J. Noncommut. Geom. 10 (2016), 207–219
DOI 10.4171/JNCG/231

Journal of Noncommutative Geometry

On the Baum–Connes conjecture for Gromov monster groups

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Abstract. We present a geometric approach to the Baum–Connes conjecture with coefficients for Gromov monster groups via a theorem of Khoskham and Skandalis. Secondly, we use recent results concerning the a-T-menability at infinity of large girth expanders to exhibit a family of coefficients for a Gromov monster group for which the Baum–Connes conjecture is an isomorphism.

Mathematics Subject Classification (2010). 22A22, 46L80; 20F65, 19K56, 05C80.

Keywords. Expander graphs, non-exact groups, topological groupoids, Baum–Connes conjecture.

1. Introduction

In [7], Gromov introduced the first examples of non-exact groups. The idea behind the construction involves taking spaces that do not have Yu's Property A, such as sequences of expander graphs, and embedding them into a group using small cancellation theory. More recently, Osajda [11] constructed further examples of groups using more refined small cancellation techniques. Osajda's examples are interesting as the expander sequence used in the construction is coarsely embedded: these groups will be the focus of this paper.

The particular types of space that appear in the constructions of both Gromov and Osajda come from sequences of finite graphs that have the property that their girth, the length of the shortest simple cycle, tends to infinity in throughout the family. The coarse geometry of this type of sequence is well understood: it is known that expander sequences of large girth do not satisfy the coarse Baum–Connes assembly conjecture, and in particular their coarse Baum–Connes assembly map is known to be injective but not surjective - this is a result of Willett and Yu, from [16]. This result was proved again by the author in [6] using groupoid techniques.

The groupoid techniques used in [6] were first introduced by Skandalis, Tu and Yu in [14], where it was shown that for any uniformly discrete metric space of bounded geometry X, there is a groupoid, denoted $G(X)$, that encodes the coarse Baum–Connes assembly map of X as a groupoid Baum–Connes assembly map for $G(X)$.

One benefit of this approach is that the groupoid $G(X)$ can often be easier to study as it comes with a large toolkit of operator algebraic and K-theoretic results concerning the Baum–Connes conjecture that can be readily accessed: for example, the work of Tu [15] on a-T-menability significant and quite relevant to both this paper, to [6] and to the original results of Skandalis, Tu and Yu from [14].

The main idea we use here is quite simple: a uniformly discrete metric space with bounded geometry X that is coarsely embedded into a finitely generated group Γ should inherit a large proportion of the geometry of the group. We illustrate this by constructing, from the group structure, a collection of maps that both generate the metric structure and control the coarse groupoid $G(X)$ in this case.

Armed with these maps, we appeal to results of Skandalis and Khoshkam [10] to produce a Morita equivalence between the coarse groupoid $G(X)$ and a transformation groupoid involving the discrete group Γ that generated the family of maps. This provides a direct method of converting statements about the groupoid $G(X)$ into statements concerning actions of the discrete group Γ. In particular, we prove:

Theorem 1.1. *Let Γ be a finitely generated discrete group that contains a coarsely embedded large girth expander X of uniformly bounded vertex degree. Then there is a locally compact Γ-space $\Omega_{\beta X}$ such that $G(X)$ is Morita equivalent to $\Omega_{\beta X} \rtimes \Gamma$.*

We use this result to derive some results of Willett and Yu (see Corollary 1.7 in [16] for the relevant statement) concerning the assembly map with coefficients for groups that coarsely contain large girth expanders:

Theorem 1.2. *Let Γ be a group satisfying the hypothesis of the Theorem above. Then the Baum–Connes conjecture for Γ with coefficients in $C_0(\Omega_{\beta X})$ fails to be a surjection, but is an injection.*

By introducing a reduction of the coarse groupoid $G(X)$ known as the boundary groupoid (denoted in this text by $G(X)|_{\partial\beta X}$), and by proving a permanence result for groupoid reductions (see Lemma 3.9 in the text), we are able to improve this to obtain positive results for another family of coefficients:

Theorem 1.3. *Let Γ be a finitely generated discrete group that contains a coarsely embedded large girth expander X of uniformly bounded vertex degree. Then there is a locally compact Hausdorff Γ-space $\Omega_{\partial\beta X}$ such that the boundary groupoid $G(X)|_{\partial\beta X}$ is Morita equivalent to $\Omega_{\partial\beta X} \rtimes \Gamma$. Moreover this groupoid is a-T-menable.*

Using results of Tu concerning the Baum–Connes conjecture for a-T-menable groupoids from [15] we conclude:

Theorem 1.4. *Let Γ satisfy the hypothesis of the previous Theorem. Then the Baum–Connes conjecture with coefficients in any $(\Omega_{\partial\beta X} \rtimes \Gamma)$-$C^*$-algebra is an isomorphism.*

This result can also be obtained from recent results of Baum, Guentner and Willett as a consequence of their work on exact C^*-crossed products (see Theorem 7.8 in [1]).

As a additional consequence of the techniques in this paper, it is possible to provide another proof that groups that coarsely contain large girth expander sequences are not K-exact (this can be shown using the techniques first introduced by Higson, Lafforgue and Skandalis in Section 7 of [9]). Whilst this proof is different, the ideas are essentially present in [9] and we include this here for completeness.

As mentioned earlier by using a groupoid approach (instead of to jumping right to the operator algebraic picture) the simple geometric and topological arguments in this paper allow us to apply many results from the established literature surrounding the Baum–Connes assembly conjecture for groupoids. This is a strength as it reduces the arguments to homological algebra, but also a weakness as it relies on the results of Tu concerning groupoid Baum–Connes conjecture.

In the following two sections, we describe some basic facts about étale groupoids and their topology, define the coarse groupoid of Skandalis, Tu and Yu [14] and the Morita equivalence result of Khoshkam and Skandalis [10]. We then show that this result applies to the coarse groupoid $G(X)$ in the particular situation that the metric space X is coarsely embedded into a finitely generated discrete group Γ. The final sections of the paper then give proofs of the previously mentioned Baum–Connes statements.

2. The coarse groupoid

In this section we recall some fundamental notions on groupoids, as well as outlining the construction of the coarse groupoid first given by Skandalis, Tu and Yu in [14]. We will always use $\mathcal{G}$ to denote a abstract small groupoid.

A groupoid $\mathcal{G}$ is a *topological groupoid* if both $\mathcal{G}$ and $\mathcal{G}^{(0)}$ are topological spaces, and the maps $r, s, ^{-1}$ and the composition are all continuous. A Hausdorff, locally compact topological groupoid $\mathcal{G}$ is *proper* if (r, s) is a proper map, *principal* if (r, s) is injective and *étale* or *r-discrete* if the map r is a local homeomorphism. When $\mathcal{G}$ is étale, s and the product are also local homeomorphisms.

As they will be necessary later, we collect here some particular facts (with references) concerning the topology of a étale groupoid. Recall that a *slice* is a subset $U \subset \mathcal{G}$ on which both the source and range maps are injective.

Proposition 2.1. *Let $\mathcal{G}$ be a locally compact Hausdorff étale groupoid. Then:*

- *$\mathcal{G}^{(0)}$ is open in $\mathcal{G}$ (Proposition 3.2 in [3])*
- *The set of open slices forms a basis of the topology for $\mathcal{G}$ (Proposition 3.5 in [3])*
- *If, in addition, $\mathcal{G}^{(0)}$ is totally disconnected then the set of clopen slices of $\mathcal{G}$ form a basis for the topology of $\mathcal{G}$ (Proposition 4.1 in [4]).*

We now make precise the class of metric spaces that we will study.

Definition 2.2. Let X be a metric space. Then X is said to be *uniformly discrete* if there exists $c > 0$ such that for every pair of distinct points $x \neq y \in X$ the distance $d(x, y) > c$. Additionally, X is said have *bounded geometry* if for every $R > 0$ there exists $N_R > 0$ such that for every $x \in X$ the cardinality of the ball of radius R about x is smaller than N_R.

Let X be a uniformly discrete bounded geometry metric space. We want to define a groupoid with the property that it captures the coarse information associated to X. To do this effectively we need to define what we mean by a *coarse structure* that is associated to a metric. The details of this can be found in Chapter 2 of [13].

Definition 2.3. Let X be a set and let $\mathcal{E}$ be a collection of subsets of $X \times X$. If $\mathcal{E}$ has the following properties:

(1) $\mathcal{E}$ is closed under finite unions;

(2) $\mathcal{E}$ is closed under taking subsets;

(3) $\mathcal{E}$ is closed under the induced product and inverse that comes from the pair groupoid product on $X \times X$;

(4) $\mathcal{E}$ contains the diagonal.

Then we say $\mathcal{E}$ is a *coarse structure* on X and we call the elements of $\mathcal{E}$ *entourages*. If in addition $\mathcal{E}$ contains all finite subsets then we say that $\mathcal{E}$ is *weakly connected*.

Example 2.4. Let X be a metric space. Then consider the collection $\mathcal{S}$ of the R-neighbourhoods of the diagonal in $X \times X$; that is, for every $R > 0$ the set:

$$\Delta_R = \{(x, y) \in X \times X | d(x, y) \leq R\}$$

Let $\mathcal{E}$ be the coarse structure generated by $\mathcal{S}$. This is called the *metric coarse structure* on X. If X is a uniformly discrete metric space of bounded geometry then this coarse structure is uniformly locally finite, proper and weakly connected - that is this coarse structure is of the type studied in [14].

Let X be a uniformly discrete metric space with bounded geometry. We denote by βX the Stone–Čech compactification of X (similarly with $\beta(X \times X)$).

Define $G(X) := \bigcup_{R>0} \overline{\Delta_R} \subseteq \beta(X \times X)$[1]. Then $G(X)$ is a locally compact, Hausdorff topological space. To equip it with a product and inverse we would ideally consider the natural extension of the pair groupoid product on $\beta X \times \beta X$. We remark that the map (r, s) from $X \times X$ extends first to an inclusion into $\beta X \times \beta X$ and universally to $\beta(X \times X)$, giving a map $\overline{(r, s)} : \beta(X \times X) \to \beta X \times \beta X$. We can restrict this map to each entourage $E \in \mathcal{E}$ allowing us to map the set $G(X)$ to $\beta X \times \beta X$. The following is Lemma 2.7b) from [14]:

[1]Equally, we could have used the definition $G(X) := \bigcup_{E \in \mathcal{E}} \overline{E}$, where $\mathcal{E}$ is the metric coarse structure and the closure operation takes place in the topology of $\beta(X \times X)$.

Lemma 2.5. *Let X be a uniformly discrete bounded geometry metric space, let E be any entourage from the metric coarse structure on X and let $\overline{E}$ its closure in $\beta(X \times X)$. Then the inclusion $E \to X \times X$ extends to a topological embedding $\overline{E} \to \beta X \times \beta X$ via $\overline{(r, s)}$.* □

Using this Lemma, we can conclude that the pair groupoid operations on $\beta X \times \beta X$ restrict to give continuous operations on $G(X)$ and we equip $G(X)$ with this induced product and inverse.

Additionally, as βX is totally disconnected, it follows from point 3) in Proposition 2.1 that a basis for the topology of $G(X)$ is formed of clopen slices.

2.1. Different generators. Let X be a uniformly discrete metric space with bounded geometry. In this section we provide a different generating set for the metric coarse structure of X by supposing that it is coarsely embedded into a discrete group Γ.

Definition 2.6. A *partial translation* of a metric space X is a bijection between subsets of X whose graph is a controlled subset in the metric coarse structure: a map: $t : U \to V$ (for $U, V \subseteq X$) is a partial translation if and only if there exists $R > 0$ such that $d(x, t(x)) \leq R$ for every $x \in U$.

This concept is important from the perspective of coarse structures on metric spaces as every entourage of the metric coarse structure is contained in the graphs of finitely many partial translations (see Lemma 4.10 in [13]). In the situation that the metric space X is coarsely embedded into a finitely generated discrete group Γ we can obtain, through restriction, a natural collection of partial translations that generate the metric coarse structure.

Proposition 2.7. *Let Γ be a finitely generated discrete group with a left invariant word metric and let $f : X \to \Gamma$ be an injective coarse embedding. Then the restriction of the right multiplication action of Γ on itself to X generates the metric coarse structure of X.*

Proof. We identify X as a subset of the Cayley graph of Γ with the induced metric, which is of the same coarse type as the original metric of X because f is a coarse embedding.

Consider the right action of Γ on itself as a collection of bijections of Γ defined by the formula:

$$t_g : \Gamma \to \Gamma, x \mapsto xg^{-1}$$

We can now restrict these maps to X, where they may not be defined everywhere. If for each $g \in \Gamma$ we denote the set of points in X with image under t_g also in X by D_g then we have:

$$t_g : D_g \to D_{g^{-1}}, x \mapsto xg^{-1}$$

Let $\mathcal{T}_X$ denote the collection of t_g restricted to X. These maps are partial translations of X: for every $g \in \Gamma$, the map t_g moves elements of X at most the length of g as

$d(x, t_g(x)) = d(x, xg^{-1}) = d(e, g^{-1})$, where the last inequality follows from the left invariance of the metric.

To see that they generate the metric coarse structure, observe that as the action of Γ on itself is transitive we have, for every $R > 0$, that every pair $(x, y) \in \Delta_R$ can be written as (x, xg^{-1}), where the length of g is at most R. Thus, we have the decomposition:

$$\Delta_R = \bigsqcup_{|g| \leq R} \{(x, t_g(x)) | x \in X\} = \bigsqcup_{|g| \leq R} gr(t_g)$$

where $gr(t_g)$ is the graph of the map t_g and $|.|$ represents the length of the element of g in the metric of Γ. □

Restrictions of this type appear in the literature as a example of a *partial translation structure* (see for instance the work in [2] for a good introduction).

3. The Morita equivalence results of Khoshkam and Skandalis

In this section we outline the technique that is used by Khoshkam and Skandalis in [10] to construct Morita equivalences between general groupoids and transformation groupoids arising from group actions on topological spaces. The central piece of notation we require is the following:

Definition 3.1. Let $\mathcal{G}$ be a locally compact groupoid and let Γ be a discrete group. A homomorphism of groupoids $\mathcal{G} \to \Gamma$ will be called a *cocycle*.

The main idea of [10] is that strong properties of cocycles determine Morita equivalences.

Definition 3.2. Let $\rho : \mathcal{G} \to \Gamma$ be a cocycle. We say it is:

(1) *transverse* if the map $\Gamma \times \mathcal{G} \to \Gamma \times \mathcal{G}^{(0)}$, $(g, \gamma) \mapsto (g\rho(\gamma), s(\gamma))$ is open.

(2) *closed* if the map $\gamma \mapsto ((r(\gamma), \rho(\gamma), s(\gamma))$ is closed.

(3) *faithful* if the map $\gamma \mapsto ((r(\gamma), \rho(\gamma), s(\gamma))$ is injective.

(4) *(T,C,F)* if it satisfies properties 1,2 and 3.

Remark 3.3. If Γ is a discrete group then to prove that a cocycle ρ is transverse it is enough to check if the map $\gamma \mapsto (\rho(\gamma), s(\gamma)$ is open [10].

Following [10] we can construct a locally compact Hausdorff Γ-space Ω from a (T,C,F)-cocycle in the following manner. Considering the space $\mathcal{G}^{(0)} \times \Gamma$ equipped with the product topology we define a relation $\sim$ on $\mathcal{G}^{(0)} \times \Gamma$: $(x, g) \sim (y, h)$ if there exists $\gamma \in \mathcal{G}$ with $s(\gamma) = x, r(\gamma) = y$ and $\rho(\gamma) = h^{-1}g$.

Denote the quotient of $\mathcal{G}^{(0)} \times \Gamma$ under $\sim$, equipped with the quotient topology, by Ω. The closed condition on the cocycle makes this space Hausdorff.

As it will become relevent to keep track of the unit space in this construction, we will denote the space Ω defined above by $\Omega_{\mathcal{G}^{(0)}}$ whenever there is ambiguity.

The main result of Khoskham and Skandalis [10] we require is the following:

Theorem 3.4. *Let $\rho : \mathcal{G} \to \Gamma$ be a continuous, faithful, closed, transverse cocycle. Then the space Ω defined above is a locally compact Hausdorff space and there is a Morita equivalence of $\mathcal{G}$ with $\Omega \rtimes \Gamma$.* □

Our aim will be to use this result for the groupoid $G(X)$ when X is a space that satisfies the hypothesis of Proposition 2.7.

3.1. Cocycles for G(X). Suppose we are in the situation described by Proposition 2.7, that is we have a uniformly discrete bounded geometry metric space X that is coarsely embedded into a finitely generated discrete group Γ. Then we can construct a cocycle from $G(X)$ to Γ using the natural decomposition $G(X) = \cup_{g\in\Gamma}\overline{gr(t_g)}$, that maps any element $\gamma \in G(X)$ first to a pair $(\omega, t_g(\omega)) \in \beta X \times \beta X$, and then maps this pair to $g \in \Gamma$. Denote this map by ρ.

Theorem 3.5. *Let X be a uniformly discrete metric space of bounded geometry and let $X \hookrightarrow \Gamma$ be a coarse embedding of X into a finitely generated discrete group Γ equipped with its usual left invariant word metric. Then the map ρ from the coarse groupoid $G(X)$ to the group Γ is (T,C,F).*

Proof. Let $\gamma \in G(X)$, $g, h \in \Gamma$. We have to check the three properties:

- Transverse: We must show that the map $(g, \gamma) \mapsto (g\rho(\gamma), s(\gamma))$ is open, and it is enough to check this on a basis elements, which we know from from the definition of the product topology and Proposition 2.1 is made up of sets of the form $\{g\} \times U$, where U is a clopen slice of $G(X)$. This maps to the set $\{g\rho(\gamma)\}_{\gamma\in U} \times s(U)$ under the map above. However this set is open as the set $s(U)$ is homeomorphic to U and is open in βX.
- Closed: We must show that the map $(r, \rho, s) : G(X) \to \beta X \times \Gamma \times \beta X$ is closed. Again, it is enough to check this on a basis for $G(X)$, which we know from Proposition 2.1 consists of clopen slices. Let $V \subset G(X)$ be a clopen slice. The conclusion follows as $s(V)$, $r(V)$ are closed in βX as they are homeomorphic to V, and as every subset of Γ is closed the desired result follows.
- Faithful: Suppose for a contradiction that $(r, \rho, s)(\gamma) = (r, \rho, s)(\gamma^{'})$, but that $\gamma \neq \gamma^{'}$. As $G(X)$ is principal, we know that γ and $\gamma^{'}$ are composable (because γ and $\gamma^{'}$ have the same source and range) and thus $\rho(\gamma^{-1}\gamma^{'})$ is not the identity in γ. This contradicts our initial assumption that $\rho(\gamma) = \rho(\gamma^{'})$ as ρ is a groupoid homomorphism. □

From Theorem 3.4 we obtain the following Corollary:

Corollary 3.6. *If X satisfies the hypothesis of Theorem 3.5, then there is a locally compact, Hausdorff Γ-space $\Omega_{\beta X}$ such that the groupoid $G(X)$ is Morita equivalent to the transformation groupoid $\Omega_{\beta X} \rtimes \Gamma$.* □

3.2. A permanence result for groupoid reductions. In this section we show that the property of having a (T,C,F)-cocycle to a group passes to groupoid reduction, which is a relevant step in using these techniques in combination with homological properties of the groupoid Baum–Connes assembly map.

Definition 3.7. A subset of $F \subseteq \mathcal{G}^{(0)}$ is said to be *saturated* if for every element of $\gamma \in \mathcal{G}$ with $s(\gamma) \in F$ we have $r(\gamma) \in F$. For such a subset we can form subgroupoid of $\mathcal{G}$, denoted by $\mathcal{G}_F$ which has unit space F and $\mathcal{G}_F^{(2)} = \{\gamma \in \mathcal{G} | s(\gamma) \text{ and } r(\gamma) \in F\}$.

We remark also that this concept is sometimes called *invariance* in the groupoid literature.

Lemma 3.8. *Let $\mathcal{G}$ be an étale locally compact Hausdorff groupoid with a (T,C,F)-cocycle ρ to Γ. Then relation $\sim$ on $\mathcal{G}^{(0)} \times \Gamma$ preserves saturated subsets of $\mathcal{G}^{(0)}$*

Proof. Let U be a saturated subset of $\mathcal{G}^{(0)}$ and let $x \in U$, $y \in U^c$. Assume for a contradiction that $(x, g) \sim (y, h)$ in $\mathcal{G}^{(0)} \times \Gamma$. Then there exists a $\gamma \in \mathcal{G}$ such that $s(\gamma) = x$, $r(\gamma) = y$ and $\rho(\gamma) = g^{-1}h$, but as U is saturated no such γ exists. □

Now we can show how (T,C,F)-cocycles interact with groupoid reduction.

Lemma 3.9. *Let $\mathcal{G}$ be an étale locally compact Hausdorff groupoid and let F be a closed saturated subset of $\mathcal{G}^{(0)}$. If $\mathcal{G}$ admits a (T,C,F)-cocycle ρ onto a discrete group Γ then so do $\mathcal{G}_F$ and $\mathcal{G}_{F^c}$.*

Proof. Observe that $\mathcal{G}_F$ is a closed subgroupoid of $\mathcal{G}$ and $\mathcal{G}_{F^c}$ is its open complement. We consider them as topological groupoids in their own right using the subspace topology. We now check that these topologies are compatible with the subspace topologies in the appropriate places in the definition of (T,C,F).

1. (1) Transverse: It is enough to show that $\{(\rho(\gamma), s(\gamma)) : \gamma \in \mathcal{G}_{F^c}\}$ is open; this follows as it is precisely the intersection of $\{(\rho(\gamma), s(\gamma)) : \gamma \in \mathcal{G}\}$ with $\Gamma \times F^c$. The same holds for $\mathcal{G}_F$.
2. (2) Closed: We must show $P : \gamma \mapsto (r(\gamma), \rho(\gamma), s(\gamma))$ is closed. Let V be a closed subset of $\mathcal{G}_{F^c}$. Then, as $\mathcal{G}_{F^c}$ is equipped with the subspace topology, there is a V' in $\mathcal{G}$ that is closed and such that $V = V' \cap \mathcal{G}^{F^c}$. Now by saturation, we can conclude $P(V) = P(V') \cap (F^c \times \Gamma \times F^c)$. Hence $P(V)$ is closed (from the definition of the subspace topology coming from $\mathcal{G}^{(0)} \times \Gamma \times \mathcal{G}^{(0)}$).
3. (3) Faithful: as the map $P : \gamma \mapsto (r(\gamma), \rho(\gamma), s(\gamma))$ is injective, it is clear that its restriction to either $\mathcal{G}_{F^c}$ or $\mathcal{G}_F$ will also be injective. □

If we suppose now that we have a groupoid $\mathcal{G}$ that admits a (T,C,F)-cocycle to a group Γ. Lemma 3.9 enables us to apply Theorem 3.4 compatably to the reductions arising from any closed (or open) invariant subset $F \subset \mathcal{G}^{(0)}$.

Remark 3.10. For a uniformly discrete metric space X of bounded geometry there are natural reductions of $G(X)$ that are interesting to consider. It is easy to see that the set X is an open *saturated* subset of βX and in particular this means that the Stone–Čech boundary $\partial\beta X$ is saturated. We remark additionally that the groupoid $G(X)|_X$ is the pair groupoid $X \times X$ (as the coarse structure is weakly connected).

Definition 3.11. The *boundary groupoid* associated to X is the groupoid reduction $G(X)|_{\partial\beta X}$.

We collect the output of Lemma 3.9 applied to $G(X)$ with saturated pair X and $\partial\beta X$ below for clarity:

Proposition 3.12. *Let X be a uniformly discrete metric space of bounded geometry, let $G(X)$ be its coarse groupoid and let Γ be a discrete group. If $G(X)$ admits a (T,C,F)-cocycle to Γ, then so do the pair groupoid $G(X)|_X = X \times X$ and the boundary groupoid $G(X)|_{\partial\beta X}$.* □

In the situation of Proposition 2.7, i.e that X is coarsely embedded into a finitely generated group Γ, Proposition 3.12 will allow us to apply Theorem 3.4 when working with both $X \times X$ and $\mathcal{G}(X)|_{\partial\beta X}$.

4. Results on Gromov monsters

We begin this section by recalling some definitions concerning expansion and girth of finite graphs, as well as some of the main results shown about sequences of these spaces. The aim in this section is to prove the positive results about the Baum–Connes conjecture with coefficients for a group Γ that coarsely contains a large girth expander sequence as was outlined in the introduction.

Definition 4.1. Let $\{X_i\}_{i\in\mathbb{N}}$ be a sequence of finite metric spaces that are uniformly discrete with bounded geometry uniformly in the index i, such that $|X_i| \to \infty$ in i. Then we can form the *coarse disjoint union* X with underlying set $\sqcup X_i$, metric d given by the metric on each component and setting $d(X_i, X_j) \to \infty$ as $i + j \to \infty$. Any such metric is proper and unique up to coarse equivalence.

If each of the X_i are finite graphs, then the girth of X_i is understood to be the length of the shortest simple cycle in X_i. A sequence of graphs $\{X_i\}$ has *large girth* if the girth of X_i tends to infinity as i does.

Let $\{X_i\}$ be a sequence of finite graphs and let Δ_i be the graph Laplacian operator for each X_i, that is the bounded linear operator in $\mathfrak{B}(\ell^2(X_i))$ defined by the formula:

$$\Delta_i(\delta_x) = \sum_{d(x,y)=1} \delta_x - \delta_y$$

where $\{\delta_x\}$ is the standard orthonormal basis indexed by X_i and the distance on X_i is given by shortest path length.

The definition of a expander sequence is given in terms of the spectrum of these operators:

Definition 4.2. Let $\{X_i\}$ be a sequence of finite graphs and let X be the associated coarse disjoint union. Then the space X (or the sequence $\{X_i\}$) is an *expander* if:

(1) There exists $k \in \mathbb{N}$ such that all the vertices of each X_i have degree at most k.

(2) $|X_i| \to \infty$ as $i \to \infty$.

(3) There exists $c > 0$ such that $spectrum(\Delta_i) \subseteq \{0\} \cup [c, 1]$ for all i.

We remark that this last condition can be phrased as saying that

$$spectrum(\Delta) \subseteq \{0\} \cup [c, 1]$$

where Δ is the orthogonal sum of the Δ_i in $\mathfrak{B}(\ell^2(X))$.

These spaces are relevant here because:

(1) They fail to satisfy the coarse Baum–Connes conjecture. This was first outlined by Higson in [8], and for certain expanders it was shown more generally by Higson, Lafforgue and Skandalis in [9] that some aspect of the coarse assembly conjecture fails. More refined results concerning large girth families were shown by Willett and Yu in [16] and later using a groupoid technique by Wright and the author in [6].

(2) They coarsely embed into finitely generated discrete groups. This was shown by Osajda in [11], building on ideas of Gromov [7].

We now obtain results concerning Baum–Connes for the groups constructed by Osajda [11] mentioned above.

Theorem 4.3. *Let Γ be a finitely generated discrete group that contains a coarsely embedded large girth expander X of uniformly bounded vertex degree. Then there is a locally compact Γ-space $\Omega_{\beta X}$ such that $G(X)$ is Morita equivalent to $\Omega_{\beta X} \rtimes \Gamma$.*

Proof. This result follows from Theorem 3.5 and Theorem 3.4. □

As mentioned above, we know that for a large girth expander X of uniformly bounded vertex degree the Baum–Connes conjecture for $G(X)$ is injective, but not surjective (using Theorems 4.6 and 3.35 from [6] or using the meta-Theorem 1.5 from [16] and Proposition 4.8 from [14]). This translates, via Theorem 4.3, to:

Theorem 4.4. *Let Γ be a group satisfying the hypothesis of Theorem 4.3. Then the Baum–Connes conjecture for Γ with coefficients in $C_0(\Omega_{\beta X})$ fails to be a surjection, but is an injection.* □

4.1. A different proof of non-K-exactness and the failure of the Baum–Connes conjecture. We will describe the failure of the Baum–Connes conjecture via a K-exactness argument and the Morita equivalence results that were proved in previous sections. The K-exactness method is adapted from Section 9 of [9].

Theorem 4.5. *Let Γ be a finitely generated discrete group that coarsely contains a large girth expander X. Then Γ is not K-exact.*

Proof. Using Proposition 3.12 and the construction that goes into Theorem 3.4, we obtain a short exact sequence of commutative Γ-C^*-algebras:

$$0 \to C_0(\Omega_X) \to C_0(\Omega_{\beta X}) \to C_0(\Omega_{\partial\beta X}) \to 0.$$

We will show that this sequence fails to remain exact after completing using the reduced crossed product by Γ in each term.

In order to do this, we observe here that the combination of Proposition 3.12 and Theorem 3.5 with Theorem 3.4 shows that the transformation groupoid $\Omega_Y \rtimes \Gamma$ are Morita equivalent to the groupoid $G(X)|_Y$, where $Y = X, \beta X$ or $\partial\beta X$.

These Morita equivalences induce strong Morita equivalences of C^*-algebras: as Γ is countable and βX is σ-compact we can, for $Y = X, \beta X$ or $\partial\beta X$, deduce that each of the spaces Ω_Y are also σ-compact, hence the cross product algebras are all σ-unital. Using the results of Rieffel [12] we have long exact sequences of K-theory groups in which all the vertical maps are isomorphisms:

$$\begin{array}{ccccccc}
\cdots \to & K_0(C_0(\Omega_X) \rtimes_r \Gamma) & \to & K_0(C_0(\Omega_{\beta X}) \rtimes_r \Gamma) & \to & K_0(C_0(\Omega_{\partial\beta X}) \rtimes_r \Gamma) & \to \cdots \\
 & \cong\uparrow & & \cong\uparrow & & \cong\uparrow & \\
\cdots \to & K_0(\mathcal{K}(\ell^2(X))) & \to & K_0(C_r^*(G(X))) & \to & K_0(C_r^*(G(X)|_{\partial\beta X})) & \to \cdots
\end{array}$$

Where $\mathcal{K}(\ell^2(X))$ is the compact operators on $\ell^2(X)$.

We can conclude the result by observing that the bottom line is not exact as a sequence of abelian groups by appealing to known results concerning the groupoid $G(X)$ and the reductions (either Section 7 in [9] or a diagram chase and Theorem 4.6 of [6]). It follows therefore that the sequence:

$$0 \to C_0(\Omega_X) \rtimes_r \Gamma \to C_0(\Omega_{\beta X}) \rtimes_r \Gamma \to C_0(\Omega_{\partial\beta X}) \rtimes_r \Gamma \to 0$$

is not exact in the middle term. □

4.2. Positive results for Gromov monsters. To prove positive results we will use ideas from [6] that are recalled below.

Definition 4.6. A uniformly discrete metric space X with bounded geometry is said to be *a-T-menable at infinity* if the coarse boundary groupoid $G(X)_{\partial\beta X}$ is a-T-menable in the sense of [15], i.e it admits a (locally) proper negative type function to $\mathbb{R}$.

Examples of spaces that are a-T-menable at infinity are spaces that coarsely embed into Hilbert space, or more generally fibred coarsely embed into Hilbert space [5]. We recall the outcome of [6] in the following Proposition:

Proposition 4.7. *Let X be a large girth expander with vertex degree uniformly bounded above. Then X is a-T-menable at infinity.* □

Using the Theorem 3.4 and Proposition 4.7 we will prove that the groupoid $\Omega_{\partial\beta X} \rtimes \Gamma$ is a-T-menable. From here, using results of Tu, we can conclude that the Baum–Connes conjecture holds for this groupoid with any coefficients.

Theorem 4.8. *Let Γ be a finitely generated group that coarsely contains a large girth expander X with uniformly bounded vertex degree. Then the groupoid $\Omega_{\partial\beta X} \rtimes \Gamma$ is a-T-menable.*

Proof. By Proposition 3.12 and Theorem 3.4 the groupoid $G(X)|_{\partial\beta X}$ is Morita equivalent to $\Omega_{\partial\beta X} \rtimes \Gamma$. The result now follows as a-T-menability for groupoids is an invariant of Morita equivalences (see [15]). □

This has a natural corollary:

Corollary 4.9. *Let Γ be a finitely generated group that coarsely contains a large girth expander X. Then the Baum–Connes conjecture for Γ with coefficients in any $(\Omega_{\partial\beta X} \rtimes \Gamma)$-$C^*$-algebra is an isomorphism.* □

We remark that different techniques that rely on a-T-menability at infinity were considered in [1] to obtain a similar result.

References

[1] P. Baum, E. Guentner and R. Willett, Expanders, exact crossed products, and the Baum–Connes conjecture, 2013. arXiv:1311.2343

[2] J. Brodzki, G. A. Niblo and N. J. Wright, Property A, partial translation structures, and uniform embeddings in groups, *J. Lond. Math. Soc. (2)*, **76** (2007), no. 2, 479–497. Zbl 1139.46045 MR 2363428

[3] R. Exel, Inverse semigroups and combinatorial C^*-algebras, *Bull. Braz. Math. Soc. (N.S.)*, **39** (2008), no. 2, 191–313. Zbl 1173.46035 MR 2419901

[4] R. Exel, Reconstructing a totally disconnected groupoid from its ample semigroup, *Proc. Amer. Math. Soc.*, **138** (2010), no. 8, 2991–3001. Zbl 1195.22002 MR 2644910

[5] M. Finn-Sell, Fibred coarse embeddings, a-t-menability and the coarse analogue of the novikov conjecture, *Journal of Functional Analysis*, **267** (2014), no. 10, 3758–3782. Zbl 1298.22004 MR 3266245

[6] M. Finn-Sell and N. Wright, The coarse Baum-Connes conjecture, boundary groupoids and expander graphs, *Adv. Math.*, **259C** (2014), 306–338. Zbl 1288.19005 MR 3197659

[7] M. Gromov, Random walk in random groups, *Geom. Funct. Anal.*, **13** (2003), no. 1, 73–146. Zbl 1122.20021 MR 1978492

[8] N. Higson, Counterexamples to the coarse Baum–Connes conjecture, *preprint*, 1999.

[9] N. Higson, V. Lafforgue and G. Skandalis, Counterexamples to the Baum-Connes conjecture, *Geom. Funct. Anal.*, **12** (2002), no. 2, 330–354. Zbl 1014.46043 MR 1911663

[10] M. Khoshkam and G. Skandalis, Regular representation of groupoid C^*-algebras and applications to inverse semigroups, *J. Reine Angew. Math.*, **546** (2002), 47–72. Zbl 1029.46082 MR 1900993

[11] D. Osajda, Small cancellation labellings of some infinite graphs and applications, *preprint*, 2014. arXiv:1406.5015

[12] M. Rieffel, Morita equivalence for operator algebras, in *Operator algebras and applications, Part I (Kingston, Ont., 1980)*, 285–298, Proc. Sympos. Pure Math., 38, Amer. Math. Soc., Providence, R.I., 1982. Zbl 0541.46044 MR 0679708

[13] J. Roe, *Lectures on coarse geometry*, University Lecture Series, 31, American Mathematical Society, Providence, RI, 2003. Zbl 1042.53027 MR 2007488

[14] G. Skandalis, J. L. Tu and G. Yu, The coarse Baum–Connes conjecture and groupoids, *Topology*, **41** (2002), no. 4, 807–834. Zbl 1033.19003 MR 1905840

[15] J. L. Tu, La conjecture de Baum–Connes pour les feuilletages moyennables, *K-Theory*, **17** (1999), no. 3, 215–264. Zbl 0939.19001 MR 1703305

[16] R. Willett and G. Yu, Higher index theory for certain expanders and Gromov monster groups. I, *Adv. Math.*, **229** (2012), no. 3, 1380–1416. Zbl 1243.46060 MR 2871145

Received 31 October, 2014; revised 15 February, 2015

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