

## Analytic and topological index maps with values in the $K$ -theory of mapping cones

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**Abstract.** Index maps taking values in the  $K$ -theory of a mapping cone are defined and discussed. The resulting index theorem can be viewed in analogy with the Freed–Melrose index theorem. The framework of geometric  $K$ -homology is used in a fundamental way. In particular, an explicit isomorphism from a geometric model for  $K$ -homology with coefficients in a mapping cone,  $C_\phi$ , to  $KK(C(X), C_\phi)$  is constructed.

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### 1. Introduction

The Baum–Douglas or  $(M, E, f)$  model for  $K$ -homology is a fundamental tool in the study of index theory. Since its introduction in [1], it has been used to study both classical and exotic index theory. In particular, it is useful to construct variants of the Baum–Douglas model which are associated to various index problems; for example, models associated to non-integer valued index maps are of interest. We refer to the Baum–Douglas model and its variants as geometric models and assume the reader is familiar with the original  $(M, E, f)$ -model, see any of [1–3, 8, 18, 22].

This paper is a continuation of [6]; the setup is as follows. Let  $X$  be a finite CW-complex,  $\phi : B_1 \rightarrow B_2$  be a unital  $*$ -homomorphism between unital  $C^*$ -algebras and  $C_\phi$  be the mapping cone of  $\phi$ . In [6], a geometric model of the Kasparov group  $KK^*(C(X), C_\phi)$  was constructed. We denote the resulting abelian group by  $K_*(X; \phi)$ . (A more detailed review of notation can be found at the end of the introduction). The main results of the present paper are as follows:

- (1) the construction of an explicit (i.e., defined at the level of cycles) isomorphism  $\lambda : K_*(X; \phi) \rightarrow KK^{*+1}(C(X), C_\phi)$  modelled on the classical topological index map;
- (2) the construction of an explicit isomorphism (i.e., analytic index map) from

$K_*(X; \phi)$  to  $KK^{*+1}(C(X), C_\phi)$  using higher Atiyah–Patodi–Singer index theory (see [17]) in the special case when  $X$  is a point and  $\phi_* : K_*(B_1) \rightarrow K_*(B_2)$  is injective;

- (3) a proof of the equality of these two maps when they are both defined, see Theorem 4.7 (this is an index theorem).

The starting point for the construction in [6] was not only the work of Baum and Douglas [1, 2], but also Higson and Roe [12]. The particular case when  $\phi$  is the unital inclusion of the complex numbers into a  $\text{II}_1$ -factor is relevant for  $\mathbb{R}/\mathbb{Z}$ -valued index theory; in this example, the map at the level of  $K$ -theory is the inclusion of the integers into the reals. Further motivation for the construction of this particular geometric model can be found in the introduction of [6]. It should also be mentioned that the isomorphism from  $K_*(X; \phi)$  to  $KK^{*+1}(C(X), C_\phi)$  considered in [6] was rather indirect. Hence the desire for an explicit isomorphism.

The construction of this isomorphism (Item (1) above) is via neat embeddings of manifolds with boundary into half-spaces. As such, in the case when  $X$  is a point, it can be viewed as analogous to the classical topological index map. Based on this analogy, we refer to the isomorphism  $K_*(X; \phi) \rightarrow KK^{*+1}(C(X), C_\phi)$  obtained via this embedding process as the topological index when  $X$  is a point.

There is also an analytic index (Item (2) above) defined under the condition that  $\phi_* : K_*(B_1) \rightarrow K_*(B_2)$  is injective. This (rather restrictive) condition ensures that the higher Atiyah–Patodi–Singer index can be defined. It is satisfied in the special case when  $\phi$  is the unital inclusion of the complex number into a  $\text{II}_1$ -factor and for other examples that model geometric  $K$ -homology with coefficients (see [6, Example 5.3]). The equality of the two index maps is the content of Theorem 4.7. This index theorem is analogous to the Freed–Melrose index theorem [9, 10].

My motivation for considering index theory with values in the  $K$ -theory of mappings cones is based on its relationship with  $\mathbb{R}/\mathbb{Z}$ -valued index theory. There are other interesting examples. Recently, see [5], Chang, Weinberger, and Yu have considered the following framework. Let  $\pi_1$  and  $\pi_2$  be two discrete finitely generated groups and  $\alpha : \pi_1 \rightarrow \pi_2$  be a group homomorphism. If  $C^*(\pi_i)$  denotes the full group  $C^*$ -algebra of  $\pi_i$ , then (from  $\alpha$ ) we obtain a  $*$ -homomorphism  $\tilde{\alpha} : C^*(\pi_1) \rightarrow C^*(\pi_2)$ . Index maps (in particular, a version of the Baum–Connes assembly map) taking value in  $K_*(C_{\tilde{\alpha}})$  are considered in [5]; these constructions are analytic in nature. In joint work with Magnus Goffeng (see [7]), we explore the connection between the work in [5] and the more geometric results of the current paper. In particular, we consider an analytically defined index map without assuming  $\phi_* : K_*(B_1) \rightarrow K_*(B_2)$  is injective.

The prerequisites for the paper are as follows. Beyond knowledge of the Baum–Douglas model, we assume the reader is familiar the basic properties of Hilbert  $C^*$ -module bundles, see for example [19, Section 2]; the bundles we consider are always locally trivial. Section 4 builds on properties of higher Atiyah–Patodi–Singer

index theory (see [17] and references therein for details). A number of constructions considered here require the framework of  $KK$ -theory (see [13]). In particular, we generalize a number of constructions from [4] to our setting.

Throughout the paper,  $N$  denotes a  $\text{II}_1$ -factor,  $B_1$  and  $B_2$  unital  $C^*$ -algebras,  $\phi : B_1 \rightarrow B_2$  a unital  $*$ -homomorphism,  $C_\phi$  the mapping cone of  $\phi$  and  $X$  a finite CW-complex. The suspension of a  $C^*$ -algebra,  $A$ , is denoted by  $SA$ . If  $B$  is a unital  $C^*$ -algebra, then the  $C^*$ -algebra of continuous  $B$ -valued function on  $X$  is denoted by  $C(X, B)$ . Finitely generated projective Hilbert  $B$ -module bundles over  $X$  that are locally trivial will be refer to as  $B$ -bundles over  $X$ . The Grothendieck group of (isomorphism classes of)  $B$ -bundles over  $X$  is denoted by  $K^0(X; B)$ . It is well known (for example, [19, Proposition 2.17]) that

$$K^0(X; B) \cong K_0(C(X, B)) \cong K_0(C(X) \otimes B).$$

Given a  $B$ -bundle over  $X$ , we have classes  $[E] \in K^0(C(X) \otimes B)$  and  $[[E]] \in KK^0(C(X), C(X) \otimes B)$  (see for example [18, Section 3.4]).

A cycle in the Baum–Douglas model is a triple,  $(M, E, f)$ , where  $M$  is a compact  $\text{spin}^c$ -manifold,  $E$  is a vector bundle over  $M$ , and  $f : M \rightarrow X$  is a continuous map; we let  $K_*^{\text{geo}}(X)$  denote the abelian group obtained from these cycles. More generally, given a unital  $C^*$ -algebra,  $B$ , one can obtain a model for  $KK^*(C(X), B)$  (denoted by  $K_*^{\text{geo}}(X; B)$ ) by replacing the vector bundle  $E$  with a  $B$ -bundle, see [22] for details. The precise definition of the cycles used to define  $K_*(X; \phi)$  is given in Definition 2.3, while the group is defined in Definition 2.6. The topological index is denoted by  $\text{ind}_{\text{top}}$ . Subscript notation is also used in the case of Dirac type operators to specify which manifold it is acting on and if it is twisted by a  $B$ -bundle.

## 2. Review of the geometric model

We review the constructions and main results of [6].

**Definition 2.1.** Let  $W$  be a locally compact space,  $Z$  a closed subspace of  $W$ , and  $\phi : B_1 \rightarrow B_2$  a unital  $*$ -homomorphism between unital  $C^*$ -algebras. Then

$$C_0(W, Z; \phi) := \{(f, g) \in C_0(W, B_2) \oplus C_0(Z, B_1) \mid f|_Z = \phi \circ g\}$$

We note that  $C_0(W, Z; \phi)$  is a  $C^*$ -algebra; it fits into the following pullback diagram:

$$\begin{array}{ccc} C_0(W, Z; \phi) & \longrightarrow & C_0(Z, B_1) \\ \downarrow & & \phi_* \downarrow \\ C_0(W, B_2) & \xrightarrow{|_Z} & C_0(Z, B_2) \end{array}$$

A prototypical example is the case when  $W$  is a manifold with boundary and  $Z = \partial W$ . In particular, the mapping cone of  $\phi$  (denoted by  $C_\phi$ ) is obtained by taking  $W = [0, 1)$  and  $Z = pt$ . The  $K_0$ -group of  $C_0(W, Z; \phi)$  is denoted by  $K^0(W, Z; \phi)$ . If  $g : W \rightarrow W'$  is a continuous map such that  $g(Z) \subseteq Z'$ , then we obtain a  $*$ -homomorphism,  $\tilde{g} : C_0(W', Z'; \phi) \rightarrow C_0(W, Z; \phi)$  and hence a map at the level of  $K$ -theory groups. We also have a  $K^0(W)$ -module structure on  $K^0(W, Z; \phi)$  obtained via

$$g \cdot (f_W, f_Z) := (g \cdot f_W, g|_Z \cdot f_Z)$$

where  $g \in C(W)$  and  $(f_W, f_Z) \in C_0(W, Z; \phi)$ . We will also make use of

$$C_b(W, Z, \phi) := \{(f, g) \in C_b(W, B_2) \oplus C_b(Z, B_1) \mid f|_Z = \phi \circ g\}$$

where  $C_b(X, B)$  denotes the bounded  $B$ -valued function on  $X$ . Of course, if  $W$  is compact, then  $C_b(W, Z; \phi) = C_0(W, Z; \phi)$ .

**Definition 2.2** (Cycles with vector bundle data [6, Definition 4.2]). A cycle (over  $X$  with respect to  $\phi$  using bundle data) is given by,  $(W, (E_{B_2}, F_{B_1}, \alpha), f)$ , where

- (1)  $W$  is a smooth, compact  $\text{spin}^c$ -manifold with boundary;
- (2)  $E_{B_2}$  is a smooth  $B_2$ -bundle over  $W$ ;
- (3)  $F_{B_1}$  is a smooth  $B_1$ -bundle over  $\partial W$ ;
- (4)  $\alpha : E_{B_2}|_{\partial W} \cong \phi_*(F_{B_1}) := F_{B_1} \otimes_\phi B_2$  is an isomorphism of  $B_2$ -bundles;
- (5)  $f : W \rightarrow X$  is a continuous map.

**Definition 2.3** (Cycles with  $K$ -theory data [6, Definition 4.3]). A cycle (over  $X$  with respect to  $\phi$  using  $K$ -theory data) is a triple,  $(W, \xi, f)$ , where:

- (1)  $W$  is a smooth, compact  $\text{spin}^c$ -manifold with boundary;
- (2)  $\xi \in K^0(W, \partial W; \phi)$ ;
- (3)  $f : W \rightarrow X$  is a continuous map.

The manifold,  $W$ , in a cycle need not be connected. We also let  $\xi_{\partial W}$  and  $\xi_W$  denote the images of  $\xi$  under the maps  $p_1 : K^0(W, \partial W; \phi) \rightarrow K^0(\partial W; B_1)$  and  $p_2 : K^0(W, \partial W; \phi) \rightarrow K^0(W; B_2)$  respectively. The opposite of a cycle,  $(W, \xi, f)$ , is the same data except  $W$  is given the opposite  $\text{spin}^c$ -structure. It is denote by  $-(W, \xi, f)$ . The disjoint union of cycles,  $(W, \xi, f)$  and  $(\tilde{W}, \tilde{\xi}, \tilde{f})$  is given by the cycle  $(W \dot{\cup} \tilde{W}, \xi \dot{\cup} \tilde{\xi}, f \dot{\cup} \tilde{f})$ . Two cycles,  $(W, \xi, f)$  and  $(\tilde{W}, \tilde{\xi}, \tilde{f})$  are isomorphic if there exists a diffeomorphism,  $h : W \rightarrow \tilde{W}$  such that  $h$  preserves the  $\text{spin}^c$ -structure,  $h^*(\tilde{\xi}) = \xi$ , and  $\tilde{f} \circ h = f$ . Throughout, a ‘‘cycle’’ more precisely refers to an isomorphism class of a cycle.

**Definition 2.4.** A bordism (with respect to  $X$  and  $\phi$ ) is given by  $(Z, W, \eta, g)$  where

- (1)  $Z$  is a compact  $\text{spin}^c$ -manifold with boundary;

- (2)  $W \subseteq \partial Z$  is a regular domain (see for example [6, Definition 4.4]);
- (3)  $\eta \in K^0(Z, \partial Z - \text{int}(W); \phi)$ ;
- (4)  $g : Z \rightarrow X$  is a continuous map.

The “boundary” of a bordism,  $(Z, W, \eta, F)$ , is given by  $(W, \eta|_W, g|_W)$ . This notion of bordism leads to an equivalence relation which we denote by  $\sim_{\text{bor}}$ .

**Definition 2.5.** Let  $(W, \xi, f)$  be a cycle and  $V$  a  $\text{spin}^c$ -vector bundle of even rank over  $W$ . Then, the vector bundle modification of  $(W, \xi, f)$  by  $V$  is defined to be  $(W^V, \pi^*(\xi) \otimes_{\mathbb{C}} \beta_V, f \circ \pi)$  where

- (1)  $\mathbf{1}$  is the trivial real line bundle over  $W$  (i.e.,  $W \times \mathbb{R}$ );
- (2)  $W^V = S(V \oplus \mathbf{1})$  (i.e., the sphere bundle of  $V \oplus \mathbf{1}$ );
- (3)  $\beta_V$  is the “Bott element” in  $K^0(W^V)$  (see [18, Section 2.5]);
- (4)  $\otimes_{\mathbb{C}}$  denotes the  $K^0(W^V)$ -module structure of  $K^0(W^V, \partial W^V; \phi)$ ;
- (5)  $\pi : W^V \rightarrow W$  is the bundle projection.

The vector bundle modification of  $(W, \xi, f)$  by  $V$  is often denoted by  $(W, \xi, f)^V$ .

**Definition 2.6.** Let  $\sim$  be the equivalence relation generated by bordism and vector bundle modification and let

$$K_*(X; \phi) = \{(W, \xi, f)\} / \sim$$

A cycle  $(W, \xi, f)$  is even (resp. odd) if the connected components of  $W$  are all even (resp. odd) dimensional. Then,  $K_0(X; \phi)$  is even cycles modulo  $\sim$  and  $K_1(X; \phi)$  is likewise only with odd cycles.

**Theorem 2.7** (see [6, Proposition 4.13 and Theorem 4.19]). *The set  $K_*(X; \phi)$  with the operation of disjoint union is an abelian group. Moreover, if  $X$  is a finite CW-complex, then the following sequence is exact:*

$$\begin{array}{ccccc} K_0(X; B_1) & \xrightarrow{\phi_*} & K_0(X; B_2) & \xrightarrow{r} & K_0(X; \phi) \\ \uparrow \delta & & & & \downarrow \delta \\ K_1(X; \phi) & \xleftarrow{r} & K_1(X; B_2) & \xleftarrow{\phi_*} & K_1(X; B_1) \end{array}$$

where the maps are defined as follows:

- (1)  $\phi_* : K_*(X; B_1) \rightarrow K_*(X; B_2)$  takes a cycle  $(M, F_{B_1}, f)$  to  $(M, \phi_*(F_{B_1}), f)$ .
- (2)  $r : K_*(X; B_2) \rightarrow K_*(X; \phi)$  takes a cycle  $(M, E_{B_2}, f)$  to  $(M, (E_{B_2}, \emptyset, \emptyset), f)$ .
- (3)  $\delta : K_*(X; \phi) \rightarrow K_{*+1}(X; B_1)$  takes a cycle  $(W, (E_{B_2}, F_{B_1}, \alpha), f)$  to  $(\partial W, F_{B_1}, f|_{\partial W})$ .

**3. An index map via the mapping cone and imbeddings**

Let  $H^n = \{(x_1, \dots, x_n) \in \mathbb{R}^n \mid x_n \geq 0\}$ . The next lemma is a consequence of Bott periodicity and the definitions of the objects involved; its proof is left to the reader.

**Lemma 3.1.** *Let  $k \in \mathbb{N}$  and  $X$  a finite CW-complex. Then*

$$\begin{aligned}
 KK^0(C(X), C_0(H^{2k}, \mathbb{R}^{2k-1}; \phi)) &\cong KK^0(C(X), SC_\phi) \\
 KK^0(C(X), C_0(H^{2k+1}, \mathbb{R}^{2k}; \phi)) &\cong KK^0(C(X), C_\phi)
 \end{aligned}$$

*In particular,  $K^0(H^{2k}, \mathbb{R}^{2k-1}; \phi) \cong K^0(SC_\phi)$  and  $K^0(H^{2k+1}, \mathbb{R}^{2k}; \phi) \cong K^0(C_\phi)$ .*

**Definition 3.2.** Let  $W$  and  $W'$  be spin<sup>c</sup>-manifolds with boundary with dimensions equal modulo two and  $i : W \rightarrow W'$  a  $K$ -oriented neat embedding. The push-forward map induced by  $i$  (denoted  $i!$ ) is given by the composition of the Thom isomorphism and the map given by identifying the normal bundle associated with  $i$  with a neighbourhood of  $W'$ . Thus, the push-forward of  $i$  defines a map

$$i! : K^0(W, \partial W; \phi) \rightarrow K^0(W', \partial W'; \phi)$$

This map has two important properties. Firstly, as a map from cocycles of the form,  $(E_{B_2}, F_{B_1}, \alpha)$  (see Definition 2.2 and [6]) to  $K$ -theory classes, it is given as follows:

$$\begin{aligned}
 i!(E_{B_2}, F_{B_1}, \alpha) \mapsto & [((\pi_W)^*(E_{B_2}) \otimes \beta_W, (\pi_{\partial W})^*(F_{B_1}) \otimes \beta_{\partial W}, \tilde{\alpha} \otimes \text{id})] \\
 & - [((\pi_W)^*(E_{B_2}) \otimes \tilde{\beta}_W, (\pi_{\partial W})^*(F_{B_1}) \otimes \tilde{\beta}_{\partial W}, \tilde{\alpha} \otimes \text{id})] \quad (3.1)
 \end{aligned}$$

where

- (1)  $\pi_W$  (resp.  $\pi_{\partial W}$ ) is the projection map from the normal bundle (resp. normal bundle restricted to the boundary) to  $W$  (resp.  $\partial W$ );
- (2)  $[\beta_W] - [\tilde{\beta}_W]$  is the Thom class of a normal bundle of  $W$  inside  $W'$  and  $\beta_{\partial W}$  (resp.  $\tilde{\beta}_{\partial W}$ ) is the restriction of  $\beta_W$  (resp.  $\tilde{\beta}_W$ ) to the boundary. The reader should note that the bundles which form the Thom class are not unique, but the resulting  $K$ -theory class (i.e., the image of the map  $i!$ ) is unique;
- (3)  $\tilde{\alpha}$  is the isomorphism from  $(\pi_W)^*(E_{B_2})|_{\partial W'}$  to  $(\pi_{\partial W})^*(F_{B_1}) \otimes_\phi B_2$  given by

$$(w, e) \mapsto (w, \alpha(e))$$

Notice that the range of this map is, in fact,  $(\pi_{\partial W})^*(F_{B_1} \otimes_\phi B_2)$ . However, this bundle can be identified with  $(\pi_{\partial W})^*(F_{B_1}) \otimes_\phi B_2$ ;

Secondly, the map can be realized via the Kasparov product with an element in  $KK^0(C_0(W, \partial W; \phi), C_0(W', \partial W'; \phi))$ . The construction of this element is as follows. Let  $\nu_W$  be a normal bundle for  $i(W) \subseteq W'$ . Then,

$$i! := ((\beta \otimes_{\mathbb{C}} [\tilde{\pi}]) \otimes_{C_0(\nu_W)} \otimes_{C_0(\nu_W, \partial \nu_W; \phi)} [l]) \otimes_{C_0(\nu_W, \partial \nu_W; \phi)} [\theta] \quad (3.2)$$

where

- (1)  $\beta \in KK(\mathbb{C}, C_0(\nu_W))$  is the Thom class. It is defined in [4, Appendix 4]; note that we are using the  $K$ -theory class rather than the class in  $KK(C(W), C_0(\nu_W))$ .
- (2)  $[\tilde{\pi}] \in KK(C_0(W, \partial W; \phi), C_b(\nu_W, \partial\nu_W; \phi))$  is the  $KK$ -theory class obtained from the  $*$ -homomorphism  $\tilde{\pi} : C_0(W, \partial W; \phi) \rightarrow C_b(\nu_W, \partial\nu_W; \phi)$  defined via  $(f_W, g_W) \mapsto (f_W \circ \pi, g_W \circ \pi|_{\partial\nu_W})$  where  $\pi : \nu_W \rightarrow W$  is the bundle projection.
- (3)  $[l] \in KK(C_0(\nu_W) \otimes C_b(\nu_W, \partial\nu_W; \phi), C_0(\nu_W, \partial\nu_W; \phi))$  is the  $KK$ -theory class obtained from the  $*$ -homomorphism

$$\iota : C_0(\nu_W) \otimes C_b(\nu_W, \partial\nu_W; \phi) \rightarrow C_0(\nu_W, \partial\nu_W; \phi)$$

defined via  $h \otimes (f_{\nu_W}, g_{\partial\nu_W}) \mapsto (h \cdot f_{\nu_W}, h|_{\partial\nu_W} \cdot g_{\partial\nu_W})$ ; here  $\cdot$  denotes pointwise multiplication.

- (4)  $[\theta] \in KK(C_0(\nu_W, \partial\nu_W; \phi), C_0(W', \partial W'; \phi))$  is the  $KK$ -theory class obtained from the  $*$ -homomorphism  $\theta : C_0(\nu_W, \partial\nu_W; \phi) \rightarrow C_0(W', \partial W'; \phi)$  given by extension by zero.

The reader familiar with pullbacks for  $C^*$ -algebras will notice that the definitions of the  $*$ -homomorphisms above (e.g.,  $\tilde{\pi}$ ,  $\iota$ , and  $\theta$ ) are obtained naturally from the fact that the  $C^*$ -algebras involved are pullbacks. We will often suppress the algebras over which the Kasparov products are taken and use subscript notation when more than one push-forward map is required. In this notation, Equation 3.2 takes the form

$$i! = (\beta_{\nu_W}) \otimes [\tilde{\pi}_{\nu_W}] \otimes [l_{\nu_W}] \otimes [\theta_{\nu_W}]$$

**Proposition 3.3.** *Let  $i : W \hookrightarrow W'$  be a neat embedding. Then, the map  $i!$  is given by taking the Kasparov product with the class,  $[i!]$ . Moreover,  $i!$  fits into the following commutative diagram:*

$$\begin{array}{ccccccc} \rightarrow & K^0(W, \partial W; \phi) & \rightarrow & K^0(W; B_2) \oplus K^0(\partial W; B_1) & \xrightarrow{r_W} & K^0(\partial W; B_2) & \rightarrow \\ & i! \downarrow & & i_W! \oplus i_{\partial W}! \downarrow & & i_{\partial W}! \downarrow & \\ \rightarrow & K^0(W, \partial W'; \phi) & \rightarrow & K^0(W; B_2) \oplus K^0(W'; B_1) & \xrightarrow{r_{W'}} & K^0(\partial W'; B_2) & \rightarrow \end{array}$$

The horizontal morphisms are given by  $KK$ -classes associated to the following  $*$ -homomorphisms:

- (1)  $C_0(W, \partial W; \phi) \rightarrow C_0(W, B_2)$  defined via  $(f, g) \mapsto f$ ;
- (2)  $C_0(W, \partial W; \phi) \rightarrow C_0(W, B_2)$  defined via  $(f, g) \mapsto g$ ;
- (3)  $C_0(W, B_2) \rightarrow C_0(\partial W, B_2)$  defined via  $f \mapsto f|_{\partial W}$ ;
- (4)  $C_0(\partial W, B_1) \rightarrow C_0(\partial W, B_2)$  defined via  $f \mapsto \phi \circ f$ .

The vertical morphisms are given by the standard push-forward classes in  $KK$ -theory.

*Proof.* For the proof of the first statement in the theorem, let  $(E_{B_2}, F_{B_1}, \alpha)$  be a cocycle and let  $\Gamma(M; E_A)$  denote the continuous section of  $E_A$  where  $E_A$  is a  $A$ -bundle over  $M$ . In this notation, the Kasparov cycle associated to  $(E_{B_2}, F_{B_1}, \alpha)$  is given by  $\xi = (\mathcal{E}, \rho, 0)$  where

$$\mathcal{E} = \{(s_W, s_{\partial W}) \in \Gamma(W; E_{B_2}) \oplus \Gamma(\partial W; F_{B_1}) \mid (s_W)|_{\partial W} = \alpha \circ (s_{\partial W} \otimes \text{Id}_{B_2})\}$$

and  $\rho$  is the unital inclusion of the complex number. The product  $\xi \otimes_{C_0(W, \partial W; \phi)} [i!]$  can be explicitly computed and (as the reader can verify) is equal to the Kasparov cycle associated to the  $i!(E_{B_2}, F_{B_1}, \alpha)$ .

The second statement follows from the action of  $i!$  on cocycles of the form,  $(E_{B_2}, F_{B_1}, \alpha)$ , discussed above (see Equation 3.1).  $\square$

Our goal is the definition of a map,  $\lambda : K_*(X; \phi) \rightarrow KK^*(C(X), SC_\phi)$ . For the even case, given a cycle  $(W, \xi, f)$  in  $K_0(X; \phi)$ , there exists (for  $k$  sufficiently large) a  $K$ -oriented neat embedding,  $i : W \rightarrow H^{2k}$  and associated  $KK$ -theory element  $[i!] \in KK(C_0(W, \partial W; \phi), C_0(H^{2k}, \mathbb{R}^{2k-1}; \phi))$ . There are also  $KK$ -elements associated to  $\xi$  and  $f : W \rightarrow X$ ; namely

- (1)  $[[\xi]] := \xi \otimes [i_W] \in KK(C(W), C_0(W, \partial W; \phi))$  where  $[i_W]$  is the  $KK$ -theory class obtained from the  $*$ -homomorphism

$$\iota : C(W) \otimes C_0(W, \partial W; \phi) \rightarrow C_0(W, \partial W; \phi)$$

defined via  $h \otimes (f_W, g_{\partial W}) \mapsto (h \cdot f_W, h|_{\partial W} \cdot g_{\partial W})$ ; we often denote  $[i_W]$  by  $[i]$ ;

- (2)  $[f] \in KK(C(X), C(W))$  is the  $KK$ -element naturally associated to the  $*$ -homomorphism  $\tilde{f} : C(X) \rightarrow C(W)$  induced from  $f$  (i.e.,  $\tilde{f}(g) := g \circ f$ );

Combining these three  $KK$ -theory elements gives the desired map. More precisely, we have the following definition.

**Definition 3.4.** Let  $\lambda : K_0(X; \phi) \rightarrow KK^0(C(X), SC_\phi)$  be the map defined at the level of cycles via

$$\lambda(W, \xi, f) := [f] \otimes_{C(W)} [[\xi]] \otimes_{C_0(W, \partial W; \phi)} [i!] \otimes_{C_0(H^{2k}, \mathbb{R}^{2k-1}; \phi)} \mathcal{B}$$

where  $\mathcal{B}$  denotes the  $KK$ -theory class which gives the map

$$KK(C(X), C_0(H^{2k}, \mathbb{R}^{2k-1}; \phi)) \cong KK(C(X), C_0(H^2, \mathbb{R}; \phi)) = KK(C(X), SC_\phi)$$

obtained via Bott periodicity. The map from  $K_1(X; \phi)$  to  $KK(C(X), C_\phi)$  is defined in a similar way; one uses a neat embedding into  $H^{2k+1}$  (for  $k$  sufficiently large). Since Bott periodicity is a natural isomorphism, we often omit the map induced from  $\mathcal{B}$ .

A proof that the map  $\lambda$  is well defined is required. It is standard to show that the map is well defined at the level of cycles (i.e., independent of the choice of embedding, normal bundle, etc). That it respects the equivalence relation used to define  $K_*(X; \phi)$  is more involved.

In particular, further notation and three lemmas are required. The first two lemmas are based on [4, Lemmas 3.5 and 3.6] (the proof of the latter is in Appendix B.2 of [4]). As such, the proofs of the lemmas stated here are similar to those for these lemmas. The final lemma concerns functorial properties of the push-forward. Again, the proofs is similar to the standard case. The fact that the maps are embeddings simplifies the proofs of these lemmas.

**Lemma 3.5.** *Let  $(W, \xi, f)$  be a cycle in  $K_*(X; \phi)$  and  $V$  an even rank  $spin^c$ -vector bundle over  $W$ . Also let  $s : W \rightarrow S(V \oplus \mathbf{1})$  be the north-pole section of  $W$  into  $S(V \oplus \mathbf{1})$  (i.e.,  $s(w) := (z(w), 1) \in S(V \oplus \mathbf{1})$  where  $z$  is the zero section). Then,*

$$(W, \xi, f)^V = (S(V \oplus \mathbf{1}), s!(\xi), f \circ \pi)$$

*Proof.* Denote  $S(V \oplus \mathbf{1})$  by  $Z$ . The vector bundle,  $V$ , gives a normal bundle of  $s(W) \subseteq Z$ . Therefore,

$$s! = ([F_V] - [F_V^\infty]) \otimes [\tilde{\pi}_V] \otimes [\iota_V] \otimes [\theta_V]$$

where  $F_V$  and  $F_V^\infty$  are the vector bundle used to define the Thom isomorphism (see [4, Proposition A.10] for details).

The  $K$ -theory class associated to the cycle  $(W, \xi, f)^V$  is given by

$$\begin{aligned} \pi_Z^*(\xi) \cdot ([F_Z] - [F_Z^\infty]) &= \xi \otimes [\tilde{\pi}_Z] \otimes ([F_Z] - [F_Z^\infty]) \otimes [\iota_Z] \\ &= \xi \otimes ([F_Z] - [F_Z^\infty]) \otimes [\tilde{\pi}_Z] \otimes [\iota_Z] \\ &= \xi \otimes ([F_V] - [F_V^\infty]) \otimes [\varphi] \otimes [\tilde{\pi}_Z] \otimes [\iota_Z] \end{aligned}$$

where  $\varphi : C_0(V) \rightarrow C(Z)$  is the natural inclusion. The reader can check that

$$\iota_Z \circ (\text{id} \otimes \tilde{\pi}_Z) \circ (\varphi \otimes \text{id}) = \theta_V \circ \iota_V \circ (\text{id} \otimes \tilde{\pi}_V)$$

as  $*$ -homomorphisms from  $C_0(V) \otimes C_0(W, \partial W; \phi)$  to  $C_0(Z, \partial Z; \phi)$ . The equality of these  $*$ -homomorphisms implies that

$$[\varphi] \otimes [\tilde{\pi}_Z] \otimes [\iota_Z] = [\tilde{\pi}_V] \otimes [\iota_V] \otimes [\theta_V]$$

This implies the result. □

**Lemma 3.6.** *Let  $W$  and  $W'$  be smooth, compact  $spin^c$ -manifolds with boundary,  $i : (W, \partial W) \rightarrow (W', \partial W')$  be a neat embedding and  $\xi \in K^0(W, \partial W; \phi)$ . Then,*

$$[[\xi \otimes_{C_0(W, \partial W; \phi)} [i!]]] = [i] \otimes_{C(W)} [[\xi]] \otimes_{C_0(W, \partial W; \phi)} [i!].$$

*Proof.* The reader should recall that by definition

$$[[\xi]] = [\iota_W(\xi)] = [\xi] \otimes_{C_0(W, \partial W; \phi)} [\iota_W].$$

As such, we must show that

$$\begin{aligned} & (\xi \otimes_{C_0(W, \partial W; \phi)} [i!]) \otimes_{C_0(W', \partial W'; \phi)} [\iota_{W'}] \\ &= [i] \otimes_{C(W)} ([\xi] \otimes_{C_0(W, \partial W; \phi)} [\iota_W]) \otimes_{C_0(W, \partial W; \phi)} [i!]. \end{aligned}$$

Let  $p : \mathbb{C} \rightarrow C(W)$  denote the  $*$ -homomorphism defined via  $\lambda \in \mathbb{C} \mapsto \lambda \cdot 1_W$ . It follows from the commutivity of the Kasparov product over  $\mathbb{C}$  and direct calculation that

$$\xi = [p] \otimes \iota_W(\xi)$$

where  $[p] \in KK^0(\mathbb{C}, C(W))$  is the  $KK$ -class associated to  $p$ .

Thus,  $\iota_{W'}(\xi \otimes i!) = \iota_{W'}([p] \otimes \iota_W(\xi) \otimes i!)$ . It follows that if  $\iota_W(\xi) \otimes i! = (E, \rho, T)$ , then  $\iota_{W'}(\xi \otimes i!) = (E, \rho', T)$  where  $\rho'$  is the composition of the inclusion  $C(W') \rightarrow C_0(W', \partial W'; \phi)$  and right action of  $C_0(W', \partial W'; \phi)$ .

The details are as follows. The Hilbert module in the  $KK$ -cycle  $\iota_{W'}([p] \otimes \iota_W(\xi) \otimes i!)$  is given by

$$(C(W) \otimes C(W')) \otimes_{C(W) \otimes C(W')} (E \otimes_{\mathbb{C}} C(W')) \otimes_{\iota_{W'}} C_0(W', \partial W'; \phi)$$

As the reader can verify, the map defined on elementary tensors via

$$f_W \otimes g_{W'} \otimes e \otimes h_{W'} \otimes a \mapsto f_W \cdot e \cdot (g_{W'} h_{W'} a)$$

gives a Hilbert  $C_0(W', \partial W'; \phi)$ -module isomorphism to  $E$ . Moreover, the representation of  $C(W')$  on  $E$  is the composition of the inclusion  $C(W') \rightarrow C_0(W', \partial W'; \phi)$  and right action of  $C_0(W', \partial W'; \phi)$ . The operator  $T$  in the original Kasparov cycle for  $\iota_W(\xi) \otimes i!$  also respects this Hilbert module isomorphism.

To proceed further, additional notation is required. Given a locally compact space  $Y$  and  $C^*$ -algebra  $A$ , let  $C_b(Y; A)$  be the continuous bounded  $A$ -valued functions on  $Y$  and

$$C_b(\nu_W, \partial \nu_W; \phi) := \{(f, g) \in C_b(\nu_W; B) \oplus C_b(\partial \nu_W; A) \mid f|_{\partial \nu_W} = \phi \circ g\}$$

Let  $\pi_{\nu_W} : \nu_W \rightarrow W$  denote the projection map and  $\rho_0 : C(W) \rightarrow C_b(\nu_W)$  denote the  $*$ -homomorphism given by  $f \mapsto f \circ \pi_{\nu_W}$ .

Using the definition of  $i!$ , the class in  $KK$ -theory,  $\xi \otimes [\iota_W] \otimes i!$ , can be represented by a Kasparov cycle,  $(E, \rho, T)$ , with the following properties:

- (1)  $E$  is a Hilbert  $C_0(\nu_W, \partial \nu_W; \phi)$ -module (since the Hilbert module in the definition of  $i!$  is constructed from a Hilbert  $C_0(\nu_W, \partial \nu_W; \phi)$ -module and the inclusion  $\theta : C_0(\nu_W, \partial \nu_W; \phi) \rightarrow C_0(W', \partial W'; \phi)$ );

(2)  $T$  commutes with the action of  $\sigma : C_b(v_W) \rightarrow \mathcal{L}(E)$  via multipliers of  $C_0(v_W)$ ;

(3) The map  $C(W) \rightarrow \mathcal{L}(E)$  is induced from  $\psi_0 : C(W) \rightarrow C_b(v_W)$ .

Let  $h : v_W \times [0, 1] \rightarrow v_W$  be the map defined by  $(x, t) \rightarrow tx$ . Then

$$\rho_t : C(W') \xrightarrow{\text{restriction}} C_b(v_W) \xrightarrow{oh^{(t,t)}} C_b(v_W) \xrightarrow{\sigma} \mathcal{L}(E)$$

defines a homotopy from  $\psi_0 \circ i \circ \sigma$  to the restriction map  $C(N) \rightarrow C_b(v_W)$  composed with  $\sigma$ . These three properties imply that  $(E, \rho_t, T)$  is a  $KK$ -homotopy from  $(E, \rho \circ i, T)$  and  $(E, \rho', T)$ .  $\square$

**Lemma 3.7.** *Let  $(W, \partial W)$ ,  $(W', \partial W')$  and  $(\tilde{W}, \partial \tilde{W})$  be smooth  $spin^c$ -manifolds. If  $s : (W, \partial W) \rightarrow (W', \partial W')$  and  $i : (W', \partial W') \rightarrow (\tilde{W}, \partial \tilde{W})$  are neat embeddings, then*

$$[s!] \otimes_{C_0(W', \partial W')} [i!] = [(i \circ s)!] \in KK(C_0(W, \partial W; \phi), C_0(\tilde{W}, \partial \tilde{W}; \phi))$$

*Proof.* We leave the proof to the reader. In fact, we will only need a weaker result: If  $\xi \in K^0(W, \partial W; \phi)$ , then

$$(\xi \otimes \iota_W) \otimes (i \circ s)! = (\xi \otimes \iota_W) \otimes (s! \otimes i!)$$

This equality follows from a short  $KK$ -theory computation using the fact that the push-forward is functorial on  $K$ -theory and the previous lemma.  $\square$

**Proposition 3.8.** *Let  $(W, \xi, f)$  be a cycle in  $K_*(X; \phi)$  and  $V$  a  $spin^c$ -vector bundle over  $W$  with even dimensional fibers. Then*

$$\lambda((W, \xi, f)^V) = \lambda(W, \xi, f) \text{ in } KK^*(C(X), SC_\phi)$$

*Proof.* Let  $Z = S(V \oplus \mathbf{1})$ ,  $i_Z : Z \rightarrow H^n$  be a neat embedding (we take  $n$  even for even cycles and  $n$  odd for odd cycles),  $s : W \rightarrow Z$  be the neat embedding of  $W$  into  $Z$  via the north pole section of  $Z$ , and  $\pi : V \rightarrow W$  denote the projection map. The definition of  $\lambda$ , the fact that  $\pi \circ s = id_W$ , and the previous three lemmas imply that

$$\begin{aligned} \lambda((W, \xi, f)^V) &= [f] \otimes [\pi] \otimes [s!(\xi)] \otimes [i_Z!] \\ &= [f] \otimes [\pi] \otimes \iota_Z(s!(\xi)) \otimes [i_Z!] \\ &= [f] \otimes [\pi] \otimes [s] \otimes \iota_W(\xi) \otimes [s!] \otimes [i_Z!] \\ &= [f] \otimes [\pi \circ s] \otimes \iota_W(\xi) \otimes [(i_Z \circ s)!] \\ &= [f] \otimes [[\xi]] \otimes [(i_Z \circ s)!] \\ &= \lambda(W, \xi, f) \end{aligned}$$

The last equality follows since  $i_Z \circ s$  is a neat embedding (of  $W$  into  $H^n$ ) and the independence of the definition of  $\lambda$  on the choice of embedding.  $\square$

The bordism relation is considered next, but first some additional notation is introduced. Recall that

$$H^{2k} = \{(x_1, \dots, x_{2k}) \in \mathbb{R}^{2k} \mid x_{2k} \geq 0\}$$

and let

$$H_-^{2k} := \{(x_1, \dots, x_{2k}) \in \mathbb{R}^{2k} \mid x_{2k} \leq 0\}$$

We will make use of the  $C^*$ -algebras  $C_0(H^{2k}, \mathbb{R}^{2k-1}; \phi)$  and  $C_0(\mathbb{R}^{2k}, H_-^{2k}; \phi)$  along with the natural maps

- (1)  $R : C_0(\mathbb{R}^{2k}, H_-^{2k}; \phi) \rightarrow C_0(H^{2k}, \mathbb{R}^{2k-1}; \phi)$  defined by restriction;
- (2)  $I : C_0(\mathbb{R}^{2k}; B_2) \rightarrow C_0(\mathbb{R}^{2k}, H_-^{2k}; \phi)$  defined via  $f \mapsto (\tilde{f}, 0)$  where

$$\tilde{f} = \begin{cases} f(x) & : x \in H^{2k} \\ 0 & : x \in H_-^{2k} \end{cases}$$

(the well-definedness of  $\tilde{f}$  follows from the fact that  $f$  vanishes at  $\infty$ );

- (3)  $\tilde{I} : C_0(\mathbb{R}^{2k}; B_2) \rightarrow C_0(H^{2k}, \mathbb{R}^{2k-1}; \phi)$  defined via  $(f, g) \mapsto (f, 0)$ .

It follows from these definitions that  $R \circ I = \tilde{I}$ .

**Proposition 3.9.** *If  $(W, \xi, f)$  is a boundary in the sense of Definition 2.4, then  $\lambda(W, \xi, f)$  is trivial in  $KK^*(C(X), SC_\phi)$ .*

*Proof.* We prove the result for even cycles; the odd case is similar. The reader should recall the notation introduced immediately before the proposition. Let  $(W, \xi, f)$  be a cycle in  $K_0(X; \phi)$  which is the boundary of  $((Z, W), \eta_Z, g)$ . Fix an embedding  $j : \partial Z \hookrightarrow \mathbb{R}^{2k}$  such that the restriction of  $j$  to  $W \subseteq \partial Z$  is a neat embedding of  $W \rightarrow H^{2k}$ . Denote  $j|_W$  by  $i$ . Let  $\nu_j$  be a normal bundle for  $j(\partial Z) \subseteq \mathbb{R}^{2k}$ . Then  $\nu_i := \nu_j|_{H^{2k}}$  is a normal bundle for  $i(W) \subseteq H^{2k}$ .

By definition,  $\lambda(W, \xi, f) = [f] \otimes [[\xi]] \otimes [i!] \in KK^0(C(X), C_0(H^{2k}, \mathbb{R}^{2k-1}; \phi))$ . Let  $(M, \eta, h)$  denote  $(\partial Z, (\eta_Z)_{B_2}|_{\partial Z}, g|_{\partial Z})$  and

$$\lambda_{B_2}(M, \eta, h) := [h] \otimes_{C(M)} [[\eta]] \otimes_{C(M)} [j!] \in KK^0(C(X), C_0(\mathbb{R}^{2k}) \otimes B_2)$$

Standard results (see for example, [22]) imply that  $\lambda_{B_2}$  is a well-defined map from  $K_0(X; B)$  to  $KK^0(C(X), B_2)$ . In particular,  $\lambda_{B_2}$  vanishes on boundaries. Hence  $\lambda_{B_2}(M, \eta, h) = 0$  (since  $(M, \eta, h)$  is a boundary in  $K_*(X; B_2)$ ). This observation reduces the proof to showing that

$$\lambda(W, \xi, f) = \tilde{I}_*(\lambda_{B_2}(M, \eta, h)) \tag{3.3}$$

where  $\tilde{I}_* : KK^0(C(X), C_0(\mathbb{R}^{2k}) \otimes B_2) \rightarrow KK^0(C(X), C_0(H^{2k}, \mathbb{R}^{2k-1}; \phi))$  is the map on  $KK$ -theory induced from the  $*$ -homomorphism,  $\tilde{I}$ .

Let  $N \in \mathbb{N}$  be sufficiently large so that the normal bundle  $\nu_j$  translated by  $(0, \dots, 0, N)$  is contained in  $\text{int}(H^{2k})$ . For  $t \in [0, 1]$ , let  $j_t$  denote the embedding of  $M$  into  $\mathbb{R}^{2k}$  defined via  $j_t(m) := j(m) + (0, \dots, 0, Nt)$ . For each  $t$ , let

$$\lambda_{j_t}(M, \eta, h) = [h] \otimes_{C(M)} [[\tilde{\eta}_t]] \otimes_{C_0(M, W_t; \phi)} [j_t!] \in KK^0(C(X), C_0(\mathbb{R}^{2k}, H^{2k-1}; \phi))$$

where  $W_t := j_t(M) \cap H_-^{2k}$  and  $[[\tilde{\eta}_t]] \in KK(C(M), C_0(M, W_t; \phi))$  the image of  $\eta$  under the map induced from the  $*$ -homomorphism,  $C_0(M, W; \phi) \rightarrow C_0(M, W_t; \phi)$  defined via  $(f, g) \mapsto (f, g|_{W_t})$ . It follows from the definitions of  $I, R, j_t$ , etc that

$$R_*(\lambda_{j_0}(M, \eta, h)) = \lambda(W, \xi, f) \text{ and } \lambda_{j_1}(M, \eta, h) = I_*(\lambda_{B_2}(M, \eta, h))$$

Moreover,  $\lambda_{j_t}(M, \eta, h)$  defines a homotopy between the  $KK$ -cycles  $\lambda_{j_0}(M, \eta, h)$  and  $\lambda_{j_1}(M, \eta, h)$ . Hence

$$\begin{aligned} \lambda(W, \xi, f) &= R_*(\lambda_{j_0}(M, \eta, h)) \\ &\sim R_*(\lambda_{j_1}(M, \eta, h)) \\ &= (R \circ I)_*(\lambda_{B_2}(M, \eta, h)) \\ &= \tilde{I}_*(\lambda_{B_2}(M, \eta, h)) \end{aligned}$$

As noted in Equation 3.3, this implies the result. □

**Theorem 3.10.** *If  $X$  is a finite CW-complex, then the map  $\lambda : K_*(X; \phi) \rightarrow KK^*(C(X), SC_\phi)$  is an isomorphism.*

*Proof.* The main step is to show that the following diagram commutes:

$$\begin{array}{ccccccccc} \rightarrow & K_0(X; B_1) & \xrightarrow{\phi_*} & K_0(X; B_2) & \xrightarrow{r} & K_0(X; \phi) & \xrightarrow{\delta} & K_1(X; B_1) & \rightarrow \\ & \lambda_{B_1} \downarrow & & \lambda_{B_2} \downarrow & & \lambda \downarrow & & \lambda_{B_1} \downarrow & \\ \rightarrow & KK^0(C(X), B_1) & \xrightarrow{\phi_*} & KK^0(C(X), B_2) & \xrightarrow{r_*^{\text{ana}}} & KK^0(C(X), SC_\phi) & \xrightarrow{\delta_*^{\text{ana}}} & KK^1(C(X), B_1) & \rightarrow \end{array}$$

where

- (1) The first exact sequence is from Theorem 2.7;
- (2) The vertical maps,  $\lambda_{B_i}$  ( $i = 1, 2$ ), are defined at the level of cycles via  $\lambda_{B_i}(M, E_{B_i}, f) = [f] \otimes_{C(M)} [[E_{B_i}]] \otimes_{C(M)} [D_M]$  (see [22] for details);
- (3) The second exact sequence is the long exact sequence in  $KK$ -theory obtained from the short exact sequence of  $C^*$ -algebras

$$0 \rightarrow SB_2 \rightarrow C_\phi \rightarrow B_1 \rightarrow 0$$

Again, the details of commutativity are given for even cycles, but the odd case is similar. That  $\lambda_{B_2} \circ \phi_* = \phi_* \circ \lambda_{B_1}$  is standard. With the goal of showing that

$r^{\text{ana}} \circ \lambda_{B_2} = \lambda \circ r$  in mind, let  $(M, E_{B_2}, f)$  be a geometric cycle in  $K_0(X; B_2)$ . Then

$$(r^{\text{ana}} \circ \lambda_{B_2})(M, E_{B_2}, f) = r^{\text{ana}}([f] \otimes [[E_{B_2}]] \otimes [i!])$$

where  $i : M \rightarrow \mathbb{R}^{2k}$  is an embedding. But  $r^{\text{ana}}$  is given by the inclusion of  $C_0(\mathbb{R}^{2k}) \otimes B_2 \rightarrow C_0(H^{2k}, \mathbb{R}^{2k-1}; \phi)$ . It is induced from the natural inclusion,  $\hat{r} : \mathbb{R}^{2k} \hookrightarrow H^{2k}$ . However, the map  $i \circ \hat{r}$  is a (neat) embedding of  $M \rightarrow H^{2k}$ . Using this embedding in the definition of  $\lambda$ , leads to the result.

Next, the proof that  $\lambda_{B_1} \circ \delta = \delta^{\text{ana}} \circ \lambda$  is considered. Let  $(W, \xi, f)$  be a cycle in  $K_0(X; \phi)$  and  $i : W \hookrightarrow H^{2k}$  a neat embedding. Then

$$\lambda_{B_1}(\delta(W, \xi, f)) = \lambda_{B_1}(\partial W, \xi_{B_1}, f|_{\partial W}) = [f|_{\partial W}] \otimes [[\xi_{B_1}]] \otimes [i|_{\partial W}!]$$

Whereas

$$(\delta^{\text{ana}} \circ \lambda)(W, \xi, f) = \delta^{\text{ana}}([f] \otimes [[\xi]] \otimes [i!]) = [f] \otimes [[\xi]] \otimes [i!] \otimes [ev_{\mathbb{R}^{2k}}]$$

where  $ev_{\mathbb{R}^{2k}} : C_0(H^{2k}, \mathbb{R}^{2k-1}; \phi) \rightarrow C_0(\mathbb{R}^{2k-1}) \otimes B_1$  is given by  $(f, g) \rightarrow g$ . To compare these  $KK$ -classes, three  $*$ -homomorphisms are required; they are

- (1)  $\gamma : C_0(W, \partial W; \phi) \rightarrow C(\partial W) \otimes B_1$  is defined via  $(f, g) \mapsto g$ ;
- (2)  $\iota_W : C(W) \otimes C_0(W, \partial W; \phi) \rightarrow C_0(W, \partial W; \phi)$  is defined above in the discussion following Equation 3.2;
- (3)  $r_W : C(W) \rightarrow C(\partial W)$  is the restriction to the boundary (i.e.,  $r_W(f) = f|_{\partial W}$ );

The  $KK$ -classes associated to these  $*$ -homomorphisms satisfy the following

- (1)  $[f|_{\partial W}] = [f] \otimes_{C(W)} [r_W]$ ;
- (2)  $[r_W] \otimes [[\xi_{B_1}]] = [[\xi]] \otimes \gamma$ ;
- (3)  $[i!] \otimes [ev_{\mathbb{R}^{2k}}] = [\gamma] \otimes [i|_{\partial W}!]$ .

The proofs of these equalities follows from standard properties of  $KK$ -theory. The first equality is standard. In regards to the second (i.e., showing that  $[r_W] \otimes [[\xi_{B_1}]] = [[\xi]] \otimes \gamma$ ), we consider the case when  $\xi$  is given by a triple  $(E_{B_2}, F_{B_1}, \alpha)$  (rather than a formal difference of such triples); the general case easily follows. If  $E_A$  is a  $A$ -bundle over  $M$ , then let  $\Gamma(M; E_A)$  denote the continuous sections of  $E_A$ .

Using this notation, the Kasparov cycle  $[(E_{B_2}, F_{B_1}, \alpha)]$  is given by  $(\mathcal{E}, \rho, 0)$  where

$$\mathcal{E} = \{(s_W, s_{\partial W}) \in \Gamma(W; E_{B_2}) \oplus \Gamma(\partial W; F_{B_1}) \mid (s_W)|_{\partial W} = \gamma \circ (s_{\partial W} \otimes \text{Id}_{B_2})\}$$

and, for  $g \in C(W)$ ,

$$\rho(g) \cdot (s_W, s_{\partial W}) := (g \cdot s_W, g|_{\partial W} \cdot s_{\partial W})$$

On the other hand, the Kasparov cycle  $[(F_{B_1})]$  is given by

$$(\Gamma(\partial W; F_{B_1}), \varphi, 0)$$

where  $\varphi$  is the representation of  $C(\partial W)$  via pointwise multiplication. The Kasparov products  $[r_W] \otimes [[F_{B_1}]]$  and  $[(E_{B_2}, F_{B_1}, \alpha)] \otimes [\gamma]$  can be explicitly computed. The methods used in the proof of Lemma 3.4.4 in [18] can be used to prove the equality of these  $KK$ -elements; the details are left to the reader.

Finally, that  $[i!] \otimes [ev_{\mathbb{R}^{2k}}] = [\gamma] \otimes [i|_{\partial W}]$  follows from the commutative diagram considered in Remark 3.3.

These computations imply that

$$\begin{aligned} [f|_{\partial W}] \otimes [[\xi_{B_1}]] \otimes [i|_{\partial W}] &= [f] \otimes [r_W] \otimes [[\xi_{B_1}]] \otimes [i|_{\partial W}] \\ &= [f] \otimes [[\xi]] \otimes [\gamma] \otimes [i|_{\partial W}] \\ &= [f] \otimes [[\xi]] \otimes [i!] \otimes [ev_{\mathbb{R}^{2k}}] \end{aligned}$$

This completes the proof that the diagram given at the beginning of the proof commutes. The Five Lemma and the fact that  $\lambda_{B_1}$  and  $\lambda_{B_2}$  are isomorphisms for  $X$  a finite CW-complex (see for example [22]) then imply that  $\lambda$  is also an isomorphism.  $\square$

**Definition 3.11.** Let  $(W, \xi, f)$  be a cycle in  $K_p(X; \phi)$  ( $p = 0$  or  $1$ ). Then, for  $k$  sufficiently large, there exists,  $i : W \rightarrow H^{2k+p}$ , a  $K$ -oriented neat embedding of  $W$  into the halfspace  $H^{2k+p}$ . The topological index of  $(W, \xi, f)$  is defined to be

$$\text{ind}_{\text{top}}(W, \xi, f) := i!(\xi)$$

Using Bott periodicity, we can (and will) consider this as an element in  $K_p(SC_\phi)$ .

**Corollary 3.12.** *The topological index map is well-defined (as a map from  $K_p(X; \phi)$  to  $K_p(H^2, \mathbb{R}; \phi)$ ). Moreover, in the case when  $X$  is a point, the topological index map is an isomorphism.*

*Proof.* The first statement follows from the fact that the topological index map is given by the composition,  $c^* \circ \lambda$ , where  $c : \mathbb{C} \rightarrow C(X)$  is the natural inclusion and  $\lambda$  is the isomorphism in Theorem 3.10. The second statement follows as a special case of Theorem 3.10.  $\square$

#### 4. An index map via boundary conditions

Our goal is the construction of an analytic index map from  $K_*(X; \phi)$  to  $K_{*+1}(C_\phi)$ . This index map will be defined under the assumption that  $\phi_* : K_*(B_1) \rightarrow K_*(B_2)$  is injective; an important example is the case when  $\phi$  is the unital inclusion of the complex numbers into a  $\text{II}_1$ -factor. We make use of higher Atiyah–Patodi–Singer index theory. In the next subsection, we discuss the relationship between the higher Atiyah–Patodi–Singer index and vector bundle modification. This discussion is written in a self-contained manner as its main result is of some independent interest;

it also serves as an introduction to higher Atiyah–Patodi–Singer index theory and the notation required for the second subsection. The reader is directed to [15] and [21] and references therein for further details on this theory.

**4.1. Higher Atiyah–Patodi–Singer index theory and vector bundle modification.**

Following [21], we introduce some notation. Let  $W$  be a connected, compact, Riemannian  $\text{spin}^c$ -manifold with boundary with a product structure in a neighborhood of the boundary. Also let  $W_{\text{cyl}}$  denote the manifold obtained from  $W$  by attaching a cylindrical end to the boundary of  $W$ . In other words, there exists  $\epsilon > 0$ , submanifold  $Z_r \subseteq W_{\text{cyl}}$ , and  $\text{spin}^c$ -preserving isometry  $e : Z_r \rightarrow (-\epsilon, \infty) \times \partial W$  such that

$$W = W_{\text{cyl}} - e^{-1}((0, \infty) \times \partial W)$$

We also let  $Z := \mathbb{R} \times \partial W$ ,  $U_\epsilon := e^{-1}((-\epsilon, 0]) \subseteq W$ , and  $p$  denote the projection  $U_\epsilon \rightarrow \partial W$ . In an abuse of notation, we refer to  $\partial W \times (0, \infty)$  when working with  $e^{-1}((0, \infty) \times \partial W)$ .

Let  $B$  be a unital  $C^*$ -algebra,  $E_B$  be a  $B$ -bundle over  $W$  and  $S_W$  be the spinor bundle associated with the  $\text{spin}^c$ -structure on  $W$ . Then,  $\mathcal{E} := S_W \otimes_{\mathbb{C}} E_B$  has a natural Dirac  $B$ -bundle structure in the sense of [21, Section 2]. We denote the Clifford connection on this bundle by  $\nabla$  and assume that this construction respects the product structure of  $\partial W \subseteq W$ . In particular,  $\mathcal{E}|_{U_\epsilon} = p^*(\mathcal{E}|_{\partial W})$ .

Let  $\not{D}_{\partial W}$  denote the Dirac operator associated to the bundle  $S_{\partial W} \otimes E_B|_{\partial W}$ . In [21] (also see [15]), a number of operators are associated to the data introduced in the previous two paragraphs. First, however, we must perturb the operator on the boundary. Let  $A$  be a selfadjoint operator in  $\mathcal{B}(L^2(\partial W; S_{\partial W} \otimes (E_B|_{\partial W})))$  such that  $\not{D}_{\partial W} + A$  is invertible. The existence of  $A$  follows from the vanishing of the index of  $\not{D}_{\partial W}$  (see [15] for further details). In fact, we could assume that  $A$  is a smoothing operator. Following the notation of [21], let  $D_W(A)$  be the operator on  $W$  associated to higher Atiyah–Patodi–Singer boundary conditions and  $D_{W_{\text{cyl}}}(A)$  be the Dirac operator on  $W_{\text{cyl}}$  perturbed on the cylinder by  $A$ . A detailed discussion of these operators (in particular, their construction) can be found in [21, Section 2].

Since the latter operator is of more importance in this work, we only give the details of its construction. Let  $\tilde{\mathcal{E}}$  denote the extension of  $\mathcal{E}$  from  $W$  to  $W_{\text{cyl}}$ ,  $\not{D}$  denote the Dirac operator associated to it, and  $\chi : W \rightarrow [0, 1]$  be a function which satisfies

- (1)  $\text{supp}(\chi) \subseteq \partial W \times (-\frac{3\epsilon}{4}, \infty)$ ;
- (2) For each  $w \in \partial W \times (0, \infty)$ ,  $f(\chi) = 1$ ;

Denote the Clifford action by  $c$  and the coordinate in the normal direction by  $x_1$ . Then  $D_{W_{\text{cyl}}}(A)$  is defined to be the closure (on  $L^2(W_{\text{cyl}}; \tilde{\mathcal{E}})$ ) of the operator

$$\not{D} - c(dx_1)\chi A$$

It has an associated index in the  $K$ -theory of  $B$ . The reader can find further details on this construction in [21].

Our goal is to consider vector bundle modification as it relates to higher index theory for manifolds with boundary. As such, let  $V$  be a  $\text{spin}^c$ -vector bundle over  $W$  with even-dimensional fibers. Further, assume that  $V$  respects the product structure of  $\partial W \subseteq W$ . Using the vector bundle modification operation, we obtain from  $W$  and  $V$  a  $\text{spin}^c$ -manifold  $\hat{W} := S(V \oplus \mathbf{1})$  where  $\mathbf{1}$  denotes the trivial real line bundle over  $W$ ; note that  $\hat{W}$  is a fiber bundle over  $W$ . Moreover, since  $W$  is connected, the fiber is  $S^{2k}$  for some  $k \in \mathbb{N}$ . By extending the vector bundle  $V$  to  $W_{\text{cyl}}$ , we can also consider the vector bundle modification of  $W_{\text{cyl}}$ . We denote the resulting manifold by  $\hat{W}_{\text{cyl}}$ .

The vector bundle modification operation affects the bundle data on  $W$  as follows. Let  $\beta$  denote the Bott bundle over  $\hat{W}$ ; it is a vector bundle and its construction can be found in [1]. Then the Hilbert  $B$ -bundle on  $\hat{W}$  is given by  $\pi^*(E_B) \otimes_{\mathbb{C}} \beta$  where  $\pi : \hat{W} \rightarrow W$  is the projection map. By the two out of three property of  $\text{spin}^c$ -vector bundles (see for example [3]), there is a  $\text{spin}^c$ -structure on  $\hat{W}$ . We let  $S_{\hat{W}}$  denote the spinor bundle associated with the  $\text{spin}^c$ -structure and  $\hat{\mathcal{E}}$  denote the  $B$ -Dirac bundle  $S_{\hat{W}} \otimes \pi^*(E_B) \otimes_{\mathbb{C}} \beta$ . These constructions can also be applied to  $\hat{W}_{\text{cyl}}$ . In an abuse of notation, we denote the Bott bundle over  $\hat{W}_{\text{cyl}}$  also by  $\beta$  and the  $B$ -Dirac bundle over  $\hat{W}_{\text{cyl}}$  also by  $\hat{\mathcal{E}}$ . Based on this discussion, we can construct the associated operators discussed in the preceding paragraphs (this time on the manifolds  $\hat{W}$  and  $\hat{W}_{\text{cyl}}$ ). However, the construction of these operators involved the choice of operator  $A$ . We would like to construct from a choice of  $A$  on the base  $W$  a natural choice of such an operator for  $\hat{W}$ .

The desired construction and the main result of this subsection are the content of the next proposition. The proof requires the following lemma which is a well-known result in  $KK$ -theory (cf. [3, Lemma 2.7] in the case of analytic  $K$ -homology).

**Lemma 4.1.** *Let  $(\mathcal{E}, \rho, F)$  be a Kasparov cycle representing a class in  $KK^0(A, B)$  and suppose that  $T \in \mathcal{L}(\mathcal{E})$  is a self-adjoint, odd-graded involution which commutes with action of  $A$  and anticommutes with  $F$ . Then the class (in  $KK^0(A, B)$ ) of  $(\mathcal{E}, \rho, F)$  is zero.*

**Proposition 4.2.** *We use the notation introduced in the previous few paragraphs. For example,  $W$  denotes a compact  $\text{spin}^c$ -manifold with boundary,  $E_B$  a  $B$ -bundle, and  $V$  a  $\text{spin}^c$ -vector bundle over  $W$  with even dimensional fibers. Then, given a choice of Dirac operator on  $W$  and selfadjoint operator  $A$  (see above), there exists a Dirac operator on  $\hat{W}$  and selfadjoint operator  $\hat{A}$  such that*

$$\text{ind}(D_W(A)) = \text{ind}(D_{\hat{W}}(\hat{A})) \in K_*(B)$$

*The reader should note that the Dirac operator on  $\hat{W}$  and  $\hat{A}$  are defined in the proof.*

**Remark 4.3.** A word concerning this proposition seems in order. Perhaps most importantly, the proposition does not imply that the higher Atiyah–Patodi–Singer index is invariant under vector bundle modification. The specific choice of spectral

section and Dirac operator on the manifold  $\hat{W}$  are important to the proof. These operators are constructed via a partition of unity argument.

In this regard, the statement of this proposition is unsatisfying in a number of ways. In particular, one would hope to find a *canonical* construction of a spectral section on the modified manifold given one on the base; our construction of the spectral section is quite ad hoc. Despite this, the result suffices for our purposes.

*Proof.* The structure of the proof is as follows. By [21, Proposition 2.1], the proof will be complete upon showing that the operators  $D_{W_{\text{cyl}}}(A)$  and  $D_{\hat{W}_{\text{cyl}}}(\hat{A})$  have the same index; of course, the construction of  $\hat{A}$  and the Dirac operator are also required. Apart from these constructions, the proof consists of two steps

- (1) proving the result in the case when  $V$  is a trivial vector bundle. The reader should note that in this case,  $\hat{W} = W \times S^{2k}$ ;
- (2) using a partition of unity argument to treat the case of general  $V$ .

As such, the steps in the proof are the same as those in the proof of Proposition 3.6 in [3]. The case when  $W$  is even dimensional is considered in detail; the odd case is left to the reader.

The case when  $V$  is a trivial bundle is considered first. In this case,  $\hat{W} = W \times S^{2k}$  and we can take the product of the Dirac operators to form the Dirac operator on  $W \times S^{2k}$ . The identification

$$L^2(\partial W \times S^{2k}; (S_{\partial W} \otimes E_B|_{\partial W}) \boxtimes \beta) \cong L^2(\partial W; S_{\partial W} \otimes E_B|_{\partial W}) \hat{\otimes} L^2(S^{2k}; \beta)$$

will be used throughout. In particular, we apply it to define  $\hat{A} := (A \otimes I)$ . It follows that  $\hat{A}$  is selfadjoint and  $\not\partial_{\partial W \times S^{2k}} + \hat{A}$  is invertible. To see that the latter of these statements holds, one (using the fact that we are working in a graded situation (see for example [11, Section A.2])) notes that

$$(\not\partial_{\partial W \times S^{2k}} + \hat{A})^2 = (\not\partial_{\partial W} + A)^2 \otimes I + I \otimes \not\partial_{S^{2k}}^2$$

where  $\not\partial_{\partial W}$  and  $\not\partial_{S^{2k}}$  are respectively the Dirac operators on  $\partial W$  and  $S^{2k}$ . That this operator is invertible follows since  $(\not\partial_{\partial W} + A)^2$  is invertible and both operators are positive. The invertibility of the original operator follows since it is selfadjoint; in particular,

$$(\not\partial_{\partial W \times S^{2k}} + \hat{A})^2 = (\not\partial_{\partial W \times S^{2k}} + \hat{A})^* (\not\partial_{\partial W \times S^{2k}} + \hat{A})$$

Let  $\hat{\otimes}$  denote the (graded) algebraic tensor product and  $\mathcal{S}$  denote the spinor bundle of  $S^{2k}$ . Then, on

$$C_c^\infty(M_{\text{cyl}}; \mathcal{E}) \hat{\otimes} C^\infty(S^{2k}; \mathcal{S} \otimes \beta) \subset C_c^\infty(M_{\text{cyl}} \times S^{2k}; \hat{\mathcal{E}})$$

the twisted Dirac operator on  $W_{\text{cyl}} \times S^{2k}$  has the form

$$\not\partial_{W_{\text{cyl}}} \hat{\otimes} I + I \hat{\otimes} \not\partial_{S^{2k}} .$$

In fact, operator  $\not{D}_{W_{\text{cyl}} \times S^{2k}} - c(dx_1)\hat{\chi}\hat{A}$  also decomposes in this way. That is, on  $C_c^\infty(M_{\text{cyl}}; \mathcal{E}) \hat{\otimes} C^\infty(S^{2k}; \mathcal{S} \otimes \beta)$ , it is equal to

$$(\not{D}_{W_{\text{cyl}}} + c(dx_1)\chi A) \hat{\otimes} I + I \hat{\otimes} \not{D}_{S^{2k}}.$$

Here, the reader should note that  $\hat{\chi}$  and  $\chi$  are related as follows:  $\hat{\chi} : W_{\text{cyl}} \times S^{2k} \rightarrow [0, 1]$  is defined via  $\hat{\chi}(w, z) := \chi(w)$ . The closure of the above operator (denoted by  $D_{W_{\text{cyl}} \times S^{2k}}(\hat{A})$ ) therefore has the form

$$D_{W_{\text{cyl}} \times S^{2k}}(\hat{A}) = D_{W_{\text{cyl}}}(A) \hat{\otimes} I + I \hat{\otimes} D_{S^{2k}}$$

as an operator on

$$L^2(W_{\text{cyl}} \times S^{2k}; \hat{\mathcal{E}}) \cong L^2(W_{\text{cyl}}; \mathcal{E}) \hat{\otimes} L^2(S^{2k}; \mathcal{S} \otimes \beta).$$

We now apply techniques from [3]. Namely, the Hilbert module on which the operator  $D_{W_{\text{cyl}} \times S^{2k}}(\hat{A})$  acts (as an unbounded operator) decomposes as follows

$$\begin{aligned} L^2(W_{\text{cyl}} \times S^{2k}; \hat{\mathcal{E}}) &\cong L^2(W_{\text{cyl}}; \mathcal{E}) \hat{\otimes} L^2(S^{2k}; \mathcal{S} \otimes \beta) \\ &\cong (L^2(W_{\text{cyl}}; \mathcal{E}) \hat{\otimes} \ker(D_{S^{2k}})) \oplus (L^2(W_{\text{cyl}}; \mathcal{E}) \hat{\otimes} \ker(D_{S^{2k}})^\perp) \end{aligned}$$

Moreover, the operator respects this decomposition. That is, if  $P$  denotes the projection onto  $L^2(W_{\text{cyl}}; \mathcal{E}) \hat{\otimes} \ker(D_{S^{2k}})$ , then

$$D_{W_{\text{cyl}} \times S^{2k}}(\hat{A}) = PD_{W_{\text{cyl}} \times S^{2k}}(\hat{A})P + P^\perp D_{W_{\text{cyl}} \times S^{2k}}(\hat{A})P^\perp$$

The operator  $PD_{W_{\text{cyl}} \times S^{2k}}(\hat{A})P$  acts as  $D_{W_{\text{cyl}}}(A)$  on  $L^2(W_{\text{cyl}}; \mathcal{E}) \hat{\otimes} \ker(D_{S^{2k}})$ ; to see this, note that  $\ker(D_{S^{2k}})$  is one dimensional and is given by the span of an even section (see [3, Proposition 3.11]).

This reduces the proof (of the special case when  $V$  is trivial) to showing that  $\text{ind}(P^\perp D_{W_{\text{cyl}} \times S^{2k}}(\hat{A})P^\perp) = 0$ . To this end, consider the operator  $\gamma \otimes T$  where  $\gamma$  is the grading operator and  $T$  is the partial isometry in the polar decomposition of  $D_{S^{2k}}$ . As the reader can verify (see also [3, Section 4]), this operator is an odd graded involution on  $L^2(M; \mathcal{E}) \hat{\otimes} \ker(D_{S^{2k}})^\perp$ . Moreover,  $\gamma \otimes T$  anti-commutes with  $P^\perp D_{W_{\text{cyl}} \times S^{2k}}^{\text{prod}}(\hat{A})P^\perp$ . Lemma 4.1 implies that  $\text{ind}(P^\perp D_{M \times S^{2k}} P^\perp)$  is zero. This completes the proof in the case when  $V$  is a trivial vector bundle.

The general case is now considered. As such, let  $V$  be a general  $\text{spin}^c$ -vector bundle with even-dimensional fibers. We must construct the Dirac operator and the operator,  $\hat{A}$ .

We begin with the Dirac operator on the boundary of  $\hat{W}$ . Again, the reader should compare our construction with the one in the proof of Proposition 3.6 in [3]. We denote the principal  $\text{Spin}^c(2k)$ -bundle associated to the  $\text{spin}^c$ -structure of  $\partial \hat{W}$

by  $\mathcal{P}_{\partial\hat{W}}$ . The Dirac operator on the boundary (twisted by the relevant Hilbert module bundle),  $D_{\partial\hat{W}}$  acts on a Hilbert module which is naturally isomorphic to

$$(L^2(\mathcal{P}, \pi^*(S_{\partial W} \otimes E_B|_{\partial W})) \hat{\otimes} L^2(S^{2k}; S_{S^{2k}} \otimes \beta))^{\text{Spin}^c(2k)} \quad (4.1)$$

where

- (1)  $\pi : \mathcal{P} \rightarrow \partial W$  is the projection map;
- (2)  $S_W$  and  $S_{S^{2k}}$  are the spinor bundles over  $\partial W$  and  $S^{2k}$  respectively;
- (3)  $\beta$  is the Bott bundle over  $S^{2k}$  (for example, see [1]).

Let  $D_{S^{2k}}$  denote the Dirac operator on  $S^{2k}$  twisted by the Bott bundle. In the proof of Proposition 3.6 in [3], an equivariant, first order, formally self-adjoint differential operator acting on  $L^2(\mathcal{P}; \pi^*(S_{\partial W}))$  is constructed. Let  $R$  denote the operator obtained by twisting this operator by  $\pi^*(E_B|_{\partial W})$ ; results in [3] imply that

$$D_{\partial\hat{W}} = R \hat{\otimes} I + I \hat{\otimes} D_{S^{2k}}$$

The construction of  $R$  depends on the following data (which we list and fix)

- (1) a finite open cover,  $\{U_j\}_{j \in J}$ , of  $\mathcal{P}$  such that  $\mathcal{P}|_{U_j}$  is trivial for each  $j \in J$ ;
- (2) specific choices of trivializations,  $\mathcal{P}|_{U_j} \cong \text{spin}^c(2k) \times U_j$ ;
- (3) a smooth partition of unity subordinate to the cover.

We define  $\hat{A}$  to be  $\pi^*(A) \hat{\otimes} I$ . It is clear that  $\hat{A}$  is a selfadjoint operator, but we must show that the operator

$$D_{\partial\hat{W}} + \hat{A} = (R + \pi^*(A)) \hat{\otimes} I + I \hat{\otimes} D_{S^{2k}}$$

is invertible.

The details are as follows. The operator  $D_{\partial\hat{W}}$  respects the Hilbert module decomposition

$$(L^2(\mathcal{P}, \pi^*(S_{\partial W} \otimes E_B)) \hat{\otimes} L^2(S^{2k}; S_{S^{2k}} \otimes \beta))^{\text{Spin}^c(2k)} \cong \mathcal{E} \oplus \mathcal{E}^\perp$$

where  $\mathcal{E} \cong L^2(\mathcal{P}; \pi^*(S_{\partial W} \otimes E_B))^{\text{Spin}^c(2k)} \otimes \ker(D_{S^{2k}})$ ; the reader can find more details on this decomposition in the proof of Proposition 3.6 of [3]. Combining this identification and the fact that  $\ker(D_{S^{2k}})$  is one dimensional and given by the span of an even section (see [3, Proposition 3.11]), we have that  $D_{\partial\hat{W}} + \hat{A}$  acts as  $D_{\partial W} + A$  on the factor  $\mathcal{E}$ ; hence, it is invertible on this factor.

On the second factor, we have that  $I \hat{\otimes} D_{S^{2k}}$  is invertible (this observation uses the fact that the spectrum of  $D_{S^{2n}}$  is discrete). Using an argument similar to the one used to show invertibility in the case of a trivial  $V$ , it follows that the restriction of  $D_W$  to the second factor is invertible.

This completes the constructions on the boundary; for  $\hat{W}_{\text{cyl}}$ , we proceed as follows. Let  $\mathcal{P}_{\text{cyl}}$  denote the principal  $\text{Spin}^c(2k)$ -bundle associated with the  $\text{spin}^c$ -structure of  $\hat{W}_{\text{cyl}}$ . The Dirac operator acts on

$$(L^2(\mathcal{P}_{\text{cyl}}, \pi^*(S_{W_{\text{cyl}}} \otimes \tilde{E}_B)) \hat{\otimes} L^2(S^{2k}; S_{S^{2k}} \otimes \beta))^{\text{Spin}^c(2k)}$$

where

- (1)  $\pi_{W_{\text{cyl}}} : \mathcal{P}_{\text{cyl}} \rightarrow W_{\text{cyl}}$  is the projection map;
- (2)  $S_{W_{\text{cyl}}}$  is the spinor bundle on  $W_{\text{cyl}}$ ;
- (3)  $\tilde{E}_B$  is the extension of  $E_B$  from  $W$  to  $W_{\text{cyl}}$ ;
- (4) the other data (e.g.,  $S_{S^{2k}}, \beta$ , etc) is as in Equation (4.1) on the previous page.

The construction of the equivariant first order formally self-adjoint differential operator from [3] discussed above can be applied here also (see [3] for further details); it leads to the following:

$$D_{\hat{W}} = R_{\hat{W}} \hat{\otimes} I + I \hat{\otimes} D_{S^{2k}}$$

where

- (1)  $D_{\hat{W}}$  is the Dirac operator on  $W_{\text{cyl}}$  twisted by  $\tilde{E}_B$  (recall that  $\tilde{E}_B$  is the extension of  $E_B$  to  $W_{\text{cyl}}$ );
- (2)  $R_{\hat{W}}$  is the operator constructed in the proof of Proposition 3.6 in [3] (twisted by the relevant bundle);
- (3)  $D_{S^{2k}}$  is the Dirac operator on  $S^{2k}$  twisted by the Bott bundle.

As in the construction on the boundary,  $R_{\hat{W}}$  depends on the choice of a finite open cover, trivialisations, and a partition of unity subordinate to the cover. In addition, we require that this data is compatible with the choices made in the construction of  $R$ . For example, for the open cover (which we denote by  $\{V_i\}_{i \in I}$ ) used to construct  $\hat{R}$ , we assume that

- (1) the bundle,  $\mathcal{P}_{\text{cyl}}|_{V_i}$  is trivial for each  $i \in I$ ;
- (2) for each  $i \in I$ ,  $V_i \cap \partial(\hat{W} \times (-\epsilon, \infty))$  is empty or equal to  $U_j \times (-\epsilon, \infty)$  for some  $j \in J$ ;

The reader should note that although  $\hat{W}_{\text{cyl}}$  is not compact such a cover exists. Similar assumptions are required for the other data used to define  $R_{\hat{W}}$ .

With all this data fixed, we can form operator  $D_{\hat{W}}(\hat{A}) = D_{\hat{W}} - c(dx_1)\hat{\chi}\hat{A}$  where  $\hat{\chi}$  and  $c(dx_1)$  are defined as in the case of a trivial  $V$ . This operator takes the form

$$R_{\hat{W}}(\hat{A}) \otimes I + I \otimes D_{S^{2k}}$$

where the operator  $R_{\hat{W}}(\hat{A})$  is given by  $R_{\hat{W}} - c(dx_1)\hat{\chi}\pi^*(A)$ . Using this decomposition, the proof given in the case when  $V$  is trivial can be generalized to the case of a non-trivial  $V$ ; the details are left to the reader.  $\square$

**4.2. The analytic index map.** For this development, it is more convenient to work with cycles of the form given in Definition 2.2 (i.e., cycles containing bundle data). In fact, we need only consider cycles in  $K_*(pt; \phi)$  since the general index map will be defined by

$$\text{ind}_{\text{ana}} : K_*(X; \phi) \rightarrow K_*(pt; \phi) \rightarrow K_{*+1}(C_\phi)$$

where the first map is defined at the level of cycles via  $(W, (E_{B_2}, F_{B_1}, \alpha), f) \mapsto (W, (E_{B_2}, F_{B_1}, \alpha))$  and the definition of the second map is the main objective of this section; the second map will also be denoted simply as  $\text{ind}_{\text{ana}}$ . To be precise, the geometric data considered in this section is the following. Let

- (1)  $W$  be a compact  $\text{spin}^c$ -manifold with boundary;
- (2)  $E_{B_2}$  be a  $B_2$ -bundle over  $W$ ;
- (3)  $F_{B_1}$  be a  $B_1$ -bundle over  $\partial W$ ;
- (4)  $\alpha : F_{B_1} \otimes_\phi B_2 \rightarrow E_{B_2}$  is an isomorphism;

The starting point for defining this index is the vanishing of index of the boundary operator (see for example [14]). We define the analytic index map from the  $K_*(X; \phi)$  to  $K_{*+1}(C_\phi)$  under the assumption that

$$\phi_* : K_*(B_1) \rightarrow K_*(B_2)$$

is injective. We also note that a relevant example is the case when  $\phi$  is the unital inclusion of the complex numbers into a  $\text{II}_1$ -factor.

Additional geometric data must be fixed to define the higher Atiyah–Patodi–Singer index. Let

- (1)  $g$  denote a Riemannian metric on  $W$  which is a product metric in a neighborhood of  $\partial W$ ;
- (2)  $\nabla_{F_{B_1}}$  a connection compatible with  $g|_{\partial W}$ ;
- (3)  $\nabla_{E_{B_2}}$  a connection which is compatible with  $g$ ,  $\nabla_{F_{B_1}}$ , and the bundle isomorphism  $\alpha$ ;
- (4)  $P$  a spectral section for the operator on the boundary (i.e.,  $D_{\partial W, F_{B_1}}$ );

With all this data fixed, results from [14] imply that there is a well-defined index

$$\text{ind}(D_{W, E_{B_2}}^P) \in K_*(B_2)$$

As an element of  $K_*(B_2)$ , it depends on these choices (e.g., the metric, connections, and spectral section). However, we will show that the image of this class under  $r_* : K_*(B_2) \rightarrow K_{*+1}(C_\phi)$  is independent of these choices.

To do so, a number of properties of the higher Atiyah–Patodi–Singer index are required. These properties are that the higher Atiyah–Patodi–Singer index, spectral flow, and difference construction of spectral sections are each functorial.

The functorial properties of this index are discussed in [17, Appendix C] while for spectral flow and the difference construction the reader can see [20].

To state these properties precisely, additional notation is required. Recall that  $\phi : B_1 \rightarrow B_2$  is a unital  $*$ -homomorphism and  $W$  is a compact  $\text{spin}^c$ -manifold with boundary. Further assume that  $F'_{B_1}$  is a  $B_1$ -bundle over all of  $W$ . Let  $P$  and  $Q$  be spectral sections for  $D_{\partial W, F'_{B_1}|_{\partial W}}$ . The following three properties will be used

$$\phi_*(\text{ind}^{B_1}(D_{W, F'_{B_1}}^P)) = \text{ind}^{B_2}(D_{W, F'_{B_1}}^{\phi_*(P)}) \tag{4.2}$$

$$\phi_*(\text{sf}(D_{\partial W, F'_{B_1}|_{\partial W}, t}; P, Q)) = \text{sf}(D_{\partial W, F'_{B_1}|_{\partial W} \otimes_{\phi} B_2, t}; \phi_*(P), \phi_*(Q)) \tag{4.3}$$

$$\phi_*([P - Q]) = [\phi_*(P) - \phi_*(Q)] \tag{4.4}$$

where

- (1)  $D_{M, E}^P$  denotes the Dirac operator on  $M$  twisted by  $E$  with the boundary conditions associated to the spectral section  $P$ ;
- (2)  $\text{ind}$  denotes the higher Atiyah–Patodi–Singer index;
- (3)  $\text{sf}(\cdot)$  denotes spectral flow (see [20] for further details);
- (4)  $[P - Q] \in K_*(B_1)$  denotes the difference class of  $P$  and  $Q$  (again further details can be found in [15] or [20]);

**Definition 4.4.** Let  $(W, (E_{B_2}, F_{B_1}, \alpha), f)$  be a cycle in  $K_*(X; \phi)$  such that

$$\text{ind}(D_{\partial W, F_{B_1}}) = 0 \in K_{*+1}(B_1)$$

Then,  $\text{ind}_{\text{ana}}(W, (E_{B_2}, F_{B_1}, \alpha), f) := r_*(\text{ind}(D_{W, E_{B_2}}^P)) \in K_{*+1}(C_\phi)$  where  $P$  is any spectral section for  $D_{\partial W, F_{B_1}}$  and  $r_* : K_*(B_2) \rightarrow K_{*+1}(C_\phi)$  is the map on  $K$ -theory induced from the  $*$ -homomorphism  $r : SB_2 \rightarrow C_\phi$ .

**Proposition 4.5.** Let  $(W, (E_{B_2}, F_{B_1}, \alpha), f)$  be a cycle in  $K_*(X; \phi)$  and assume that  $\text{ind}(D_{\partial W, F_{B_1}}) = 0$ . Then, the map  $\text{ind}_{\text{ana}}$  is well-defined as map on (isomorphism classes of) cycles.

*Proof.* A proof that the index map is well-defined at the level of cycles amounts to showing the right-hand side of the equation is independent of the choice of metric, connection, and spectral section used to define the higher Atiyah–Patodi–Singer index. We begin with a special case; let

- (1)  $\{g_t\}_{t \in [0,1]}$  be a one parameter family of Riemannian metrics on  $W$ ;
- (2)  $\nabla_{F_{B_1}, t}$  be a one parameter family of connections on  $F_{B_1}$  which is compatible with  $g_t|_{\partial W}$ ;
- (3)  $\nabla_{E_{B_2}, t}$  be a one parameter family of connections on  $E_{B_2}$  which is compatible with  $g_t$  and with the family of connections  $\nabla_{F_{B_1}, t}$ ;

(4)  $\hat{P}_t$  be a one parameter family of spectral sections for  $D_{\partial W, F_{B_1}}$ .

Set  $P = \phi_*(\hat{P}_t)$ . By functorial properties of spectral sections and the fact that  $E_{B_2}|_{\partial W} \cong F_{B_1} \otimes_{\phi} B_2$ , both  $P_0$  and  $P_1$  are spectral sections for  $D_{\partial W, E_{B_1}|_{\partial W}}$ . Using this data, the following indices are well defined:

$$\text{ind}(D_{W, E_{B_2}}^{P_0}) \text{ and } \text{ind}(D_{W, E_{B_2}}^{P_1})$$

Then, [14, Proposition 8] implies that

$$\text{ind}(D_{W, E_{B_2}}^{P_0}) - \text{ind}(D_{W, E_{B_2}}^{P_1}) = \text{sf}(\{D_{\partial W, (E_{B_2})|_{\partial W}, t}; P_0, P_1\}) \in K_*(B_2)$$

where  $\text{sf}(D_{\partial W, E|_{\partial W}, t}; P_0, P_1)$  is the spectral flow of the family of operators on the boundary (again see [14]). Functorial properties of spectral flow (i.e., Equation 4.3) imply that  $\text{sf}(D_{\partial W, E_{B_2}, t}; P_0, P_1)$  is in the image of  $\phi_*$ . Exactness (i.e.,  $r_* \circ \phi_* = 0$ ) leads to

$$r_*(\text{ind}(D_{W, E_{B_2}}^{P_0})) - r_*(\text{ind}(D_{W, E_{B_2}}^{P_1})) = 0 \in K_{*+1}(C_{\phi})$$

This completes the proof of the special case.

The only difference in the general case is that we cannot assume that the spectral sections,  $\hat{P}_0$  and  $\hat{P}_1$ , are joined via a one-parameter family. However, there does exist a family of spectral section  $\hat{Q}_t$ . As above, set  $P_0 = \phi_*(\hat{P}_0)$ ,  $P_1 = \phi_*(\hat{P}_1)$ , and  $Q_t = \phi_*(\hat{Q}_t)$ . Then, using [14, Proposition 8 and Theorem 8], we have

$$\begin{aligned} &\text{ind}(D_{W, E_{B_2}}^{P_0}) - \text{ind}(D_{W, E_{B_2}}^{P_1}) \\ &= \text{ind}(D_{W, E_{B_2}}^{P_0}) - \text{ind}(D_{W, E_{B_2}}^{P_1}) - \text{ind}(D_{W, E_{B_2}}^{Q_0}) + \text{ind}(D_{W, E_{B_2}}^{Q_1}) \\ &\quad + \text{sf}(\{D_{\partial W, (E_{B_2})|_{\partial W}, t}; Q_0, Q_1\}) \\ &= [Q_0 - P_0] + [P_1 - Q_0] + \text{sf}(\{D_{\partial W, (E_{B_2})|_{\partial W}, t}; Q_0, Q_1\}) \end{aligned}$$

Applying  $r_*$  to this equation and using the functorial properties of the difference classes and spectral flow leads to

$$\begin{aligned} &r_*(\text{ind}(D_{W, E_{B_2}}^{P_0})) - r_*(\text{ind}(D_{W, E_{B_2}}^{P_1})) \\ &= (r_* \circ \phi_*)([Q_0 - P_0] + [P_1 - Q_0] + \text{sf}(\{D_{\partial W, (F_{B_1})|_{\partial W}, t}; \hat{Q}_0, \hat{Q}_1\})) \end{aligned}$$

Exactness then implies the result. □

**Theorem 4.6.** *If  $\phi_* : K_*(B_1) \rightarrow K_*(B_2)$  is injective, then the analytic index map (see Definition 4.4) induces a well-defined map  $K_*(X; \phi) \rightarrow K_{*+1}(C_{\phi})$ .*

*Proof.* The injective of  $\phi_*$  implies that the conditions of Proposition 4.5 are satisfied for any cycle in  $K_*(X; \phi)$ . Thus the index map is well-defined at the level of cycles. We need to show that the map respects the three relations.

**Disjoint union/direct sum.** This follows from basic properties of the higher Atiyah–Patodi–Singer index.

**Bordism.** Let  $(Z, W, (E'_{B_2}, F'_{B_1}, \alpha'), g)$  be a bordism and  $(W, (E_{B_2}, F_{B_1}, \alpha), f)$  denote its boundary. Denote by  $(M, V_{B_1}, h)$  the  $K_*(X; B_1)$ -bordism obtained by restricting the given data to the  $\text{spin}^c$  manifold with boundary,  $\partial W - \text{int}(W)$ . Let  $P$  and  $Q$  be spectral sections for  $D_{\partial W, F_{B_1}}$  and  $D_{\partial M, V_{B_1}|_{\partial M}}$  respectively and  $\tilde{P}$  and  $\tilde{Q}$  denote the spectral sections (for  $D_{\partial W, F_{B_1} \otimes_{\phi} B_2}$  and  $D_{\partial M, V_{B_1}|_{\partial M} \otimes_{\phi} B_2}$  respectively) obtained via the  $*$ -homomorphism  $\phi$ . Using [14, Theorem 8] and the functorial properties listed above, the indices on the various manifolds (we suppress the bundle data from the notation) involved are related via

$$\begin{aligned} \text{ind}^{B_2}(D_{\tilde{W}}^{\tilde{P}}) + \phi_*(\text{ind}^{B_1}(D_M^{I-Q})) &= \text{ind}^{B_2}(D_{\tilde{W}}^{\tilde{P}}) + \text{ind}^{B_2}(D_M^{I-\tilde{Q}}) \\ &= \text{ind}_{AS}^{B_2}(D_{W \cup M}) + [\tilde{P} - \tilde{Q}] \\ &= \text{ind}_{AS}^{B_2}(D_{W \cup M}) + \phi_*([P - Q]) \end{aligned}$$

The fact that  $r_* \circ \phi_* = 0$  implies that

$$\mu_{\text{ana}}(W, (E_{B_2}, F_{B_1}, \alpha), f) = r_*(\text{ind}^{B_2}(D_W)) = r_*(\text{ind}_{AS}^{B_2}(D_{W \cup M}))$$

Finally, the bordism invariance of the Mishchenko–Fomenko index and the fact that  $W \cup M = \partial Z$  (the bundles respect this bordism) imply that the right-hand side of this equation vanishes. This proves the required bordism invariance.

**Vector bundle modification.** Let  $(W, (E_{B_2}, F_{B_1}, \alpha), f)$  denote a cycle and  $V$  a  $\text{spin}^c$ -vector bundle of even rank over  $W$ . Since the higher Atiyah–Patodi–Singer index respects disjoint union, we may assume that  $W$  is connected. Using Proposition 4.2, we have that

$$\text{ind}(D_W(A)) = \text{ind}(D_{\hat{W}}(\hat{A})) \in K_*(B_2)$$

where we have used the notation of Proposition 4.2. However, the definition of  $\mu_{\text{ana}}$  is independent of the choice of spectral section (see Proposition 4.5). As such,

$$\begin{aligned} \mu_{\text{ana}}(W, (E_{B_2}, F_{B_1}, \alpha), f) &= r_*(\text{ind}^{B_2}(D_W(A))) \\ &= r_*(\text{ind}^{B_2}(D_{\hat{W}}(\hat{A}))) \\ &= \mu_{\text{ana}}(W, (E_{B_2}, F_{B_1}, \alpha), f)^V \quad \square \end{aligned}$$

**Theorem 4.7.** *Suppose that  $\phi_* : K_*(B_1) \rightarrow K_*(B_2)$  is injective (so that analytic index is well-defined). Then the topological index and analytic index are equal. In particular, in the case of  $X = \text{pt}$ , the analytic index is an isomorphism.*

*Proof.* The second statement in the theorem follows from the first and the fact that the topological index is an isomorphism in the case of a point. To prove the first statement, note that both the topological index and analytic index factor through the map

$$K_*(X; \phi) \rightarrow K_*(pt; \phi)$$

defined at the level of cycles via  $(W, \xi, f) \mapsto (W, \xi)$ . Thus, we need only show that they give the same isomorphism from  $K_*(pt; \phi)$  to  $K_{*+1}(C_\phi)$ . Using Theorem 2.7, we have that exactness and the injectivity of  $\phi_*$  imply that the map  $r : K_*(pt; B_2) \rightarrow K_{*+1}(pt; \phi)$  is onto. This implies that given a cycle  $(W, \xi) \in K_*(pt; \phi)$  there exists *closed* compact  $\text{spin}^c$ -manifold  $M$  and  $\eta \in K^0(M; B_2)$  such that  $r(M, \eta) \sim (W, \xi)$  (in the group  $K_{*+1}(pt; \phi)$ ). The theorem now follows, since both the topological and analytic index of  $(W, \xi)$  are equal to  $\tilde{r} \circ \text{ind}_{K_*(B_2)}(M, \eta)$  where  $\text{ind}_{K_*(B_2)}(M, \eta)$  denotes the Mishchenko–Fomenko index and  $\tilde{r} : K_*(B_2) \rightarrow K_{*+1}(C_\phi)$  is the map on  $K$ -theory induced from the natural  $*$ -homomorphism  $SB_2 \rightarrow C_\phi$ .  $\square$

**Remark 4.8.** Under the assumptions in the statement of the previous theorem, its proof implies that *any* index map  $K_*(X; \phi) \rightarrow KK^*(\mathbb{C}, SC_\phi)$  which agrees with the Mishchenko–Fomenko index on cycles without boundary is equal to the topological index map. In particular, this statement holds (up to a factor of  $-1$ ) for the index map discussed in [6] for the special case when  $\phi$  is the unital inclusion of the complex number into a  $\text{II}_1$ -factor. Note that since the index map discussed in [6] takes values in  $\mathbb{R}/\mathbb{Z}$ , we must fix the isomorphism from  $KK(\mathbb{C}, SC_\phi)$  to  $\mathbb{R}/\mathbb{Z}$  to be the one compatible with isomorphism from  $KK(\mathbb{C}, N)$  to  $\mathbb{R}$  defined via the trace of the  $\text{II}_1$ -factor,  $N$ .

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