Simplicity of twisted C^*-algebras of higher-rank graphs and crossed products by quasifree actions

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Dedicated to George A. Elliott on the occasion of his 70th birthday

Abstract. We characterise simplicity of twisted C^*-algebras of row-finite k-graphs with no sources. We show that each 2-cocycle on a cofinal k-graph determines a canonical second-cohomology class for the periodicity group of the graph. The groupoid of the k-graph then acts on the cartesian product of the infinite-path space of the graph with the dual group of the centre of any bicharacter representing this second-cohomology class. The twisted k-graph algebra is simple if and only if this action is minimal. We apply this result to characterise simplicity for many twisted crossed products of k-graph algebras by quasifree actions of free abelian groups.

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1. Introduction

Higher-rank graphs, or k-graphs, are combinatorial objects introduced by the first two authors in [22] as graph-based models for the higher-rank Cuntz–Krieger algebras studied by Robertson and Steger [42]. These C^*-algebras have been widely studied [4, 7, 11, 12], and their fundamental structure theory is by now fairly well understood. They provide interesting examples in noncommutative geometry [33, 48] and have been used to establish weak semiprojectivity [49] and calculate nuclear dimension [44] for UCT Kirchberg algebras.

Recently [24], we introduced a cohomology theory for k-graphs, and studied the associated twisted k-graph algebras $C^*(\Lambda, c)$ [25, 26]. These include many examples that do not arise naturally from untwisted k-graph C^*-algebras (see [26, Example 7.10], [34], and Theorem 5.1 below). So twisted k-graph C^*-algebras could

serve as useful models of various classes of classifiable C^*-algebras, particularly as they are always nuclear and belong to the UCT class [26, Corollary 8.7].

It is therefore important to understand when twisted k-graph C^*-algebras are simple. Simplicity of untwisted k-graph algebras was characterised in [41], and [26, Corollary 8.2] shows that if $C^*(\Lambda)$ is simple, so is every $C^*(\Lambda, c)$. But the converse fails: regarding $\mathbb{N}^2$ as a 2-graph T_2, the associated 2-graph algebra $C^*(T_2) \cong C(\mathbb{T}^2)$ is not simple, but for each irrational number θ there is a cocycle c_θ such that $C^*(T_2, c_\theta)$ is the irrational-rotation algebra A_θ. This elementary example indicates that characterising simplicity of $C^*(\Lambda, c)$ is a subtle problem — an indication confirmed by the partial results in [45]. In this paper, we present a complete solution to this problem: Theorem 3.4 gives a necessary and sufficient condition for $C^*(\Lambda, c)$ to be simple.

The broad strokes of our solution are as follows. We show that $C^*(\Lambda, c)$ is isomorphic to the C^*-algebra of a Fell bundle $\mathcal{B}_\Lambda$ over a topologically principal amenable étale quotient $\mathcal{H}_\Lambda$ of the k-graph groupoid introduced in [22]. Results of Ionescu and Williams [16] show that if $\mathcal{B}$ is Fell bundle over a groupoid $\mathcal{H}$, then $\mathcal{H}$ acts on the primitive ideal space of the C^*-subalgebra $C^*(\mathcal{H}^{(0)}; \mathcal{B}) \subseteq C^*(\mathcal{H}; \mathcal{B})$ sitting over the unit space of $\mathcal{H}$. We prove that if $\mathcal{H}$ is topologically principal, amenable and étale, then $C^*(\mathcal{H}; \mathcal{B})$ is simple if and only if this action is minimal. In particular, $C^*(\mathcal{H}_\Lambda; \mathcal{B}_\Lambda)$ is simple if and only if the Ionescu–Williams action of $\mathcal{H}_\Lambda$ is minimal. Our task is then to identify the action; it turns out that this is quite intricate.

The k-graph has a periodicity group $\mathrm{Per}(\Lambda) \subseteq \mathbb{Z}^k$ [6]. We use groupoid technology to show that c determines a class $[\omega] \in H^2(\mathrm{Per}(\Lambda), \mathbb{T})$. We then identify a map from $\mathcal{H}_\Lambda$ to the primitive ideal space of the noncommutative torus A_ω that becomes a homomorphism when $\mathrm{Prim}(A_\omega)$ is regarded as a quotient of $\mathbb{T}^k$. The map $\mathcal{H}_\Lambda \to \mathrm{Prim}(A_\omega)$ determines an action θ of $\mathcal{H}_\Lambda$ on $\mathcal{H}_\Lambda^{(0)} \times \mathrm{Prim}(A_\omega)$. We prove that there is a homeomorphism between this cartesian-product space and $\mathrm{Prim}(C^*(\mathcal{H}_\Lambda^{(0)}; \mathcal{B}_\Lambda))$ which intertwines the action θ with the action described in the preceding paragraph, and deduce our main result.

Even the action θ is not easy to compute in a generic example. So we develop some key examples where it *can* be computed. Firstly, if A_ω is simple, then θ is minimal precisely when Λ is cofinal. So we show how to decide efficiently whether A_ω is simple. Secondly, we show how to recover many twisted crossed products of k-graph algebras by quasifree actions of $\mathbb{Z}^l$ as twisted $(k + l)$-graph algebras, and show that our main theorem yields a very satisfactory characterisation of simplicity of many such twisted crossed products.

The paper is organised as follows. In Section 2 we establish the background that we need regarding k-graphs and their groupoids. This includes a fairly general technical result giving a sufficient condition for the interior of the isotropy of a Hausdorff étale groupoid to be closed. The main point is that the interior of the isotropy in the

groupoid $\mathcal{G}_\Lambda$ of a cofinal k-graph Λ is always closed, and relatively easy to describe: Corollary 2.2 says that if $\mathrm{Per}(\Lambda)$ is the periodicity group of the k-graph discussed in [6], then the interior $\mathcal{I}_\Lambda$ of the isotropy in $\mathcal{G}_\Lambda$ can be identified canonically with $\mathcal{G}_\Lambda^{(0)} \times \mathrm{Per}(\Lambda)$. Moreover, we can form the quotient groupoid $\mathcal{H}_\Lambda := \mathcal{G}_\Lambda/\mathcal{I}_\Lambda$, and this quotient is itself a topologically principal amenable étale groupoid.

In Section 3, we study carefully the relationship between cocycles on a k-graph Λ, and dynamics associated to the corresponding k-graph groupoid. In [22], the k-graph algebra $C^*(\Lambda)$ is identified with the C^*-algebra of a groupoid $\mathcal{G}_\Lambda$ with unit space Λ^∞, the space of infinite paths in Λ. We showed in [26] that each $C^*(\Lambda, c)$ can be realised as a twisted C^*-algebra $C^*(\mathcal{G}_\Lambda, \sigma_c)$ of the same groupoid. The restriction of the groupoid cocycle σ_c to each fibre of $\mathcal{I}_\Lambda$ determines a 2-cocycle of $\mathrm{Per}(\Lambda)$. We show that these 2-cocycles are all cohomologous, and deduce in Proposition 3.1 that σ_c is cohomologous to a cocycle σ whose restriction to $\mathcal{I}_\Lambda$ is of the form $1_{\Lambda^\infty} \times \omega$ for some bicharacter ω of $\mathrm{Per}(\Lambda)$. We then have $C^*(\Lambda, c) \cong C^*(\mathcal{G}_\Lambda, \sigma)$, and we seek to characterise simplicity of the latter. As in [29], the primitive-ideal space of each fibre of $C^*(\mathcal{I}_\Lambda, \sigma)$ can be identified with the character space of the kernel $Z_\omega \subseteq \mathrm{Per}(\Lambda)$ of the antisymmetric bicharacter associated to ω. So $\mathrm{Prim}\, C^*(\mathcal{I}_\Lambda, \sigma)$ can be identified with $\Lambda^\infty \times \widehat{Z}_\omega$. We construct from σ a $\mathrm{Per}(\Lambda)\widehat{}$-valued 1-cocycle r^σ on $\mathcal{G}_\Lambda$. We then have the wherewithal to state our main theorem, Theorem 3.4, although the proof must wait until the end of the subsequent section. We show in Lemma 3.5 how to compute the bicharacter ω, and hence the group Z_ω appearing in the main theorem, without passing to the groupoid $\mathcal{G}_\Lambda$. We then show that r^σ determines an action θ of the quotient $\mathcal{H}_\Lambda$ of $\mathcal{G}_\Lambda$ by the interior of its isotropy on $\Lambda^\infty \times \widehat{Z}_\omega$. Theorem 3.4 can be recast as saying that $C^*(\Lambda, c)$ is simple if and only if θ is minimal (see Corollary 4.8).

In Section 4 we prove Theorem 3.4 using the technology of Fell bundles. We use ideas from [8] (see also [37]) to recover $C^*(\Lambda, c)$ as the C^*-algebra of a Fell bundle $\mathcal{B}_\Lambda$ over the quotient $\mathcal{H}_\Lambda$ of $\mathcal{G}_\Lambda$ discussed above. We show that the restriction $C^*(\mathcal{H}_\Lambda^{(0)}; \mathcal{B}_\Lambda)$ of $C^*(\mathcal{H}_\Lambda; \mathcal{B}_\Lambda)$ to the unit space of $\mathcal{H}_\Lambda$ is isomorphic to $C_0(\Lambda^\infty) \otimes C^*(\mathrm{Per}(\Lambda), \omega)$. Since $\mathrm{Per}(\Lambda)$ is a free abelian group, the C^*-algebra $C^*(\mathrm{Per}(\Lambda), \omega)$ is a noncommutative torus, and has primitive ideal space $\Lambda^\infty \times \widehat{Z}_\omega$ (see, for example, [40]). Results of Ionescu and Williams [16] show that conjugation in $\mathcal{B}_\Lambda$ determines an action of $\mathcal{H}_\Lambda$ on the primitive-ideal space of $C^*(\mathcal{H}_\Lambda^{(0)}; \mathcal{B}_\Lambda)$ and hence on $\Lambda^\infty \times \widehat{Z}_\omega$. By identifying specific elements in the fibres of $\mathcal{B}_\Lambda$ that implement the Ionescu–Williams action, we prove that it matches up with the action θ of Section 3. In Lemma 4.6 and Corollary 4.7, we adapt the standard argument in [38] to prove that if $\mathcal{B}$ is a Fell bundle over a topologically principal amenable étale groupoid $\mathcal{H}$, then $C^*(\mathcal{H}; \mathcal{B})$ is simple if and only if the Ionescu–Williams action of $\mathcal{H}$ is minimal. We then prove our main theorem by applying this result to the bundle $\mathcal{B}_\Lambda$ over $\mathcal{H}_\Lambda$.

In Section 5, we investigate a broad class of examples of twisted higher-rank graph C^*-algebras where the hypotheses of our main result are readily checkable.

We show how a $\mathbb{T}^l$-valued 1-cocycle on a k-graph combined with a bicharacter ω of $\mathbb{Z}^l$ can be combined to obtain a 2-cocycle on the Cartesian-product $(k + l)$-graph $\Lambda \times \mathbb{N}^l$ for which the associated twisted $(k + l)$-graph C^*-algebra is isomorphic to a twisted crossed product of $C^*(\Lambda)$ by $\mathbb{Z}^l$. We demonstrate that the hypotheses of our main theorem can be effectively checked for these examples, and obtain a usable characterisation of simplicity of crossed products arising in this way when Λ is aperiodic.

We finish in Section 6 by presenting a number of concrete examples of our main result, showing how all of its working parts interact and demonstrating that each of the ingredients of the statement is genuinely necessary to obtain a satisfactory characterisation of simplicity. We also present a somewhat unrelated example which we believe is nevertheless interesting in its own right: a 3-graph all of whose twisted C^*-algebras (including the untwisted one) are simple, but for which the twisted C^*-algebras are not all mutually isomorphic.

2. Background

Throughout the paper, we regard $\mathbb{N}^k$ as a monoid under addition, with identity 0 and generators $e_1, \ldots, e_k$. For $m, n \in \mathbb{N}^k$, we write m_i for the ith coordinate of m, and define $m \vee n \in \mathbb{N}^k$ by $(m \vee n)_i = \max\{m_i, n_i\}$.

Given a small category $\mathcal{C}$, we write $\mathcal{C}^{*2} = \{(\lambda, \mu) \in \mathcal{C} \times \mathcal{C} : s(\lambda) = r(\mu)\}$ for the collection of composable pairs in $\mathcal{C}$. A $\mathbb{T}$-valued 2-cocycle[1] c on $\mathcal{C}$ is a map $c : \mathcal{C}^{*2} \to \mathbb{T}$ such that $c(\lambda, s(\lambda)) = c(r(\lambda), \lambda) = 1$ for all λ and $c(\mu, \nu)c(\lambda, \mu\nu) = c(\lambda, \mu)c(\lambda\mu, \nu)$ for composable λ, μ, ν. If $b : \mathcal{C} \to \mathbb{T}$ is a function with $b(\alpha) = 1$ for every identity morphism α of $\mathcal{C}$, then $\delta^1 b(\mu, \nu) := b(\mu)b(\nu)\overline{b(\mu\nu)}$ defines a 2-cocycle called the 2-coboundary associated to b. Two 2-cocycles c, c' are cohomologous if $(\mu, \nu) \mapsto \overline{c(\mu, \nu)}c'(\mu, \nu)$ is a 2-coboundary.

If $\mathcal{C}$ is a discrete group, then any function $c : \mathcal{C} \times \mathcal{C} \to \mathbb{T}$ for which the functions $c(\cdot, \alpha)$ and $c(\beta, \cdot)$ are homomorphisms is a cocycle; such cocycles are called *bicharacters*. If, in addition, $\mathcal{C}$ is abelian, every 2-cocycle is cohomologous to a bicharacter (see [20, Theorem 7.1]). For more background on 2-cocycles and bicharacters on abelian groups see [2, 20, 29] (note that in the first two references 2-cocycles are called multipliers).

2.1. k-graphs and twisted C^*-algebras. Let Λ be a countable small category and $d : \Lambda \to \mathbb{N}^k$ be a functor. Write $\Lambda^n := d^{-1}(n)$ for $n \in \mathbb{N}^k$. Then Λ is a k-graph (see [22]) if d satisfies the *factorisation property*: $(\mu, \nu) \mapsto \mu\nu$ is a bijection of $\{(\mu, \nu) \in \Lambda^m \times \Lambda^n : s(\mu) = r(\nu)\}$ onto Λ^{m+n} for each $m, n \in \mathbb{N}^k$. We then have $\Lambda^0 = \{\text{id}_o : o \in \text{Obj}(\Lambda)\}$, and so we regard the domain and codomain maps as maps

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[1]In [26] these were called *categorical cocycles*, in contradistinction to cubical cocycles.

$s, r : \Lambda \to \Lambda^0$. Recall from [31] that for $v, w \in \Lambda^0$ and $X \subseteq \Lambda$, we write

$$vX := \{\lambda \in X : r(\lambda) = v\}, \quad Xw := \{\lambda \in X : s(\lambda) = w\},$$
$$\text{and} \quad vXw = vX \cap Xw.$$

A k-graph Λ is *row-finite with no sources* if $0 < |v\Lambda^n| < \infty$ for all $v \in \Lambda^0$ and $n \in \mathbb{N}^k$. See [22] for further details regarding the basic structure of k-graphs.

Let Λ be a row-finite k-graph with no sources. We recall the definition of the *infinite path space* Λ^∞ given in [22, Definition 2.1]. We write Ω_k for the k-graph $\{(m, n) \in \mathbb{N}^k : m \le n\}$ with $r(m, n) = (m, m)$, $s(m, n) = (n, n)$, $(m, n)(n, p) = (m, p)$ and $d(m, n) = n - m$. We identify Ω_k^0 with $\mathbb{N}^k$ by $(m, m) \mapsto m$. We define Λ^∞ to be the collection of all k-graph morphisms $x : \Omega_k \to \Lambda$. For $p \in \mathbb{N}^k$, we define $T^p : \Lambda^\infty \to \Lambda^\infty$ by $(T^p x)(m, n) := x(m + p, n + p)$ for all $(m, n) \in \Omega_k$. (Traditionally, as in [22], these shift maps T^p have been denoted σ^p, but we will use σ in this paper to denote a 2-cocycle on $\mathcal{G}_\Lambda$.) For $x \in \Lambda^\infty$ we denote $x(0)$ by $r(x)$. For $\lambda \in \Lambda$ and $x \in \Lambda^\infty$ with $r(x) = s(\lambda)$, there is a unique element $\lambda x \in \Lambda^\infty$ such that $(\lambda x)(0, d(\lambda)) = \lambda$ and $T^{d(\lambda)}(\lambda x) = x$.

As in [22, Definition 4.7], we say that Λ is *cofinal* if, for every $x \in \Lambda^\infty$ and every $v \in \Lambda^0$, there exists $n \in \mathbb{N}^k$ such that $v\Lambda x(n) \ne \emptyset$. We say that Λ is *aperiodic* if it satisfies the "aperiodicity condition" of [22, Definition 4.3]: for every $v \in \Lambda^0$ there exists $x \in \Lambda^\infty$ with $r(x) = v$ and $T^m(x) \ne T^n(x)$ whenever $m \ne n$.

Given a k-graph Λ, the group of all 2-cocycles on Λ (as described above) is denoted $\underline{Z}^2(\Lambda, \mathbb{T})$. Let Λ be a row-finite k-graph with no sources, and fix $c \in \underline{Z}^2(\Lambda, \mathbb{T})$. A Cuntz–Krieger (Λ, c)-family in a C^*-algebra B is a function $t : \lambda \mapsto t_\lambda$ from Λ to B such that

(CK1) $\{t_v : v \in \Lambda^0\}$ is a collection of mutually orthogonal projections;

(CK2) $t_\mu t_\nu = c(\mu, \nu) t_{\mu\nu}$ whenever $s(\mu) = r(\nu)$;

(CK3) $t_\lambda^* t_\lambda = t_{s(\lambda)}$ for all $\lambda \in \Lambda$; and

(CK4) $t_v = \sum_{\lambda \in v\Lambda^n} t_\lambda t_\lambda^*$ for all $v \in \Lambda^0$ and $n \in \mathbb{N}^k$.

$C^*(\Lambda, c)$ is then defined to be the universal C^*-algebra generated by a Cuntz–Krieger (Λ, c)-family (see [26, Notation 5.4]).

2.2. Groupoids. A groupoid is a small category $\mathcal{G}$ with inverses. We use standard groupoid notation as in, for example, [38]. So $\mathcal{G}^{(0)}$ is the set of identity morphisms of $\mathcal{G}$, called the unit space, and $\mathcal{G}^{(2)}$ denotes the set $\mathcal{G}^{*2}$ of composable pairs in $\mathcal{G}$. The groupoid $\mathcal{G}$ is an étale Hausdorff groupoid if it has a locally compact Hausdorff topology under which all operations in $\mathcal{G}$ are continuous (when $\mathcal{G}^{(2)} \subseteq \mathcal{G} \times \mathcal{G}$ is given the relative topology) and the range and source maps $r, s : \mathcal{G} \to \mathcal{G}^{(0)}$ are local homeomorphisms. It then makes sense to talk about continuous cocycles on $\mathcal{G}$. We write $Z^2(\mathcal{G}, \mathbb{T})$ for the group of continuous $\mathbb{T}$-valued 2-cocycles on $\mathcal{G}$ and say that two

continuous 2-cocycles are cohomologous if they differ by a continuous 2-coboundary — that is, the coboundary $\delta^1 b$ associated to a continuous map $b : \mathcal{G} \to \mathbb{T}$ such that $b|_{\mathcal{G}^{(0)}} \equiv 1$. A 1-cocycle on $\mathcal{G}$ with values in a group G is a map $\rho : \mathcal{G} \to G$ that carries composition in $\mathcal{G}$ to the group operation in G. Given $u \in \mathcal{G}^{(0)}$ we write $\mathcal{G}^u$ for $\{\gamma \in \mathcal{G} : r(\gamma) = u\}$, $\mathcal{G}_u$ for $\{\gamma \in \mathcal{G} : s(\gamma) = u\}$ and $\mathcal{G}_u^u = \mathcal{G}^u \cap \mathcal{G}_u$. The *isotropy* of $\mathcal{G}$ is the set $\bigcup_{u \in \mathcal{G}^{(0)}} \mathcal{G}_u^u$ of elements of $\mathcal{G}$ whose range and source coincide. A groupoid is *minimal* if $r(\mathcal{G}_u)$ is dense in $\mathcal{G}^{(0)}$ for every unit $u \in \mathcal{G}^{(0)}$. It is *topologically principal* if $\{u \in \mathcal{G}^{(0)} : \mathcal{G}_u^u = \{u\}\}$ is dense in $\mathcal{G}^{(0)}$.

It will be important later to know that the interior of the isotropy in the groupoid associated to a cofinal k-graph is closed. This will follow from the following fairly general result.

Proposition 2.1. *Let $\mathcal{G}$ be an étale groupoid, let G be a countable discrete abelian group, and let $c \in Z^1(\mathcal{G}, G)$. Suppose that $\mathcal{G}$ is minimal and that for all x, the restriction of c to $\mathcal{G}_x^x$ is injective. Let $\mathcal{I}$ denote the interior of the isotropy of $\mathcal{G}$. For $x, y \in \mathcal{G}^{(0)}$, we have $c(\mathcal{I} \cap \mathcal{G}_x^x) = c(\mathcal{I} \cap \mathcal{G}_y^y)$. The set H defined by $H := c(\mathcal{I} \cap \mathcal{G}_x^x)$ for any $x \in \mathcal{G}^{(0)}$ is a subgroup of G, and $s \times c$ induces an isomorphism from $\mathcal{I}$ to $\mathcal{G}^{(0)} \times H$. In particular the interior of the isotropy of $\mathcal{G}$ is closed.*

Proof. For $x \in \mathcal{G}^{(0)}$ set $\mathcal{I}_x := \mathcal{I} \cap \mathcal{G}_x^x$ and note that $H_x := c(\mathcal{I}_x)$ is a subgroup of G. Fix $x, y \in \mathcal{G}^{(0)}$; we prove that $H_x = H_y$. By symmetry it suffices to show that $H_x \subset H_y$, so we fix $h \in H_x$ and prove that $h \in H_y$. Fix $\alpha \in \mathcal{I}_x$ such that $c(\alpha) = h$. Since G is discrete and c is continuous, there is an open neighbourhood U of α such that $U \subseteq \mathcal{I} \cap c^{-1}(h)$. Since c is injective on each $\mathcal{G}_x^x$, this U is a bisection. Since $\mathcal{G}$ is minimal, the set $s(\mathcal{G}^y)$ is dense in $\mathcal{G}^{(0)}$, and so there exists $\gamma \in \mathcal{G}^y$ such that $s(\gamma) \in s(U)$. The unique element β of $\mathcal{I}_{s(\gamma)} \cap U$ satisfies $c(\beta) = h$, and then $\gamma \beta \gamma^{-1} \in \mathcal{I}_y$ satisfies $c(\gamma \beta \gamma^{-1}) = c(\gamma) c(\beta) \overline{c(\gamma)} = h$. So $h \in H_y$ as required.

Thus the map $s \times c$ yields an isomorphism from the interior of the isotropy of $\mathcal{G}$ to $\mathcal{G}^{(0)} \times H$. Since $\mathcal{I}$ is the intersection of the closed set $c^{-1}(H)$ with the isotropy of $\mathcal{G}$, which is also closed, we deduce that $\mathcal{I}$ is closed. $\qquad\square$

Given an étale groupoid $\mathcal{G}$ and a 2-cocycle $\sigma \in Z^2(\mathcal{G}, \mathbb{T})$, it is straightforward to check that $C_c(\mathcal{G})$ is a *-algebra under the operations

$$(fg)(\gamma) = \sum_{\eta\zeta=\gamma} \sigma(\eta, \zeta) f(\eta) g(\zeta) \quad \text{and} \quad f^*(\gamma) = \overline{\sigma(\gamma, \gamma^{-1}) f(\gamma^{-1})}$$

for $f, g \in C_c(\mathcal{G})$. The twisted groupoid C^*-algebra $C^*(\mathcal{G}, \sigma)$ is then defined to be the closure of $C_c(\mathcal{G})$ under the maximal C^*-norm (see [38] for more details).

2.3. k-graph groupoids. Following [22, Definition 2.7] we associate a groupoid $\mathcal{G}_\Lambda$ to each row-finite k-graph Λ with no sources by putting

$$\mathcal{G}_\Lambda := \{(x, l - m, y) \in \Lambda^\infty \times \mathbb{Z}^k \times \Lambda^\infty : l, m \in \mathbb{N}^k, T^l x = T^m y\}.$$

For $\mu, \nu \in \Lambda$ with $s(\mu) = s(\nu)$ define $Z(\mu, \nu) \subset \mathcal{G}_\Lambda$ by

$$Z(\mu, \nu) := \{(\mu x, d(\mu) - d(\nu), \nu x) : x \in \Lambda^\infty, r(x) = s(\mu)\}.$$

For $\lambda \in \Lambda$, we define $Z(\lambda) := Z(\lambda, \lambda)$.

The sets $Z(\mu, \nu)$ form a basis of compact open sets for a locally compact Hausdorff topology on $\mathcal{G}_\Lambda$ under which it is an étale groupoid with structure maps $r(x, l - m, y) = (x, 0, x), s(x, l-m, y) = (y, 0, y),$ and $(x, l-m, y)(y, p-q, z) = (x, l - m + p - q, z)$ (see [22, Proposition 2.8]). The $Z(\lambda)$ are then a basis for the relative topology on $\mathcal{G}_\Lambda^{(0)} \subseteq \mathcal{G}_\Lambda$. We identify $\mathcal{G}_\Lambda^{(0)} = \{(x, 0, x) : x \in \Lambda^\infty\}$ with Λ^∞. Following [22, Proposition 2.8, Corollary 3.5] $\mathcal{G}_\Lambda$ is an amenable étale groupoid. Moreover $\mathcal{G}_\Lambda$ is minimal if and only if Λ is cofinal [22, Proof of Proposition 4.8].

Suppose now that Λ is cofinal. As in [6], we define a relation on Λ by $\mu \sim \nu$ if and only if $s(\mu) = s(\nu)$ and $\mu x = \nu x$ for all $x \in s(\mu)\Lambda^\infty$. This is an equivalence relation on Λ which respects range, source and composition. By [6, Theorem 4.2(1)], the set $\mathrm{Per}(\Lambda) := \{d(\mu) - d(\nu) : \mu, \nu \in \Lambda \text{ and } \mu \sim \nu\} \subseteq \mathbb{Z}^k$ is a subgroup of $\mathbb{Z}^k$. Since Λ is cofinal, [6, Lemma 4.6] gives

$$\mathrm{Per}(\Lambda) \subseteq \{m \in \mathbb{Z}^k : (x, m, x) \in \mathcal{G}_\Lambda \text{ for all } x \in \Lambda^\infty\}.$$

Since $\mathrm{Per}(\Lambda)$ is a subgroup of $\mathbb{Z}^k$, it is also a finitely generated free abelian group and so $\mathrm{Per}(\Lambda) \cong \mathbb{Z}^l$ for some integer $l \leq k$.

Corollary 2.2. *Let Λ be a cofinal row-finite k-graph with no sources. Let $\mathcal{I}_\Lambda$ denote the interior of the isotropy in $\mathcal{G}_\Lambda$. Then $\mathcal{I}_\Lambda$ is closed and*

$$\mathcal{I}_\Lambda = \{(x, m, x) : x \in \Lambda^\infty, m \in \mathrm{Per}(\Lambda)\} \cong \Lambda^\infty \times \mathrm{Per}(\Lambda).$$

Moreover, $\mathcal{H}_\Lambda := \mathcal{G}_\Lambda / \mathcal{I}_\Lambda$ is an amenable, topologically principal, locally compact, Hausdorff, étale groupoid.

Proof. Note that $\mathcal{G}_\Lambda$ is a minimal étale groupoid and the restriction of the canonical cocycle $c \in Z^1(\mathcal{G}_\Lambda, \mathbb{Z}^k)$, given by $c(x, n, y) = n$ to $(\mathcal{G}_\Lambda)_x^x$ is injective for each $x \in \Lambda^\infty$. Hence by Proposition 2.1 $\mathcal{I}_\Lambda$ is closed.

By definition of the topology on $\mathcal{G}_\Lambda$, the set $\{(x, m, x) : x \in \Lambda^\infty, m \in \mathrm{Per}(\Lambda)\}$ is contained in $\mathcal{I}_\Lambda$, the interior of the isotropy of $\mathcal{G}_\Lambda$. Conversely, if $\alpha \in \mathcal{I}_\Lambda$, then $\alpha = (x, m, x)$ for some $x \in \Lambda^\infty$ and $m \in \mathbb{Z}^k$, and Proposition 2.1 applied to the cocycle c of the preceding paragraph shows that $(y, m, y) \in \mathcal{I}_\Lambda$ for every y. In particular $m \in \mathrm{Per}(\Lambda)$. So $\mathcal{I}_\Lambda = \{(x, m, x) : x \in \Lambda^\infty, m \in \mathrm{Per}(\Lambda)\}$ as claimed.

It is routine to check that $\mathcal{G}_\Lambda / \mathcal{I}_\Lambda$ is a locally compact Hausdorff étale groupoid (see, for example, [47, Proposition 2.5]). Since $c(\mathcal{I}_\Lambda) = \mathrm{Per}\,\Lambda$, there exists $\tilde{c} \in Z^1(\mathcal{H}_\Lambda, \mathbb{Z}^k / \mathrm{Per}\,\Lambda)$ such that $\tilde{c}([(x, n, y)]) = n + \mathrm{Per}\,\Lambda$. The groupoid $\tilde{c}^{-1}(0)$ is isomorphic to $c^{-1}(0)$ which is amenable by, for example, [52, Lemma 6.7]. Since $\mathbb{Z}^k / \mathrm{Per}(\Lambda)$ is abelian and hence amenable, it follows from [50, Proposition 9.3]

that $\mathcal{H}_\Lambda$ is amenable.   By construction of $\mathcal{I}_\Lambda$, the interior of the isotropy of $\mathcal{H}_\Lambda$ is trivial, and therefore $\mathcal{H}_\Lambda$ is topologically principal by, for example, [39, Proposition 3.6]. $\square$

We frequently identify $\mathcal{I}_\Lambda$ with $\Lambda^\infty \times \mathrm{Per}(\Lambda)$ as in Corollary 2.2.

Let $\Lambda\ {}_s*_s\ \Lambda := \{(\mu, \nu) \in \Lambda \times \Lambda : s(\mu) = s(\nu)\}$. Lemma 6.6 of [26] shows that there exists $\mathcal{P} \subseteq \Lambda\ {}_s*_s\ \Lambda$ such that

$$(\lambda, s(\lambda)) \in \mathcal{P} \text{ for all } \lambda \quad \text{and} \quad \mathcal{G}_\Lambda = \bigsqcup_{(\mu,\nu)\in\mathcal{P}} Z(\mu, \nu). \tag{2.1}$$

Given such a set $\mathcal{P}$, for $g \in \mathcal{G}_\Lambda$ we write (μ_g, ν_g) for the element of $\mathcal{P}$ with $g \in Z(\mu_g, \nu_g)$. Fix $c \in \underline{Z}^2(\Lambda, \mathbb{T})$. Let $\tilde{d} : \mathcal{G}_\Lambda \to \mathbb{Z}^k$ be the canonical 1-cocycle $\tilde{d}(x, n, y) = n$. Lemma 6.3 of [26] shows that for composable $g, h \in \mathcal{G}_\Lambda$, there exist $\alpha, \beta, \gamma \in \Lambda$ and $y \in \Lambda^\infty$ such that

$$\nu_g \alpha = \mu_h \beta, \quad \mu_g \alpha = \mu_{gh} \gamma \quad \text{and} \quad \nu_h \beta = \nu_{gh} \gamma;$$

and

$$g = (\mu_g \alpha y, \tilde{d}(g), \nu_g \alpha y), \quad h = (\mu_h \beta y, \tilde{d}(h), \nu_h \beta y)$$
$$\text{and} \quad gh = (\mu_{gh} \gamma y, \tilde{d}(gh), \nu_{gh} \gamma y). \tag{2.2}$$

The formula

$$\sigma_c(g, h) = c(\mu_g, \alpha)\overline{c(\nu_g, \alpha)}c(\mu_h, \beta)\overline{c(\nu_h, \beta)}c(\mu_{gh}, \gamma)c(\nu_{gh}, \gamma) \tag{2.3}$$

does not depend on the choice of α, β, γ, and determines a continuous groupoid 2-cocycle on $\mathcal{G}_\Lambda$. If σ_c' is obtained from c in the same way with respect to another collection $\mathcal{P}'$, then σ_c and σ_c' are cohomologous. Corollary 7.7 of [26] shows that there is an isomorphism $C^*(\Lambda, c) \cong C^*(\mathcal{G}_\Lambda, \sigma_c)$ that carries each s_λ to the characteristic function $1_{Z(\lambda, s(\lambda))}$.

3. An action of the k-graph groupoid associated to a k-graph 2-cocycle

We consider a row-finite k-graph Λ with no sources. Lemma 7.2 of [45] says that if Λ is not cofinal, then $C^*(\Lambda, c)$ is nonsimple for every $c \in \underline{Z}^2(\Lambda, \mathbb{T})$. Since we are interested here in when $C^*(\Lambda, c)$ *is* simple, we shall therefore assume henceforth that Λ is cofinal.

Recall that if σ is a continuous $\mathbb{T}$-valued 2-cocycle on a groupoid $\mathcal{G}$, then there is a groupoid extension $\mathcal{G}^{(0)} \times \mathbb{T} \to \mathcal{G} \times_\sigma \mathbb{T} \to \mathcal{G}$, where $\mathcal{G} \times_\sigma \mathbb{T}$ is equal to $\mathcal{G} \times \mathbb{T}$ as a set, with unit space $(\mathcal{G} \times_\sigma \mathbb{T})^{(0)} = \mathcal{G}^{(0)} \times \{1\}$, range map $r(g, z) = (r(g), 1)$, source map $s(g, z) = (s(g), 1)$ and operations

$$(\alpha, w)(\beta, z) = (\alpha\beta, \sigma(\alpha, \beta)zw) \quad \text{and} \quad (\alpha, w)^{-1} = (\alpha^{-1}, \overline{\sigma(\alpha, \alpha^{-1})}\overline{w}).$$

Given $\sigma \in Z^2(\mathcal{G}_\Lambda, \mathbb{T})$ we write $\mathcal{I}_\Lambda \times_\sigma \mathbb{T}$ for $\mathcal{I}_\Lambda \times \mathbb{T}$ regarded as a subgroupoid of $\mathcal{G}_\Lambda \times_\sigma \mathbb{T}$. We often implicitly identify $(\mathcal{G}_\Lambda \times_\sigma \mathbb{T})^{(0)}$ with $\mathcal{G}_\Lambda^{(0)}$.

Proposition 3.1. *Let Λ be a cofinal row-finite k-graph with no sources, and suppose $c \in \underline{Z}^2(\Lambda, \mathbb{T})$. Let $\mathcal{P} \subseteq \Lambda_s *_s \Lambda$ be as in (2.1), and let $\sigma_c \in Z^2(\mathcal{G}_\Lambda, \mathbb{T})$ be as in (2.3). For $x \in \Lambda^\infty$, define $\sigma_c^x \in Z^2(\mathrm{Per}(\Lambda), \mathbb{T})$ by $\sigma_c^x(p, q) = \sigma_c((x, p, x), (x, q, x))$. Then there is a bicharacter ω of $\mathrm{Per}(\Lambda)$ such that σ_c^x is cohomologous to ω for every $x \in \Lambda^\infty$. For any such bicharacter ω, there exists a cocycle $\sigma \in Z^2(\mathcal{G}_\Lambda, \mathbb{T})$ such that σ is cohomologous to σ_c and $\sigma|_{\mathcal{I}_\Lambda} = \omega \times 1_{\Lambda^\infty}$.*

To prove the proposition, we first prove some lemmas.

Lemma 3.2. *Let Λ be a row-finite k-graph with no sources, and take $\sigma \in Z^2(\mathcal{G}_\Lambda, \mathbb{T})$. For $\alpha = (x, m, y) \in \mathcal{G}_\Lambda$ and $p \in \mathrm{Per}(\Lambda)$, define $r_\alpha^\sigma : \mathrm{Per}(\Lambda) \to \mathbb{T}$ by*

$$r_\alpha^\sigma(p) = \sigma\big(\alpha, (y, p, y)\big)\sigma\big((x, m + p, y), \alpha^{-1}\big)\overline{\sigma\big(\alpha, \alpha^{-1}\big)}. \tag{3.1}$$

For $x \in \Lambda^\infty$ define $\sigma_x \in Z^2(\mathrm{Per}(\Lambda), \mathbb{T})$ by $\sigma_x(p, q) = \sigma((x, p, x), (x, q, x))$. For each $p \in \mathrm{Per}(\Lambda)$, the map $r^\sigma(p) : \mathcal{G}_\Lambda \to \mathbb{T}$ is continuous, and we have

$$r_\alpha^\sigma(p + q) = \sigma_{r(\alpha)}(p, q)\overline{\sigma_{s(\alpha)}(p, q)}r_\alpha^\sigma(p)r_\alpha^\sigma(q). \tag{3.2}$$

If $\sigma_x = \sigma_y$ for all x, y, then $\alpha \mapsto r_\alpha^\sigma$ is a continuous $\mathrm{Per}(\Lambda)\widehat{\ }$-valued 1-cocycle on $\mathcal{G}_\Lambda$.

Proof. The map $\alpha \mapsto r_\alpha^\sigma(p)$ is continuous because σ is continuous.

A straightforward calculation in the central extension $\mathcal{G}_\Lambda \times_\sigma \mathbb{T}$ shows that for $w, z \in \mathbb{T}$, we have

$$(\alpha, w)((y, p, y), z)(\alpha, w)^{-1} = ((x, p, x), r_\alpha^\sigma(p)z).$$

Computing further in the central extension, we have

$$
\begin{aligned}
((x, p + q, x), & r_\alpha^\sigma(p + q)) \\
&= (\alpha, 1)((y, p + q, y), 1)(\alpha, 1)^{-1} \\
&= (\alpha, 1)((y, p, y), \overline{\sigma_y(p, q)})(\alpha, 1)^{-1}(\alpha, 1)((y, q, y), 1)(\alpha, 1)^{-1} \\
&= ((x, p, x), r_\alpha^\sigma(p)\overline{\sigma_y(p, q)})((x, q, x), r_\alpha^\sigma(q)) \\
&= \big((x, p + q, x), \sigma_x(p, q)\overline{\sigma_y(p, q)}r_\alpha^\sigma(p)r_\alpha^\sigma(q)\big),
\end{aligned}
$$

giving (3.2).

Now suppose that $\sigma_x = \sigma_y$ for all x, y. Then (3.2) implies immediately that $r_\alpha^\sigma \in \mathrm{Per}(\Lambda)\widehat{\ }$ for all α. Take $p \in \mathrm{Per}(\Lambda)$ and composable $\alpha, \beta \in \mathcal{G}_\Lambda$. Let $y = s(\beta)$. Computing again in $\mathcal{G}_\Lambda \times_\sigma \mathbb{T}$, we have

$$
\begin{aligned}
\big(r(\alpha), r_\alpha^\sigma(p)r_\beta^\sigma(p)\big) &= (\alpha, 1)(\beta, 1)\big((y, p, y), 1\big)(\beta, 1)^{-1}(\alpha, 1)^{-1} \\
&= (\alpha\beta, \sigma(\alpha, \beta))\big((y, p, y), 1\big)(\alpha\beta, \sigma(\alpha, \beta))^{-1} \\
&= \big(r(\alpha), r_{\alpha\beta}^\sigma(p)\big).
\end{aligned}
$$

So $\alpha \mapsto r_\alpha^\sigma$ is a 1-cocycle on $\mathcal{G}_\Lambda$. $\qquad\square$

Given a cocycle $\omega \in Z^2(\mathrm{Per}(\Lambda), \mathbb{T})$, we write ω^* for the cocycle $(p, q) \mapsto \overline{\omega(q, p)}$. By [29, Proposition 3.2] (see also [20, Lemma 7.1]), the map $\omega\omega^*$ is a bicharacter of $\mathrm{Per}(\Lambda)$ which is antisymmetric[2] in the sense that $(\omega\omega^*)(p, q) = \overline{(\omega\omega^*)(q, p)}$. Proposition 3.2 of [29] implies that $\omega \mapsto \omega\omega^*$ descends to an isomorphism of $H^2(\mathrm{Per}(\Lambda), \mathbb{T})$ onto the group $X^2(\mathrm{Per}(\Lambda), \mathbb{T})$ of all antisymmetric bicharacters of $\mathrm{Per}(\Lambda)$.

Lemma 3.3. *Let Λ be a cofinal row-finite k-graph with no sources, and suppose $c \in \underline{Z}^2(\Lambda, \mathbb{T})$. Let σ_c be a continuous cocycle on $\mathcal{G}_\Lambda$ of the form (2.3), so that $C^*(\mathcal{G}_\Lambda, \sigma_c) \cong C^*(\Lambda, c)$. For each $x \in \Lambda^\infty$, let $\sigma_c^x \in Z^2(\mathrm{Per}(\Lambda), \mathbb{T})$ be the cocycle given by $\sigma_c^x(p, q) = \sigma_c((x, p, x), (x, q, x))$. Then the cohomology class of σ_c^x is independent of x.*

Proof. The formula (2.3) shows that σ_c is locally constant as a function from $\mathcal{G}_\Lambda \times \mathcal{G}_\Lambda \to \mathbb{T}$. Restricting σ_c to $\mathcal{I}_\Lambda$, we obtain cocycles σ_c^x on the groups $(\mathcal{I}_\Lambda)_x = \{(x, p, x) : p \in \mathrm{Per}(\Lambda)\} \cong \mathrm{Per}(\Lambda)$. For $x \in \Lambda^\infty$, let ω_x be the bicharacter of $\mathrm{Per}(\Lambda)$ given by

$$\omega_x(p, q) := \sigma_c^x\big((x, p, x), (x, q, x)\big)\overline{\sigma_c^x\big((x, q, x), (x, p, x)\big)}.$$

Fix free abelian generators $g_1, \ldots, g_l$ for $\mathrm{Per}(\Lambda)$. Since each ω_x is a bicharacter, it is determined by the values $\omega_x(g_i, g_j)$. Fix $x \in \Lambda^\infty$. Since σ_c is locally constant, for each i, j there is a neighbourhood $U_{i,j}$ of x such that $\sigma_c^x(g_i, g_j) = \sigma_c((x, g_i, x), (x, g_j, x))$ is constant on $U_{i,j}$. So for $y \in U := \bigcap_{i,j} U_{i,j}$ we have $\omega_y(g_i, g_j) = \omega_x(g_i, g_j)$ for all i, j, and hence $\omega_y = \omega_x$. Now [29, Proposition 3.2] implies that the cohomology class (in $H^2(\mathrm{Per}(\Lambda), \mathbb{T})$) of σ_c^x is locally constant with respect to x. Since Λ is cofinal, $\mathcal{G}_\Lambda$ is minimal, and so every orbit in $\mathcal{G}_\Lambda$ is dense; so to see that the cohomology class of σ_c^x is globally constant, it suffices to show that it is constant on orbits. For $x \in \Lambda^\infty$, the subgroup $\{((x, p, x), z) : p \in \mathrm{Per}(\Lambda), z \in \mathbb{T}\} \subseteq (\mathcal{G}_\Lambda \times_{\sigma_c} \mathbb{T})_x^x$, which is isomorphic to the group extension $\mathrm{Per}(\Lambda) \times_{\sigma_c^x} \mathbb{T}$ of $\mathrm{Per}(\Lambda)$ by $\mathbb{T}$. Conjugation by any α in $\mathcal{G}_\Lambda \times_{\sigma_c} \mathbb{T}$ is an isomorphism $\mathrm{Ad}_\alpha : A_{s(\alpha)} \cong A_{r(\alpha)}$. For $\gamma \in \mathcal{G}_\Lambda$, let $r_\gamma^{\sigma_c} : \mathrm{Per}(\Lambda) \to \mathbb{T}$ be the map of Lemma 3.2. Fix $\alpha = (\gamma, w) \in \mathcal{G}_\Lambda \times_{\sigma_c} \mathbb{T}$, $p \in \mathrm{Per}(\Lambda)$ and $z \in \mathbb{T}$, and let $x = r(\gamma)$ and $y = s(\gamma)$. A quick calculation gives $\mathrm{Ad}_\alpha((y, p, y), z) = ((x, p, x), r_\gamma^{\sigma_c}(p)z)$. If $p = 0$, then (3.1) collapses to give $\mathrm{Ad}_\alpha((y, 0, y), z) = ((x, 0, x), z)$ because $\sigma_c((x, m, y), (y, 0, y)) = 1$. If $q : \mathcal{I}_\Lambda \times_{\sigma_c} \mathbb{T} \to \Lambda^\infty \times \mathrm{Per}(\Lambda)$ is the quotient map $((x, p, x), z) \mapsto (x, p)$, then

$$q(\mathrm{Ad}_\alpha((y, p, y), z)) = q\big((x, p, x), r_\gamma^{\sigma_c}(p)z\big) = (x, p),$$

so that Ad_α descends through q to the map $(y, p) \mapsto (x, p)$. Thus conjugation by α

---

[2]In [29], the word "symplectic" is used instead of antisymmetric.

determines an isomorphism

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & \mathbb{T} & \longrightarrow & A_y & \longrightarrow & \mathrm{Per}(\Lambda) & \longrightarrow & 0 \\
& & \Big\downarrow \mathrm{id} & & \Big\downarrow \mathrm{Ad}_\alpha & & \Big\downarrow \mathrm{id} & & \\
0 & \longrightarrow & \mathbb{T} & \longrightarrow & A_x & \longrightarrow & \mathrm{Per}(\Lambda) & \longrightarrow & 0
\end{array}
$$

of extensions. Hence σ_c^x is cohomologous to σ_c^y; so the cohomology class of σ_c^x is constant on orbits. $\qquad\square$

Proof of Proposition 3.1. Let σ_c be a continuous cocycle on $\mathcal{G}_\Lambda$ constructed in §2.3 so that $C^*(\mathcal{G}_\Lambda, \sigma_c) \cong C^*(\Lambda, c)$. By Lemma 3.3, the cohomology class of σ_c^x is independent of x. So there exists a cocycle $\omega \in Z^2(\mathrm{Per}(\Lambda), \mathbb{T})$ whose cohomology class agrees with that of σ_c^x for each x. We may assume that ω is a bicharacter.

The map $\tilde{c}_x : \mathrm{Per}(\Lambda) \times \mathrm{Per}(\Lambda) \to \mathbb{T}$ defined by

$$(p,q) \mapsto \tilde{c}_x(p,q) := \sigma_c((x,p,x),(x,q,x))\overline{\omega(p,q)}$$

is a coboundary on $\mathrm{Per}(\Lambda)$ for each x. Fix free abelian generators $g_1, \ldots, g_l$ for $\mathrm{Per}(\Lambda)$. Fix $x \in \Lambda^\infty$, and define $b_x(0) = b_x(g_j) = 1 \in \mathbb{T}$ for all j, and then define b_x on $\mathrm{Per}(\Lambda)$ inductively by

$$b_x(m)\overline{b_x(m + g_i)} = \tilde{c}_x(g_i, m) \qquad \text{whenever } m \in \langle g_j : j \leq i \rangle. \tag{3.3}$$

Since $x \mapsto \tilde{c}_x(p,q)$ is continuous for each p, q, an induction argument shows that $x \mapsto b_x(p)$ is continuous for each p.

We claim that each $\delta^1 b_x = \tilde{c}_x$. To see this, choose for each x a cochain $\tilde{b}_x : \mathrm{Per}(\Lambda) \to \mathbb{T}$ such that $\delta^1 \tilde{b}_x = \tilde{c}_x$. The map $a_x(m) := \prod_{i=1}^l \overline{\tilde{b}_x(g_i)}^{m_i}$ is a 1-cocycle on $\mathrm{Per}(\Lambda)$, and so $\delta^1 a_x = 1$. Hence $\delta^1(a_x \tilde{b}_x) = \delta^1 \tilde{b}_x = \tilde{c}_x$. We have $(a_x \tilde{b}_x)(0) = (a_x \tilde{b}_x)(g_i) = 1$ for all i, and for $i \leq l$ and $m \in \langle g_j : j \leq i \rangle$, we have

$$(a_x \tilde{b}_x)(m)\overline{(a_x \tilde{b}_x)(m + g_i)} = (a_x \tilde{b}_x)(g_i)(a_x \tilde{b}_x)(m)\overline{(a_x \tilde{b}_x)(m + g_i)}$$
$$= \delta^1(a_x \tilde{b}_x)(g_i, m) = \tilde{c}_x(g_i, m).$$

So b_x and $a_x \tilde{b}_x$ both take 0 and each g_i to 1 and satisfy (3.3). Hence $b_x = a_x \tilde{b}_x$, and $\delta^1 b_x = \tilde{c}_x$ for all x.

Since $x \mapsto b_x(p)$ is continuous for each p, the map $b : (x, p, x) \mapsto b_x(p)$ is a continuous cochain on $\mathcal{I}_\Lambda$. Since $\mathcal{I}_\Lambda$ is clopen in $\mathcal{G}_\Lambda$, we can extend b to a cochain $\tilde{b}$ on all of $\mathcal{G}_\Lambda$ by setting $\tilde{b}|_{\mathcal{G}_\Lambda \setminus \mathcal{I}_\Lambda} \equiv 1$.

Now $\delta^1 \tilde{b}$ is a continuous coboundary on $\mathcal{G}_\Lambda$, so $\sigma := \sigma_c \cdot \overline{\delta^1 \tilde{b}}$ represents the same cohomology class as σ_c. Hence $C^*(\mathcal{G}_\Lambda, \sigma) \cong C^*(\mathcal{G}_\Lambda, \sigma_c) \cong C^*(\Lambda, c)$ [38, Proposition II.1.2]. We have $\sigma((x,p,x),(x,q,x)) = \omega(p,q)$ for $p, q \in \mathrm{Per}(\Lambda)$ by construction of $\tilde{b}$. Each σ_c^x is cohomologous to ω by choice of ω. $\qquad\square$

Given an abelian group A and a cocycle $\omega \in Z^2(A, \mathbb{T})$, we write

$$Z_\omega := \{p \in A : (\omega\omega^*)(p, q) = 1 \text{ for all } q \in A\} \tag{3.4}$$

for the kernel of the homomorphism $p \mapsto (\omega\omega^*)(p, \cdot)$ from A to $\widehat{A}$ induced by $\omega\omega^*$. Thus Z_ω is a subgroup of A.

We now have all the ingredients needed to state our main theorem.

Theorem 3.4. *Let Λ be a row-finite cofinal k-graph with no sources, and suppose that $c \in \underline{Z}^2(\Lambda, \mathbb{T})$. Suppose that $\sigma \in Z^2(\mathcal{G}_\Lambda, \mathbb{T})$ and $\omega \in Z^2(\mathrm{Per}(\Lambda), \mathbb{T})$ satisfy $C^*(\mathcal{G}_\Lambda, \sigma) \cong C^*(\Lambda, c)$ and $\sigma|_{\mathcal{I}_\Lambda} = \omega \times 1_{\Lambda^\infty}$ (such a pair σ, ω exist by Proposition 3.1). Let $r^\sigma : \mathcal{G}_\Lambda \to \mathrm{Per}(\Lambda)\widehat{}$ be the cocycle of Lemma 3.2. Then $C^*(\Lambda, c)$ is simple if and only if $\{(r(\gamma), r_\gamma^\sigma|_{Z_\omega}) : \gamma \in (\mathcal{G}_\Lambda)_x\}$ is dense in $\Lambda^\infty \times \widehat{Z}_\omega$ for all $x \in \Lambda^\infty$. In particular, if ω is nondegenerate, then $C^*(\Lambda, c)$ is simple.*

The proof of the main theorem will occupy the remainder of this section and most of the next. Before we get started, we provide a practical method for computing Z_ω without reference to $\mathcal{G}_\Lambda$. To see why this is useful, observe that to apply our main theorem, it is typically necessary to compute a cocycle σ on $\mathcal{G}_\Lambda$ with the required properties, and this is not so easy to do. (We discuss a class of examples where this is possible in Section 5.) But the last statement of the theorem says that if we know that the centre of the bicharacter ω is trivial, then no computations in $\mathcal{G}_\Lambda$ are necessary.

In the following result, for $m \in \mathbb{Z}^k$, we write m^+ and m^- for $m \vee 0$ and $(-m) \vee 0$. So $m = m^+ - m^-$ and $m^+ \wedge m^- = 0$.

Lemma 3.5. *Let Λ be a row-finite cofinal k-graph with no sources, and suppose that $c \in \underline{Z}^2(\Lambda, \mathbb{T})$. Let $g_1, \ldots, g_l$ be free generators for $\mathrm{Per}(\Lambda)$. There exists $v \in \Lambda^0$ such that $T^{g_i^+}(x) = T^{g_i^-}(x)$ for all $x \in Z(v)$ and $1 \leq i \leq l$. Let $N = \sum_{i=1}^l g_i^+ + g_i^-$, and fix $\lambda \in v\Lambda^N$. For $1 \leq i \leq l$, factorise $\lambda = \mu_i \tau_i = \nu_i \rho_i$ where $d(\mu_i) = g_i^+$ and $d(\nu_i) = g_i^-$. For $1 \leq i, j \leq l$, factorise $\lambda = \mu_{ij} \tau_{ij} = \nu_{ij} \rho_{ij}$ where $d(\mu_{ij}) = (g_i + g_j)^+$ and $d(\nu_{ij}) = (g_i + g_j)^-$. Let ω be the bicharacter of $\mathrm{Per}(\Lambda)$ such that*

$$\omega(g_i, g_j) := c(\mu_i, \tau_i)\overline{c(\nu_i, \rho_i)}c(\mu_j, \tau_j)\overline{c(\nu_j, \rho_j)}c(\mu_{ij}, \tau_{ij})c(\nu_{ij}, \rho_{ij})$$

for $1 \leq i, j \leq l$. Then there exists $\sigma \in Z^2(\mathcal{G}_\Lambda, \mathbb{T})$ and a bicharacter ω' of $\mathrm{Per}(\Lambda)$ such that $C^*(\Lambda, c) \cong C^*(\mathcal{G}_\Lambda, \sigma)$, $\sigma|_{\mathcal{I}_\Lambda} = \omega' \times 1_{\Lambda^\infty}$, and $Z_{\omega'} = Z_\omega$. In particular, if $Z_\omega = \{0\}$ then $C^*(\Lambda, c)$ is simple.

Proof. We claim that there is a set $\mathcal{P} \subseteq \Lambda \,_s*_s \Lambda$ such that

$$\{(\lambda, s(\lambda)) : \lambda \in \Lambda\} \cup \{(\mu_i, \nu_i) : i \leq l\} \cup \{(\mu_{ij}, \nu_{ij}) : i \neq j\} \subseteq \mathcal{P},$$

and such that $\mathcal{G}_\Lambda = \bigsqcup_{(\mu,\nu) \in \mathcal{P}} Z(\mu, \nu)$.

To see this, we argue as in [26, Lemma 6.6] to see that the $Z(\lambda, s(\lambda))$ are mutually disjoint and $\bigsqcup_\lambda Z(\lambda, s(\lambda))$ is clopen in $\mathcal{G}_\Lambda$. Let

$$\mathcal{P}_0 := \{(\lambda, s(\lambda)) : \lambda \in \Lambda\} \cup \{(\mu_i, \nu_i) : i \leq l\} \cup \{(\mu_{ij}, \nu_{ij}) : 1 \leq i \leq j \leq l\}.$$

For each $i \leq l$ either $(\mu_i, \nu_i) = (\mu_i, s(\mu_i))$, or $d(\mu_i) - d(\nu_i) \notin \mathbb{N}^k$ and so $Z(\mu_i, \nu_i) \cap Z(\lambda, s(\lambda)) = \emptyset$ for all λ; and likewise for each (μ_{ij}, ν_{ij}). Each $Z(\mu_i, \nu_i) \subseteq \Lambda^\infty \times \{g_i\} \times \Lambda^\infty$, and each $Z(\mu_{ij}, \nu_{ij}) \subseteq \Lambda^\infty \times \{g_i + g_j\} \times \Lambda^\infty$. So the $Z(\mu_i, \nu_i)$ and the $Z(\mu_{ij}, \nu_{ij})$ collectively are mutually disjoint. Hence the $Z(\mu, \nu)$ where $(\mu, \nu) \in \mathcal{P}_0$ are mutually disjoint, and $\bigsqcup_{(\mu,\nu)\in\mathcal{P}_0} Z(\mu, \nu)$ is clopen in $\mathcal{G}_\Lambda$. Now the argument of the final two paragraphs of [26, Lemma 6.6] shows that we can extend $\mathcal{P}_0$ to the required collection $\mathcal{P}$.

The formula (2.3) yields a cocycle $\sigma_c \in Z^2(\mathcal{G}_\Lambda, \mathbb{T})$, and the construction of $\mathcal{P}_0$ shows that

$$\sigma_c((x, g_i, x), (x, g_j, x)) = \omega(g_i, g_j) \quad \text{for all } x \in Z(\lambda) \text{ and } i, j \leq l.$$

Corollary 7.9 of [26] implies that $C^*(\Lambda, c) \cong C^*(\mathcal{G}_\Lambda, \sigma_c)$. Proposition 3.1 applied to this $\mathcal{P}$ gives a bicharacter ω' of $\operatorname{Per}(\Lambda)$ and a cocycle σ on $\mathcal{G}_\Lambda$ such that $C^*(\mathcal{G}_\Lambda, \sigma) \cong C^*(\mathcal{G}_\Lambda, \sigma_c) \cong C^*(\Lambda, c)$ and ω' is cohomologous to σ_c^x and hence to ω. Thus $Z_\omega = Z_{\omega'}$ by [29, Proposition 3.2]. The final statement follows from Theorem 3.4. □

We finish the section by showing that r^σ induces an action θ of the quotient $\mathcal{H}_\Lambda$ of $\mathcal{G}_\Lambda$ by the interior of its isotropy on the space $\Lambda^\infty \times \widehat{Z}_\omega$. In particular, we prove that θ is minimal if and only if $\{(r(\gamma), r_\gamma^\sigma|_{Z_\omega}) : \gamma \in (\mathcal{G}_\Lambda)_x\}$ is dense in $\Lambda^\infty \times \widehat{Z}_\omega$ for all $x \in \Lambda^\infty$ as in Theorem 3.4. In the next section we will realise $C^*(\Lambda, c)$ as the C^*-algebra of a Fell bundle over a quotient groupoid $\mathcal{H}_\Lambda$ of $\mathcal{G}_\Lambda$, and show that the action, given by [16], of $\mathcal{H}_\Lambda$ on the primitive ideal space of the C^*-algebra over the unit space in this bundle is isomorphic as a groupoid action to θ. We then prove our main theorem by showing that minimality of the action described in [16] characterises simplicity of the C^*-algebra of the Fell bundle.

Given Λ, c and ω as in Proposition 3.1, and with Z_ω as in (3.4), we can form the quotient $H := \operatorname{Per}(\Lambda)/Z_\omega$. We then have $\widehat{H} \cong Z_\omega^\perp \leq \operatorname{Per}(\Lambda)^\widehat{\ }$, so we regard $\widehat{H}$ as a subgroup of $\operatorname{Per}(\Lambda)^\widehat{\ }$.

Lemma 3.6. *Let Λ be a row-finite cofinal k-graph with no sources, and suppose that $c \in \underline{Z}^2(\Lambda, \mathbb{T})$. Suppose that $\sigma \in Z^2(\mathcal{G}_\Lambda, \mathbb{T})$ is cohomologous to σ_c and that $\omega \in Z^2(\operatorname{Per}(\Lambda), \mathbb{T})$ satisfies $\sigma|_{\mathcal{I}_\Lambda} = 1_{\Lambda^\infty} \times \omega$ as in Proposition 3.1. Let r^σ be the $\operatorname{Per}(\Lambda)^\widehat{\ }$-valued 1-cocycle on $\mathcal{G}_\Lambda$ given by Lemma 3.2. For $\alpha \in \mathcal{I}_\Lambda$, we have $r_\alpha^\sigma \in Z_\omega^\perp$. Let $\pi : \mathcal{G}_\Lambda \to \mathcal{H}_\Lambda := \mathcal{G}_\Lambda/\mathcal{I}_\Lambda$ be the quotient map. There is a continuous $\widehat{Z}_\omega$-valued 1-cocycle $\tilde{r}^\sigma$ on $\mathcal{H}_\Lambda$ such that $\tilde{r}_{\pi(\alpha)}^\sigma(p) = r_\alpha^\sigma(p)$ for all $\alpha \in \mathcal{G}_\Lambda$ and $p \in Z_\omega$. There is an action θ of $\mathcal{H}_\Lambda$ on $\Lambda^\infty \times \widehat{Z}_\omega$ such that*

$$\theta_\alpha(s(\alpha), \chi) = (r(\alpha), \tilde{r}_\alpha^\sigma \cdot \chi) \text{ for all } \alpha \in \mathcal{H}_\Lambda \text{ and } \chi \in \widehat{Z}_\omega.$$

Proof. Suppose that $p \in \mathrm{Per}(\Lambda)$ and $q \in Z_\omega$. Calculating in $\mathcal{G}_\Lambda \times_\sigma \mathbb{T}$, we have

$$
\begin{aligned}
((x, p, x), 1)&((x, q, x), 1)((x, p, x), 1)^{-1} \\
&= \left((x, q, x), \sigma\big((x, p, x), (x, q, x)\big)\overline{\sigma\big((x, q, x), (x, p, x)\big)}\right) \\
&= \left((x, q, x), \omega\omega^*(p, q)\right).
\end{aligned}
$$

Hence $r^\sigma_{(x,p,x)}(q) = (\omega\omega^*)(p, q) = 1$ since $q \in Z_\omega$.

Suppose that $\pi(\alpha) = \pi(\beta)$. Then $\alpha = \beta\gamma$ for some $\gamma \in \mathcal{I}_\Lambda$. So for $p \in Z_\omega$ we have $r^\sigma_\alpha(p) = r^\sigma_\beta(p)r^\sigma_\gamma(p) = r^\sigma_\beta(p)$, showing that $\tilde{r}^\sigma$ is well defined. It is continuous by definition of the quotient topology, and is a 1-cocycle because r^σ is. For $p, p' \in Z_\omega$, we have

$$
\tilde{r}^\sigma_{\pi(\alpha)}(p + p') = r^\sigma_\alpha(p + p') = r^\sigma_\alpha(p)r^\sigma_\alpha(p') = \tilde{r}^\sigma_{\pi(\alpha)}(p)\tilde{r}^\sigma_{\pi(\alpha)}(p').
$$

The final statement follows immediately. □

4. A Fell bundle associated to a *k*-graph 2-cocycle

To identify the twisted k-graph algebra $C^*(\Lambda, c)$ with the C^*-algebra of a Fell bundle, we must first construct the bundle. We start by describing what will become the C^*-algebra sitting over the unit-space in this bundle: the twisted C^*-algebra of $\mathcal{I}_\Lambda$ sitting inside that of $\mathcal{G}_\Lambda$.

Lemma 4.1. *Let* Λ *be a row-finite cofinal* k*-graph with no sources, and take* $c \in \underline{Z}^2(\Lambda, \mathbb{T})$. *Suppose that* $\sigma \in Z^2(\mathcal{G}_\Lambda, \mathbb{T})$ *is cohomologous to* σ_c *and that* $\omega \in Z^2(\mathrm{Per}(\Lambda), \mathbb{T})$ *satisfies* $\sigma|_{\mathcal{I}_\Lambda} = 1_{\Lambda^\infty} \times \omega$ *as in Proposition 3.1. There is a* $*$*-homomorphism*

$$
\Phi : C_c(\Lambda^\infty) \otimes C^*(\mathrm{Per}(\Lambda), \omega) \to C_c(\mathcal{I}_\Lambda, \sigma)
$$

such that $\Phi(f \otimes u_p)(x, q, x) = \delta_{p,q} f(x)$ *for all* $f \in C_c(\Lambda^\infty)$ *and* $p, q \in \mathrm{Per}(\Lambda)$. *This* $*$*-homomorphism extends to an isomorphism*

$$
\Phi : C_0(\Lambda^\infty) \otimes C^*(\mathrm{Per}(\Lambda, \omega)) \to C^*(\mathcal{I}_\Lambda, \sigma).
$$

Proof. Recall that we identify $\mathcal{I}_\Lambda$ with $\Lambda^\infty \times \mathrm{Per}(\Lambda)$ as in Corollary 2.2. For each $p \in \mathrm{Per}(\Lambda)$, the characteristic function $U_p := 1_{\Lambda^\infty \times \{p\}}$ is a unitary multiplier of $C^*(\mathcal{I}_\Lambda, \sigma)$. By construction of σ, we have $U_p U_q = \omega(p, q)U_{p+q}$ for all p, q, so the universal property of $C^*(\mathrm{Per}(\Lambda), \omega)$ gives a homomorphism $C^*(\mathrm{Per}(\Lambda), \omega) \to \mathcal{M}C^*(\mathcal{I}_\Lambda, \sigma)$ satisfying $u_p \mapsto U_p$. The U_p clearly commute with $C_0(\Lambda^\infty)$, so the universal property of the tensor product gives a homomorphism $C_0(\Lambda^\infty) \otimes C^*(\mathrm{Per}(\Lambda), \omega) \to C^*(\mathcal{I}_\Lambda, \sigma)$ satisfying $f \otimes u_p \mapsto U_p f$. For $x \in \Lambda^\infty$, we have $(\mathcal{I}_\Lambda)_x \cong \mathrm{Per}(\Lambda)$, and $\sigma|_{(\mathcal{I}_\Lambda)_x} = \omega$ by Proposition 3.1. Hence the regular

representation of $C^*(\mathcal{I}_\Lambda, \sigma)$ on $\ell^2((\mathcal{I}_\Lambda)_x)$ is unitarily equivalent to the canonical faithful representation of $C^*(\mathrm{Per}(\Lambda), \omega)$ on $\ell^2(\mathrm{Per}(\Lambda))$. Thus the homomorphism $\rho : f \otimes u_p \mapsto U_p f$ is a fibrewise-injective homomorphism of $C_0(\Lambda^\infty)$-algebras, and is injective on the central copy of $C_0(\Lambda^\infty)$. Since the norm on a $C_0(X)$-algebra is the same as the supremum norm on the algebra of sections of the associated upper-semicontinuous bundle [51], it follows that ρ is injective. Since its range contains $C_0(\Lambda^\infty \times \{p\})$ for each p, it is also surjective. $\qquad\square$

Given a compact abelian group G, and an action β of a closed subgroup H of G on a C^*-algebra C, the *induced algebra* $\mathrm{Ind}_H^G C$ is defined by

$$\mathrm{Ind}_H^G C = \{f : G \to C \mid f(g - h) = \beta_h(f(g)) \text{ for } g \in G \text{ and } h \in H\}$$

with pointwise operations. Note that we use additive notation to emphasize that G must be abelian. This algebra carries an induced action lt of G given by translation: $\mathrm{lt}_g(f)(g') = f(g' - g)$. For $h \in H$, we have $\mathrm{lt}_h(f)(g) = \beta_h(f(g))$.

If C is simple, then $\mathrm{Prim}(\mathrm{Ind}_H^G C)$ is homeomorphic to G/H: for $g + H \in G/H$, the corresponding primitive ideal I_{g+H} is the ideal $\{f \in \mathrm{Ind}_H^G C : f|_{g+H} = 0\}$ (see, for example, [36, Proposition 6.6]).

Now let A be a discrete abelian group and let ω be a $\mathbb{T}$-valued 2-cocycle on A. Let $Z_\omega \subseteq A$ be the centre of ω as in (3.4). The antisymmetric bicharacter $\omega\omega^*$ descends to an antisymmetric bicharacter $(\omega\omega^*)\check{}$ of $B := A/Z_\omega$. There is a cocycle $\tilde{\omega} \in Z^2(B, \mathbb{T})$ such that

$$\tilde{\omega}\tilde{\omega}^*(a + Z_\omega, a' + Z_\omega) = (\omega\omega^*)(a, a') \text{ for all } a, a' \in A.$$

By construction, the antisymmetric bicharacter $(\omega\omega^*)\check{}$ is nondegenerate in the sense that $g + Z_\omega \mapsto (\omega\omega^*)\check{}(g + Z_\omega, \cdot)$ is injective (as a map from B to its dual $\widehat{B}$).

There is an action $\beta^A : \widehat{A} \to \mathrm{Aut}(C^*(A, \omega))$ such that $\beta_t^A(U_a) = \chi(a)U_a$ for $\chi \in \widehat{A}$. Recall from [29, Lemma 5.11 and Theorem 6.3][3] that there is an $\widehat{A}$-equivariant isomorphism

$$C^*(A, \omega) \cong \mathrm{Ind}_{\widehat{B}}^{\widehat{A}} C^*(B, \tilde{\omega}),$$

and that $C^*(B, \tilde{\omega})$ is simple. Hence $\mathrm{Prim}(C^*(A, \omega)) \cong \widehat{A}/\widehat{B} \cong \widehat{Z_\omega}$ [36, Proposition 6.6]. In particular, in the situation of Lemma 4.1, we have

$$\mathrm{Prim}(C^*(\mathcal{I}_\Lambda, \sigma)) \cong \mathrm{Prim}\left(C_0(\Lambda^\infty) \otimes C^*(\mathrm{Per}(\Lambda), \omega)\right) \cong \Lambda^\infty \times \widehat{Z_\omega}.$$

Now resume the hypotheses and notation of Lemma 4.1. We construct a Fell bundle over $\mathcal{H}_\Lambda$. We describe the fibres of the bundle and the multiplication and involution operations first, and then prove in Proposition 4.2 that there is a topology compatible with these operations under which we obtain the desired Fell bundle. We

---

[3]There is a typographical error in the statement of [29, Theorem 6.3]: G/G_z should read G_z.

write $C^*(\mathcal{H};\mathcal{B})$ for the full C^*-algebra of a Fell bundle $\mathcal{B}$ over a groupoid $\mathcal{H}$, and $C_r^*(\mathcal{H};\mathcal{B})$ for the corresponding reduced C^*-algebra. We make the convention that $C^*(\mathcal{H}^{(0)};\mathsf{B})$ denotes the full C^*-algebra of the restriction of $\mathcal{B}$ to the unit space; this $C^*(\mathcal{H}^{(0)};\mathsf{B})$ is a $C_0(\mathcal{H}^{(0)})$-algebra. For background on Fell bundles over étale groupoids, see [21, 28].

We identify both $\mathcal{H}_\Lambda^0$ and $\mathcal{G}_\Lambda^{(0)}$ with Λ^∞. We continue to write $\pi : \mathcal{G}_\Lambda \to \mathcal{H}_\Lambda$ for the quotient map, and we often write $[\gamma]$ for $\pi(\gamma)$, and regard it as an equivalence class in $\mathcal{G}_\Lambda$; that is $[\gamma] = \{\alpha \in \mathcal{G}_\Lambda : \pi(\alpha) = \pi(\gamma)\} = \gamma \cdot \mathcal{I}_\Lambda$.

We define $A_x = C^*(\mathrm{Per}(\Lambda),\omega)$ for $x \in \Lambda^\infty$, and we write $\mathcal{A}_\Lambda$ for the trivial bundle $\Lambda^\infty \times C^*(\mathrm{Per}(\Lambda),\omega)$. So $C^*(\mathcal{I}_\Lambda,\sigma) \cong C^*(\Lambda^\infty;\mathcal{A}_\Lambda)$, and A_x is the fibre of $\mathcal{A}_\Lambda$ over x. For $[\gamma] \in \mathcal{H}_\Lambda$, we let $B_{[\gamma]}^\circ := C_c([\gamma])$ as a vector space over $\mathbb{C}$, and we define multiplication $B_{[\alpha]}^\circ \times B_{[\beta]}^\circ \to B_{[\alpha\beta]}^\circ$ where $s(\alpha) = r(\beta)$, and involution $B_{[\alpha]}^\circ \to B_{[\alpha^{-1}]}^\circ$ by

$$(f * g)(\gamma) = \sum_{\substack{\eta\zeta=\gamma,\\ \eta\in[\alpha],\zeta\in[\beta]}} \sigma(\eta,\zeta)f(\eta)g(\zeta) \quad \text{and} \quad f^*(\gamma) = \overline{\sigma(\gamma,\gamma^{-1})f(\gamma^{-1})}.$$

We regard each $B_{[x]}^\circ$ (where $x \in \Lambda^\infty = \mathcal{H}_\Lambda^{(0)}$) as a dense subspace of A_x, and endow it with the norm inherited from A_x. We write $B_x := \overline{B_{[x]}^\circ} \cong A_x$, and identify B_x and A_x. For $[\gamma] \in \mathcal{H}_\Lambda$, we define an $A_{s(\gamma)}$-valued inner product on $B_{[\gamma]}^\circ$ by $\langle f,g\rangle_{s(\gamma)} := f^* * g$. Then we obtain a norm on $B_{[\gamma]}^\circ$ by $\|f\| = \|\langle f,f\rangle_{s(\gamma)}\|^{1/2}$, and we write $B_{[\gamma]}$ for the completion of $B_{[\gamma]}^\circ$ in this norm. Note that $B_{[\gamma]}$ is a full right $A_{s(\gamma)}$-Hilbert module. It is straightforward to show that $\|f * g\| \le \|f\|\|g\|$ for all $f \in B_{[\alpha]}^\circ$ and $g \in B_{[\beta]}^\circ$ when $s(\alpha) = r(\beta)$, and involution extends to an isometric conjugate linear map $B_{[\alpha]} \to B_{[\alpha^{-1}]}$.

Proposition 4.2. *With notation as above there is a unique topology on $\mathcal{B}_\Lambda = \bigsqcup_{[\gamma]\in\mathcal{H}_\Lambda} B_{[\gamma]}$ under which it is a Banach bundle and such that for each $f \in C_c(\mathcal{G}_\Lambda,\sigma)$, the section $[\gamma] \mapsto f|_{[\gamma]}$ is norm continuous. Under this topology, the space $\mathcal{B}_\Lambda$ is a saturated continuous Fell bundle over $\mathcal{H}_\Lambda$.*

Proof. The first assertion follows from [13, Proposition 10.4] once we show that the sections $[\gamma] \mapsto f|_{[\gamma]}$ associated to elements $f \in C_c(\mathcal{G}_\Lambda,\sigma)$ are norm continuous and the ranges of these sections are pointwise dense. Let $f \in C_c(\mathcal{G}_\Lambda,\sigma)$. Then the restriction $[x] \mapsto \|f|_{[x]}\|$ is continuous on $\mathcal{H}_\Lambda^{(0)} = \Lambda^\infty$ because $C^*(\mathcal{I}_\Lambda,\sigma) \cong C_0(\Lambda^\infty) \otimes C^*(\mathrm{Per}(\Lambda),\omega)$ by Lemma 4.1. To show $[\gamma] \mapsto \|f|_{[\gamma]}\|$ is continuous, note that $\|f|_{[\gamma]}\| = \|(f^* * f)|_{[s(\gamma)]}\|^{1/2}$. Since $[x] \mapsto \|(f^* * f)|_{[x]}\|$ is continuous on $\mathcal{H}_\Lambda^{(0)}$, and since the source map $[\gamma] \mapsto [s(\gamma)]$ in $\mathcal{H}_\Lambda$ is continuous, it follows that $[\gamma] \mapsto \|f|_{[\gamma]}\|$ is continuous.

It is straightforward to check that for each $b \in B_{[\gamma]}^\circ$, there exists $f \in C_c(\mathcal{G}_\Lambda,\sigma)$ such that $b = f|_{[\gamma]}$. Hence, for each $[\gamma]$, $\{f|_{[\gamma]} : f \in C_c(\mathcal{G}_\Lambda,\sigma)\}$ is dense in $B_{[\gamma]}$. $\qquad\square$

We now establish that $C^*(\Lambda, c)$ can be identified with the C^*-algebra of the Fell bundle we have just constructed. Recall that if $\mathcal{B}$ is a Fell bundle over a groupoid $\mathcal{G}$, then $C^*(\mathcal{G}; \mathcal{B})$, the C^*-algebra of the bundle, is a universal completion of the algebra of compactly supported sections of the bundle, and $C_r^*(\mathcal{G}; \mathcal{B})$ is the corresponding reduced C^*-algebra.

Theorem 4.3. *Let Λ be a row-finite cofinal k-graph with no sources, and take $c \in \underline{Z}^2(\Lambda, \mathbb{T})$. Let $\mathcal{H}_\Lambda$ denote the quotient of $\mathcal{G}_\Lambda$ by the interior $\mathcal{I}_\Lambda$ of its isotropy, and let $\mathcal{B}_\Lambda$ be the Fell bundle over $\mathcal{H}_\Lambda$ described in Proposition 4.2. Then $C^*(\mathcal{H}_\Lambda; \mathcal{B}_\Lambda) = C_r^*(\mathcal{H}_\Lambda; \mathcal{B}_\Lambda)$, and there is an isomorphism $\pi : C^*(\Lambda, c) \cong C^*(\mathcal{H}_\Lambda; \mathcal{B}_\Lambda)$ such that $\pi(s_\lambda)([\gamma]) = 1_{Z(\lambda, s(\lambda))}|_{[\gamma]}$ for all $\lambda \in \Lambda$ and $\gamma \in \mathcal{G}_\Lambda$.*

Proof. Corollary 2.2 says that $\mathcal{H}_\Lambda$ is amenable. Hence $C^*(\mathcal{H}_\Lambda; \mathcal{B}_\Lambda) = C_r^*(\mathcal{H}_\Lambda; \mathcal{B}_\Lambda)$ by [46, Theorem 1]. Define elements t_λ of $C_c(\mathcal{H}_\Lambda; \mathcal{B}_\Lambda)$ by $t_\lambda([\gamma]) := 1_{Z(\lambda, s(\lambda))}|_{[\gamma]} \in B_{[\gamma]}^\circ \subseteq B_{[\gamma]}$. Theorem 6.7 of [26] shows that the $1_{Z(\lambda, s(\lambda))}$ form a Cuntz–Krieger (Λ, c)-family in $C^*(\mathcal{G}_\Lambda, \sigma)$. It follows that the t_λ constitute a Cuntz–Krieger (Λ, c)-family in $C^*(\mathcal{H}_\Lambda; \mathcal{B}_\Lambda)$. So the universal property of $C^*(\Lambda, c)$ yields a homomorphism $\pi : C^*(\Lambda, c) \to C^*(\mathcal{H}_\Lambda; \mathcal{B}_\Lambda)$.

To see that π is injective, we aim to apply the gauge-invariant uniqueness theorem [26, Corollary 7.7]. The projections $\{t_v : v \in \Lambda^0\}$ are nonzero because the $Z(v)$ are nonempty, so we just need to show that there is an action β of $\mathbb{T}^k$ on $C^*(\mathcal{H}_\Lambda; \mathcal{B}_\Lambda)$ such that $\beta_z(t_\lambda) = z^{d(\lambda)} t_\lambda$ for all $\lambda \in \Lambda$. Let $\tilde{d} : \mathcal{G}_\Lambda \to \mathbb{Z}^k$ be the canonical cocycle $(x, m, y) \mapsto m$. For $z \in \mathbb{T}^k$, $[\gamma] \in \mathcal{H}_\Lambda$ and $f \in B_{[\gamma]}^\circ$ define $\beta_z^{[\gamma]}(f) \in B_{[\gamma]}^\circ$ by $\beta_z^{[\gamma]}(f)(\alpha) = z^{\tilde{d}(\alpha)} f(\alpha)$. Simple calculations show that $\beta_z^{[\gamma]}(f) * \beta_z^{[\gamma']}(g) = \beta_z^{[\gamma\gamma']}(f * g)$ and that $\beta_z^{[\gamma^{-1}]}(f^*) = \beta_z^{[\gamma]}(f)^*$. For $x \in \mathcal{G}_\Lambda^{(0)}$, the map $\beta_z^{[x]} : C_c(\mathcal{I}_\Lambda, \sigma) \to C_c(\mathcal{I}_\Lambda, \sigma)$ extends to the canonical action of $\mathbb{T}^k$ on $A_x = C^*(\mathrm{Per}(\Lambda), \omega)$, and so is isometric. It follows that the $\beta_z^{[\gamma]}$ are isometric for the norms on the fibres $B_{[\gamma]}$ of $\mathcal{B}_\Lambda$. For $f \in C_c(\mathcal{G}_\Lambda, \sigma)$ supported on a basic open set $Z(\mu, \nu)$, the map $[\gamma] \mapsto \beta_z^{[\gamma]}(f)$ is clearly continuous. Since such f span $C_c(\mathcal{G}_\Lambda, \sigma)$, and by definition of the topology on $\mathcal{B}$ (see Proposition 4.2), it follows that if $p : \mathcal{B}_\Lambda \to \mathcal{H}_\Lambda$ is the bundle map, then $\xi \mapsto \beta_z^{p(\xi)}(\xi)$ is continuous. So for fixed $z \in \mathbb{T}^k$, the collection $\beta_z^{[\gamma]}$ determines an automorphism of the bundle $\mathcal{B}_\Lambda$, and hence induces an automorphism β_z of $C^*(\mathcal{H}_\Lambda; \mathcal{B}_\Lambda)$. It is routine to check that $z \mapsto \beta_z$ is an action of $\mathbb{T}^k$ on $C^*(\mathcal{H}_\Lambda; \mathcal{B}_\Lambda)$ such that $\beta_z(f)([\gamma]) = \beta_z^{[\gamma]}(f([\gamma]))$ for $f \in C_c(\mathcal{H}_\Lambda; \mathcal{B}_\Lambda)$ and $z \in \mathbb{T}^k$. In particular, each $\beta_z(t_\lambda) = z^{d(\lambda)} t_\lambda$, and so the gauge-invariant uniqueness theorem [26, Corollary 7.7] shows that π is injective as required.

It remains to show that π is surjective. For this, it suffices to fix $\gamma, \gamma' \in \mathcal{G}_\Lambda$ such that $[\gamma] \neq [\gamma']$, and show that the set $\{\pi(a)([\gamma]) : a \in C^*(\Lambda, c), \pi(a)[\gamma'] = 0\}$ is dense in $B_{[\gamma]}$. Since π is linear, it will suffice to show that for each $\alpha \in [\gamma]$ there exists $a \in C^*(\Lambda, c)$ such that $\pi(a)([\gamma']) = 0$ and $\pi(a)([\gamma]) = \delta_\alpha$. Fix $\alpha \in [\gamma]$, say

$\tilde{d}(\alpha) = m$. Choose a compact open bisection $U \subseteq \tilde{d}^{-1}(m)$. If $m \notin \mathrm{Per}(\Lambda) + \tilde{d}(\gamma')$, then $U \cap [\gamma'] = \emptyset$; otherwise, since $[\gamma'] \neq [\alpha]$, either $r(\gamma') \neq r(\alpha)$ or $s(\gamma') \neq s(\alpha)$, and so we may shrink U to ensure that $U \cap [\gamma'] = \emptyset$. By definition of the topology on $\mathcal{G}_\Lambda$, there exist $\mu, \nu \in \Lambda$ such that $\alpha \in Z(\mu, \nu) \subseteq U$. The function $t_\mu t_\nu^*$ takes values in $\mathbb{T}$ on $Z(\mu, \nu)$, and so there is a complex scalar $z \in \mathbb{T}$ such that $a := z s_\mu s_\nu^*$ satisfies $\pi(a)([\gamma]) = \delta_\alpha$ and $\pi(a)([\gamma']) = 0$ as claimed. $\qquad\square$

To prove Theorem 3.4, we aim to identify the action of $\mathcal{H}_\Lambda$ on $\mathrm{Prim}(C^*(\mathcal{H}_\Lambda^{(0)}; \mathcal{B}_\Lambda))$ obtained from [16] with the action θ of Lemma 3.6. To do this, we first give an explicit description of the action from [16].

Lemma 4.4. *Let $\mathcal{B}$ be a Fell bundle over a groupoid $\mathcal{G}$ with unital fibres over $\mathcal{G}^{(0)}$. Let $\gamma \in \mathcal{G}$ and suppose that $u \in B_\gamma$ is unitary in the sense that $u^* u = 1_{A_{s(\gamma)}}$ and $u u^* = 1_{A_{r(\gamma)}}$. For any ideal I of $A_{s(\gamma)}$, we have*

$$u I u^* = \overline{\mathrm{span}}\{x a y^* : x, y \in B_\gamma, a \in I\}.$$

In particular, $I \mapsto u I u^*$ is the map from $\mathrm{Prim}(A_{s(\gamma)})$ to $\mathrm{Prim}(A_{r(\gamma)})$ described in [16, Lemma 2.1].

Proof. We clearly have $u I u^* \subseteq \overline{\mathrm{span}}\{x a y^* : x, y \in B_\gamma, a \in I\}$. For the reverse inclusion, fix $x_i, y_i \in B_\gamma$ and $a_i \in I$, and observe that

$$\sum_i x_i a_i y_i^* = \sum_i u u^* x_i a y_i u u^* = u \Big(\sum_i u^* x_i a y_i u \Big) u^* \in u I u^*. \qquad\square$$

We can now identify the action described in [16] with that of Lemma 3.6.

Theorem 4.5. *Let Λ be a row-finite cofinal k-graph with no sources, and suppose that $c \in \underline{Z}^2(\Lambda, \mathbb{T})$. Take ω as in Proposition 3.1, and let $\mathcal{H}_\Lambda$ denote the quotient of $\mathcal{G}_\Lambda$ by the interior $\mathcal{I}_\Lambda$ of its isotropy. Let $\mathcal{B}_\Lambda$ be the Fell bundle over $\mathcal{H}_\Lambda$ described in Proposition 4.2, and let $\mathcal{A}_\Lambda$ denote the bundle of C^*-algebras obtained by restricting $\mathcal{B}_\Lambda$ to $\mathcal{H}_\Lambda^{(0)}$. There is a homeomorphism $i : \mathrm{Prim}(C^*(\mathcal{H}_\Lambda^{(0)}; \mathcal{B}_\Lambda)) \to \Lambda^\infty \times \widehat{Z_\omega}$ that intertwines the action θ of $\mathcal{H}_\Lambda$ on $\Lambda^\infty \times \widehat{Z_\omega}$ described in Lemma 3.6 and the action of $\mathcal{H}_\Lambda$ on $\mathrm{Prim}(C^*(\mathcal{H}_\Lambda^{(0)}; \mathcal{B}_\Lambda))$ described in [16].*

Proof. Put $H := \mathrm{Per}(\Lambda)/Z_\omega$. Take $x \in \Lambda^\infty$. Then

$$A_x = C^*(\mathrm{Per}(\Lambda), \omega) \cong \overline{\mathrm{span}}\{\delta_{(x,p,x)} : p \in \mathrm{Per}(\Lambda)\},$$

and there is an isomorphism $i_x : A_x \to \mathrm{Ind}_{\widehat{H}}^{\widehat{\mathrm{Per}(\Lambda)}} C^*(H, \omega)$ such that

$$i_x(\delta_{(x,p,x)})(\chi) = \chi(p) U_{p + Z_\omega}$$

for $\chi \in \mathrm{Per}(\Lambda)\widehat{}$. In particular i_x is equivariant for the action of $\mathrm{Per}(\Lambda)\widehat{}$ on A_x given by $\chi \cdot \delta_{(x,p,x)} = \chi(p)\delta_{(x,p,x)}$ and the action lt of $\mathrm{Per}(\Lambda)\widehat{}$ on $\mathrm{Ind}_{\widehat{H}}^{\mathrm{Per}(\Lambda)\widehat{}} C^*(H, \omega)$ by translation.

Take $\gamma \in \mathcal{G}_\Lambda$ and $p \in \mathrm{Per}(\Lambda)$. A straightforward calculation in $\mathcal{B}_\Lambda$ shows that

$$\delta_\gamma * \delta_{(s(\gamma),p,s(\gamma))} * \delta_\gamma^* = r_\gamma^\sigma(p)\delta_{(r(\gamma),p,r(\gamma))}.$$

The element δ_γ is a unitary in the fibre $B_{[\gamma]}$, and so Lemma 4.4 shows that conjugation by δ_γ in $\mathcal{B}_\Lambda$ implements the homeomorphism $\alpha_{[\gamma]}$ of [16] from $\mathrm{Prim}(A_{s(\gamma)}) \subseteq \mathrm{Prim}(C^*(\mathcal{H}_\Lambda^{(0)}; \mathcal{B}_\Lambda))$ to $\mathrm{Prim}(A_{r(\gamma)})$. We have $i_{r(\gamma)} \circ \mathrm{Ad}_{\delta_\gamma} \circ i_{r(\gamma)}^{-1} = \mathrm{lt}_{r_\gamma^\sigma}$, and it follows that for $\chi \in \mathrm{Per}(\Lambda)\widehat{}$ the primitive ideal $I_{\chi\widehat{H}}$ of $\mathrm{Ind}_{\widehat{H}}^{\mathrm{Per}(\Lambda)\widehat{}} C^*(H, \omega)$ consisting of functions that vanish on $\chi\widehat{H}$ satisfies

$$\left(i_{r(\gamma)} \circ \mathrm{Ad}_{\delta_\gamma} \circ i_{r(\gamma)}^{-1}\right)(I_{\chi\widehat{H}}) = I_{r_\gamma^\sigma \cdot \chi\widehat{H}}.$$

That is, the induced map $(i_{r(\gamma)})_* \alpha_{[\gamma]} (i_{r(\gamma)})_*^{-1}$ on $\mathrm{Prim}\left(\mathrm{Ind}_{\widehat{H}}^{\mathrm{Per}(\Lambda)\widehat{}} C^*(H, \omega)\right)$ is translation by $\tilde{r}_{[\gamma]}^\sigma$. Lemma 4.1 yields a homeomorphism

$$i : \mathrm{Prim}(C^*(\mathcal{H}^{(0)}; \mathcal{B}_\Lambda)) \to \Lambda^\infty \times \widehat{Z}_\omega$$

such that $i|_{\mathrm{Prim}(A_x)} = i_x$, and this homeomorphism does the job. $\qquad\square$

Recall from [21, Proposition 3.6] that if $\mathcal{B}$ is a Fell bundle over an étale[4] groupoid $\mathcal{H}$, then restriction of compactly supported sections to $\mathcal{H}^{(0)}$ extends to a faithful conditional expectation $P : C_r^*(\mathcal{H}; \mathcal{B}) \to C^*(\mathcal{H}^{(0)}; \mathcal{B})$.

Lemma 4.6. *Suppose that $\mathcal{H}$ is an étale topologically principal groupoid and that $\mathcal{B}$ is a Fell bundle over $\mathcal{H}$. If I is a nonzero ideal of $C_r^*(\mathcal{H}; \mathcal{B})$, then $I \cap C^*(\mathcal{H}^{(0)}; \mathcal{B}) \neq \{0\}$.*

Proof. Our argument is based on that of [1, Proposition 2.4] (see also [3, Lemma 4.2] or [23, Lemma 3.5]).

Let I be a nonzero ideal of $C_r^*(\mathcal{H}; \mathcal{B})$. Let $P : C_r^*(\mathcal{H}; \mathcal{B}) \to C^*(\mathcal{H}^{(0)}; \mathcal{B})$ be the faithful conditional expectation discussed above. Choose $a \in I^+$ such that $\|P(a)\| = 1$. Choose $b \in \Gamma_c(\mathcal{H}; \mathcal{B}) \cap C_r^*(\mathcal{H}; \mathcal{B})^+$ such that $\|a - b\| < 1/4$, so that $\|P(b)\| > 3/4$. Then $b - P(b) \in \Gamma_c(\mathcal{H}; \mathcal{B})$; thus, $K := \mathrm{supp}(b - P(b))$ is compact and contained in $\mathcal{H} \setminus \mathcal{H}^{(0)}$. Let $U = \{u \in \mathcal{H}^{(0)} : \|P(b)(u)\| > 3/4\}$. By [3, Lemma 3.3], there exists an open set V such that $V \subseteq U$ and $VKV = \emptyset$.

Fix a continuous function $h : \mathbb{R} \to [0, 1]$ such that h is identically 0 on $(-\infty, 1/2]$ and identically 1 on $[3/4, \infty)$. Then

$$h(P(b)) \, P(b) \, h(P(b)) \geq \frac{1}{2} h(P(b))^2.$$

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[4]The statement in [21] says "r-discrete", but étale is meant.

Choose $g \in C_c(\mathcal{H}^{(0)}) \subseteq \mathcal{M}C_r^*(\mathcal{H}; \mathcal{B})$ such that $\|g\|_\infty = 1$ and $\mathrm{supp}(g) \subseteq V$. Let $f := gh(P(b))$. Since $\|h(P(b))(u)\| = 1$ for all $u \in U$, and since $V \subseteq U$, we have $f \neq 0$.

Since $\mathrm{supp}(f) \subseteq \mathrm{supp}(g) \subseteq V$, we have $f(b - P(b))f = 0$. Hence

$$fbf = f\big(P(b) + (b - P(b))\big)f = fP(b)f = gh(P(b))\,P(b)\,h(P(b))g$$
$$\geq \frac{1}{2}g^2(h(P(b)))^2 = \frac{1}{2}f^2.$$

Thus

$$faf \geq fbf - \frac{1}{4}f^2 = fP(b)f - \frac{1}{4}f^2 \geq \frac{1}{4}f^2.$$

Since $faf \in I$ and I is hereditary, we deduce that $\frac{1}{4}f^2 \in I \cap C^*(\mathcal{H}^{(0)}; \mathcal{B})\backslash\{0\}$. $\qquad\square$

Lemma 4.6 allows us to characterise the simplicity of $C^*(\mathcal{H}; \mathcal{B})$ when $\mathcal{H}$ is amenable and topologically principal.

Corollary 4.7. *Suppose that $\mathcal{H}$ is a topologically principal amenable groupoid and that $\mathcal{B}$ is a Fell bundle over $\mathcal{H}$. Then $C^*(\mathcal{H}; \mathcal{B})$ is simple if and only if the action of $\mathcal{H}$ on $\mathrm{Prim}(C^*(\mathcal{H}^{(0)}; \mathcal{B}))$ described in [16, Section 2] is minimal.*

Proof. We denote the action of $\mathcal{H}$ on $\mathrm{Prim}(C^*(\mathcal{H}^{(0)}; \mathcal{B}))$ by ϑ. First suppose that ϑ is not minimal. Then there exists ρ in $\mathrm{Prim}(C^*(\mathcal{H}^{(0)}; \mathcal{B}))$ such that $\mathcal{H} \cdot \rho$ is not dense in $\mathrm{Prim}(C^*(\mathcal{H}^{(0)}; \mathcal{B}))$. Hence $X := \overline{\mathcal{H} \cdot \rho}$ is a nontrivial closed invariant subspace of $\mathrm{Prim}(C^*(\mathcal{H}^{(0)}; \mathcal{B}))$. The set I_X of sections of $\mathcal{B}|_{\mathcal{H}^{(0)}}$ that vanish on X is a nontrivial $\mathcal{H}_\Lambda$-invariant ideal of the C^*-algebra $C^*(\mathcal{H}^{(0)}; \mathcal{B})$ sitting over the unit space of $\mathcal{H}$. Thus the ideal of $C^*(\mathcal{H}; \mathcal{B})$ generated by I_X is a proper nontrivial ideal [16, Theorem 3.7] and so $C^*(\mathcal{H}; \mathcal{B})$ is not simple.

Now suppose that ϑ is minimal. Let I be a nonzero ideal of $C^*(\mathcal{H}; \mathcal{B})$. Theorem 1 of [46] shows that $C^*(\mathcal{H}; \mathcal{B}) = C_r^*(\mathcal{H}; \mathcal{B})$, so Lemma 4.6 implies that $I_0 := I \cap C^*(\mathcal{H}^{(0)}; \mathcal{B})$ is nonzero. Let X denote the set of primitive ideals of $C^*(\mathcal{H}^{(0)}; \mathcal{B})$ that contain I_0. Since I is nonzero, X is nonempty, and it is closed by definition of the topology on $\mathrm{Prim}(C^*(\mathcal{H}^{(0)}; \mathcal{B}))$. Lemma 2.1 of [16] implies that X is also ϑ-invariant. Since ϑ is minimal, it follows that $X = \mathrm{Prim}(C^*(\mathcal{H}^{(0)}; \mathcal{B}))$, and so $I_0 = C^*(\mathcal{H}^{(0)}; \mathcal{B})$. Since $C^*(\mathcal{H}^{(0)}; \mathcal{B})$ contains an approximate identity for $C_r^*(\mathcal{H}; \mathcal{B})$, it follows that $I = C^*(\mathcal{H}; \mathcal{B})$ as required. $\qquad\square$

Since we established in Corollary 2.2 that $\mathcal{H}_\Lambda$ is always topologically principal and amenable, we obtain an immediate corollary.

Corollary 4.8. *Let Λ be a row-finite cofinal k-graph with no sources, and suppose that $c \in \underline{Z}^2(\Lambda, \mathbb{T})$. Let θ be the action of $\mathcal{H}_\Lambda$ on $\Lambda^\infty \times \mathrm{Per}(\Lambda)\widehat{}$ obtained from any choice of σ in Lemma 3.6. Then $C^*(\Lambda, c)$ is simple if and only if θ is minimal.*

Proof. Theorem 4.3 shows that $C^*(\Lambda, c)$ is simple if and only if $C^*(\mathcal{H}_\Lambda; \mathcal{B}_\Lambda)$ is simple. Corollary 2.2 shows that $\mathcal{H}_\Lambda$ is amenable and topologically principal, and so Corollary 4.7 implies that $C^*(\mathcal{H}_\Lambda; \mathcal{B}_\Lambda)$ is simple if and only if the action of $\mathcal{H}_\Lambda$ on $\mathrm{Prim}(C^*(\mathcal{H}_\Lambda^{(0)}; \mathcal{B}_\Lambda))$ is minimal. Theorem 4.5 shows that this action is minimal if and only if θ is minimal, giving the result. $\qquad\square$

We can now prove our main theorem.

Proof of Theorem 3.4. By Corollary 4.8, it is enough to show that the set

$$\{(r(\gamma), r_\gamma^\sigma|_{Z_\omega}) : \gamma \in (\mathcal{G}_\Lambda)_x\} \tag{4.1}$$

is dense in $\Lambda^\infty \times \widehat{Z}_\omega$ for every $x \in \Lambda^\infty$ if and only if the action θ of $\mathcal{H}_\Lambda$ on $\Lambda^\infty \times \mathrm{Per}(\Lambda)\widehat{\ }$ is minimal. Clearly (4.1) is dense for every x if and only if $\{(r(\gamma), r_\gamma^\sigma|_{Z_\omega} \cdot \chi) : \gamma \in (\mathcal{G}_\Lambda)_x\}$ is dense in $\Lambda^\infty \times Z_\omega$ for every x and every $\chi \in \widehat{Z}_\omega$, which is precisely minimality of θ. $\qquad\square$

5. Twists induced by torus-valued 1-cocycles, and crossed-products by quasifree actions

In this section we describe a class of examples of twisted $(k + l)$-graph C^*-algebras arising from $\mathbb{T}^l$-valued 1-cocycles on aperiodic k-graphs. We describe the simplicity criterion obtained from our main theorem for these examples. We then show that these twisted C^*-algebras can also be interpreted as twisted crossed-products of k-graph algebras by quasifree actions, giving a characterisation of simplicity for the latter. See also [5, Theorem 2.1] for the case where $l = 1$.

Recall that we write T_l for $\mathbb{N}^l$ when regarded as an l-graph with degree map the identity functor.

Given a cocycle $\omega \in Z^2(\mathbb{Z}^l, \mathbb{T})$ and an action α of $\mathbb{Z}^l$ on a C^*-algebra A, we write $A \times_{\alpha, \omega} \mathbb{Z}^l$ for the twisted crossed product, which is the universal C^*-algebra generated by unitary multipliers $\{u_n \in \mathcal{M}(A \times_{\alpha, \omega} \mathbb{Z}^l) : n \in \mathbb{Z}^l\}$ and a homomorphism $\pi : A \to A \times_{\alpha, \omega} \mathbb{Z}^l$ such that $u_m \pi(a) u_m^* = \pi(\alpha_m(a))$ for all m, a and such that $u_m u_n = \omega(m, n) u_{m+n}$ for all m, n. For further details on twisted crossed products, see [30].

Let Λ be a row-finite k-graph with no sources. Consider a 1-cocycle $\phi \in \underline{Z}^1(\Lambda, \mathbb{T}^l)$. Define a 2-cocycle $c_\phi \in \underline{Z}^2(\Lambda \times T_l, \mathbb{T})$ by

$$c_\phi((\lambda, m), (\mu, n)) = \phi(\mu)^m \quad \text{for composable } \lambda, \mu \in \Lambda \text{ and } m, n \in \mathbb{N}^l = T_l. \tag{5.1}$$

There is a continuous cocycle $\tilde{\phi} \in Z^1(\mathcal{G}_\Lambda, \mathbb{T}^l)$ such that

$$\tilde{\phi}(\mu x, d(\mu) - d(\nu), \nu x) = \phi(\mu)\overline{\phi(\nu)} \quad \text{for all } x \in \Lambda^\infty \text{ and } \mu, \nu \in \Lambda r(x).$$

Let ω be a bicharacter of $\mathbb{Z}^l$. There is a cocycle $c_{\phi,\omega} \in \underline{Z}^2(\Lambda \times T_l, \mathbb{T})$ such that

$$c_{\phi,\omega}((\lambda, m), (\mu, n)) = \phi(\mu)^m \omega(m, n). \tag{5.2}$$

The next theorem is the main result of this section. It characterises simplicity of $C^*(\Lambda \times T_l, c_{\phi,\omega})$ in terms of $\tilde{\phi}$ and the centre Z_ω of ω described in (3.4), under the simplifying assumption that Λ is aperiodic.

Theorem 5.1. *Let Λ be a row-finite k-graph with no sources. Take $\phi \in \underline{Z}^1(\Lambda, \mathbb{T}^l)$, and let ω be a bicharacter of $\mathbb{Z}^l$. Let $\tilde{\phi} \in Z^1(\mathcal{G}_\Lambda, \mathbb{T})$ and $c_{\phi,\omega} \in \underline{Z}^2(\Lambda \times T_l, \mathbb{T})$ be as above. Then*

(1) *there is an action β of $\mathbb{Z}^l$ on $C^*(\Lambda)$ such that $\beta_m(s_\lambda) = \phi(\lambda)^m s_\lambda$ for all $\lambda \in \Lambda$ and $m \in \mathbb{Z}^l$;*

(2) *there is an isomorphism $\rho : C^*(\Lambda \times T_l, c_{\phi,\omega}) \cong C^*(\Lambda) \times_{\beta,\omega} \mathbb{Z}^l$ such that*

$$\rho(s_{(\lambda,m)}) = \pi(s_\lambda)u_m \text{ for all } \lambda \in \Lambda \text{ and } m \in T_l; \text{ and}$$

(3) *if Λ is aperiodic, then $C^*(\Lambda \times T_l, c_{\phi,\omega})$ is simple if and only if $\{(r(\gamma), \tilde{\phi}(\gamma)|_{Z_\omega}) : \gamma \in (\mathcal{G}_\Lambda)_x\}$ is dense in $\Lambda^\infty \times \widehat{Z}_\omega$ for all $x \in \Lambda^\infty$.*

Remark 5.2. The observant reader may be surprised to note that there is no cofinality hypothesis on the preceding theorem. But a moment's reflection shows that it is still there, just hidden: each of the conditions that $C^*(\Lambda \times T_l, c_{\phi,\omega})$ is simple, and that each $\{(r(\gamma), \tilde{\phi}(\gamma)|_{Z_\omega}) : \gamma \in (\mathcal{G}_\Lambda)_x\}$ is dense in $\Lambda^\infty \times \widehat{Z}_\omega$ implies that Λ is cofinal.

We prove (1) and (2) here and defer the proof of (3) to the end of the section.

Proof of Theorem 5.1(1) and (2). For part (1) observe that for each $m \in \mathbb{Z}^l$ the set $\{\phi(\lambda)^m s_\lambda : \lambda \in \Lambda\}$ is a Cuntz–Krieger Λ-family, so induces a homomorphism $\beta_m : C^*(\Lambda) \to C^*(\Lambda)$ such that $\beta_m(s_\lambda) = \phi(\lambda)^m s_\lambda$. Clearly $\beta_0 = \text{id}$ and $\beta_m \circ \beta_n = \beta_{m+n}$, so β is an action.

For part (2) we first check that $\{\pi(s_\lambda)u_m : (\lambda, m) \in \Lambda \times T_l\}$ is a Cuntz–Krieger $(\Lambda \times T_l, c_{\phi,\omega})$-family. For $\lambda \in \Lambda$ and $m \in T_l$ let $t_{(\mu,m)} = \pi(s_\lambda)u_m$. It is easy to check that (CK1), (CK3) and (CK4) hold because each β_m fixes each $s_\lambda s_\lambda^*$. To check (CK2) we compute

$$\begin{aligned}
t_{(\lambda,m)}t_{(\mu,n)} &= \pi(s_\lambda)u_m\pi(s_\mu)u_n = \pi(s_\lambda)u_m\pi(s_\mu)u_m^* u_m u_n \\
&= \pi(s_\lambda)\pi(\beta_m(s_\mu))\omega(m, n)u_{m+n} = \phi(\mu)^m\omega(m, n)\pi(s_{\lambda\mu})u_{m+n} \\
&= \phi(\mu)^m\omega(m, n)t_{(\lambda\mu,m+n)} = c_{\phi,\omega}((\lambda, m)(\mu, n))t_{(\lambda,m)(\mu,n)}.
\end{aligned}$$

Now the universal property of $C^*(\Lambda \times T_l, c_{\phi,\omega})$ gives a homomorphism ρ satisfying the desired formula. The gauge-invariant uniqueness theorem [26, Corollary 7.7] shows that ρ is injective. The map ρ is surjective because its image contains all the generators of the twisted crossed product. $\square$

In order to prove (3) we first do some preparatory work to identify the action θ of Lemma 3.6 in terms of the cocycle $\tilde{\phi} \in Z^2(\mathcal{G}_\Lambda, \mathbb{T})$. Before that, though, a comment on the hypotheses of Theorem 5.1 is in order.

Remark 5.3. Note that the aperiodicity hypothesis in Theorem 5.1 is needed in our proof and simplifies the statement, but is not a necessary condition for $C^*(\Lambda \times T_l, c_{\phi,\omega})$ to be simple. To see this, consider $\Lambda = T_1$, which is $\mathbb{N}$ regarded as a 1-graph. Put $l = 1$, define $\phi : \Lambda \to \mathbb{T}$ by $\phi(1) = e^{2\pi i \theta}$ and take ω to be trivial. Then Λ is cofinal, but certainly not aperiodic. We have $\Lambda \times T_1 \cong T_2$, and $c_{\phi,\omega} \in \underline{Z}^2(T_2, \mathbb{T})$ is given by $c_{\phi,\omega}(m,n) := e^{2\pi i \theta m_2 n_1}$, so $C^*(\Lambda \times T_1, c_{\phi,\omega})$ is the rotation algebra A_θ, which is simple whenever θ is irrational.

Let $\Gamma := \Lambda \times T_l$. The proof of Theorem 5.1 (3) requires quite a bit of preliminary work. To begin preparations, observe that the projection map $\pi : \Gamma \to \Lambda$ given by $(\lambda, m) \mapsto \lambda$ induces a homeomorphism

$$\pi_\infty : \Gamma^\infty \to \Lambda^\infty \quad \text{such that } \pi_\infty(x)(m,n) = \pi(x((m,0),(n,0))).$$

There is an isomorphism $\mathcal{G}_\Gamma \cong \mathcal{G}_\Lambda \times \mathbb{Z}^l$ given by

$$\begin{aligned}
\big((\alpha,m)x, &(d(\alpha),m) - (d(\beta),n), (\beta,n)x\big) \\
&\mapsto \big((\alpha\pi_\infty(x), d(\alpha) - d(\beta), \beta\pi_\infty(x)), m - n\big).
\end{aligned} \tag{5.3}$$

Suppose that Λ is cofinal and aperiodic, and fix a subset $\mathcal{P}$ of $\Lambda \,_s*_s \Lambda$ such that $(\mu, s(\mu)) \in \mathcal{P}$ for all $\mu \in \Lambda$ and such that $\{Z(\mu,\nu) : (\mu,\nu) \in \mathcal{P}\}$ is a partition of $\mathcal{G}_\Lambda$. For $g \in \mathcal{G}_\Lambda$, let $(\mu_g, \nu_g) \in \mathcal{P}$ be the unique pair such that $g \in Z(\mu_g, \nu_g)$. Observe that $\operatorname{Per}(\Gamma) = \{0\} \times \mathbb{Z}^l \subseteq \mathbb{Z}^{k+l}$. Recall that $\mathcal{H}_\Gamma$ is the quotient $\mathcal{G}_\Gamma/\mathcal{I}_\Gamma$ of the $(k+l)$-graph groupoid by the interior of its isotropy.

Proposition 5.4. *Let Λ be a row-finite cofinal aperiodic k-graph with no sources, and let $\Gamma := \Lambda \times T_l$. Then $\operatorname{Per}(\Gamma) = \{0\} \times \mathbb{Z}^l \subseteq \mathbb{Z}^{k+l}$, the identification of $\mathcal{G}_\Gamma$ with $\mathcal{G}_\Lambda \times \mathbb{Z}^l$ described above carries $\mathcal{I}_\Gamma$ to $\mathcal{G}_\Lambda^{(0)} \times \mathbb{Z}^l$, and $(g,m) \cdot \mathcal{I}_\Gamma \mapsto g$ gives an isomorphism $\mathcal{H}_\Gamma \cong \mathcal{G}_\Lambda$. Let $\phi : \Lambda \to \mathbb{T}^l$ be a 1-cocycle, and let ω be a bicharacter of $\mathbb{Z}^l$. Let $c_{\phi,\omega} \in \underline{Z}^2(\Gamma, \mathbb{T})$ be the 2-cocycle of (5.2). There exists a cocycle $\sigma \in Z^2(\mathcal{G}_\Gamma, \mathbb{T})$ such that $\sigma|_{\mathcal{I}_\Gamma} = 1_{\Lambda^\infty} \times \omega$, $C^*(\mathcal{G}_\Gamma, \sigma) \cong C^*(\Gamma, c_{\phi,\omega})$ and the action θ of $\mathcal{G}_\Lambda$ on $\Lambda^\infty \times \widehat{Z}_\omega$ of Lemma 3.6 satisfies*

$$\theta_g(s(g), \chi) = (r(g), \overline{\tilde{\phi}(g)|_{Z_\omega}} \chi). \tag{5.4}$$

Before proving Proposition 5.4, we establish two technical lemmas that will help in the proof. The first of these shows that the passage from a k-graph cocycle to a groupoid cocycle described in (2.3) has no effect on cohomology when the k-graph in question is $\mathbb{N}^l$ and its groupoid is $\mathbb{Z}^l$.

Recall that for $m \in \mathbb{Z}^l$, we write $m^+ := m \vee 0$ and $m^- := (-m) \vee 0$; so $m = m^+ - m^-$ and $m^+ \wedge m^- = 0$.

Lemma 5.5. *Let ω be a bicharacter of $\mathbb{Z}^l$. Then $\omega|_{T_l \times T_l}$ belongs to $\underline{Z}^2(T_l, \mathbb{T})$. The l-graph T_l has a unique infinite path x, and there is an isomorphism $\mathcal{G}_{T_l} \cong \mathbb{Z}^l$ given by $(x, m, x) \mapsto m$. Let $\mathcal{P} = \{(m^+, m^-) : m \in \mathbb{Z}^l\}$. Then $(\lambda, s(\lambda)) \in \mathcal{P}$ for all $\lambda \in T_l$, and $\mathcal{G}_{T_l} = \bigsqcup_{(\mu, \nu) \in \mathcal{P}} Z(\mu, \nu)$. Let $\sigma_\omega \in Z^2(\mathcal{G}_{T_l}, \mathbb{T})$ be the 2-cocycle obtained from ω and $\mathcal{P}$ as in (2.3). Then σ_ω is cohomologous to ω when regarded as a cocycle on $\mathbb{Z}^l$.*

Proof. Every bicharacter of $\mathbb{Z}^l$ is a 2-cocycle, so ω restricts to a cocycle on T_l. It is clear that $\mathcal{G}_{T_l} \cong \mathbb{Z}^l$ as claimed. We identify $\mathcal{G}_{T_l}$ with $\mathbb{Z}^l$ for the rest of the proof.

For $m \in \mathcal{G}_{T_l}$, the pair $(\mu_m, \nu_m) = (m^+, m^-)$ is the unique element of $\mathcal{P}$ such that $m \in Z(\mu_m, \nu_m)$. So for $m, n \in \mathcal{G}_{T_l}$, equation (2.3) gives

$$\sigma_\omega(m, n) = \omega(m^+, \alpha)\overline{\omega(m^-, \alpha)}\omega(n^+, \beta)\overline{\omega(n^-, \beta)}\overline{\omega((m+n)^+, \gamma)}\omega((m+n)^-, \gamma)$$

for any choice of $\alpha, \beta, \gamma \in T_l$ such that $m^+ + \alpha = (m+n)^+ + \gamma$, $n^- + \beta = (m+n)^- + \gamma$ and $m^- + \alpha = n^+ + \beta$.

We show that $\sigma_\omega(e_i, e_j) = \omega(e_i, e_j)$ for all $i, j \leq l$. For this observe that for $m = e_i$ and $n = e_j$, the elements $\alpha = e_j$, $\beta = 0$ and $\gamma = 0$ satisfy the above conditions, so

$$\sigma_\omega(e_i, e_j) = \omega(e_i, e_j)\overline{\omega(0, e_j)}\omega(e_j, 0)\overline{\omega(0, 0)}\omega(e_i + e_j, 0)\omega(0, 0) = \omega(e_i, e_j)$$

as claimed.

Hence $(\sigma_\omega \sigma_\omega^*)(e_i, e_j) = (\omega\omega^*)(e_i, e_j)$ for all i, j. These are bicharacters by [29, Proposition 3.2] and so we have $\sigma_\omega \sigma_\omega^* = \omega\omega^*$. Thus [29, Proposition 3.2] implies that σ_ω is cohomologous to ω as claimed. $\qquad\square$

Our second technical result shows how to obtain a partition $\mathcal{Q}$ satisfying (2.1) for the $(k+l)$-graph $\Lambda \times T_l$ from a partition $\mathcal{P}$ satisfying (2.1) for Λ.

Lemma 5.6. *Let Λ be a row-finite k-graph with no sources. Suppose that $\mathcal{P} \subseteq \Lambda \,_s*_s \Lambda$ satisfies $(\mu, s(\mu)) \in \mathcal{P}$ for all μ, and $\mathcal{G}_\Lambda = \bigsqcup_{(\mu, \nu) \in \mathcal{P}} Z(\mu, \nu)$. Let $\mathcal{Q} := \{((\mu, m^+), (\nu, m^-)) : (\mu, \nu) \in \mathcal{P}, m \in \mathbb{Z}^l\}$. Then $(\alpha, s(\alpha)) \in \mathcal{Q}$ for all $\alpha \in \Gamma$, and $\mathcal{G}_\Gamma = \bigsqcup_{(\alpha, \beta) \in \mathcal{Q}} Z(\alpha, \beta)$.*

Proof. Consider an element $\alpha = (\mu, m) \in \Gamma$. We have $m^+ = m$ and $m^- = 0$, and $(\alpha, s(\alpha)) = ((\mu, m), (s(\mu), 0))$. Since $(\mu, s(\mu)) \in \mathcal{P}$, we have

$$((\mu, m), (s(\mu), 0)) = ((\mu, m^+), (s(\mu), m^-)) \in \mathcal{Q}.$$

Let $\rho : \mathcal{G}_\Gamma \to \mathcal{G}_\Lambda \times \mathbb{Z}^l$ denote the map (5.3). Then

$$\mathcal{G}_\Gamma = \rho^{-1}(\mathcal{G}_\Lambda \times \mathbb{Z}^l) = \bigsqcup_{(\mu,\nu)\in\mathcal{P}} \rho^{-1}(Z(\mu,\nu) \times \mathbb{Z}^l)$$

$$= \bigsqcup_{(\mu,\nu)\in\mathcal{P}} \bigsqcup_{m\in\mathbb{Z}^l} \rho^{-1}(Z(\mu,\nu) \times \{m\})$$

$$= \bigsqcup_{(\mu,\nu)\in\mathcal{P}} \bigsqcup_{m\in\mathbb{Z}^l} Z((\mu,m^+),(\nu,m^-)) = \bigsqcup_{(\alpha,\beta)\in\mathcal{Q}} Z(\alpha,\beta). \quad \square$$

Proof of Proposition 5.4. Since the homeomorphism $\pi_\infty : \Gamma^\infty \to \Lambda^\infty$ intertwines the shift maps, we have $T^{(p,m)}(x) = T^{(q,n)}(x)$ if and only if $T^p(\pi_\infty(x)) = T^q(\pi_\infty(x))$. Hence $\mathrm{Per}(\Gamma) = \mathrm{Per}(\Lambda) \times \mathbb{Z}^l = \{0\} \times \mathbb{Z}^l$ because Λ is aperiodic. The next two assertions are straightforward to check using the definition of the isomorphism $\mathcal{G}_\Gamma \cong \mathcal{G}_\Lambda \times \mathbb{Z}^l$.

We identify $\mathcal{G}_\Gamma$ with $\mathcal{G}_\Lambda \times \mathbb{Z}^l$ for the remainder of this proof. Choose $\mathcal{P} \subseteq \Lambda_s *_s \Lambda$ such that $(\lambda, s(\lambda)) \in \mathcal{P}$ for all λ and $\mathcal{G}_\Lambda = \bigsqcup_{(\mu,\nu)\in\mathcal{P}} Z(\mu,\nu)$. For $g \in \mathcal{G}_\Lambda$, write (μ_g, ν_g) for the element of $\mathcal{P}$ with $g \in Z(\mu_g, \nu_g)$. Let

$$\mathcal{Q} = \{(\mu,m^+),(\nu,m^-) : (\mu,\nu) \in \mathcal{P}, m \in \mathbb{Z}^l\}$$

as in Lemma 5.6. For $(g,m) \in \mathcal{G}_\Gamma$, let $\tilde{\mu}_{(g,m)} := (\mu_g, m^+)$, and $\tilde{\nu}_{(g,m)}) = (\nu_g, m^-)$. Then $(\tilde{\mu}_{(g,m)}, \tilde{\nu}_{(g,m)})$ is the unique element of $\mathcal{Q}$ such that $(g,m) \in Z(\tilde{\mu}_{(g,m)}, \tilde{\nu}_{(g,m)})$. For $((x,0,x),m) \in \mathcal{I}_\Gamma$, we have

$$\tilde{\mu}_{((x,0,x),m)} = (r(x),m^+) \quad \text{and} \quad \tilde{\nu}_{((x,0,x),m)} = (r(x),m^-).$$

Let $\tilde{\omega}$ denote the 2-cocycle on Γ given by $((\mu,m),(\nu,n)) \mapsto \omega(m,n)$, and recall that $c_\phi \in \underline{Z}^2(\Lambda \times T_l, \mathbb{T})$ is given by (5.1). Let σ_{c_ϕ} be the 2-cocycle on $\mathcal{G}_\Gamma$ obtained from (2.3) applied to $c_\phi \in \underline{Z}^2(\Gamma, \mathbb{T})$ and $\mathcal{Q}$, and let $\sigma_{\tilde{\omega}}$ be the 2-cocycle on $\mathcal{G}_\Gamma$ obtained in the same way from the cocycle $((\mu,m),(\nu,n)) \mapsto \omega(m,n)$ on Γ. Using that $c_{\phi,\omega} = c_\phi \tilde{\omega}$ and the definitions (2.3) of σ_{c_ϕ}, $\sigma_{\tilde{\omega}}$ and $\sigma_{c_{\phi,\omega}}$, it is easy to see that $\sigma_{c_{\phi,\omega}} = \sigma_{c_\phi}\sigma_{\tilde{\omega}}$. Let $\sigma_\omega \in Z^2(\mathcal{G}_{T_l}, \mathbb{T})$ be the cocycle obtained from ω as in Lemma 5.5. The formulas for $\sigma_{\tilde{\omega}}$ and σ_ω show that the identification $\mathcal{G}_\Gamma \cong \mathcal{G}_\Lambda \times \mathbb{Z}^l \cong \mathcal{G}_\Lambda \times \mathcal{G}_{T_l}$ carries $\sigma_{\tilde{\omega}}$ to $1_{Z^2(\mathcal{G}_\Lambda,\mathbb{T})} \times \sigma_\omega$. Thus Lemma 5.5 shows that $\sigma_{c_{\phi,\omega}}$ is cohomologous to the cocycle $\sigma \in Z^2(\mathcal{G}_\Gamma, \mathbb{T})$ given by

$$\sigma((g,m),(h,n)) = \sigma_{c_\phi}((g,m),(h,n))\omega(m,n).$$

Corollary 7.9 of [26] shows that $C^*(\Gamma, c_{\phi,\omega}) \cong C^*(\mathcal{G}_\Gamma, \sigma_{c_{\phi,\omega}})$. Proposition II.1.2 of [38] says that cohomologous groupoid cocycles determine isomorphic twisted

groupoid C^*-algebras, and so we have $C^*(\Gamma, c_{\phi,\omega}) \cong C^*(\mathcal{G}_\Gamma, \sigma)$. We have $\sigma|_{\mathcal{I}_\Gamma} = 1_{\Lambda^\infty} \times \omega$ by construction, so it remains to calculate the action $\tilde{r}^\sigma$ of $\mathcal{G}_\Lambda$ on $\widehat{Z}_\omega$ described in Lemma 3.6.

For this, we first claim that σ_{c_ϕ} satisfies

$$\sigma_{c_\phi}((g, m), (h, n)) = \big(\overline{\phi(\mu_g)}\phi(\mu_{gh})\big)^m \big(\overline{\phi(\nu_h)}\phi(\nu_{gh})\big)^n. \tag{5.5}$$

To see this, choose $\alpha, \beta, \gamma \in \Lambda$ and $y \in \Lambda^\infty$ satisfying (2.3). Then there exist m_α, m_β and $m_\gamma \in \mathbb{N}^l$ such that $\tilde{\alpha} = (\alpha, m_\alpha)$, $\tilde{\beta} = (\beta, m_\beta)$ and $\tilde{\gamma} = (\gamma, m_\gamma)$ satisfy equations (2.3) with respect to the $\tilde{\mu}$'s and $\tilde{\nu}$'s. So

$$
\begin{aligned}
\sigma_{c_\phi}((g, m), (h, n)) &= c_\phi(\tilde{\mu}_g, \tilde{\alpha})\overline{c_\phi}(\tilde{\nu}_g, \tilde{\alpha})c_\phi(\tilde{\mu}_h, \tilde{\beta})\overline{c_\phi}(\tilde{\nu}_h, \tilde{\beta})\overline{c_\phi}(\tilde{\mu}_{gh}, \tilde{\gamma})c_\phi(\tilde{\nu}_{gh}, \tilde{\gamma}) \\
&= \phi(\alpha)^{m^+}\overline{\phi(\alpha)}^{m^-}\phi(\beta)^{n^+}\overline{\phi(\beta)}^{n^-}\overline{\phi(\gamma)}^{(m+n)^+}\phi(\gamma)^{(m+n)^-} \\
&= \big(\phi(\alpha)\overline{\phi(\gamma)}\big)^m\big(\phi(\beta)\overline{\phi(\gamma)}\big)^n. \tag{5.6}
\end{aligned}
$$

We have $\phi(\mu_g)\phi(\alpha) = \phi(\mu_g\alpha) = \phi(\mu_{gh}\gamma) = \phi(\mu_{gh})\phi(\gamma)$, and rearranging gives $\phi(\alpha)\overline{\phi(\gamma)} = \overline{\phi(\mu_g)}\phi(\mu_{gh})$, and similarly, $\phi(\beta)\overline{\phi(\gamma)} = \overline{\phi(\nu_h)}\phi(\nu_{gh})$. Substituting this into (5.6), we obtain (5.5).

Now, fix $p \in Z_\omega$ and $(g, n) = ((x, m, y), n) \in \mathcal{G}_\Gamma$. Using (5.5) at the second equality, we calculate:

$$
\begin{aligned}
r^\sigma_{(g,n)}(p) &= \sigma\big((g, n), ((y, 0, y), p)\big)\sigma\big((g, n + p), (g^{-1}, -n)\big)\overline{\sigma\big((g, n), (g^{-1}, -n)\big)} \\
&= \Big[\big(\overline{\phi(\mu_g)}\phi(\mu_g)\big)^n\big(\overline{\phi(\nu_{s(g)})}\phi(\nu_g)\big)^p\Big] \\
&\qquad \cdot \Big[\big(\overline{\phi(\mu_g)}\phi(\mu_{gg^{-1}})\big)^{n+p}\big(\overline{\phi(\nu_{g^{-1}})}\phi(\nu_{gg^{-1}})\big)^{-n}\Big] \\
&\qquad\quad \cdot \Big[\overline{\big(\overline{\phi(\mu_g)}\phi(\mu_{gg^{-1}})\big)^n\big(\overline{\phi(\nu_{g^{-1}})}\phi(\nu_{gg^{-1}})\big)^{-n}}\Big] \\
&\qquad\qquad \cdot \omega(n, p)\omega(n + p, -n)\overline{\omega(n, -n)} \\
&= \phi(\nu_g)^p\overline{\phi(\mu_g)}^{n+p}\phi(\nu_{g^{-1}})^n\phi(\mu_g)^n\phi(\nu_{g^{-1}})^{-n}\omega(n, p)\overline{\omega(p, n)} \\
&= \big(\overline{\phi(\mu_g)}\phi(\nu_g)\big)^p(\omega\omega^*)(n, p).
\end{aligned}
$$

Since $p \in Z_\omega$, we have $(\omega\omega^*)(n, p) = 1$, and so $r^\sigma_{(g,n)}(p) = \overline{\tilde{\phi}(g)}^p$. Since $(g, n) \mapsto g$ induces an isomorphism $\mathcal{H}_\Gamma \cong \mathcal{G}_\Lambda$, we deduce that $\tilde{r}^\sigma_g(p) = \overline{\tilde{\phi}(g)}^p$ for $g \in \mathcal{G}_\Lambda$ and $p \in Z_\omega$. So (5.4) follows from the definition of θ. $\qquad\square$

Proof of part (3) of Theorem 5.1. As observed in Remark 5.2, both conditions appearing in statement (3) imply that Λ is cofinal, so we may assume that this is the case. The cocycle σ of Proposition 5.4 satisfies the hypotheses of Theorem 3.4. So Theorem 3.4 combined with Proposition 5.4 says that $C^*(\Gamma, c_{\phi,\omega})$ is simple if and only if the action θ of $\mathcal{G}_\Lambda$ on $\Lambda^\infty \times Z_\omega$ described in (5.4) is minimal.

The condition described in the statement of Theorem 5.1(3) is precisely the statement that $\mathcal{G}_\Lambda \cdot (x, 1)$, the orbit of $(x, 1)$ under θ, is dense for all x. Since $\mathcal{G}_\Lambda \cdot (x, z) = \{(y, zw) : (y, w) \in \mathcal{G}_\Lambda \cdot (x, 1)\}$, the result follows. $\square$

As a special case of Theorem 5.1, we obtain the following.

Corollary 5.7. *Let E be a strongly connected graph which is not a simple cycle. Let ϕ be a function from E^1 to $\mathbb{T}$. There is an action $\beta : \mathbb{Z} \to \operatorname{Aut}(C^*(E))$ such that $\beta_n(s_e) = \phi(e)^n s_e$ for all $e \in E^1$ and $n \in \mathbb{Z}$. If there is a cycle $\mu = \mu_1 \cdots \mu_n$ such that $\prod_i \phi(\mu_i) = e^{2\pi i \theta}$ for some $\theta \notin \mathbb{Q}$, then $C^*(E) \times_\beta \mathbb{Z}$ is simple.*

Proof. We will apply Theorem 5.1 to the 1-graph E^* and the extension of ϕ to a 1-cocycle $\phi : E^* \to \mathbb{T}$. Fix $x \in E^\infty$. By Theorem 5.1, it suffices to show that

$$\overline{\{(r(\gamma), \tilde{\phi}(\gamma)) : \gamma \in \mathcal{G}_E, s(\gamma) = x\}} = E^\infty \times \mathbb{T}.$$

Choose a basic open set $Z(\eta)$ of E^∞ and $z \in \mathbb{T}$. It suffices to show that there are elements γ_n with $s(\gamma_n) = x$ and $r(\gamma_n) \in Z(\eta)$ such that $\tilde{\phi}(\gamma_n) \to z$.

Since E is strongly connected, there exist $\alpha \in s(\eta) E^* r(\mu)$ and $\beta \in r(\mu) E^* r(x)$. The elements $\gamma_n := (\eta \alpha \mu^n \beta x, d(\eta \alpha \mu^n \beta), x) \in \mathcal{G}_E$ satisfy $s(\gamma_n) = x$, $r(\gamma_n) \in Z(\eta)$, and $\tilde{\phi}(\gamma_n) := e^{2\pi i n \theta} \tilde{\phi}(\gamma_0)$. Since θ is irrational, these values are dense in $\mathbb{T}$. $\square$

We conclude the section by giving a version of Theorem 5.1 whose statement does not require groupoid technology. Let Λ be a row-finite k-graph with no sources, and take $\phi \in \underline{Z}^1(\Lambda, \mathbb{T}^l)$ and ω a bicharacter of $\mathbb{Z}^l$. Regard each $\phi(\lambda)$ as a character of $\mathbb{Z}^l$ so that $\phi(\lambda)|_{Z_\omega}$ is a character of Z_ω. Then Λ acts on $\Lambda^\infty \times \widehat{Z}_\omega$ by partial homeomorphisms $\{\vartheta_\lambda : \lambda \in \Lambda\}$ where each $\operatorname{dom}(\vartheta_\lambda) = Z(s(\lambda)) \times \widehat{Z}_\omega$, and

$$\vartheta_\lambda(x, \chi) = (\lambda x, \phi(\lambda)|_{Z_\omega} \chi) \text{ for } (x, \chi) \in Z(s(\lambda)) \times \widehat{Z}_\omega.$$

Define a relation $\sim$ on $\Lambda^\infty \times \widehat{Z}_\omega$ by

$(x, \chi) \sim (x', \chi')$ if and only if there exist $(y, \rho) \in \Lambda^\infty \times \widehat{Z}_\omega$ and $\lambda, \mu \in \Lambda$, such that $s(\lambda) = s(\mu) = r(y)$, $\vartheta_\lambda(y, \rho) = (x, \chi)$ and $\vartheta_\mu(y, \rho) = (x', \chi')$.

Then $\sim$ is an equivalence relation; indeed, it is the equivalence relation induced by ϑ. We write $[x, \chi]$ for the equivalence class of (x, χ) under $\sim$.

Corollary 5.8. *Let Λ be an aperiodic row-finite k-graph with no sources and take $\phi \in \underline{Z}^1(\Lambda, \mathbb{T}^l)$. Let $c_{\phi, \omega} \in \underline{Z}^2(\Lambda \times T_l, \mathbb{T})$ be as defined in (5.2). Then $C^*(\Lambda \times T_l, \phi_{c, \omega})$ is simple if and only if $[x, 1]$ is dense in $\Lambda^\infty \times \widehat{Z}_\omega$ for all $x \in \Lambda^\infty$.*

Proof. It is routine to check that $[x, 1]$ is equal to the set

$$\{(r(\lambda), \tilde{\phi}(\gamma)|_{Z_\omega} : \gamma \in (\mathcal{G}_\Lambda)_x\}$$

of Theorem 5.1 part (3). $\square$

6. Examples

First we discuss an example of a nondegenerate 2-cocycle on $\mathbb{Z}^2$ pulled back over the degree map to a cocycle on a cofinal 2-graph for which the twisted C^*-algebra is not simple. This shows that the hypothesis of [45, Theorem 7.1] that $c|_{\text{Per}(\Lambda)}$ is nondegenerate cannot be relaxed to the weaker assumption that c is nondegenerate as a cocycle on $\mathbb{Z}^k$.

Example 6.1. Let $\theta \in \mathbb{R} \setminus \mathbb{Q}$, and define a 1-cocycle ϕ on Ω_1 by $\phi(m, n) = e^{2\pi i(n-m)\theta}$. As in the preceding section, construct $c_\phi \in \underline{Z}^2(\Omega_1 \times T_1, \mathbb{T})$ by $c_\phi((\alpha, m), (\beta, n)) = \phi(\beta)^m$. Then $c_\phi = \varpi \circ d$ where $d : \Omega_1 \times T_1 \to \mathbb{N}^2$ is the degree map, and $\varpi \in Z^2(\mathbb{Z}^2, \mathbb{T})$ is the cocycle $\varpi(m, n) = e^{2\pi i\theta m_2 n_1}$. Since θ is irrational, the antisymmetric bicharacter associated to ϖ is nondegenerate. Note that $\text{Per}(\Lambda) \cong \mathbb{Z}$ and so the restriction of ϖ to $\text{Per}(\Lambda)$ is degenerate.

We have $\mathcal{G}_{\Omega_1} \cong \mathbb{N} \times \mathbb{N}$ as an equivalence relation. So $\mathcal{G}_{\Omega_1}^{(0)} \times \mathbb{T} \cong \mathbb{N} \times \mathbb{T}$, and under this identification, the action θ described in Proposition 5.4 boils down to

$$\theta_{(m,n)}(n, z) = (m, e^{2\pi i\theta(n-m)}z).$$

This is not minimal because each orbit intersects each $\{n\} \times \mathbb{T}$ in a singleton. So the resulting twisted C^*-algebra is not simple.

We present a second example showing that the same pulled-back cocycle as used in Example 6.1 can yield a simple C^*-algebra if connectivity in the underlying 1-graph Λ is slightly more complicated than in Ω_1.

Example 6.2. Let Λ be a strongly connected 1-graph with at least one edge and suppose that Λ is not a simple cycle. Then Λ is cofinal and aperiodic (see, for example, [23]) and contains a nontrivial cycle λ. A simple example is $\Lambda = B_2$ the bouquet of 2 loops, so that $C^*(\Lambda) = \mathcal{O}_2$.

Take $\theta \in \mathbb{R} \setminus \mathbb{Q}$, and define a 1-cocycle ϕ on Λ by $\phi(\lambda) = e^{2\pi i d(\lambda)\theta}$. As in the preceding section, construct $c_\phi \in \underline{Z}^2(\Lambda \times T_1, \mathbb{T})$ by $c_\phi((\alpha, m), (\beta, n)) = \phi(\beta)^m$. Since $\theta \in \mathbb{R} \setminus \mathbb{Q}$, the paths $\mu = \lambda$ and $\nu = r(\lambda)$ satisfy $r(\mu) = r(\nu)$ and $s(\mu) = s(\nu)$, and $\phi(\mu)\overline{\phi(\nu)} = e^{2\pi i d(\lambda)\theta}$ where $d(\lambda)\theta$ is irrational. So Corollary 5.7 shows that $C^*(\Lambda \times T_1, c_\phi)$ is simple.

Our next example shows that it is possible for the bicharacter ω of $\text{Per}(\Lambda)$ determined by c as in Proposition 3.1 to be degenerate and still have $C^*(\Lambda, c)$ simple.

Example 6.3. Let Λ be the 1-graph with vertex v and edges $\{e, f\}$, i.e. the bouquet B_2 of two loops. Fix $\theta \in \mathbb{R} \setminus \mathbb{Q}$ and define $\phi : \Lambda^1 \to \mathbb{T}$ by $\phi(e) = 1$ and $\phi(f) = e^{2\pi i\theta}$. As pointed out in [10, Page 917] one can check that the action $\beta = \beta^\phi$ of $\mathbb{Z}$ on $C^*(\Lambda)$ as given in part (1) of Theorem 5.1 is outer. Hence by [18, Theorem 3.1] $C^*(\Lambda) \times_\beta \mathbb{Z}$ is simple (and purely infinite by [19, Lemma 10]). Since $\text{Per}(\Lambda \times T_1) \cong \mathbb{Z}$ the bicharacter ω of Proposition 3.1 is degenerate. Note that we could also deduce simplicity of $C^*(\Lambda) \times_\beta \mathbb{Z}$ from Corollary 5.8.

The next example illustrates that the interaction between the action θ, the group $\mathrm{Per}(\Lambda)$ and the subgroup Z_ω can be fairly complicated.

Example 6.4. Consider the 4-graph $\Lambda = B_2 \times T_3$, where B_2 is the bouquet of two loops as in the preceding example, and T_3 denotes $\mathbb{N}^3$ regarded as a 3-graph. Choose irrational numbers θ and ρ. Choose any $\phi : B_2^1 \to \mathbb{T}^3$ such that $\phi(e)_1 = 1$ and $\phi(f)_1 = e^{2\pi i \theta}$, and let $c_\phi \in \underline{Z}^2(B_2 \times T_1, \mathbb{T}^3)$ be as in (5.1). Define a bicharacter ω of $\mathbb{Z}^3$ by $\omega(p, q) = e^{2\pi i \rho q_2 p_3}$. Let $c_{\phi,\omega}$ be the cocycle $c_{\phi,\omega}((\alpha, m, p), (\beta, n, q)) = c_\phi((\alpha, m), (\beta, n))\omega(p, q)$ of (5.2). We have $\mathrm{Per}(\Lambda) = \{0\} \times \mathbb{Z}^3 \subset \mathbb{Z}^4$. It is routine to check using Lemma 3.5 that $Z_\omega = \{0\} \times \mathbb{Z} \times \{0\} \subseteq \mathrm{Per}(\Lambda)$, so is a proper nontrivial subgroup of $\mathrm{Per}(\Lambda)$. The argument of Corollary 5.7 shows that $\{(r(\gamma), \tilde{\phi}(\gamma)|_{Z_\omega}) : \gamma \in (\mathcal{G}_{B_2})_x\}$ is dense in $B_2^\infty \times \widehat{Z}_\omega$ for all $x \in B_2^\infty$. So Theorem 5.1(3) shows that $C^*(\Lambda, c_{\phi,\omega})$ is simple. Note that $\{(r(\gamma), \tilde{\phi}(\gamma)) : \gamma \in (\mathcal{G}_{B_2})_x\}$ may not be dense in $B_2^\infty \times \widehat{\mathrm{Per}(\Lambda)}$ for any x; for example if $\phi(e)_3 = \phi(f)_3 = 1$.

Finally, we present an example which is not directly related to our simplicity theorems here, but is of independent interest. Corollary 8.2 of [26] implies that if $C^*(\Lambda)$ is simple, then each $C^*(\Lambda, c)$ is simple. Moreover, Theorem 5.4 of [25] shows that $C^*(\Lambda)$ and $C^*(\Lambda, c)$ very frequently (perhaps always) have the same K-theory. From this and the Kirchberg–Phillips theorem [17, 35] we deduce that if $C^*(\Lambda)$ is purely infinite and simple, then all its twisted C^*-algebras for cocycles that lift to $\mathbb{R}$-valued cocycles coincide. On the other hand, the twisted C^*-algebras of T_2, for example, are rotation algebras and so different choices of cocycle yield different twisted C^*-algebras; but in this instance the untwisted algebra is not simple. We were led to ask whether there exists a k-graph whose twisted C^*-algebras (including the untwisted one) are all simple but are not all identical. We present a 3-graph with this property.

Example 6.5. Consider the 3-coloured graph with vertices $\{v_n : n = 1, 2, \ldots\}$, blue edges $\{e_{n,i} : n \in \mathbb{N} \setminus \{0\}, i \in \mathbb{Z}/n^2\mathbb{Z}\}$, red edges $\{f_n : n \in \mathbb{N} \setminus \{0\}\}$ and green edges $\{g_n : n \in \mathbb{N} \setminus \{0\}\}$, with $r(e_{n,i}) = r(f_n) = s(f_n) = r(g_n) = s(g_n) = v_n$ and $s(e_{n,i}) = v_{n+1}$ for all n, i. The graph can be drawn as follows. (For those with monochrome printers: the solid edges are blue, the dashed edges are red, and the edges drawn in a dash-dot-dash pattern are green.)
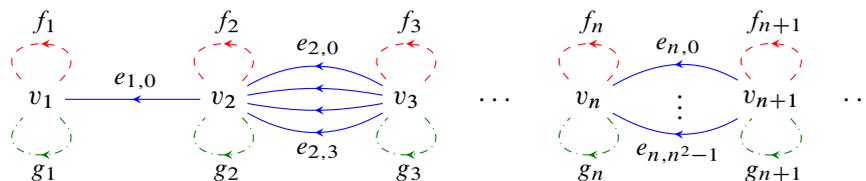
Specify commuting squares by $f_n e_{n,i} = e_{n,i+1} f_{n+1}$, $g_n e_{n,i} = e_{n,i+n} g_{n+1}$ and $f_n g_n = g_n f_n$ (addition in the subscripts of the $e_{n,i}$ takes place in the cyclic group $\mathbb{Z}/n^2\mathbb{Z}$). It is easy to check that this is a complete and associative collection of

commuting squares as in [15], and so determines a 3-graph Λ, in which each $d(e_{n,i}) = (1, 0, 0)$, each $d(f_n) = (0, 1, 0)$ and $d(g_n) = (0, 0, 1)$. It is clear that Λ is cofinal. We claim that it is also aperiodic. For this, [27, Remark 3.2 and Theorem 3.4] imply that it is enough to show that if $\mu, \nu \in \Lambda$ satisfy

$$r(\mu) = r(\nu), \quad s(\mu) = s(\nu) \quad \text{and} \quad d(\mu) \wedge d(\nu) = 0 \tag{6.1}$$

and if $\mu\alpha\Lambda \cap \nu\alpha\Lambda \neq \emptyset$ for all $\alpha \in s(\mu)\Lambda^0$ then $\mu = \nu = r(\mu)$. Fix μ, ν satisfying (6.1). Then $\mu = f_m^p$ and $\nu = g_m^q$ (or vice versa) for some $p, q \in \mathbb{N}$. Let $n = \max\{p, q\}$ so that $m + n > p, q$, and let $\alpha = e_{m,0}e_{m+1,0} \cdots e_{m+n,0}$. The factorisation property implies that $\mu\alpha$ has the form $e_{m,i_m}e_{m+1,i_{m+1}} \cdots e_{m+n,p} f_{m+n+1}^p$ and $\nu\alpha$ has the form $e_{m,j_m}e_{m+1,j_{m+1}} \cdots e_{m+n,q(m+n)} g_{m+n+1}^q$. So $\mu\alpha\Lambda \cap \nu\alpha\Lambda \neq \emptyset$ forces $p = q(m + n)$ in $\mathbb{Z}/(m + n)^2\mathbb{Z}$. Since $m + n \geq p, q$, this forces $p = q = 0$ so that $\mu = \nu = v_m$ as required. So Λ is aperiodic as claimed.

It now follows from [26, Corollary 8.2] that $C^*(\Lambda, c)$ is simple for every $c \in \underline{Z}^2(\Lambda, \mathbb{T})$. Let $\theta \in [0, 1)$ and define $c_\theta \in \underline{Z}^2(\Lambda, \mathbb{T})$ by $c_\theta(\mu, \nu) = e^{2\pi i\theta d(\mu)_3 d(\nu)_2}$. We show that $C^*(\Lambda, c_\theta)$ and $C^*(\Lambda, c_\rho)$ are nonisomorphic whenever θ and ρ are rationally independent[5].

For each $n = 1, 2, \ldots$ the Cuntz–Krieger relations give $s_{f_n}s_{f_n}^* = s_{f_n}^*s_{f_n} = s_{g_n}s_{g_n}^* = s_{g_n}^*s_{g_n} = p_{v_n}$, so s_{f_n} and s_{g_n} are unitaries in $C^*(\{s_{f_n}, s_{g_n}\})$. The definition of c_θ gives $s_{g_n}s_{f_n} = e^{2\pi i\theta}s_{f_n}s_{g_n}$. This is the defining relation for the rotation algebra A_θ and so $C^*(\{s_{f_n}, s_{g_n}\}) \cong A_\theta$. We observe that we can express the corner $p_{v_1}C^*(\Lambda, c_\theta)p_{v_1}$ as the direct limit of the C^*-algebras $p_{v_1}C^*(\Lambda_n, c_\theta)p_{v_1}$, where Λ_n is the locally-convex 3-graph $\{v_1, \ldots, v_n\}\Lambda\{v_1, \ldots, v_n\}$. Each of these subalgebras is canonically isomorphic to $M_{q_n}(A_\theta)$, where $q_n = \prod_{i=1}^{n-1} i^2$ (note $q_1 = q_2 = 1$), so its K_0-group is isomorphic to $\mathbb{Z}^2$.

We claim that if θ is irrational, then the map on K_0 induced by the inclusion map

$$p_{v_1}C^*(\Lambda_n, c_\theta)p_{v_1} \hookrightarrow p_{v_1}C^*(\Lambda_{n+1}, c_\theta)p_{v_1},$$

is given by multiplication by n^2. This is clear by direct computation using the Cuntz–Krieger relation for the class $[p_{v_1}]$ of the identity, and therefore for the class of a minimal projection in $M_{q_n}(\mathbb{C}1) \subseteq M_{q_n}(A_\theta)$. The other generating K_0-class is that of the matrix $p_n \in A_\theta \subseteq M_{q_n}(A_\theta)$ with $(1, 1)$-entry equal to the Powers–Rieffel projection and all other entries zero. The trace on $M_{q_{n+1}}(A_\theta)$ carries p_{n+1} to θ/q_{n+1}, while its restriction to the image of $M_{q_n}(A_\theta)$ must agree with the trace on $M_{q_n}(A_\theta)$ so carries the image of p_n to θ/q_n; so $[\iota(p_n)]_0 = (q_{n+1}/q_n)[p_{n+1}]_0 = n^2[p_{n+1}]_0$.

We deduce from the continuity of K-theory as a functor into the category of ordered abelian groups (see for example [43, Theorem 6.3.2]) that when θ is

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[5]In an earlier version of the paper, we proved only that $C^*(\Lambda, \theta)$ and $C^*(\Lambda)$ are nonisomorphic when θ is irrational. We thank the anonymous referee for suggesting that we expand our analysis to encompass the relationship between these algebras for different irrational values of θ. We believe that the end product is cleaner and more informative even in in the situation we had originally considered.

irrational, $K_0(C^*(\Lambda, c_\theta))$ is isomorphic as an ordered abelian group to $\mathbb{Q} + \theta\mathbb{Q}$ with the order inherited from $\mathbb{R}$, and hence isomorphic as a group to $\mathbb{Q}^2$. Since each c_θ is of exponential form, Theorem 5.4 of [25] (see also [14]) shows that $K_0(C^*(\Lambda, c_\theta)) \cong \mathbb{Q}^2$ for all θ.

Since the tracial-state space of each $p_{v_1} C^*(\Lambda_n) p_{v_1}$ is compact and nonempty, and since the $p_{v_1} C^*(\Lambda_n) p_{v_1}$ are nested, a compactness argument shows that $p_{v_1} C^*(\Lambda, c_\theta) p_{v_1}$ admits a trace τ (or one can directly construct a trace using the approach of [32, Proposition 3.8]). The functional τ_* on $K_0(p_{v_1} C^*(\Lambda, c_\theta) p_{v_1})$ induced by any trace τ has range $\bigcup \tau_*|_{K_0(p_{v_1} C^*(\Lambda_n, c_\theta) p_{v_1})}$. Since each $p_{v_1} C^*(\Lambda_n, c_\theta) p_{v_1} \cong M_{q_n}(A_\theta)$, uniqueness of the map on $K_0(A_\theta)$ induced by a trace on A_θ (see [9, Lemma 2.3]) implies that every trace on $p_{v_1} C^*(\Lambda, c_\theta) p_{v_1}$ induces the same functional $\tau_* : K_0(p_{v_1} C^*(\Lambda_n, c_\theta) p_{v_1}) \to \mathbb{R}$, and that this τ_* has range $\bigcup_n (\mathbb{Z} + \theta\mathbb{Z})/q_n = \mathbb{Q} + \theta\mathbb{Q}$.

It follows that the Morita-equivalence classes (and so in particular the isomorphism classes) of $C^*(\Lambda, c_\theta)$ and $C^*(\Lambda, c_\rho)$ are distinct for rationally independent θ and ρ. In particular if θ is irrational then $C^*(\Lambda, c_\theta)$ and $C^*(\Lambda) = C^*(\Lambda, c_0)$ are not isomorphic.

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