

The Noether problem for Hopf algebras

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Abstract. In previous work, Eli Aljadeff and the first-named author attached an algebra \mathcal{B}_H of rational fractions to each Hopf algebra H . The generalized Noether problem is the following: for which finite-dimensional Hopf algebras H is \mathcal{B}_H the localization of a polynomial algebra? A positive answer to this question when H is the algebra of functions on a finite group G implies a positive answer to the classical Noether problem for G . We show that the generalized Noether problem has a positive answer for all finite-dimensional pointed Hopf algebras over a field of characteristic zero (we actually give a precise description of \mathcal{B}_H for such a Hopf algebra).

A theory of polynomial identities for comodule algebras over a Hopf algebra H gives rise to a universal comodule algebra whose subalgebra of coinvariants \mathcal{V}_H maps injectively into \mathcal{B}_H . In the second half of this paper, we show that \mathcal{B}_H is a localization of \mathcal{V}_H when H is a finite-dimensional pointed Hopf algebra in characteristic zero. We also report on a result by Uma Iyer showing that the same localization result holds when H is the algebra of functions on a finite group.

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1. Introduction

Let G be a finite group and k a field. Consider the purely transcendental extension $K = k(t_g \mid g \in G)$ of k generated by indeterminates t_g indexed by the elements of G . The group G acts on K by left multiplication: $h \cdot t_g = t_{hg}$ ($g, h \in G$). Let $L = K^G$ be the subfield of G -invariant elements of K . Emmy Noether posed the following problem in [14]: *is L a purely transcendental extension of k ?* A positive answer ensures that G can be realized as the Galois group of a Galois extension of k ; it also implies the existence of a generic polynomial for G over k (at least when k is infinite).

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There is an abundant bibliography on Noether's problem. Answers depend on the group and on the base field. For instance, by Fischer [3] the extension L/k is purely transcendental if G is abelian and k contains a primitive e -th root of unity, where e is the exponent of G . If k has not enough roots of unity, for instance when $k = \mathbb{Q}$ is the field of rationals, then Swan [19] showed that there are cyclic groups G such that L is not purely transcendental over k . For non-abelian groups, the answer to the problem may be negative even over an algebraically closed field: by Saltman [17] this is the case for certain meta-abelian p -groups when k is the field of complex numbers.

In this note we extend Noether's problem to the framework of finite-dimensional Hopf algebras over the base field k . To such a Hopf algebra H Eli Aljadeff and the first-named author associated an algebra \mathcal{B}_H of rational fractions, which is a finitely generated smooth domain of Krull dimension equal to the dimension of H (see [2]). The algebra \mathcal{B}_H is called the *generic base algebra* associated to H ; in the terminology of non-commutative geometry it is the "base space" of a "non-commutative fiber bundle" whose fibers are the forms of H .

The generalized Noether problem (GNP) which we introduce in this paper is the following: *is \mathcal{B}_H the localization of a polynomial algebra?* A positive answer to (GNP) for the Hopf algebra of k -valued functions on a finite group G implies a positive answer to the classical Noether problem for G and k . Our first main result (Theorem 4.4) states that (GNP) has a positive answer for all *finite-dimensional pointed* Hopf algebras over a field of characteristic zero. Actually in Theorem 4.6 we prove more precisely that for such a Hopf algebra H ,

$$\mathcal{B}_H = k[u_1^{\pm 1}, \dots, u_\ell^{\pm 1}, u_{\ell+1}, \dots, u_n],$$

where n is the dimension of H and ℓ is the order of the group G of group-like elements of H , and where u_1, \dots, u_n are monomials whose degrees are bounded by an integer defined in terms of a certain abelian quotient of G . The latter statement is a Hopf algebra analogue of the fact, due to Noether [13], that the algebra of G -invariant polynomials in $k[t_g \mid g \in G]$ is generated by homogeneous polynomials of degree $\leq \text{card } G$.

A theory of polynomial identities for comodule algebras had also been set up in [2], giving rise to the so-called universal comodule algebra \mathcal{U}_H , an analogue of the relatively free algebra of the classical theory of polynomial identities. The subalgebra \mathcal{V}_H of H -coinvariants of \mathcal{U}_H maps injectively into \mathcal{B}_H . In Section 5 we show that \mathcal{B}_H is a localization of \mathcal{V}_H when H is a finite-dimensional pointed Hopf algebra over a field of characteristic zero (see Theorem 5.4). Uma Iyer showed likewise that \mathcal{B}_H is a localization of \mathcal{V}_H when H is the algebra of k -valued functions on a finite group whose order is prime to the characteristic of k ; with her permission we state and prove her result (Theorem 5.5) in Section 5.5.

Throughout the paper we fix a field k . All linear maps are to be k -linear and unadorned tensor products mean tensor products over k .

2. Pointed Hopf algebras

In this section we prove some facts on pointed Hopf algebras needed in the proofs of the main results.

2.1. Hopf algebras and comodule algebras. By algebra we mean an associative unital k -algebra and by coalgebra a coassociative counital k -coalgebra. We denote the coproduct of a coalgebra by Δ and its counit by ε . We shall also make use of a Heyneman–Sweedler-type notation for the image

$$\Delta(x) = x_1 \otimes x_2$$

of an element x of a coalgebra C under the coproduct, and we write

$$\Delta^{(2)}(x) = x_1 \otimes x_2 \otimes x_3$$

for the image of x under the iterated coproduct $\Delta^{(2)} = (\Delta \otimes \text{id}_C) \circ \Delta$.

Given a Hopf algebra H , we denote its counit by ε and its antipode by S . We also denote the augmentation ideal $\ker(\varepsilon : H \rightarrow k)$ by H^+ and the group of group-like elements of H by $G(H)$.

Recall that a *right H -comodule algebra* over a Hopf algebra H is an algebra A equipped with a right H -comodule structure whose (coassociative, counital) *coaction* $\delta : A \rightarrow A \otimes H$ is an algebra map. The subalgebra $A^{\text{co-}H}$ of *right coinvariants* of an H -comodule algebra A is the following subalgebra of A :

$$A^{\text{co-}H} = \{a \in A \mid \delta(a) = a \otimes 1\}.$$

2.2. Finite-dimensional pointed Hopf algebras. In this subsection we present three technical results for finite-dimensional pointed Hopf algebras.

Recall that a Hopf algebra H is *pointed* if any simple subcoalgebra is one-dimensional. Group algebras $k[G]$, enveloping algebras $U(\mathfrak{g})$ of Lie algebras, Drinfeld–Jimbo quantum enveloping algebras $U_q(\mathfrak{g})$ and their finite-dimensional quotients $u_q(\mathfrak{g})$ are important examples of pointed Hopf algebras.

See [20, Chap. VIII] and [12, Chap. 5] for basic properties of pointed Hopf algebras.

Lemma 2.1. *Any finite-dimensional commutative pointed Hopf algebra over a field of characteristic zero is a group algebra.*

Proof. Since H is pointed, by scalar extension to the algebraic closure \bar{k} of k , we may suppose that k is algebraically closed; note that $G(H) = G(H \otimes \bar{k})$. It then follows from [19, Th. 13.1.2] that H is reduced, hence $H = O_k(G)$ is the Hopf algebra of k -valued functions on some finite group G . For the dual Hopf algebra $H^* = k[G]$, the pointedness of H means that any simple H^* -module is one-dimensional, which is only possible if G is abelian. By Fourier transform, we have $H = k[\widehat{G}]$, where \widehat{G} is the group of characters of G . \square

Given a Hopf algebra H , let H_{ab} be the largest commutative Hopf algebra quotient of H and let $q : H \rightarrow H_{\text{ab}}$ be the canonical Hopf algebra surjection.

Lemma 2.2. *Let H be a finite-dimensional pointed Hopf algebra. Assume that H_{ab} is a group algebra. Then there exists a right $k[G(H)]$ -module coalgebra retraction $\gamma : H \rightarrow k[G(H)]$ of the inclusion $k[G(H)] \subset H$ such that the composite*

$$H \xrightarrow{\gamma} k[G(H)] \xrightarrow{q|_{k[G(H)]}} H_{\text{ab}}$$

coincides with q .

By Lemma 2.1 and the fact that any Hopf algebra quotient of a pointed Hopf algebra is pointed, the assumption in the previous lemma is always satisfied when the base field is of characteristic zero.

Proof. Set $G = G(H)$. The quotient $R = H/H(k[G])^+$ is a quotient coalgebra of H . Let $H \rightarrow R; h \mapsto \bar{h}$ denote the quotient coalgebra map. By the cosemisimplicity of $k[G]$ it follows from [10, Lemma 4.2] (or [11, Cor. 3.11]) that there is a right $k[G]$ -module coalgebra retraction $\gamma : H \rightarrow k[G]$ of the inclusion. Then the map

$$\lambda : R \rightarrow k[G] \otimes R; \quad \lambda(\bar{h}) = \gamma(h_1) S(\gamma(h_3)) \otimes \bar{h}_2 \tag{2.1}$$

defines a left $k[G]$ -comodule coalgebra structure on R , and the associated smash-coproduct coalgebra $R \triangleright\blacktriangleleft k[G]$ is isomorphic to H via the unit-preserving, right $k[G]$ -module coalgebra isomorphism

$$\xi : H \xrightarrow{\cong} R \triangleright\blacktriangleleft k[G]; \quad \xi(h) = \bar{h}_1 \otimes \gamma(h_2). \tag{2.2}$$

Note that this kind of isomorphism from H to a smash-coproduct of R by $k[G]$ arises, in this manner, uniquely from a retraction $H \rightarrow k[G]$ as above. Identify H with $R \triangleright\blacktriangleleft k[G]$ via the isomorphism (2.2). Since γ is then identified with the quotient $H \rightarrow H/R^+k[G] = k[G]$, one sees that $q|_{k[G]} \circ \gamma = q$ if and only if

$$q \text{ vanishes on } R^+ = R^+ \otimes k. \tag{2.3}$$

We will now choose a new retraction γ so that this last condition is satisfied.

Denote the coradical filtrations on H and on R respectively by

$$k[G] = H_0 \subset H_1 \subset H_2 \subset \dots, \quad k = R_0 \subset R_1 \subset R_2 \subset \dots.$$

Note that each H_n is a right $k[G]$ -module subcoalgebra in H , and the isomorphism (2.2) restricts to

$$H_n \xrightarrow{\cong} R_n \triangleright\blacktriangleleft k[G], \quad n = 0, 1, \dots.$$

On the associated gradings, there is an induced isomorphism of graded right $k[G]$ -module coalgebras. It coincides with the canonical isomorphism of graded Hopf algebras

$$\text{gr } H = \bigoplus_{n \geq 0} H_n/H_{n-1} \xrightarrow{\cong} \text{gr } R \rtimes k[G], \quad \text{gr } R = \bigoplus_{n \geq 0} R_n/R_{n-1}$$

onto the biproduct $\text{gr } R \rtimes k[G]$ (see [15, Th. 3]) which arises from the projection $\text{gr } H \rightarrow H_0 = k[G]$. Indeed, the composition of the induced isomorphism with $\varepsilon \otimes \text{id}$ coincides with the projection. For the biproduct we naturally identify the graded coalgebra $\text{gr } R$ with $\text{gr } H/(\text{gr } H)k[G]^+$, and thereby regard it as a graded Hopf algebra object in the braided category of Yetter-Drinfeld modules over $k[G]$.

Since H_{ab} is cosemisimple by the assumption, $q : H \rightarrow H_{\text{ab}}$ is a filtered Hopf algebra map, where H_{ab} is trivially coradically filtered, meaning that $(H_{\text{ab}})_n = H_{\text{ab}}$ for all $n \geq 0$. Hence, q induces a graded Hopf algebra map $\text{gr } q : \text{gr } H \rightarrow H_{\text{ab}} = H_{\text{ab}}(0)$ which vanishes in positive degrees. It follows that $q(R_n^+) = q(R_n^+)$ for all $n > 0$. Therefore, for (2.3) to hold, it is enough for the following to hold:

$$q \text{ vanishes on } R_1^+ = R_1 \cap R^+. \tag{2.4}$$

Note that R_1^+ consists of all primitive elements in the coalgebra R . It has a basis v_1, \dots, v_r such that $\lambda(v_i) = g_i \otimes v_i$ for some $g_i \in G$; see (2.1). Note that v_i is $(1, g_i)$ -primitive in H , namely $\Delta(v_i) = g_i \otimes v_i + v_i \otimes 1$ on H . Since $q(v_i)$ is $(1, q(g_i))$ -primitive in the group algebra H_{ab} , we have $q(v_i) = c_i(1 - q(g_i))$ for some $c_i \in k$. Define $w_i = v_i - c_i(1 - g_i)$; this element remains $(1, g_i)$ -primitive. Note that

$$q(w_i) = 0, \quad 1 \leq i \leq r. \tag{2.5}$$

We see that $v_i \mapsto w_i$ and $1 \mapsto 1$ give rise to a unit-preserving, right $k[G]$ -module coalgebra isomorphism

$$R_1 \rtimes k[G] \xrightarrow{\cong} H_1.$$

By [10, Th. 4.1] (or [11, Th. 3.10]) the corresponding retraction $H_1 \rightarrow k[G]$ of $k[G] \subset H_1$, namely the inverse of the last isomorphism composed with $\varepsilon \otimes \text{id} : R_1 \rtimes k[G] \rightarrow k[G]$ extends to a right $k[G]$ -module coalgebra retraction $H \rightarrow k[G]$ of $k[G] \subset H$. Choose the latter retraction as the new γ . Then one sees from (2.5) that the desired condition (2.4) is satisfied. \square

Let A be a commutative pointed Hopf algebra containing a left coideal subalgebra B . We denote by Q the quotient Hopf algebra $A/(B^+)$, where (B^+) is the ideal of A generated by $B^+ = B \cap A^+$. The natural projection turns A into a right Q -comodule algebra. Let $G(B) = G(A) \cap B$ be the monoid consisting of all group-like elements of A contained in B .

Lemma 2.3. *With the previous notation, the left coideal subalgebra $A^{\text{co-}\mathcal{Q}}$ consisting of the right \mathcal{Q} -coinvariant elements of A is the localization*

$$B [g^{-1} \mid g \in G(B)]$$

of B by $G(B)$.

Proof. By [9, Th. 1.3], $C = A^{\text{co-}\mathcal{Q}}$ is the smallest left coideal subalgebra of A containing B such that $G(C) = G(A) \cap C$ is a group. One easily checks that $B [g^{-1} \mid g \in G(B)]$ is such a left coideal subalgebra. \square

3. The generic algebra associated to a Hopf algebra

We now introduce the objects needed to state the generalized Noether problem in Section 4.

3.1. The free commutative Hopf algebra generated by a coalgebra. By Takeuchi [21, Chap. IV], given a coalgebra C , there is (up to isomorphism) a unique commutative Hopf algebra $S(t_C)_\Theta$ together with a coalgebra map

$$t : C \rightarrow S(t_C)_\Theta$$

such that for any coalgebra map $f : C \rightarrow H'$ from C to a commutative Hopf algebra H' , there is a unique Hopf algebra map $\tilde{f} : S(t_C)_\Theta \rightarrow H'$ such that $f = \tilde{f} \circ t$. The commutative Hopf algebra $S(t_C)_\Theta$ is called the *free commutative Hopf algebra* generated by C .

Let us give an explicit construction of $S(t_C)_\Theta$. Pick a copy t_C of the underlying vector space of C and denote the identity map from C to t_C by $x \mapsto t_x$ ($x \in C$). Let $S(t_C)$ be the symmetric algebra over the vector space t_C . By [2, Lemma A.1] there is a unique linear map $x \mapsto t_x^{-1}$ from C to the field of fractions $\text{Frac } S(t_C)$ of $S(t_C)$ such that for all $x \in C$,

$$t_{x_1} t_{x_2}^{-1} = t_{x_1}^{-1} t_{x_2} = \varepsilon(x) 1.$$

Then $S(t_C)_\Theta$ is the subalgebra of $\text{Frac } S(t_C)$ generated by all elements t_x and t_x^{-1} , where x runs over C . The coproduct, counit, antipode of $S(t_C)_\Theta$ are determined for all $x \in C$ by

$$\Delta(t_x) = t_{x_1} \otimes t_{x_2} \quad \text{and} \quad \Delta(t_x^{-1}) = t_{x_2}^{-1} \otimes t_{x_1}^{-1}, \quad (3.1)$$

$$\varepsilon(t_x) = \varepsilon(t_x^{-1}) = \varepsilon(x), \quad (3.2)$$

$$S(t_x) = t_x^{-1} \quad \text{and} \quad S(t_x^{-1}) = t_x \quad . \quad (3.3)$$

The map $t : C \rightarrow S(t_C)_\Theta$ is defined by $x \mapsto t_x$ ($x \in C$).

By [21, Th. 61 and Cor. 64] the algebra $S(t_C)_\Theta$ is a localization of $S(t_C)$. More precisely, if $(D_\alpha)_\alpha$ is a family of finite-dimensional subcoalgebras of C such that $\sum_\alpha D_\alpha$ contains the coradical of C , then $S(t_C)_\Theta$ is obtained from $S(t_C)$ by inverting certain group-like elements $\Theta_\alpha \in S(t_{D_\alpha}) \subset S(t_C)$:

$$S(t_C)_\Theta = S(t_C) \left[\left(\frac{1}{\Theta_\alpha} \right)_\alpha \right]. \tag{3.4}$$

In the sequel we shall need the following lemma.

Lemma 3.1. *If the coalgebra C is pointed, then so is $S(t_C)_\Theta$.*

Proof. Recall the following facts from [12, Lemma 5.1.10 and Cor. 5.3.5].

- (i) If C and D are coalgebras with respective coradicals C_0 and D_0 , then the coradical $(C \otimes D)_0$ of the tensor product $C \otimes D$ of coalgebras is contained in $C_0 \otimes D_0$.
- (ii) Given a coalgebra surjection $f : C \rightarrow D$, we have $D_0 \subset f(C_0)$.

Note that a bialgebra B generated by a subcoalgebra C is pointed if C is pointed. Indeed, by (i), and by (ii) applied to the product map $C^{\otimes n} \rightarrow C^n$, we have

$$B_0 = \left(\sum_{n \geq 0} C^n \right)_0 = \sum_{n \geq 0} (C^n)_0 \subset \sum_{n \geq 0} C_0^n.$$

Thus, B_0 is included in the subalgebra generated by C_0 , and is generated by the group-like elements of C if C is pointed. Applying this to $B = S(t_C)$ and $C = t_C$, we deduce that $S(t_C)$ is pointed if C is pointed.

Now by (3.4) the algebra $S(t_C)_\Theta$ is obtained from $S(t_C)$ by inverting a (central) multiplicative subset T consisting of group-like elements. We consider the pointed coalgebra $k[T^{-1}]$ spanned by the symbols t^{-1} , $t \in T$, which are supposed to be group-like. Thus by (i) above, the tensor product of coalgebras $S(t_C) \otimes k[T^{-1}]$ is pointed. We conclude by applying (ii) to the coalgebra surjection $S(t_C) \otimes k[T^{-1}] \rightarrow S(t_C)_\Theta$ given by $P \otimes t^{-1} \mapsto Pt^{-1}$ ($P \in S(t_C), t \in T$). \square

3.2. The H_{ab} -coaction and the subalgebra \mathcal{C}_H . Let H be a Hopf algebra and $S(t_H)_\Theta$ be the free commutative Hopf algebra generated by the coalgebra underlying H , as defined in the previous subsection. We denote by H_{ab} the largest commutative Hopf algebra quotient of H . Applying the universal property of $S(t_H)_\Theta$ to the canonical surjection of Hopf algebras $q : H \rightarrow H_{\text{ab}}$, we obtain a unique Hopf algebra surjection

$$\tilde{q} : S(t_H)_\Theta \rightarrow H_{\text{ab}} \tag{3.5}$$

such that $\tilde{q}(t_x) = q(x)$ and $\tilde{q}(t_x^{-1}) = S(q(x)) = q(S(x))$ for all $x \in H$.

Using \tilde{q} , we may equip $S(t_H)_\Theta$ with a right H_{ab} -comodule algebra structure. Its coaction is the algebra map δ_S defined as the following composition:

$$\delta_S : S(t_H)_\Theta \xrightarrow{\Delta} S(t_H)_\Theta \otimes S(t_H)_\Theta \xrightarrow{\text{id} \otimes \tilde{q}} S(t_H)_\Theta \otimes H_{\text{ab}}. \quad (3.6)$$

We have $\delta_S(t_x) = t_{x_1} \otimes q(x_2)$ for all $x \in H$.

The previous formula shows that δ_S sends $S(t_H)$ to $S(t_H) \otimes H_{\text{ab}}$, which implies that $S(t_H)$ is an H_{ab} -comodule subalgebra of $S(t_H)_\Theta$.

Let \mathcal{C}_H be the subalgebra of right H_{ab} -coinvariants of $S(t_H)_\Theta$:

$$\mathcal{C}_H = S(t_H)_\Theta^{\text{co-}H_{\text{ab}}} = \{a \in S(t_H)_\Theta \mid \delta_S(a) = a \otimes 1\}.$$

It contains the subalgebra of right H_{ab} -coinvariants of $S(t_H)$:

$$S(t_H)^{\text{co-}H_{\text{ab}}} \subset \mathcal{C}_H = S(t_H)_\Theta^{\text{co-}H_{\text{ab}}}.$$

Since $S(t_H)_\Theta$ is a localization of $S(t_H)$, we may wonder whether $\mathcal{C}_H = S(t_H)_\Theta^{\text{co-}H_{\text{ab}}}$ is likewise a localization of $S(t_H)^{\text{co-}H_{\text{ab}}}$. The following provides an answer.

Proposition 3.2. *The algebra \mathcal{C}_H is a localization of $S(t_H)^{\text{co-}H_{\text{ab}}}$.*

For the proof we need the following lemma.

Lemma 3.3. *Each element Θ_α as in (3.4) may be chosen so that $\tilde{q}(\Theta_\alpha) = 1$.*

Proof. According to [21, Sect. 11], Θ_α is given by the determinant of a square matrix $(t_{x_{i,j}})$, where $x_{i,j} \in D_\alpha$ and

$$\Delta(x_{i,j}) = \sum_{\ell} x_{i,\ell} \otimes x_{\ell,j} \quad \text{and} \quad \varepsilon(x_{i,j}) = \delta_{i,j}$$

for all i, j . These formulas imply that in $S(t_H)$ the matrix $M = (t_{x_{i,j}})$ satisfies the equations $\Delta(M) = (M \otimes I)(I \otimes M)$ and $\varepsilon(M) = I$, where I is an identity matrix. This implies that $\Theta_\alpha = \det M$ is group-like, as is well known.

Let us now consider the image $S(x_{i,j})$ of $x_{i,j}$ under the antipode of H . The transpose M' of the matrix $(t_{S(x_{i,j})})$ satisfies the same formulas as above. Therefore, $\Theta'_\alpha = \det M'$ is a group-like element of $S(t_{S(D_\alpha)})$. We may replace D_α by $D_\alpha + S(D_\alpha)$ and Θ_α by $\Theta_\alpha \Theta'_\alpha$. It remains to prove that $\tilde{q}(\Theta_\alpha \Theta'_\alpha) = 1$. But this follows from the fact that the matrices

$$(q(x_{i,j})) = (\tilde{q}(t_{x_{i,j}})) \quad \text{and} \quad (q(S(x_{i,j}))) = (\tilde{q}(t_{S(x_{i,j})}))$$

with entries in H_{ab} are inverses of each other. \square

Proof of Proposition 3.2. Choose $(\Theta_\alpha)_\alpha$ as in Lemma 3.3. Let \bar{P} be an element of $\mathcal{C}_H = S(t_H)_\Theta^{\text{co-}H_{\text{ab}}}$. Then $\bar{P} \Theta \in S(t_H)$ for some finite product Θ of the elements Θ_α ; the element Θ is group-like and $\tilde{q}(\Theta) = 1$. Define $P = \bar{P} \Theta$. Since $\delta_S(\bar{P}) = \bar{P} \otimes 1$ and $\delta_S(\Theta) = \Theta \otimes \tilde{q}(\Theta) = \Theta \otimes 1$, we have $\delta_S(P) = P \otimes 1$, which means that P belongs to $S(t_H)^{\text{co-}H_{\text{ab}}}$. \square

3.3. The generic base algebra. Under some conditions on H , the commutative algebra \mathcal{C}_H has an alternative description, which we now present.

For any pair (x, y) of elements of a Hopf algebra H consider the following elements of $S(t_H)_\Theta$:

$$\sigma(x, y) = t_{x_1} t_{y_1} t_{x_2 y_2}^{-1} \quad \text{and} \quad \sigma^{-1}(x, y) = t_{x_1 y_1} t_{x_2}^{-1} t_{y_2}^{-1}. \quad (3.7)$$

Following [2, Sect. 5] and [6, Sect. 3], we define the *generic base algebra* \mathcal{B}_H attached to the Hopf algebra H to be the subalgebra of $S(t_H)_\Theta$ generated by all elements $\sigma(x, y)$ and $\sigma^{-1}(x, y)$. Since \mathcal{B}_H sits inside $S(t_H)_\Theta$, it is a domain.

The generic base algebra \mathcal{B}_H is the algebra of coinvariants of a right H -comodule algebra \mathcal{A}_H parametrizing the “forms” of H (for details, see [2, 6] and also Section 5.3 of this paper).

If H is *finite-dimensional*, then by Theorem 3.6 and Corollary 3.7 of [8] the following holds:

- (a) \mathcal{B}_H is a finitely generated smooth Noetherian domain of Krull dimension equal to $\dim H$;
- (b) the embedding $\mathcal{B}_H \subset S(t_H)_\Theta$ turns $S(t_H)_\Theta$ into a finitely generated projective \mathcal{B}_H -module.

We now relate \mathcal{B}_H to the algebra \mathcal{C}_H of H_{ab} -coinvariants introduced in the previous subsection.

Proposition 3.4. *Let H be a Hopf algebra.*

- (a) *We have $\mathcal{B}_H \subset \mathcal{C}_H$.*
- (b) *The equality $\mathcal{B}_H = \mathcal{C}_H$ holds if one of the following conditions is satisfied:*
 - (i) *H is finite-dimensional;*
 - (ii) *H is cocommutative;*
 - (iii) *H is pointed and each element of the kernel of the natural homomorphism $G(H)_{\text{ab}} \rightarrow G(H_{\text{ab}})$ is of finite order;*
 - (iv) *H is commutative.*

Here $G(H)_{\text{ab}}$ denote the abelianization of the group $G(H)$ of group-like elements of H . Note that in view of (i), Conditions (ii)–(iv) above are only relevant for infinite-dimensional Hopf algebras.

Proof. (a) It suffices to show that all elements $\sigma(x, y)$ and $\sigma^{-1}(x, y)$ are H_{ab} -coinvariants. We shall check this for $\sigma(x, y)$, the proof for $\sigma^{-1}(x, y)$ being similar. By [8, Lemma 3.3],

$$\delta_S(\sigma(x, y)) = t_{x_1} t_{y_1} t_{x_3 y_3}^{-1} \otimes \tilde{q}(\sigma(x_2, y_2)).$$

Now,

$$\begin{aligned}\tilde{q}(\sigma(x, y)) &= \tilde{q}(t_{x_1}) \tilde{q}(t_{y_1}) \tilde{q}(t_{x_2 y_2}^{-1}) = \tilde{q}(t_{x_1}) \tilde{q}(t_{y_1}) \tilde{q}(S(t_{x_2 y_2})) \\ &= q(x_1) q(y_1) q(S(x_2 y_2)) = q((x_1 y_1) S(x_2 y_2)) \\ &= \varepsilon(xy) q(1).\end{aligned}$$

Consequently,

$$\delta_S(\sigma(x, y)) = t_{x_1} t_{y_1} \varepsilon(x_2 y_2) t_{x_3 y_3}^{-1} \otimes q(1) = \sigma(x, y) \otimes q(1),$$

which shows that $\sigma(x, y)$ belongs to \mathcal{C}_H .

(b) (i)–(iii) By Theorems 3.6, 3.8, and 3.9 of [8], $S(t_H)_\Theta$ is faithfully flat as a \mathcal{B}_H -module. Then by [22, Th. 3], $\mathcal{B}_H = S(t_H)_\Theta^{\text{co-}Q}$, where Q is the quotient Hopf algebra of $S(t_H)_\Theta$ by the ideal (\mathcal{B}_H^+) generated by $\mathcal{B}_H^+ = \mathcal{B}_H \cap \ker \varepsilon$. By [8, Prop. 3.1], there is a natural identification $Q \cong H_{\text{ab}}$, which allows us to conclude.

(iv) As is shown in Part (a) of the proof of [8, Th. 3.13], the mapping $c \otimes x \mapsto ct_x$ gives an isomorphism $\mathcal{C}_H \otimes H \cong S(t_H)_\Theta$. Moreover, since the natural inclusion $\mathcal{B}_H \otimes H \subset \mathcal{C}_H \otimes H$, composed with the isomorphism, gives a surjection by [8, Lemma 3.2], we have $\mathcal{B}_H = \mathcal{C}_H$. \square

4. The generalized Noether problem

We now state the problem which gives the title to the paper. We provide answers for certain classes of Hopf algebras.

4.1. The problem and its relationship with the classical Noether problem.

Problem 4.1 (GNP). *Given a field k and a finite-dimensional Hopf algebra H over k , is the algebra \mathcal{B}_H a localization of a polynomial algebra in finitely many variables?*

We call Problem (GNP) the *generalized Noether problem*. Let us first show how (GNP) is related to the classical Noether problem for finite groups.

Let G be a finite group and $H = O_k(G)$ the dual Hopf algebra of the group algebra $k[G]$. The elements of H can be seen as k -valued functions on G ; denote by e_g the function which vanishes everywhere on G , except at the element g , where $e_g(g) = 1$. On the basis $\{e_g\}_{g \in G}$ the coproduct, the counit and the antipode of H are given by

$$\Delta(e_g) = \sum_{h \in G} e_{gh^{-1}} \otimes e_h, \quad S(e_g) = e_{g^{-1}}, \quad (4.1)$$

$\varepsilon(e_g) = 1$ if $g = 1$ is the identity element of G , and $\varepsilon(e_g) = 0$ otherwise.

Proposition 4.2. *If (GNP) has a positive answer for the Hopf algebra $H = O_k(G)$, then so has the classical Noether problem for G and k .*

In this sense, (GNP) is an extension of Noether’s problem.

Proof. If we set $t_g = t_{e_{g^{-1}}}$ for any $g \in G$, then $S(t_H) = k[t_g \mid g \in G]$ is the bialgebra with coproduct given for $g \in G$ by

$$\Delta(t_g) = \sum_{h \in G} t_{hg} \otimes t_{h^{-1}}. \tag{4.2}$$

By [2, Ex. B.5], the algebra $S(t_H)_\Theta$ is the following localization of $S(t_H)$:

$$S(t_H)_\Theta = k[t_g \mid g \in G] \left[\frac{1}{\Theta_G} \right],$$

where $\Theta_G = \det(t_{gh^{-1}})_{g,h \in G}$ is Dedekind’s group determinant.

Under the algebra map $\tilde{q} : S(t_H)_\Theta \rightarrow H_{\text{ab}} = H$ of (3.5), the algebra $S(t_H)_\Theta$ becomes a right H -comodule algebra with coaction (3.6). Now it is well known that any right H -comodule algebra structure on an algebra A is the same as a left action $(h, a) \mapsto h \cdot a$ of G on A by algebra automorphisms: the H -coaction δ on A is related to the G -action by

$$\delta(a) = \sum_{h \in G} h \cdot a \otimes e_h. \tag{4.3}$$

Moreover, the subalgebra of coinvariants $A^{\text{co-}H}$ coincides with the subalgebra A^G of G -invariant elements of A . Since for $H = O_k(G)$ we have $\tilde{q}(t_g) = e_{g^{-1}}$ for all $g \in G$, it follows from (3.6) and from (4.2) that

$$\delta_S(t_g) = \sum_{h \in G} t_{hg} \otimes e_h.$$

Comparing this formula with (4.3), we see that the corresponding left action of G on $S(t_H)_\Theta$ is given by $h \cdot t_g = t_{hg}$, which is precisely the one presented in the introduction. One easily checks that the square Θ_G^2 of the Dedekind determinant is G -invariant, so that by Proposition 3.4,

$$\mathcal{B}_H = \mathcal{C}_H = (S(t_H)_\Theta)^G = k[t_g \mid g \in G]^G \left[\frac{1}{\Theta_G^2} \right].$$

If $\mathcal{B}_H = k[t_g \mid g \in G]^G [1/\Theta_G^2]$ is the localization of a polynomial algebra in finitely many variables, then its fraction field is a purely transcendental extension of k . This fraction field is also the fraction field of $k[t_g \mid g \in G]^G$. Now the latter is $k(t_g \mid g \in G)^G$: indeed, if $F = P/Q$ is a G -invariant fraction with polynomial numerator and denominator, then it can be written as $F = P'/Q'$, where $P' = P \prod_{g \neq 1} (g \cdot Q)$ and $Q' = \prod_{g \in G} (g \cdot Q)$ are invariant polynomials. Therefore, the field $k(t_g \mid g \in G)^G$ is a purely transcendental extension of k . \square

4.2. Positive answers to (GNP). Before stating the main result of this section, we present a class of Hopf algebras for which it is easy to provide a positive answer to (GNP).

Example 4.3. Let $H = k[G]$ be the Hopf algebra of a finite group G . It is easy to see that the maximal commutative Hopf algebra quotient is the group algebra $H_{\text{ab}} = k[G_{\text{ab}}]$, where G_{ab} is the abelianization of G . By Proposition 3.4, $\mathcal{B}_H = \mathcal{C}_H$. It is the group algebra of a finitely generated free abelian group; more precisely, by [1, Prop. 9 and 14] (see also [4, Prop. A.1]),

$$\mathcal{B}_{k[G]} = \mathcal{C}_{k[G]} = S(t_{k[G]})_{\Theta}^{\text{co-}k[G_{\text{ab}}]} = k[Y_G],$$

where Y_G is the kernel of the homomorphism $\mathbb{Z}^G \rightarrow G_{\text{ab}}; t_g \mapsto \bar{g}$ ($g \in G$). (Here \mathbb{Z}^G is the free abelian group generated by the symbols t_g ($g \in G$) and \bar{g} denotes the image of $g \in G$ in G_{ab} .) Since Y_G is a finite index subgroup of \mathbb{Z}^G , it is a free abelian group of the same rank $\text{card } G$. (A basis of Y_G is described in Lemma 4.7). Therefore $\mathcal{B}_{k[G]} = k[Y_G]$ is an algebra of Laurent polynomials in finitely many ($\text{card } G$) variables, which shows that (GNP) has a positive answer for $H = k[G]$.

The main result of this section is the following.

Theorem 4.4. *Let H be a finite-dimensional pointed Hopf algebra such that H_{ab} is a group algebra. Then (GNP) has a positive answer for H .*

Recall from Lemma 2.1 that the hypothesis on H_{ab} is satisfied when the base field k is of characteristic zero.

The theorem had previously been established for special classes of finite-dimensional pointed Hopf algebras: for finite group algebras by Aljadeff, Haile and Natapov [1] as detailed in Example 4.3; furthermore, for the Taft algebras, for the Hopf algebras $E(n)$ and for certain monomial Hopf algebras by Iyer and the first-named author (see [4, Th. 2.1, Th. 3.1, Th. 4.1]). In all these cases, H_{ab} is a group algebra.

Proof. We set $G = G(H)$. Let $R = H/H(k[G])^+$ and choose $\gamma : H \rightarrow k[G]$ as in Lemma 2.2. Define $\lambda : R \rightarrow k[G] \otimes R$ as in (2.1), and construct thereby the smash-coproduct coalgebra $R \triangleright \blacktriangleleft k[G]$. Identify H with $R \triangleright \blacktriangleleft k[G]$ via the $k[G]$ -module coalgebra isomorphism ξ given by (2.2). Since $H = R^+G \oplus k[G]$, we have

$$S(t_H)_{\Theta} = S(t_{R^+G}) \otimes S(t_{k[G]})_{\Theta}. \tag{4.4}$$

Note that this is an identification of $S(t_{k[G]})_{\Theta}$ -algebras.

Consider the Hopf algebra surjections

$$\tilde{\gamma} : S(t_H)_{\Theta} \rightarrow S(t_{k[G]})_{\Theta} \quad \text{and} \quad \tilde{q} : S(t_H)_{\Theta} \rightarrow H_{\text{ab}}$$

induced from the coalgebra maps γ and q , respectively.

Regard $S(t_H)_\Theta$ as a right $S(t_{k[G]})_\Theta$ -comodule algebra along $\tilde{\gamma}$, and let A denote the (left coideal) subalgebra of $S(t_H)_\Theta$ consisting of all right $S(t_{k[G]})_\Theta$ -coinvariants. Since $\tilde{\gamma}$ is a Hopf algebra retraction of the inclusion $S(t_{k[G]})_\Theta \subset S(t_H)_\Theta$, the product map gives a natural isomorphism

$$A \otimes S(t_{k[G]})_\Theta = S(t_H)_\Theta \tag{4.5}$$

of right $S(t_{k[G]})_\Theta$ -comodules and of $S(t_{k[G]})_\Theta$ -algebras. Recall from Lemma 2.2 that $q = q|_{k[G]} \circ \gamma$. It follows that

$$\tilde{q} = \tilde{q}|_{S(t_{k[G]})_\Theta} \circ \tilde{\gamma}. \tag{4.6}$$

Regard $S(t_{k[G]})_\Theta$ as a right H_{ab} -comodule algebra along $\tilde{q}|_{S(t_{k[G]})_\Theta}$. Then since A is right \tilde{q} -coinvariant, it follows from (4.6) that the isomorphism given in (4.5) restricts to an isomorphism

$$A \otimes S(t_{k[G]})_\Theta^{\text{co-}H_{\text{ab}}} = S(t_H)_\Theta^{\text{co-}H_{\text{ab}}} = \mathcal{B}_H. \tag{4.7}$$

Now $S(t_{k[G]})_\Theta^{\text{co-}H_{\text{ab}}}$, being the algebra of coinvariants along a Hopf algebra map, is a Hopf subalgebra of $S(t_{k[G]})_\Theta$; the latter being the group algebra of a finitely generated free abelian group (see Example 4.3), so is $S(t_{k[G]})_\Theta^{\text{co-}H_{\text{ab}}} \cong k[\mathbb{Z}^\ell]$ for some non-negative integer ℓ . We remark that $\ell = \text{card } G$, since the short exact sequence

$$S(t_{k[G]})_\Theta^{\text{co-}H_{\text{ab}}} \rightarrow S(t_{k[G]})_\Theta \rightarrow H_{\text{ab}}$$

of commutative Hopf algebras restricts on the level of group-like elements to the short exact sequence

$$0 \rightarrow \mathbb{Z}^\ell \rightarrow \mathbb{Z}^G \rightarrow G(H_{\text{ab}}) \rightarrow 0 \tag{4.8}$$

of abelian groups with finite cokernel.

It remains to prove that A is a polynomial algebra with finitely many variables. But this follows since one sees from (4.4) and (4.5) that

$$S(t_{R+G}) \cong S(t_H)_\Theta / (S(t_{k[G]})_\Theta^+) \cong A. \tag{4.9}$$

In conclusion, the isomorphism $\mathcal{B}_H \cong S(t_{R+G}) \otimes k[\mathbb{Z}^\ell]$ provides a positive answer to (GNP). We remark that, if we set $n = \dim H$, then $S(t_{R+G}) \cong k[\mathbb{N}^{n-\ell}]$, since we see from $H = R^+G \oplus k[G]$ that $\dim R^+G = n - \ell$. \square

In the previous proof we have actually established the more precise isomorphism

$$\mathcal{B}_H \cong k[\mathbb{Z}^\ell] \otimes k[\mathbb{N}^{n-\ell}], \quad n = \dim H \text{ and } \ell = \text{card } G(H).$$

We will come back to it in Section 4.3.

Remark 4.5 (A variant of (GHP)). By Proposition 3.4, $\mathcal{B}_H = S(t_H)_{\Theta}^{\text{co-}H_{\text{ab}}}$ if H is finite-dimensional. As was seen in Section 3.2, $S(t_H)$ is an H_{ab} -comodule subalgebra of $S(t_H)_{\Theta}$, so that we can consider the coinvariant subalgebra $S(t_H)^{\text{co-}H_{\text{ab}}}$. A variant of (GNP) is the following: is $S(t_H)^{\text{co-}H_{\text{ab}}}$ a polynomial algebra in finitely many variables? The following example shows that this variant may have a negative answer at the same time as (GNP) has a positive one.

Let H be the four-dimensional Sweedler algebra. Proceeding as in the proof of [4, Th. 2.1], it is easy to check that $S(t_H)^{\text{co-}H_{\text{ab}}}$ is spanned by the monomials $t_1^a t_x^b t_y^c t_z^d$ ($a, b, c, d \in \mathbb{N}$) such that $b + c$ is even. Hence, $S(t_H)^{\text{co-}H_{\text{ab}}}$ is the subalgebra of $k[t_1, t_x, t_y, t_z]$ generated by $t_1, t_x^2, t_x t_y, t_y^2, t_z$; this is not a polynomial algebra, whereas $\mathcal{B}_H = S(t_H)_{\Theta}^{\text{co-}H_{\text{ab}}}$, which is obtained from the previous algebra by inverting t_1 and t_x^2 , is a localization of the polynomial algebra $k[t_1, t_x^2, t_x t_y, t_z]$.

4.3. Bounding the degrees of generators. Given a finite group G , consider the polynomial algebra $S(t_G) = k[t_g \mid g \in G]$ in the variables t_g ($g \in G$) and let G act on the variables t_g , hence on $S(t_G)$, as in the introduction. Let $S(t_G)^G$ be the subalgebra of $S(t_G)$ of G -invariant polynomials. We denote by $\beta(G)$ the smallest integer β such that $S(t_G)^G$ is generated by homogeneous polynomials of degree $\leq \beta$.

In [13] Emmy Noether proved that $\beta(G) \leq \text{card } G$. It is easy to check that, if the group G is cyclic, then $\beta(G) = \text{card } G$. In her thesis [18], Barbara Schmid proved a conjecture by Kraft, namely $\beta(G) < \text{card } G$ if G is not cyclic.

We now prove a similar result for a finite-dimensional pointed Hopf algebra H such that H_{ab} is a group algebra. Set $\bar{G} = G(H_{\text{ab}})$; equivalently, $H_{\text{ab}} = k[\bar{G}]$.

Define two non-negative integers d, r related to the finite abelian group \bar{G} as follows: if \bar{G} is trivial, set $d = r = 0$; if \bar{G} is not trivial, let

$$\bar{G} = \mathbb{Z}/p_1^{e_1} \times \cdots \times \mathbb{Z}/p_r^{e_r}, \quad r \geq 1, \quad p_i \text{ primes}, \quad e_i \geq 1 \tag{4.10}$$

be the primary decomposition of \bar{G} , and set $d = p_1^{e_1} + \cdots + p_r^{e_r} (\geq 2)$.

Theorem 4.6. *Under the hypotheses of Theorem 4.4 and with the above notation, we have*

$$\mathcal{B}_H = k[u_1^{\pm 1}, \dots, u_{\ell}^{\pm 1}, u_{\ell+1}, \dots, u_n],$$

where $n = \dim H$ and $\ell = \text{card } G(H)$ and where u_1, \dots, u_{ℓ} are monomials in the variables t_g of degree $\leq d - r + 1$ and $u_{\ell+1}, \dots, u_n$ are monomials of degree ≤ 2 .

Note that $d - r + 1 \leq \text{card } \bar{G}$, but $d - r + 1$ may be much smaller: for instance, if \bar{G} is a p -group of order p^e , then $d - r + 1$ ranges from p^e when $\bar{G} \cong \mathbb{Z}/p^e$ is cyclic down to $e(p - 1) + 1$ when $\bar{G} \cong (\mathbb{Z}/p)^e$ is elementary abelian.

For the proof of Theorem 4.6 we shall make use of the fact that $S(t_{k[G]})_{\Theta}^{\text{co-}H_{\text{ab}}}$ is a subalgebra of \mathcal{B}_H inside $S(t_{k[G]})_{\Theta}$ (see proof of Theorem 4.4) and of the two lemmas below.

To state the first lemma we need the following notation. By Item (ii) in the proof of Lemma 3.1, the surjection of pointed Hopf algebras $H \rightarrow H_{\text{ab}}$ restricts to a surjection of groups $G \rightarrow \bar{G}$. Suppose that the group \bar{G} is non-trivial. For $1 \leq i \leq r$ let s_i be a generator of the summand $\mathbb{Z}/p_i^{e_i}$ in the primary decomposition (4.10) of \bar{G} , and fix an element $\sigma_i \in G$ whose image in \bar{G} is s_i . Given an element $g \in G$, its image in \bar{G} is of the form $s_1^{f_1(g)} \cdots s_r^{f_r(g)}$ ($0 \leq f_i(g) < p_i^{e_i}$, $1 \leq i \leq r$). For $g \in G$ different from the identity element e and of the elements $\sigma_1, \dots, \sigma_r$, let $I = \{i \in \{1, \dots, r\} \mid f_i(g) \neq 0\}$ and set

$$u_g = t_g \prod_{i \in I} t_{\sigma_i}^{p_i^{e_i} - f_i(g)} \in S(t_{k[G]}) \subset S(t_H).$$

Lemma 4.7. *The set consisting of the monomials (in the t -variables) $t_e, t_{\sigma_1}^{p_1^{e_1}}, \dots, t_{\sigma_r}^{p_r^{e_r}}$ and of the above monomials u_g is a (multiplicatively written) basis of the kernel \mathbb{Z}^ℓ appearing in the short exact sequence of groups (4.8).*

Proof. These elements clearly belong to the kernel \mathbb{Z}^ℓ . Their matrix with respect to the suitably ordered basis $(t_g)_{g \in G}$ of \mathbb{Z}^G is triangular with determinant equal to $p_1^{e_1} \cdots p_r^{e_r} = \bar{G}$. Therefore, they form a basis of \mathbb{Z}^ℓ . \square

Lemma 4.8. *As an $S(t_{k[G]})_{\Theta}^{\text{co-}H_{\text{ab}}}$ -algebra, \mathcal{B}_H is freely generated by $n - \ell$ monomials of degree ≤ 2 in $S(t_H)$.*

Proof. We return to the situation of the proof of Theorem 4.4: we have the identification $H = R \rtimes k[G]$ of right $k[G]$ -module coalgebras, under which the right $k[G]$ -module coalgebra map $\gamma : H \rightarrow k[G]$ is identified with $\varepsilon \otimes \text{id} : R \otimes k[G] \rightarrow k[G]$.

We claim that $R = R \otimes k$ precisely consists of those elements of H which are right $k[G]$ -coinvariant along γ , that is,

$$R = \{h \in H \mid (\text{id} \otimes \gamma) \circ \Delta(h) = h \otimes 1\}.$$

Indeed, this holds for smash coproducts in general. In our situation, recall from the proof of Lemma 2.2 that R is a left $k[G]$ -comodule coalgebra with respect to $\lambda : R \rightarrow k[G] \otimes R$. For $x \in R$ and $g \in G$, the coproduct $\Delta(x \otimes g)$ on H is given by

$$\Delta(x \otimes g) = (x_1 \otimes (x_2)^{(-1)}g) \otimes ((x_2)^{(0)} \otimes g),$$

where $\lambda(x) = x^{(-1)} \otimes x^{(0)}$. Since $(\text{id} \otimes \varepsilon) \circ \lambda(x) = \varepsilon(x)1$, it follows that

$$(\text{id} \otimes \gamma) \circ \Delta(x \otimes g) = (x \otimes g) \otimes g.$$

Thus the right $k[G]$ -comodule structure on $H = R \otimes k[G]$ is the natural one given by the right tensor factor $k[G]$. This proves the claim.

Now choose arbitrarily a k -basis $\{x_i\}_i$ of R^+ . Then we have the k -basis $\{x_i g\}_{i,g}$ of R^+G , where g runs over G . Note $\dim R^+G = n - \ell$. One sees from the claim above that the coproduct $\Delta(t_{x_i g})$ on $S(t_H)_\Theta$, composed with $\text{id} \otimes \tilde{\gamma}$, turns into

$$(\text{id} \otimes \tilde{\gamma}) \circ \Delta(t_{x_i g}) = t_{x_i g} \otimes t_g .$$

Obviously, $\{t_{x_i g}\}_{i,g}$ is a set of free generators of $S(t_{R^+G})$. Each $t_{x_i g}$ is congruent, modulo $(S(t_{k[G]})_\Theta^+)$, to $t_{x_i g}(t_g)^{-1}$, which is seen, by the last equation, to be in A . Recall from (4.9) and (4.7) the isomorphisms

$$S(t_{R^+G}) \cong S(t_H)_\Theta / (S(t_{k[G]})_\Theta^+) \cong A, \quad A \otimes S(t_{k[G]})_\Theta^{\text{co-}H_{\text{ab}}} \cong \mathcal{B}_H .$$

It follows that $\{t_{x_i g}(t_g)^{-1}\}_{i,g}$ is a set of free generators of the k -algebra A , and is a set of free generators of the $S(t_{k[G]})_\Theta^{\text{co-}H_{\text{ab}}}$ -algebra \mathcal{B}_H . Multiplying $t_{x_i g}(t_g)^{-1}$ by the unit $t_g t_{g^{-1}}$ in $S(t_{k[G]})_\Theta^{\text{co-}H_{\text{ab}}}$, we obtain the set $\{t_{x_i g} t_{g^{-1}}\}_{i,g}$ of free generators of the $S(t_{k[G]})_\Theta^{\text{co-}H_{\text{ab}}}$ -algebra \mathcal{B}_H . The set consists of monomials of degree 2 in $S(t_H)$, among which $t_{x_i} t_e$ may be replaced by the degree-one monomial t_{x_i} (since t_e is a unit in $S(t_{k[G]})_\Theta^{\text{co-}H_{\text{ab}}}$). Thus the lemma follows. \square

Proof of Theorem 4.6. It follows from Lemma 4.8 that inside $S(t_{k[G]})_\Theta$ we have $\mathcal{B}_H = S(t_{k[G]})_\Theta^{\text{co-}H_{\text{ab}}}[u_{\ell+1}, \dots, u_n]$, where $u_{\ell+1}, \dots, u_n$ are monomials of $S(t_H)$ of degree ≤ 2 .

Now by the proof of Theorem 4.4 the subalgebra $S(t_{k[G]})_\Theta^{\text{co-}H_{\text{ab}}}$ is the algebra of the kernel \mathbb{Z}^ℓ of the surjection $\mathbb{Z}^G \rightarrow \bar{G} = G(H_{\text{ab}})$ appearing in the short exact sequence of groups (4.8).

If \bar{G} is trivial, then $\mathbb{Z}^\ell = \mathbb{Z}^G$, hence $S(t_{k[G]})_\Theta^{\text{co-}H_{\text{ab}}} = S(t_{k[G]})_\Theta$, which is the algebra of Laurent polynomials in the monomials t_g ($g \in G$), which are of degree $1 = d - r + 1$.

Otherwise, $S(t_{k[G]})_\Theta^{\text{co-}H_{\text{ab}}}$ is the algebra of Laurent polynomials in the basis elements described in Lemma 4.7. To complete the proof, it is enough to bound the degree of these elements. Now each monomial u_g is of degree

$$1 + \sum_{i \in I} (p_i^{e_i} - f_i(g)),$$

which is smaller than or equal to

$$1 + \sum_{i \in I} (p_i^{e_i} - 1) \leq 1 + \sum_{i=1}^r (p_i^{e_i} - 1) = d - r + 1 .$$

We conclude by observing that the degrees of the remaining monomials $t_e, t_{\sigma_1}^{p_1^{e_1}}, \dots, t_{\sigma_r}^{p_r^{e_r}}$ are not larger than $d - r + 1$. \square

Remark 4.9. Let H be a pointed Hopf algebra such that $H_{\text{ab}} = k[\bar{G}]$ is a group algebra. As observed above, the canonical surjection of pointed Hopf algebras $H \rightarrow H_{\text{ab}}$ restricts to a surjection $k[G] \rightarrow k[\bar{G}]$, where $G = G(H)$. Denoting the abelianization of G by G_{ab} , we see that the surjection $k[G] \rightarrow k[\bar{G}]$ in turn induces a surjection $\varphi : k[G_{\text{ab}}] \rightarrow H_{\text{ab}} = k[\bar{G}]$.

The map φ is an isomorphism if the Hopf algebra embedding $k[G] \subset H$ splits as a Hopf algebra map. Indeed, in this case, a splitting $H \rightarrow k[G]$ composed with the canonical map $k[G] \rightarrow k[G_{\text{ab}}]$ induces a Hopf algebra map $H_{\text{ab}} = k[\bar{G}] \rightarrow k[G_{\text{ab}}]$. The latter map is an inverse of φ since both maps are induced from the identity on $k[G]$.

In general $\varphi : k[G_{\text{ab}}] \rightarrow H_{\text{ab}}$ is not an isomorphism, as is seen from the following examples. Consider the pointed Hopf algebra $H = U_q(\mathfrak{sl}_2)$ as given in [5, p. 122]. The group $G = G(H)$ is infinite cyclic, hence $G_{\text{ab}} \cong \mathbb{Z}$. Now the relations (1.10)–(1.12) in *loc. cit.* imply that $E = 0$, $F = 0$ and $K = K^{-1}$ in H_{ab} . Therefore, $H_{\text{ab}} \cong k[\mathbb{Z}/2]$.

Similarly let $H = \bar{U}_q$ be the finite-dimensional Hopf algebra defined in [5, p. 136] as the quotient of $U_q(\mathfrak{sl}_2)$ by the relations $E^e = F^e = 0$ and $K^e = 1$ for some integer $e \geq 2$. For this Hopf algebra, we have $G = G_{\text{ab}} \cong \mathbb{Z}/e$ whereas $H_{\text{ab}} \cong k$ if the integer e is odd and $H_{\text{ab}} \cong k[\mathbb{Z}/2]$ if e is even. Thus, the surjection $\varphi : k[G_{\text{ab}}] \rightarrow H_{\text{ab}}$ cannot be an isomorphism when $e \geq 3$.

5. Polynomial identities

A theory of polynomial identities for comodule algebras was worked out in [2]. It leads naturally to a “universal H -comodule algebra” \mathcal{U}_H , whose definition will be recalled below. The subalgebra of H -coinvariants \mathcal{V}_H of \mathcal{U}_H maps injectively into the generic base algebra \mathcal{B}_H defined in Section 3.3. In this section we consider another localization problem for \mathcal{B}_H , which is motivated by the fact that a positive answer to it has important consequences (detailed below) for the “universal deformation space” $\mathcal{A}_H = \mathcal{B}_H \otimes_{\mathcal{V}_H} \mathcal{U}_H$.

5.1. The universal comodule algebra. Starting from a Hopf algebra H , we fix another copy X_H of the underlying vector space of H ; we denote the identity map from H to X_H by $x \mapsto X_x$.

Consider the tensor algebra $T(X_H)$ over the vector space X_H ; it is an algebra of *non-commutative polynomials*. The algebra $T(X_H)$ possesses a right H -comodule algebra structure that extends the natural right H -comodule algebra structure on H : its coaction $\delta_T : T(X_H) \rightarrow T(X_H) \otimes H$ is given by

$$\delta_T(X_x) = X_{x_1} \otimes x_2, \quad (x \in H).$$

We equip $S(t_H) \otimes H$ with the trivial right H -comodule algebra structure whose coaction is given by $\text{id} \otimes \Delta$. The subalgebra of H -coinvariant elements of $S(t_H) \otimes H$ is $S(t_H) \otimes 1$, which we may identify with $S(t_H)$.

By [2, Lemma 4.2] the algebra map $\mu : T(X_H) \rightarrow S(t_H) \otimes H$ defined for $x \in H$ by

$$\mu(X_x) = t_{x_1} \otimes x_2 \tag{5.1}$$

is a right H -comodule algebra map, which is universal in the sense that any H -comodule algebra map $T(X_H) \rightarrow H$ factors through μ (see [2, Th. 4.3]).

Now let I_H be the kernel of the comodule algebra map $\mu : T(X_H) \rightarrow S(t_H) \otimes H$ defined by (5.1). The kernel I_H is a two-sided ideal, right H -subcomodule of $T(X_H)$. Any element $P \in I_H$ is an *identity for H* in the sense that it vanishes under any right H -comodule algebra map $T(X_H) \rightarrow H$. See [2, 6, 7] for a theory of such comodule algebra identities.

Consider the quotient right H -comodule algebra

$$\mathcal{U}_H = T(X_H)/I_H .$$

We call \mathcal{U}_H the *universal H -comodule algebra* (this corresponds to the “relatively free algebra” in the classical literature on polynomial identities; see [16]).

By definition of I_H , the map $\mu : T(X_H) \rightarrow S(t_H) \otimes H$ induces an injection of right H -comodule algebras

$$\bar{\mu} : \mathcal{U}_H = T(X_H)/I_H \hookrightarrow S(t_H) \otimes H .$$

5.2. The algebra of coinvariants \mathcal{V}_H . We denote the subalgebra of right H -coinvariants of the universal comodule algebra \mathcal{U}_H by \mathcal{V}_H :

$$\mathcal{V}_H = \mathcal{U}_H^{\text{co-}H} .$$

The algebra \mathcal{V}_H is a central subalgebra of \mathcal{U}_H . Note that an element \bar{P} of \mathcal{U}_H is in \mathcal{V}_H if and only if $\bar{\mu}(\bar{P})$ belongs to $S(t_H) \otimes 1$.

Using Lemma 8.1 of [2] and following the proof of Proposition 9.1 of *loc. cit.*, one shows that the map $\bar{\mu}$ send \mathcal{V}_H into the subalgebra $\mathcal{B}_H \cap S(t_H)$ of \mathcal{B}_H (here we identify $S(t_H)$ with the subalgebra $S(t_H) \otimes 1$ of $S(t_H) \otimes H$).

By [8, Cor. 4.4], the generic base algebra \mathcal{B}_H of a Hopf algebra H (defined in Section 3.3) has another set of generators, namely the elements

$$p_x = t_{x_1} t_{S(x_2)} , \quad q_{x,y} = t_{x_1} t_{y_1} t_{S(x_2 y_2)} , \tag{5.2}$$

$$p'_x = t_{S(x_1)}^{-1} t_{x_2}^{-1} , \quad q'_{x,y} = t_{S(x_1 y_1)}^{-1} t_{x_2}^{-1} t_{y_2}^{-1} , \tag{5.3}$$

when x, y run over all elements of H . The essential virtue of these generators over the older generators $\sigma(x, y)$ and $\sigma^{-1}(x, y)$ of (3.7) is that p_x and $q_{x,y}$ (resp. p'_x and $q'_{x,y}$) are polynomials in the t -variables (resp. in the t^{-1} -variables), whereas the formulas for $\sigma(x, y)$ and $\sigma^{-1}(x, y)$ mix the t -variables and the t^{-1} -variables.

Lemma 5.1. *Let H be a Hopf algebra. The elements p_x and $q_{x,y}$ ($x, y \in H$) belong to $\bar{\mu}(\mathcal{V}_H)$.*

Proof. Consider the following elements of $T(X_H)$:

$$P_x = X_{x_1} X_{S(x_2)}, \quad Q_{x,y} = X_{x_1} X_{y_1} X_{S(x_2 y_2)},$$

where $x, y \in H$. By [2, Lemma 2.1] they are coinvariant elements of $T(X_H)$, and by [8, Lemma 4.1] we have $\mu(P_x) = p_x$ and $\mu(Q_{x,y}) = q_{x,y}$. \square

Let \mathcal{W}_H be the subalgebra of $\bar{\mu}(\mathcal{V}_H)$ generated by all elements p_x and $q_{x,y}$, where x, y run over H .

Lemma 5.2. *Let H be a Hopf algebra.*

- (i) *The algebra \mathcal{W}_H is a left coideal subalgebra of $S(t_H)_\Theta$.*
- (ii) *The Hopf ideal (\mathcal{W}_H^+) of $S(t_H)_\Theta$ generated by $\mathcal{W}_H^+ = \mathcal{W}_H \cap S(t_H)_\Theta^+$ coincides with (\mathcal{B}_H^+) .*

Proof. (i) Using the formula (3.1) for the coproduct Δ of $S(t_H)_\Theta$, we obtain

$$\Delta(p_x) = t_{x_1} t_{S(x_3)} \otimes p_{x_2} \quad \text{and} \quad \Delta(q_{x,y}) = t_{x_1} t_{y_1} t_{S(x_3 y_3)} \otimes q_{x_2, y_2}.$$

Since \mathcal{W}_H is generated by the elements p_x and $q_{x,y}$, the conclusion follows.

(ii) Let $R = S(t_H)_\Theta / (\mathcal{W}_H^+)$. Since $\mathcal{W}_H \subset \mathcal{B}_H$, we have a Hopf algebra surjection $R \rightarrow S(t_H)_\Theta / (\mathcal{B}_H^+)$. From

$$t_{S_H(x)} = S_R(t_x), \quad t_x t_y = S_R(t_{S_H(xy)}) \in R,$$

it follows that

$$t_x t_y = S_R(S_R(t_{xy})) = t_{xy} \in R.$$

Here, S_H (resp. S_R) denotes the antipode of H (resp. of R). The result for $x = y = 1$, together with the existence of t_1^{-1} , shows that $t_1 = 1 \in R$. Therefore, the map $t : H \rightarrow S(t_H)_\Theta$ composed with the natural projection $S(t_H)_\Theta \rightarrow R$ yields a Hopf algebra surjection, which in turn factors through a Hopf algebra surjection $H_{\text{ab}} \rightarrow R$. Since the composite of the last surjection with $R \rightarrow S(t_H)_\Theta / (\mathcal{B}_H^+)$ is an isomorphism by [8, Prop. 3.1], the desired result follows. \square

5.3. The second localization problem. In the sequel we identify \mathcal{V}_H with its image $\bar{\mu}(\mathcal{V}_H)$ in \mathcal{B}_H . We now raise another question.

Problem 5.3 (loc). *Given a Hopf algebra H , is the algebra \mathcal{B}_H a localization of the subalgebra \mathcal{V}_H ?*

Problem (loc) is motivated by the fact proved in [2, Sect. 7] and in [8, Sect. 3] that, if \mathcal{B}_H a localization of \mathcal{V}_H , then the *central localization*

$$\mathcal{A}_H = \mathcal{B}_H \otimes_{\mathcal{V}_H} \mathcal{U}_H$$

of the universal comodule algebra \mathcal{U}_H satisfies the following two properties.

- (i) The extension $\mathcal{B}_H \subset \mathcal{A}_H$ is a *cleft H -Galois extension*; in particular, there is a left \mathcal{B}_H -module, right H -comodule isomorphism

$$\mathcal{A}_H \cong \mathcal{B}_H \otimes H.$$

Thus, after localization, \mathcal{U}_H becomes a free module of rank $\dim H$ over its subalgebra of coinvariants.

- (ii) The comodule algebra \mathcal{A}_H is a “versal deformation space” for the *forms* of H in the following sense. Any cleft right H -comodule algebra A that is a form of H (i.e., such that $k' \otimes_k A \cong k' \otimes_k H$ for some field extension k' of k) is isomorphic to a comodule algebra of the form $\mathcal{A}_H/\mathfrak{m}\mathcal{A}_H$, where \mathfrak{m} is some maximal ideal of \mathcal{B}_H . Conversely, if in addition $S(t_H)_\Theta$ is faithfully flat over \mathcal{B}_H , then for any maximal ideal \mathfrak{m} of \mathcal{B}_H , the comodule algebra $\mathcal{A}_H/\mathfrak{m}\mathcal{A}_H$ is a form of H . (The faithful flatness assumption is satisfied in a number of cases, including the case when H is finite-dimensional, see [8, Sect. 3.2] and below.)

In the language of non-commutative geometry, \mathcal{A}_H is a “non-commutative fiber bundle” over the generic base algebra \mathcal{B}_H .

In view of the statements of Section 5.2, to obtain a positive answer to Problem (loc), it suffices to check that the elements p'_x and $q'_{x,y}$ defined by (5.3) are all fractions of elements of \mathcal{V}_H . This is easily verified if x, y are certain special elements of H . Indeed, by elementary computations, if x, y are both group-like elements, then

$$p'_x = \frac{1}{p_x} \quad \text{and} \quad q'_{x,y} = \frac{1}{q_{x,y}}.$$

Similarly, if y is group-like and x is skew-primitive with $\Delta(x) = g \otimes x + x \otimes h$ and $\varepsilon(x) = 0$ for some group-like elements g, h , then

$$p'_x = -\frac{p_x}{p_g p_h} \quad \text{and} \quad q'_{x,y} = -\frac{q_{x,y}}{q_{g,y} q_{h,y}}.$$

Our second main result is the following.

Theorem 5.4. *Let H be a pointed Hopf algebra such that $S(t_H)_\Theta$ is faithfully flat over \mathcal{B}_H , then \mathcal{B}_H is a localization of \mathcal{V}_H .*

The faithful flatness assumption is verified for instance if the pointed Hopf algebra H is cocommutative by [8, Th. 3.13], if $k[G(H)] \hookrightarrow H$ splits as an algebra

map by [8, Remark 3.14(a)], if H is finite-dimensional, or if each element of the kernel of the canonical group epimorphism $G(H)_{\text{ab}} \rightarrow G(H_{\text{ab}})$ is of finite order by [8, Th. 3.9].

Proof. Since $\mathcal{W}_H \subset \mathcal{V}_H \subset \mathcal{B}_H$, it is enough to prove that \mathcal{B}_H is a localization of \mathcal{W}_H . It follows from the assumptions and from [8, Lemma 3.11] that

$$\mathcal{B}_H = S(t_H)_{\Theta}^{\text{co-}Q},$$

where $Q = S(t_H)_{\Theta}/(\mathcal{B}_H^+)$. Now, it follows from Lemma 5.2 (ii) that

$$Q = S(t_H)_{\Theta}/(\mathcal{W}_H^+).$$

We conclude by applying Lemma 2.3 to the Hopf algebra $A = S(t_H)_{\Theta}$, which is pointed by Lemma 3.1, and to the left coideal subalgebra $B = \mathcal{W}_H$. \square

5.4. A commutative square. Recall from Section 3.2 that $S(t_H)$ has an H_{ab} -comodule algebra structure with the coaction δ_S defined by (3.6).

Let $\pi : T(X_H) \rightarrow S(t_H)$ be the abelianization map, which is the algebra map determined by $\pi(X_x) = t_x$ for all $x \in H$. It easily follows from the definitions that the square

$$\begin{array}{ccc} T(X_H) & \xrightarrow{\pi} & S(t_H) \\ \downarrow \delta_T & \circlearrowleft & \downarrow \delta_S \\ T(X_H) \otimes H & \xrightarrow{\pi \otimes q} & S(t_H) \otimes H_{\text{ab}} \end{array} \tag{5.4}$$

commutes.

From this square we deduce that π sends the subalgebra $T(X_H)^{\text{co-}H}$ of H -coinvariants to the subalgebra $S(t_H)^{\text{co-}H_{\text{ab}}}$ of H_{ab} -coinvariants:

$$\pi : T(X_H)^{\text{co-}H} \rightarrow S(t_H)^{\text{co-}H_{\text{ab}}}.$$

We can also express $(\text{id} \otimes q) \circ \mu$ as the diagonal in the square:

$$(\text{id} \otimes q) \circ \mu = \delta_S \circ \pi. \tag{5.5}$$

5.5. The algebra of functions on a finite group. Uma Iyer (January, 2014) proved the following result, which provides a positive answer to Problem (loc) when H is the algebra of functions on a finite group.

Theorem 5.5. *If $H = O_k(G)$ is the Hopf algebra of k -valued functions on a finite group G and the characteristic of k does not divide the order of G , then \mathcal{B}_H is a localization of \mathcal{V}_H .*

Since H is a commutative Hopf algebra, the canonical map $q : H \rightarrow H_{\text{ab}}$ is the identity and $\tilde{q} : S(t_H)_\Theta \rightarrow H$ is given by $\tilde{q}(t_x) = x$ for all $x \in H$.

The coaction δ_S now turns $S(t_H)_\Theta$ and $S(t_H)$ into H -comodule algebras. The commutative square (5.4) means that $\pi : T(X_H) \rightarrow S(t_H)$ is an H -comodule algebra map and that it induces a map on the subalgebras of coinvariant elements

$$\pi : T(X_H)^{\text{co-}H} \rightarrow S(t_H)^{\text{co-}H} .$$

It follows from (5.5) that the map $\mu : T(X_H) \rightarrow S(t_H) \otimes H$ is given in this case by

$$\mu = \delta_S \circ \pi . \tag{5.6}$$

Since $\bar{\mu} : \mathcal{U}_H \rightarrow S(t_H) \otimes H$ is a comodule algebra injection and the subalgebra of H -coinvariants in $S(t_H) \otimes H$ is $S(t_H) = S(t_H) \otimes 1$, we obtain the inclusion

$$\mathcal{V}_H \subset S(t_H) .$$

Lemma 5.6. *Under the hypotheses of the theorem, the algebra \mathcal{V}_H is the subalgebra of H -coinvariants in $S(t_H)$:*

$$\mathcal{V}_H = S(t_H)^{\text{co-}H} .$$

Proof. Let $P \in T(X_H)^{\text{co-}H}$. By definition, $\delta_T(P) = P \otimes 1$. Since π is a comodule algebra map, we have $\delta_S(\pi(P)) = \pi(P) \otimes 1$. It follows from this and from (5.6) that

$$\mu(P) = \delta_S(\pi(P)) = \pi(P) \otimes 1 .$$

In other words, the map μ restricted to $T(X_H)^{\text{co-}H}$ coincides with π and thus sends $T(X_H)^{\text{co-}H}$ into $S(t_H)^{\text{co-}H}$:

$$\mu = \pi : T(X_H)^{\text{co-}H} \rightarrow S(t_H)^{\text{co-}H} .$$

This proves the inclusion $\mathcal{V}_H \subset S(t_H)^{\text{co-}H}$.

Under the hypotheses of the theorem, H is cosemisimple. So the surjection $\pi : T(X_H) \rightarrow S(t_H)$ splits H -colinearly, implying that $T(X_H)^{\text{co-}H} \rightarrow S(t_H)^{\text{co-}H}$ is surjective. Thus, $S(t_H)^{\text{co-}H} = \mu(T(X_H)^{\text{co-}H}) = \bar{\mu}(\mathcal{V}_H)$. □

Proof of Theorem 5.5. By Propositions 3.2 and 3.4, $\mathcal{B}_H = \mathcal{C}_H$ is a localization of the algebra $S(t_H)^{\text{co-}H_{\text{ab}}} = S(t_H)^{\text{co-}H}$. We conclude with Lemma 5.6. □

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