

The Standard Model in noncommutative geometry and Morita equivalence

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Abstract. We discuss some properties of the spectral triple $(A_F, H_F, D_F, J_F, \gamma_F)$ describing the internal space in the noncommutative geometry approach to the Standard Model, with $A_F = \mathbb{C} \oplus \mathbb{H} \oplus M_3(\mathbb{C})$. We show that, if we want H_F to be a Morita equivalence bimodule between A_F and the associated Clifford algebra, two terms must be added to the Dirac operator; we then study its relation with the orientability condition for a spectral triple. We also illustrate what changes if one considers a spectral triple with a degenerate representation, based on the complex algebra $B_F = \mathbb{C} \oplus M_2(\mathbb{C}) \oplus M_3(\mathbb{C})$.

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1. Introduction

In the spectral action approach to (quantum) field theory, the space of the theory is the product of an ordinary spin manifold M with a finite noncommutative space (cf. [11, 23] and references therein). States of the system are represented by unit vectors in $L^2(M, S) \otimes H$, where $L^2(M, S)$ are square integrable sections of the spinor bundle $S \rightarrow M$ and H is a finite-dimensional Hilbert space representing the internal degrees of freedom of a particle. The algebra containing the observables is the tensor product of smooth functions $C^\infty(M)$ on M with certain finite dimensional algebra A . More precisely, one has an “almost commutative” geometry described by a product of spectral triples, with Dirac operator constructed from the Dirac operator of M and certain selfadjoint operator (a Hermitian matrix) D on H .

A deep algebraic characterization of the space of Dirac spinor fields $L^2(M, S)$ on a spin manifold is as the Morita equivalence bimodule between $C(M)$ and the algebra $\mathcal{C}\ell(M)$ of sections of the Clifford bundle of M . It is natural to investigate if also the finite-dimensional spectral triple of the Standard Model (A, H, D) describes a (noncommutative) spin manifold, and in particular if the elements of H are in some sense “spinors”. This condition — which we name “property (M)” in Def. 2.4 — can be precisely formulated again in terms of Morita equivalence involving A and

certain noncommutative analogue of $\mathcal{C}\ell(M)$, and is satisfied in some basic examples like e.g. Einstein–Yang–Mills systems.

We investigate the consequence of such a requirement on the finite non-commutative geometry that should describe the Standard Model of elementary particles. We shall show that in order to satisfy such a condition, we are forced to introduce two additional terms in the Dirac operator, and consider a non-standard grading. In order to get the correct experimental value of the Higgs mass, various modifications of the original model have been proposed: to enlarge the Hilbert space thus introducing new fermions [22]; to turn one of the elements of the internal Dirac operator into a field by hand [4] rather than getting it as a fluctuation of the metric; to break (relax) the 1st order condition [6, 7], thus allowing the presence of new terms in the Dirac operator; to enlarge the algebra [14] and use a twisted spectral triple [15] with bounded twisted commutators. In the present paper from the Morita condition and a different grading we get two extra fields (without breaking any of the other conditions). We postpone to future work a discussion of the physical implications and, in particular, how the Higgs mass is modified.

Besides the original model, which is built around the real algebra:

$$A_F = \mathbb{C} \oplus \mathbb{H} \oplus M_3(\mathbb{C}) , \quad (1.1)$$

where \mathbb{H} denotes the division ring of quaternions, we shall also consider the complex algebra

$$B_F = \mathbb{C} \oplus M_2(\mathbb{C}) \oplus M_3(\mathbb{C}) , \quad (1.2)$$

which has an interesting interpretation from quantum group theory. Namely, it is the semisimple part [12] of a certain quotient of $U_q(sl(2))$ for q a 3rd root of unity. As explained in [13], the dual compact quantum group Q fits into the exact sequence

$$1 \rightarrow Q \rightarrow SL_q(2) \rightarrow SL(2, \mathbb{C}) \rightarrow 1 .$$

Recall that $SL(2, \mathbb{C})$ is a double covering of the restricted Lorentz group. One might argue that trading a commutative space for an almost commutative one, the Lorentz group should be replaced by a compact quantum group covering it, which takes into account the symmetries of the internal space as well. (For preliminary studies of Hopf-algebra symmetries of A_F/B_F see [12, 13, 17]; for compact quantum group symmetries see [1, 2].)

We show that a minimal modification in the representation allows to replace A_F by B_F without changing the content of the theory. In particular, at the representation level the complexification $\pi(A_F)_{\mathbb{C}}$ of $\pi(A_F)$ is the minimal unitalization of the degenerate representation $\pi(B_F)$ (the representations, here denoted by the same symbol which we will omit later on, are introduced in §3); adding the identity operator (which commutes with the Dirac operator) doesn't produce new fields.

The plan of the paper is the following. In §2 we review some basic ideas of noncommutative geometry [8, 16, 19], with a view to applications to gauge theory [11, 23]. In §3, we review the derivation of the finite spectral triple of the Standard Model and discuss an alternative based on the complex algebra B_F (§3.3). In §4, we describe the most general Dirac operator satisfying the 1st order condition (which is necessary for the “property (M)” in Def. 2.4), and in §5 two possible grading operators; the Dirac operator of Chamseddine–Connes [3, 5, 9, 11] appears in §5.3. In §6, we discuss the natural condition for a spectral triple to be “spin^c”, based on Morita equivalence, and derive some necessary conditions for this to be satisfied; we show that in order to satisfy these condition one has to introduce two additional terms in the Dirac operator of Chamseddine–Connes, one mixing e_R with $\bar{\nu}_R$ and one mixing leptons with quarks (for a study of lepto-quarks in this setting, one can see [21]). The last term is also necessary in order to have an irreducible spectral triple, cf. §7.2. In §7.1, we study the problem of orientability for the modified Dirac operator. In §7.3 we discuss the irreducibility of the Pati–Salam model. We conclude in §8 with some final remarks.

2. Mathematical set-up

Let M be a closed oriented Riemannian manifold, $C(M)$ and $C^\infty(M)$ the algebras of complex-valued continuous resp. smooth functions, and $\mathcal{C}\ell(M)$ the algebra of continuous sections of the bundle of (complexified) Clifford algebras: as a $C(M)$ -module, it is equivalent to the module of continuous sections of the bundle $\Lambda^\bullet T_C^*M \rightarrow M$, but with product defined by the Clifford multiplication. The manifold M is spin^c if and only if there exists a Morita equivalence $\mathcal{C}\ell(M)$ - $C(M)$ bimodule Σ (see e.g. §1 of [24]). Such a Σ is automatically projective and finitely generated, hence by Serre–Swan theorem $\Sigma = \Gamma(S)$ is the module of sections of some complex vector bundle $S \rightarrow M$, the *spinor bundle* in the conventional picture from differential geometry.

Once we have S , we can introduce the Dirac operator \not{D} , a self-adjoint operator on the Hilbert space $L^2(M, S)$ of square integrable sections of $S \rightarrow M$ [24, §1.4]. Let π be the representation of $C(M)$ on $L^2(M, S)$ by pointwise multiplication and c the representation of $\mathcal{C}\ell(M)$ by Clifford multiplication (see e.g. [16] or [24] for the details). The data

$$(C^\infty(M), \pi, L^2(M, S), \not{D}) \tag{2.1}$$

is the prototypical example of commutative spectral triple, and one can indeed prove under some additional assumptions that any commutative spectral triple comes from such a construction [10, Thm. 1.2]. The spectral triple (2.1) is \mathbb{Z}_2 -graded if M is even dimensional.

There is an algebraic characterization for spin manifolds as well: a spin^c manifold M is spin if and only if there exists a real structure for the spectral triple (2.1) (whose definition we recall below in the finite-dimensional case).

Let us observe that, for any $f \in C^\infty(M)$, $i[\not{D}, \pi(f)] = c(df)$ is the operator of Clifford multiplication by df and such operators generate $\mathcal{Cl}(M)$. In the even case, the grading γ belongs to $\mathcal{Cl}(M)$.

For later use, we recall the definition of spectral triple in the finite-dimensional case, adapted to our purposes.

Definition 2.1. A finite dimensional spectral triple (A, π, H, D) is given by a finite dimensional complex Hilbert space H , a Hermitian operator D on H , and a real or complex C^* -algebra with a faithful $*$ -representation $\pi : A \rightarrow \text{End}_{\mathbb{C}}(H)$. The spectral triple is *even* if H is \mathbb{Z}_2 -graded, $\pi(A)$ is even and D is odd; we denote by γ the grading operator. The spectral triple is *real* if there is an antilinear isometry J on H — called the *real structure* — satisfying

$$J^2 = \epsilon \text{id}_H, \quad JD = \epsilon' DJ, \quad J\gamma = \epsilon'' \gamma J \quad (\text{only in the even case})$$

for some $\epsilon, \epsilon', \epsilon'' \in \{\pm 1\}$, together with the 0th order condition

$$[\pi(a), J\pi(b)J^{-1}] = 0 \quad \forall a, b \in A,$$

and the 1st order condition:

$$[[D, \pi(a)], J\pi(b)J^{-1}] = 0 \quad \forall a, b \in A. \tag{2.2}$$

In order to simplify the notations, we will often omit the representation symbol π and set $\gamma := 1$ if we have an odd spectral triple. Note that we don’t lose generality by assuming that the representation is faithful. Note also that H is complex even when A is real. The values of $\epsilon, \epsilon', \epsilon''$ determine the *KO-dimension* of the spectral triple (according to the table that is, for example, in [24, §3.8]).

Definition 2.2. Let (A, H, D, γ) be a spectral triple (with $\gamma := 1$ if the spectral triple is odd) and $\Omega^1(A) := \text{Span}\{a[D, b] : a, b \in A\}$. We call $\mathcal{Cl}(A)_o$ the complex $*$ -algebra generated by A and Ω^1 , and $\mathcal{Cl}(A)_e$ the complex $*$ -algebra generated by $\mathcal{Cl}(A)_o$ and γ .

This is similar to Definition 3.19 of [20] ($\mathcal{Cl}(A)_o$ is their $\mathcal{C}_{\mathcal{D}}(A)$ in the even case, while in the odd case they double the Hilbert space to get a \mathbb{Z}_2 -graded algebra).

Let $A^\circ := JAJ^{-1}$ be the opposite algebra (thought of as a subalgebra of $\text{End}_{\mathbb{C}}(H)$). Recall that a linear map $\pi_D : (A \otimes A^\circ) \otimes A^{\otimes n} \rightarrow \text{End}_{\mathbb{C}}(H)$ is given by

$$c = \sum_{\text{finite}} (a_0^i \otimes b_0^i) \otimes a_1^i \otimes \cdots \otimes a_n^i \mapsto \pi_D(c) := \sum_{\text{finite}} a_0^i b_0^i [D, a_1^i] \cdots [D, a_n^i],$$

for all $a_j^i \in A$ and $b_0^i \in A^\circ$. By restriction (take $b_0^i = 1$) we get a surjective map

$$\pi_D : \bigoplus_{n \geq 0} A^{\otimes n+1} \rightarrow \mathcal{Cl}(A)_o$$

which we denote by the same symbol. Note that γ is in the image of the latter map if and only if the two Clifford algebras coincide: $\mathcal{Cl}(A)_e = \mathcal{Cl}(A)_o$. This in particular happens when the spectral triple is orientable, cf. below.

Definition 2.3. Let $n \geq 0$. A spectral triple is *orientable* (resp. *orientable in a weak sense*), with global dimension $\leq n$, if there exists a Hochschild cycle with coefficients in A (resp. in $A \otimes A^\circ$) such that $\pi_D(c) = \gamma$.

Note that c defines a class $[c] \in HH_n(A)$ (resp. $[c] \in H_n(A, A \otimes A^\circ)$). For a finite-dimensional real or complex C^* -algebra, $HH_n(A)$ and $H_n(A, A \otimes A^\circ)$ are zero if $n > 0$ (we thank U. Krähmer for this remark). On the other hand, since π_D is only defined on chains, rather than on homology classes ($[c] = 0 \not\Rightarrow \pi_D(c) = 0$), it still makes sense to study orientability for arbitrary $n \geq 0$.

The 0th and 1st order conditions imply that H is a $\mathcal{Cl}(A)_e$ - A° bimodule. Indeed a and $[D, a]$ commute with b° for all $a \in A, b^\circ \in A^\circ$, and γ commutes with A° since it commutes with A and $J\gamma = \epsilon''\gamma J$. Inspired by the example (2.1) we give then the following definition (much similar to the “condition 5” of [20]):

Definition 2.4. A spectral triple (A, H, D, J, γ) has the *property (M)* (resp. *property (M) with grading*) if H is a Morita equivalence bimodule between A° and $\mathcal{Cl}(A)_o$ (resp. $\mathcal{Cl}(A)_e$).

Since $\mathcal{Cl}(A)_o \subset \mathcal{Cl}(A)_e$, clearly the “property (M) with grading” is weaker. The two conditions are equivalent if the spectral triple is odd (so $\gamma = 1$) or orientable.

Example 2.5. If $H = A, J(a) = a^*$ and $D = 0$, the spectral triple has the property (M).

2.1. The gauge group of a real spectral triple. Let (A, π, H, D, J) be a real spectral triple, and assume that A is a unital and π is a unital representation. Let $U(A)$ be the group of unitary elements of A . Due to the 0th order condition, the map $\rho : U(A) \rightarrow \text{Aut}_{\mathbb{C}}(H)$ given by

$$\rho(u) := uJuJ^{-1}, \tag{2.3}$$

is a representation, called *adjoint representation*.

The gauge group $G(A)$ of a real spectral triple is defined as

$$G(A) := \{uJuJ^{-1} : u \in U(A)\}.$$

Example 2.6. In the spectral triple $(M_n(\mathbb{C}), M_n(\mathbb{C}), 0, J)$ of the Einstein Yang–Mills system the algebra acts by left multiplication, $J(a) = a^*$ is the Hermitian conjugation, and the gauge group is $G(A) = \text{PU}(n)$. This spectral triple has the property (M), cf. Example 2.5.

2.2. Spectral triples with a degenerate representation. A necessary and sufficient condition for the map ρ in (2.3) to send $U(A)$ into invertible operators is that $\rho(1) = 1$ (then automatically, $\rho(u^{-1}) = \rho(u)^{-1}$). A sufficient condition is that π is a unital representation, that is $\pi(1) = 1$. For a spectral triple with a degenerate representation, the unit of A is not the identity operator on H , and (2.3) is in general not a representation of the unitary group $U(A)$. Here we explain how to bypass this problem.

Degenerate representations appear for example when one tries to sum a real spectral triple with one which has no real structure. Let $(A, \bar{\pi}_0, \bar{H}_0, 0)$ and $(A, \pi_1, H_1, 0, J_1)$ be two finite-dimensional spectral triples, the latter one real with $J_1^2 = 1$, and both with the same algebra A and null Dirac operator. Then we can define a new real spectral triple $(A, \pi, H, 0, J)$ as follows. We set

$$H := H_0 \oplus \bar{H}_0 \oplus H_1 ,$$

where $H_0 = (\bar{H}_0)^*$ is the dual space. We define

$$\pi(a)(x, y, z) = (0, \bar{\pi}_0(a)y, \pi_1(a)z) , \quad J(x, y, z) = (y^*, x^*, J_1z) ,$$

for all $x \in H_0, y \in \bar{H}_0, z \in H_1$. Note that the representation π is degenerate. If we extend $\bar{\pi}_0$ and π_1 trivially to H (as zero on $H_0 \oplus H_1$ resp. $H_0 \oplus \bar{H}_0$), then we can simply write:

$$\pi = \bar{\pi}_0 + \pi_1 .$$

Since π is degenerate, the map $u \mapsto \pi(u)J\pi(u)J$ is not a representation of $U(A)$ in $\text{Aut}(H)$ (it doesn’t map $1 \mapsto 1$, and u into an invertible operator). A unitary representation ρ of $U(A)$ on H is given by

$$\rho(u) := \bar{\pi}_0(u) + J\bar{\pi}_0(u)J + \pi_1(u)J\pi_1(u)J . \tag{2.4}$$

Indeed $\bar{\pi}_0(1) = \text{id}_{\bar{H}_0}, \pi_1(1) = \text{id}_{H_1}$ and $J\bar{\pi}_0(1)J = \text{id}_{H_0}$. So $\rho(1) = 1$. Moreover, $\bar{\pi}_0, J\bar{\pi}_0(\cdot)J, \pi_1$ and $J\pi_1(\cdot)J$ are mutually commuting, hence ρ is multiplicative, and from $\rho(u)\rho(u^*) = \rho(uu^*) = 1$ we deduce that the representation is also unitary.

Basically, we are considering the direct sum of three representations of $U(A)$: the fundamental associated to $\bar{\pi}_0$ and its dual, and the adjoint representation of π_1 .

In §3.3 we exhibit a possible choice of the above data, such that $U(A)$ contains (strictly) the gauge group G_{SM} of the Standard Model (modulo a finite subgroup), and $\rho|_{G_{SM}}$ gives the correct representation.

3. From particles to algebras

In this section we give a review of the derivation of the data (A_F, H_F, J_F) from physical considerations, and collect at the end few results about the algebra and its

commutant that will be useful in the following sections. In some sense, these data reflect the “topology” of the finite noncommutative manifold describing the internal space of the Standard Model, while the Dirac operator encodes the metric properties. In §3.3 we explain how to get the same gauge group from a spectral triple based on the complex algebra (1.2) with a degenerate representation. For simplicity, we work with only one generation of leptons/quarks.

3.1. The gauge group of the Standard Model. Let

$$\widetilde{G}_{SM} := U(1) \times SU(2) \times SU(3)$$

be the usual gauge group of the Standard Model, let H be the finite-dimensional Hilbert space representing the internal degrees of freedom of elementary fermions. Let us recall what is the representation of \widetilde{G}_{SM} . We have a decomposition $H = F \oplus F^*$, with F^* the dual space of F . The vector space F (for *fermions*) has basis

$$\begin{pmatrix} \nu_L \\ e_L \end{pmatrix} \quad \begin{pmatrix} u_L^c \\ d_L^c \end{pmatrix}_{c=1,2,3} \quad \begin{matrix} \nu_R & \{u_R^c\}_{c=1,2,3} \\ e_R & \{d_R^c\}_{c=1,2,3} \end{matrix}$$

where ν stands for neutrino, e for electron, u^c for up-quark and d^c for down-quark with color $c = 1, 2, 3$, L, R stands for left-handed resp. right-handed. We will use the label \uparrow for the first particle in each column (neutrino or quark up) and \downarrow for the second one (electron or quark down). Left-handed doublets carry the fundamental representation of $SU(2)$, while right handed particles are $SU(2)$ -invariant; in particular, the \uparrow particle in each doublet has *weak isospin* $I_{3,w} = 1/2$ and the \downarrow has *weak isospin* $I_{3,w} = -1/2$. The $SU(2)$ -singlets have *weak isospin* $I_{3,w} = 0$. Each one of the color triplets carry the fundamental representation of $SU(3)$, the other particles being $SU(3)$ -invariant. Each particle carries a 1-dimensional representation $\lambda \rightarrow \lambda^{3Y_w}$ of $U(1)$, where $Y_w \in \frac{1}{3}\mathbb{Z}$ is the *weak hypercharge*; it is computed from the formula $Q = I_{3,w} + \frac{1}{2}Y_w$ where Q is the charge of the particle. The value of $3Y_w$ is given by the following table:

particle	ν_L, e_L	u_L^c, d_L^c	ν_R	e_R	u_R^c	d_R^c
$3Y_w$	-3	1	0	-6	4	-2

The final representation is actually the direct sum of n copies of $H = F \oplus F^*$, where n is the number of generations ($n = 3$ according to our current knowledge). For simplicity, the factor taking into account generations will be neglected.

For the computations, it will be convenient to encode the complex vector space F of dimension 16 as $F \simeq M_4(\mathbb{C})$. Namely we arrange the particles in a 4×4 matrix as follows

$$\begin{bmatrix} \nu_R & u_R^1 & u_R^2 & u_R^3 \\ e_R & d_R^1 & d_R^2 & d_R^3 \\ \nu_L & u_L^1 & u_L^2 & u_L^3 \\ e_L & d_L^1 & d_L^2 & d_L^3 \end{bmatrix}.$$

We put in the first column leptons, in the other three the quarks according to the color. In the rows we put in the order: $\uparrow R, \downarrow R, \uparrow L, \downarrow L$.

Let e_{ij} the 4×4 matrix with 1 in position (i, j) and zero everywhere else. Matrices $\{e_{ij}\}_{i,j=1}^4$ form an orthonormal basis of $M_4(\mathbb{C})$ for the inner product associated to the trace $\langle a, b \rangle = \text{Tr}(a^*b)$. With this notation, for example, the state associated to the unit vector e_{31} represents a left handed neutrino.

In the dual representation F^* , one has:

$$\begin{bmatrix} \bar{\nu}_R & \bar{e}_R & \bar{\nu}_L & \bar{e}_L \\ \bar{u}_R^1 & \bar{d}_R^1 & \bar{u}_L^1 & \bar{d}_L^1 \\ \bar{u}_R^2 & \bar{d}_R^2 & \bar{u}_L^2 & \bar{d}_L^2 \\ \bar{u}_R^3 & \bar{d}_R^3 & \bar{u}_L^3 & \bar{d}_L^3 \end{bmatrix}.$$

Elements of H are then of the form $a \oplus b$ with $a, b \in M_4(\mathbb{C})$.

Endomorphisms of $F \simeq F^* \simeq M_4(\mathbb{C})$ are given by $M_4(\mathbb{C}) \otimes M_4(\mathbb{C})$, where the first factor acts on $F = M_4(\mathbb{C})$ via row-by-column multiplication from the left, and the second via row-by-column multiplication from the right. From the weak hypercharge table we get the following representation π_{SM} of \tilde{G}_{SM} on H :

$$\begin{aligned} \pi_{SM}(\lambda, q, m) = & \left[\begin{array}{cc|cc} \lambda^3 & 0 & 0 & 0 \\ 0 & \bar{\lambda}^3 & 0 & 0 \\ \hline 0 & 0 & & q \\ 0 & 0 & & \end{array} \right] \otimes \left[\begin{array}{c|ccc} \bar{\lambda}^3 & 0 & 0 & 0 \\ \hline 0 & & & \\ 0 & & \lambda m^* & \\ \hline 0 & & & \end{array} \right] \\ & \oplus \left[\begin{array}{c|ccc} \lambda^3 & 0 & 0 & 0 \\ \hline 0 & & & \\ 0 & & \bar{\lambda} m & \\ \hline 0 & & & \end{array} \right] \otimes \left[\begin{array}{cc|cc} \bar{\lambda}^3 & 0 & 0 & 0 \\ 0 & \lambda^3 & 0 & 0 \\ \hline 0 & 0 & & q^* \\ 0 & 0 & & \end{array} \right] \end{aligned}$$

for all $\lambda \in U(1), q \in SU(2)$ and $m \in SU(3)$. Here the first summand acts on F and the second one on F^* .

Computing the kernel of π_{SM} one sees that the relevant group is not exactly \tilde{G}_{SM} but a quotient by a finite subgroup. Let $\mathbb{Z}_6 := \{\mu \in \mathbb{C} : \mu^6 = 1\}$ the group of 6-th roots of unity. There is an injective morphism of groups:

$$\mathbb{Z}_6 \ni \mu \mapsto (\mu, \mu^3 1_2, \mu^4 1_3) \in \tilde{G}_{SM}. \tag{3.1}$$

One easily checks that the image is exactly the kernel of π_{SM} , so that there is an exact sequence

$$1 \rightarrow \mathbb{Z}_6 = \ker \pi_{SM} \rightarrow \tilde{G}_{SM} \rightarrow \text{Im } \pi_{SM} \rightarrow 1$$

We want to identify $\text{Im } \pi_{SM} = \tilde{G}_{SM}/\mathbb{Z}_6$. Let $G_{SM} := \text{S}(\text{U}(2) \times \text{U}(3))$ be the group of $\text{SU}(5)$ matrices of the form

$$\begin{bmatrix} 2 \times 2 \text{ block} & 0 \\ 0 & 3 \times 3 \text{ block} \end{bmatrix}$$

There is a surjective morphism of groups

$$\tilde{G}_{SM} \ni (\lambda, q, m) \mapsto \begin{bmatrix} \lambda^3 q & \\ & \bar{\lambda}^2 \bar{m} \end{bmatrix} \in G_{SM}, \tag{3.2}$$

where $\bar{m} = (m^*)^t$. An element (λ, q, m) is in the kernel of the map $\tilde{G}_{SM} \rightarrow G_{SM}$ if and only if $q = \bar{\lambda}^3 1_2$ and $m = \bar{\lambda}^2 1_3$. But $\det(q) = \det(m) = \bar{\lambda}^6$ must be 1, hence $\lambda \in \mathbb{Z}_6$ and

$$\text{Im } \pi_{SM} \simeq G_{SM}.$$

The representation can be linearized as follows. Let $J : H \rightarrow H$ be the antilinear operator $J(a \oplus b) := b^* \oplus a^*$, transforming a particle into its antiparticle. We can write

$$\pi_{SM}(\lambda, q, m) = \tilde{\pi}(\tilde{\lambda}, q, \tilde{m}) J \tilde{\pi}(\tilde{\lambda}, q, \tilde{m}) J^{-1}$$

where $\tilde{m} := \bar{\lambda} m \in \text{U}(3)$, $\tilde{\lambda} := \lambda^3$ and

$$\tilde{\pi}(\tilde{\lambda}, q, \tilde{m}) = \left[\begin{array}{cc|cc} \tilde{\lambda} & 0 & 0 & 0 \\ 0 & \tilde{\lambda}^* & 0 & 0 \\ \hline 0 & 0 & & q \\ 0 & 0 & & \end{array} \right] \otimes 1 \oplus \left[\begin{array}{c|ccc} \tilde{\lambda} & 0 & 0 & 0 \\ \hline 0 & & & \\ 0 & & \tilde{m} & \\ \hline 0 & & & \end{array} \right] \otimes 1$$

The latter can be now extended in an obvious way, by \mathbb{R} -linearity, as a representation of the real algebra A_F in (1.1), where we think of quaternions as matrices in $M_2(\mathbb{C})$ of the form

$$\begin{bmatrix} \alpha & \beta \\ -\bar{\beta} & \bar{\alpha} \end{bmatrix}, \quad \alpha, \beta \in \mathbb{C},$$

so that with this identification $\text{U}(\mathbb{H}) = \text{SU}(2)$.

3.2. The data $(A_F, A_F^\circ, H_F, J_F)$. With the identifications as in the previous section the Hilbert space becomes $H_F = M_{8 \times 4}(\mathbb{C})$, with elements:

$$v = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}, \quad v_1, v_2 \in M_4(\mathbb{C}),$$

and inner product $\langle v, w \rangle = \text{Tr}(v^*w)$. Linear operators on H_F are finite sums $L = \sum_i a_i \otimes b_i$, with $a_i \in M_8(\mathbb{C})$ acting via row-by-column multiplication from the left and $b_i \in M_4(\mathbb{C})$ acting via row-by-column multiplication from the right. One easily checks that the adjoint of L is $L^* = \sum_i a_i^* \otimes b_i^*$, with a_i^*, b_i^* denoting Hermitian conjugation.

The real structure J_F is the operator

$$J_F \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} v_2^* \\ v_1^* \end{bmatrix}. \tag{3.3}$$

We identify $A_F = \mathbb{C} \oplus \mathbb{H} \oplus M_3(\mathbb{C})$ with the subalgebra of elements $a \otimes 1 \in \text{End}_{\mathbb{C}}(H_F)$, with a of the form:

$$a = \left[\begin{array}{c|cc} \lambda & 0 & 0 & 0 \\ 0 & \bar{\lambda} & 0 & 0 \\ \hline 0 & 0 & & q \\ 0 & 0 & & \end{array} \right] \left[\begin{array}{c|ccc} \lambda & 0 & 0 & 0 \\ \hline 0 & & & \\ 0 & & m & \\ 0 & & & \end{array} \right], \tag{3.4}$$

with $\lambda \in \mathbb{C}, q \in \mathbb{H}$ and $m \in M_3(\mathbb{C})$ (with zeros on the off-diagonal blocks).

We denote by $A_F^\circ = J_F A_F J_F$ the subalgebra of elements $\text{End}_{\mathbb{C}}(H_F)$ of the form:

$$a^\circ = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \otimes \left[\begin{array}{c|ccc} \lambda & 0 & 0 & 0 \\ \hline 0 & & & \\ 0 & & m & \\ 0 & & & \end{array} \right] + \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \otimes \left[\begin{array}{c|cc} \lambda & 0 & 0 & 0 \\ 0 & \bar{\lambda} & 0 & 0 \\ \hline 0 & 0 & & q \\ 0 & 0 & & \end{array} \right].$$

(On the first factor of each tensor $0, 1 \in M_4(\mathbb{C})$ are the zero and the identity matrix.)

If $A \subset \text{End}_{\mathbb{C}}(H_F)$ is a real $*$ -subalgebra, we denote by $A_{\mathbb{C}}$ the complex linear span of the elements in A ; note that A and $A_{\mathbb{C}}$ have the same commutant in $\text{End}_{\mathbb{C}}(H_F)$. The map $a \mapsto a^\circ = J_F \bar{a} J_F$ (here $\bar{a} = (a^*)^t$) gives two isomorphisms $A_F \rightarrow A_F^\circ$ and $(A_F)_{\mathbb{C}} \rightarrow (A_F^\circ)_{\mathbb{C}}$.

Lemma 3.1. *The commutant of the algebra of elements (3.4) in $M_8(\mathbb{C})$ is the algebra C_F with elements*

$$\left[\begin{array}{cccc} q_{11} & & & q_{12} \\ & \alpha & & \\ & & \beta_{12} & \\ q_{21} & & & q_{22} \\ & & & & \delta_{13} \end{array} \right], \tag{3.5}$$

where the β -block is 2×2 , the δ -block is 3×3 , and all other framed blocks are 1×1 ($\alpha, \beta, \delta \in \mathbb{C}, q = (q_{ij}) \in M_2(\mathbb{C})$). All other blocks are zero (zeroes are omitted).

The commutant of A_F in $\text{End}_{\mathbb{C}}(H)$ is $A'_F = C_F \otimes M_4(\mathbb{C})$.

Proof. By direct computation. □

Note that $A'_F \simeq M_4(\mathbb{C})^{\oplus 3} \oplus M_8(\mathbb{C})$. The map $x \mapsto J_F \bar{x} J_F$ is an isomorphism between A'_F and $(A_F^\circ)'$. From this, we get the following result.

Lemma 3.2. *The commutant $(A_F^\circ)'$ of A_F° has elements*

$$a \otimes e_{11} + \begin{bmatrix} b & \\ & c \end{bmatrix} \otimes e_{22} + \begin{bmatrix} b & \\ & d \end{bmatrix} \otimes (e_{33} + e_{44}) \tag{3.6}$$

with $a \in M_8(\mathbb{C}), b, c, d \in M_4(\mathbb{C})$.

3.3. The data (B_F, \bar{H}_0, H_1) . In this section we explain how to get the same gauge group from a spectral triple based on the complex algebra (1.2) with a degenerate representation.

Let us put particles into a row vector and a 3×4 matrix as follows

$$\left[e_R \quad d_R^1 \quad d_R^2 \quad d_R^3 \right], \quad \begin{bmatrix} v_R & u_R^1 & u_R^2 & u_R^3 \\ v_L & u_L^1 & u_L^2 & u_L^3 \\ e_L & d_L^1 & d_L^2 & d_L^3 \end{bmatrix},$$

Thus a particle is represented by a vector in $\mathbb{C}^3 \oplus M_{3 \times 4}(\mathbb{C})$, with inner product given on each summand by $\langle v, w \rangle = \text{Tr}(v^* w)$. Antiparticles belong to the dual space.

The Hilbert space is then $H = H_0 \oplus \bar{H}_0 \oplus H_1$, where elements of $H_0 \simeq \mathbb{C}^4$ are row vectors, elements of $\bar{H}_0 \simeq \mathbb{C}^4$ are column vectors, and H_1 has elements

$$\begin{bmatrix} a \\ b \end{bmatrix}, \quad a \in M_{3 \times 4}(\mathbb{C}), b \in M_{4 \times 3}(\mathbb{C}).$$

The real structure J is the antilinear operator:

$$J \left(v \oplus w \oplus \begin{bmatrix} a \\ b \end{bmatrix} \right) = w^* \oplus v^* \oplus \begin{bmatrix} b^* \\ a^* \end{bmatrix}. \tag{3.7}$$

We define two unital $*$ -representations

$$\bar{\pi}_0 : B_F \rightarrow \text{End}_{\mathbb{C}}(\bar{H}_0) \quad \text{and} \quad \pi_1 : B_F \rightarrow \text{End}_{\mathbb{C}}(H_1)$$

of the algebra $B_F = \mathbb{C} \oplus M_2(\mathbb{C}) \oplus M_3(\mathbb{C})$ in (1.2) as follows

$$\bar{\pi}_0(\lambda, q, m) = \left[\begin{array}{c|ccc} \lambda & 0 & 0 & 0 \\ \hline 0 & & & \\ 0 & & m & \\ 0 & & & \end{array} \right],$$

$$\pi_1(\lambda, q, m) = \left[\begin{array}{c} \left[\begin{array}{c|cc} \lambda & 0 & 0 \\ \hline 0 & & \\ 0 & & q \end{array} \right] \\ \\ \left[\begin{array}{c|ccc} \lambda & 0 & 0 & 0 \\ \hline 0 & & & \\ 0 & & m & \\ 0 & & & \end{array} \right] \end{array} \right],$$

both acting via row-by-column multiplication from the left. Here $\lambda \in \mathbb{C}$, $q \in M_2(\mathbb{C})$ and $m \in M_3(\mathbb{C})$ are 2×2 and 3×3 blocks, and the off-diagonal 3×4 and 4×3 blocks are zero.

An (injective) representation ρ of $U(A) = U(1) \times U(2) \times U(3)$ is given by (2.4). One can check that ρ , composed with the map

$$\tilde{G}_{SM} = U(1) \times SU(2) \times SU(3) \xrightarrow{\varphi} U(A), \quad (\lambda, q, m) \mapsto (\lambda^6, \lambda^3 q, \lambda^2 m),$$

gives the correct representation of \tilde{G}_{SM} (in particular, each particle has the correct weak hypercharge). The kernel φ is given again by the elements in (3.1), so that the range of φ is $G_{SM} \simeq \tilde{G}_{SM}/\mathbb{Z}_6$. The map

$$U(A) \supset G_{SM} \ni (\lambda, q, m) \mapsto \begin{bmatrix} q & \\ & \bar{m} \end{bmatrix} \in S(U(2) \times U(3))$$

is an isomorphism. We then recover G_{SM} as the subgroup of $U(A)$ satisfying the unimodularity condition

$$\det \bar{\pi}_0(u) = \det \pi_1(u) = 1 .$$

The relation with A_F is as follows. Let $\{a_i\}_{i=1}^4$ be the rows of $a \in M_4(\mathbb{C})$ and $\{b_j\}_{j=1}^4$ the columns of $b \in M_4(\mathbb{C})$. With the isometry

$$H \ni a_2 \oplus b_2 \oplus \left[\begin{array}{c|cc} a_1 & & \\ a_3 & & \\ a_4 & & \\ \hline b_1 & b_3 & b_4 \end{array} \right] \longrightarrow \begin{bmatrix} a \\ b \end{bmatrix} \in M_{8 \times 4}(\mathbb{C})$$

we transform J in (3.7) into the real structure J_F in (3.3), and π into the representation (denoted by the same symbols):

$$\pi(\lambda, q, m) = \left[\begin{array}{c} \left[\begin{array}{cc|cc} \lambda & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \hline 0 & 0 & & q \\ 0 & 0 & & \end{array} \right] \\ \left[\begin{array}{c|ccc} \lambda & 0 & 0 & 0 \\ \hline 0 & & & \\ 0 & & m & \\ 0 & & & \end{array} \right] \end{array} \right] \otimes 1, \quad (3.8)$$

where $\lambda \in \mathbb{C}$, $q \in M_2(\mathbb{C})$ and $m \in M_3(\mathbb{C})$.

Note that the only difference between the matrix in (3.8) and the one in (3.4) is the zero in position (2, 2) replacing $\bar{\lambda}$. More precisely, the algebra $(A_F)_{\mathbb{C}}$ is the minimal unitalization of $\pi(B_F)$ in $\text{End}_{\mathbb{C}}(H_F)$, and $(A_F^{\circ})_{\mathbb{C}}$ is the unitalization of $A^{\circ} := J_F \pi(B_F) J_F$.

Adding the identity doesn't change the commutant, nor Ω^1 . Thus, the results in the next section which we state for the algebra A_F are valid for B_F as well.

4. The 1st order condition

In this section, we describe the most general Dirac operator satisfying the 1st order condition, which is the crucial one for a study of the property (M). To keep things general, at the beginning we make no assumption regarding the other axioms (parity, KO-dimension, etc.). We will then impose the additional requirement $J_F D_F = D_F J_F$, with the plus sign on the right hand side dictated by the physical content of the theory (the mass terms in the spectral action come from elements commuting with J_F). It turns out that for any D_F satisfying the 1st order condition, there is one commuting with J_F which gives the same Clifford algebra (so, the condition

$J_F D_F = D_F J_F$ does not create any particular problem). In the next sections, we will discuss the issue of the grading and the property (M), with or without grading.

The next proposition was originally stated in [18, §3.4], and proved by decomposing the A -bimodule H into irreducible ones and determining the corresponding matrix elements of D . Here, without assuming the orientability condition, we present an alternative proof that doesn't make use of such a decomposition.

Proposition 4.1. *Let H be an $A \otimes A^\circ$ -bimodule (i.e. $[a, b^\circ] = 0 \forall a \in A$ and $b^\circ \in A^\circ$). Then $D \in \text{End}_{\mathbb{C}}(H)$ satisfies the 1st order condition — i.e. $[[D, a], b^\circ] = 0 \forall a \in A, b^\circ \in A^\circ$ — if and only if it is of the form*

$$D = D_0 + D_1$$

where $D_0 \in (A^\circ)'$ and $D_1 \in A'$.

We need a preliminary Lemma.

Lemma 4.2. *Let H be finite-dimensional and V any $*$ -subalgebra of $\text{End}(H)$. Then, there exists a direct complement W of V in $\text{End}(H)$ satisfying $[V, W] \subseteq W$.*

Proof. Modulo an isomorphism, we can assume $H = \mathbb{C}^n$ for some n , and $\text{End}(H) = M_n(\mathbb{C})$. Let $W = V^\perp$ be the orthogonal complement with respect to the Hilbert–Schmidt inner product: $\langle v, w \rangle_{\text{HS}} := \text{Tr}(v^* w) \forall v, w \in M_n(\mathbb{C})$. For all $a, b \in V$ and $c \in V^\perp$, using the cyclic property of the trace, we derive:

$$\langle a, [b, c] \rangle_{\text{HS}} = \langle [b^*, a], c \rangle_{\text{HS}} = 0,$$

where in last step we noticed that $[b^*, a] \in V$, since V is a $*$ -algebra, and then the inner product is zero. Thus $[b, c] \in V^\perp$, and $[V, V^\perp] \subset V^\perp$. \square

Proof of Prop. 4.1. The “if” part is trivial; we now prove the “only if”. We want to prove that the 1st order condition implies $D \in A' + (A^\circ)'$, where by $A' + (A^\circ)'$ we mean the vector space generated by the commutants A' and $(A^\circ)'$ (not the algebra).

We apply Lemma 4.2 to $V = A'$ and decompose $D = D_0 + D_1$ with $D_0 \in A'$ and $D_1 \in W$. From the 1st order condition:

$$[a, [D_1, b^\circ]] = [[D_1, a], b^\circ] - [D_1, [a, b^\circ]] = [[D, a], b^\circ] + 0 = 0,$$

for all $a \in A, b^\circ \in A^\circ$. Hence $[D_1, b^\circ] \in A'$ for all $b^\circ \in A^\circ$. But $A^\circ \subseteq A'$ and from Lemma 4.2 we also have $[D_1, b^\circ] \in W$. Since the sum $\text{End}(H) = A' \oplus W$ is direct, it must be $[D_1, b^\circ] = 0$. This means that $D_1 \in (A^\circ)'$. \square

Note that, contrary to [18], here the decomposition in Prop. 4.1 is not necessarily unique. Uniqueness of the decomposition in [18] follows from the orientability condition. However, we'll see that in the Standard Model example, the spectral triple is not orientable (cf. §7.1) and $A'_F \cap (A^\circ_F)'$ is not zero.

We now come back to the Standard Model. In the rest of the paper, we employ A_F, A_F°, H_F, J_F as defined in §3.2, but the same results are valid for the algebra B_F and the representation discussed in §3.3.

Proposition 4.3. *An operator $D_F = D_F^*$ as in Prop. 4.1 commutes with J_F if and only if it is of the form:*

$$D_F = D_0 + J_F D_0 J_F$$

with $D_0 = D_0^* \in (A_F^\circ)'$.

Proof. $x \mapsto J_F \bar{x} J_F$ gives a bijection $A_F' \rightarrow (A_F^\circ)'$. The condition $J_F D_F J_F = D_F$ gives

$$(J_F D_0 J_F - D_1) + (J_F D_1 J_F - D_0) = 0 .$$

Since the first term is in A_F' and the second in $(A_F^\circ)'$, the sum is zero if and only if both $J_F D_0 J_F - D_1$ and $J_F D_1 J_F - D_0$ belong to $A_F' \cap (A_F^\circ)'$. Called $D' = D_1 - J_F D_0 J_F$, one has the decomposition

$$D_F = D_0 + J_F D_0 J_F + D' .$$

From $J_F D_F J_F - D_F = J_F D' J_F - D'$ one deduces that J_F and D' must commute. So $D_F = (D_0 + D'/2) + J_F (D_0 + D'/2) J_F$ and we get the decomposition (4.2), after renaming $D_0 + D'/2 \rightarrow D_0$.

Decompose $D_0 = S + iT$ with S and T selfadjoint. Since J_F is antilinear and $J_F = J_F^*$:

$$D_F - D_F^* = 2i(T - J_F T J_F)$$

which must be zero. But this implies

$$D_F = S + J_F S J_F + i(T - J_F T J_F) = S + J_F S J_F .$$

Renaming $S \rightsquigarrow D_0$ (which now is selfadjoint) we conclude the proof. □

Remark 4.4. Note that the D_1 term does not contribute to $\mathcal{C}\ell(A_F)$ (it commutes with A_F). Then, for any Dirac operator as in Prop. 4.1, we can find one commuting with J_F (replacing D_1 by $J_F D_0 J_F$) without changing the Clifford algebra $\mathcal{C}\ell(A_F)$. In particular, the property (M) puts constraints only on D_0 .

It is useful to reformulate Prop. 4.1 and Prop. 4.3 as follows. Let

$$D_R := (\Upsilon_R e_{51} + \tilde{\Upsilon}_R e_{15}) \otimes e_{11} , \tag{4.1}$$

with $\Upsilon_R \in \mathbb{C}$. Note that $D_R \in A_F' \cap (A_F^\circ)'$ and $J_F D_R = D_R J_F$.

Proposition 4.5. *The most general $D_F = D_F^*$ satisfying the 1st order condition is*

$$D_F = D_0 + D_1 + D_R \tag{4.2}$$

where $D_0 = D_0^* \in (A_F^\circ)'$ and $D_1 = D_1^* \in A_F'$ have null entry in direction of $e_{15} \otimes e_{11}$ and $e_{51} \otimes e_{11}$, and $D_1 = J_F D_0 J_F$ if D_F and J_F commute.

In this way we isolated all the terms which do not contribute to Ω^1 . For any $a \in A_F$ and D_F as in (4.2), $[D_F, a] = [D_0, a]$.

5. The grading operator

Lemma 5.1. *Let γ_F be a grading operator. Any odd Dirac operator satisfying the 1st order condition can be written in the form $D_F = D_0 + D_1 + \kappa D_R$ as in Prop. 4.5, with both D_0 and D_1 odd operators and $\kappa = 0$ or 1 depending on the parity of D_R .*

Proof. From Prop. 4.1, we can write $D_F = D_0 + D_1 + T_0 + T_1$ where $D_0, T_0 \in (A'_F)'$, $D_1, T_1 \in A'_F$, D_0, D_1 are odd and T_0, T_1 are even. From

$$\gamma_F D_F \gamma_F + D_F = 2(T_0 + T_1) = 0$$

we deduce $T_0 + T_1 = 0$, so that $D_F = D_0 + D_1$ with both D_0 and D_1 odd operators. □

Lemma 5.2. *Let γ_F be a grading operator either commuting or anticommuting with J_F . Any odd Dirac operator satisfying the 1st order condition and commuting with J_F can be written in the form $D_F = D_0 + J_F D_0 J_F + \kappa D_R$ as in Prop. 4.5, with D_0 an odd operator and $\kappa = 0$ or 1 depending on the parity of D_R .*

Proof. It follows from Lemma 5.1. Since $D_1 = J_F D_0 J_F$, the condition $\gamma_F D_0 \gamma_F = -D_0$ implies $\gamma_F D_1 \gamma_F = -D_1$. □

We now study the form of D_0 for Dirac operators of the type described by Lemma 5.1 or 5.2 for two natural choices of the grading operator (we ignore D_1 , cf. Remark 4.4). It is worth noticing that both such gradings anticommute with J_F , and then give real spectral triples of KO-dimension 6.

5.1. The standard grading. The grading in [3, 5, 9, 11] (the chirality operator) is:

$$\gamma_F = \begin{bmatrix} 1_2 & & \\ & -1_2 & \\ & & 0_4 \end{bmatrix} \otimes 1_4 + \begin{bmatrix} 0_4 & \\ & -1_4 \end{bmatrix} \otimes \begin{bmatrix} 1_2 & \\ & -1_2 \end{bmatrix}. \tag{5.1}$$

It follows from Lemma 3.2 that any D_0 anticommuting with (5.1) has the form:

$$D_0 = \begin{bmatrix} & * & * & & \otimes & \otimes & \otimes \\ & * & * & * & \otimes & \otimes & \otimes \\ * & * & & & & & \\ * & * & & & & & \\ \otimes & & & & & & \\ \otimes & & & & & & \\ \otimes & & & & & & \end{bmatrix} \otimes e_{11} + \begin{bmatrix} & * & * \\ & * & * \\ * & * \\ * & * \end{bmatrix} \otimes (1 - e_{11}),$$

where the asterisks indicate the only positions where one can have non-zero matrix entries. The circled entries (\otimes) are the ones that are not allowed by the non-standard grading (5.2).

Let $Z(A)$ be the center of A and $Z(B)$ be the center of B . Note that

$$A \cap B \subset Z(A) \cap Z(B) \subset A' \cap B' \tag{6.1}$$

and that $Z(A) = Z(A') = A \cap A'$, and similarly for B .

Recall that H is a Morita equivalence A - B° -bimodule iff $A = B'$, which is equivalent to the condition $A' = B$ (by von Neumann Bicommutant Theorem: $A'' = A$ and $B'' = B$ in the finite-dimensional case).

Lemma 6.1. *If H is a Morita equivalence A - B° -bimodule, then the inclusions (6.1) are equalities.*

Proof. It follows trivially from $Z(A) = A \cap A'$ and $A' = B$, and similar for $Z(B)$. □

Proposition 6.2.

- i) *If D_F and γ_F are as in §5.1, the property (M), with or without grading, is not satisfied.*
- ii) *Let D_F and γ_F be as in §5.2. If the property (M), with or without grading, holds then*
 - *each summand in D_0 must have at least one circled coefficient (\otimes) different from zero;*
 - *in each of the first two summands, in both the 1st and 2nd row there must be at least one non-zero element;*
 - *in the first summand: at least one element in the 5th row must be non-zero and at least one element in the upper-right block must be non-zero.*

Proof. It is enough to give the proof for the property (M) with grading (the weaker one).

We apply Lemma 6.1 to $A = \mathcal{C}l(A_F)_e$, $B = (A_F^\circ)_\mathbb{C}$ and $H = H_F$. Let D_0 and γ_F be as in §5.1 or §5.2. Note that A is generated by A_F , $[D_0, A_F]$ and γ_F . Moreover, due to the 1st order condition, A and B are mutually commuting.

Any operator $X \in A'_F \cap (A_F^\circ)'$ commuting with D_0 and γ_F belongs to $A' \cap B'$ (since it also commutes with $[D_0, A_F]$). If we can exhibit such an X and prove that $X \notin Z(B)$, then the property (M) with grading is not satisfied. Note that $Z(B)$ has elements:

$$a^\circ = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \otimes \left[\begin{array}{c|ccc} \lambda & 0 & 0 & 0 \\ \hline 0 & & & \\ 0 & & \alpha 1_3 & \\ 0 & & & \end{array} \right] + \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \otimes \left[\begin{array}{cc|cc} \lambda & 0 & 0 & 0 \\ \hline 0 & \lambda' & 0 & 0 \\ 0 & 0 & & \\ 0 & 0 & & \beta 1_2 \end{array} \right],$$

with $\lambda, \lambda', \alpha, \beta \in \mathbb{C}$. For D_0, γ_F as in §5.1, the operator $X = e_{55} \otimes (1 - e_{11})$ does the job:

- 1) it commutes with D_0 and γ_F ,
- 2) it belongs to $A'_F = C_F \otimes M_4(\mathbb{C})$ ($e_{55} \in C_F$: take $q_{22} = 1$ and all other coefficients zero in (3.5)),
- 3) it belongs to $(A_F^\circ)'$ (cf. Lemma 3.2),
- 4) and it does not belong to $Z(B)$.

Let now D_0 and γ_F be as in §5.2. Concerning the first summand:

- if all the circled terms (\otimes) are zero, then $X = (e_{66} + e_{77} + e_{88}) \otimes e_{11}$ satisfies the conditions (1–4) above;
- if all the elements in the 1st resp. 2nd row are zero (and then also in 1st resp. 2nd column, by hermiticity), then $X = e_{11} \otimes e_{11}$ resp. $X = e_{22} \otimes e_{11}$ satisfies (1–4);
- if all the elements in the 5th row are zero, similarly by hermiticity $X = e_{55} \otimes e_{11}$ satisfies the conditions (1–4) above;
- if all the elements in the upper-right blocks are zero, then $X = (e_{11} + e_{22} + e_{33} + e_{44}) \otimes e_{11}$ satisfies the conditions (1–4) above.

Concerning the second summand:

- if all the elements in the 1st resp. 2nd row are zero (and then also in 1st resp. 2nd column, by hermiticity), then $X = e_{11} \otimes (1 - e_{11})$ resp. $X = e_{22} \otimes (1 - e_{11})$ satisfies the conditions (1–4) above;

Concerning the third resp. fourth summands:

- if the circled terms (\otimes) are zero, $X = e_{55} \otimes e_{22}$ resp. $X = e_{55} \otimes (e_{33} + e_{44})$ satisfies the conditions (1–4) above. \square

Corollary 6.3. *Let D_F be as in §5.3 and γ_F one of the two gradings (5.1) or (5.2). If $\Delta = 0$ or $\Omega = 0$, the property (M), with or without grading, is not satisfied.*

The operator D_F in §5.3, with $\Omega \neq 0$ and $\Delta \neq 0$, represents a minimal modification of the Dirac operator of [3, 5, 9, 11] which satisfies all the conditions in Prop. 6.2. We will now show that for such an operator, the Morita condition is satisfied.

6.1. Morita with a grading. This section is devoted to prove the following theorem.

Theorem 6.4. *Let γ_F be as in §5.2, D_F as in §5.3 with all coefficients different from zero, and assume that at least one of the following conditions holds:*

1. $\Upsilon_v \neq \pm \Upsilon_u$,
2. $\Upsilon_e \neq \pm \Upsilon_d$.

Then, the spectral triple:

- i) does not satisfy the property (M);
- ii) it satisfies the property (M) with grading.

As a corollary, $\mathcal{C}\ell(A_F)_o \neq \mathcal{C}\ell(A_F)_e$, so $\gamma_F \notin \mathcal{C}\ell(A_F)_o$.

We need a preliminary lemma. From now on, we assume that the hypothesis of Thm. 6.4 are satisfied.

Lemma 6.5. *The A_F -bimodule Ω^1 is generated by the four elements:*

$$\begin{aligned}\omega_v &= e_{31} \otimes (\Upsilon_v e_{11} + \Upsilon_u(1 - e_{11})) , & \xi &= e_{52} \otimes e_{11} , \\ \omega_e &= e_{42} \otimes (\Upsilon_e e_{11} + \Upsilon_d(1 - e_{11})) , & \eta &= e_{56} \otimes 1 ,\end{aligned}$$

and their adjoints.

Proof. A linear basis of $(A_F)_\mathbb{C}$ is given by the elements:

$$\begin{aligned}X_{ij} &:= e_{ij} \otimes 1 \text{ with } i, j = 3, 4, & Y &:= e_{22} \otimes 1 , \\ Z_{kl} &:= e_{kl} \otimes 1 \text{ with } k, l = 6, 7, 8, & T &:= (e_{11} + e_{55}) \otimes 1 .\end{aligned}$$

For any projection $p^2 = p = p^*$, the commutator

$$[D_F, p] = [D_F, p^2] = p[D_F, p] + [D_F, p]p$$

is a linear combination of $p[D_F, p]$ and its adjoint $-[D_F, p]p$. Hence $X_{33}[D_F, X_{33}]$ and $X_{44}[D_F, X_{44}]$ can be taken as generators, instead of $[D_F, X_{33}]$ and $[D_F, X_{44}]$. An explicit computation gives:

$$-X_{33}[D_F, X_{33}] = \omega_v , \quad -X_{44}[D_F, X_{44}] = \omega_e .$$

Note that $[D_F, X_{34}]$ is also the adjoint of $[D_F, X_{43}]$, and

$$[D_F, X_{43}] = (X_{34}\omega_e)^* - X_{43}\omega_v$$

is still generated by ω_v, ω_e and adjoints. Next

$$[D_F, Y]Y = \omega_e + \Omega\xi , \quad [D_F, Z_{66}]Z_{66} = \Delta\eta .$$

Since $\Omega, \Delta \neq 0$, this proves that $\xi, \eta \in \Omega^1$.

Furthermore $[D_F, Z_{6k}] = \Delta\eta Z_{6k}$ and $[D_F, Z_{k6}] = -[D_F, Z_{6k}]^*$ are combinations of η and η^* for all $k = 7, 8$, and $[D_F, Z_{jk}] = 0$ if $j, k = 7, 8$. Finally

$$-T[D_F, T] = \omega_v^* + \Omega\xi + \Delta\eta ,$$

proving that the elements $\omega_v, \omega_e, \xi, \eta$ and their adjoints are a generating family for Ω^1 . \square

Proof of Theorem 6.4. We now prove that: (i) $\mathcal{Cl}(A_F)'_0 \not\supseteq (A_F^\circ)_{\mathbb{C}}$ (it is strictly greater), i.e. the property (M) is not satisfied. (ii) $\mathcal{Cl}(A_F)'_e = (A_F^\circ)_{\mathbb{C}}$, i.e. the property (M) with grading is satisfied.

$\mathcal{Cl}(A_F)'_0$ is given by the set of elements in Lemma 3.1 that commute with the generators in Lemma 6.5. A tensor $\sum x_{ij} \otimes e_{ij}$, with each x_{ij} as in (3.5), commutes with η and η^* iff $q_{12} = q_{21} = 0$ and $q_{22} = \delta$. Hence, the most general $\phi \in A'_F$ commuting with η, η^* is:

$$\phi = e_{11} \otimes a + e_{22} \otimes b + (e_{33} + e_{44}) \otimes c + \left(\sum_{i=5}^8 e_{ii}\right) \otimes d$$

with $a, b, c, d \in M_4(\mathbb{C})$ arbitrary matrices. Its commutator with ξ and ξ^* vanishes iff

$$de_{11} = e_{11}b, \quad e_{11}d = be_{11}. \tag{6.2}$$

Its commutator with ω_ν, ω_e and their adjoints vanishes iff:

$$Ea = cE, \quad aE = Ec, \quad Fb = cF, \quad bF = Fc, \tag{6.3}$$

where

$$E := \left[\begin{array}{c|ccc} \Upsilon_\nu & 0 & 0 & 0 \\ \hline 0 & & & \\ 0 & & \Upsilon_u 1_3 & \\ 0 & & & \end{array} \right], \quad F := \left[\begin{array}{c|ccc} \Upsilon_e & 0 & 0 & 0 \\ \hline 0 & & & \\ 0 & & \Upsilon_d 1_3 & \\ 0 & & & \end{array} \right],$$

are invertible by hypothesis. It follows from (6.3) that c commutes with both E^2 and F^2 . If the hypothesis of Theorem 6.4 are satisfied, at least one of the matrices E^2, F^2 is not proportional to the identity. Its commutation with c implies that

$$c = \left[\begin{array}{c|ccc} \lambda & 0 & 0 & 0 \\ \hline 0 & & & \\ 0 & & m & \\ 0 & & & \end{array} \right],$$

for some $\lambda \in \mathbb{C}$ and $m \in M_3(\mathbb{C})$. But then c commutes with E and F as well, and it follows from (6.3) that $a = E^{-1}cE = c$ and $b = F^{-1}cF = c$. Now, b commutes with e_{11} as well, and from (6.2) we get

$$d = \left[\begin{array}{c|ccc} \lambda & 0 & 0 & 0 \\ \hline 0 & & & \\ 0 & & m' & \\ 0 & & & \end{array} \right],$$

with the same λ as before, and with $m' \in M_3(\mathbb{C})$. Thus, $\mathcal{Cl}(A_F)'_0$ has elements

$$\phi = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \otimes \left[\begin{array}{c|ccc} \lambda & 0 & 0 & 0 \\ \hline 0 & & & \\ 0 & & m & \\ 0 & & & \end{array} \right] + \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \otimes \left[\begin{array}{c|ccc} \lambda & 0 & 0 & 0 \\ \hline 0 & & & \\ 0 & & m' & \\ 0 & & & \end{array} \right], \tag{6.4}$$

with $\lambda \in \mathbb{C}$ and $m, m' \in M_3(\mathbb{C})$, and is strictly greater than $(A_F^\circ)_{\mathbb{C}}$.

Imposing the extra condition $[\phi, \gamma_F] = 0$, we reduce one $M_3(\mathbb{C})$ to $\mathbb{C} \oplus M_2(\mathbb{C})$. Indeed $[\phi, \gamma_F] = 0$ iff d commutes with the matrix

$$\begin{bmatrix} 1_2 & \\ & -1_2 \end{bmatrix},$$

i.e. m' belongs to $\mathbb{C} \oplus M_2(\mathbb{C}) \subset M_3(\mathbb{C})$. This proves that $\mathcal{C}l(A_F)'_e = (A_F^\circ)_{\mathbb{C}}$. \square

6.2. Morita without the grading. Let D_0 be as in §5.3 and

$$\tilde{D}_F = \tilde{D}_0 + J_F \tilde{D}_0 J_F + D_R,$$

with

$$\tilde{D}_0 := D_0 + \Gamma(e_{57} + e_{75}) \otimes e_{22}.$$

Note that this is still of the type described in §5.2. Here we have three additional parameters with respect to [3, 5, 9, 11]: $\Omega \in \mathbb{C}$ and $\Delta \in \mathbb{R}$ as in §5.3, and the new one $\Gamma \in \mathbb{R}$.

Theorem 6.6. *Let $\Upsilon_v, \Upsilon_e, \Upsilon_u, \Upsilon_d, \Omega, \Delta, \Gamma$ be all different from zero, and at least one of the following two conditions satisfied:*

1. $\Upsilon_v \neq \pm \Upsilon_u$,
2. $\Upsilon_e \neq \pm \Upsilon_d$.

Then $(A_F, H_F, \tilde{D}_F, J_F)$ satisfies the property (M).

Lemma 6.7. *If $\Omega, \Delta, \Gamma \neq 0$, the A_F -bimodule Ω^1 is generated by the elements in Lemma 6.5 plus the element*

$$\zeta = e_{57} \otimes e_{22}$$

and its adjoint.

Proof. Repeating the proof of Lemma 6.5, the only change is:

$$\begin{aligned} [D_F, Z_{77}]Z_{77} &= \Gamma\zeta, & [D_F, Z_{78}] &= \Gamma\zeta Z_{78}, \\ -T[D_F, T] &= \omega_v^* + \Omega\xi + \Delta\eta + \Gamma\zeta, & \text{and } [D_F, Z_{87}] &= -[D_F, Z_{78}]^*. \end{aligned} \quad \square$$

Proof of Theorem 6.6. $\mathcal{C}l(A_F)'_0$ now is the set of elements ϕ in (6.4) which in addition commute with ζ . But $[\phi, \zeta] = 0$ iff $m' \in \mathbb{C} \oplus M_2(\mathbb{C}) \subset M_3(\mathbb{C})$, so $\mathcal{C}l(A_F)'_0 = (A_F^\circ)_{\mathbb{C}}$ and the property (M) holds. \square

7. Some remarks on orientability and irreducibility

7.1. Orientability. A classification of finite-dimensional spectral triples satisfying, among other axioms, the orientability condition is in [18]; in fact, in the classification of Dirac operators such assumption plays a crucial role: for example, the uniqueness

of the decomposition in §3.4 follows immediately from the orientability condition. In our case (in the Standard Model with neutrino mixing), the sum $A'_F + (A_F^\circ)'$ is not direct, the term D_R in (4.1) being an example of non-trivial element in the intersection. This already suggests that the orientability condition in global dimension zero is not satisfied. In fact, we can say something more.

Proposition 7.1. *Let either*

- (1) D_F and γ_F be as in Theorem 6.4, or
- (2) γ_F be the standard grading in §5.1 and D_F any operator of the type described in §5.1.

In both cases, there is no chain $c \in A_F^{\otimes n+1}$ such that $\pi_D(c) = \gamma_F$, for any $n \geq 0$.

Proof. (1) $\gamma_F \notin \mathcal{C}\ell(A_F)_0$, as stated in Theorem 6.4.

(2) Let $X := e_{55} \otimes e_{23}$. This operator commutes with A_F and D_0 , hence with any element of $\mathcal{C}\ell(A_F)_0$. But it anticommutes with γ_F , proving that $\gamma_F \notin \mathcal{C}\ell(A_F)_0$. \square

A stronger statement holds for Chamseddine–Connes Dirac operator.

Proposition 7.2. *Let γ_F as in (5.1) or (5.2) and D_F as in §5.3. If $\Upsilon_\nu, \Omega, \Delta$ are all zero, then there is no chain $c \in (A_F \otimes A_F^\circ) \otimes A_F^{\otimes n}$ such that $\pi_D(c) = \gamma_F$, for any $n \geq 0$.*

Proof. The element $X = e_{15} \otimes e_{11}$ commutes with A_F, A_F° and D_0 , hence with any element in the image of the map π_D , but it anticommutes with γ_F , hence $\gamma_F \notin \text{Im}(\pi_D)$. \square

For 0-chains, it follows from the argument in [18] that, no matter which Dirac operator one chooses, the orientability conditions cannot be satisfied.

Proposition 7.3. *For γ_F as in (5.1) or (5.2) there is no $c \in A_F \otimes A_F^\circ$ such that $\pi_D(c) = \gamma_F$.*

Proof. The operator $X := e_{15} \otimes e_{11}$ belongs to $A'_F \cap (A_F^\circ)'$, but anticommutes with γ_F . Hence γ_F is not in the algebra generated by A_F and A_F° . \square

For the spectral triple of Theorem 6.6, on the other hand, since $\gamma_F \in (A_F^\circ)'$ and $(A_F^\circ)' = \mathcal{C}\ell(A_F)_0$ due to the property (M), it immediately follows that $\gamma_F \in \mathcal{C}\ell(A_F)_0$.

Proposition 7.4. *Let the spectral triple be as in Theorem 6.6. Then there is a $c \in \bigoplus_{n \geq 0} A_F^{\otimes n+1}$ such that $\pi_D(c) = \gamma_F$.*

Of course, this gives no clue on whether c in previous proposition is a simple tensor (so, a chain) or possibly a cycle.

7.2. Irreducibility. We say that a real spectral triple (A, H, D, J) is *irreducible* if there is no proper subspace of H , other than $\{0\}$, which carries a subrepresentation of A and is stable under D, J and (in the even case) γ . Equivalently, it is irreducible if there is no non-trivial projection $p = p^* = p^2 \in \text{End}_{\mathbb{C}}(H)$ (so, other than 0 and 1), which commute with A, D, J and γ in the even case [16, Def. 11.2].

Let D_F be the operator in §5.3, and γ_F one of the gradings in (5.1) or (5.2). If $\Delta = 0$ (and possibly $\Omega \neq 0$), then $(A_F, H_F, D_F, \gamma_F, J_F)$ is clearly reducible. Take:

$$p = \left(\sum_{i=1}^4 e_{ii} \right) \otimes e_{11} + e_{55} \otimes 1$$

the operator projecting on the subspace of H_F containing only leptons. It clearly commutes with A_F, D_F, γ_F and J_F .

In order to have irreducibility, we need in D_F a term mixing leptons and quarks.

Proposition 7.5. *The even spectral triple of Theorem 6.4 and the odd spectral triple of Theorem 6.6 are both irreducible.*

Proof. For the even spectral triple of Theorem 6.4, if p is a projection commuting with A_F, D_F, γ_F, J_F . Then it belongs to $\mathcal{C}\ell(A_F)'_e = A_F^\circ$ (property (M)). Similarly, for the odd spectral triple of Theorem 6.6, one proves that p must belong to $\mathcal{C}\ell(A_F)'_o = (A_F^\circ)_{\mathbb{C}}$. But it also commutes with $A_F^\circ = J_F A_F J_F$, so it belongs to the center of $(A_F^\circ)_{\mathbb{C}}$. Hence:

$$p = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \otimes \left[\begin{array}{c|ccc} \lambda & 0 & 0 & 0 \\ \hline 0 & & & \\ 0 & & \beta 1_3 & \\ 0 & & & \end{array} \right] + \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \otimes \left[\begin{array}{cc|cc} \lambda & 0 & 0 & 0 \\ 0 & \lambda' & 0 & 0 \\ \hline 0 & 0 & & \\ 0 & 0 & & \delta 1_2 \end{array} \right],$$

with $\lambda, \lambda', \beta, \delta \in \mathbb{C}$. Since

$$J_F p J_F = \{ \lambda(e_{11} + e_{55}) + \lambda' e_{22} + \delta(e_{33} + e_{44}) + \beta(e_{55} + e_{66} + e_{77}) \} \otimes 1,$$

if p commutes with J_F it is proportional to the identity, hence $p = 0$ or $p = 1$. \square

Let us mention that other inequivalent definitions of irreducibility can be used. For example, the one adopted in §18.3 of [11] says that a real spectral triple is irreducible if H carries an irreducible representation of A and J . Such a condition is stronger than the one used by us, and is the condition leading to the algebra $M_2(\mathbb{H}) + M_4(\mathbb{C})$. This is later reduced to A_F (which allows to introduce a grading and leads to the original Dirac operator of [3, 5, 9, 11]), thus loosing the irreducibility property. In the next section, we discuss the intermediate algebra A^{ev} of the Pati–Salam model.

7.3. On the Pati–Salam model. The Pati–Salam model is a grand unified theory with gauge group $\text{Spin}(4) \times \text{Spin}(6) \simeq \text{SU}(2) \times \text{SU}(2) \times \text{SU}(4)$. The relevant algebra is now $A^{\text{ev}} = \mathbb{H} \oplus \mathbb{H} \oplus M_4(\mathbb{C})$, which we identify with the subalgebra of elements $a \otimes 1 \in \text{End}_{\mathbb{C}}(H_F)$, with a of the form:

$$a = \left[\begin{array}{c|c} \left[\begin{array}{c|c} x & \\ \hline & y \end{array} \right] & \\ \hline & m \end{array} \right],$$

with $x, y \in \mathbb{H}$ (and we think of them as 2×2 complex matrices) and $m \in M_4(\mathbb{C})$. All off-diagonal blocks are zero.

The data $(A^{\text{ev}}, H_F, J_F, \gamma_F, D_F)$, with D_F as in §5.3 and γ_F as in (5.1), satisfies all the conditions of a real spectral triple except for the 1st order condition [6,7] (and then the property (M) cannot be satisfied). On the other hand, it is a simple check to verify that irreducibility, in the stronger sense of §18.3 of [11] (so, without γ_F and D_F) is satisfied.

Lemma 7.6. *The commutant $(A^{\text{ev}})'$ has elements $\sum a \otimes b$ with $b \in M_4(\mathbb{C})$ arbitrary and $a \in M_8(\mathbb{C})$ of the form*

$$a = \left[\begin{array}{c|c|c} \alpha_{1_2} & & \\ \hline & \beta_{1_2} & \\ \hline & & \delta_{1_4} \end{array} \right],$$

where the α and β -blocks are 2×2 , the δ -block is 4×4 , and $\alpha, \beta, \delta \in \mathbb{C}$.

Proof. By direct computation. □

Proposition 7.7. *There is no non-trivial projections on H_F commuting with A^{ev} and J_F .*

Proof. It follows from previous lemma that any p commuting with A^{ev} has the form

$$p = (e_{11} + e_{22}) \otimes \alpha + (e_{33} + e_{44}) \otimes \beta + \left(\sum_{i=5}^8 e_{ii} \right) \otimes \delta,$$

where now $\alpha, \beta, \delta \in M_4(\mathbb{C})$ are three projections. Since (in 4×4 blocks):

$$J_F p J_F = \begin{bmatrix} 0 & 0 \\ 0 & \alpha \end{bmatrix} \otimes (e_{11} + e_{22}) + \begin{bmatrix} 0 & 0 \\ 0 & \beta \end{bmatrix} \otimes (e_{33} + e_{44}) + \begin{bmatrix} \delta & 0 \\ 0 & 0 \end{bmatrix} \otimes 1,$$

we deduce that p commutes with J_F if and only if $\alpha = \beta = \delta$ are proportional to the identity, and then $p = 0$ or $p = 1$ is a trivial projection. □

Orientability (in the weak sense) is also easy to check, since in (5.1) the first summand belongs to A^{ev} and the second is minus the conjugated by J_F . So, $\gamma_F \in A^{\text{ev}} + J_F A^{\text{ev}} J_F$ (which implies weak orientability, since every 0-chain is a cycle).

8. Conclusions

In this paper we studied the property (M), cf. Def. 2.4, which is a possible natural mathematical generalization of the notion of spin-manifold and of Dirac spinors to noncommutative geometry. Although the original Chamseddine–Connes' spectral triple is shown not to satisfy this property, we find that it is enough to add two terms to the Dirac operator D_F and slightly change the grading in order to satisfy it. The new terms in the Dirac operator will generate of course new fields (they are introduced with the purpose of enlarging the module of 1-forms, and then the Clifford algebra; generators are given in Lemma 6.5). Then study of whether (and how) they contribute to the action functional of the model is however beyond the scope of this paper. Obviously the non zero (for the property (M)) constants Ω , Δ in front of the new terms can be arbitrarily small and so below the current experimental observation threshold. Their fine-tuning in order to get the correct value of the Higgs mass could be studied in future work.

Of the two terms, the Ω -term is compatible with both the original and the modified grading of §5.1 and §5.2. The Δ -term on the other hand is compatible (anticommutes) only with the modified grading. Such a term may potentially (see [21]) mix quarks and leptons, and although it may seem exotic, it is also necessary for the irreducibility of the spectral triple (cf. §7.2): without this term, the leptonic and quark sectors of H_F carry each one a sub-spectral triple. A third additional term (with coefficient Γ) in D_F is instead necessary (though non sufficient) if one wants the spectral triple to be also orientable, cf. Prop. 7.4 for the precise statement.

Concerning the grading, the one in (5.2) is minimal a modification the one in (5.1): they agree on leptons and have opposite sign on baryons (quarks). A study of the physical consequences of this modification are beyond the scope of this paper. The internal grading contributes to the grading of the full spectral triple, product of the finite-dimensional one with the canonical spectral triple of a 4-dimensional spin manifold. The full grading is used to project out from the Hilbert space unphysical degrees of freedom (and partially solve the quadrupling of degrees of freedom, cf. [11] for the details). Thus, changing the grading in principle could affect this part of the theory, which should be studied in the future.

We close by stressing again that the aim of this paper was a mathematical study of the property (M), and few related issues, but a detailed analysis of the physical aspects of our model (for example, understanding what happens to the Higgs mass) goes beyond the scope of the paper and is postponed to future works.

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