \mathcal{O}(W_q(k,l)) \to \widetilde{\oplus_{s=1}^{l} \mathscr{K}}$ factorizes through the unital $*$-subalgebra $\widetilde{\oplus_{s=1}^{l} \mathscr{L}^1}$.*

Proof. Let $s \in \{1,\ldots,l\}$. We only need to show that $\pi_s(z_0^l (z_1^*)^k), \pi_s(z_1 z_1^*) \in \mathscr{L}^1$. With notation $a := z_0^l (z_1^*)^k$ and $b := z_1 z_1^*$, the operator $\pi_s(b) : l^2(\mathbb{N}_0) \to l^2(\mathbb{N}_0)$ is positive and diagonal with eigenvalues $\{q^{2s}\, q^{2lp}\}_{p=0}^{\infty}$ each of multiplicity 1.

It is immediate to show that $\pi_s(b)^{1/2} \in \mathscr{L}^1$. Indeed, from (5.1),

$$\mathrm{Tr}(\pi_s(b)^{1/2}) = \sum_{p=0}^{\infty} q^s\, q^{lp} = q^s\, (1-q^l)^{-1} < \infty ,$$

having restricted the deformation parameter to $q \in (0,1)$. From $\pi_s(b)^{1/2} \in \mathscr{L}^1$ the inclusion $\pi_s(b) \in \mathscr{L}^1$ follows as well.

To obtain that $\pi_s(a) \in \mathscr{L}^1$ we need to verify that $|\pi_s(a)| \in \mathscr{L}^1$. Recall that

$$a^* a = b^k \cdot \prod_{m=1}^{l} (1 - q^{-2m} b) .$$

Using this relation, we may compute the absolute value:

$$|\pi_s(a)| = \pi_s(b)^{k/2} \cdot \Big(\prod_{m=1}^{l} (1 - q^{-2m} \pi_s(b)) \Big)^{1/2} .$$

Since $\mathscr{L}^1$ is an ideal in $\mathscr{L}(l^2(\mathbb{N}_0))$ we may thus conclude that $|\pi_s(a)| \in \mathscr{L}^1$. □

6. Quantum lens spaces

We define 3-dimensional quantum lens spaces $\mathcal{O}\big(L_q(dlk;k,l)\big)$ as fixed point algebras for the action of a finite cyclic group on the coordinate algebra of the quantum 3-sphere. We show that these spaces are quantum principal bundles over quantum weighted projective spaces. Our examples are more general than those of [5]. As said, the enveloping C^*-algebras of the lens spaces will be given as Pimsner algebras.

6.1. Coordinate algebras. Let $k, l \in \mathbb{N}$ be coprime positive integers. For each $d \in \mathbb{N}$ define the action of the cyclic group $\mathbb{Z}/(dlk)\mathbb{Z}$ on the quantum sphere S_q^3,

$$\alpha^{1/d} : \mathbb{Z}/(dlk)\mathbb{Z} \times \mathcal{O}(S_q^3) \to \mathcal{O}(S_q^3) ,$$

by letting on generators:

$$\alpha^{1/d}(1, z_0) := \exp\left(\frac{2\pi \mathrm{i}}{dl}\right) z_0 \qquad \text{and} \qquad \alpha^{1/d}(1, z_1) := \exp\left(\frac{2\pi \mathrm{i}}{dk}\right) z_1 . \quad (6.1)$$

Definition 6.1. The coordinate algebra for the *quantum lens space* $L_q(dlk;k,l)$ is the fixed point algebra of the action $\alpha^{1/d}$. This unital $*$-algebra is denoted by $\mathcal{O}\big(L_q(dlk;k,l)\big)$. Thus

$$\mathcal{O}\big(L_q(dlk;k,l)\big) := \big\{ x \in \mathcal{O}(S_q^3) \mid \alpha^{1/d}(1,x) = x \big\} .$$

The elements $z_0^l (z_1^*)^k$ and $z_1 z_1^*$, generating the weighted projective space algebra $\mathcal{O}(W_q(k,l))$, are clearly invariant leading, for any $d \in \mathbb{N}$, to an algebra inclusion

$$\mathcal{O}(W_q(k,l)) \hookrightarrow \mathcal{O}\big(L_q(dlk;k,l)\big) .$$

Next, for each $n \in \mathbb{N}_0$, consider the subspaces of $\mathcal{O}(S^3_q)$ given by

$$\begin{aligned}\mathcal{A}_{(n)}(k,l) &:= \sum_{j=0}^{n} (z_0^*)^{lj} (z_1^*)^{k(n-j)} \cdot \mathcal{O}(W_q(k,l))\,, \\ \mathcal{A}_{(-n)}(k,l) &:= \sum_{j=0}^{n} (z_0)^{lj} (z_1)^{k(n-j)} \cdot \mathcal{O}(W_q(k,l))\,.\end{aligned} \tag{6.2}$$

By construction these subspaces are in fact right-modules over $\mathcal{O}(W_q(k,l))$.

Recall that the algebra $\mathcal{O}(S^3_q)$ admits [23] a vector space basis given by the vectors $\{e_{p,r,s} \mid p \in \mathbb{Z},\ r,s \in \mathbb{N}_0\}$, where

$$e_{p,r,s} = \begin{cases} z_0^p z_1^r (z_1^*)^s & \text{for} \quad p \geq 0 \\ (z_0^*)^{-p} z_1^r (z_1^*)^s & \text{for} \quad p \leq 0 \end{cases}.$$

Lemma 6.2. *Let $n \in \mathbb{Z}$. It holds that*

$$\begin{aligned} e_{p,r,s} \in \mathcal{A}_{(n)}(k,l) &\Leftrightarrow pk + (r-s)l = -nkl \\ &\Leftrightarrow \sigma_w^{k,l}(e_{p,r,s}) = w^{-nkl} e_{p,r,s}\,, \ \forall\, w \in S^1\,. \end{aligned}$$

As a consequence, it holds that

$$x \in \mathcal{A}_{(n)}(k,l) \Leftrightarrow \sigma_w^{k,l}(x) = w^{-nkl} x\,, \ \forall\, w \in S^1\,.$$

Proof. Clearly one has that

$$\begin{aligned} e_{p,r,s} \in \mathcal{A}_{(n)}(k,l) &\Rightarrow pk + (r-s)l = -nkl \\ &\Leftrightarrow \sigma_w^{k,l}(e_{p,r,s}) = w^{-nkl} e_{p,r,s}\,, \ \forall\, w \in S^1\,. \end{aligned}$$

Thus, it only remains to prove the implication

$$pk + (r-s)l = -nkl \Rightarrow e_{p,r,s} \in \mathcal{A}_{(n)}(k,l)\,.$$

Then, suppose $pk+(r-s)l = -nkl$. Since $k,l \in \mathbb{N}$ are coprime there exists integers $d_0, d_1 \in \mathbb{Z}$ such that $p = d_0 l$ and $(r-s) = d_1 k$. Furthermore, $d_0 + d_1 = -n$. Suppose first that $(r-s)\,,\ p \geq 0$. Then,

$$e_{p,r,s} = z_0^p z_1^{(r-s)} (z_1 z_1^*)^s = z_0^{ld_0} z_1^{kd_1} (z_1 z_1^*)^s \in \mathcal{A}_{(-d_0-d_1)}(k,l) = \mathcal{A}_{(n)}(k,l)\,.$$

Suppose next that $p \geq 0$ and $(r-s) \leq 0$. Then,

$$e_{p,r,s} = z_0^p (z_1^*)^{s-r} (z_1 z_1^*)^r = z_0^{ld_0} (z_1^*)^{-d_1 k} (z_1 z_1^*)^r\,.$$

We now have two sub-cases: Either $d_0 \geq -d_1$ or $-d_1 \geq d_0$. When $d_0 \geq -d_1$, it follows from the above that

$$e_{p,r,s} = z_0^{l(d_0+d_1)} z_0^{-d_1 l} (z_1^*)^{-d_1 k} (z_1 z_1^*)^r \in \mathcal{A}_{(n)}(k,l)\,.$$

On the other hand, if $-d_1 \geq d_0$, we have that

$$e_{p,r,s} = z_0^{ld_0} (z_1^*)^{kd_0} (z_1^*)^{(-d_1 - d_0)k} (z_1 z_1^*)^r \in \mathscr{A}_{(n)}(k,l)\,.$$

The remaining two cases (when $p \leq 0$ and $(r-s) \geq 0$ and when $p\,,\ (r-s) \leq 0$) follow by similar arguments. This proves the lemma. □

Proposition 6.3. *The subspaces $\{\mathscr{A}_{(dn)}(k,l)\}_{n\in\mathbb{Z}}$ gives $\mathcal{O}(L_q(dlk;k,l))$ the structure of a $\mathbb{Z}$-graded unital $*$-algebra.*

Proof. We need to prove that the vector space sum provides a bijection

$$\oplus_{n\in\mathbb{Z}} \mathscr{A}_{(dn)}(k,l) \to \mathcal{O}(L_q(dlk;k,l))\,.$$

Suppose thus that $\sum_{n\in\mathbb{Z}} x_n = 0$ where $x_n \in \mathscr{A}_{(dn)}(k,l)$ for all $n \in \mathbb{Z}$ and $x_n = 0$ for all but finitely many $n \in \mathbb{Z}$. It then follows from Lemma 6.2 that the terms x_n lie in different homogeneous spaces for the circle action $\{\sigma_w^{k,l}\}_{w\in S^1}$ on $\mathcal{O}(S_q^3)$. We may then conclude that $x_n = 0$ for all $n \in \mathbb{Z}$. This proves the claimed injectivity.

Next, let $x \in \mathcal{O}(L_q(dlk;k,l))$. Without loss of generality we may take $x = e_{p,r,s}$ for some $p \in \mathbb{Z}$ and $r,s \in \mathbb{N}_0$. The fact that $x \in \mathcal{O}(L_q(dlk;k,l))$ then means that

$$p/(dl) + (r-s)/(dk) \in \mathbb{Z} \;\Leftrightarrow\; pk + (r-s)l \in (dkl)\,\mathbb{Z}\,.$$

It then follows from Lemma 6.2 that $e_{p,r,s} \in \sum_{n\in\mathbb{Z}} \mathscr{A}_{(dn)}(k,l)$. This proves surjectivity.

Finally, let $x \in \mathscr{A}_{(dn)}(k,l)$ and $y \in \mathscr{A}_{(dm)}(k,l)$. It only remains to prove that $x^* \in \mathscr{A}_{(-dn)}(k,l)$ and $xy \in \mathscr{A}_{(d(n+m))}(k,l)$. But these properties also follow immediately from Lemma 6.2 since $\sigma_w^{k,l}$ is a $*$-automorphism of $\mathcal{O}(S_q^3)$ for each $w \in S^1$. □

6.2. Lens spaces as quantum principal bundles. The right-modules $\mathscr{A}_{(1)}(k,l)$ and $\mathscr{A}_{(-1)}(k,l)$ play a central role. Recall from (6.2) that they are given by

$$\mathscr{A}_{(1)}(k,l) := (z_1^*)^k \cdot \mathcal{O}(W_q(k,l)) + (z_0^*)^l \cdot \mathcal{O}(W_q(k,l)) \qquad \text{and}$$
$$\mathscr{A}_{(-1)}(k,l) := z_1^k \cdot \mathcal{O}(W_q(k,l)) + z_0^l \cdot \mathcal{O}(W_q(k,l))\,.$$

Proposition 6.4. *There exist elements*

$$\xi_1, \xi_2, \beta_1, \beta_2 \in \mathscr{A}_{(1)}(k,l) \qquad \textit{and} \qquad \eta_1, \eta_2, \alpha_1, \alpha_2 \in \mathscr{A}_{(-1)}(k,l)$$

such that

$$\xi_1\eta_1 + \xi_2\eta_2 = 1 = \alpha_1\beta_1 + \alpha_2\beta_2\,.$$

Proof. Firstly, a repeated use of the defining relations of the algebra $\mathcal{O}(S_q^3)$ leads to

$$(z_0^*)^l z_0^l = \prod_{m=1}^{l} (1 - q^{-2m} z_1 z_1^*) .$$

Then, define the polynomial $F \in \mathbb{C}[X]$ by the formula

$$F(X) := \Big(1 - \prod_{m=1}^{l} (1 - q^{-2m} X)\Big)/X .$$

Since $z_1 z_1^* = z_1^* z_1$ one has that

$$(z_0^*)^l z_0^l + z_1^* \, F(z_1 z_1^*) \, z_1 = 1 .$$

In particular, this implies that

$$\begin{aligned} 1 &= \big((z_0^*)^l z_0^l + z_1^* \, F(z_1 z_1^*) \, z_1\big)^k = \sum_{j=0}^{k} \big((z_0^*)^l z_0^l\big)^j \, \big(z_1^* \, F(z_1 z_1^*) \, z_1\big)^{k-j} \binom{k}{j} \\ &= (z_1^*)^k \big(F(z_1 z_1^*)\big)^k z_1^k + \sum_{j=1}^{k} \big((z_0^*)^l z_0^l\big)^j \, \big(1 - (z_0^*)^l z_0^l\big)^{k-j} \binom{k}{j} \\ &= (z_1^*)^k \big(F(z_1 z_1^*)\big)^k z_1^k + (z_0^*)^l \left\{ \sum_{j=1}^{k} \big(z_0^l (z_0^*)^l\big)^{j-1} \big(1 - z_0^l (z_0^*)^l\big)^{k-j} \binom{k}{j} \right\} z_0^l . \end{aligned}$$

Define now the polynomial $G \in \mathbb{C}[X]$ by the formula

$$G(X) := (1 - (1 - X)^k)/X = \sum_{j=1}^{k} X^{j-1} (1 - X)^{k-j} \binom{k}{j} , \tag{6.3}$$

so that

$$\sum_{j=1}^{k} \big(z_0^l (z_0^*)^l\big)^{j-1} \big(1 - z_0^l (z_0^*)^l\big)^{k-j} \binom{k}{j} = G\big(z_0^l (z_0^*)^l\big) .$$

And this enables us to write the above identities as

$$1 = (z_1^*)^k \big(F(z_1 z_1^*)\big)^k z_1^k + (z_0^*)^l G\big(z_0^l (z_0^*)^l\big) z_0^l . \tag{6.4}$$

Notice that both $F(z_1 z_1^*)$ and $G\big(z_0^l (z_0^*)^l\big)$ belong to $\mathcal{O}(W_q(k,l))$. We thus define

$$\begin{aligned} \xi_1 &:= (z_1^*)^k \big(F(z_1 z_1^*)\big)^k , \qquad & \eta_1 &:= z_1^k , \\ \xi_2 &:= (z_0^*)^l \, G\big(z_0^l (z_0^*)^l\big) , \qquad & \eta_2 &:= z_0^l , \end{aligned}$$

and this proves the first half of the proposition.

To prove the second half, we consider instead the identity

$$z_0^l (z_0^*)^l = \prod_{m=0}^{l-1} (1 - q^{2m} z_1^* z_1) ,$$

which again follows by a repeated use of the defining identities for $\mathcal{O}(S_q^3)$.

The polynomial $\widetilde{F} \in \mathbb{C}[X]$ is now given by the formula

$$\widetilde{F}(X) := \Big(1 - \prod_{m=0}^{l-1} (1 - q^{2m} X)\Big)/X .$$

and we obtain that

$$z_0^l (z_0^*)^l + z_1 \tilde{F}(z_1 z_1^*) z_1^* = 1 .$$

By taking k^{th} powers and computing as above, this yields that

$$1 = z_1^k \big(\tilde{F}(z_1 z_1^*)\big)^k (z_1^*)^k + z_0^l \left\{ \sum_{j=1}^{k} \binom{k}{j} \big((z_0^*)^l z_0^l\big)^{j-1} \big(1 - (z_0^*)^l z_0^l\big)^{k-j} \right\} (z_0^*)^l .$$

This identity may be rewritten as

$$1 = z_1^k \big(\widetilde{F}(z_1 z_1^*)\big)^k (z_1^*)^k + z_0^l G\big((z_0^*)^l z_0^l\big) (z_0^*)^l ,$$

where $G \in \mathbb{C}[X]$ is again the one defined by (6.3).
Since both $\widetilde{F}(z_1 z_1^*)$ and $G\big((z_0^*)^l z_0^l\big)$ belong to $\mathcal{O}(W_q(k,l))$ we define

$$\alpha_1 := z_1^k \big(\widetilde{F}(z_1 z_1^*)\big)^k , \qquad \beta_1 := (z_1^*)^k ,$$
$$\alpha_2 := z_0^l \, G\big((z_0^*)^l z_0^l\big) , \qquad \beta_2 := (z_0^*)^l .$$

This ends the proof of the present proposition. □

The next proposition is now an immediate consequence of Proposition 6.3, Proposition 6.4, Theorem 4.3, and Proposition 4.6.

Proposition 6.5. *The triple* $\big(\mathcal{O}(L_q(dlk); k, l), \mathcal{O}(U(1)), \mathcal{O}(W_q(k,l))\big)$ *is a quantum principal* $U(1)$*-bundle for each* $d \in \mathbb{N}$.

6.3. C^*-completions. We fix $k, l \in \mathbb{N}$ to be coprime positive integers. Let $d \in \mathbb{N}$. With $C(S_q^3)$ the C^*-algebra of continuous functions on the quantum sphere S_q^3, the action of the cyclic group $\mathbb{Z}/(dlk)\mathbb{Z}$ given on generators in (6.1) results into an action

$$\alpha^{1/d} : \mathbb{Z}/(dkl)\mathbb{Z} \times C(S_q^3) \to C(S_q^3) .$$

Definition 6.6. The C^*-algebra of continuous functions on the *quantum lens space* $L_q(dlk;k,l)$ is the fixed point algebra of this action. It is denoted by $C(S_q^3)^{1/d}$. Thus

$$C(S_q^3)^{1/d} := \left\{x \in C(S_q^3) \mid \alpha^{1/d}(1,x) = x\right\}.$$

Lemma 6.7. *The C^*-quantum lens space $C(S_q^3)^{1/d}$ is the closure of the algebraic quantum lens space $\mathcal{O}(L_q(dkl;k,l))$ with respect to the universal C^*-norm on $\mathcal{O}(S_q^3)$.*

Proof. This follows by applying the bounded operator $E_{1/d} : C(S_q^3) \to C(S_q^3)^{1/d}$,

$$E_{1/d} : x \mapsto \frac{1}{dkl} \sum_{m=1}^{dkl} \alpha^{1/d}([m],x),$$

with $[m]$ denoting the residual class in $\mathbb{Z}/(dkl)\mathbb{Z}$ of the integer m. □

Alternatively, and in parallel with Definition 5.3, we could define the C^*-quantum lens space as the universal enveloping C^*-algebra of the algebraic quantum lens space $\mathcal{O}(L_q(dkl;k,l))$. We will denote this C^*-algebra by $C(L_q(dkl;k,l))$.

Lemma 6.8. *For all $d \in \mathbb{N}$, the identity map $\mathcal{O}(L_q(dkl;k,l)) \to \mathcal{O}(L_q(dkl;k,l))$ induces an isomorphisms of C^*-algebras,*

$$C(S_q^3)^{1/d} \simeq C(L_q(dkl;k,l)).$$

Proof. We use Theorem 3.10. Indeed, let $d \in \mathbb{N}$ and let $\|\cdot\| : \mathcal{O}(S_q^3) \to [0,\infty)$ and $\|\cdot\|' : \mathcal{O}(L_q(dkl;k,l)) \to [0,\infty)$ denote the universal C^*-norms of the two different unital $*$-algebras in question. We then have $\|x\| \leq \|x\|'$ for all $x \in \mathcal{O}(L_q(dkl;k,l))$ since the inclusion $\mathcal{O}(L_q(dkl;k,l)) \to \mathcal{O}(S_q^3)$ induce a $*$-homomorphism $C(L_q(dkl;k,l)) \to C(S_q^3)^{1/d}$. But we also have $\|x\|' \leq \|x\|$ since the restriction $\|\cdot\| : \mathcal{O}(W_q(k,l)) \to [0,\infty)$ is the maximal C^*-norm on $\mathcal{O}(W_q(k,l))$ by Lemma 5.6. □

From now on, to lighten the notation, denote by $B := C(W_q(k,l))$ the C^*-quantum weighted projective line. Furthermore, let E denote the Hilbert C^*-module over B obtained as the closure of the module $\mathscr{A}_{(1)}(k,l)$ in the universal C^*-norm on the quantum sphere $\mathcal{O}(S_q^3)$. As usual, we let $\phi : B \to \mathscr{L}(E)$ denote the $*$-homomorphism induced by the left multiplication $B \times C(S_q^3) \to C(S_q^3)$.

We are ready to realize the C^*-quantum lens spaces as Pimsner algebras.

Theorem 6.9. *For all $d \in \mathbb{N}$, there is an isomorphism of C^*-algebras,*

$$\mathcal{O}_{E^{\widehat{\otimes}_\phi d}} \simeq C(S_q^3)^{1/d},$$

given by

$$S_{\xi_1 \otimes \ldots \otimes \xi_d} \mapsto \xi_1 \cdot \ldots \cdot \xi_d \quad \text{for all} \quad \xi_1, \ldots, \xi_d \in E.$$

Proof. Recall from Proposition 6.3 that, for all $d \in \mathbb{N}$, it holds that

$$\mathcal{O}(L_q(dlk;k,l)) \simeq \oplus_{n\in\mathbb{Z}} \mathscr{A}_{(dn)}(k,l)\,.$$

Let us denote by $\{\rho_w\}_{w\in S^1}$ the associated circle action on $\mathcal{O}(L_q(dlk;k,l))$. Then, we have $\|\rho_w(x)\| \leq \|x\|$ for all $x \in \mathcal{O}(L_q(dlk;k,l))$ and all $w \in S^1$, where $\|\cdot\|$ is the norm on $C(S_q^3)^{1/d}$ (the restriction of the maximal C^*-norm on $C(S_q^3)$). To see this, choose a $z \in S^1$ such that $z^{dkl} = w$. Then $\sigma_z^{(k,l)}(x) = \rho_w(x)$, where the weighted circle action $\sigma^{(k,l)} : S^1 \times C(S_q^3) \to C(S_q^3)$ is the one defined at the beginning of §5.1.

An application of Theorem 3.9 now shows that

$$\mathcal{O}_{E^{\widehat{\otimes}_\phi d}} \simeq C(S_q^3)^{1/d}$$

for all $d \in \mathbb{N}$, provided that $\{\rho_w\}_{w\in S^1}$ satisfies the conditions of Assumption 3.1. To this end, taking into account the analysis of the coordinate algebra $\mathcal{O}(L_q(lk;k,l))$ provided in §6.1, the only non-trivial thing to check is that the collections

$$\langle E,E\rangle := \operatorname{span}\big\{\xi^*\eta \mid \xi,\eta \in E\big\} \qquad \text{and} \qquad \langle E^*,E^*\rangle := \operatorname{span}\big\{\xi\eta^* \mid \xi,\eta \in E\big\}$$

are dense in $C(W_q(k,l))$. But this follows at once from Proposition 6.4. □

7. KK-theory of quantum lens spaces

We now combine the results obtained until this point and, using methods coming from the Pimsner algebra constructions, we are able to compute the KK-theory of the quantum lens spaces $L_q(dkl;k,l)$ for any coprime $k,l \in \mathbb{N}$ and any $d \in \mathbb{N}$.

As before we let E denote the Hilbert C^*-module over the quantum weighted projective line $C(W_q(k,l))$ which is obtained as the closure of $\mathscr{A}_{(1)}(k,l)$ in $C(S_q^3)$.

The two polynomials in $\mathcal{O}(W_q(k,l))$ in the proof of Proposition 6.4, written as

$$\begin{aligned} (F(z_1z_1^*))^k &= \Big(\big(1-(z_0^*)^l z_0^l\big)/(z_1z_1^*)\Big)^k \qquad \text{and} \\ G\big(z_0^l(z_0^*)^l\big) &= \big(1-(1-z_0^l(z_0^*)^l)^k\big)/(z_0^l(z_0^*)^l)\,, \end{aligned}$$

are manifestly positive, since $\|z_1z_1^*\| \leq 1$ and thus also $\|z_0^l(z_0^*)^l\|, \|(z_0^*)^l z_0^l\| \leq 1$ in $C(W_q(k,l))$. Thus it makes sense to take their square roots:

$$\begin{aligned} \xi_1 &:= F(z_1z_1^*)^{k/2} = \Big(\big(1-(z_0^*)^l z_0^l\big)/(z_1z_1^*)\Big)^{k/2} \in C(W_q(k,l)) \qquad \text{and} \\ \xi_0 &:= G\big(z_0^l(z_0^*)^l\big)^{1/2} = \Big(\big(1-(1-z_0^l(z_0^*)^l)^k\big)/(z_0^l(z_0^*)^l)\Big)^{1/2} \in C(W_q(k,l))\,. \end{aligned}$$

Next, define the morphism of Hilbert C^*-modules $\Psi : E \to C(W_q(k,l))^2$ by

$$\Psi : \eta \mapsto \begin{pmatrix} \xi_1 z_1^k \, \eta \\ \xi_0 z_0^l \, \eta \end{pmatrix},$$

whose adjoint $\Psi^* : C(W_q(k,l))^2 \to E$ is then given by

$$\Psi^* : \begin{pmatrix} x \\ y \end{pmatrix} \mapsto (z_1^*)^k \xi_1 \, x + (z_0^*)^l \xi_0 \, y \,.$$

It then follows from (6.4) that $\Psi^*\Psi = \mathrm{id}_E$. The associated orthogonal projection is

$$P := \Psi\Psi^* = \begin{pmatrix} \xi_1 \, (z_1 z_1^*)^k \, \xi_1 & \xi_1 \, z_1^k (z_0^*)^l \, \xi_0 \\ \xi_0 \, z_0^l (z_1^*)^k \, \xi_1 & \xi_0 \, z_0^l (z_0^*)^l \, \xi_0 \end{pmatrix} \in M_2(C(W_q(k,l)))\,. \tag{7.1}$$

7.1. Fredholm modules over quantum weighted projective lines. We recall [7, Chap. IV] that an *even Fredholm module* over a $*$-algebra $\mathscr{A}$ is a datum (H, ρ, F, γ) where H is a Hilbert space of a representation ρ of $\mathscr{A}$, the operator F on H is such that $F^2 = F$ and $F^2 = 1$, with a $\mathbb{Z}/2\mathbb{Z}$-grading γ, $\gamma^2 = 1$, which commutes with the representation and such that $\gamma F + F\gamma = 0$. Finally, for all $a \in \mathscr{A}$ the commutator $[F, \rho(a)]$ is required to be compact. The Fredholm module is said to be 1-*summable* if the commutator $[F, \rho(a)]$ is trace class for all $a \in \mathscr{A}$.

Now, the quantum sphere S_q^3 is the 'underlying manifold' of the quantum group $\mathrm{SU}_q(2)$. The latter's counit when restricted to the subalgebra $\mathcal{O}(W_q(k,l))$ yields a one-dimensional representation $\varepsilon : \mathcal{O}(W_q(k,l)) \to \mathbb{C}$, given on generators by,

$$\varepsilon(z_1 z_1^*) = \varepsilon(z_0^l (z_1^*)^k) := 0\,, \quad \varepsilon(1) = 1\,.$$

Next, let $H := l^2(\mathbb{N}_0) \otimes \mathbb{C}^2$. We use the subscripts "+" and "−" to indicate that the corresponding spaces are thought of as being even or odd respectively, for a $\mathbb{Z}/2\mathbb{Z}$-grading γ: $H_\pm$ will be two copies of H. For each $s \in \{1, \ldots, l\}$, with the $*$-representation π_s given in (5.1), define the even $*$-homomorphism

$$\rho_s : \mathcal{O}(W_q(k,l)) \to \mathscr{L}\big(H_+ \oplus H_-\big)\,, \quad \rho_s : x \mapsto \begin{pmatrix} \pi_s(\Psi x \Psi^*) & 0 \\ 0 & \varepsilon(\Psi x \Psi^*) \end{pmatrix}.$$

We are slightly abusing notation here: the element $\Psi x \Psi^*$ is a 2×2 matrix, hence π_s and ε have to be applied component-wise. Next, define

$$F = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \gamma = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{7.2}$$

Lemma 7.1. *The datum $\mathscr{F}_s := \big(H_+ \oplus H_-, \rho_s, F, \gamma\big)$, defines an even* 1*-summable Fredholm module over the coordinate algebra $\mathcal{O}(W_q(k,l))$.*

Proof. It is enough to check that $\pi_s(\Psi z_1 z_1^* \Psi^*), \pi_s(\Psi z_0^l (z_1^*)^k \Psi^*) \in \mathscr{L}^1(H)$ and furthermore that $\pi_s(P) - \varepsilon(P) \in \mathscr{L}^1(H)$, for P the projection in (7.1).

That the two operators involving the generators $z_1 z_1^*$ and $z_0^l (z_1^*)^k$ lie in $\mathscr{L}^1(H)$ follows easily from Lemma 5.7. To see that $\pi_s(P) - \varepsilon(P) \in \mathscr{L}^1(H)$ note that

$$\varepsilon(P) = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} .$$

The desired inclusion then follows since Lemma 5.7 yields that the operators $\pi_s(z_1 z_1^*)^k$, $\pi_s(z_0^l (z_1^*)^k)$, and $\pi_s(1 - z_0^l (z_0^*)^l)$ are of trace class. □

For $s = 0$, we take

$$\rho_0 := \begin{pmatrix} \varepsilon & 0 \\ 0 & 0 \end{pmatrix} : C(W_q(k,l)) \to \mathscr{L}(\mathbb{C} \oplus \mathbb{C})$$

and define the even 1-summable Fredholm module

$$\mathscr{F}_0 := \big(\mathbb{C}_+ \oplus \mathbb{C}_-, \rho_0, F, \gamma\big) .$$

Remark 7.2. The 1-summable $l + 1$ Fredholm modules over $\mathcal{O}(W_q(k,l))$ we have defined are different from the 1-summable Fredholm modules defined in [5, §4]. The present Fredholm modules are obtained by "twisting" the Fredholm modules in [5] with the Hilbert C^*-module E.

7.2. Index pairings. Recall the representations π_s of $C(W_q(k,l))$ given in (5.1).

For each $r \in \{1, \dots, l\}$, let $p_r \in C(W_q(k,l))$ denote the orthogonal projection defined by the requirement

$$\pi_s(p_r) = \begin{cases} e_{00} & \text{for } s = r \\ 0 & \text{for } s \neq r \end{cases} , \tag{7.3}$$

where $e_{00} : l^2(\mathbb{N}_0) \to l^2(\mathbb{N}_0)$ denotes the orthogonal projection onto the closed subspace $\mathbb{C}e_0 \subseteq l^2(\mathbb{N}_0)$. For $r = 0$, let $p_0 = 1 \in C(W_q(k,l))$. The classes of these $l + 1$ projections $\{p_r,\ r = 0, 1, \dots, l\}$ form a basis for the group $K_0(C(W_q(k,l)))$ given in Corollary 5.4.

On the other hand we have the classes in the K-homology group $K^0(C(W_q(k,l)))$ represented by the even 1-summable Fredholm modules $\mathscr{F}_s$, $s = 0, \dots, l$, which we described in the previous paragraph.

We are interested in computing the index pairings

$$\langle [\mathscr{F}_s], [p_r] \rangle := \tfrac{1}{2} \operatorname{Tr}\big(\gamma F [F, \rho_s(p_r)]\big) \in \mathbb{Z} , \qquad \text{for } r, s \in \{0, \dots, l\} .$$

Proposition 7.3. *It holds that:*

$$\langle [\mathscr{F}_s], [p_r] \rangle = \begin{cases} 1 & \text{for } s = r \\ 1 & \text{for } r = 0 \\ 0 & \text{else} \end{cases} .$$

Proof. Suppose first that $r, s \in \{1, \ldots, l\}$. We then have:

$$\langle [\mathscr{F}_s], [p_r] \rangle = \mathrm{Tr}\big(\pi_s(\Psi p_r \Psi^*)\big) ,$$

and the above operator trace is well-defined since $\pi_s(\Psi p_r \Psi^*)$ is an orthogonal projection in $M_2(\mathscr{K})$ and it is therefore of trace class. We may then compute as follows:

$$\begin{aligned} \mathrm{Tr}\big(\pi_s(\Psi p_r \Psi^*)\big) &= \mathrm{Tr}\big(\pi_s(\xi_1 z_1^k p_r (z_1^*)^k \xi_1)\big) + \mathrm{Tr}\big(\pi_s(\xi_0 z_0^l p_r (z_0^*)^l \xi_0)\big) \\ &= \mathrm{Tr}\big(\pi_s(p_r (z_1^*)^k \xi_1^2 z_1^k)\big) + \mathrm{Tr}\big(\pi_s(p_r (z_0^*)^l \xi_0^2 z_0^l)\big) \\ &= \mathrm{Tr}\big(\pi_s(p_r)\big) = \delta_{sr} , \end{aligned}$$

where the second identity follows from [20, Cor. 3.8] and $\delta_{sr} \in \{0, 1\}$ denotes the Kronecker delta.

If $r \in \{1, \ldots, l\}$ and $s = 0$, then $\rho_0(p_r) = 0$ and thus $\langle [\mathscr{F}_0], [p_r] \rangle = 0$.

Next, suppose that $r = s = 0$. Then

$$\langle [\mathscr{F}_0], [p_0] \rangle = \mathrm{Tr} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = 1 .$$

Finally, suppose that $r = 0$ and $s \in \{1, \ldots, l\}$. We then compute

$$\begin{aligned} \langle [\mathscr{F}_s], [p_0] \rangle &= \mathrm{Tr}\big(\pi_s(P) - \varepsilon(P)\big) = \mathrm{Tr}\big(\pi_s(\xi_1^2 (z_1 z_1^*)^k)\big) + \mathrm{Tr}\big(\pi_s(\xi_0 z_0^l (z_0^*)^l \xi_0) - 1\big) \\ &= \mathrm{Tr}\big(\pi_s(1 - (z_0^*)^l z_0^l)^k\big) - \mathrm{Tr}\big(\pi_s(1 - z_0^l (z_0^*)^l)^k\big) . \end{aligned}$$

We will prove in the next lemma that this quantity is equal to 1. This will complete the proof of the present proposition. □

Lemma 7.4. *It holds that:*

$$\mathrm{Tr}\big(\pi_s(1 - (z_0^*)^l z_0^l)^k\big) - \mathrm{Tr}\big(\pi_s(1 - z_0^l (z_0^*)^l)^k\big) = \mathrm{Tr}\big(\pi_s([z_0^l, (z_0^*)^l])\big) = 1 .$$

Proof. Notice firstly that $\pi_s\big(1 - (z_0^*)^l z_0^l\big), \pi_s\big(1 - z_0^l (z_0^*)^l\big) \in \mathscr{L}^1(l^2(\mathbb{N}_0))$ by Lemma 5.7. It then follows by induction that

$$\mathrm{Tr}\big(\pi_s(1 - (z_0^*)^l z_0^l)^k\big) - \mathrm{Tr}\big(\pi_s(1 - z_0^l (z_0^*)^l)^k\big) = \mathrm{Tr}\big(\pi_s([z_0^l, (z_0^*)^l])\big) .$$

Indeed, with $x := z_0^l$, for all $j \in \{2, 3, \ldots\}$, one has that,

$$\begin{aligned}
&\mathrm{Tr}\big(\pi_s(1 - x^* x)^j\big) - \mathrm{Tr}\big(\pi_s(1 - x x^*)^j\big) \\
&\quad = \mathrm{Tr}\big(\pi_s(1 - x^* x)^{j-1}\big) - \mathrm{Tr}\big(\pi_s(x x^*(1 - x x^*)^{j-1})\big) - \mathrm{Tr}\big(\pi_s(1 - x x^*)^j\big) \\
&\quad = \mathrm{Tr}\big(\pi_s(1 - x^* x)^{j-1}\big) - \mathrm{Tr}\big(\pi_s(1 - x x^*)^{j-1}\big) .
\end{aligned}$$

It therefore suffices to show that $\mathrm{Tr}\big(\pi_s([z_0^l, (z_0^*)^l])\big) = 1$. Now, one has:

$$[z_0^l, (z_0^*)^l] = \sum_{m=0}^{l} (-1)^m q^{m(m-1)} \binom{l}{m}_{q^2} (1 - q^{-2ml})\, (z_1 z_1^*)^m$$

where the notation $\binom{l}{m}_{q^2}$ refers to the q^2-binomial coefficient, defined by the identity

$$\prod_{m=1}^{l} (1 + q^{2(m-1)} Y) = \sum_{m=0}^{l} q^{m(m-1)} \binom{l}{m}_{q^2} Y^m$$

in the polynomial algebra $\mathbb{C}[Y]$. Then, as in [5, Prop. 4.3] one computes:

$$\begin{aligned}
\mathrm{Tr}\big(\pi_s([z_0^l, (z_0^*)^l])\big) &= \sum_{m=1}^{l} (-1)^m q^{m(m-1)} \binom{l}{m}_{q^2} (1 - q^{-2ml}) \frac{q^{2ms}}{1 - q^{2ml}} \\
&= 1 - \sum_{m=0}^{l} (-1)^m q^{m(m-1)} \binom{l}{m}_{q^2} q^{2m(s-l)} \\
&= 1 - \prod_{m=1}^{l} (1 - q^{2(s-m)}) = 1 ,
\end{aligned}$$

since, due to $s \in \{1, \ldots, l\}$ one of the factors in the product must vanish. □

Remark 7.5. The non-vanishing of the pairings in Proposition 7.3 for $r = 0$ means that the class of the projection P in (7.1) is non-trivial in $K_0(C(W_q(k,l)))$. (In this case the pairings are computing the couplings of the Fredholm modules of [5, §4] with the projection P.) Geometrically this means that the line bundle $\mathscr{A}_{(1)}(k,l)$ over $\mathcal{O}(W_q(k,l))$ and as a consequence the quantum principal $U(1)$-bundle $\mathcal{O}(W_q(k,l))) \hookrightarrow \mathcal{O}(L_q(dlk); k,l)$ are non-trivial.

7.3. Gysin sequences. To ease the notation, we now let $C(W_q) := C(W_q(k,l))$ and $C(L_q(d)) := C(L_q(dkl; k,l))$. Also as before we let E denote the Hilbert C^*-module over $C(W_q)$ obtained as the closure of $\mathscr{A}_{(1)}(k,l)$ in $C(S_q^3)$. The $*$-homomorphism $\phi : C(W_q) \to \mathscr{L}(E)$ is induced by the product on $C(S_q^3)$.

For each $d \in \mathbb{N}$, let $[E^{\widehat{\otimes} d}] \in KK(C(W_q), C(W_q))$ denote the class of the Hilbert C^*-module $E^{\widehat{\otimes}_\phi d}$ as in Definition 2.5. Recall from Theorem 6.9 that the Pimnser algebra $\mathcal{O}_{E^{\widehat{\otimes}_\phi d}}$ can be identified with $C(L_q(d))$:

$$\mathcal{O}_{E^{\widehat{\otimes}_\phi d}} \simeq C(L_q(d)) .$$

Then, given any separable C^*-algebra B, by Theorem 2.7 we obtain two six term exact sequences:

$$\begin{array}{ccccc}
KK_0(B, C(W_q)) & \xrightarrow{1-[E^{\widehat{\otimes} d}]} & KK_0(B, C(W_q)) & \xrightarrow{i_*} & KK_0\big(B, C(L_q(d))\big) \\
{\scriptstyle [\partial]}\big\uparrow & & & & \big\downarrow{\scriptstyle [\partial]} \\
KK_1(B, C(L_q(d))) & \xleftarrow[i_*]{} & KK_1(B, C(W_q)) & \xleftarrow[1-[E^{\widehat{\otimes} d}]]{} & KK_1(B, C(W_q))
\end{array} \tag{7.4}$$

and

$$\begin{array}{ccccc}
KK_0(C(W_q), B) & \xleftarrow[1-[E^{\widehat{\otimes} d}]]{} & KK_0(C(W_q), B) & \xleftarrow[i^*]{} & KK_0\big(C(L_q(d)), B\big) \\
\big\downarrow{\scriptstyle [\partial]} & & & & {\scriptstyle [\partial]}\big\uparrow \\
KK_1\big(C(L_q(d)), B\big) & \xrightarrow{i^*} & KK_1(C(W_q), B) & \xrightarrow{1-[E^{\widehat{\otimes} d}]} & KK_1(C(W_q), B)
\end{array} . \tag{7.5}$$

We will refer to these two sequences as the *Gysin sequences* (in KK-theory) for the quantum lens space $L_q(dkl; k, l)$.

Remark 7.6. For $B = \mathbb{C}$, the first sequence above was first constructed in [2] for quantum lens spaces in any dimension n (and not just for $n = 1$) but with weights all equal to one; so that the 'base space' was a quantum projective space.

7.4. Computing the KK-theory of quantum lens spaces. We recall from [5, Prop. 5.1] that $C(W_q)$ is isomorphic to $\widetilde{\mathscr{K}^l}$ (see also §5.2). In particular, this means that $C(W_q)$ is KK-equivalent to $\mathbb{C}^{l+1}$.

To show this equivalence explicitly, for each $s \in \{0, \dots, l\}$ we define a KK-class $[\Pi_s] \in KK(C(W_q), \mathbb{C})$ via the Kasparov module $\Pi_s \in \mathbb{E}(C(W_q), \mathbb{C})$ given by:

$$\Pi_s := \big(l^2(\mathbb{N}_0)_+ \oplus l^2(\mathbb{N}_0)_-, \widetilde{\pi}_s, F, \gamma\big) \qquad \text{for } s \neq 0$$

and $$\Pi_0 := (\mathbb{C}, \varepsilon, 0) \qquad \text{for } s = 0 ,$$

with F and γ the canonical operators in (7.2). The representation is

$$\widetilde{\pi}_s = \begin{pmatrix} \pi_s & 0 \\ 0 & \varepsilon \end{pmatrix} ,$$

with the representation π_s given by (5.1) and ε is (induced by) the counit.

Furthermore, for each $r \in \{0, \ldots, l\}$ we define the KK-class $[I_r] \in KK(\mathbb{C}, C(W_q))$ by the Kasparov module

$$I_r := \big(C(W_q), i_r, 0\big) \in \mathbb{E}(\mathbb{C}, C(W_q)),$$

where $i_r : \mathbb{C} \to C(W_q)$ is the $*$-homomorphism defined by $i_r : 1 \mapsto p_r$ with the orthogonal projections $p_r \in C(W_q)$ given in (7.3).

Upon collecting these classes as

$$[\Pi] := \oplus_{s=0}^{l} [\Pi_s] \in KK(C(W_q), \mathbb{C}^{l+1}) \quad \text{and}$$
$$[I] := \oplus_{r=0}^{l} [I_r] \in KK(\mathbb{C}^{l+1}, C(W_q)),$$

it follows that $[I]\widehat{\otimes}_{C(W_q)}[\Pi] = [1_{\mathbb{C}^{l+1}}]$ and that $[\Pi]\widehat{\otimes}_{\mathbb{C}^{l+1}}[I] = [1_{C(W_q)}]$, from stability of KK-theory (see [4, Cor. 17.8.8]).

We need a final tensoring with the Hilbert C^*-module E. This yields a class

$$[I_r]\widehat{\otimes}_{C(W_q)}[E]\widehat{\otimes}_{C(W_q)}[\Pi_s] \in KK(\mathbb{C}, \mathbb{C}),$$

for each $s, r \in \{0, \ldots, l\}$. Then, we let $M_{sr} \in \mathbb{Z}$ denote the corresponding integer in $KK(\mathbb{C}, \mathbb{C}) \simeq \mathbb{Z}$, with $M := \{M_{sr}\}_{s,r=0}^{l} \in M_{l+1}(\mathbb{Z})$ the corresponding matrix.

As a consequence the six term exact sequence in (7.4) becomes

$$\begin{array}{ccccc} \oplus_{r=0}^{l} K^0(B) & \xrightarrow{1-M^d} & \oplus_{s=0}^{l} K^0(B) & \longrightarrow & KK_0\big(B, C(L_q(d))\big) \\ \uparrow & & & & \downarrow \\ KK_1(B, C(L_q(d))) & \longleftarrow & \oplus_{s=0}^{l} K^1(B) & \xleftarrow[1-M^d]{} & \oplus_{r=0}^{l} K^1(B) \end{array}, \qquad (7.6)$$

while, with $M^t \in M_{l+1}(\mathbb{Z})$ denoting the matrix transpose of $M \in M_{l+1}(\mathbb{Z})$, the six term exact sequence in (7.5) becomes

$$\begin{array}{ccccc} \oplus_{s=0}^{l} K_0(B) & \xleftarrow[1-(M^t)^d]{} & \oplus_{r=0}^{l} K_0(B) & \longleftarrow & KK_0\big(C(L_q(d)), B\big) \\ \downarrow & & & & \uparrow \\ KK_1\big(C(L_q(d)), B\big) & \longrightarrow & \oplus_{r=0}^{l} K_1(B) & \xrightarrow{1-(M^t)^d} & \oplus_{s=0}^{l} K_1(B) \end{array}. \qquad (7.7)$$

In order to proceed we therefore need to compute the matrix $M \in M_{l+1}(\mathbb{Z})$.

Lemma 7.7. *The Kasparov product* $[E]\widehat{\otimes}_{C(W_q)}[\Pi_s] \in KK(C(W_q), \mathbb{C})$ *is represented by the Fredholm module* $\mathscr{F}_s$ *in Lemma 7.1 for each* $s \in \{0, \ldots, l\}$.

Proof. Recall firstly that the class $[E] \in KK(C(W_q), C(W_q))$ is represented by the Kasparov module

$$\big(E, \phi, 0\big) \in \mathbb{E}(C(W_q), C(W_q)),$$

where $\phi : C(W_q) \to \mathscr{L}(E)$ is induced by the product on the algebra $C(S_q^3)$.

It then follows from the observations in the beginning of §7 that $(E, \phi, 0)$ is equivalent to the Kasparov module

$$\big(C(W_q)^2, \Psi\phi\Psi^*, 0\big) \in \mathbb{E}(C(W_q), C(W_q)) .$$

Suppose next that $s = 0$. The Kasparov product $[E]\widehat{\otimes}_{C(W_q)}[\Pi_0]$ is then represented by the Kasparov module

$$\big(C(W_q)^2\widehat{\otimes}_\varepsilon\mathbb{C}, \Psi\phi\Psi^* \otimes 1, 0\big) \in \mathbb{E}(C(W_q), \mathbb{C}) ,$$

which is equivalent to the Kasparov module

$$\left(\mathbb{C}_+ \oplus \mathbb{C}_- , \begin{pmatrix} \varepsilon & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\right) .$$

This proves the claim of the lemma in this case.

Suppose thus that $s \in \{1, \dots, l\}$. The Kasparov product $[E]\widehat{\otimes}_{C(W_q)}[\Pi_s]$ is then represented by the Kasparov module given by the $\mathbb{Z}/2\mathbb{Z}$-graded Hilbert space

$$\big(C(W_q)^2\widehat{\otimes}_{\pi_s} l^2(\mathbb{N}_0)\big)_+ \oplus \big(C(W_q)^2\widehat{\otimes}_\varepsilon l^2(\mathbb{N}_0)\big)_- \simeq H_+ \oplus H_-$$

with associated $*$-homomorphism

$$\rho_s = \begin{pmatrix} \pi_s(\Psi\phi\Psi^*) & 0 \\ 0 & \varepsilon(\Psi\phi\Psi^*) \end{pmatrix} : C(W_q) \to \mathscr{L}\big(H_+ \oplus H_-\big) ,$$

and with Fredholm operator F and grading γ the canonical ones in (7.2). This proves the claim of the lemma in these cases as well. □

The results of Lemma 7.7 and Proposition 7.3 now yield the following:

Proposition 7.8. *The matrix $M = \{M_{sr}\} \in M_{l+1}(\mathbb{Z})$ has entries*

$$M_{sr} = \langle[\mathscr{F}_s], [I_r]\rangle = \begin{cases} 1 & \text{for } s = r \\ 1 & \text{for } r = 0 \\ 0 & \text{else} \end{cases} .$$

A combination of Proposition 7.8 and the six term exact sequences in (7.6) and (7.7) then allows us to compute the K-theory and the K-homology of the quantum lens space $L_q(dlk; k, l)$ for all $d \in \mathbb{N}$.

When $B = \mathbb{C}$, the sequence in (7.6) reduces to

$$0 \longrightarrow K_1(C(L_q(d)) \longrightarrow \mathbb{Z}^{l+1} \xrightarrow{1-M^d} \mathbb{Z}^{l+1} \longrightarrow K_0(C(L_q(d)) \longrightarrow 0$$

while the one in (7.7) becomes

$$0 \longleftarrow K^1(C(L_q(d)) \longleftarrow \mathbb{Z}^{l+1} \xleftarrow{1-(M^t)^d} \mathbb{Z}^{l+1} \longleftarrow K^0(C(L_q(d)) \longleftarrow 0 .$$

Let us use the notation $\iota : \mathbb{Z} \to \mathbb{Z}^l$, $1 \mapsto (1, \dots, 1)$ for the diagonal inclusion and let $\iota^t : \mathbb{Z}^l \to \mathbb{Z}$ denote the transpose, $\iota^t : (m_1, \dots, m_l) \mapsto m_1 + \dots + m_l$.

Theorem 7.9. *Let $k, l \in \mathbb{N}$ be coprime and let $d \in \mathbb{N}$. Then*

$$K_0\big(C(L_q(dlk;k,l))\big) \simeq \mathrm{Coker}(1-M^d) \simeq \mathbb{Z} \oplus \big(\mathbb{Z}^l/\mathrm{Im}(d\cdot\iota)\big)$$
$$K_1\big(C(L_q(dlk;k,l))\big) \simeq \mathrm{Ker}(1-M^d) \simeq \mathbb{Z}^l$$

and

$$K^0\big(C(L_q(dlk;k,l))\big) \simeq \mathrm{Ker}(1-(M^t)^d) \simeq \mathbb{Z} \oplus \big(\mathrm{Ker}(\iota^t)\big)$$
$$K^1\big(C(L_q(dlk;k,l))\big) \simeq \mathrm{Coker}(1-(M^t)^d) \simeq \mathbb{Z}/(d\mathbb{Z}) \oplus \mathbb{Z}^l\,.$$

We finish by stressing that the results on the K-theory and K-homology of the lens spaces $L_q(dlk;k,l)$ are different from the ones obtained for instance in [13]. In fact our lens spaces are not included in the class of lens spaces considered there. Thus, for the moment, there seems to be no alternative method which results in a computation of the KK-groups of these spaces.

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