J. Noncommut. Geom. 10 (2016), 379–404
DOI 10.4171/JNCG/236

Journal of Noncommutative Geometry

The noncommutative infinitesimal equivariant index formula, Part II

Yong Wang*

Abstract. In this paper, we prove that infinitesimal equivariant Chern–Connes characters are well defined. We decompose an equivariant index as a pairing of infinitesimal equivariant Chern–Connes characters with the Chern character of an idempotent matrix. We compute the limit of infinitesimal equivariant Chern–Connes characters when the time goes to zero by using the Getzler symbol calculus and then extend these theorems to the family case. We also prove that infinitesimal equivariant eta cochains are well defined and prove the noncommutative infinitesimal equivariant index formula for manifolds with boundary.

Mathematics Subject Classification (2010). 58J20, 19K56.

Keywords. Infinitesimal equivariant Chern–Connes characters, Getzler symbol calculus, infinitesimal equivariant eta cochains, infinitesimal equivariant family Chern–Connes characters.

1. Introduction

The Atiyah–Bott–Segal–Singer index formula is a generalization of the Atiyah-Singer index theorem to manifolds admitting group actions. In [6, 22, 24], various heat kernel proofs of the equivariant index theorem have been given and each method has its own advantage. For manifolds with boundary, the equivariant extension of the Atiyah–Patodi–Singer index theorem was given by Donnelly in [13]. In the equivariant Atiyah–Patodi–Singer index theorem, the equivariant eta invariant appears and the regularity of the equivariant eta invariant was proved by Zhang in [29]. An infinitesimal version of the equivariant index formula was established in [6] and a direct heat kernel proof was given by Bismut in [7]. The infinitesimal equivariant index formula for manifolds with boundary was established in [19] with the introduction of the infinitesimal equivariant eta invariant.

The counterpart of the index formula in the noncommutative geometry is the computation of the Chern–Connes character [11, 18, 20]. The JLO character was computed in [12] and [9] by using the Getzler symbol calculus in [17]. In [2, 10]

*This work was supported by NSFC No.11271062 and NCET-13-0721.

and [24], these authors gave the computations of the equivariant JLO characters associated to a G-equivariant θ-summable Fredholm module. In [26], we defined the truncated infinitesimal equivariant Chern–Connes characters and computed the limit of the truncated infinitesimal equivariant Chern–Connes characters when the time goes to zero.

Compared with [26], there are several improvements in the present paper. In (2.2) in [26], we defined truncated infinitesimal equivariant Chern–Connes characters. It is only well defined when it is a polynomial of Lie algebra elements. In this paper, we drop off the truncated order J (see (2.2)) and this consequently requires much better estimates (see Lemma 2.2). As in [18], we decompose an equivariant index as a pairing of infinitesimal equivariant Chern–Connes characters with the Chern character of an idempotent matrix. Compared with Corollary 2.13 in [26], we drop off the limit on the right hand side of Corollary 2.13. Next we compute the limit of infinitesimal equivariant Chern–Connes characters when the time goes to zero by using the Getzler symbol calculus. Since we have dropped off the truncated order, (2.15) in [26] does not hold for our infinitesimal equivariant Chern–Connes characters. So we can not directly apply the method of Theorem 2.12 in [26]. Instead, we first apply the Getzler symbol calculus to prove the existence of the limit of infinitesimal equivariant Chern–Connes characters when time goes to zero (Theorem 3.9) and then use Theorem 2.12 in [26] to get the result. On the direction, in Section 3 in [26], we define the truncated infinitesimal equivariant eta cochains. Again in this paper we drop off the truncated order and then give a proof of the regularity at zero of infinitesimal equivariant eta cochains by using the method in [24]. That is, we prove that (3.5) in [26] holds for any k. This allows us to establish the noncommutative infinitesimal equivariant index formula for manifolds with boundary (see Theorem 4.9). In this paper, we also define family infinitesimal equivariant Chern–Connes characters and give the family generalization of the above theorems which does not appear in [26].

This paper is organized as follows: In Section 2, we prove that infinitesimal equivariant Chern–Connes characters are well defined. Then we decompose the equivariant index as a pairing of infinitesimal equivariant Chern–Connes characters with the Chern character of an idempotent matrix. In Section 3, We compute the limit of infinitesimal equivariant Chern–Connes characters when the time goes to zero by using the Getzler symbol calculus. In Section 4, we prove that infinitesimal equivariant eta cochains are well defined and prove the noncommutative infinitesimal equivariant index formula for manifolds with boundary. In Section 5, we extend results in Sections 2 and 3 to the family case.

2. The infinitesimal equivariant JLO cocycle and the index pairing

Let M be a compact oriented even dimensional Riemannian manifold without boundary with a fixed spin structure and S be the bundle of spinors on M. Denote by D the associated Dirac operator on $H = L^2(M;S)$, the Hilbert space of L^2-sections of the bundle S. Let $c(df) : S \to S$ denote the Clifford action with $f \in C^\infty(M)$. Suppose that G is a compact connected Lie group acting on M by orientation-preserving isometries preserving the spin structure and $\mathfrak{g}$ is the Lie algebra of G. Then G commutes with the Dirac operator. For $X \in \mathfrak{g}$, let $X_M(p) = \frac{d}{dt}|_{t=0}e^{-tX}p$ be the Killing field induced by X. Let $c(X)$ denote the Clifford action by X_M, and $\mathfrak{L}_X$ denote the Lie derivative respectively. Define $\mathfrak{g}$-equivariant modifications of D and D^2 for $X \in \mathfrak{g}$ as follows:

$$D_X := D - \frac{1}{4}c(X); \quad H_X := D_{-X}^2 + \mathfrak{L}_X = (D + \frac{1}{4}c(X))^2 + \mathfrak{L}_X. \tag{2.1}$$

Then H_X is the equivariant Bismut Laplacian. Let $\mathbb{C}[\mathfrak{g}^*]$ denote the space of formal power series in $X \in \mathfrak{g}$ and ψ_t be the rescaling operator on $\mathbb{C}[\mathfrak{g}^*]$ which is defined by $X \to \frac{X}{t}$ for $t > 0$.

Let

$$A = C_G^\infty(M) = \{f \in C^\infty(M) \mid f(g \cdot x) = f(x),\ g \in G,\ x \in M\},$$

then the data $(A, H, D + \frac{1}{4}c(X), G)$ defines a non selfadjoint perturbation of finitely summable (hence θ-summable) equivariant unbounded Fredholm module (A, H, D, G) in the sense of [21] (for details, see [10] and [21]). For $(A, H, D + \frac{1}{4}c(X), G)$, *the infinitesimal equivariant JLO cochain* $\mathrm{ch}^{2k}(D, X)$ can be defined by the formula:

$$\mathrm{ch}^{2k}(D,X)(f^0,\ldots,f^{2k}) := \int_{\triangle_{2k}} \mathrm{Str}\Big[e^{-\mathfrak{L}_X} f^0 e^{-\sigma_0(D+\frac{1}{4}c(X))^2} c(df^1) \cdot e^{-\sigma_1(D+\frac{1}{4}c(X))^2} \cdots c(df^{2k}) e^{-\sigma_{2k}(D+\frac{1}{4}c(X))^2}\Big] d\mathrm{Vol}_{\triangle_{2k}}, \tag{2.2}$$

where $\triangle_{2k} = \{(\sigma_0,\ldots,\sigma_{2k}) \mid \sigma_0 + \cdots \sigma_{2k} = 1\}$ is the $2k$-simplex. For an integer $J \geq 0$, denote by $\mathbb{C}[\mathfrak{g}^*]_J$ the space of polynomials of degree $\leq J$ in $X \in \mathfrak{g}$ and let $(\cdot)_J : \mathbb{C}[\mathfrak{g}^*] \to \mathbb{C}[\mathfrak{g}^*]_J$ be the natural projection. Fix basis $e_1,\ldots,e_n$ of $\mathfrak{g}$ and let $X = x_1e_1 + \cdots x_ne_n$. A J-degree polynomial on X is namely a J-degree polynomial on $x_1,\ldots,x_n$. Now we prove that $\mathrm{ch}^{2k}(D,X)(f^0,\ldots,f^{2k})$ is well defined.

Let H be a Hilbert space. For $q \geq 0$, denote by $\|.\|_q$ the Schatten p-norm on the Schatten ideal L^p. Let $L(H)$ denote the Banach algebra of bounded operators on H.

Lemma 2.1 ([25]). (i) $\mathrm{Tr}(AB) = \mathrm{Tr}(BA)$, *for* $A,\ B \in L(H)$ *and* $AB, BA \in L^1$.

(ii) *For* $A \in L^1$, *we have*

$$|\mathrm{Tr}(A)| \leq \|A\|_1, \quad \|A\| \leq \|A\|_1.$$

(iii) *For $A \in L^q$ and $B \in L(H)$, we have*

$$\|AB\|_q \le \|B\|\|A\|_q, \quad \|BA\|_q \le \|B\|\|A\|_q.$$

(iv) *(Hölder inequality) If $\frac{1}{r} = \frac{1}{p} + \frac{1}{q}$, $p, q, r > 0$, $A \in L^p$, $B \in L^q$, then $AB \in L^r$ and $\|AB\|_r \le \|A\|_p\|B\|_q$.*

Let $H_X = D^2 + F_X$, where F_X is a first order differential operator with degree ≥ 1 coefficients depending on X.

Lemma 2.2. *For any $1 \ge u > 0$, $t > 0$, we have:*

$$\|e^{-utH_X}\|_{u^{-1}} \le 2e^{\frac{t}{2}}\{1 + [\|(1+D^2)^{-\frac{1}{2}}F_X\|^2 e^{-1}\pi ut]^{\frac{1}{2}}\} \cdot e^{\|(1+D^2)^{-\frac{1}{2}}F_X\|^2 e^{-1}\pi ut}(\mathrm{tr}[e^{-\frac{tD^2}{2}}])^u. \quad (2.3)$$

Proof. By the Duhamel principle, it is that

$$\|e^{-utH_X}\|_{u^{-1}} = \left\| \sum_{m\ge 0}(-ut)^m \int_{\triangle_m} e^{-v_0utD^2}F_X e^{-v_1utD^2} \cdot F_X \cdots e^{-v_{m-1}utD^2}F_X e^{-v_m utD^2}dv \right\|_{u^{-1}}. \quad (2.4)$$

Also $\|(-ut)^m \int_{\triangle_m} e^{-v_0utD^2}F_X e^{-v_1utD^2}F_X \cdots e^{-v_{m-1}utD^2}F_X e^{-v_m utD^2}dv\|_{u^{-1}}$ is continuous and bounded by (2.7) in [26]. By the measure of the boundary of $\triangle_m$ being zero, we can estimate (2.4) in the interior of $\triangle_m$, that is $v_j > 0$. It holds that

$$\|e^{-\frac{v_j}{2}utD^2}F_X\| \le (v_j ut)^{-\frac{1}{2}}e^{-\frac{1-v_j ut}{2}}\|(1+D^2)^{-\frac{1}{2}}F_X\|, \quad (2.5)$$

where we use that F_X is a first order differential operator and the equality

$$\sup\{(1+x)^{\frac{l}{2}}e^{-\frac{utx}{2}}\} = (ut)^{-\frac{l}{2}}e^{-\frac{l-ut}{2}}. \quad (2.6)$$

By the Hölder inequality, (2.4) and (2.5), the conditions that $0 < u \le 1$ and $v_0 + \cdots + v_{m-1} \le 1$, we have

$$\|e^{-utH_X}\|_{u^{-1}} \le e^{\frac{t}{2}} \sum_{m\ge 0} e^{-\frac{m}{2}}(ut)^{\frac{m}{2}}\|(1+D^2)^{-\frac{1}{2}}F_X\|^m \cdot \int_{\triangle_m} v_0^{-\frac{1}{2}} \cdots v_{m-1}^{-\frac{1}{2}}dv(\mathrm{tr}[e^{-\frac{tD^2}{2}}])^u. \quad (2.7)$$

It holds that (see line 7 in [3, p. 21])

$$\int_{\triangle_m} v_0^{-\frac{1}{2}} \cdots v_{m-1}^{-\frac{1}{2}}dv = \frac{\pi^{\frac{m}{2}}}{\frac{m}{2}\Gamma(\frac{m+1}{2})}. \quad (2.8)$$

By $\Gamma(x+1)=x\Gamma(x)$, $\Gamma(n)=(n-1)!$ and $\Gamma(\frac{1}{2})=\sqrt{\pi}$, then $\Gamma(\frac{m+1}{2})=(\frac{m-1}{2})!$ when m is odd; $\Gamma(\frac{m+1}{2})=\frac{(m-1)!!\sqrt{\pi}}{2^{\frac{m}{2}}}$ when m is even. By (2.8) and

$$\lim_{m\to+\infty}\frac{(2m-1)!!}{(2m)!!}=0, \tag{2.9}$$

we know that the series (2.7) is absolutely convergent. When m is odd, then

$$\frac{\pi^{\frac{m}{2}}}{\frac{m}{2}\Gamma(\frac{m+1}{2})}\le\frac{2\pi^{\frac{m}{2}}}{(\frac{m+1}{2})!}. \tag{2.10}$$

When m is even, then

$$\frac{\pi^{\frac{m}{2}}}{\frac{m}{2}\Gamma(\frac{m+1}{2})}\le\frac{2\pi^{\frac{m}{2}}}{(\frac{m}{2})!}. \tag{2.11}$$

By (2.7), (2.8), (2.10) and (2.11), we have

$$\begin{aligned}\|e^{-utH_X}\|_{u^{-1}}\le 2e^{\frac{t}{2}}\Bigg[&\sum_{m\ \text{even}}\frac{(\|(1+D^2)^{-\frac{1}{2}}F_X\|^2e^{-1}\pi ut)^{\frac{m}{2}}}{(\frac{m}{2})!}\\&+\sum_{m\ \text{odd}}\frac{(\|(1+D^2)^{-\frac{1}{2}}F_X\|^2e^{-1}\pi ut)^{\frac{m}{2}}}{(\frac{m+1}{2})!}\Bigg](\mathrm{tr}[e^{-\frac{tD^2}{2}}])^u.\end{aligned} \tag{2.12}$$

Therefore, (2.3) can be obtained. □

By (2.2), (2.3) and the Hölder inequality as well as $\mathrm{Vol}_{\Delta_{2k}}=\frac{1}{(2k)!}$, for $t=1$ and $\sigma_l\le 1$, we get

$$\begin{aligned}|\mathrm{ch}^{2k}(D,X)(f^0,\dots,f^{2k})|\le{}&\frac{1}{(2k)!}\|f^0\|\Bigg(\prod_{j=1}^{2k}\|df^j\|\Bigg)\\&\cdot[2e^{\frac{1}{2}}(1+(\|(1+D^2)^{-\frac{1}{2}}F_X\|^2e^{-1}\pi)^{\frac{1}{2}})]^{2k+1}\\&\cdot e^{\|(1+D^2)^{-\frac{1}{2}}F_X\|^2e^{-1}\pi}(\mathrm{tr}[e^{-\frac{D^2}{2}}]).\end{aligned} \tag{2.13}$$

Thus, $\mathrm{ch}^{2k}(D,X)$ is well defined. Recall that an even cochain $\{\Phi_{2n}\}$ is called entire if $\sum_n\|\Phi_{2n}\|n!z^n$ is entire, where $\|\Phi\|:=\sup_{\|f^j\|_1\le 1}\{|\Phi(f^0,f^1,\dots,f^{2k})|\}$. By (2.13), then $\{\mathrm{ch}^{2k}(D,X)\}$ is an entire cochain. Let $p\in M_r(\mathbb{C}^\infty(M))$ and $p=p^2=p^*$ and $p(gx)=p(x)$. Define the Chern character of p by (see [18])

$$\mathrm{ch}(p):=\mathrm{Tr}(p)+\sum_l(-1)^l\frac{(2l)!}{2\cdot l!}\mathrm{Tr}(2p-1,p,\dots,p)_{2l}. \tag{2.14}$$

By (2.13), $\langle\mathrm{ch}^*(D,X),\mathrm{ch}(p)\rangle$ is convergent.

Similarly to Theorem A in [18], we have

Proposition 2.3. (1) *The infinitesimal equivariant Chern–Connes character is closed:*

$$(B+b)(\mathrm{ch}^*(D,X))=0. \tag{2.15}$$

(2) *Let* $D_\tau = D+\tau V$ *and* $D_{-X,\tau}=D_{-X}+\tau V$ *and* V *is a bounded operator which commutes with* e^{-X}, *then there exists a cochain* $\mathrm{ch}^*(D_\tau, X, V)$ *such that*

$$\frac{d}{d\tau}\mathrm{ch}^*(D_\tau,X)=-(B+b)\mathrm{ch}^*(D_\tau,X,V). \tag{2.16}$$

By the Serre–Swan theorem, we denote the vector bundle over M with the fibre $p(x)(\mathbb{C}^r)$ at $x\in M$ by $\mathrm{Im}p$. Let $D_{\mathrm{Im}p}$ be the Dirac operator twisted by the bundle $\mathrm{Im}p$. By Proposition 2.3, $(B+b)\mathrm{ch}(p)=0$ and Proposition 8.11 in [4], we have by taking $V=(2p-1)[D,p]$ that (see Section 3 in [18])

Theorem 2.4. *The following index formula holds*

$$\mathrm{Ind}_{e^{-X}}(D_{\mathrm{Im}p,+})=\langle \mathrm{ch}^*(D,X),\mathrm{ch}(p)\rangle\,. \tag{2.17}$$

In Theorem 2.4, X is unnecessarily small.

3. The computations of infinitesimal equivariant Chern–Connes characters

In this section, we will compute infinitesimal equivariant Chern–Connes characters by Theorem 2.12 in [26] and the Getzler symbol calculus in [17] and [9]. Recall the Getzler symbol calculus in [17] and [9]. Let E be a vector bundle over the compact manifold M and $\pi: T^*M\to M$ be the natural map and $E^0=\pi^*(\mathrm{Hom}(E,E))$ be the pull-back of the bundle $\pi^*(\mathrm{Hom}(E,E))$ to a bundle over T^*M.

Definition 3.1. A section $p\in E^0$ is called a symbol of order l if for every multi-index α and β we have the estimates:

$$\|\partial_x^\alpha\partial_\xi^\beta p(x,\xi)\|\le C_{\alpha\beta}(1+|\xi|)^{m-|\beta|}. \tag{3.1}$$

We denote by $\Sigma^l(E)$ the symbols of order l.

By the representative theorem of the Clifford algebra $Cl(T^*M)\simeq \mathrm{Hom}(S(TM))$ and the isomorphism $Cl(T^*M)\simeq \wedge(T^*M)$, note a map $\overline{\sigma}$ defined by

$$\overline{\sigma}:\ \mathrm{Hom}(S(TM)\otimes E)\simeq \mathrm{Hom}E\otimes Cl(T^*M)\simeq \mathrm{Hom}E\otimes\wedge(T^*M), \tag{3.2}$$

and $\overline{\theta}$ is the inverse of $\overline{\sigma}$. Let $\mathbb{L}=\pi^*(\mathrm{Hom(E)}\otimes\wedge(T^*M))\otimes\mathbb{C}[\mathfrak{g}^*]$ and $X\in\mathfrak{g}$.

Definition 3.2. A section $p \in \mathbb{L}$ is called a s-symbol of order l if

$$p = \sum_{j=0}^{\dim M} \Big(\sum_{|\alpha| \geq 0} p_{j,\alpha} X^{\alpha} \Big) \otimes \omega_j, \tag{3.3}$$

where $\omega_j \in \Omega^j(M)$, $p_{j,\alpha} \in \Sigma^{l-j-2|\alpha|}(E)$ and $\sum_{|\alpha|\geq 0} \| p_{j,\alpha}(x,\xi) \| |X^{\alpha}|$ is convergent. We denote the collection of s-symbol of order l by $S\Sigma^l(E, X)$.

Let x_0 be a fixed point in M and $T_{x_0}M$ be the tangent space and exp be the exponential map respectively. Let h be a function that is identically one in a neighborhood of the diagonal of $M \times M$ such that the exponential map is a diffeomorphism on the support of h. Let $(x_0, x) \in \operatorname{supp}(h)$. Let

$$\tau(x_0, x) : (S(TM) \otimes E)_{x_0} \to (S(TM) \otimes E)_x$$

be a parallel translation about $\nabla^{S(TM)\otimes E}$ along the unique geodesic from x_0 to x. If $s \in \Gamma(S(TM) \otimes E)$, then we define

$$\widehat{s}_{x_0}(x) = h(x_0, x)\tau(x, x_0)s(x). \tag{3.4}$$

We write $\widehat{s}_{x_0}(Y)$ instead of $\widehat{s}_{x_0}(\exp_{x_0} Y)$.

Let θ_X be the one-form associated with X_M which is defined by $\theta_X(Y) = g(X, Y)$ for the vector field Y. Let $\nabla^{S,X}$ be the Clifford connection $\nabla^S - \frac{1}{4}\theta_X$ on the spinors bundle and $\triangle_X$ be the Laplacian on $S(TM)$ associated with $\nabla^{S,X}$. Let $\mu(X)(\cdot) = \nabla^{TM}_{\cdot} X_M$. Let $U = \{x \in T_{x_0}M \mid \|x\| < \varepsilon\}$, where ε is smaller than the injectivity radius of the manifold M at x_0. Define $\alpha : U \times \mathfrak{g} \to \mathbb{C}$ via the formula

$$\alpha_X(x) := -\frac{1}{4}\int_0^1 (\iota(\mathcal{R})\theta_X)(tx)t^{-1}dt, \quad \rho(X, x) = e^{\alpha_X(x)}, \tag{3.5}$$

where $\mathcal{R} = \sum_{i=1}^n x_i \frac{\partial}{\partial x_i}$. Then $\rho(X, 0) = 1$. Recall [4, Lemma 8.13] that the following identity holds

$$H_X = -g^{ij}(x)\Big(\nabla^X_{\partial_i}\nabla^X_{\partial_j} - \sum_k \Gamma^k_{ij}\nabla^X_{\partial_k}\Big) + \frac{1}{4}r_M, \tag{3.6}$$

where r_M is the scalar curvature and Γ^k_{ij} is the connection coefficient of ∇^L.

Let $\tau^X(x_0, x) : (S(TM) \otimes E)_{x_0} \to (S(TM) \otimes E)_x$ be parallel translation about $\nabla^{S\otimes E, X}$ along the unique geodesic from x_0 to x. If $s \in \Gamma(S(TM) \otimes E)$, then we define

$$\widehat{s}^X_{x_0}(x) = h(x_0, x)\tau^X(x, x_0)s(x). \tag{3.7}$$

Then

$$\widehat{s}^X_{x_0}(x) = \rho\widehat{s}_{x_0}(x). \tag{3.8}$$

where $\rho = \rho(X, x)$ is defined by (3.5).

Definition 3.3. Let $p \in S\Sigma(E,X)$ and $s \in \Gamma(S(TM)\otimes E)$, then we define

$$\theta(p)(s)(x_0) = \int\limits_{T_{x_0}M\times T^*_{x_0}M} e^{-\sqrt{-1}\langle Y,\xi\rangle}\overline{\theta}(p)(x_0,\xi,X)\widehat{s}^X_{x_0}(Y)dYd\xi. \tag{3.9}$$

Remark. The operator $\theta(p)$ is well defined since $\sum_{|\alpha|\geq 0}\|p_{j,\alpha}(x,\xi)\||X^\alpha|$ and $e^{|\alpha_X(x)|}$ are convergent. The operator $\theta(p)$ depends on the choice of the cut off function h, but the result does not depend on the cut off function for computations of infinitesimal equivariant Chern–Connes characters. We denote by $Op(E,X)$ all such operators with smoothing operators.

Definition 3.4. Given $s \in \Gamma(S(TM)\otimes E)$, define $\overline{s}^X_{x_0}(x) = h(x_0,x)\tau^X(x_0,x)s(x_0)$ and $\overline{s}_{x_0}(x) = h(x_0,x)\tau(x_0,x)s(x_0)$, then $\overline{s}^X_{x_0}(x) = \rho^{-1}\overline{s}_{x_0}(x)$. Let $P \in Op(E,X)$ and $s \in \Gamma(S(TM)\otimes E)$. Define $\sigma(P) \in \mathrm{End}(E)_{x_0}\otimes\Omega(M)\otimes\mathbb{C}[\mathfrak{g}^*]$ by

$$\sigma(P)(x_0,\xi,X) = \overline{\sigma}P_y(e^{\sqrt{-1}\left\langle \exp^{-1}_{x_0}(y),\xi\right\rangle}\overline{s}^X_{x_0}(y))|_{y=x_0}. \tag{3.10}$$

Lemma 3.5. *Let $P = \sum_\alpha P_\alpha X^\alpha \in Op(E,X)$. If $\sum_\alpha\|P_\alpha\|_1|X^\alpha|$ is convergent, then $\sigma(P)$ is convergent.*

Proof. Since $\sum_\alpha\|P_\alpha\|_1|X^\alpha|$ and $e^{|\alpha_X(x)|}$ are convergent, this comes from Definition 3.4 and $|e^{\sqrt{-1}\left\langle \exp^{-1}_{x_0}(y),\xi\right\rangle}| = 1$ and $|h(x_0,x)| \leq 1$ and $\tau(x_0,x)$ being an isometry. □

Lemma 3.6. *Let $Y = \sum c_i\partial_i,\ Z = \sum d_j\partial_j$ with $c_i,\ d_j \in \mathbb{R}$. we have*

$$\sigma(\nabla^X_Y)(x,\xi) = \sqrt{-1}\,\langle Y,\xi\rangle_x\,, \tag{3.11}$$

$$\sigma(\nabla^X_Y\nabla^X_Z)(x,\xi) = -\langle Y,\xi\rangle\langle Z,\xi\rangle + \frac{1}{4}\left\langle R^L(Y,Z)\partial_k,\partial_l\right\rangle f^k\wedge f^l + \frac{1}{4}\left\langle \mu^X(Y),Z\right\rangle, \tag{3.12}$$

where f^k is the dual base of ∂_k.

Proof. By Definition 3.4, We have

$$\sigma(\nabla^X_Y)(x_0,\xi) = \overline{\sigma}[\nabla^X_Y(e^{\sqrt{-1}\left\langle \exp^{-1}_{x_0}(y),\xi\right\rangle}\rho^{-1}\overline{s}_{x_0}(y))]|_{y=x_0}. \tag{3.13}$$

By

$$\left(d - \frac{1}{4}\theta_X\right)_{\partial_j}(\rho^{-1})|_{x=x_0} = 0; \quad \nabla_Y(\overline{s}_{x_0}(x))|_{x=x_0} = 0, \tag{3.14}$$

similarly to the computations of Example 1 in [9], we get (3.11). We know that $\rho\nabla^X_Y\nabla^X_Z\rho^{-1} = \rho\nabla^X_Y\rho^{-1}\rho\nabla^X_Z\rho^{-1}$. By the appendix II in [1], we have

$$\nabla_Y\nabla_Z\overline{s}_{x_0}(y)|_{y=x_0} = \frac{1}{4}\left\langle R^L(Y,Z)\partial_k,\partial_l\right\rangle f^k\wedge f^l s(x_0). \tag{3.15}$$

In the trivialization of $S(TM)$, the conjugate $\rho(X,x)(\nabla^{S,X}_{\partial_i})\rho(X,x)^{-1}$ is given by Lemma 8.13 in [4] which is

$$\rho(X,x)(\nabla^{S,X}_{\partial_i})\rho(X,x)^{-1} = \partial_i + \frac{1}{4}\sum_{j,a<b}\left\langle R(\partial_i,\partial_j)e_a,e_b\right\rangle c(e_a)c(e_b)x^j - \frac{1}{4}\mu^M_{ij}(X)x^j + \sum_{j<k} f_{ijk}(x)c(e_j)c(e_k) + g_i(x) + \langle h_i(x), X\rangle, \quad (3.16)$$

where $f_{ijk}(x) = O(|x|^2)$, $g_i(x) = O(|x|)$, and $h_i(x) = O(|x|^2)$. By (3.15) and (3.16), similarly to the computations of Example 2 in [9], we have (3.12). $\square$

Proposition 3.7. *The following equality holds*

$$\sigma(H_X) = |\xi|^2 + \frac{1}{4}r_M. \quad (3.17)$$

The operator t^2H_X is an asymptotic pseudodifferential operator (see Definition 3.5 in [9]).

Proof. By Lemma 3.6 and (3.6) and $g^{ij}(x_0) = \delta^{ij}$, $\Gamma^k_{ij}(x_0) = 0$ and $R^L(Y,Y) = \left\langle \mu^X(Y), Y\right\rangle = 0$, we get Proposition 3.7. $\square$

Definition 3.8. If $p(x,\xi,X) \in S\Sigma(E,X)$, then

$$p_t(x,\xi,X) = \sum_{j=0}^{\dim M}\left(\sum_{|\alpha|\geq 0} p_{j,\alpha}(x,t\xi)t^{2|\alpha|}X^\alpha\right)\otimes \omega_j t^j, \quad (3.18)$$

Let $\psi_t : X \to \frac{X}{t}$ be the rescaling operator on the Lie algebra.

Theorem 3.9. *For $P = \sum P_\alpha X^\alpha \in OP(S\Sigma^{-\infty}(E,X))$ and $t > 0$, then*

$$\psi_t^2\mathrm{Tr_s}(P) = (2\pi)^{-n}\left(\frac{2}{\sqrt{-1}}\right)^{\frac{n}{2}}\int_M\int_{T^*_{x_0}M}\mathrm{Tr_s}\sigma(P)_{\frac{1}{t}}(x_0,\xi)d\xi dx. \quad (3.19)$$

If $P = P_t$ and P_t is an asymptotic pseudodifferential operator and $\sigma(P_t)(x,\xi)$ tends to zero when $|\xi|$ tends to infinity, then

$$\psi_t^2\mathrm{Tr_s}(P_t) = b_0 + O(t), \quad (3.20)$$

where b_0 is a constant.

Proof. By Theorem 3.7 in [17], we have for any $s > 0$ that

$$\mathrm{Tr_s}(P_\alpha) = (2\pi)^{-n}\left(\frac{2}{\sqrt{-1}}\right)^{\frac{n}{2}}\int_M\int_{T^*_{x_0}M}\mathrm{Tr_s}\sigma_G(P_\alpha)_s(x_0,\xi)d\xi dx, \quad (3.21)$$

where

$$\sigma_G(P)(x_0,\xi,X)=\overline{\sigma}P_y(e^{\sqrt{-1}\langle\exp_{x_0}^{-1}(y),\xi\rangle}\overline{s}_{x_0}(y))|_{y=x_0}. \tag{3.22}$$

Since $\sigma(P_t)(x,\xi)$ tends to zero when $|\xi|$ tends to infinity, by using the equality which will be proved in the following Lemma 3.11

$$\int_{T^*_{x_0}M}\mathrm{Tr_s}\sigma_G(P)_s(x_0,\xi)d\xi dx=\int_{T^*_{x_0}M}\mathrm{Tr_s}\sigma_G(\rho P\rho^{-1})_s(x_0,\xi)d\xi dx, \tag{3.23}$$

we have for $\rho(x_0)=1$ that

$$\begin{aligned}\int_{T^*_{x_0}M}\mathrm{Tr_s}\sigma_G(P_\alpha)_s(x_0,\xi)d\xi&=\int_{T^*_{x_0}M}\mathrm{Tr_s}\sigma_G(P_\alpha\rho^{-1})_s(x_0,\xi)d\xi\\&=\int_{T^*_{x_0}M}\mathrm{Tr_s}\sigma(P_\alpha)_s(x_0,\xi)d\xi.\end{aligned} \tag{3.24}$$

So

$$\mathrm{Tr_s}(P_\alpha X^\alpha)=(2\pi)^{-n}\Big(\frac{2}{\sqrt{-1}}\Big)^{\frac{n}{2}}\int_M\int_{T^*_{x_0}M}\mathrm{Tr_s}\sigma\Big(P_\alpha\frac{X^\alpha}{s^{2|\alpha|}}\Big)_s(x_0,\xi)d\xi dx. \tag{3.25}$$

Let $s=\frac{1}{t}$, then

$$\mathrm{Tr_s}(P_\alpha X^\alpha)=(2\pi)^{-n}\Big(\frac{2}{\sqrt{-1}}\Big)^{\frac{n}{2}}\int_M\int_{T^*_{x_0}M}\mathrm{Tr_s}\sigma(P_\alpha X^\alpha t^{2|\alpha|})_{\frac{1}{t}}(x_0,\xi)d\xi dx. \tag{3.26}$$

So

$$\psi_t^2\mathrm{Tr_s}(P_\alpha X^\alpha)=(2\pi)^{-n}\Big(\frac{2}{\sqrt{-1}}\Big)^{\frac{n}{2}}\int_M\int_{T^*_{x_0}M}\mathrm{Tr_s}\sigma(P_\alpha X^\alpha)_{\frac{1}{t}}(x_0,\xi)d\xi dx. \tag{3.27}$$

By taking the sum $\sum_\alpha$, we get (3.19). By Definitions 3.8 and Definition 3.5 in [9], for the asymptotic pseudodifferential operator P_t, we have

$$\sigma(P_t)=\sum_{l=0}^{+\infty}t^l p_l(x,\xi,X)_t, \tag{3.28}$$

so

$$\sigma(P_t)_{\frac{1}{t}}=\sum_{l=0}^{+\infty}t^l p_l(x,\xi,X), \tag{3.29}$$

By (3.19) and (3.29), we get (3.20). □

Let μ^M be the Riemannian moment of X defined by $\mu^M(X)Y = -\nabla_Y X^M$. Let $F_{\mathfrak{g}}^M(X) = \mu^M + R$ be the equivariant Riemannian curvature of M. The equivariant $\widehat{A}$-genus of the tangent bundle of M is defined by

$$\widehat{A}(F_{\mathfrak{g}}^M(X)) = \det\left(\frac{F_{\mathfrak{g}}^M(X)/2}{\sinh(F_{\mathfrak{g}}^M(X)/2)}\right)^{\frac{1}{2}}.$$

Theorem 3.10. *When $2k \leq \dim M$ and X is small which means that $\|X_M\|$ is sufficiently small, then for $f^j \in C_G^\infty(M)$,*

$$\lim_{t\to 0} \psi_t \mathrm{ch}^{2k}(\sqrt{t}D, X)(f^0, \dots, f^{2k}) = \frac{1}{(2k)!}(2\pi\sqrt{-1})^{-n/2} \cdot \int_M f^0 \wedge df^1 \wedge \cdots \wedge df^{2k} \widehat{A}(F_{\mathfrak{g}}^M(X)) d\mathrm{Vol}_M. \quad (3.30)$$

Proof. In Theorem 3.9, let $P_t = t^{2k} f^0 e^{-\sigma_0 t^2 H_X} c(df^1) \cdots c(df^{2k}) e^{-\sigma_{2k} t^2 H_X}$, then by Proposition 3.7, similarly to Lemma 3.13 in [9], we have P_t is an asymptotic pseudodifferential operator. By (3.20) and taking the J-jet, we have

$$\lim_{t\to 0} \psi_t^2 \mathrm{Tr}_s(P_t)_J = b_{0,J}. \quad (3.31)$$

By Theorem 2.12 in [26], we have

$$\lim_{t\to 0} \psi_t^2 \mathrm{Tr}_s(P_t)_J = \frac{1}{(2k)!}(2\pi\sqrt{-1})^{-n/2} \cdot \int_M f^0 \wedge df^1 \wedge \cdots \wedge df^{2k} \widehat{A}(F_{\mathfrak{g}}^M(X))_J d\mathrm{Vol}_M. \quad (3.32)$$

By (3.31) and (3.32) and when J goes to infinity, we obtain

$$b_0 = \frac{1}{(2k)!}(2\pi\sqrt{-1})^{-n/2} \int_M f^0 \wedge df^1 \wedge \cdots \wedge df^{2k} \widehat{A}(F_{\mathfrak{g}}^M(X)) d\mathrm{Vol}_M. \quad (3.33)$$

By (3.20) and (3.33), when t goes to zero, we get (3.30). □

Lemma 3.11. *The equality (3.23) holds.*

Proof. Considering the equalities (70) and (71) in [23] (Note that these formulas hold for any pseudodifferential operators defined by (3.22) and not only for asymptotic pseudodifferential operators), let $N = 0$, then

$$\sigma_G(P\rho^{-1})(x,\xi) = \sigma_G(P)(x,\xi)\rho^{-1} + r_0(\xi). \quad (3.34)$$

where $r_0(\xi)$ is defined by

$$r_0(\xi)=\frac{\sqrt{-1}}{(2\pi)^n}\sum_{j=1}^{n}\int_{T^*_{x_0}M(y)}\int_0^1\frac{\partial}{\partial y_j}a(\xi+sy)ds\cdot y_j[\mathcal{F}(f^\psi)](y)dy, \quad (3.35)$$

and the Fourier transform $\mathcal{F}$ and f^ψ are defined by (7) and (8) in [23] respectively, a is the symbol of P. By (0.2) in [17], we have the leading symbol of $e^{-t^2D^2}$ is $e^{-t^2|\xi|^2}$. As in (2.4), using the Duhamel principle, we expanse the operator P_t and the leading symbol of P_t is the product of $e^{-t^2|\xi|^2}$ and a polynomial on ξ. Without loss of generality, we assume $a=e^{-|\xi|^2}$. The following two well-known theorems are necessary:

I. *Let $f(x,y)$ be continues on the domain $x\geq a,\ y\geq b$ and $\int_b^{+\infty}f(x,y)dy$ be uniformly convergent about x on any finite interval included in $[a,+\infty]$ and $\int_a^{+\infty}f(x,y)dx$ be uniformly convergent about y on any finite interval included in $[b,+\infty]$. We assume that the integral $\int_b^{+\infty}[\int_a^{+\infty}|f(x,y)|dx]dy$ or $\int_a^{+\infty}[\int_b^{+\infty}|f(x,y)|dy]dx$ exists, then*

$$\int_a^{+\infty}\left[\int_b^{+\infty}f(x,y)dy\right]dx=\int_b^{+\infty}\left[\int_a^{+\infty}f(x,y)dx\right]dy=\text{finite number}. \quad (3.36)$$

II. *There exists $\beta>0$, such that $|f(x,y)|\leq F(x)$ for any $x>\beta$ and $y\in I$ and that $\int_a^{+\infty}F(x)dx$ exists, then $\int_a^{+\infty}f(x,y)dx$ is uniformly convergent.*

By (3.35), we consider

$$\int_{T^*_{x_0}M(\xi)}r_0(\xi)d\xi=\frac{\sqrt{-1}}{(2\pi)^n}\sum_{j=1}^{n}\int_{T^*_{x_0}M(\xi)}\int_{T^*_{x_0}M(y)}\int_0^1\frac{\partial a}{\partial\xi_j}|_{\xi+sy}sds \cdot y_j[\mathcal{F}(f^\psi)](y)dyd\xi. \quad (3.37)$$

Since the Schwartz function $[\mathcal{F}(f^\psi)](y)$ is integral on $T^*_{x_0}M(y)$, we take some estimates on the right hand side of (3.37) in the polar coordinates of $T^*_{x_0}M(\xi)$ and $T^*_{x_0}M(y)$ and then we can verify that the right hand side of (3.37) satisfies the conditions of Theorem I. Using $\int_{T^*_{x_0}M(\xi)}\frac{\partial}{\partial\xi_j}[e^{-|\xi|^2}\xi^\beta]d\xi=0$ and (3.37), we get $\int_{T^*_{x_0}M}r_0(\xi)=0$. Therefore we get (3.23). □

Let

$$\mathrm{Ch}(\mathrm{Im}(p))=\sum_{k=0}^{\infty}\left(-\frac{1}{2\pi\sqrt{-1}}\right)^k\frac{1}{k!}\mathrm{Tr}[p(dp)^{2k}]. \quad (3.38)$$

We have

Corollary 3.12. *When X is small, then*

$$\mathrm{Ind}_{e^{-X}}(D_{\mathrm{Im}p,+}) = (2\pi\sqrt{-1})^{-n/2}\int_M \widehat{A}(F_{\mathfrak{g}}^M(X))\mathrm{Ch}(\mathrm{Im}p). \tag{3.39}$$

Proof. Using the same discussions as those in [18], we have the homotopy property of $\mathrm{ch}^*(D,X)$ for tD_{-X}. So by (2.17), we have

$$\mathrm{Ind}_{e^{-X}}(D_{\mathrm{Im}p,+}) = \langle \mathrm{ch}^*(tD_{-X}), \mathrm{ch}(p)\rangle\,, \tag{3.40}$$

where

$$\begin{aligned}\mathrm{ch}^{2k}(tD_{-X})(f^0,\ldots,f^{2k}) := t^{2k}\int_{\Delta_{2k}} \mathrm{Str}\Big[e^{-\mathfrak{L}_X} f^0 e^{-\sigma_0 t^2(D+\frac14 c(X))^2} c(df^1)\\ \cdot e^{-\sigma_1 t^2(D+\frac14 c(X))^2}\cdots c(df^{2k}) e^{-\sigma_{2k} t^2(D+\frac14 c(X))^2}\Big] d\mathrm{Vol}_{\Delta_{2k}},\end{aligned} \tag{3.41}$$

In (3.40), let $e^{-X} = e^{-t^2X}$ and use $(\psi_t)^2$ acting on (3.40), then we get

$$\mathrm{Ind}_{e^{-X}}(D_{\mathrm{Im}p,+}) = \Big\langle \widetilde{\mathrm{ch}}^*(tD,X), \mathrm{ch}(p)\Big\rangle, \tag{3.42}$$

where

$$\begin{aligned}\widetilde{\mathrm{ch}}^{2k}(tD,X)(f^0,\ldots,f^{2k}) := t^{2k}\int_{\Delta_{2k}} \mathrm{Str}\Big[f^0 e^{-\sigma_0 t^2 H_{\frac{X}{t^2}}} c(df^1)\\ \cdots c(df^{2k}) e^{-\sigma_{2k} t^2 H_{\frac{X}{t^2}}}\Big] d\mathrm{Vol}_{\Delta_{2k}}.\end{aligned} \tag{3.43}$$

Since $\mathrm{Ind}_{e^{-X}}(D_{\mathrm{Im}p,+})$ is independent of t, taking the limit as $t\to 0$ in (3.42), we get by Theorem 3.10 that

$$\mathrm{Ind}_{e^{-X}}(D_{\mathrm{Im}p,+}) = (2\pi\sqrt{-1})^{-n/2}\int_M \widehat{A}(F_{\mathfrak{g}}^M(X))\mathrm{Ch}(\mathrm{Im}p)d\mathrm{Vol}_M. \tag{3.44}$$

□

4. The infinitesimal equivariant eta cochains

In this section, we prove the limit of truncated infinitesimal equivariant eta cochains exists when J goes to infinity. By the Duhamel principle and (2.5), we have

$$\begin{aligned}
&\|D_{-X}e^{-utH_X}\|_{u^{-1}} \\
&\leq \sum_{m\geq 0}(ut)^m \int_{\triangle_m} \|D_{-X}(1+D^2)^{-\frac{1}{2}}\| \|(1+D^2)^{\frac{1}{2}}e^{-\frac{\sigma_0}{2}utD^2}\| \|e^{-\frac{\sigma_0}{2}utD^2}\|_{(u\sigma_0)^{-1}} \\
&\qquad \cdot \|F_X(1+D^2)^{-\frac{1}{2}}\| \|(1+D^2)^{\frac{1}{2}}e^{-\frac{\sigma_1}{2}utD^2}\| \|e^{-\frac{\sigma_1}{2}utD^2}\|_{(u\sigma_1)^{-1}} \\
&\qquad\qquad \cdots \|F_X(1+D^2)^{-\frac{1}{2}}\| \|(1+D^2)^{\frac{1}{2}}e^{-\frac{\sigma_m}{2}utD^2}\| \|e^{-\frac{\sigma_m}{2}utD^2}\|_{(u\sigma_m)^{-1}}d\sigma \\
&\leq \|D_{-X}(1+D^2)^{-\frac{1}{2}}\|(eut)^{-\frac{1}{2}} \sum_{m\geq 0}(e^{-1}ut\|F_X(1+D^2)^{-\frac{1}{2}}\|^2)^{\frac{m}{2}} \\
&\qquad\qquad\qquad \cdot e^{\frac{ut}{2}}(\mathrm{tr}e^{-\frac{t}{2}D^2})^u \int_{\triangle_m} \sigma_0^{-\frac{1}{2}}\cdots\sigma_m^{-\frac{1}{2}}d\sigma \\
&\leq \|D_{-X}(1+D^2)^{-\frac{1}{2}}\|(ut)^{-\frac{1}{2}}2e^{\frac{ut}{2}}\{1+[\|F_X(1+D^2)^{-\frac{1}{2}}\|^2e^{-1}\pi ut]^{\frac{1}{2}}\} \\
&\qquad\qquad\qquad \cdot e^{\|F_X(1+D^2)^{-\frac{1}{2}}\|^2\pi ut}(\mathrm{tr}e^{-\frac{t}{2}D^2})^u,
\end{aligned} \tag{4.1}$$

where

$$\int_{\triangle_m}\sigma_0^{-\frac{1}{2}}\cdots\sigma_m^{-\frac{1}{2}}d\sigma = \frac{\pi^{\frac{m+1}{2}}}{\Gamma(\frac{m}{2}+1)}, \tag{4.2}$$

and

$$\frac{\pi^{\frac{m+1}{2}}}{\Gamma(\frac{m}{2}+1)} = \frac{\pi^{\frac{m+1}{2}}}{(\frac{m}{2})!}, \qquad \text{when } m \text{ is even}, \tag{4.3}$$

$$\frac{\pi^{\frac{m+1}{2}}}{\Gamma(\frac{m}{2}+1)} \leq \frac{2\pi^{\frac{m}{2}}}{(\frac{m-1}{2})!}, \qquad \text{when } m \text{ is odd}. \tag{4.4}$$

Now let M be a compact oriented odd dimensional Riemannian manifold without boundary with a fixed spin structure and S be the bundle of spinors on M. The fundamental setup consists with that on page 2. Let $K_t = \sqrt{t}(D + \frac{c(X)}{4t})$, then $\frac{dK_t}{dt} = \frac{1}{2\sqrt{t}}D_{\frac{X}{t}}$. For $a_0, \ldots, a_{2k} \in C_G^\infty(M)$, we define the infinitesimal equivariant cochain $\mathrm{ch}_X^{2k}(K_t, \frac{dK_t}{dt})$ by the formula:

$$\begin{aligned}&\mathrm{ch}_X^{2k}\Big(K_t, \frac{dK_t}{dt}\Big)(a_0, \ldots, a_{2k})\\&\quad = \sum_{j=0}^{2k}(-1)^j \langle a_0, [K_t, a_1], \ldots, [K_t, a_j], \frac{dK_t}{dt}, [K_t, a_{j+1}], \ldots, [K_t, a_{2k}]\rangle_t(X).\end{aligned} \tag{4.5}$$

If A_j $(0 \leq j \leq q)$ are operators on $\Gamma(M, S(TM))$, we define

$$\langle A_0, \ldots, A_q\rangle_t(X) = \int_{\triangle_q} \mathrm{tr}[e^{-L_X} A_0 e^{-\sigma_0 K_t^2} A_1 e^{-\sigma_1 K_t^2} \cdots A_q e^{-\sigma_q K_t^2}]d\sigma, \tag{4.6}$$

where $\triangle_q = \{(\sigma_0, \ldots, \sigma_q) \mid \sigma_0 + \cdots + \sigma_q = 1,\ \sigma_j \geq 0\}$ is a simplex in $\mathbf{R^q}$ and L_X is the Lie derivative generated by X on the spinors bundle.

Formally, *the infinitesimal equivariant eta cochain* for the odd dimensional manifold is defined to be an even cochain sequence by the formula:

$$\eta_X^{2k}(D) = \frac{1}{\sqrt{\pi}} \int_0^\infty \mathrm{ch}_X^{2k}\Big(K_t, \frac{dK_t}{dt}\Big)dt, \tag{4.7}$$

Then $\eta_X^0(D)(1)$ is the half of the infinitesimal equivariant eta invariant defined by Goette in [19]. In order to prove that the above expression is well defined, it is necessary to check the integrality near the two ends of the integration. Firstly, the regularity at infinity comes from the following lemma.

Lemma 4.1. *For $a_0, \ldots, a_{2k} \in C_G^\infty(M)$, we have*

$$\mathrm{ch}_X^{2k}\Big(K_t, \frac{dK_t}{dt}\Big)(a_0, \ldots, a_{2k}) = O(t^{-\frac{3}{2}}), \quad \text{as } t \to \infty. \tag{4.8}$$

Proof. Let L_0 be a fixed large number. Then $\frac{1}{\Gamma(\frac{1}{2})} \int_\varepsilon^{L_0} \mathrm{ch}_X^{2k}(K_t, \frac{dK_t}{dt})(a_0, \ldots, a_{2k})dt$ is well defined by Lemma 2.2 and (4.1). Similarly to Lemma 2.2 and (4.1), we know that Lemma 3.5 in [26] holds when J goes to infinity. So $\frac{1}{\Gamma(\frac{1}{2})} \int_{L_0}^\infty \mathrm{ch}_X^{2k}(K_t, \frac{dK_t}{dt})dt$ is well defined and Lemma 4.1 holds. □

Next, we prove the regularity at zero. Let $F_* = D_{-X}^2$ and $\widehat{F_*} = H_X - dtD_X$ where dt is an auxiliary Grassmann variable as shown in [8]. Then $t\psi_t\widehat{F_*} = tH_{\frac{X}{t}} - 2t^{\frac{3}{2}}dt\frac{dK_t}{dt}$. Let

$$\mathrm{ch}^{2k}(\widehat{F_*})(a_0, \ldots, a_{2k}) = t^k \int_{\triangle_{2k}} \psi_t \mathrm{tr}[a_0 e^{-t\sigma_0 \widehat{F_*}}[D, a_1] \cdots [D, a_{2k}]e^{-t\sigma_{2k}\widehat{F_*}}]d\sigma, \tag{4.9}$$

$$\mathrm{ch}^{2k}(F_*)(a_0, \ldots, a_{2k}) = t^k \int_{\triangle_{2k}} \psi_t \mathrm{tr}[a_0 e^{-t\sigma_0 H_X}[D, a_1] \cdots [D, a_{2k}]e^{-t\sigma_{2k} H_X}]d\sigma. \tag{4.10}$$

By the Duhamel principle and $dt^2 = 0$, we have

$$
\begin{aligned}
e^{-t\sigma_j \psi_t \widehat{F_*}} &= e^{-t\sigma_j \psi_t H_X} + \int_0^1 e^{-(1-a)t\sigma_j \psi_t H_X} \Big(2t^{\frac{3}{2}} dt \frac{dK_t}{dt}\Big) e^{-at\sigma_j \psi_t H_X} d(\sigma_j a) \\
&= e^{-t\sigma_j \psi_t H_X} + 2t^{\frac{3}{2}} dt \int_0^{\sigma_j} e^{-(\sigma_j - \xi)t\psi_t H_X} \frac{dK_t}{dt} e^{-t\xi\psi_t H_X} d\xi
\end{aligned}
\tag{4.11}
$$

By (4.5) and (4.9)–(4.11) and $dt^2 = 0$, we get

$$
\begin{aligned}
&\mathrm{ch}^{2k}(\widehat{F_*})(a_0, \dots, a_{2k}) \\
&\quad = \mathrm{ch}^{2k}(F_*)(a_0, \dots, a_{2k}) - 2t^{\frac{3}{2}} \mathrm{ch}_X^{2k}\Big(K_t, \frac{dK_t}{dt}\Big)(a_0, \dots, a_{2k}) dt.
\end{aligned}
\tag{4.12}
$$

Lemma 4.2. *The following estimate holds*

$$
\mathrm{ch}_X^{2k}\Big(K_t, \frac{dK_t}{dt}\Big) \sim O(1) \quad \textit{when } t \to 0. \tag{4.13}
$$

Proof. By (4.12), we only need to prove

$$
\mathrm{ch}^{2k}(\widehat{F_*})(a_0, \dots, a_{2k}) - \mathrm{ch}^{2k}(F_*)(a_0, \dots, a_{2k}) = O(t^{\frac{3}{2}}) dt. \tag{4.14}
$$

Let

$$
Q_{\widehat{F_*}} = a_0(\widehat{F_*} + \partial_t)^{-1} c(da_1) \cdots c(da_{2q})(\widehat{F_*} + \partial_t)^{-1}, \tag{4.15}
$$

$$
Q_{F_*} = a_0(F_* + \partial_t)^{-1} c(da_1) \cdots c(da_{2q})(F_* + \partial_t)^{-1}. \tag{4.16}
$$

By using Lemma 8.4 in [24], we have

$$
t^q \psi_t [a_0 e^{-t\sigma_0 \widehat{F_*}} [D, a_1] \cdots [D, a_{2q}] e^{-t\sigma_{2q} \widehat{F_*}}](x, y) = t^{-q} \psi_t K_{Q_{\widehat{F_*}}}(x, y, t); \tag{4.17}
$$

$$
t^q \psi_t [a_0 e^{-t\sigma_0 H_X} [D, a_1] \cdots [D, a_{2q}] e^{-t\sigma_{2q} H_X}](x, y) = t^{-q} \psi_t K_{Q_{F_*}}(x, y, t). \tag{4.18}
$$

So we only need to prove

$$
t^{-q} \psi_t \mathrm{tr}\Big[K_{Q_{\widehat{F_*}}}(x, x, t) - K_{Q_{F_*}}(x, x, t)\Big] = O(t^{\frac{3}{2}}) dt. \tag{4.19}
$$

By the trace property, we have

$$
\begin{aligned}
&t^{-q} \psi_t \mathrm{tr}\Big[K_{Q_{\widehat{F_*}}}(x, x, t) - K_{Q_{F_*}}(x, x, t)\Big] \\
&\quad = t^{-q} \psi_t \mathrm{tr}\Big[K_{Q_{h\rho \widehat{F_*} (h\rho)^{-1}}}(x, x, t) - K_{Q_{\rho H_X \rho^{-1}}}(x, x, t)\Big].
\end{aligned}
\tag{4.20}
$$

By (3.15), (3.18) and (3.24) in [26] and $dt^2 = 0$ where we use dt instead of z in [26], we have

$$\begin{aligned} t^{-q}\psi_t \left[Q_{h\rho\widehat{F}_*(h\rho)^{-1}} - Q_{\rho H_X\rho^{-1}}\right] \\ = -t^{-q} dt\psi_t \left[a_0(\partial_t + \rho H_X\rho^{-1})^{-1}u(\partial_t + \rho H_X\rho^{-1})^{-1}c(da_1)\cdots \right. \\ \cdots c(da_{2q})(\partial_t + \rho H_X\rho^{-1})^{-1} + \cdots + a_0(\partial_t + \rho H_X\rho^{-1})^{-1}\cdots \\ \left. \cdots c(da_{2q})(\partial_t + \rho H_X\rho^{-1})^{-1}u(\partial_t + \rho H_X\rho^{-1})^{-1}\right]. \end{aligned} \tag{4.21}$$

By $O_G(u) \leq 0$ and $O_G((\partial_t + \rho H_X\rho^{-1})^{-1}) = -2$, we have

$$\begin{aligned} O_G\left[(\partial_t + \rho H_X\rho^{-1})^{-1}u(\partial_t + \rho H_X\rho^{-1})^{-1}c(da_1)\cdots \right. \\ \left. \cdots c(da_{2q})(\partial_t + \rho H_X\rho^{-1})^{-1}\right] = -2q - 4, \end{aligned} \tag{4.22}$$

which has odd Clifford elements. When we drop off the truncated order J in Lemma 2.9 in [26] and consider the convergent series on X as in Definition 3.2, we know that Lemma 2.9 in [26] holds for our operator in (4.22). By (4.20)–(4.22) and Lemma 2.9 1) in [26] for $j = n$ and $m = -2q - 4$, we get (4.19). □

Remark. Similarly to Proposition 1.2 in [28], We use the symbol calculus about the connection ∇^X in Section 3 instead of the Getzler symbol calculus in Proposition 1.2 in [28], then we can give another proof of Lemma 4.2.

Again Proposition 3.8 in [26] holds, we have

Proposition 4.3. *Assume that D is invertible with λ being the smallest positive eigenvalue of $|D|$ and $\|dp\| < \lambda$, then the pairing $\langle \eta^*_X(D), \mathrm{ch}_*(p)\rangle$ is well defined.*

We also have the following theorem.

Theorem 4.4. *Assume D is invertible and $\|dp\| < \lambda$ where λ is the smallest eigenvalue of $|D|$, then we have*

$$\frac{1}{2}\eta_X(p(D \otimes I_r)p) = \langle \eta^*_X(D), \mathrm{ch}_*(p)\rangle, \tag{4.23}$$

where $\eta_X(p(D \otimes I_r)p)$ is the Goette's infinitesimal equivariant eta invariant.

Proof. We still use the same notations and discussions after Proposition 3.8 in [26]. The difference is that we add ψ_t in the definition of A. That is, let $A = d_{(u,s,t)} + \psi_t\widetilde{\mathbf{D}_{-\mathbf{X}}}$ be a superconnection on the trivial infinite dimensional superbundle with the base $[0, 1] \times \mathbf{R} \times (0, +\infty)$ and the fibre $H \otimes \mathbf{C^r} \oplus H \otimes \mathbf{C^r}$. Then we have

$$\begin{aligned} A^2 = t\psi_t\mathbf{D}^2_{-X,u} - s^2/4 - (1-u)t^{\frac{1}{2}}s\sigma[\mathbf{D}, p] + ds\sigma\left(p - \frac{1}{2}\right) \\ + t^{\frac{1}{2}}du(2p-1)[\mathbf{D}, p] + \frac{dt}{2t^{\frac{1}{2}}}\psi_t\mathbf{D}_{X,u}. \end{aligned} \tag{4.24}$$

Since we prove the regularity at zero, we can take $\varepsilon = 0$ in (3.41)–(3.45) in [26]. By the following lemma, Theorem 4.4 can be proved. □

Lemma 4.5. *Let $D_u = D + u(2p-1)[D, p]$ for $u \in [0,1]$. We assume that D be invertible and $\|dp\| < \lambda$, then we have $\eta_X(D_0) = \eta(D_1)$.*

Proof. By $\|dp\| < \lambda$, then $D_u = D + u(2p-1)[D, p]$ is invertible for $u \in [0,1]$. Similar to the discussions of Proposition 4.4 in [28], the infinitesimal equivariant eta invariant of D_u is well defined. So $\eta_X(D_u)$ is smooth. Let $A = (2p-1)dp$. Then by the definition of the infinitesimal equivariant eta invariant and the Duhamel principle, we have

$$\frac{d}{du}\eta_X(D_u) = \frac{1}{\sqrt{\pi}}\int_0^{+\infty} \mathrm{tr}[e^{-X}Ae^{-tD^2_{-\frac{X}{t},u}}]d\sqrt{t} + L, \tag{4.25}$$

where

$$L = -\frac{t^{\frac{1}{2}}}{2\sqrt{\pi}}\int_0^{+\infty}\int_0^1 \mathrm{tr}\left\{e^{-X}D_{\frac{X}{t},u}e^{-(1-s)tD^2_{-\frac{X}{t},u}}[D_{-\frac{X}{t},u}, A]_+e^{-stD^2_{-\frac{X}{t},u}}ds\right\}dt. \tag{4.26}$$

By the trace property and direct computations, then

$$\frac{\partial}{\partial t}\left(\sqrt{t}D_u + \frac{c(X)}{4\sqrt{t}}\right)^2 = \frac{1}{2}\left[D_u + \frac{c(X)}{4t}, D_u - \frac{c(X)}{4t}\right]_+, \tag{4.27}$$

$$\begin{aligned}\int_0^1 \mathrm{tr}&\left\{Ae^{-(1-s)tD^2_{-\frac{X}{t},u}}[D_{-\frac{X}{t},u}, D_{\frac{X}{t},u}]_+e^{-stD^2_{-\frac{X}{t},u}}\right\}ds\\ &= \int_0^1 \mathrm{tr}\left\{D_{\frac{X}{t},u}e^{-(1-s)tD^2_{-\frac{X}{t},u}}[D_{-\frac{X}{t},u}, A]_+e^{-stD^2_{-\frac{X}{t},u}}\right\}ds.\end{aligned} \tag{4.28}$$

By using the Duhamel principle and the Leibniz rule and (4.26)–(4.28), we get

$$\frac{\partial}{\partial u}\psi_t\mathrm{tr}[D_{X,u}e^{-t(D^2_{-X,u}+L_X)}]d\sqrt{t} = \frac{\partial}{\partial t}\mathrm{tr}[t^{\frac{1}{2}}e^{-X}Ae^{-tD^2_{-\frac{X}{t},u}}]dt. \tag{4.29}$$

So

$$\frac{d}{du}\eta_X(D_u) = \frac{1}{\sqrt{\pi}}\mathrm{tr}[t^{\frac{1}{2}}e^{-X}Ae^{-tD^2_{-\frac{X}{t},u}}]\Big|_{t=0}^{+\infty}. \tag{4.30}$$

As D_u is invertible, then

$$\lim_{t\to+\infty}\mathrm{tr}[t^{\frac{1}{2}}e^{-X}Ae^{-tD^2_{-\frac{X}{t},u}}] = 0. \tag{4.31}$$

Using Lemma 2.9 in [26] for $j = n$ and $m = -1$, similar to the discussions on Line 14 in [28, p. 164], we have

$$\begin{aligned}&\lim_{t\to 0}\operatorname{tr}[t^{\frac{1}{2}}e^{-X}Ae^{-tD^2-\frac{X}{t},u}]\\&\quad = c_0\int_M \widehat{A}(F^M_{\mathfrak{g}}(X))\operatorname{tr}\left\{(2p-1)(dp)\exp\left[\frac{\sqrt{-1}}{2\pi}(A'\wedge A'+dA')\right]\right\}\\&\quad = 0,\end{aligned}\tag{4.32}$$

where $A' = u(2p-1)dp$. Then by (4.30)–(4.32), Lemma 4.5 is proved. □

Let N be an even-dimensional compact manifold with the boundary M. We endow N with a metric which is a product in a collar neighborhood of M. Denote by D (D_M) the Dirac operator on N (M). Let $C^\infty_*(N) = \{f \in C^\infty(N) \mid f$ is independent of the normal coordinate x_n near the boundary $\}$.

Definition 4.6. The infinitesimal equivariant Chern–Connes character on N, $\tau_X = \{\tau^0_X, \tau^2_X, \ldots, \tau^{2q}_X \cdots\}$ is defined by

$$\begin{aligned}\tau^{2q}_X(f^0, f^1, \cdot, f^{2q}) &:= -\eta^{2q}_X(D_M)(f^0|_M, f^1|_M, \cdot, f^{2q}|_M)\\&\quad + \frac{1}{(2q)!(2\pi\sqrt{-1})^q}\int_M \widehat{A}(F^M_{\mathfrak{g}}(X))f^0df^1\wedge\cdots\wedge df^{2q},\end{aligned}\tag{4.33}$$

where $f^0, f^1, \cdot, f^{2q} \in C^\infty_*(N)$.

Similarly to Proposition 4.2 in [27], we have

Proposition 4.7. *The infinitesimal equivariant Chern–Connes character is $b - B$ closed (for the definitions of b, B, see [15]). That is, we have*

$$b\tau^{2q-2}_X + B\tau^{2q}_X = 0. \tag{4.34}$$

By Proposition 4.3, we have

Proposition 4.8. *Suppose that D_M is invertible with λ being the smallest positive eigenvalue of $|D_M|$. We assume that $\|d(p|_M)\| < \lambda$, then the pairing $\langle\tau^*_X, \mathrm{ch}_*(p)\rangle$ is well defined.*

We let $C_1(M) = M \times (0, 1]$, $\widetilde{N} = N \cup_{M\times\{1\}} C_1(M)$ and $\mathcal{U}$ be a collar neighborhood of M in N. For $\varepsilon > 0$, we take a metric g^ε of $\widetilde{N}$ such that on $\mathcal{U} \cup_{M\times\{1\}} C_1(M)$

$$g^\varepsilon = \frac{dr^2}{\varepsilon} + r^2 g^M.$$

Let $S = S^+ \oplus S^-$ be spinors bundle associated to $(\widetilde{N}, g^\varepsilon)$ and H^∞ be the set $\{\xi \in \Gamma(\widetilde{N}, S) \mid \xi$ and its derivatives are zero near the vertex of cone$\}$. Denote by

$L_c^2(\widetilde{N}, S)$ the L^2-completion of H^∞ (similarly define $L_c^2(\widetilde{N}, S^+)$ and $L_c^2(\widetilde{N}, S^-)$). Let

$$D_\varepsilon: \ H^\infty \to H^\infty; \quad D_{+,\varepsilon}: \ H_+^\infty \to H_-^\infty,$$

be the Dirac operators associated with $(\widetilde{N}, g^\varepsilon)$ which are Fredholm operators for the sufficiently small ε. By $\|d(p|_M)\| < \lambda$, then $pD_M p$ is invertible. Recall the Goette's infinitesimal equivariant index theorem for the twisting bundle Imp with the connection pd in [19] that

$$\mathrm{Ind}_{e^{-X}}(pD_{+,\varepsilon}p) = \sum_{r=0}^{\infty} \frac{(-1)^r}{r!(2\pi\sqrt{-1})^r} \int_N \widehat{A}(F_{\mathfrak{g}}^N(X))\mathrm{Tr}[p(dp)^{2r}] - \frac{1}{2}\eta_X(pD_M p). \tag{4.35}$$

By the Stokes theorem and the trace property and $p(dp)^2 = (dp)^2 p$, we have

$$\int_M \widehat{A}(F_{\mathfrak{g}}^M(X))\mathrm{tr}[p_M(d_M p_M)^{2k-1}] = 0. \tag{4.36}$$

By $L_X(p) = \iota_X d(p) = 0$, then $\iota_X[p(dp)^{2k-1}] = 0$. By the Stokes theorem and (4.36), we get

$$\begin{aligned}\int_N \widehat{A}(F_{\mathfrak{g}}^N(X))\mathrm{tr}[(d_N p_N)^{2k}] &= \int_N (d + \iota_X)\left[\widehat{A}(F_{\mathfrak{g}}^N(X))\mathrm{tr}[p(d_N p_N)^{2k-1}]\right] \\ &= \int_M \widehat{A}(F_{\mathfrak{g}}^M(X))\mathrm{tr}[p_M(d_M p_M)^{2k-1}] = 0.\end{aligned} \tag{4.37}$$

By Theorem 4.4 and Definition 4.6 and (2.14) and (4.37), we get

Theorem 4.9. *Suppose that D_M is invertible with λ being the smallest positive eigenvalue of $|D_M|$. We assume that $\|d(p|_M)\| < \lambda$ and $p \in M_{r\times r}(C_*^\infty(N))$, then*

$$\mathrm{Ind}_{e^{-X}}(pD_{+,\varepsilon}p) = \langle \tau_X^*(D), \mathrm{ch}_*(p)\rangle. \tag{4.38}$$

5. The infinitesimal equivariant Chern–Connes character for a family of Dirac operators

In this section, we extend Sections 2 and 3 to the family case. Let us recall the definition of the equivariant family Bismut Laplacian. Let M be a $n+q$ dimensional compact connected manifold and B_0 be a q dimensional compact connected manifold. Assume that $\pi : M \to B_0$ is a fibration and M and B_0 are oriented. Taking the orthogonal bundle of the vertical bundle TZ in TM with respect to any Riemannian metric, determines a smooth horizontal subbundle $T^H M$, i.e. $TM = T^H M \oplus TZ$. Recall that B_0 is Riemannian, so we can lift the Euclidean scalar product g_{B_0} of TB_0 to $T^H M$. And we assume that TZ is endowed with a scalar product g_Z. Thus we can introduce a new scalar product $g_{B_0} \oplus g_Z$ in TM. Denote by ∇^L the Levi-Civita connection on TM with respect to this metric. Let ∇^{B_0} denote the Levi-Civita connection on TB_0 and still denote by ∇^{B_0} the pullback connection on $T^H M$. Let $\nabla^Z = P_Z(\nabla^L)$, where P_Z denotes the projection to TZ. Let $\nabla^{\oplus} = \nabla^{B_0} \oplus \nabla^Z$ and $\omega = \nabla^L - \nabla^{\oplus}$ and T be the torsion tensor of $\nabla^{\oplus}$. Now we assume that the bundle TZ is spin. Let $S(TZ)$ be the associated spinors bundle and ∇^Z can be lifted to give a connection on $S(TZ)$. Let D be the tangent Dirac operator.

Let G be a compact Lie group which acts fiberwise on M. We will consider that G acts as identity on B_0. We assume that the action of G lifts to $S(TZ)$ and the G-action commutes with D. Let E be the vector bundle $\pi^*(\wedge T^* B_0) \otimes S(TZ)$. This bundle carries a natural action m_0 of the degenerate Clifford module $Cl_0(M)$. Define the connection for $X \in \mathfrak{g}$ whose Killing vector field is in TZ,

$$\nabla^{E,-X,\oplus} := \pi^* \nabla^{B_0} \otimes 1 + 1 \otimes \nabla^{S,-X}, \tag{5.1}$$

$$\omega(Y)(U, V) := g(\nabla^L_Y U, V) - g(\nabla^{\oplus}_Y U, V), \tag{5.2}$$

$$\nabla^{E,-X,0}_Y := \nabla^{E,-X,\oplus}_Y + \frac{1}{2} m_0(\omega(Y)), \tag{5.3}$$

for $Y, U, V \in TM$. Then the equivariant Bismut superconnection acting on $\Gamma(M, \pi^* \wedge (T^* B_0) \otimes S(TZ))$ is defined by

$$B^{-X} = \sum_{i=1}^{n} c(e_i^*) \nabla^{E,-X,0}_{e_i} + \sum_{j=1}^{q} f_j^* \wedge \nabla^{E,-X,0}_{f_j}; \quad B^{-X} = B + \frac{1}{4} c(X). \tag{5.4}$$

where $e_1, \ldots, e_n$ and $f_1, \ldots, f_q$ are orthonormal basis of TZ and TB_0 respectively, and B is the Bismut superconnection defined by

$$\nabla^{E,\oplus} := \pi^* \nabla^{B_0} \otimes 1 + 1 \otimes \nabla^S; \tag{5.5}$$

$$\nabla^{E,0}_Y := \nabla^{E,\oplus}_Y + \frac{1}{2} m_0(\omega(Y)); \tag{5.6}$$

$$B = \sum_{i=1}^{n} c(e_i^*) \nabla^{E,0}_{e_i} + \sum_{j=1}^{\overline{q}} c(f_j^*) \nabla^{E,0}_{f_j}. \tag{5.7}$$

Define the equivariant family Bismut Laplacain as follows:

$$H_{B,X} = (B^{-X})^2 + L^E_X, \tag{5.8}$$

where L^E_X is the Lie derivative induced by X on the bundle E. Then

$$H_{B,X} = D^2 + F_+ + \widetilde{F}_+, \tag{5.9}$$

where $D^2_{-X} = D^2 + F_+$ and $\widetilde{F}_+ = H_{B,X} - D^2_{-X}$ is a first order differential operator along the fibre with coefficients in $\Omega_{\geq 1}(B_0)$.

Definition 5.1. The infinitesimal equivariant family JLO cochain $\mathrm{ch}^{2k}(B,X)$ can be defined by the formula for $f^0,\dots,f^{2k}$ in $C^\infty_G(M)$:

$$\mathrm{ch}^{2k}(B,X)(f^0,\dots,f^{2k}) := \int_{\Delta_{2k}} \mathrm{Str}\Big[f^0 e^{-\sigma_0 H_{B,X}} c(df^1) e^{-\sigma_1 H_{B,X}} \cdots \cdots c(df^{2k}) e^{-\sigma_{2k} H_{B,X}}\Big] d\,\mathrm{Vol}_{\Delta_{2k}}, \quad (5.10)$$

where Str is taking the trace along the fibre.

Similarly to Section 2, we can prove that (5.10) is well defined and $\langle \mathrm{ch}^*(B,X), \mathrm{ch}\, p\rangle$ is convergent by the following lemma.

Lemma 5.2. *For any* $1 \geq u > 0$, *we have:*

$$\|e^{-uH_{B,X}}\|_{u^{-1}} \leq C_0 e^{\|F_X(1+D^2)^{-\frac{1}{2}}\|\pi u} (\mathrm{tr}[e^{-\frac{D^2}{2}}])^u, \quad (5.11)$$

where the constant C_0 *is independent of* u.

Proof. By (5.10) and the Duhamel principle, we have

$$e^{-uH_{B,X}} = e^{-uH_X} + \sum_{r>0}^{\dim B_0} I_r, \quad (5.12)$$

where

$$I_r = \int_{\Delta_r} e^{-s_0 u H_X} \widetilde{F}_+ e^{-s_1 u H_X} \cdots \widetilde{F}_+ e^{-s_r u H_X} ds. \quad (5.13)$$

In (4.1), we use $\widetilde{F}_+$ and su instead of D_{-X} and u respectively and let $t = 1$, then we have

$$\begin{aligned} \|\widetilde{F}_+ e^{-suH_X}\|_{(su)^{-1}} &\leq 2(su)^{-\frac{1}{2}} \|\widetilde{F}_+(1+D^2)^{-\frac{1}{2}}\| e^{\frac{su}{2}} \\ &\quad \cdot \{1 + [\|F_X(1+D^2)^{-\frac{1}{2}}\|^2 e^{-1}\pi su]^{\frac{1}{2}}\} \\ &\quad \cdot e^{\|F_X(1+D^2)^{-\frac{1}{2}}\|^2 \pi su} (\mathrm{tr} e^{-\frac{1}{2}D^2})^{su}. \end{aligned} \quad (5.14)$$

By Lemma 2.2 and (5.12)–(5.14) and the Hölder inequality, we get Lemma 5.2. □

Similarly to Propositions 4.11 and 4.12 in [3], we have

Proposition 5.3. (1) *The infinitesimal equivariant family Chern–Connes character is closed:*

$$(B + b + d_{B_0})(\mathrm{ch}^*(B,X)) = 0. \quad (5.15)$$

(2) *Let* $B_\tau = B^{-X} + \tau V$ *and* V *is a bounded operator which commutes with* e^{-X}, *then there exists a cochain* $\mathrm{ch}^*(B_\tau, X, V)$ *such that*

$$\frac{d}{d\tau}\mathrm{ch}^*(B_\tau, X) = -[b + B + d_{B_0}]\mathrm{ch}^*(B_\tau, X, V). \quad (5.16)$$

By taking $V = (2p-1)[B, p]$, we get

Theorem 5.4. *The following index formula holds in the cohomology of B_0*

$$\mathrm{Ch}_{e^{-X}}[\mathrm{Ind}(D_{\mathrm{Im}p,+,z})] = \langle \mathrm{ch}^*(B, X), \mathrm{ch}(p)\rangle\,. \tag{5.17}$$

Let ϕ_t be the rescaling operator on $\Omega(B_0)$ defined by $dy_j \to \frac{dy_j}{\sqrt{t}}$ for $t > 0$. By the method in Section 4 in [26], similarly to Theorem 2.12 in [26], we get

Lemma 5.5. *When $2k \leq n$ and X is small, then for $f^j \in C_G^\infty(M)$,*

$$\lim_{t\to 0} \phi_t \psi_t \mathrm{ch}^{2k}(\sqrt{t}B, X)(f^0, \dots, f^{2k})_J = \frac{1}{(2k)!}(2\pi\sqrt{-1})^{-n/2}\int_Z f^0 \wedge df^1 \wedge \cdots \wedge df^{2k}\widehat{A}(F_{\mathfrak{g}}^Z(X))_J. \tag{5.18}$$

Extending Theorem 3.9 to the family case, we have by Lemma 5.5 by

Theorem 5.6. *When $2k \leq n$ and X is small, then for $f^j \in C_G^\infty(M)$,*

$$\lim_{t\to 0} \phi_t \psi_t \mathrm{ch}^{2k}(\sqrt{t}B, X)(f^0, \dots, f^{2k}) = \frac{1}{(2k)!}(2\pi\sqrt{-1})^{-n/2}\int_Z f^0 \wedge df^1 \wedge \cdots \wedge df^{2k}\widehat{A}(F_{\mathfrak{g}}^Z(X)). \tag{5.19}$$

By Theorems 5.4 and 5.6 and the following homotopy property, similarly to Corollary 3.11, we have

Corollary 5.7. *When X is small, then*

$$\mathrm{Ch}_{e^{-X}}[\mathrm{Ind}(D_{\mathrm{Im}p,+,z})] = (2\pi\sqrt{-1})^{-n/2}\int_Z \widehat{A}(F_{\mathfrak{g}}^Z(X))\mathrm{Ch}(\mathrm{Im}p). \tag{5.20}$$

Let $B_t = \sqrt{t}\phi_t\psi_t(B^{-X})$ and $\mathcal{F}_t = B_t^2$. Then we have the homotopy formula:

Proposition 5.8. *There is a cochain $\mathrm{ch}(B_t, \frac{dB_t}{dt}, X)$ such that the following formula holds*

$$\frac{d\,\mathrm{ch}(B_t, X)}{dt} = -(b + B + d_{B_0})\mathrm{ch}\Big(B_t, \frac{dB_t}{dt}, X\Big). \tag{5.21}$$

Proof. We know that B_t is a superconnection on the infinite dimensional bundle $C^\infty(M, E) \to B_0$ which we write $\mathcal{E} \to B_0$. Let $\widetilde{B_0} = B_0 \times \mathbb{R}_+$, and $\widetilde{\mathcal{E}}$ be the superbundle $\pi^*\mathcal{E}$ over $\widetilde{B_0}$, which is the pull-back to $\widetilde{B_0}$ of $\mathcal{E}$. Define a superconnection $\widehat{B}$ on $\widetilde{\mathcal{E}}$ by the formula

$$(\widehat{B}\beta)(x,t) = (B_t\beta(\cdot,t))(x) + dt \wedge \frac{\partial\beta(x,t)}{\partial t}. \tag{5.22}$$

The curvature $\widehat{\mathcal{F}}$ of $\widehat{B}$ is

$$\widehat{\mathcal{F}} = \mathcal{F}_t - \frac{dB_t}{dt} \wedge dt, \tag{5.23}$$

where $\mathcal{F}_t = B_t^2$ is the curvature of B_t. By the Duhamel principle, then

$$e^{-\widehat{\mathcal{F}}} = e^{-\mathcal{F}_t} + \left(\int_0^1 e^{-u\mathcal{F}_t} \frac{dB_t}{dt} e^{-(1-u)\mathcal{F}_t} du\right) \wedge dt. \tag{5.24}$$

Let $f^0, \ldots, f^{2k}$ be in $C_G^\infty(M)$, then $[\widehat{B}, f^j] = [B_t, f^j]$. We replace K_t in (4.5) and (4.6) by the above B_t, then we define the cochain $\operatorname{ch}(B_t, \frac{dB_t}{dt}, X)$. So by (5.24), we get on $C_G^\infty(M)$ that

$$\operatorname{ch}(\widehat{B}, X) = \operatorname{ch}(B_t, X) + \operatorname{ch}\Big(B_t, \frac{dB_t}{dt}, X\Big) dt. \tag{5.25}$$

Similarly to (5.15), we have

$$(b + B + d_{\widetilde{B_0}})\operatorname{ch}(\widehat{B}, X) = 0; \quad (b + B + d_{B_0})\operatorname{ch}(B_t, X) = 0. \tag{5.26}$$

By (5.25) and (5.26), we get Proposition 5.8. □

Acknowledgements. The author would like to thank Profs. Weiping Zhang and Huitao Feng for very helpful suggestions and discussions. The author would like to thank the referee for careful reading and helpful comments.

References

[1] M. Atiyah, R. Bott and V. Patodi, On the heat equation and the index theorem, *Invent. Math.*, **19** (1973), 279–330. Zbl 0257.58008 MR 0650828

[2] F. Azmi, The equivariant Dirac cyclic cocycle, *Rocky Mountain J. Math.*, **30** (2000), 1171–1206. Zbl 0982.58017 MR 1810162

[3] M. Benameur and A. Carey, Higher spectral flow and an entire bivariant JLO cocycle, *J. K-theory*, **11** (2013), 183–232. Zbl 1276.58002 MR 3034288

[4] N. Berline, E. Getzler and M. Vergne, *Heat kernels and Dirac operators*, Springer-Verlag, Berlin, 1992. Zbl 0744.58001 MR 1215720

[5] N. Berline and M. Vergne, The equivariant index and Kirillov character formula, *Amer. J. Math.*, **107** (1985), 1159–1190. Zbl 0604.58046 MR 0805808

[6] N. Berline and M. Vergne, A computation of the equivariant index of the Dirac operators, *Bull. Soc. Math. France*, **113** (1985), 305–345. Zbl 0592.58044 MR 0834043

[7] J. M. Bismut, The infinitesimal Lefschetz formulas: a heat equation proof, *J. Func. Anal.*, **62** (1985), 435–457. Zbl 0572.58021 MR 0794778

[8] J. M. Bismut and D. S. Freed, The analysis of elliptic families, II., *Commun. Math. Phys.*, **107** (1986), 103–163. Zbl 0657.58038 MR 0861886

[9] J. Block and J. Fox, Asymptotic pseudodifferential operators and index theory, *Contemp. Math.*, **105** (1990), 1–32. Zbl 0704.58048 MR 1047274

[10] S. Chern and X. Hu, Equivariant Chern character for the invariant Dirac operators, *Michigan Math. J.*, **44** (1997), 451–473. Zbl 0911.58035 MR 1481113

[11] A. Connes, Entire cyclic cohomology of Banach algebras and characters of θ-summable Fredholm module, *K-Theory*, **1** (1988), 519–548. Zbl 0657.46049 MR 0953915

[12] A. Connes and H. Moscovici, Cyclic cohomology, the Novikov conjecture and hyperbolic groups, *Topology*, **29** (1990), 345–388. Zbl 0759.58047 MR 1066176

[13] H. Donnelly, Eta invariants for G-space, *Indiana Univ. Math. J.*, **27** (1978), 889–918. Zbl 0402.58006 MR 0511246

[14] H. Feng, A note on the noncommutative Chern character (in Chinese), *Acta Math. Sinica*, **46** (2003), 57–64. Zbl 1041.53032 MR 1971713

[15] H. Figueroa, J. Gracia-Bondía and J. Várilly, *Elements of noncommutative geometry*, Birkhäuser, Boston, 2001. Zbl 0958.46039 MR 1789831

[16] E. Getzler, The odd Chern character in cyclic homology and spectral flow, *Topology*, **32** (1993), 489–507. Zbl 0801.46088 MR 1231957

[17] E. Getzler, Pseudodifferential operators on supermanifolds and the Atiyah–Singer index theorem, *Comm. Math. Phys.*, **92** (1983), no. 2, 163–178. Zbl 0543.58026 MR 0728863

[18] E. Getzler and A. Szenes, On the Chern character of theta-summable Fredholm modules, *J. Func. Anal.*, **84** (1989), 343–357. Zbl 0686.46044 MR 1001465

[19] S. Goette, Equivariant eta invariants and eta forms, *J. reine angew Math.*, **526** (2000), 181–236. Zbl 0974.58021 MR 1778304

[20] A. Jaffe, A. Lesniewski and K. Osterwalder, Quantum K-theory: The Chern character, *Comm. Math. Phys.*, **118** (1988), 1–14. Zbl 0656.58048 MR 0954672

[21] S. Klimek and A. Lesniewski, Chern character in equivariant entire cyclic cohomology, *K-Theory*, **4** (1991), 219–226. Zbl 0744.46064 MR 1106953

[22] J. D. Lafferty, Y. L. Yu and W. P. Zhang, A direct geometric proof of Lefschetz fixed point formulas, *Trans. AMS.*, **329** (1992), 571–583. Zbl 0747.58016 MR 1022168

[23] M. Pflaum, The normal symbol on Riemannian manifolds, *New York J. Math.*, **4** (1998), 97–125. Zbl 0903.35099 MR 1640055

[24] R. Ponge and H. Wang, Noncommutative Geometry and Conformal Geometry, II. Connes–Chern character and the local equivariant index theorem, *J. Noncommut. Geom.*, **10** (2016), no. 1, 307–378.

[25] B. Simon, *Trace ideals and their applications*, Lecture Note 35, London Math. Soc., 1979. Zbl 1074.47001 MR 2154153

[26] Y. Wang, The noncommutative infinitesimal equivariant index formula, *J. K-Theory*, **14** (2014), 73–102. Zbl 06334061 MR 3238258

[27] Y. Wang, The equivariant noncommutative Atiyah–Patodi–Singer index theorem, *K-Theory*, **37** (2006), 213–233. Zbl 1117.58011 MR 2273457

[28] F. Wu, The Chern–Connes character for the Dirac operators on manifolds with boundary, *K-Theory*, **7** (1993), 145–174. Zbl 0787.58041 MR 1235286

[29] W. P. Zhang, A note on equivariant eta invariants, *Proc. AMS.*, **108** (1990), 1121–1129. Zbl 0688.58039 MR 1004426

Received 25 November, 2014; revised 02 March, 2015

Y. Wang, School of Mathematics and Statistics, Northeast Normal University,
Changchun Jilin 130024, China

E-mail: wangy581@nenu.edu.cn