

Hopf-cyclic cohomology of quantum enveloping algebras

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Abstract. In this paper we calculate both the periodic and non-periodic Hopf-cyclic cohomology of Drinfeld–Jimbo quantum enveloping algebra $U_q(\mathfrak{g})$ for an arbitrary semi-simple Lie algebra \mathfrak{g} with coefficients in a modular pair in involution. We obtain this result by showing that the coalgebra Hochschild cohomology of these Hopf algebras are concentrated in a single degree determined by the rank of the Lie algebra \mathfrak{g} .

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1. Introduction

In this paper we calculate the Hopf-cyclic cohomology of Drinfeld–Jimbo quantum enveloping algebra $U_q(\mathfrak{g})$ for an arbitrary semi-simple Lie algebra \mathfrak{g} with coefficients in a modular pair in involution (MPI) ${}^\sigma k_\varepsilon$. This cohomology was previously calculated only for sl_2 by Crainic in [7]. We also verified the original calculations of Moscovici–Rangipour [26] of the Hopf-cyclic cohomology of the Connes–Moscovici Hopf algebras \mathcal{H}_1 and \mathcal{H}_{1S} with coefficients in the trivial MPI ${}^1k_\varepsilon$ using our new cohomological machinery.

The calculation of the Hopf-cyclic cohomology for Connes–Moscovici Hopf algebras \mathcal{H}_n is a big challenge. These Hopf algebras are designed to calculate the characteristic classes of codimension- n foliations [4]. The intricate calculations of Hopf-cyclic cohomology of these Hopf algebras by Moscovici and Rangipour used crucially the fact that these Hopf algebras are bicrossed product Hopf algebras [13, 24, 26–28]. On top of previously calculated explicit classes of \mathcal{H}_1 , explicit representatives of Hopf-cyclic cohomology classes of \mathcal{H}_2 are recently obtained in [32] via a cup product with a SAYD-twisted cyclic cocycle. More recently, Moscovici gave a geometric approach for \mathcal{H}_n in [25] using explicit quasi-isomorphisms between the Hopf-cyclic complex of \mathcal{H}_n , Dupont’s simplicial de Rham DG-algebra [9], and the Bott complex [1].

The cohomological machinery we developed in this paper, allowed us to replicate the results of [26] on the Hopf-cyclic cohomology of the Hopf algebra \mathcal{H}_1 , and its Schwarzian quotient \mathcal{H}_{1S} . To this end, we start by calculating in Proposition 4.3 the cohomologies of the weight 1 subcomplex of the coalgebra Hochschild complexes of these Hopf algebras using their canonical grading, but without appealing their bicrossed product structure. Then we use the Cartan homotopy formula for \mathcal{H}_1 , as developed in [26], to obtain the periodic Hopf-cyclic cohomology groups with coefficients in the trivial, which happens to be the only finite dimensional, SAYD module for these Hopf algebras [31].

On the quantum enveloping algebra side there are several computations [11, 17, 29, 30] on the Ext-groups. However, the literature on Hopf-cyclic cohomology, or even coalgebraic cohomology of any variant, of quantum enveloping algebras is rather meek. The only results we are aware of are both for $U_q(\mathfrak{sl}_2)$: one for the ordinary Hopf-cyclic cohomology by Crainic [7], and one for the dual Hopf-cyclic cohomology by Khalkhali and Rangipour [21].

The central result we achieve in this paper is the computation of both the periodic and non-periodic Hopf-cyclic cohomology of the quantized enveloping algebra $U_q(\mathfrak{g})$ in full, for an arbitrary semisimple Lie algebra \mathfrak{g} . In Theorem 4.8 we first calculate the coalgebra Hochschild cohomology of $U_q(\mathfrak{g})$ with coefficients in the comodule ${}^\sigma k_\varepsilon$ of Klimyk–Schmüdgen [22, Prop. 6.6], which is in fact an MPI over $U_q(\mathfrak{g})$. We observe that the Hochschild cohomology is concentrated in a single degree determined by the rank of the Lie algebra \mathfrak{g} , and finally we calculate the periodic and non-periodic Hopf-cyclic cohomology groups of $U_q(\mathfrak{g})$ in Theorem 4.10.

One of the important implications of Theorem 4.8 is that we now have candidates for noncommutative analogues of the Haar functionals for $U_q(\mathfrak{g})$. The fact that coalgebra Hochschild cohomology of $U_q(\mathfrak{sl}_2)$ is concentrated only in a single degree was first observed by Crainic in [7]. The dual version of the statement, that is the algebra Hochschild homology of $k_q[SL(N)]$ the quantized coordinate ring of $SL(N)$ with coefficients twisted by the modular automorphism σ of the Haar functional is also concentrated in a single degree, is proven by Hadfield and Kraehmer in [12]. Kraehmer used this fact to prove an analogue of the Poincaré duality for Hochschild homology and cohomology for $k_q[SL(N)]$ in [23]. We plan on investigating the ramification of the fact that Hochschild cohomology of $U_q(\mathfrak{g})$ is concentrated in a single degree, and its connections with the dimension-drop phenomenon and twisted Calabi–Yau coalgebras [16], in a future paper.

2. Preliminaries

In this section we recall basic material that will be needed in the sequel. More explicitly, in the first subsection our objective is to recall the coalgebra Hochschild cohomology. To this end we also bring the definitions of the cobar complex of a

coalgebra, and hence the Cotor-groups. The second subsection, on the other hand, is devoted to a very brief summary of the Hopf-cyclic cohomology with coefficients.

2.1. Cobar and Hochschild complexes. In this subsection we recall the definition of the cobar complex of a coalgebra \mathcal{C} , and it is followed by the definition of the Cotor-groups associated to a coalgebra \mathcal{C} and a pair (V, W) of \mathcal{C} -comodules of opposite parity.

Let \mathcal{C} be a coassociative coalgebra. Following [3, 8] and [20], the cobar complex of \mathcal{C} is defined to be the differential graded space

$$\mathbf{CB}^*(\mathcal{C}) := \bigoplus_{n \geq 0} \mathcal{C}^{\otimes n+2}$$

with the differential

$$\begin{aligned} d : \mathbf{CB}^n(\mathcal{C}) &\longrightarrow \mathbf{CB}^{n+1}(\mathcal{C}) \\ d(c^0 \otimes \cdots \otimes c^{n+1}) &= \sum_{j=0}^n (-1)^j c^0 \otimes \cdots \otimes \Delta(c^j) \otimes \cdots \otimes c^{n+1}. \end{aligned}$$

Let $\mathcal{C}^e := \mathcal{C} \otimes \mathcal{C}^{\text{cop}}$ be the enveloping coalgebra of \mathcal{C} . In case \mathcal{C} is counital, the cobar complex $\mathbf{CB}^*(\mathcal{C})$ yields a \mathcal{C}^e -injective resolution of the (left) \mathcal{C}^e -comodule \mathcal{C} , [8].

Following the terminology of [19], for a pair (V, W) of two \mathcal{C} -comodules of opposite parity (say, V is a right \mathcal{C} -comodule, and W is a left \mathcal{C} -comodule), we call the complex

$$(\mathbf{CB}^*(V, \mathcal{C}, W), d), \quad \mathbf{CB}^*(V, \mathcal{C}, W) := V \square_{\mathcal{C}} \mathbf{CB}^*(\mathcal{C}) \square_{\mathcal{C}} W$$

where $d : \mathbf{CB}^n(V, \mathcal{C}, W) \longrightarrow \mathbf{CB}^{n+1}(V, \mathcal{C}, W)$

$$\begin{aligned} d(v \otimes c^1 \otimes \cdots \otimes c^n \otimes w) &= v_{\langle 0 \rangle} \otimes v_{\langle 1 \rangle} \otimes c^1 \otimes \cdots \otimes c^n \otimes w \\ &\quad + \sum_{j=1}^n (-1)^j v \otimes c^1 \otimes \cdots \otimes \Delta(c^j) \otimes \cdots \otimes c^n \otimes w \\ &\quad + (-1)^{n+1} v \otimes c^1 \otimes \cdots \otimes c^n \otimes w_{\langle -1 \rangle} \otimes w_{\langle 0 \rangle}, \end{aligned} \quad (2.1)$$

the two-sided (cohomological) cobar complex of the coalgebra \mathcal{C} .

The Cotor-groups of a pair (V, W) of \mathcal{C} -comodules of opposite parity are defined by

$$\text{Cotor}_{\mathcal{C}}^*(V, W) := H_*(V \square_{\mathcal{C}} \bar{Y}(\mathcal{C}) \square_{\mathcal{C}} W, d),$$

where $\bar{Y}(\mathcal{C})$ is an injective resolution of \mathcal{C} via \mathcal{C} -bicomodules. In case \mathcal{C} is a counital coalgebra one has

$$\text{Cotor}_{\mathcal{C}}^*(V, W) = H_*(\mathbf{CB}^*(V, \mathcal{C}, W), d). \quad (2.2)$$

We next recall the Hochschild cohomology of a coalgebra \mathcal{C} with coefficients in the \mathcal{C} -bicomodule (equivalently \mathcal{C}^e -comodule) V , from [8], as the homology of the complex

$$\mathbf{CH}^*(\mathcal{C}, V) = \bigoplus_{n \geq 0} \mathbf{CH}^n(\mathcal{C}, V), \quad \mathbf{CH}^n(\mathcal{C}, V) := V \otimes \mathcal{C}^{\otimes n}$$

with the differential $b : \mathbf{CH}^n(\mathcal{C}, V) \rightarrow \mathbf{CH}^{n+1}(\mathcal{C}, V)$

$$\begin{aligned} b(v \otimes c^1 \otimes \cdots \otimes c^n) &= v_{\langle 0 \rangle} \otimes v_{\langle 1 \rangle} \otimes c^1 \otimes \cdots \otimes c^n \\ &\quad + \sum_{k=1}^n (-1)^k v \otimes c^1 \otimes \cdots \otimes \Delta(c^k) \otimes \cdots \otimes c^n \\ &\quad + (-1)^{n+1} v_{\langle 0 \rangle} \otimes c^1 \otimes \cdots \otimes c^n \otimes v_{\langle -1 \rangle}. \end{aligned} \quad (2.3)$$

Identification

$\mathbf{CB}^n(\mathcal{C}) \cong \mathcal{C}^e \otimes \mathcal{C}^{\otimes n}$, $n > 0$, $c^0 \otimes \cdots \otimes c^{n+1} \mapsto (c^0 \otimes c^{n+1}) \otimes c^1 \otimes \cdots \otimes c^n$ as left \mathcal{C}^e -comodules, where the left \mathcal{C}^e -comodule structure on $\mathcal{C}^{\otimes n+2}$ is given by $\nabla(c^0 \otimes \cdots \otimes c^{n+1}) = (c^0_{(1)} \otimes c^{n+1}_{(2)}) \otimes (c^0_{(2)} \otimes c^1 \otimes \cdots \otimes c^n \otimes c^{n+1}_{(1)})$, and on $\mathcal{C}^e \otimes \mathcal{C}^{\otimes n}$ by $\nabla((c \otimes c') \otimes (c^1 \otimes \cdots \otimes c^n)) = (a_{(1)} \otimes c'_{(2)}) \otimes (a_{(2)} \otimes c'_{(1)}) \otimes (c^1 \otimes \cdots \otimes c^n)$, yields

$$(\mathbf{CH}^*(\mathcal{C}, V), b) \cong (V \square_{\mathcal{C}^e} \mathbf{CB}^*(\mathcal{C}), d).$$

Hence, in case \mathcal{C} is counital one can interpret the Hochschild cohomology of \mathcal{C} , with coefficients in V , in terms of Cotor-groups as

$$HH^*(\mathcal{C}, V) = H_*(\mathbf{CH}^*(\mathcal{C}, V), b) = \text{Cotor}_{\mathcal{C}^e}^*(V, \mathcal{C}),$$

or more generally,

$$HH^*(\mathcal{C}, V) = H_*(V \square_{\mathcal{C}^e} \bar{Y}(\mathcal{C}), d)$$

for any injective resolution $\bar{Y}(\mathcal{C})$ of \mathcal{C} via left \mathcal{C}^e -comodules.

2.2. Hopf-cyclic cohomology of Hopf algebras. In this subsection we recall the basics of the Hopf-cyclic cohomology theory for Hopf algebras from [4, 6]. To this end we start with the coefficient spaces for this homology theory, the stable anti-Yetter–Drinfeld (SAYD) modules.

Let \mathcal{H} be a Hopf algebra. A right \mathcal{H} -module, left \mathcal{H} -comodule V is called an anti-Yetter–Drinfeld (AYD) module over \mathcal{H} if

$$\blacktriangledown(v \cdot h) = S(h_{(3)})v_{\langle -1 \rangle} h_{(1)} \otimes v_{\langle 0 \rangle} \cdot h_{(2)},$$

for any $v \in V$ and any $h \in \mathcal{H}$, and V is called stable if

$$v_{\langle 0 \rangle} \cdot v_{\langle -1 \rangle} = v$$

for any $v \in V$.

In particular, the field k , regarded as an \mathcal{H} -module by a character $\delta : \mathcal{H} \rightarrow k$, and a \mathcal{H} -comodule via a group-like $\sigma \in \mathcal{H}$, is an AYD module over \mathcal{H} if

$$S_\delta^2 = \text{Ad}_\sigma, \quad S_\delta(h) = \delta(h_{(1)})S(h_{(2)}),$$

and it is stable if

$$\delta(\sigma) = 1.$$

Such a pair (δ, σ) is called a modular pair in involution (MPI), [5, 15].

Let V be a right-left SAYD module over a Hopf algebra \mathcal{H} . Then

$$C^*(\mathcal{H}, V) = \bigoplus_{n \geq 0} C^n(\mathcal{H}, V), \quad C^n(\mathcal{H}, V) := V \otimes \mathcal{H}^{\otimes n}$$

is a cocyclic module [14], via the face operators

$$\begin{aligned} d_i &: C^n(\mathcal{H}, V) \rightarrow C^{n+1}(\mathcal{H}, V), \quad 0 \leq i \leq n+1 \\ d_0(v \otimes h^1 \otimes \cdots \otimes h^n) &= v \otimes 1 \otimes h^1 \otimes \cdots \otimes h^n, \\ d_i(v \otimes h^1 \otimes \cdots \otimes h^n) &= v \otimes h^1 \otimes \cdots \otimes h^{i(1)} \otimes h^{i(2)} \otimes \cdots \otimes h^n, \\ d_{n+1}(v \otimes h^1 \otimes \cdots \otimes h^n) &= v_{\langle 0 \rangle} \otimes h^1 \otimes \cdots \otimes h^n \otimes v_{\langle -1 \rangle}, \end{aligned}$$

the degeneracy operators

$$\begin{aligned} s_j &: C^n(\mathcal{H}, V) \rightarrow C^{n-1}(\mathcal{H}, V), \quad 0 \leq j \leq n-1 \\ s_j(v \otimes h^1 \otimes \cdots \otimes h^n) &= v \otimes h^1 \otimes \cdots \otimes \varepsilon(h^{j+1}) \otimes \cdots \otimes h^n, \end{aligned}$$

and the cyclic operator

$$\begin{aligned} t &: C^n(\mathcal{H}, V) \rightarrow C^n(\mathcal{H}, V), \\ t(v \otimes h^1 \otimes \cdots \otimes h^n) &= v_{\langle 0 \rangle} \cdot h^{1(1)} \otimes S(h^{1(2)}) \cdot (h^2 \otimes \cdots \otimes h^n \otimes v_{\langle -1 \rangle}). \end{aligned}$$

The total cohomology of the associated first quadrant bicomplex $(CC^{*,*}(\mathcal{H}, V), b, B)$, [27], where

$$CC^{p,q}(\mathcal{H}, V) := \begin{cases} C^{q-p}(\mathcal{H}, V), & \text{if } q \geq p \geq 0, \\ 0, & \text{if } p > q, \end{cases}$$

with the coalgebra Hochschild coboundary

$$b : CC^{p,q}(\mathcal{H}, V) \rightarrow CC^{p,q+1}(\mathcal{H}, V), \quad b := \sum_{i=0}^q (-1)^i d_i,$$

and the Connes boundary operator

$$B : CC^{p,q}(\mathcal{H}, V) \rightarrow CC^{p-1,q}(\mathcal{H}, V), \quad B := \left(\sum_{i=0}^p (-1)^{ni} t^i \right) s_{p-1} t,$$

is called the Hopf-cyclic cohomology of the Hopf algebra \mathcal{H} with coefficients in the SAYD module V , and is denoted by $HC(\mathcal{H}, V)$.

Finally, the periodic Hopf-cyclic cohomology is defined similarly as the total complex of the bicomplex

$$CC^{p,q}(\mathcal{H}, V) := \begin{cases} C^{q-p}(\mathcal{H}, V), & \text{if } q \geq p \\ 0, & \text{if } p > q, \end{cases}$$

and is denoted by $HP(\mathcal{H}, V)$.

3. The cohomological machinery

This section contains the main computational tool of the present paper, namely, given a coalgebra coextension $C \rightarrow D$ with a coflatness condition, we compute the coalgebra Hochschild cohomology of C by means of the Hochschild cohomology of D on the E_1 -term of a spectral sequence.

Let a coextension $\pi : C \rightarrow D$ be given. We first introduce the auxiliary coalgebra $Z := C \oplus D$ with the comultiplication

$$\Delta(y) = y_{(1)} \otimes y_{(2)} \quad \text{and} \quad \Delta(x) = x_{(1)} \otimes x_{(2)} + \pi(x_{(1)}) \otimes x_{(2)} + x_{(1)} \otimes \pi(x_{(2)})$$

and the counit

$$\varepsilon(x + y) = \varepsilon(y),$$

for any $x \in C$ and $y \in D$.

Next, let V be a C -bicomodule, and let C be coflat both as a left and a right D -comodule. Then consider the decreasing filtration

$$G_p^{p+q} = \begin{cases} \bigoplus_{n_0+\dots+n_p=q} V \otimes Z^{\otimes n_0} \otimes D \otimes \dots \otimes Z^{\otimes n_{p-1}} \otimes D \otimes Z^{\otimes n_p}, & p \geq 0 \\ 0, & p < 0 \end{cases}$$

on $\mathbf{CH}^*(Z, V)$. In the associated spectral sequence we get

$$E_0^{i,j} = G_i^{i+j} / G_{i+1}^{i+j} = \bigoplus_{n_0+\dots+n_i=j} V \otimes C^{\otimes n_0} \otimes D \otimes \dots \otimes C^{\otimes n_{i-1}} \otimes D \otimes C^{\otimes n_i},$$

which gives us

$$E_1^{0,j} = HH^j(C, V).$$

On the horizontal differential however, by the definition of the filtration we use only the D -bicomodule structure on C . Hence, by the coflatness assumption

$$E_2^{i,j} = 0, \quad i > 0.$$

As a result, the spectral sequence collapses and we get

$$HH^n(Z, V) \cong HH^n(C, V), \quad n \geq 0. \quad (3.1)$$

Alternatively, one can use the short exact sequence

$$0 \rightarrow D \xrightarrow{i} Z \xrightarrow{p} C \rightarrow 0 \quad \text{where } i : y \mapsto (0, y), p : (x, y) \mapsto x$$

of coalgebras, and [10, Lemma 4.10], to conclude (3.1).

Now consider $\mathbf{CH}^*(Z, V)$, this time with the decreasing filtration

$$F_p^{n+p} = \begin{cases} \bigoplus_{n_0+\dots+n_p=n} V \otimes Z^{\otimes n_0} \otimes C \otimes \dots \otimes Z^{n_{p-1}} \otimes C \otimes Z^{\otimes n_p}, & p \geq 0 \\ 0, & p < 0. \end{cases}$$

The associated spectral sequence is

$$E_0^{i,j} = F_i^{i+j} / F_{i+1}^{i+j} = \bigoplus_{n_0+\dots+n_i=j} V \otimes D^{\otimes n_0} \otimes C \otimes \dots \otimes D^{n_{i-1}} \otimes C \otimes D^{\otimes n_i},$$

and by the coflatness assumption, on the vertical direction it computes

$$HH^j(D, C^{\square_D i} \square_D V).$$

We can summarize our discussion in the following theorem.

Theorem 3.1. *Let $\pi : C \rightarrow D$ be a coalgebra projection, V a C -bicomodule, and C be coflat both as a left and a right D -comodule. Then there is a spectral sequence, whose E_1 -term is*

$$E_1^{i,j} = HH^j(D, C^{\square_D i} \square_D V),$$

converging to $HH^{i+j}(C, V)$.

4. Computations

In this section we will apply the cohomological machinery developed in Section 3 to compute the Hopf-cyclic cohomology groups of quantized enveloping algebras, Connes–Moscovici Hopf algebra \mathcal{H}_1 and its Schwarzian quotient \mathcal{H}_{1S} .

To this end, we will compute the Cotor-groups with MPI coefficients, from which we will obtain coalgebra Hochschild cohomology groups in view of [7, Lemma 5.1]. Therefore we note the following analogue of Theorem 3.1.

Theorem 4.1. *Let $\pi : C \rightarrow D$ be a coalgebra projection, and $V = V' \otimes V''$ a C -bicomodule such that the left C -comodule structure is given by V' and the right C -comodule structure is given by V'' . Let also C be coflat both as a left and a right D -comodule. Then there is a spectral sequence, whose E_1 -term is of the form*

$$E_1^{i,j} = \text{Cotor}_D^j(V'', C^{\square_D i} \square_D V'),$$

converging to $HH^{i+j}(C, V)$.

Proof. The proof follows from Theorem 3.1 in view of the definition (2.1) of the coboundary map of a cobar complex, and (2.2). \square

Let us also recall the principal coextensions from [33], see also [2]. Let H be a Hopf algebra with a bijective antipode, and C a left H -module coalgebra. Moreover, H^+ being the augmentation ideal of H (that is, $H^+ := \ker \varepsilon$), define the (quotient) coalgebra $D := C/H^+C$.

Then, by [33, Theorem II],

- (a) C is a projective left H -module,
- (b) $\text{can} : H \otimes C \longrightarrow C \square_D C, h \otimes c \mapsto h \cdot c_{(1)} \otimes c_{(2)}$ is injective,

if and only if

- (a) C is faithfully flat left (and right) D -comodule,
- (b) $\text{can} : H \otimes C \longrightarrow C \square_D C$ is an isomorphism.

We will use this set-up to meet the hypothesis of Theorem 3.1 (and Theorem 4.1).

4.1. Connes–Moscovici Hopf algebras. In this subsection we will compute the periodic Hopf-cyclic cohomology groups of the Connes–Moscovici Hopf algebra \mathcal{H}_1 and its Schwarzian quotient \mathcal{H}_{1S} using the spectral sequence introduced in Theorem 4.1, and the Cartan homotopy formula developed in [26].

We will use our main machinery to recover the results of [26] on the periodic Hopf-cyclic cohomology of the Connes–Moscovici Hopf algebra \mathcal{H}_1 . Therefore, in this subsection we will take a quick detour to the Hopf algebra \mathcal{H}_1 of codimension 1, and its Schwarzian quotient \mathcal{H}_{1S} from [4, 6, 26].

Let $F\mathbb{R} \longrightarrow \mathbb{R}$ be the frame bundle over \mathbb{R} , equipped with the flat connection whose fundamental vertical vector field is

$$Y = y \frac{\partial}{\partial y},$$

and the basic horizontal vector field is

$$X = y \frac{\partial}{\partial x},$$

in local coordinates of $F\mathbb{R}$. They act on the crossed product algebra $\mathcal{A} = C_c^\infty(F\mathbb{R}) \rtimes \text{Diff}(\mathbb{R})$, a typical element of which is written by $fU_\varphi^* := f \rtimes \varphi^{-1}$, via

$$Y(fU_\varphi^*) = Y(f)U_\varphi^*, \quad X(fU_\varphi^*) = X(f)U_\varphi^*.$$

Then \mathcal{H}_1 is the unique Hopf algebra that makes \mathcal{A} to be a (left) \mathcal{H}_1 -module algebra. To this end one has to introduce the further differential operators

$$\delta_n(fU_\varphi^*) := y^n \frac{d}{dx^n} (\log \varphi'(x)) fU_\varphi^*, \quad n \geq 1,$$

and the Hopf algebra structure of \mathcal{H}_1 is given by

$$\begin{aligned} [Y, X] &= X, & [Y, \delta_n] &= n\delta_n, & [X, \delta_n] &= \delta_{n+1}, & [\delta_n, \delta_m] &= 0, \\ \Delta(Y) &= Y \otimes 1 + 1 \otimes Y, \\ \Delta(\delta_1) &= \delta_1 \otimes 1 + 1 \otimes \delta_1, \\ \Delta(X) &= X \otimes 1 + 1 \otimes X + \delta_1 \otimes Y, \\ \varepsilon(X) &= \varepsilon(Y) = \varepsilon(\delta_n) = 0, \\ S(X) &= -X + \delta_1 Y, & S(Y) &= -Y, & S(\delta_1) &= -\delta_1. \end{aligned}$$

The ideal generated by the Schwarzian derivative

$$\delta'_2 := \delta_2 - \frac{1}{2}\delta_1^2,$$

is a Hopf ideal (an ideal, a coideal and is stable under the antipode), therefore the quotient \mathcal{H}_{1S} becomes a Hopf algebra, called the Schwarzian Hopf algebra. As an algebra \mathcal{H}_{1S} is generated by X, Y, Z , and the Hopf algebra structure is given by

$$\begin{aligned} [Y, X] &= X, & [Y, \delta_n] &= n\delta_n, & [X, Z] &= \frac{1}{2}Z^2, \\ \Delta(Y) &= Y \otimes 1 + 1 \otimes Y, \\ \Delta(Z) &= Z \otimes 1 + 1 \otimes Z, \\ \Delta(X) &= X \otimes 1 + 1 \otimes X + Z \otimes Y, \\ \varepsilon(X) &= \varepsilon(Y) = \varepsilon(Z) = 0, \\ S(X) &= -X + ZY, & S(Y) &= -Y, & S(Z) &= -Z. \end{aligned}$$

Hence

$$\mathcal{F} := \text{Span} \{ \delta_{\alpha_1}^{n_1} \cdots \delta_{\alpha_p}^{n_p} \mid p, \alpha_1, \dots, \alpha_p \geq 1, n_1, \dots, n_p \geq 0 \} \subseteq \mathcal{H}_1$$

is a Hopf subalgebra of \mathcal{H}_1 . Finally let us note, by [31], that the only modular pair in involution (MPI) on \mathcal{H}_1 is $(\delta, 1)$ of [4], where $\delta : g\ell_1^{\text{aff}} \rightarrow k$ is the trace of the adjoint representation of $g\ell_1^{\text{aff}}$ on itself.

Let $C := \mathcal{H}_1$, and let us consider the Hopf subalgebra $\mathcal{F} \subseteq C$. Then

$$D = C/\mathcal{F}^+C = \mathcal{U} := U(g\ell_1^{\text{aff}}),$$

see [27, Lemma 3.19], or [6, Section 5]. Hence, by [33, Thm. II] we conclude that C is (faithfully) coflat as left and right D -comodule. Therefore, the hypothesis of Theorem 4.1 is satisfied.

Since $\sigma k = k$, we have

$$C^{\square_D} \square_D k = \mathcal{F}^{\otimes i},$$

and hence by Theorem 3.1,

$$E_1^{i,j} = HH^j(\mathcal{U}, \mathcal{F}^{\otimes i}) \Rightarrow HH^{i+j}(\mathcal{H}_1, k).$$

Moreover, since the \mathcal{U} -coaction on \mathcal{F} is trivial, we have

$$E_1^{i,j} = HH^j(\mathcal{U}, k) \otimes \mathcal{F}^{\otimes i} \Rightarrow HH^{i+j}(\mathcal{H}_1, k).$$

As a result of the Cartan homotopy formula [26, Coroll. 3.9] for \mathcal{H}_1 , one has [26, Coroll. 3.10] as recalled below. We adopt the same notation from [26], and we denote by $HP(\mathcal{H}_{1\mathfrak{h}}[p], k)$ the periodic Hopf-cyclic cohomology, with trivial coefficients, of the weight p subcomplex of \mathcal{H}_1 , with respect to the grading given by the adjoint action of $Y \in \mathcal{H}_1$.

Corollary 4.2. *The periodic Hopf-cyclic cohomology groups of \mathcal{H}_1 are computed by the weight 1 subcomplex, i.e.*

$$HP(\mathcal{H}_{1\mathfrak{h}}[1], k) = HP(\mathcal{H}_1, k), \quad HP(\mathcal{H}_{1\mathfrak{h}}[p], k) = 0, \quad p \neq 1.$$

Since our spectral sequence respects the weight, in view of [18, Thm. 18.7.1] we check only

$$\begin{aligned} E_1^{1,0} &= \langle \mathbf{1} \otimes \delta_1 \otimes \mathbf{1} \rangle \in k \otimes C \square_D k, \\ E_1^{1,1} &= \langle \mathbf{1} \otimes \bar{Y} \otimes \delta_1 \otimes \mathbf{1} \rangle \in k \otimes D \otimes C \square_D k, \\ E_1^{0,1} &= \langle \mathbf{1} \otimes \bar{X} \otimes \mathbf{1} \rangle \in k \otimes D \otimes k, \\ E_1^{0,2} &= \langle \mathbf{1} \otimes \overline{X \wedge Y} \otimes \mathbf{1} \rangle \in k \otimes D \otimes D \otimes k, \end{aligned}$$

as the weight 1 subcomplex. Here by $\bar{x} \in D$ we mean the element $x \in \mathcal{H}_1$ viewed in D .

Let $d_0 : E_0^{i,j} \rightarrow E_0^{i,j+1}$ be the vertical, and $d_1 : E_1^{i,j} \rightarrow E_1^{i+1,j}$ be the horizontal coboundary. Then we first have

$$d_1(\mathbf{1} \otimes \delta_1 \otimes \mathbf{1}) = \mathbf{1} \otimes 1 \otimes \delta_1 \otimes \mathbf{1} - \mathbf{1} \otimes \Delta(\delta_1) \otimes \mathbf{1} + \mathbf{1} \otimes \delta_1 \otimes 1 \otimes \mathbf{1} = 0.$$

Next, we similarly observe

$$d_1(\mathbf{1} \otimes \bar{X} \otimes \bar{Y} \otimes \mathbf{1}) = \mathbf{1} \otimes 1 \otimes \bar{X} \otimes \bar{Y} \otimes \mathbf{1} - \mathbf{1} \otimes \bar{X} \otimes \bar{Y} \otimes 1 \otimes \mathbf{1},$$

and

$$\begin{aligned} d_0(\mathbf{1} \otimes X \otimes \bar{Y} \otimes \mathbf{1} + \mathbf{1} \otimes \bar{X} \otimes Y \otimes 1 \otimes \mathbf{1} + \frac{1}{2} \mathbf{1} \otimes \delta_1 \otimes \bar{Y}^2 \otimes \mathbf{1}) \\ = \mathbf{1} \otimes \bar{X} \otimes \bar{Y} \otimes 1 \otimes \mathbf{1} - \mathbf{1} \otimes 1 \otimes \bar{X} \otimes \bar{Y} \otimes \mathbf{1}. \end{aligned}$$

On the other hand, we have

$$d_1(\mathbf{1} \otimes \bar{Y} \otimes \bar{X} \otimes \mathbf{1}) = \mathbf{1} \otimes \mathbf{1} \otimes \bar{Y} \otimes \bar{X} \otimes \mathbf{1} - \mathbf{1} \otimes \bar{Y} \otimes \bar{X} \otimes \mathbf{1} \otimes \mathbf{1},$$

and

$$\begin{aligned} d_0(\mathbf{1} \otimes Y \otimes \bar{X} \otimes \mathbf{1} + \mathbf{1} \otimes \bar{Y} \otimes X \otimes \mathbf{1} - \frac{1}{2} \mathbf{1} \otimes \bar{Y}^2 \otimes \delta_1 \otimes \mathbf{1} - \mathbf{1} \otimes \bar{Y} \otimes \delta_1 Y \otimes \mathbf{1}) \\ = \mathbf{1} \otimes \bar{Y} \otimes \bar{X} \otimes \mathbf{1} \otimes \mathbf{1} - \mathbf{1} \otimes \mathbf{1} \otimes \bar{Y} \otimes \bar{X} \otimes \mathbf{1}. \end{aligned}$$

Therefore

$$d_1(\mathbf{1} \otimes \overline{X \wedge Y} \otimes \mathbf{1}) = 0.$$

Finally we calculate

$$d_1(\mathbf{1} \otimes \bar{X} \otimes \mathbf{1}) = \mathbf{1} \otimes \mathbf{1} \otimes \bar{X} \otimes \mathbf{1} + \mathbf{1} \otimes \bar{X} \otimes \mathbf{1} \otimes \mathbf{1}.$$

We also note that

$$d_0(\mathbf{1} \otimes X \otimes \mathbf{1}) = -\mathbf{1} \otimes \bar{X} \otimes \mathbf{1} \otimes \mathbf{1} - \mathbf{1} \otimes \mathbf{1} \otimes \bar{X} \otimes \mathbf{1} - \mathbf{1} \otimes \delta_1 \otimes \bar{Y} \otimes \mathbf{1},$$

and

$$d_0(\mathbf{1} \otimes \delta_1 Y \otimes \mathbf{1}) = -\mathbf{1} \otimes \bar{Y} \otimes \delta_1 \otimes \mathbf{1} - \mathbf{1} \otimes \delta_1 \otimes \bar{Y} \otimes \mathbf{1}.$$

Hence,

$$\mathbf{1} \otimes \bar{Y} \otimes \delta_1 \otimes \mathbf{1} = d_1(\mathbf{1} \otimes \bar{X} \otimes \mathbf{1}) + d_0(\mathbf{1} \otimes X \otimes \mathbf{1} - \mathbf{1} \otimes \delta_1 Y \otimes \mathbf{1}).$$

As a result, on the E_2 -term we will see

$$E_2^{1,0} = \langle \mathbf{1} \otimes \delta_1 \otimes \mathbf{1} \rangle, \quad E_2^{0,2} = \langle \mathbf{1} \otimes \overline{X \wedge Y} \otimes \mathbf{1} \rangle.$$

Transgression of these cocycles yields [26, Prop. 4.3] as follows.

Proposition 4.3. *The Hochschild cohomology of the weight 1 subcomplex of \mathcal{H}_1 is generated by*

$$[\delta_1] \in HH^1(\mathcal{H}_1, k), \quad [X \otimes Y - Y \otimes X - \delta_1 Y \otimes Y] \in HH^2(\mathcal{H}_1, k).$$

Consequently, we recover [26, Thm. 4.4].

Theorem 4.4. *The periodic Hopf-cyclic cohomology of \mathcal{H}_1 with coefficients in the SAYD module ${}^1k_\varepsilon$ is given by*

$$HP^{\text{odd}}(\mathcal{H}_1, k) = \langle \delta_1 \rangle, \quad HP^{\text{even}}(\mathcal{H}_1, k) = \langle X \otimes Y - Y \otimes X - \delta_1 Y \otimes Y \rangle.$$

On the Schwarzian quotient we similarly recover [26, Thm. 4.5] as follows.

Theorem 4.5. *The periodic Hopf-cyclic cohomology of \mathcal{H}_{1S} with coefficients in the SAYD module ${}^1k_\varepsilon$ is given by*

$$HP^{\text{odd}}(\mathcal{H}_{1S}, k) = \langle Z \rangle, \quad HP^{\text{even}}(\mathcal{H}_{1S}, k) = \langle X \otimes Y - Y \otimes X - ZY \otimes Y \rangle.$$

4.2. Quantum enveloping algebras. In this subsection we will compute the (periodic and non-periodic) Hopf-cyclic cohomology groups of the quantized enveloping algebras $U_q(\mathfrak{g})$. Our strategy will be to realize it as a principal coextension.

Let us first recall Drinfeld–Jimbo quantized enveloping algebras of Lie algebras from [22, Subsect. 6.1.2].

Let \mathfrak{g} be a finite dimensional complex semi-simple Lie algebra, $\alpha_1, \dots, \alpha_\ell$ a fixed ordered sequence of simple roots, and $A = [a_{ij}]$ the Cartan matrix. Let also q be a fixed nonzero complex number such that $q_i^2 \neq 1$, where $q_i := q^{d_i}$, $1 \leq i \leq \ell$, and $d_i = (\alpha_i, \alpha_i)/2$.

Then the Drinfeld–Jimbo quantized enveloping algebra $U_q(\mathfrak{g})$ is the Hopf algebra with 4ℓ generators E_i, F_i, K_i, K_i^{-1} , $1 \leq i \leq \ell$, and the relations

$$\begin{aligned} K_i K_j &= K_j K_i, & K_i K_i^{-1} &= K_i^{-1} K_i = 1, \\ K_i E_j K_i^{-1} &= q_i^{a_{ij}} E_j, & K_i F_j K_i^{-1} &= q_i^{-a_{ij}} F_j, \\ E_i F_j - F_j E_i &= \delta_{ij} \frac{K_i - K_i^{-1}}{q_i - q_i^{-1}}, \\ \sum_{r=0}^{1-a_{ij}} (-1)^r \begin{bmatrix} 1-a_{ij} \\ r \end{bmatrix}_{q_i} E_i^{1-a_{ij}-r} E_j E_i^r &= 0, & i \neq j, \\ \sum_{r=0}^{1-a_{ij}} (-1)^r \begin{bmatrix} 1-a_{ij} \\ r \end{bmatrix}_{q_i} F_i^{1-a_{ij}-r} F_j F_i^r &= 0, & i \neq j, \end{aligned}$$

where

$$\begin{bmatrix} n \\ r \end{bmatrix}_q = \frac{(n)_q!}{(r)_q! (n-r)_q!}, \quad (n)_q := \frac{q^n - q^{-n}}{q - q^{-1}}.$$

The rest of the Hopf algebra structure of $U_q(\mathfrak{g})$ is given by

$$\begin{aligned} \Delta(K_i) &= K_i \otimes K_i, & \Delta(K_i^{-1}) &= K_i^{-1} \otimes K_i^{-1} \\ \Delta(E_i) &= E_i \otimes K_i + 1 \otimes E_i, & \Delta(F_j) &= F_j \otimes 1 + K_j^{-1} \otimes F_j \\ \varepsilon(K_i) &= 1, & \varepsilon(E_i) &= \varepsilon(F_i) = 0 \\ S(K_i) &= K_i^{-1}, & S(E_i) &= -E_i K_i^{-1}, & S(F_i) &= -K_i F_i. \end{aligned}$$

Let us also recall, from [22], the Hopf-subalgebras

$$\begin{aligned} U_q(\mathfrak{b}_+) &= \text{Span} \{ E_1^{p_1} \dots E_\ell^{p_\ell} K_1^{q_1} \dots K_\ell^{q_\ell} \mid r_1, \dots, r_\ell \geq 0, q_1, \dots, q_\ell \in \mathbb{Z} \}, \\ U_q(\mathfrak{b}_-) &= \text{Span} \{ K_1^{q_1} \dots K_\ell^{q_\ell} F_1^{r_1} \dots F_\ell^{r_\ell} \mid p_1, \dots, p_\ell \geq 0, q_1, \dots, q_\ell \in \mathbb{Z} \}, \end{aligned}$$

of $U_q(\mathfrak{g})$.

A modular pair in involution for the Hopf algebra $U_q(\mathfrak{g})$ is given by [22, Prop. 6.6]. Let $K_\lambda := K_1^{n_1} \dots K_\ell^{n_\ell}$ for any $\lambda = \sum_i n_i \alpha_i$, where $n_i \in \mathbb{Z}$. Then, $\rho \in \mathfrak{h}^*$ being

the half-sum of the positive roots of \mathfrak{g} , by [22, Prop. 6.6] we have

$$S^2(a) = K_{2\rho} a K_{2\rho}^{-1}, \quad \forall a \in U_q(\mathfrak{g}).$$

Thus, $(\varepsilon, K_{2\rho})$ is a MPI for the Hopf algebra $U_q(\mathfrak{g})$.

For the Hopf subalgebra $H := U_q(\mathfrak{b}_-) \subseteq U_q(\mathfrak{g}) =: C$, we obtain $D = C/CH^+ = U_q(\mathfrak{b}_+)$. Then by [33, Thm. II] we conclude that C is (faithfully) coflat as left and right D -comodule, and hence the hypothesis of Theorem 4.1 is satisfied.

Lemma 4.6. *Let C and D be as above, and $\mu = K_1^{p_1} \cdots K_\ell^{p_\ell}$, $p_1, \dots, p_\ell \geq 0$. Then we have*

$$\text{Cotor}_D^n(k, {}^\mu k) = \begin{cases} k^{\oplus \frac{(p_1 + \dots + p_\ell)!}{p_1! \cdots p_\ell!}}, & \text{if } n = p_1 + p_2 + \dots + p_\ell, \\ 0, & \text{if } n \neq p_1 + p_2 + \dots + p_\ell. \end{cases}$$

Proof. We apply Theorem 4.1 to the coextension

$$\begin{aligned} \pi : D &\longrightarrow W := \text{Span}\{K_1^{m_1} \cdots K_\ell^{m_\ell} \mid m_1, \dots, m_\ell \in \mathbb{Z}\} \\ E_1^{r_1} \cdots E_\ell^{r_\ell} K_1^{m_1} \cdots K_\ell^{m_\ell} &\mapsto \begin{cases} K_1^{m_1} \cdots K_\ell^{m_\ell}, & \text{if } r_1 = r_2 = \dots = r_\ell = 0, \\ 0, & \text{otherwise} \end{cases} \end{aligned}$$

to have a spectral sequence, converging to $\text{Cotor}_D(k, {}^\mu k)$, whose E_1 -term is

$$E_1^{i,j} = \text{Cotor}_W^j(k, \underbrace{D \square_W \cdots \square_W D}_{i \text{ many}} \square_W {}^\mu k).$$

Since

$$\begin{aligned} &\underbrace{D \square_W \cdots \square_W D}_{i \text{ many}} \square_W {}^\mu k \\ &= \text{Span} \left\{ E_{a_s}^{b_s} \cdots E_{\alpha_s}^{\beta_s} \mu K_{a_1}^{-b_1} \cdots K_{\alpha_1}^{-\beta_1} \cdots K_{a_s}^{-b_s} \cdots K_{\alpha_s}^{-\beta_s} \otimes \cdots \right. \\ &\quad \left. \cdots \otimes \underbrace{\mu K_{a_1}^{-b_1} \cdots K_{\alpha_1}^{-\beta_1} \otimes \cdots \otimes \mu K_{a_1}^{-b_1} \cdots K_{\alpha_1}^{-\beta_1}}_{i_2 \text{ many}} \right. \\ &\quad \left. \otimes E_{a_1}^{b_1} \cdots E_{\alpha_1}^{\beta_1} \mu K_{a_1}^{-b_1} \cdots K_{\alpha_1}^{-\beta_1} \otimes \underbrace{\mu \otimes \cdots \otimes \mu}_{i_1 \text{ many}} \otimes \mathbf{1} \right\}, \end{aligned}$$

$i \geq s \geq 0, \ell \geq a_1, \dots, a_s, \dots, \alpha_1, \dots, \alpha_s \geq 1, i_1, i_2, \dots, b_1, \dots, b_s, \beta_1, \dots, \beta_s \geq 0$, the left W -coaction on a typical element is given by

$$\begin{aligned} &\nabla_W^L(E_{a_i} K_{a_i}^{-1} \cdots K_{a_1}^{-1} \mu \otimes \cdots \otimes E_{a_2} K_{a_2}^{-1} K_{a_1}^{-1} \mu \otimes E_{a_1} K_{a_1}^{-1} \mu \otimes \mathbf{1}) \\ &= K_{a_i}^{-1} \cdots K_{a_1}^{-1} \mu \otimes E_{a_i} K_{a_i}^{-1} \cdots K_{a_1}^{-1} \mu \otimes \cdots \otimes E_{a_2} K_{a_2}^{-1} K_{a_1}^{-1} \mu \otimes E_{a_1} K_{a_1}^{-1} \mu \otimes \mathbf{1}. \end{aligned} \tag{4.1}$$

Since W consists only of the group-like elements, the result follows from the W -coaction (4.1) to be trivial. \square

Lemma 4.7. *Let C and D be as above, and $\sigma = K_{2\rho}$. Then we have*

$$\text{Cotor}_D^n(k, \underbrace{C \square_D \cdots \square_D C \square_D}_{i \text{ many}} \sigma k) = \begin{cases} k^{\oplus \binom{\ell}{i}}, & \text{if } n = \ell - i, \\ 0, & \text{if } n \neq \ell - i. \end{cases}$$

Proof. Let us first note that

$$\begin{aligned} & \underbrace{C \square_D \cdots \square_D C \square_D}_{i \text{ many}} \sigma k \\ &= \text{Span} \left\{ \sigma K_{\alpha_1}^{-b_1} \cdots K_{\alpha_1}^{-\beta_1} \cdots K_{\alpha_{s-1}}^{-b_{s-1}} \cdots K_{\alpha_{s-1}}^{-\beta_{s-1}} F_{\alpha_s}^{b_s} \cdots F_{\alpha_s}^{\beta_s} \otimes \cdots \right. \\ & \quad \cdots \otimes \underbrace{\sigma K_{\alpha_1}^{-b_1} \cdots K_{\alpha_1}^{-\beta_1} \otimes \cdots \otimes \sigma K_{\alpha_1}^{-b_1} \cdots K_{\alpha_1}^{-\beta_1}}_{i_2 \text{ many}} \otimes \sigma F_{\alpha_1}^{b_1} \cdots \\ & \quad \left. \cdots F_{\alpha_1}^{\beta_1} \otimes \underbrace{\sigma \otimes \cdots \otimes \sigma}_{i_1 \text{ many}} \otimes \mathbf{1} \right\}, \end{aligned}$$

$i \geq s \geq 0, \ell \geq a_1, \dots, a_s, \dots, \alpha_1, \dots, \alpha_s \geq 1, i_1, i_2, \dots, b_1, \dots, b_s, \beta_1, \dots, \beta_s \geq 0$. Then, the left D -coaction on a typical element is given by

$$\begin{aligned} & \nabla_D^L(\sigma K_{\alpha_1}^{-1} \cdots K_{\alpha_{i-1}}^{-1} F_{\alpha_i} \otimes \cdots \otimes \sigma K_{\alpha_1}^{-1} F_{\alpha_2} \otimes \sigma F_{\alpha_1} \otimes \mathbf{1}) \\ &= \sigma K_{\alpha_1}^{-1} \cdots K_{\alpha_{i-1}}^{-1} K_{\alpha_i}^{-1} \otimes \sigma K_{\alpha_1}^{-1} \cdots K_{\alpha_{i-1}}^{-1} F_{\alpha_i} \otimes \cdots \otimes \sigma K_{\alpha_1}^{-1} F_{\alpha_2} \otimes \sigma F_{\alpha_1} \otimes \mathbf{1}. \end{aligned} \tag{4.2}$$

By the proof of Lemma 4.6, there is no repetition on the indexes appearing in (4.2). Accordingly, the result follows from Lemma 4.6. \square

Proposition 4.8. *For $\sigma := K_{2\rho}$ we have*

$$\text{Cotor}_{U_q(\mathfrak{g})}^n(k, \sigma k) = \begin{cases} k^{\oplus 2^\ell}, & \text{if } n = \ell \\ 0, & \text{if } n \neq \ell. \end{cases}$$

Proof. Letting C and D be as above, it follows from Theorem 4.1 that we have a spectral sequence converging to $\text{Cotor}_C(k, \sigma k)$ whose E_1 -term is

$$E_1^{i,j} = \text{Cotor}_D^j(k, C^{\square_D i} \square_D \sigma k).$$

By Lemma 4.7 the cocycles are computed at $j = \ell - i$, i.e. $i + j = \ell$. The number of cocycles, on the other hand, is

$$\sum_{i=0}^{\ell} \binom{\ell}{i} = 2^\ell. \quad \square$$

Remark 4.9. We note that since $C = U_q(\mathfrak{g})$ is a Hopf algebra, we could start with the Hopf subalgebra $H = C \subseteq C$ to get $D = H/H^+ = W$, and to arrive at the same result.

We are now ready to compute the (periodic) Hopf-cyclic cohomology of $U_q(\mathfrak{g})$.

Theorem 4.10. For $\sigma := K_{2\rho}$, and $\ell \equiv \epsilon \pmod{2}$, we have

$$HP^\epsilon(U_q(\mathfrak{g}), \sigma k) = k^{\oplus 2^\ell}, \quad HP^{1-\epsilon}(U_q(\mathfrak{g}), \sigma k) = 0.$$

Proof. Let C be as above. As a result of Proposition 4.8 we have

$$HH^n(C, \sigma k) = \begin{cases} k^{\oplus 2^\ell}, & \text{if } n = \ell \\ 0, & \text{if } n \neq \ell. \end{cases}$$

Hence, the Connes' SBI sequence yields

$$\begin{aligned} HC^n(C, \sigma k) &= 0, \quad n < \ell, \\ HC^{\ell+1}(C, \sigma k) &\cong HC^{\ell+3}(C, \sigma k) \cong \dots \cong 0 \\ k^{\oplus 2^\ell} &\cong HH^\ell(C, \sigma k) \cong HC^\ell(C, \sigma k) \cong HC^{\ell+2}(C, \sigma k) \cong \dots \end{aligned}$$

where the isomorphisms are given by the periodicity map

$$S : HC^p(C, \sigma k) \longrightarrow HC^{p+2}(C, \sigma k). \quad \square$$

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