Modular envelopes, OSFT and nonsymmetric (non-Σ) modular operads

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Abstract. Our *aim* is to introduce and advocate non-Σ (non-symmetric) modular operads. While ordinary modular operads were inspired by the structure of the moduli space of stable complex curves, non-Σ modular operads model surfaces with open strings outputs. An immediate *application* of our theory is a short proof that the modular envelope of the associative operad is the linearization of the terminal operad in the category of non-Σ modular operads. This gives a succinct description of this object that plays an important rôle in open *string field theory*. We also sketch further perspectives of the presented approach.

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1. Introduction

Operads have their non-Σ (non-symmetric) versions obtained by forgetting the symmetric group actions. Likewise, for the non-Σ version of cyclic operads one requires only the actions of the cyclic subgroups of the symmetric groups, see e.g. Definitions II.1.4, II.1.14 and Section II.5.1 of [32]. In both cases we thus demand less structure. As we explain in Example 2.9 below, this straightforward approach fails for modular operads whose non-Σ versions have been a mystery so far.
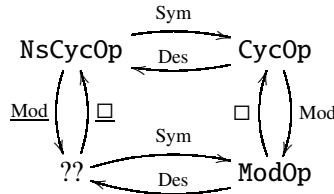
There are, fortunately, some clues and inspirations, namely modular envelopes of cyclic operads, introduced under the name modular completions by the author in [28, Definition 2]. The modular envelope $\mathrm{Mod}(*_C)$ of the terminal cyclic operad $*_C$ in the category Set of sets turned out to be the terminal modular operad in Set, see [28, p. 382]. Notice that the linearization $\mathrm{Span}(*_C)$ of $*_C$ is the cyclic operad $\mathcal{C}om$ for commutative associative algebras.

The modular envelope $\mathrm{Mod}(\underline{*}_C)$ of the terminal non-Σ cyclic Set-operad, described much later in [6, 9], turned out to be surprisingly complex. (The authors of [6, 9] worked with the linearized version, i.e. with the cyclic operad $\mathcal{A}ss$ for

associative algebras, but the linear structure is irrelevant here as everything important happens inside Set.) A derived version of this modular envelope was studied in [7, 15]. Guided by the example of $\mathrm{Mod}(*_C)$, one would expect $\mathrm{Mod}(\underline{*}_C)$ to be the (symmetrization of) the terminal Set-operad in the conjectural category of non-Σ modular operads. The description of $\mathrm{Mod}(\underline{*}_C)$ given in [9, Theorem 3.1] and its relation to the moduli space of Riemann surfaces with boundaries [6, Theorem 3.7] therefore gives some feeling what non-Σ modular operads should be.

Another natural requirement is that the conjectural category of non-Σ modular operads should fill the left bottom corner of the diagram

$$
\begin{array}{ccc}
& \xrightarrow{\text{Sym}} & \\
\mathtt{NsCycOp} & \underset{\text{Des}}{\xleftarrow{\hspace{1cm}}} & \mathtt{CycOp} \\
\text{Mod} \Big\Updownarrow \ \square & & \square \ \Big\Updownarrow \text{Mod} \\
& \xrightarrow{\text{Sym}} & \\
?? & \underset{\text{Des}}{\xleftarrow{\hspace{1cm}}} & \mathtt{ModOp}
\end{array}
$$

in which $\square : \mathtt{ModOp} \to \mathtt{CycOp}$ is the forgetful functor, $\mathrm{Mod} : \mathtt{CycOp} \to \mathtt{ModOp}$ the modular completion [28] (known today as the modular envelope), $\mathrm{Des} : \mathtt{CycOp} \to \mathtt{NsCycOp}$ the forgetful functor (the desymmetrization, not to be mistaken with Batanin's desymmetrization of [2]) and the symmetrization $\mathrm{Sym} : \mathtt{NsCycOp} \to \mathtt{CycOp}$ its left adjoint.

The category of (ordinary) modular operads contains the category of cyclic operads as the full subcategory of operads concentrated in genus 0. Requiring the same from the category of non-Σ modular operads leads to the notion of geometricity that does not have analog in the symmetric world.

Our *aim* is to introduce and advocate the notion of non-Σ (non-symmetric) modular operads. The main *definitions* are Definitions 4.1 and 5.6, and the main *result* is isomorphism (6.2a) of Theorem 6.3. As an immediate *application* we give a short, elementary proof of the description of the modular envelope $\mathrm{Mod}(\mathcal{A}ss)$ of the associative operad given in [6, 9].

Perspectives. It turns out that the elements of the modular envelope $\mathrm{Mod}(*_C)$ of the terminal cyclic Set operad $*_C$ describe isomorphism classes of oriented surfaces with holes, and likewise the non-Σ modular envelope $\underline{\mathrm{Mod}}(\underline{*}_C)$ of the terminal non-Σ cyclic Set-operad describe isomorphism classes of oriented surfaces with teethed holes. These geometric objects describe interactions in closed and open string field theory. Very crucially, Theorem 6.3 asserts that both $\mathrm{Mod}(*_C)$ and $\underline{\mathrm{Mod}}(\underline{*}_C)$ are the *terminal* objects in an appropriate category of modular operads.

In Section 7 we consider the operad $*_D$ describing associative algebras with an involution. It is a cyclic dihedral operad in the sense of [31, Section 3] that equals the Möbiusisation [4, Definition 3.32] of the terminal cyclic operad $*_C$. By [4, Theorem 3.10], its modular envelope $\mathrm{Mod}(*_D)$ consist of isomorphism classes

of *non-oriented* surfaces with teethed holes. We believe that there exists another version of modular operads (say dihedral modular operads) such that $\mathrm{Mod}(\ast_D)$ is the (symmetrization of the) *terminal* modular operad of this type.

We hope that similar reasoning applies to other objects, such as the surfaces describing interactions in open-closed string field theory that may include also D-branes, or even higher-dimensional manifolds. In each case the corresponding type of modular operad should reflect how a geometric object is composed from simper pieces, e.g. pair of pants in the case of closed string field theory. We are led to formulate

Principle. *For a large class of geometric objects there exists a version of modular operads such that the set of isomorphism classes of these objects is the terminal modular* Set-*operad of a given type.*

Several steps in this direction have already been made in [8] where various generalizations of dihedral, quaternionic and other generalizations of cyclic structures based on crossed simplicial groups of Krasauskas [25] and Fiedorowicz–Loday [10] were studied, and their relation to structured surfaces was clarified.

The nature of the above principle is similar to the cobordism hypothesis [11] as they both describe objects of geometric nature by purely categorial means, see again [8] where the relation to the structured cobordism hypothesis in dimension 2 was explicitly formulated, cf. also Remark 4.3.7 of [8].

Notations and conventions. Throughout the paper, $\mathtt{M} = (\mathtt{M}, \otimes, 1)$ will stand for a complete and cocomplete, possibly enriched, symmetric monoidal category, with the initial object $0 \in \mathtt{M}$. Typical examples will be $\Bbbk$-Mod, the category of modules over a commutative unital ring $\Bbbk$, or the cartesian category Set of sets. By Span : Set $\to$ $\Bbbk$-Mod we denote the $\Bbbk$-linear span, i.e. the left adjoint to the forgetful functor $\Bbbk$-Mod $\to$ Set. The glossary of categories introduced and used throughout the paper is given in Figure 1, various forgetful functors and their adjoints are listed in Figure 2.

An order (without an adjective) in this paper will always be a linear order of a finite set. By $\ast$ we denote a chosen one-point set.

Assumptions. We assume certain familiarity with basic notions of operad theory. There exists rich and easily accessible literature, for instance the monograph [32], overview articles [12, 29, 30] or a recent account [26]. For the reader's convenience, we recall a definition of cyclic and modular operads based on finite sets in the Appendix.

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`Set`,	the cartesian monoidal category of sets,
`k-Mod`,	the category of $\Bbbk$-modules,
`Fin`,	the category of finite sets,
`Cyc`,	the category of finite cyclically ordered sets,
`MultCyc`,	the category of finite multicyclically ordered sets,
`ModMod`,	the category of modular modules,
`NsModMod`,	the category of non-Σ modular modules,
`CycOp`,	the category of cyclic operads,
`NsCycOp`,	the category of non-Σ cyclic operads,
`ModOp`,	the category of modular operads,
`NsModOp`,	the category of non-Σ modular operads,
$\boldsymbol{\Gamma}((\mathrm{S}; g))$,	the category of non-Σ modular graphs.

Figure 1. Notation of categories.

forgetful functor:	left adjoint:
$\square : \mathtt{ModOp} \longrightarrow \mathtt{CycOp}$	$\mathrm{Mod} : \mathtt{CycOp} \longrightarrow \mathtt{ModOp}$
$\underline{\square} : \mathtt{NsModOp} \longrightarrow \mathtt{NsCycOp}$	$\underline{\mathrm{Mod}} : \mathtt{NsCycOp} \longrightarrow \mathtt{NsModOp}$
$\mathrm{Des} : \mathtt{CycOp} \longrightarrow \mathtt{NsCycOp}$	$\mathrm{Sym} : \mathtt{NsCycOp} \longrightarrow \mathtt{CycOp}$
$\mathrm{Des} : \mathtt{ModOp} \longrightarrow \mathtt{NsModOp}$	$\mathrm{Sym} : \mathtt{NsModOp} \longrightarrow \mathtt{ModOp}$
$F : \mathtt{ModOp} \longrightarrow \mathtt{ModMod}$	$\mathbb{M} : \mathtt{ModMod} \longrightarrow \mathtt{ModOp}$
$\underline{F} : \mathtt{NsModOp} \longrightarrow \mathtt{NsModMod}$	$\underline{\mathbb{M}} : \mathtt{NsModMod} \longrightarrow \mathtt{NsModOp}$
$U : \Bbbk\text{-}\mathtt{Mod} \longrightarrow \mathtt{Set}$	$\mathrm{Span} : \mathtt{Set} \longrightarrow \Bbbk\text{-}\mathtt{Mod}$

Figure 2. Forgetful functors and their left adjoints.

2. Cyclic orders and the first try

We open this section by recalling basic facts about cyclically ordered sets and their morphisms. Although we will actually need only combinations of *iso*morphisms of *totally* cyclically ordered finite sets, see Remark 3.3, we will present the definitions in full generality, putting the material of this and the following sections into a

broader context. Our exposition follows closely [8, Section 2.2], cf. also the classical sources [16, 33].

Definition 2.1. A *partial cyclic order* on a set C is a ternary relation $\mathfrak{C} \subset C^{\times 3}$ satisfying the following conditions.

   (i) The triplet (c, c, c) belongs to $\mathfrak{C}$ for any $c \in C$.

   (ii) If $(a, b, c) \in \mathfrak{C}$ and $(b, a, c) \in \mathfrak{C}$, then card$\{a, b, c\} \leq 2$.

   (iii) Let $a, b, c, d \in C$ be mutually distinct elements. If $(a, b, c) \in \mathfrak{C}$ and $(a, c, d) \in \mathfrak{C}$, then $(a, b, d) \in \mathfrak{C}$ and $(b, c, d) \in \mathfrak{C}$.

   (iv) If $(a, b, c) \in \mathfrak{C}$, then $(b, c, a) \in \mathfrak{C}$.

   (v) If $(a, b, c) \in \mathfrak{C}$, then $(a, a, c) \in \mathfrak{C}$.

If $\mathfrak{C}$ satisfies, moreover, the condition:

   (vi) for any $a, b, c \in C$, either $(a, b, c) \in \mathfrak{C}$, or $(b, a, c) \in \mathfrak{C}$,

then $\mathfrak{C}$ is called a *total* cyclic order. A *cyclically ordered set* is a couple $C = (C, \mathfrak{C})$ of a set with a (partial or total) cyclic order.

Example 2.2. Each set C with less than two elements (including the empty one) has a unique total cyclic order $\mathfrak{C} = C^{\times 3}$. The disjoint union $S = C_1 \cup \cdots \cup C_b$ of cyclically ordered sets $C_i = (C_i, \mathfrak{C}_i)$, $1 \leq i \leq b$, bears a natural induced cyclic order $\mathfrak{C}_1 \cup \cdots \cup \mathfrak{C}_b$, cf. [33, Definition 3.7]. Each subset $T \subset C$ of a cyclically ordered set $C = (C, \mathfrak{C})$ bears an induced cyclic order $\mathfrak{C}|_T$.

To define morphisms, we need the following construction. Assume that $\sigma : C' \to C''$ is a map of sets, that C'' bears a partial cyclic order $\mathfrak{C}''$, and that one is given a partial linear order $\leq_c$ on each fiber $\sigma^{-1}(c)$, $c \in C''$.

Definition 2.3. The *lexicographic order* $\mathfrak{C}' = \text{Lex}\big(\sigma, \{\leq_c\}\big)$ on C' is given by postulating that $(a, b, c) \in \mathfrak{C}'$ if $\big(\sigma(a), \sigma(b), \sigma(c)\big) \in \mathfrak{C}''$ and either

   (i) The elements $\sigma(a), \sigma(b), \sigma(c)$ are mutually distinct, or

   (ii) for some cyclic permutation $(x, y, z) \in \big\{(a, b, c), (b, c, a), (c, a, b)\big\}$ one has

$$\sigma(x) = \sigma(y) \neq \sigma(z), x \leq_{\sigma(x)} y,$$

   (iii) or $\sigma(a) = \sigma(b) = \sigma(c)$ and $x \leq_{\sigma(x)} y \leq_{\sigma(x)} z$ for some $(x, y, z) \in \big\{(a, b, c), (b, c, a), (c, a, b)\big\}$.

If $\mathfrak{C}'$ is a total cyclic order and each $\leq_c$ a total linear order, then $\text{Lex}\big(\sigma, \{\leq_c\}\big)$ is a total cyclic order. If, moreover, C' and C'' are finite sets, a simple alternative description of the lexicographic order is given in [8, Example 2.2.2(4)].

Example 2.4. If $\sigma : C' \to C''$ is a monomorphism of sets and C'' is totally ordered, then the induced lexicographic order on C' is the inverse image $(\sigma^{\times 3})^{-1}(\mathfrak{C}'')$ of $\mathfrak{C}''$ under the map $\sigma^{\times 3} : C'^{\times 3} \to C''^{\times 3}$. The opposite extreme is when $A = (A, \leq)$ is a

totally linearly ordered set and $\sigma : A \to *$ the (unique) map to a one-point set. The total lexicographic order $\mathfrak{C}$ induced on A by σ is then given by

$$(a, b, c) \in \mathfrak{C} \text{ if and only if } a \leq b \leq c \text{ or } b \leq c \leq a \text{ or } c \leq a \leq b. \qquad (2.1)$$

The following notation will often be used in the sequel.

Definition 2.5. For a totally ordered set $A = (A, \leq)$ we denote by $[A]$ the underlying set of A with the total cyclic order (2.1). We say that this cyclic order is *represented* by the linear order $\leq$.

We can finally define morphisms of cyclically ordered sets, cf. [8, Definition 2.2.9].

Definition 2.6. Let $C' = (C', \mathfrak{C}')$ and $C'' = (C'', \mathfrak{C}'')$ be partially cyclically ordered sets.

If the order $\mathfrak{C}'$ is total, then a morphism $(C', \mathfrak{C}') \to (C'', \mathfrak{C}'')$ is a couple $\big(\sigma, \{\leq_c\}\big)$ consisting of a map $\sigma : C' \to C''$ of the underlying sets and of a family $\{\leq_c\}$ of total linear orders on each fiber $\sigma^{-1}(c)$, $c \in C''$, such that $\mathrm{Lex}\big(\sigma, \{\leq_c\}\big) = \mathfrak{C}'$.

If $(C', \mathfrak{C}')$ is a partially cyclically ordered set, then a morphism is a compatible system of morphisms $(T, \mathfrak{C}'|_T) \to (C'', \mathfrak{C}'')$ given for all totally cyclically ordered subsets $T \subset C'$.

Example 2.7. A morphism of cyclically ordered sets is therefore a map of the underlying sets plus some additional data. We will however actually need only morphisms whose underlying map is an isomorphism. Since all its fibers are one-point sets, we do not need any additional data, and such a morphism will simply be an isomorphism $\sigma : C' \to C''$ of the underlying sets for which $(\sigma^{\times 3})^{-1}(\mathfrak{C}'') = \mathfrak{C}'$.

From now on, by a cyclic order (without the adjective partial) we will always mean a *total* cyclic order. While linearly ordered sets A are naturally represented by horizontal intervals oriented from the left to the right, or as the combs

whose teeth represent elements of A, we depict cyclically ordered sets C as circumferences of counterclockwise oriented circles in the plane $\mathbb{R}^2$. We call such pictures *pancakes*:

$$(2.2)$$

with the spikes representing elements of C. Our pancakes appeared as 'spiders' in [5, 32], and, in the form of $2k$-gons, as complementary regions of quasi-filling arc systems in [22, Section 4].

We will need to extend the notation of Definition 2.5 as follows. For finite ordered sets $A_1, \ldots, A_k$ we denote by $A_1 \cdots A_k$, their union with the unique order in which

$A_1 < \cdots < A_k$, and by $[A_1 \cdots A_k]$ the corresponding cyclically ordered set. If e.g. $A_1 = \{u\}$ we abbreviate $[\{u\}A_2]$ by $[uA_2]$, &c.

Remark 2.8. While $A_1 A_2 \neq A_2 A_1$ when both A_1 and A_2 are non-empty, $[A_1 A_2]$ always equals $[A_2 A_1]$. Notice that $[A'] = [A'']$ for some finite ordered sets A', A'' if and only if there are ordered sets A_1 and A_2 such that $A' = A_1 A_2$ and $A'' = A_2 A_1$.

Let Fin (resp. Cyc) denote the category of finite (resp. cyclically ordered finite) sets and their <u>iso</u>morphisms. As we recall in the Appendix, the pieces $\mathcal{P}((S))$ of an (ordinary) cyclic operad $\mathcal{P}$ are indexed by finite sets $S \in$ Fin, and their structure operations are

$$_u\circ_v : \mathcal{P}((S')) \otimes \mathcal{P}((S'')) \longrightarrow \mathcal{P}\big((S' \cup S'') \setminus \{u, v\}\big), \qquad (2.3)$$

where S' and S'' are disjoint finite sets and $u \in S'$, $v \in S''$.

We will follow the convention used in [32] and distinguish non-Σ versions of operads by <u>underlying</u>. A non-Σ cyclic operad [32, II.5.1] $\underline{\mathcal{P}}$ has its components $\underline{\mathcal{P}}((C))$ indexed by cyclically ordered sets $C \in$ Cyc, and structure operations

$$_u\circ_v : \underline{\mathcal{P}}((C')) \otimes \underline{\mathcal{P}}((C'')) \longrightarrow \underline{\mathcal{P}}\big((C' \cup C'') \setminus \{u, v\}\big) \qquad (2.4)$$

of the same type as (2.3). The codomain of (2.4) however does not make sense unless we specify a cyclic order on the set

$$(C' \cup C'') \setminus \{u, v\}, \qquad (2.5)$$

were $C' \cup C''$ is the union of the corresponding underlying sets; we will use this kind of shorthand freely. It is given by the *pancake merging at* $\{u, v\}$ as follows.

Assume that the cyclic order of C' is represented by the linear order
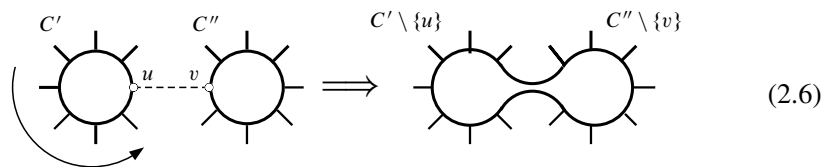
$$a_1 < a_2 < \cdots < a_k < u$$

and the cyclic order of C'' by

$$v < b_1 < b_2 < \cdots < b_l.$$

Then we equip (2.5) with the cyclic order is represented by

$$a_1 < a_2 < \cdots < a_k < b_1 < b_2 < \cdots < b_l.$$

Notice that we allow the case when $C' = \{u\}$ and $C'' = \{v\}$, then (2.5) is an empty cyclically ordered set. In the pancake world, (2.5) is realized by merging two pancakes into one:

$$(2.6)$$

Modular operads [14, Section 2], [32, Section II.5.3] have, besides (2.3), also the contractions

$$\xi_{uv} : \mathcal{P}((S; g)) \to \mathcal{P}((S \setminus \{u, v\}; g + 1)),$$

where $S \in \mathtt{Fin}$ and $u, v \in S$ are distinct elements; $\mathcal{P}$ here has an additional grading by the genus $g \in \mathbb{N}$ which is irrelevant now. It is therefore natural to expect that our conjectural non-Σ modular operad has pieces $\underline{\mathcal{P}}(C; g)$ indexed by $C \in \mathtt{Cyc}$, $g \in \mathbb{N}$ and, besides (2.4), the contractions

$$\xi_{uv} : \underline{\mathcal{P}}((C; g)) \to \underline{\mathcal{P}}((C \setminus \{u, v\}; g + 1)). \tag{2.7}$$

There is only one natural cyclic order on the subset $C \setminus \{u, v\}$ of C, namely the restriction of the cyclic order of C, so we are forced to equip (2.5) with this cyclic order. The following example shows that it does not work.

Example 2.9. Consider ordered sets X, Y and Z, and distinct symbols u', u'', v' and v''. Let

$$x \in \underline{\mathcal{P}}(([Xv'Zu']; g')) \quad \text{and} \quad y \in \underline{\mathcal{P}}(([Yu''v'']; g''))$$

be arbitrary elements. According to the definition of the cyclic order of (2.5) used in (2.4),

$$(x \,_{v'}\circ_{v''} y) \in \underline{\mathcal{P}}(([Zu'XYu'']; g' + g'')) \quad \text{and} \quad (x \,_{u'}\circ_{u''} y) \in \underline{\mathcal{P}}(([Xv'Zv''Y]; g' + g'')),$$

thus

$$\xi_{u'u''}(x \,_{v'}\circ_{v''} y) \in \underline{\mathcal{P}}(([XYZ]; g' + g'' + 1))$$

while

$$\xi_{v'v''}(x \,_{u'}\circ_{u''} y) \in \underline{\mathcal{P}}(([YXZ]; g' + g'' + 1)).$$

The standard exchange rule in Definition A.4(iv) between compositions and contractions in a modular operad must of course hold also in the non-Σ case, therefore

$$\xi_{u'u''}(x \,_{v'}\circ_{v''} y) = \xi_{v'v''}(x \,_{u'}\circ_{u''} y). \tag{2.8}$$

But this is *not possible*. If $X, Y, Z \neq \emptyset$, the cyclically ordered sets $[XYZ]$ and $[XZY]$ are non-isomorphic, so (2.8) compares elements of different spaces. This quandary will be resolved by introducing multicyclically ordered sets.

We believe that Figure 3 helps to understand the situation. It shows (from the left to the right the pancake representing the cyclically ordered set $C' = [Xv'Zu']$, the one representing $C'' = [Yu''v'']$ and two realizations of the pancakes representing the merging of $C' \cup C''$ at $\{v', v''\}$ resp. the merging $C' \cup C''$ at $\{u', u''\}$. The meaning of the dotted lines will be explained in Example 4.3.
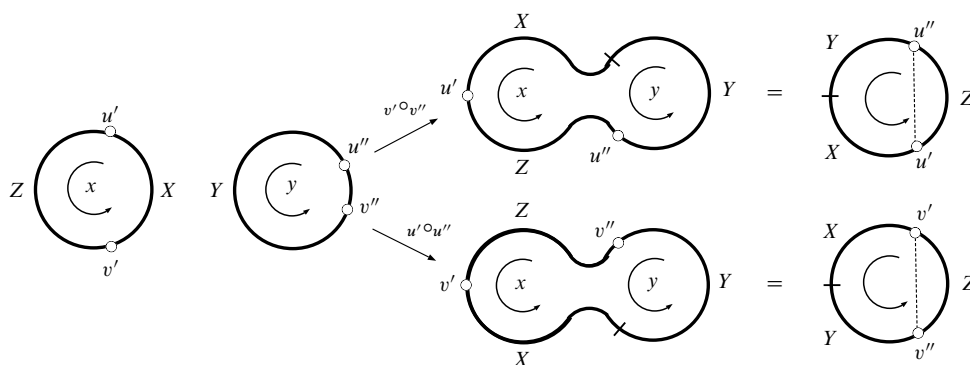
Figure 3. Naïve attempts fail: the relative positions of X and Y decorating the circumferences of the two pancakes in the last column are interchanged.

3. Multicyclic orders

In this section we introduce multicyclically ordered sets as objects appropriately indexing the pieces of non-Σ modular operads.

Definition 3.1. A *multicyclic order* on a set finite S is a disjoint decomposition $\mathsf{S} = C_1 \cup \cdots \cup C_b$ of S into $b > 0$ possibly empty totally cyclically ordered sets. A *morphism*

$$\sigma : \mathsf{S}' = C_1' \cup \cdots \cup C_{b'}' \longrightarrow \mathsf{S}'' = C_1'' \cup \cdots \cup C_{b''}'$$

is a couple (σ, u) consisting of

   (i) a morphism $\sigma = \big(\sigma, \{\leq_s\}\big) : S' \to S''$ of the underlying sets with the induced cyclic orders, and of

   (ii) a map $u : \{1, \ldots, b'\} \to \{1, \ldots, b''\}$ of the indexing sets

such that that $\sigma(C_i') \subset C_{u(i)}''$ for each $1 \leq i \leq b'$.

   Notice that a given set has infinitely many multicyclic orders, but the geometricity that we introduce in Definition 3.6 below allows only finite number of them.

Remark 3.2. It is clear that a morphism in Definition 3.1 is given by a family

$$\sigma_i = \big(\sigma_i, \{\leq_{c_i}\}\big) : C_i' \to C_{u(i)}'', \ 1 \leq i \leq b',$$

of morphisms of totally cyclically ordered sets. This offers the following alternative description. For a category $\mathcal{C}$ denote by $\mathrm{coProd}(\mathcal{C})$ the category of formal finite coproducts $A_1 \sqcup \cdots \sqcup A_s$, $s \geq 1$, of objects of $\mathcal{C}$ with the Hom-sets

$$\mathrm{coProd}(\mathcal{C})\big(A_1 \sqcup \cdots \sqcup A_s, B_1 \sqcup \cdots \sqcup B_t\big) := \prod_{1 \leq i \leq s} \coprod_{1 \leq j \leq t} \mathcal{C}(A_i, B_j).$$

Morphisms of Definition 3.1 are precisely morphism in the category $\mathrm{coProd}(\boldsymbol{\Lambda})$ generated by the category $\boldsymbol{\Lambda}$ of finite totally cyclically ordered sets.

We denote by `MultCyc` the category of multicyclically ordered sets and their **iso**morphisms. It contains the full subcategory `Cyc` of cyclically ordered finite sets and their isomorphisms embedded as multicyclically ordered sets with $b = 1$.

Remark 3.3. It follows from the observations in Remark 3.2 that

$$\texttt{MultCyc} \cong \text{coProd}(\texttt{Cyc}),$$

our basic category of multicyclic sets could therefore be introduced using only isomorphisms of totally cyclically ordered sets.

Remark 3.4. The underlying set S of each $\mathsf{S} = C_1 \cup \cdots \cup C_b \in \texttt{MultCyc}$ has a partial cyclic order $\mathfrak{C}_1 \cup \cdots \cup \mathfrak{C}_b$ induced by the cyclic orders $\mathfrak{C}_i$ of C_i, $1 \leq i \leq b$. Notice however that the correspondence $\mathsf{S} \mapsto (S, \mathfrak{C}_1 \cup \cdots \cup \mathfrak{C}_b)$ does not induce an embedding of `MultCyc` into the category of partially cyclically ordered sets. If, for instance, $\mathsf{S}' = C_1$ is a non-empty totally cyclically ordered set and $\mathsf{S}'' = C_1 \cup C_2$ with $C_2 = \emptyset$, then the underlying cyclically ordered sets agree, but S' and S'' are not isomorphic in `MultCyc`.

Definition 3.5. A *non-Σ modular module* is a functor

$$E : \texttt{MultCyc} \times \mathbb{N} \to \texttt{M},$$

where `M` is our fixed symmetric monoidal category and the natural numbers $\mathbb{N} = \{0, 1, 2, \ldots\}$ are considered as a discrete category.

Explicitly, a non-Σ modular module is a rule $(\mathsf{S}, g) \mapsto E(\!(\mathsf{S}; g)\!)$ that assigns to each multicyclically ordered S and $g \in \mathbb{N}$ an object $E(\!(\mathsf{S}, g)\!) \in \texttt{M}$, together with a functorial family of maps $E(\!(\sigma)\!) : E(\!(\mathsf{S}', g)\!) \to E(\!(\mathsf{S}'', g)\!)$ defined for each isomorphism $\sigma : \mathsf{S}' \to \mathsf{S}''$ of multicyclically ordered sets. If $\mathsf{S} = C_1 \cup \cdots \cup C_b$, we will sometimes write more explicitly $E(\!(C_1, \ldots, C_b; g)\!)$ instead of $E(\!(\mathsf{S}; g)\!)$. We call g the (operadic) *genus*.

Definition 3.6. We call a couple $(\mathsf{S}, g) \in \texttt{MultCyc} \times \mathbb{N}$ with $\mathsf{S} = C_1 \cup \cdots \cup C_b$ *geometric* if

$$G := \tfrac{1}{2}(g - b + 1) \in \mathbb{N}. \tag{3.1}$$

A non-Σ modular module E is *geometric*, if $E(\!(\mathsf{S}; g)\!) \neq 0$ implies that (S, g) is geometric.

Geometricity therefore means that $g - b + 1$ is an even non-negative integer. We will call G defined in (3.1) the *geometric genus* and b the *number of boundaries*. The reasons for this terminology will be clarified later in Section 7, see also Remark 4.5.

Example 3.7. For $g \leq 3$, only the following components of a geometric non-Σ modular module can be nontrivial: $E(\!(C_1; 0)\!)$, $E(\!(C_1, C_2; 1)\!)$, $E(\!(C_1, C_2, C_3; 2)\!)$ and $E(\!(C_1, C_2, C_3, C_4; 3)\!)$ in the geometric genus 0, and $E(\!(C_1; 2)\!)$, $E(\!(C_1, C_2; 3)\!)$ in geometric genus 1.

Let C be a cyclically ordered set. For $u, v \in C$, the set $C \setminus \{u, v\}$ has an obvious induced cyclic order given by the restriction of the original one. It naturally decomposes as

$$C \setminus \{u, v\} = I_1 \cup I_2 \tag{3.2}$$

where I_1, I_2 are the open intervals whose boundary points are u and v, considered with the induced cyclic orders. Notice that if u and v are adjacent in the cyclic order, one or both of I_1, I_2 may be empty.

If we distribute the elements of C around the circumference of a pancake so that their cyclic order is induced by the (say) counterclockwise orientation of the oven plate, then (3.2) is realized by cutting the pancake, with the knife running through u and v, as in:
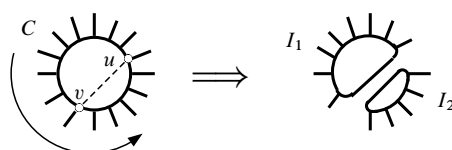
Pancake cutting together with pancake merging (2.6) induce two basic operations on multicyclically ordered sets. The *merging* starts with two multicyclically ordered sets

$$\mathsf{S}' = C'_1 \cup \cdots \cup C'_{b'} \quad \text{and} \quad \mathsf{S}'' = C''_1 \cup \cdots \cup C''_{b''},$$

whose underlying sets S' and S'' are disjoint. For $u \in C'_i$ and $v \in C''_j$, let

$$\mathsf{S}' \cup \mathsf{S}'' \setminus \{u, v\}$$

be the multicyclically ordered set whose underlying set is $S' \cup S'' \setminus \{u, v\}$, decomposed as

$$C'_1 \cup \cdots \cup \widehat{C'_i} \cup \cdots \cup C'_{b'} \cup C''_1 \cup \cdots \widehat{C''_j} \cup \cdots \cup C''_{b''} \cup \left(C'_i \cup C''_j \setminus \{u, v\}\right), \tag{3.3}$$

where $\widehat{}$ indicates that the corresponding term has been omitted, and $\left(C'_i \cup C''_j \setminus \{u, v\}\right)$ is cyclically ordered as in (2.5).

Let $\mathsf{S} = C_1 \cup \cdots \cup C_b$ is a multicyclically ordered set, $u \in C_i$, $v \in C_j$. If $i \neq j$, we define the *cut*

$$\mathsf{S} \setminus \{u, v\} \tag{3.4}$$

to be the multicyclically ordered set whose underlying set $S \setminus \{u, v\}$ decomposed as

$$S \setminus \{u, v\} = C_1 \cup \cdots \cup \widehat{C_i} \cup \cdots \widehat{C_j} \cup \cdots \cup C_b \cup \left(C_i \cup C_j \setminus \{u, v\}\right) \tag{3.5}$$

with $C_i \cup C_j \setminus \{u, v\}$ cyclically ordered as in (2.5). If $i = j$, we define (3.4) as the multicyclically ordered set given by the decomposition

$$S \setminus \{u, v\} = C_1 \cup \cdots \cup \widehat{C_i} \cup \cdots \cup C_b \cup \left(C_i \setminus \{u, v\}\right), \tag{3.6}$$

where $C_i \setminus \{u, v\}$ is the union of two multicyclically ordered sets as in (3.2). Notice that the number of cyclically ordered components of (3.3) is $b' + b'' - 1$, of (3.5) is $b - 1$ and of (3.6) is $b + 1$.

4. Biased definition of non-Σ modular operads

We formulate a definition of non-Σ modular operads biased towards the bilinear operations ${}_u\circ_v$ and contractions ξ_{uv}. Recall that $\mathtt{M}$ denotes our basic symmetric monoidal category; let τ be its commutativity constraint. Regarding multicyclically ordered sets, we use the notation introduced in §3.

Definition 4.1. A non-Σ modular operad in $\mathtt{M} = (\mathtt{M}, \otimes, 1)$ is a non-Σ modular module

$$\underline{\mathcal{P}} = \big\{ \underline{\mathcal{P}}(\!(\mathsf{S}; g)\!) \in \mathtt{M}; \ (\mathsf{S}, g) \in \mathtt{MultCyc} \times \mathbb{N} \big\}$$

together with morphisms (compositions)

$$ {}_u\circ_v : \underline{\mathcal{P}}(\!(\mathsf{S}'; g')\!) \otimes \underline{\mathcal{P}}(\!(\mathsf{S}''; g'')\!) \to \underline{\mathcal{P}}\big(\!(\mathsf{S}' \cup \mathsf{S}'' \setminus \{u, v\}; g' + g'')\!\big) \qquad (4.1)$$

defined for arbitrary disjoint multicyclically ordered sets S' and S'' with elements $u \in \mathsf{S}'$, $v \in \mathsf{S}''$ of their underlying sets, and contractions

$$ \xi_{uv} = \xi_{vu} : \underline{\mathcal{P}}(\!(\mathsf{S}; g)\!) \to \underline{\mathcal{P}}\big(\!(\mathsf{S} \setminus \{u, v\}; g + 1)\!\big) \qquad (4.2)$$

given for any multicyclically ordered set S and distinct elements $u, v \in \mathsf{S}$ of its underlying set. These data are required to satisfy the following axioms.

(i) For S', S'' and u, v as in (4.1), one has the equality

$$ {}_u\circ_v = {}_v\circ_u \, \tau $$

of maps $\underline{\mathcal{P}}(\!(\mathsf{S}'; g')\!) \otimes \underline{\mathcal{P}}(\!(\mathsf{S}''; g'')\!) \to \underline{\mathcal{P}}\big(\!(\mathsf{S}' \cup \mathsf{S}'' \setminus \{u, v\}; g' + g'')\!\big)$.

(ii) For mutually disjoint multicyclically ordered sets $\mathsf{S}_1, \mathsf{S}_2, \mathsf{S}_3$, and $a \in \mathsf{S}_1$, $b, c \in \mathsf{S}_2$, $b \neq c$, $d \in \mathsf{S}_3$, one has the equality

$$ {}_a\circ_b (id \otimes {}_c\circ_d) = {}_c\circ_d ({}_a\circ_b \otimes id) $$

of maps $\underline{\mathcal{P}}(\!(\mathsf{S}_1; g_1)\!) \otimes \underline{\mathcal{P}}(\!(\mathsf{S}_2; g_2)\!) \otimes \underline{\mathcal{P}}(\!(\mathsf{S}_3; g_3)\!) \to \underline{\mathcal{P}}\big(\!(\mathsf{S}_1 \cup \mathsf{S}_2 \cup \mathsf{S}_3 \setminus \{a, b, c, d\}; g_1 + g_2 + g_3)\!\big)$.

(iii) For a multicyclically ordered set S and mutually distinct $a, b, c, d \in \mathsf{S}$, one has the equality

$$ \xi_{ab} \, \xi_{cd} = \xi_{cd} \, \xi_{ab} $$

of maps $\underline{\mathcal{P}}(\!(\mathsf{S}; g)\!) \to \underline{\mathcal{P}}\big(\!(\mathsf{S} \setminus \{a, b, c, d\}; g + 2)\!\big)$.

(iv) For multicyclically ordered sets S', S'' and distinct $a, c \in \mathsf{S}'$, $b, d \in \mathsf{S}''$, one has the equality

$$ \xi_{ab} \, {}_c\circ_d = \xi_{cd} \, {}_a\circ_b $$

of maps $\underline{\mathcal{P}}(\!(\mathsf{S}' \cup \mathsf{S}''; g)\!) \to \underline{\mathcal{P}}\big(\!(\mathsf{S}' \cup \mathsf{S}'' \setminus \{a, b, c, d\}; g + 1)\!\big)$.

(v) For multicyclically ordered sets S', S'' and mutually distinct $a, c, d \in S'$, $b \in S''$, one has the equality

$$_a\circ_b (\xi_{cd} \otimes id) = \xi_{cd}\ _a\circ_b$$

of maps $\underline{\mathcal{P}}((S' \cup S''; g)) \to \underline{\mathcal{P}}((S' \cup S'' \setminus \{a, b, c, d\}; g + 1)).$

(vi) For arbitrary isomorphisms $\rho : S' \to D'$ and $\sigma : S'' \to D''$ of multicyclically ordered sets and u, v as in (4.1), one has the equality

$$\underline{\mathcal{P}}((\rho|_{S'\setminus\{u\}} \cup \sigma|_{S''\setminus\{v\}}))\ _u\circ_v =\ _{\rho(u)}\circ_{\sigma(v)} (\underline{\mathcal{P}}((\rho)) \otimes \underline{\mathcal{P}}((\sigma)))$$

of maps $\underline{\mathcal{P}}((S'; g')) \otimes \underline{\mathcal{P}}((S''; g'')) \to \underline{\mathcal{P}}((D' \cup D'' \setminus \{\rho(u), \sigma(v)\}; g' + g'')).$

(vii) For S, u, v as in (4.2) and an isomorphism $\rho : S \to D$ of multicyclically ordered sets, one has the equality

$$\underline{\mathcal{P}}((\rho|_{D\setminus\{\rho(u),\rho(v)\}}))\ \xi_{ab} = \xi_{\rho(u)\rho(v)}\underline{\mathcal{P}}((\rho))$$

of maps $\underline{\mathcal{P}}((S; g)) \to \underline{\mathcal{P}}((S \setminus \{\rho(u), \rho(v)\}; g + 1)).$

Remark 4.2. While $\xi_{uv} = \xi_{vu}$, the behavior of the $_u\circ_v$-operation under the interchange $u \leftrightarrow v$ is given by axiom (i). Axioms (ii)–(v) are interchange rules between $_u\circ_v$'s and the contractions, while the remaining two axioms describe how the structure operations behave under automorphisms. In (vi) and (vii) one sees the restrictions of automorphisms of multicyclically ordered sets. It is clear that they are automorphisms of the corresponding multicyclically ordered subsets.

Example 4.3. With the definition of non-Σ modular operads given above, both sides of (2.8) belong to the *same* space, namely to $\underline{\mathcal{P}}(([XY], [Z]; g'+g''+1))$. The problem risen in Example 2.9 is thus resolved by cutting the two rightmost pancakes in Figure 3 along the dotted lines.

Definition 4.4. A non-Σ modular operad $\underline{\mathcal{P}}$ is *geometric*, if its underlying non-Σ modular module is geometric.

Notice that the $_u\circ_v$-operations always preserve the geometric genus (3.1). The contractions ξ_{uv} preserve it if u, v in (4.2) belong to the same component of the multicyclically ordered set S, and raise it by 1 if they belong to different components of S. Therefore each non-Σ modular operad $\underline{\mathcal{P}}$ contains a maximal geometric suboperad.

From this point on, we assume that all non-Σ modular operads are *geometric*. With this assumption, the only nontrivial components of $\underline{\mathcal{P}}$ in (operadic) genus 0 are $\underline{\mathcal{P}}((C; 0))$, where C is a cyclically ordered set, i.e. a multicyclically ordered set with one component. It is simple to show that the collection

$$\square\underline{\mathcal{P}} := \{\underline{\mathcal{P}}((C; 0)); C \text{ is cyclically ordered}\}$$

together with $_u\circ_v$ operations (4.1) is a non-Σ cyclic operad. So we have the forgetful functor

$$\square : \texttt{NsModOp} \rightarrow \texttt{NsCycOp}. \tag{4.3}$$

In Section 6 we construct its left adjoint $\underline{\text{Mod}} : \texttt{NsCycOp} \rightarrow \texttt{NsModOp}$.

One also has the forgetful functor (the *desymmetrization*) Des : $\texttt{ModOp} \rightarrow \texttt{NsModOp}$ given by

$$\text{Des}(\mathcal{P})((\mathsf{S}; g)) := \mathcal{P}((S)),$$

where S is the underlying set of the multicyclically ordered set S. It has the left adjoint Sym : $\texttt{NsModOp} \rightarrow \texttt{ModOp}$ given by

$$\text{Sym}(\underline{\mathcal{P}})((S; g)) := \coprod \underline{\mathcal{P}}((\mathsf{S}; g)), \tag{4.4}$$

where the coproduct runs over all multicyclically ordered sets whose underlying set equals S. Notice that the geometricity guarantees that the coproduct in (4.4) is finite. We call Sym$(\underline{\mathcal{P}})$ the *symmetrization* of the non-Σ modular operad $\underline{\mathcal{P}}$.

Remark 4.5. Assuming the geometricity, the category of non-Σ cyclic operads is isomorphic to the full subcategory of non-Σ modular operads $\underline{\mathcal{P}}$ such that $\underline{\mathcal{P}}((\mathsf{S}; g)) = 0$ for $g \geq 1$. Without the geometricity assumption, this natural property that obviously holds for ordinary modular operads, will not be true. A 'geometric' explanation of the geometricity will be given in Remark 7.1.

Example 4.6. Assume that the basic monoidal category is the cartesian category Set of sets and let $*$ be a fixed one-point set. Then one has the *terminal* non-Σ modular operad $\underline{*}_M$ with

$$\underline{*}_M((\mathsf{S}; g)) := * \quad \text{for each geometric } (\mathsf{S}, g) \in \texttt{MultCyc} \times \mathbb{N},$$

with all structure operations the unique maps $* \rightarrow *$ or $* \times * \rightarrow *$.

5. Un-biased definition of non-Σ modular operads

We give an alternative definition of non-Σ modular operads as algebras over a certain monad of decorated graphs representing their pasting schemes, thus extending the table in [30, Figure 14]. This way of defining various types of operads is standard, see e.g. [14, §2.20], [32, II.1.12, II.5.3] or [30, Theorem 40], so we only emphasize the particular features of the non-Σ modular case. We start by recalling a definition of graphs suggested by Kontsevich and Manin [24] commonly used in operad theory. More refined notions of graphs already exist, see e.g. [3, Part 4], but we will not need them here.

Definition 5.1. A *graph* Γ is a finite set *Flag*(Γ) (whose elements are called *flags* or *half-edges*) together with an involution σ and a partition λ.

The *vertices Vert*(Γ) of a graph Γ are the blocks of the partition λ. The *edges edge*(Γ) are pairs of flags forming a two-cycle of σ relative to the decomposition of a permutation into disjoint cycles. The *legs Leg*(Γ) are the fixed points of σ.

We denote by *Leg*(v) the flags belonging to the block v or, in common speech, half-edges adjacent to the vertex v. The cardinality of *Leg*(v) is the *valency* of v. We say that two flags $x, y \in Flag(\Gamma)$ *meet* if they belong to the same block of the partition λ. In plain language, this means that they share a common vertex.

Definition 5.2. A *non-Σ modular graph* Γ is a connected graph as above that has, moreover, the following local structure at each vertex $v \in Vert(\Gamma)$:

  (i) a multicyclic order of the set *Leg*(v) of half-edges adjacent to v and

  (ii) a genus $g_v \in \mathbb{N}$.

Remark 5.3. Non-Σ modular graphs satisfying moreover a stability condition already appeared under the name *stable graphs* in [23, Appendix B], or as *stable ribbon graphs* in [1, Section 8], in connection with compactifications of moduli spaces of Riemann surfaces, cf. also [34, Section 1]. The relation between stable ribbon graphs and quotients of ribbon graphs was discussed in [27, Section 9]. The local multicyclic structure of graphs dual to arc families was also explicitly recognized in [18, Appendix A.1], while the global one is apparent at the set of marked points of windowed surfaces [20, Section 1].

We will denote by Leg(v) the set *Leg*(v) with the given multicyclic order and by b_v the number of cyclically ordered subsets in the corresponding decomposition. We say that Γ is *geometric* if at each $v \in Vert(\Gamma)$,

$$G_v := \tfrac{1}{2}(g_v + 1 - b_v) \in \mathbb{N}.$$

Crucially, the local structure of a non-Σ modular graph induces the same kind of structure on its external legs:

Proposition 5.4. *The set of legs of a non-Σ modular graph has an induced multicyclic order.*

Proof. By an oriented edge cycle in Γ we understand a sequence

$$(a_1, b_1), (a_2, b_2), \dots, (a_s, b_s),$$

where a_i, b_i are half-edges such that $\sigma(a_i) = b_i$ for $1 \leq i \leq s$. So, if $a_i \neq b_i$, (a_i, b_i) is an oriented edge, if $a_i = b_i$ it is a leg of Γ. We require that a_i is the immediate successor of b_{i-1}, $1 < i \leq s$, and that a_1 is the immediate successor of b_s in the cyclically ordered set to which these elements belong.

If this cyclically ordered set consists of b_{i-1} resp. of b_s only, then of course $a_i = b_{i-1}$ resp. $a_1 = b_s$ so the cycle runs back along the same half-edge. We also assume that each ordered couple (a_i, b_i) occurs exactly once so that the cycle does

not not run twice along itself. This does not exclude that $(a_i, b_i) = (b_j, a_j)$, i.e. that the cycle runs twice along the same edge, but each time in different direction.

Let $\{X_1, \ldots, X_b\}$ be the set of oriented edge cycles of Γ and

$$C_i := \{e \in Leg(\Gamma);\ (e, e) \in X_i,\ 1 \le i \le b\}. \tag{5.1}$$

Then $C_1, \ldots, C_b$ is the required disjoint decomposition of $Leg(\Gamma)$ and each individual C_i is cyclically ordered by the cyclic orientation of the corresponding edge cycle.   □

We will denote by $\mathsf{Leg}(\Gamma)$ the set $Leg(\Gamma)$ with the multicyclic order of Proposition 5.4 and by $b(\Gamma)$ the number of cyclically ordered sets in the decomposition of $\mathsf{Leg}(\Gamma)$. The (operadic) genus of a non-Σ modular graph is defined by the usual formula [32, (II.5.28)]

$$g(\Gamma) := b_1(\Gamma) + \sum_{v \in Vert(\Gamma)} g(v),$$

where $b_1(\Gamma)$ is the first Betti number of Γ, i.e. the number of independent circuits of Γ. We leave as an exercise to prove

Proposition 5.5. *If Γ is geometric then $\big(\mathsf{Leg}(\Gamma), g\big) \in \mathtt{MultCyc} \times \mathbb{N}$ is geometric, too, i.e.*

$$G(\Gamma) := \tfrac{1}{2}\big(g(\Gamma) + b(\Gamma) - 1\big) \in \mathbb{N}.$$

A *morphism* $f : \Gamma_0 \to \Gamma_1$ of graphs is given by a permutation of vertices, followed by a contraction of some edges of the graph Γ_0, leaving the legs untouched; a precise definition can be found in [32, II.5.3]. Assume that Γ_0 and Γ_1 bear a non-Σ modular structure. It is simple to see that the non-Σ modular structure of Γ_0 induces via f a non-Σ modular structure on Γ_1. We say that $f : \Gamma_0 \to \Gamma_1$ is a *morphism of non-Σ modular graphs* if this induced structure on Γ_1 coincides with the given one.

For a multicyclically ordered set S, let $\mathbf{\Gamma}((\mathsf{S}; g))$ be the category whose objects are pairs (Γ, ρ) consisting of a non-Σ modular graph Γ of genus g and an isomorphism $\rho : \mathsf{Leg}(\Gamma) \to \mathsf{S}$ of multicyclically ordered sets. Morphisms of $\mathbf{\Gamma}((S; g))$ are morphisms as above preserving the labelling of the legs. The category $\mathbf{\Gamma}((\mathsf{S}; g))$ has a terminal object $\star_{\mathsf{S},g}$, the 'non-Σ modular corolla' with no edges, one vertex v of genus g and legs labeled by S.

For a non-Σ modular module E and a non-Σ modular graph Γ, one forms the unordered product [32, Definition II.1.58]

$$E(\Gamma) := \bigotimes_{v \in Vert(\Gamma)} E((\mathsf{Leg}(v); g_v)). \tag{5.2}$$

Let $\mathtt{Iso}\mathbf{\Gamma}((\mathsf{S}; g))$ denote the subcategory of isomorphisms in $\mathbf{\Gamma}((\mathsf{S}; g))$. The correspondence $\Gamma \mapsto E(\Gamma)$ extends to a functor from the category $\mathtt{Iso}\mathbf{\Gamma}((\mathsf{S}; g))$

to M. We define the endofunctor $\underline{\mathbb{M}} : \texttt{NsModMod} \to \texttt{NsModMod}$ on the category of non-Σ modular modules as the colimit

$$\underline{\mathbb{M}}(E)((\mathsf{S}; g)) := \operatorname*{colim}_{\Gamma \in \texttt{Iso}\mathbf{\Gamma}((S;g))} E(\Gamma).$$

For each non-Σ modular graph $\Gamma \in \mathbf{\Gamma}((\mathsf{S}; g))$ one has the coprojection

$$\iota_\Gamma : E(\Gamma) \longrightarrow \underline{\mathbb{M}}(E)((\mathsf{S}, g)). \tag{5.3}$$

In particular, for the corolla $\star_{\mathsf{S},g}$, one gets the morphism

$$\iota_{\star_{\mathsf{S},g}} : E(\mathsf{S}; g) = E(\star_{\mathsf{S},s}) \longrightarrow \underline{\mathbb{M}}(E)((\mathsf{S}; g)). \tag{5.4}$$

Given a non-Σ modular module E, the second iterate $(\underline{\mathbb{M}} \circ \underline{\mathbb{M}})(E)$ is a colimit of non-Σ modular graphs whose vertices are decorated by non-Σ modular graphs decorated by E, i.e. a colimit of 'nested' graphs. Forgetting the nests gives rise to a natural transformation

$$\mu : \underline{\mathbb{M}} \circ \underline{\mathbb{M}} \longrightarrow \underline{\mathbb{M}} \text{ (the multiplication)}$$

while morphisms (5.4) form a natural transformation

$$\nu : id \longrightarrow \underline{\mathbb{M}} \text{ (the unit).}$$

Precisely as in [14, §2.17] or in the proof of [32, Theorem II.5.10] one shows that $\underline{\mathbb{M}} = (\underline{\mathbb{M}}, \mu, \nu)$ is a monad on the category $\texttt{NsModMod}$. We have the following theorem/definition.

Theorem 5.6. *Non-Σ modular operads are algebras for the monad* $\underline{\mathbb{M}} = (\underline{\mathbb{M}}, \mu, \nu)$.

Proof. A straightforward modification of the proof of [32, Theorem II.5.41]. □

A non-Σ modular operad is thus a non-Σ modular module $\underline{\mathcal{P}}$ equipped with a morphism

$$\alpha : \underline{\mathbb{M}}(\underline{\mathcal{P}}) \longrightarrow \underline{\mathcal{P}} \tag{5.5}$$

of non-Σ modular modules having the usual properties [32, Definition II.1.103]. The following claim expresses a standard property of algebras over a monad.

Proposition 5.7. *The multiplication* $\mu : (\underline{\mathbb{M}} \circ \underline{\mathbb{M}})(E) \to \underline{\mathbb{M}}(E)$ *makes* $\underline{\mathbb{M}}(E)$ *an algebra for the monad* $\underline{\mathbb{M}}$. *It is the* free non-Σ modular operad *on the non-Σ modular module E.*

Thus $\underline{\mathbb{M}}(-)$ interpreted as a functor $\texttt{NsModMod} \to \texttt{NsModOp}$ is the left adjoint to the obvious forgetful functor $\underline{F} : \texttt{NsModOp} \to \texttt{NsModMod}$.

Remark 5.8. The biased structure operations of the free operad $\underline{\mathbb{M}}(E)$ are induced by the grafting of the underlying graphs. For graphs Γ', Γ'' with legs $u \in Leg(\Gamma')$, $v \in Leg(\Gamma'')$, one has the graph $\Gamma' {}_u\circ_v \Gamma''$ obtained by grafting the free end of the half-edge u to the free end of v. Formally, $\Gamma' {}_u\circ_v \Gamma''$ is defined by

$$Flag(\Gamma' {}_u\circ_v \Gamma'') := Flag(\Gamma') \cup Flag(\Gamma''),$$

the partition of $Flag(\Gamma' {}_u\circ_v \Gamma'')$ being the union of the partitions of $Flag(\Gamma')$ and $Flag(\Gamma'')$, the involution σ on $Flag(\Gamma_1 {}_u\circ_v \Gamma_2)$ agreeing with the involution σ' of $Flag(\Gamma')$ on $Flag(\Gamma') \setminus \{u\}$, with the involution σ'' of $Flag(\Gamma'')$ on $Flag(\Gamma'') \setminus \{v\}$, and $\sigma(u) := v$. The contraction $\xi_{uv}(\Gamma)$ is, for $u, v \in Flag(\Gamma)$, defined similarly.

Example 5.9. Assume that E is a geometric non-Σ modular module such that $E((S; g)) = 0$ for $g \geq 1$. In other words, the only nontrivial pieces of E are $E((C; 0))$, where C is a cyclically ordered set. The elements of the free non-Σ modular operad $\underline{\mathbb{M}}(E)$ are the equivalence classes of decorated graphs whose vertices v are umbels with one blossom, i.e. the pancakes (2.2) with the spikes representing the half-edges in the cyclically ordered set $Leg(v)$. Free operads of this form will play a central rôle in our proof of (6.2b).

A very particular case is the geometric non-Σ modular module $\underline{*}$ in Set defined by

$$\underline{*}((S; g)) := \begin{cases} * & \text{if } g = 0, \text{ and} \\ \emptyset & \text{otherwise.} \end{cases}$$

In Section 7 we visualize the generators of $\underline{\mathbb{M}}(\underline{*})$ via cog wheels (7.1).

For a non-Σ modular operad $\underline{\mathcal{P}}$ and $\Gamma \in \mathbf{\Gamma}((S; g))$ we will call the composition

$$\alpha_\Gamma : \underline{\mathcal{P}}(\Gamma) \xrightarrow{\iota_\Gamma} (\underline{\mathbb{M}}\,\underline{\mathcal{P}})((S; g)) \xrightarrow{\alpha} \underline{\mathcal{P}}((S; g)) \tag{5.6}$$

of (5.5) with (5.3) the *contraction* along the graph Γ. The following analog of [32, Theorem II.5.42] claims that the contractions are part of a functor:

Theorem 5.10. *A geometric non-Σ modular module $\underline{\mathcal{P}}$ is a non-Σ modular operad if and only if the correspondence $\Gamma \mapsto \underline{\mathcal{P}}(\Gamma)$ is, for each geometric $(S, g) \in$ MultCyc$\times\mathbb{N}$, an object part of a functor $\alpha : \mathbf{\Gamma}((S; g)) \to \mathtt{M}$ extending (5.6). By this we mean that $\alpha(f) = \alpha_\Gamma$ for the unique morphism $f : \Gamma \to \star_{S,g}$.*

Proof. A simple modification of the proof of [32, Theorem II.5.42]. □

6. Modular envelopes

Modular envelopes of (ordinary) cyclic operads were introduced under the name modular operadic completions by the author in [28, Definition 2]. The *modular*

envelope functor Mod : CycOp $\rightarrow$ ModOp is the left adjoint to the obvious forgetful functor $\square$: ModOp $\rightarrow$ CycOp. Analogously we define the modular envelope of a non-Σ cyclic operad via the left adjoint to the non-Σ version $\underline{\square}$ of the forgetful functor considered in Section 4. We of course need to prove that this left adjoint exists:

Proposition 6.1. *The forgetful functor (4.3) has a left adjoint* $\underline{\text{Mod}}$: NsCycOp $\rightarrow$ NsModOp.

Proof. The proof will be transparent if we assume that the objects of the basic category M have elements. Then we take the free non-Σ modular operad $\underline{\mathbb{M}}(\underline{F}\,\mathcal{P})$ generated by the non-Σ cyclic collection $\underline{F}\,\mathcal{P}$ placed in the operadic genus 0 and define $\underline{\text{Mod}}(\mathcal{P})$ as the quotient

$$\underline{\text{Mod}}(\mathcal{P}) := \underline{\mathbb{M}}(\underline{F}\,\mathcal{P})/\mathcal{I} \tag{6.1}$$

of $\underline{\mathbb{M}}(\underline{F}\,\mathcal{P})$ by the operadic ideal $\mathcal{I}$ generated by

$$x_u \circ_v^{\mathcal{P}} y = x_u \circ_v^{\mathbb{M}} y,$$

where $_u\circ_v^{\mathcal{P}}$ resp. $_u\circ_v^{\mathbb{M}}$ are the $_u\circ_v$-operations in $\underline{\mathcal{P}}$ resp. $\underline{\mathbb{M}}$, $x \in \mathcal{P}(C')$, $y \in \mathcal{P}(C'')$, $u \in C'$ and $v \in C''$ for some disjoint cyclically ordered sets C', C''.

If objects of M do not have elements, we replace the quotient (6.1) by an obvious colimit. It is clear that (6.1) defines a left adjoint to (4.3). $\square$

Definition 6.2. We call $\underline{\text{Mod}}(\mathcal{P})$ the *non-Σ modular envelope* of the non-Σ cyclic operad $\mathcal{P}$.

Informally, $\underline{\text{Mod}}(\mathcal{P})$ is obtained by adding to $\mathcal{P}$ the results of contractions, splinting the cyclically ordered groups of inputs if the contraction takes place within the same group. This process is nicely visible at Doubek's construction of the modular envelope of the operad for associative algebras [9].

If the basic category M is Set, the category of cyclic (resp. non-Σ cyclic, resp. modular, resp. non-Σ modular) operads has a terminal object $*_C$ (resp. $\underline{*}_C$, resp. $*_M$ resp. $\underline{*}_M$) consisting of a chosen one-point set $*$ in each arity; the terminal non-Σ modular operad $\underline{*}_M$ has already been mentioned in Example 4.6. The main result of this section is:

Theorem 6.3. *The modular envelope of the terminal operad in the category of cyclic (resp. non-Σ cyclic)* Set-*operads is the terminal modular (resp. the terminal non-Σ modular)* Set-*operad, in formulas:*

$$\text{Mod}(*_C) \cong *_M \tag{6.2a}$$

and $$\underline{\text{Mod}}(\underline{*}_C) \cong \underline{*}_M. \tag{6.2b}$$

The cyclic operad $\mathcal{C}om$ for commutative associative algebras is the linear span of the terminal cyclic $\mathtt{Set}$-operad, that is $\mathcal{C}om = \mathrm{Span}(*_C)$. The following immediate corollary of Theorem 6.3 was stated without proof in [28, p. 382].

Theorem 6.4 ([28]). *The modular envelope* $\mathrm{Mod}(\mathcal{C}om)$ *of the cyclic operad for commutative associative algebras is the linear span of the terminal modular set-operad, i.e.*

$$\mathrm{Mod}(\mathcal{C}om)((S;g)) = \Bbbk, \ (S,g) \in \mathtt{Fin} \times \mathbb{N},$$

the maps $\mathrm{Mod}(\mathcal{C}om)((\sigma))$ *induced by morphisms in* $\underset{\cong}{\mathtt{Fin}}$ *are the identities, all* $_u \circ_v$-*operations are the canonical isomorphisms* $\Bbbk \otimes \Bbbk \xrightarrow{\cong} \Bbbk$ *and all contractions* ξ_{uv} *are the identities.*

Proof. Both $\mathrm{Span}(-)$ and $\mathrm{Mod}(-)$ are the left adjoints to forgetful functors that commute with each other, so

$$\mathrm{Span}\big(\mathrm{Mod}(\mathcal{S})\big) \cong \mathrm{Mod}\big(\mathrm{Span}(\mathcal{S})\big)$$

for each cyclic operad $\mathcal{S}$ in $\mathtt{Set}$. $\square$

Since the non-Σ cyclic operad $\underline{\mathcal{A}ss}$ is the linear span of the terminal non-Σ cyclic $\mathtt{Set}$-operad $\underline{*}_C$, we likewise obtain from Theorem 6.3:

Theorem 6.5. *The non-Σ modular envelope* $\underline{\mathrm{Mod}}(\underline{\mathcal{A}ss})$ *of the non-Σ cyclic operad* $\underline{\mathcal{A}ss}$ *for associative algebras is the linear span of the terminal non-Σ modular operad in the category of sets. Explicitly,*

$$\underline{\mathrm{Mod}}(\underline{\mathcal{A}ss})((\mathsf{S};g)) = \Bbbk$$

for each multicyclically ordered set S *and* $g \in \mathbb{N}$. *All structure operations are either the identities of* $\Bbbk$ *or the canonical isomorphisms* $\Bbbk \otimes \Bbbk \xrightarrow{\cong} \Bbbk$.

Since the symmetrization (4.4) clearly commutes with the non-Σ modular envelope functor, Theorem 6.5 implies the isomorphisms

$$\mathrm{Mod}(\mathcal{A}ss) \cong \mathrm{Sym}\big(\mathrm{Span}(\underline{*}_M)\big) \tag{6.3}$$

proved in [6, 9] though not expressed in this form there.

Let us start proving the isomorphisms (6.2a) and (6.2b) of Theorem 6.3. From here till the end of this section the basic category will be the category of sets.

Proof of (6.2a). It will be a warm-up for the proof of (6.2b) given below. The modular envelope $\mathrm{Mod}(*_C)$ is characterized by the adjunction

$$\mathtt{ModOp}\big(\mathrm{Mod}(*_C), \mathcal{P}\big) \cong \mathtt{CycOp}(*_C, \square\,\mathcal{P}) \tag{6.4}$$

that must hold for each modular $\mathtt{Set}$-operad $\mathcal{P}$. It is clear that there is a one-to-one correspondence between morphisms in $\mathtt{CycOp}(*_C, \square\,\mathcal{P})$ and families

$$\varsigma(S) \in \mathcal{P}((S;0)), \ S \in \mathtt{Fin}, \tag{6.5}$$

such that

$$\mathcal{P}(\!(\sigma)\!)\big(\varsigma(S)\big) = \varsigma(D) \text{ and } \varsigma(S') {}_u\circ_v \varsigma(S'') = \varsigma(S' \cup S'' \setminus \{u, v\}) \qquad (6.6)$$

for each S', S'', u, v, σ for which the above expressions make sense.

The theorem will obviously be proved if we exhibit a one-to-one correspondence between morphisms $*_C \to \square\,\mathcal{P}$ represented by (6.5), and families

$$\varpi(S; g) \in \mathcal{P}(\!(S; g)\!), \ S \in \mathtt{Fin}\times\mathbb{N}, \qquad (6.7)$$

such that $\varpi(S; 0) = \varsigma(S)$ for each $S \in \mathtt{Fin}$, and

$$\mathcal{P}(\!(\sigma)\!)\big(\varpi(S; g)\big) = \varpi(D; g), \ \sigma : S \to D \in \mathtt{Fin}, \qquad (6.8a)$$

$$\varpi(S'; g') {}_u\circ_v \varpi(S''; g'') = \varpi\big(S' \cup S'' \setminus \{u, v\}; g' + g''\big), \qquad (6.8b)$$

and $\qquad\qquad\quad \xi_{uv}\varpi(S; g) = \varpi\big(S \setminus \{u, v\}; g - 2\big), \qquad (6.8c)$

whenever the above objects are defined. In the light of (6.4) this is the same as to show that each family (6.5) uniquely determines a family (6.7) with $\varsigma(S; 0) = \varsigma(S)$ for each $S \in \mathtt{Fin}$.

Let Γ be a connected graph of genus g with trivial local genera g_v at each vertex, and $Leg(\Gamma) = S$. Decorate the vertices of Γ by (6.5) and denote the result by

$$\varsigma(\Gamma) = \bigotimes_{v \in Vert(\Gamma)} \varsigma\big(Leg(v)\big) \in \mathbb{M}(\mathcal{P})(\!(S; g)\!),$$

where $\mathbb{M}(\mathcal{P})$ is the free modular operad [32, Section II.5.3] on the modular module $F\mathcal{P}$. Formally correct notation would therefore be $\mathbb{M}(F\mathcal{P})$ but we want to save space. The composition (contraction) $c : \mathbb{M}(\mathcal{P})(\!(S; g)\!) \to \mathcal{P}(\!(S; g)\!)$ along the graph Γ determines

$$\varpi_\Gamma := c\big(\varsigma(\Gamma)\big) \in \mathcal{P}(\!(S; g)\!). \qquad (6.9)$$

Assume we proved, for Γ, S and g as above, that

ϖ_Γ depends only on the type of Γ, i.e. on S and g, not on the concrete Γ. (6.10)

We claim that then $\varpi(S; g) := \varpi_\Gamma$ is the requisite extension of (6.7).

Indeed, to establish (6.8a), denote by Γ' the graph Γ with the legs relabeled according to σ. Then (6.10) implies that $\mathcal{P}(\!(\sigma)\!)\varpi_\Gamma = \varpi_{\Gamma'}$. As for (6.8b), assume that $\varpi(S'; g') = \varpi_{\Gamma'}$ and $\varpi(S''; g'') = \varpi_{\Gamma''}$. Then $\varpi(S'; g') {}_u\circ_v \varpi(S''; g'')$ equals $\varpi_{\Gamma' {}_u\circ_v \Gamma''}$ with $\Gamma' {}_u\circ_v \Gamma''$ the grafting recalled in Remark 5.8, which in turn equals $\varpi(S' \cup S'' \setminus \{u, v\}; g' + g'')$. This proves (6.8b). Property (6.8c) can be discussed similarly.

Choose a maximal subtree T of Γ and denote by $K := \Gamma/T$ the result of shrinking $T \subset K$ into a corolla. Notice that K has only one vertex. By the associativity [32, Theorem II.5.42] of the contractions, $\varpi_\Gamma = \varpi_K$, it is therefore

enough to prove (6.10) for graphs K with one vertex. Graphically, such a K is a non-planar 'tick' with g 'bellies' as in
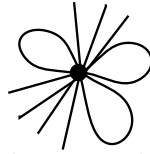
where $g = 3$. We prove (6.10) by induction on the genus g.

If $g = 0$, K is the corolla with $Leg(K) = S$ whose unique vertex is decorated by ϖ_T which equals $\varsigma(S)$, because the contraction in (6.9) uses only the cyclic part $\square\,\mathcal{P}$ of the operad $\mathcal{P}$. Therefore ϖ_T does not depend on the choice of T and is determined by a given map $*_C \to \square\,\mathcal{P}$.

Assume we have proved (6.10) for all $g' < g$. Let K' be a tick obtained by removing one belly of K. Then clearly $Leg(K') = S \cup \{u, v\}$ for some u and v, and

$$\varpi_K = \xi_{uv}(\varpi_{K'}).$$

By the induction assumption, $\varpi_{K'}$ equals $\varpi\big(S \cup \{u, v\}; g-1\big)$ and does not depend on the concrete form of K', so

$$\varpi_K = \xi_{uv}\big(\varpi(S \cup \{u, v\}; g-1)\big)$$

depends only on the finite set S and the genus g. □

Let us formulate a useful

Definition 6.6. A family (6.5) satisfying (6.6) is an *internal operad* in the cyclic operad $\square\,\mathcal{P}$. Similarly (6.7) satisfying (6.8a)–(6.8c) is an *internal operad* in the modular operad $\mathcal{P}$. These notions have obvious non-Σ analogs.

The proof of (6.2b) occupies the rest of this section. Since it follows the scheme of the proof of Theorem 6.4, we only emphasize the differences. We must to show that each internal non-Σ cyclic operad

$$\varsigma(C) \in \underline{\mathcal{P}}((C; 0)), \; C \in \mathtt{Cyc}, \tag{6.11}$$

in $\underline{\square}\,\underline{\mathcal{P}}$ uniquely extends to an internal non-Σ modular operad

$$\varpi(\mathtt{S}; g) \in \underline{\mathcal{P}}((\mathtt{S}, g)), \; \mathtt{S} \in \mathtt{MultCyc} \times \mathbb{N}$$

in $\underline{\mathcal{P}}$ such that $\varpi(C; 0) = \varsigma(C)$ for $C \in \mathtt{Cyc} \subset \mathtt{MultCyc}$.

As in the proof of Theorem 6.4, for a non-Σ modular graph Γ with $Leg(\Gamma) = \mathtt{S}$ whose local genera vanish we define $\varpi_\Gamma \in \underline{\mathcal{P}}((\mathtt{S}; g))$ using the contractions (5.6) along Γ in $\underline{\mathcal{P}}$. We then need to prove an analog of (6.10):

$$\begin{aligned} &\varpi_\Gamma \text{ depends only on the type of } \Gamma, \text{ i.e. on the multicyclically}\\ &\quad \text{ordered set } \mathtt{S} = Leg(\Gamma) \text{ and the genus } g = g(\Gamma). \end{aligned} \tag{6.12}$$

By choosing a maximal subtree T of Γ we again reduce (6.12) to graphs K with only one vertex. This time, K is not a 'tick' but a pancake

(6.13)
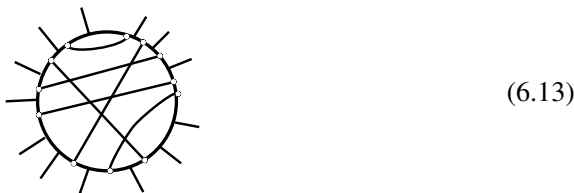
with the internal ribs marking the half-edges which have been contracted.

Example 6.7. The only pancake K with the operadic genus $g = 0$ has $b(K) = 0$. It is a circle with the circumference decorated by a cyclically ordered set $C := Leg(K)$. In this case $\varpi_K = \varsigma(C)$, so (6.10) is satisfied trivially.

Example 6.8. There is only one type of a pancake with $g = 1$, the left one in Figure 4.
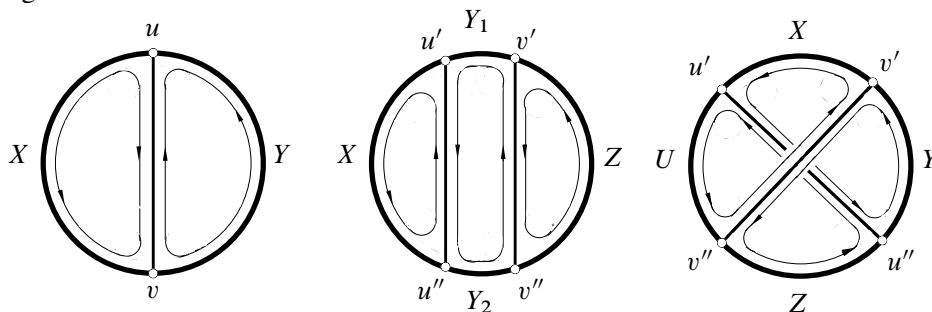
Figure 4. Three pancakes.

When its circumference is labeled by the ordered sets X and Y as in the figure, then, by definition

$$\varpi_K := \xi_{uv}\varsigma\big([XuYv]\big) \in \underline{\mathcal{P}}\big(([X], [Y]; 1)\big).$$

We prove that ϖ_K depends only on the induced cyclically ordered sets $[X]$ and $[Y]$, not on the particular orders of X and Y.

Let, for instance, X' be an ordered set such that $[X'] = [X]$ and K' be the pancake obtained from K by replacing X by X'. We will show that

$$\varpi_K = \varpi_{K'}, \tag{6.14}$$

where

$$\varpi_{K'} := \xi_{uv}\varsigma\big([X'uYv]\big) \in \underline{\mathcal{P}}\big(([X'], [Y]; 1)\big) = \underline{\mathcal{P}}\big(([X], [Y]; 1)\big).$$

As we noticed in Remark 2.8, $[X'] = [X]$ if and only if there are ordered sets X_1 and X_2 such that $X = X_1 X_2$ and $X' = X_2 X_1$. By the interchange (iv) of Definition 4.1,

$$\xi_{v'v''}\{\varsigma([u'v'X_1])\,_{u'}\circ_{u''}\varsigma([X_2v''Yu''])\} = \xi_{u'u''}\{\varsigma([u'v'X_1])\,_{v'}\circ_{v''}\varsigma([X_2v''Yu''])\}. \tag{6.15}$$

The corresponding term in the curly bracket in the left hand side equals

$$\varsigma([u'v'X_1])\,_{u'}\circ_{u''}\varsigma([X_2v''Yu'']) = \varsigma([v'X_1X_2v''Y])$$
$$= \varsigma([X_1X_2v''Yv']) = \varsigma([Xv''Yv']),$$

while the term in the right hand side is

$$\varsigma([u'v'X_1])\,_{v'}\circ_{v''}\varsigma([X_2v''Yu'']) = \varsigma([X_1u'Yu''X_2])$$
$$= \varsigma([X_2X_1u'Yu'']) = \varsigma([X'u'Yu'']),$$

thus (6.15) implies

$$\xi_{v'v''}\varsigma([Xv''Yv']) = \xi_{u'u''}\varsigma([X'u'Yu''])$$

which is (6.14). The independence of the particular order of Y can be proved similarly.

Example 6.9. There are two types of pancakes with $g = 2$. The middle one in Figure 4 has $b = 3$ and

$$\varpi_K = \xi_{u'u''}\xi_{v'v''}\varsigma([Xu'Y_1v'Zv''Y_2u'']) \in \underline{\mathcal{P}}(([X], [Y_1Y_2], [Z]; 3)). \tag{6.16a}$$

The second type in the right hand side of Figure 4 has $b = 1$ and

$$\varpi_K = \xi_{u'u''}\xi_{v'v''}\varsigma(Xv'Yu''Zv''Uu') \in \underline{\mathcal{P}}(([UZYX]; 2)). \tag{6.16b}$$

We leave as an exercise on the axioms of non-Σ modular operads to prove that the elements ϖ_K in (6.16a) resp. in (6.16b) depend only on the cyclically ordered sets $[X]$, $[Y_1Y_2]$ and $[Z]$ resp. $[UZYX]$.

We prove (6.12) by induction based on the following simple:

Lemma 6.10. *Let K be a pancake with $g > 0$.*

  (i) *If $b(K) = 1$, then there exist a pancake K' obtained by removing one rib of K such that $b(K') = 2$.*

  (ii) *If $b(K) > 1$, then there exist a pancake K' obtained from K by removing one rib that has $b(K') = b(K) - 1$.*

Proof. The case $b(K) = 1$ may happen only when $g(K)$ is even, by the geometricity (3.1). Thus by removing an *arbitrary* rib we obtain a pancake K' with $b(K') = 2$. This proves (i).

To prove (ii) we analyze the pancake (6.13) representing K. The legs of K are irrelevant for the proof so we ignore them. Let us inspect how the ribs enter the circumference of the pancake. The oriented edge cycles (5.1) are the boundaries of the regions between the ribs:
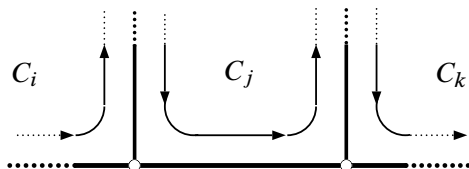
In this picture, the horizontal line represents a part of the circumference of the pancake and the vertical lines the ribs.

It is simple to see that removing a rib adjacent to two different regions decreases $b(K)$ by one. Since $b(K) > 1$ by assumption, there are at least two different edge cycles, therefore such a rib exists. This finishes the proof.   □

Let us finally start the actual inductive proof of (6.12). The cases when $g(K) \leq 2$ are analyzed in Examples 6.7–6.9. Fix $g \geq 3$, assume that we have proved (6.12) for all K's with $g(K) < g$ and prove it for K with $g(K) = g$. As in Lemma 6.10 we distinguish two cases.

Case 1: $b(K) = 1$. As there are no pancakes with $b(K) = 1$ and $g(K) = 4$, in this case in fact $g(K) \geq 4$. By Lemma 6.10.(i) and the inductive assumption,

$$\varpi_K = \xi_{u'u''} \varpi \big([X_1 u'], [u'' X_2]; g - 1\big), \tag{6.17}$$

where X_1, X_2 are ordered sets such that $C := Leg(K) = [X_1 X_2]$. We must show that the right hand side of (6.17) does not depend on the particular choices of X_1 and X_2. The choices are represented by a rib of a circle with the circumference decorated by C as in Figure 5–left.
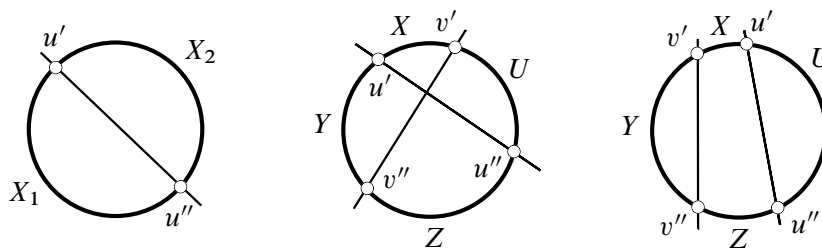

Figure 5. One rib (left), two crossing ribs (middle) and two parallel ribs (left).

Assume we have two different ribs, $C = [X_1' X_2']$ and $C = [X_1'' X_2'']$. They may either be crossing as in the middle of Figure 5, or parallel as in the rightmost picture

of Figure 5. The crossing case is parametrized by ordered sets X, Y, Z, U such that

$$X'_1 = ZY, \ X'_2 = XU, \ X''_1 = YX \ \text{and} \ X''_2 = UZ,$$

see Figure 5 again. The element

$$\varpi\big([v''Yu''Zv'Uu'X]; g-2\big) \in \underline{\mathcal{P}}\big(\!\big([v''Yu''Zv'Uu'X]; g-2\big)\!\big)$$

is then the 'equalizer' of the ribs, which is expressed by the commutative square

$$\varpi\big([v''Yu''Zv'Uu'X]; g-2\big)$$

$$\xi_{v'v''} \qquad \qquad \xi_{u'u''}$$

$$\varpi\big([X'_1 u''], [u'X'_2]; g-1\big) \qquad \varpi\big([X''_1 v''], [v'X''_2]; g-1\big).$$

$$\xi_{u'u''} \qquad \qquad \xi_{v'v''}$$

$$\varpi_K$$

Therefore

$$\xi_{u'u''}\varpi\big([X'_1 u'], [u''X'_2]; g-1\big) = \xi_{u'u''}\varpi\big([X''_1 u'], [u''X''_2]; g-1\big)$$

as we needed to show. Notice that we need to assume $g \geq 2$ in order the equalizer to exist.

The non-crossing case is parametrized by ordered sets X, Y, Z, U such that

$$X'_1 = ZYX, \ X'_2 = U, \ X''_1 = XUZ \ \text{and} \ X''_2 = Y.$$

The equalizer of these two choices is $\varpi\big([Xu'Zv'], [v''Y], [u''U]; g-2\big)$ as the reader easily verifies. This finishes the discussion of the $b(K) = 1$ case.

Case 2: $b(K) > 1$. Now K is of type $(C_1, \ldots, C_b; g)$ with $b \geq 2$. Lemma 6.10.(ii) translates to the formula

$$\varpi_K = \xi_{u', u''}\varpi\big(C_1, \ldots, \widehat{C}_i, \ldots, \widehat{C}_j, \ldots, C_b, [u'X_i u''X_j]; g-1\big), \qquad (6.18)$$

where $\widehat{}$ indicates the omission and X_i, X_j are ordered sets such that $[X_i] = C_i$ and $[X_j] = C_j$. As before we must prove that the right hand side of (6.18) does not depend on the particular choices of i, j and ordered sets X_i, X_j. So suppose that we have two different choices

$$[X'_i] = C_{i'}, \ [X'_j] = C_{j'}, \ \text{resp.} \ [X''_i] = C_{i''}, \ [X''_j] = C_{j''}$$

for some ordered sets X'_i, X'_j, X''_i, X''_j and $1 \leq i' < j' \leq b$, $1 \leq i'' < j'' \leq b$. We distinguish three cases.

Case 2(i): $\{i\,', j\,'\} = \{i\,'', j\,''\}$. We may clearly assume that $i' = i'' = 1$, $j' = j'' = 2$. Since C_k's with $k > 2$ do not affect calculations we will not explicitly mention them. We therefore have

$$C_1 = [X_1'] = [X_1''], \ C_2 = [X_2'] = [X_2''],$$

which, as observed in Remark 2.8, happens if and only if there are ordered sets X, Y, Z, U such that

$$X_1' = XY, \ X_2' = ZU, \ X_1'' = YX \ \text{ and } \ X_2'' = UZ.$$

One easily verifies that then $\varpi\big([Yv''Zu'], [u''Uv'X]; g-2\big)$ is a equalizer of these choices.

Case 2(ii): the cardinality of $\{i\,', j\,', i\,'', j\,''\}$ is 3. We may assume that $i' = 1$, $j' = i'' = 2$, $j'' = 3$, and neglect C_k's with $k > 3$. So we have two presentations

$$\varpi_K = \varpi\big([u'X_1u''X_2'], C_3; g-1\big) \ \text{ and } \ \varpi_K = \varpi\big(C_1, [v'X_2''v''X_3]; g-1\big) \quad (6.19)$$

in which $C_1 = [X_1]$, $C_2 = [X_2'] = [X_2'']$ and $C_3 = [X_3]$. By Remark 2.8, there are ordered sets Y, Z such that $X_2' = YZ$ and $X_2'' = ZY$. One easily sees that $\varpi\big(u'X_1u''Yv'X_3v''Z; g-2\big)$ is a equalizer for the two presentations in (6.19). The last case is

Case 2(iii): the cardinality of $\{i\,', j\,', i\,'', j\,''\}$ is 4. This case is simple so we leave its analysis to the reader. This finishes the proof of Theorem 6.5.

Remark 6.11. The computations in this section, namely in Example 6.8, can be used to lift the commutativity assumption in [22, Section 4.1], cf. namely Remark 4.1 of [22], proving that TFT's as functors from the category of cobordisms to vector spaces are equivalent to Frobenius algebras.

7. Surface models of Mod($\mathcal{A}ss$)

In this section we recall the approach which J. Chuang and A. Lazarev used to prove their [6, Theorem 3.7] that describes the modular envelope Mod($\mathcal{A}ss$) via the set of isomorphism classes of oriented surfaces with teethed holes. We also present a parallel approach due to R.M. Kaufmann, M. Livernet and R.C. Penner [19, 20]. Our setup nicely conveys their ideas. The above mentioned authors of course worked in the category of ordinary operads, but what they did in fact took place within non-Σ modular operads. In the second half of this section we briefly mention a non-oriented modification due to C. Braun [4]. Throughout this section, the basic monoidal category will be the cartesian category Set of sets.

Let $\underline{*}$ be the geometric non-Σ modular module of Example 5.9. Its generators $\underline{*}((C))$ will this time be visualized as oriented cog wheels

$$(7.1)$$

whose cogs are indexed by the cyclically ordered set C. The elements of the free non-Σ modular operad $\underline{\mathbb{M}}(\underline{*})$ are then obtained by gluing these wheels together along the tips of their cogs so that the orientation is preserved, see Figure 6–left.
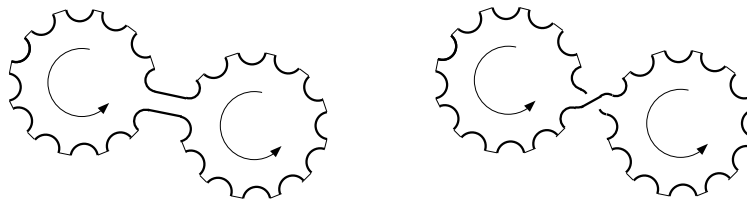

Figure 6. Oriented (left) and un-oriented (right) glueing of cog wheels.

It is an exercise in combinatorial geometry that $\underline{\mathbb{M}}(\underline{*})(C_1, \ldots, C_b; g)$ consists of all decompositions of an oriented surface of the genus

$$G = \tfrac{1}{2}(g - b + 1) \tag{7.2}$$

with b teethed boundaries whose teeth are labelled by the cyclically ordered sets $C_1, \ldots, C_b$ shown in Figure 7.
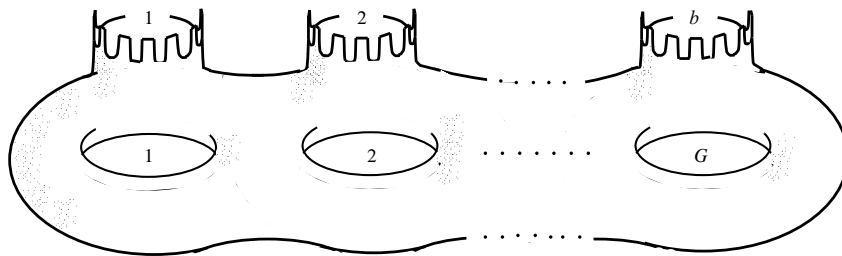

Figure 7. The oriented surface with b teethed boundaries and genus G.

By (6.1), the modular envelope $\underline{\mathrm{Mod}}(*_C)$ is the quotient of $\underline{\mathbb{M}}(\underline{*}) = \underline{\mathbb{M}}\big(\underline{F}(*_C)\big)$ by the relations that forget how a concrete surface was build from the cog wheels. As in the proof of [6, Theorem 3.7] we therefore identify, referring to the results of [17], $\underline{\mathrm{Mod}}(*_C)$ with the set of isomorphism classes of surfaces as in Figure 7. Since there is only one isomorphism class for a given geometric $(C_1, \ldots, C_b; g) \in \mathtt{MultCyc} \times \mathbb{N}$, we get (6.2b).

Remark 7.1. Notice that, given $(C_1, \ldots, C_b; g) \in \mathtt{MultCyc} \times \mathbb{N}$, there exists a surface as in Figure 7 if and only if $(C_1, \ldots, C_b; g)$ is geometric. This shall explain our terminology.

Other incarnations of the surface model of the operad $\underline{\mathrm{Mod}}(*_C)$ appeared also e.g. in [19, 20]. Let us outline its basic features. Our exposition will be merely sketched and also the notation will serve only for the purposes of this section. Denote by $\underline{\mathcal{S}ur}$ the modular operad of isomorphism classes of surfaces as in Figure 7, with the operad structure given by glueing the tips of the teeth. This operad is a version of the operad of isomorphism classes of windowed surfaces introduced in [20, Section 1], with the rôle of teeth played by marked points on the boundary.

The operad $\underline{\mathcal{A}rc}$ of isotopy classes of arc families in windowed surfaces, see [19, Section 1] or [20, Section 2], contains the suboperad $\underline{\mathcal{T}ri}$ of arc families whose complementary regions triangulate the underlying surface. One has the commutative diagram

$$
\begin{array}{ccc}
\underline{\mathrm{Mod}}(*_C) & \xrightarrow[\cong]{\omega} & \underline{\mathcal{S}ur} \\
& \beta \nwarrow \quad \alpha \nearrow \qquad \nwarrow \gamma & \\
& \underline{\mathcal{T}ri} \hookrightarrow \qquad \underline{\mathcal{A}rc} &
\end{array}
$$

in which $\gamma : \underline{\mathcal{A}rc} \to \underline{\mathcal{S}ur}$ associates to an arc family its underlying surface, and $\alpha : \underline{\mathcal{T}ri} \to \underline{\mathcal{S}ur}$ is the restriction of γ. Both maps are surjective because every surface has a triangulation. The existence of the third map $\beta : \underline{\mathcal{T}ri} \to \underline{\mathrm{Mod}}(*_C)$ follows from the freeness $\underline{\mathcal{T}ri}$ while ω exists because $\underline{\mathcal{S}ur}$ contains $*_C$ as a cyclic suboperad. The morphism β is surjective as well, since $\underline{\mathcal{T}ri}$ contains the generators of the terminal non-Σ cyclic operad $*_C$.

To establish that ω is an isomorphism, one needs to know that the kernel of α contains the kernel of β. In geometric terms this means that all triangulations of a given surface differ by a sequence of 'elementary moves' corresponding to the axioms of modular operads.

Despite its conceptual clarity, the above approaches relied on a rather deep result of [17] that the classifying space of the category of ribbon graphs of genus G with b boundary components is homeomorphic to the moduli space of Riemann surfaces of the same genus and the same number of boundary components. We therefore still believe that a direct combinatorial description of $\mathrm{Mod}(\mathcal{A}ss)$ given in [9] or here has some merit.

Remark 7.2. There are three equivalent pictures of the surface model for $\underline{\mathrm{Mod}}(*_C)$: teethed surfaces in Figure 7 and the windowed surfaces with marked points on the boundary, resp. appropriate subdivisions of these surfaces given by arc families. The third, dual picture, closer to the approach of the present article, uses graphs whose vertices correspond to the regions of the subdivision and the edges to the their common boundaries. This remark should relate e.g. [1, 18, 20] where these objects were used in the operadic context, to the present article.

A non-oriented variant of the above calculations starts with the cyclic operad $*_D$ whose component $*_D((S))$ consists of cog wheels whose cogs are indexed by the finite set S and have their tips decorated by arrows, as
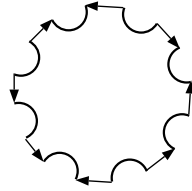
Clearly, if S has n elements, $*_D((S))$ has $2^{n-1}(n-1)!$ elements. The structure operations glue the tips of the cogs in such a way that the arrows go in the opposite directions, as in
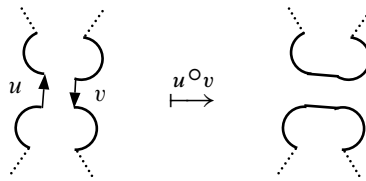
The operad $*_D$ is the Möbiusisation [4, Definition 3.32] of the terminal cyclic operad $*_C$, the subscript D referring to the dihedral structure [31, Section 3] that $*_D$ carries. Algebras over its linearization $\mathrm{Span}(*_D)$ are associative algebras with involution [4, Proposition 3.9]. The Chuang-Lazarev approach applies also to this situation, except that the sewing may not preserve the orientations now, see Figure 6–right. Indeed, C. Braun (who of course worked in $\Bbbk$-$\mathtt{Mod}$ not in $\mathtt{Set}$) proved

Theorem 7.3 ([4, Theorem 3.10]). *The component* $\mathrm{Mod}(*_D)((S; g))$ *of the modular envelope* $\mathrm{Mod}(*_D)$ *is the set of isomorphism classes of (not necessarily oriented) surfaces with b teethed holes whose teeth are labeled by S, with m handles and u crosscaps such that $g = 2m + b + u - 1$.*

As we theorized in the Introduction, we believe that $\mathrm{Mod}(*_D)$ is (related to) the terminal operad in a suitable category of dihedral operads.

A. Cyclic and modular operads

There are two versions of biased definitions of operads. The skeletal version has natural numbers as the arities, in the non-skeletal the arities are finite sets. We recall, following [9, Section 2], the non-skeletal definitions of classical cyclic and modular operads.

Definition A.1. A *cyclic module* is a functor $E : \mathtt{Fin} \to \mathtt{M}$ from the category of finite sets and their isomorphisms to our fixed symmetric monoidal category $\mathtt{M}$.

Definition A.2. A *cyclic operad* in $\mathbb{M} = (\mathbb{M}, \otimes, 1)$ is a cyclic module

$$\mathcal{P} = \big\{\mathcal{P}((S)) \in \mathbb{M}; \ S \in \mathtt{Fin}\big\}$$

together with morphisms (compositions)

$$_u\circ_v : \mathcal{P}((S')) \otimes \mathcal{P}((S'')) \to \mathcal{P}\big((S' \cup S'' \setminus \{u, v\})\big) \tag{A.1}$$

defined for arbitrary disjoint sets S' and S'' with elements $u \in S'$, $v \in S''$. These data are required to satisfy the following axioms.

(i) For S', S'' and u, v as in (A.1), one has the equality

$$_u\circ_v = \ _v\circ_u \ \tau$$

of maps $\mathcal{P}((S')) \otimes \mathcal{P}((S'')) \to \mathcal{P}\big((S' \cup S'' \setminus \{u, v\})\big)$, where τ is the symmetry constraint in $\mathbb{M}$.

(ii) For mutually disjoint sets S_1, S_2, S_3, and $a \in S_1$, $b, c \in S_2$, $b \neq c$, $d \in S_3$, one has the equality

$$_a\circ_b (id \otimes \ _c\circ_d) = \ _c\circ_d \, (\, _a\circ_b \otimes id)$$

of maps $\mathcal{P}((S_1)) \otimes \mathcal{P}((S_2)) \otimes \mathcal{P}((S_3)) \to \mathcal{P}\big((S_1 \cup S_2 \cup S_3 \setminus \{a, b, c, d\})\big)$.

(iii) For arbitrary isomorphisms $\rho : S' \to D'$ and $\sigma : S'' \to D''$ of sets and u, v as in (A.1), one has the equality

$$\mathcal{P}\big((\rho|_{S'\setminus\{u\}} \cup \sigma|_{S''\setminus\{v\}})\big) \, _u\circ_v = \ _{\rho(u)}\circ_{\sigma(v)} \ \big(\mathcal{P}((\rho)) \otimes \mathcal{P}((\sigma))\big)$$

of maps $\mathcal{P}((S')) \otimes \mathcal{P}((S'')) \to \mathcal{P}\big((D' \cup D'' \setminus \{\rho(u), \sigma(v)\})\big)$.

The category $\mathtt{Fin}$ of finite sets is equivalent to its full skeletal subcategory $\mathtt{Fin}_{sk}$ whose objects are the sets $[n] := \{1, \ldots, n\}$, $n \geq 0$, with $[0]$ interpreted as the empty set $\emptyset$. The components of the skeletal version of $\mathcal{P}$ are

$$\mathcal{P}(n) := \mathcal{P}\big(([n+1])\big), \ n \geq -1,$$

with the induced action of the symmetric group $\Sigma_{n+1} = \text{Aut}\big([n+1]\big)$. The structure operations

$$_i\circ_j : \mathcal{P}(m) \otimes \mathcal{P}(n) \to \mathcal{P}(m+n-1), \ \ 0 \leq i \leq m, \ \ 0 \leq j \leq n,$$

are induced from the equivalence of $\mathtt{Fin}$ with $\mathtt{Fin}_{sk}$.

Notice that we allow also the component $\mathcal{P}((\emptyset)) = \mathcal{P}(-1)$ and the operation

$$_u\circ_v : \mathcal{P}\big(((\{u\}))\big) \otimes \mathcal{P}\big(((\{v\}))\big) \to \mathcal{P}((\emptyset)) \text{ resp. } _0\circ_0 : \mathcal{P}(0) \otimes \mathcal{P}(0) \to \mathcal{P}(-1),$$

while the original definition [13, Theorem 2.2] always requires an 'output,' i.e. the arities must be non-empty sets (or $n \geq 0$ in the skeletal $\mathcal{P}(n)$). We do not demand operadic units.

Definition A.3. A *modular module* is a functor

$$E : \mathtt{Fin} \times \mathbb{N} \to \mathtt{M},$$

where the natural numbers $\mathbb{N} = \{0, 1, 2, \ldots\}$ are considered as a discrete category.

Definition A.4. A modular operad in $\mathtt{M} = (\mathtt{M}, \otimes, 1)$ is a modular module

$$\mathcal{P} = \{\mathcal{P}((S; g)) \in \mathtt{M}; \ (S, g) \in \mathtt{Fin} \times \mathbb{N}\} \tag{A.2}$$

together with morphisms (compositions)

$$_u \circ_v : \mathcal{P}((S'; g')) \otimes \mathcal{P}((S''; g'')) \to \mathcal{P}((S' \cup S'' \setminus \{u, v\}; g' + g''))$$

defined for arbitrary disjoint sets S' and S'' with elements $u \in S'$, $v \in S''$, and contractions

$$\xi_{uv} = \xi_{vu} : \mathcal{P}((S; g)) \to \mathcal{P}((S \setminus \{u, v\}; g + 1))$$

given for any set S and distinct elements $u, v \in S$. These data are required to satisfy the following axioms.

(i) For S', S'' and u, v as in (A.2), one has the equality

$$_u \circ_v = \ _v \circ_u \tau$$

of maps $\mathcal{P}((S'; g')) \otimes \mathcal{P}((S''; g'')) \to \mathcal{P}((S' \cup S'' \setminus \{u, v\}; g' + g''))$.

(ii) For mutually disjoint sets S_1, S_2, S_3, and $a \in S_1$, $b, c \in S_2$, $b \neq c$, $d \in S_3$, one has the equality

$$_a \circ_b (id \otimes \ _c \circ_d) = \ _c \circ_d (_a \circ_b \otimes id)$$

of maps $\mathcal{P}((S_1; g_1)) \otimes \mathcal{P}((S_2; g_2)) \otimes \mathcal{P}((S_3; g_3)) \to \mathcal{P}((S_1 \cup S_2 \cup S_3 \setminus \{a, b, c, d\}; g_1 + g_2 + g_3))$.

(iii) For a set S and mutually distinct $a, b, c, d \in S$, one has the equality

$$\xi_{ab} \, \xi_{cd} = \xi_{cd} \, \xi_{ab}$$

of maps $\mathcal{P}((S; g)) \to \mathcal{P}((S \setminus \{a, b, c, d\}; g + 2))$.

(iv) For sets S', S'' and distinct $a, c \in S'$, $b, d \in S''$, one has the equality

$$\xi_{ab} \ _c \circ_d = \xi_{cd} \ _a \circ_b$$

of maps $\mathcal{P}((S' \cup S''; g)) \to \mathcal{P}((S' \cup S'' \setminus \{a, b, c, d\}; g + 1))$.

(v) For sets S', S'' and mutually distinct $a, c, d \in S'$, $b \in S''$, one has the equality

$$_a \circ_b (\xi_{cd} \otimes id) = \xi_{cd} \ _a \circ_b$$

of maps $\mathcal{P}((S' \cup S''; g)) \to \mathcal{P}((S' \cup S'' \setminus \{a, b, c, d\}; g + 1))$.

(vi) For arbitrary isomorphisms $\rho : S' \to D'$ and $\sigma : S'' \to D''$ of sets and u, v as in (4.1), one has the equality

$$\mathcal{P}\big(\!\big(\rho|_{S'\setminus\{u\}} \cup \sigma|_{S''\setminus\{v\}}\big)\!\big)\, {}_u\!\circ_v \;=\; {}_{\rho(u)}\!\circ_{\sigma(v)} \;\big(\mathcal{P}(\!(\rho)\!) \otimes \mathcal{P}(\!(\sigma)\!)\big)$$

of maps $\mathcal{P}(\!(S'; g')\!) \otimes \mathcal{P}(\!(S''; g'')\!) \to \mathcal{P}\big(\!\big(D' \cup D'' \setminus \{\rho(u), \sigma(v)\}; g' + g''\big)\!\big)$.

(vii) For S, u, v as in (4.2) and an isomorphism $\rho : S \to D$ of sets, one has the equality

$$\mathcal{P}\big(\!\big(\rho|_{D\setminus\{\rho(u),\rho(v)\}}\big)\!\big)\, \xi_{ab} = \xi_{\rho(u)\rho(v)}\mathcal{P}(\!(\rho)\!)$$

of maps $\mathcal{P}(\!(S; g)\!) \to \mathcal{P}\big(\!\big(S \setminus \{\rho(u), \rho(v)\}; g + 1\big)\!\big)$.

Informally, cyclic operads are modular operads without the contractions and the genus grading. In the seminal paper [14] where modular operads were introduced, the *stability* demanding that

$$\mathcal{P}(\!(S; g)\!) = 0 \ \text{ if } \operatorname{card}(S) < 3 \text{ and } g = 0, \text{ or } \operatorname{card}(S) = 0 \text{ and } g = 1$$

was assumed, but we do not require this property. As a matter of fact, our main examples of non-Σ modular operads are not stable, though stable versions of our results can easily be formulated and proved.

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