The Batalin–Vilkovisky structure over the Hochschild cohomology ring of a group algebra

Yuming Liu[*,‡] and Guodong Zhou[**,‡]

Abstract. We realize explicitly the well-known additive decomposition of the Hochschild cohomology ring of a group algebra at the chain level. As a result, we describe the cup product, the Batalin–Vilkovisky operator and the Lie bracket in the Hochschild cohomology ring of a group algebra.

Mathematics Subject Classification (2010). 16E40, 18G10; 20C05.

Keywords. Additive decomposition, Batalin–Vilkovisky structure, cup product, group cohomology, Hochschild cohomology ring, normalized bar resolution, setwise self-homotopy.

1. Introduction

Let k be a field and G a finite group. Then the Hochschild cohomology ring of the group algebra kG admits an additive decomposition:

$$HH^*(kG) \simeq \bigoplus_{x \in X} H^*(C_G(x), k)$$

where X is a set of representatives of conjugacy classes of elements of G and $C_G(x)$ is the centralizer of $x \in G$. The proof of this isomorphism can be found in [2] or [17]. The usual proof is abstract rather than giving an explicit isomorphism. For example, one of the key steps is to use the so-called Eckmann–Shapiro Lemma, one needs to construct some comparison maps between two projective resolutions in order to write it down explicitly, and this is usually difficult. In [17], Siegel and Witherspoon used techniques and notations from group representation theory to

interpret the above additive decomposition explicitly. For our purpose, we need to give an explicit isomorphism at the chain level.

A priori, the additive decomposition gives an isomorphism of graded vector spaces. The left side has a graded commutative algebra structure given by the cup product, a graded Lie algebra structure given by the Gerstenhaber Lie bracket [8], and a Batalin–Vilkovisky (BV) algebra structure given by the $\triangle$ operator [19]. It would be interesting to describe these structures in terms of pieces from the right side.

For graded algebra structure, it was done by Holm for abelian groups using computations [10], then Cibils and Solotar gave a conceptual proof in [4]. The general case was dealt with by Siegel and Witherspoon [17], they described the cup product formula by notations from group representation theory. Our goal in the present paper is to represent the cup product, the Lie bracket and the BV operator in the Hochschild cohomology ring in terms of the additive decomposition. This is based on the explicit construction of an isomorphism in the additive decomposition (although there is no canonical choice for such an isomorphism).

The main obstruction in realizing an isomorphism in the additive decomposition comes from the fact that, it is usually difficult to construct the comparison map between two projective resolutions of modules. There is a way to simplify such construction, namely, one can reduce it to construct a setwise self-homotopy over one projective resolution, which is often much easier. This method was already used in a recent paper by the second author jointly with Le [12]. For convenience, we shall give a brief introduction to this idea in Section 2.

The article is organized as follows. In Section 2, we recall Le and Zhou's method on constructing comparison maps. In Section 3 and 4, we review the definitions of various structures over Hochschild cohomology and group cohomology, using the normalized bar resolutions. We always use the normalized bar resolutions since they are easy to describe and can greatly simplify the computations.

In Section 5, we give a way to realize explicitly the additive decomposition of the Hochschild cohomology of a group algebra. The main line of our method follows from [17]. In Section 6, we shall use some idea from [4] to give another way to realize the additive decomposition.

We give the cup product formula in Section 7. Our formula shows that the group cohomology $H^*(G, k)$ can be seen as a subalgebra of the Hochschild cohomology $HH^*(kG)$ at the complex level, and that the additive decomposition naturally gives an isomorphism of graded $H^*(G, k)$-modules.

We deal with the $\triangle$ operator and the graded Lie bracket in the next section. In particular, we show that the operator $\triangle$ restricts to each summand under the additive decomposition, and that $H^*(G, k)$ is indeed a BV subalgebra of $HH^*(kG)$.

In the final section, we use our formulae to compute the BV structure of the Hochschild cohomology ring for symmetric group of degree 3 over $\mathbb{F}_3$. This paper, with [11], should be the first attempts of concrete computation for the BV structure of a non-commutative algebra.

2. How to construct comparison morphisms [12, Appendix]

Definition 2.1 (cf. [1]). Let A be an algebra over a field k. Let

$$C^* : \cdots \longrightarrow C_{n+1} \xrightarrow{d_{n+1}} C_n \xrightarrow{d_n} C_{n-1} \longrightarrow \cdots$$

be a chain complex of A-modules. If there are maps (just as maps between sets) $s_n : C_n \longrightarrow C_{n+1}$ such that $s_{n-1} d_n + d_{n+1} s_n = id_{C_n}$ for all n, then the maps $\{s_n\}$ are called a setwise self-homotopy over the complex C^*.

Remark 2.2. (i) There is a setwise self-homotopy over a complex C^* of A-modules if and only if C^* is an exact complex, that is, C^* is a zero object in the derived category $D(\mathrm{Mod}\, A)$. Compare this with the usual self-homotopy, which is equivalent to saying that C^* is split exact, and hence it is a zero object in the homotopy category $K(\mathrm{Mod}\, A)$.
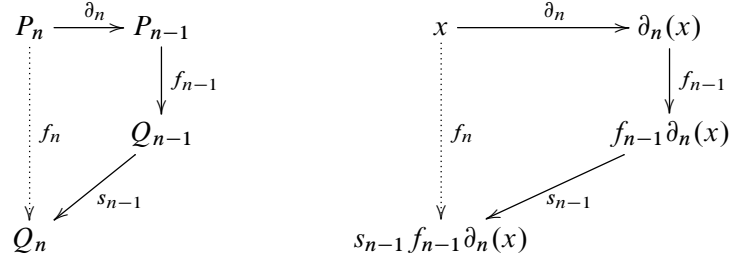
(ii) Usually a setwise self-homotopy can be taken to be linear maps, so it is a self-homotopy in the usual sense in the category of complexes of k-vector spaces. In case that the exact complex is a right bounded complex of A-A-bimodules, a setwise self-homotopy can even be chosen as homomorphisms of one-sided modules.

We will show how to use a setwise self-homotopy to construct a comparison map. Let M and N be two A-modules, and let $f : M \longrightarrow N$ be an A-module homomorphism. Suppose that $P^* = (P_i, \partial_i)$ is a free resolution of M, and that $Q^* = (Q_i, d_i)$ is a projective resolution of N. Suppose further that there is a setwise self-homotopy $s = \{s_n\}$ over Q^* (including N):

$$\begin{array}{ccccccccc}
\cdots \xrightarrow{d_3} & Q_2 & \xrightarrow{d_2} & Q_1 & \xrightarrow{d_1} & Q_0 & \xrightarrow{d_0} & N & \longrightarrow 0 \\
& \downarrow^{id} & {}^{s_2}\swarrow & \downarrow^{id} & {}^{s_1}\swarrow & \downarrow^{id} & {}^{s_0}\swarrow & \downarrow^{id} & {}^{s_{-1}} \\
\cdots \xrightarrow{d_3} & Q_2 & \xrightarrow{d_2} & Q_1 & \xrightarrow{d_1} & Q_0 & \xrightarrow{d_0} & N & \longrightarrow 0.
\end{array}$$

For each $i \geq 0$, choose a basis X_i for the free A-module P_i (the ith term of P^*). We define inductively the maps $f_i : X_i \longrightarrow Q_i$ as follows: for $x \in X_0$, $f_0(x) = s_{-1} f \partial_0(x)$; for $i > 1$ and for $x \in X_i$, $f_i(x) = s_{i-1} f_{i-1} \partial_i(x)$. Extending A-linearly the maps f_i we get A-homomorphisms $f_i : P_i \longrightarrow Q_i$. It is easy to

verify that $\{f_i\}$ gives a chain map between the complexes P^* and Q^*. We illustrate the above procedure in the following diagram:

$$
\begin{array}{ccc}
P_n \xrightarrow{\ \partial_n\ } P_{n-1} & \qquad & x \xrightarrow{\ \partial_n\ } \partial_n(x) \\[2mm]
\Big\downarrow f_{n-1} & & \Big\downarrow f_{n-1} \\[2mm]
\ f_n \quad Q_{n-1} & & f_n \qquad f_{n-1}\partial_n(x) \\[2mm]
\swarrow s_{n-1} & & \swarrow s_{n-1} \\[2mm]
Q_n & & s_{n-1}f_{n-1}\partial_n(x)
\end{array}
$$

We shall use the following standard homological fact.

Lemma 2.3. *Let A and B be two rings and let $F : \mathrm{Mod}\,A \longrightarrow \mathrm{Mod}\,B$ be an additive contravariant (resp., covariant) functor. If C^* and D^* are two projective resolutions of an A-module M, then the cochain complexes FC^* and FD^* of B-modules are homotopic. In particular, if $\varphi : C^* \longrightarrow D^*$ and $\psi : D^* \longrightarrow C^*$ are two chain maps inducing identity maps $id_M : M \longrightarrow M$, then $F\varphi : FD^* \longrightarrow FC^*$ (resp. $F\varphi : FC^* \longrightarrow FD^*$) and $F\psi : FC^* \longrightarrow FD^*$ (resp., $F\psi : FD^* \longrightarrow FC^*$) are inverse homotopy equivalences.*

3. Reminder on Hochschild cohomology

In this section, we recall the definitions of various structures over Hochschild cohomology. For the cup product and the Lie bracket in the Hochschild cohomology ring, we refer to Gerstenhaber's original paper [8]; for the Batalin–Vilkovisky algebra structure, we refer to Tradler [19].

Let k be a field and A an associative k-algebra with identity 1_A. Denote by $\overline{A}$ the quotient space $A/(k \cdot 1_A)$. We shall write $\otimes$ for $\otimes_k$ and $A^{\otimes n}$ for the n-fold tensor product $A \otimes \cdots \otimes A$. The *normalized bar resolution* $(\mathrm{Bar}_*(A), d_*)$ of A is a free resolution of A as A-A-bimodules, where

$$\mathrm{Bar}_{-1}(A) = A,$$

and for $n \geq 0$,

$$\mathrm{Bar}_n(A) = A \otimes \overline{A}^{\otimes n} \otimes A,$$

$$d_0 : \mathrm{Bar}_0(A) = A \otimes A \longrightarrow A, \quad a_0 \otimes a_1 \longmapsto a_0 a_1 \text{(multiplication map)},$$

and for $n \geq 1$, $d_n : \mathrm{Bar}_n(A) \longrightarrow \mathrm{Bar}_{n-1}(A)$ sends $a_0 \otimes \overline{a_1} \otimes \cdots \otimes \overline{a_n} \otimes a_{n+1}$ to

$$
a_0 a_1 \otimes \overline{a_2} \otimes \cdots \otimes \overline{a_n} \otimes a_{n+1} + \sum_{i=1}^{n-1} (-1)^i a_0 \otimes \cdots \otimes \overline{a_i a_{i+1}} \otimes \cdots \otimes a_{n+1}
$$
$$
+ (-1)^n a_0 \otimes \overline{a_1} \otimes \cdots \otimes \overline{a_{n-1}} \otimes a_n a_{n+1}.
$$

The normalized bar resolution is a natural quotient complex of the usual bar resolution. The exactness of the normalized bar resolution is an easy consequences of the following fact: there is a setwise self-homotopy $s_n : \mathrm{Bar}_n(A) \longrightarrow \mathrm{Bar}_{n+1}(A)$ over $\mathrm{Bar}_*(A)$ given by

$$
s_n(a_0 \otimes \overline{a_1} \otimes \cdots \otimes \overline{a_n} \otimes a_{n+1}) = 1 \otimes \overline{a_0} \otimes \overline{a_1} \otimes \cdots \otimes \overline{a_n} \otimes a_{n+1}.
$$

Notice that here each s_n is just a right A-module homomorphism. For simplicity, in the following we will write a_i for $\overline{a_i}$.

Let $_A M_A$ be an A-A-bimodule. Remember that any A-A-bimodule can be identified with a left module over the enveloping algebra $A^e = A \otimes A^{\mathrm{op}}$. We have the *Hochschild cohomology complex* $(C^*(A, M), \delta_*)$:

$$
C^n(A, M) = \mathrm{Hom}_{A^e}(\mathrm{Bar}_n(A), M) \simeq \mathrm{Hom}_k(\overline{A}^{\otimes n}, M), \quad \text{for } n \geq 0,
$$

$$
\delta_n : C^n(A, M) \longrightarrow C^{n+1}(A, M), f \longmapsto \delta_n(f),
$$

where $\delta_n(f)$ sends $a_1 \otimes \cdots \otimes a_{n+1}$ to

$$
a_1 f(a_2 \otimes \cdots \otimes a_{n+1}) + \sum_{i=1}^{n} (-1)^i f(a_1 \otimes \cdots \otimes a_i a_{i+1} \otimes \cdots \otimes a_{n+1})
$$
$$
+ (-1)^{n+1} f(a_1 \otimes \cdots \otimes a_n) a_{n+1}.
$$

For $n \geq 0$, the *degree-n Hochschild cohomology group* of the algebra A with coefficients in M is defined to be

$$
HH^n(A, M) = H^n(C^*(A, M)) \simeq \mathrm{Ext}_{A^e}^n(A, M).
$$

If in particular, $A = kG$ the group algebra of a finite group G, then the Hochschild cohomology complex $(C^*(A, M), \delta_*)$ has the following form:

$$
C^n(kG, M) \simeq \mathrm{Hom}_k(\overline{kG}^{\otimes n}, M) \simeq \mathrm{Map}(\overline{G}^n, M), \quad \text{for } n \geq 0,
$$

where $\overline{G} = G - \{1\}$ and $\mathrm{Map}(\overline{G}^{\times n}, M)$ denotes all the maps between the sets $\overline{G}^{\times n}$ and M, and the differential is given by

$$
\delta_n : \mathrm{Map}(\overline{G}^{\times n}, M) \longrightarrow \mathrm{Map}(\overline{G}^{\times n+1}, M), \quad f \longmapsto \delta_n(f),
$$

where $\delta_n(f)$ sends $(g_1, \ldots, g_{n+1}) \in \overline{G}^{n+1}$ to

$$g_1 f(g_2, \ldots, g_{n+1})$$
$$+ \sum_{i=1}^{n} (-1)^i f(g_1, \ldots, g_i g_{i+1}, \ldots, g_{n+1}) + (-1)^{n+1} f(g_1, \ldots, g_n)g_{n+1}.$$

When $M = A$ with the obvious A-A-bimodule structure, we write $C^n(A)$ (resp. $HH^n(A)$) for $C^n(A, A)$ (resp. $HH^n(A, A)$). Let $f \in C^n(A)$, $g \in C^m(A)$. Then the *cup product* $f \cup g \in C^{n+m}(A)$ is defined as follows:

$$f \cup g : \overline{A}^{\otimes(n+m)} \longrightarrow A, \quad a_1 \otimes \cdots \otimes a_{n+m} \longmapsto f(a_1 \otimes \cdots \otimes a_n)g(a_{n+1} \otimes \cdots \otimes a_{n+m}).$$

This cup product is associative and induces a well-defined product over

$$HH^*(A) = \bigoplus_{n \geq 0} HH^n(A) = \bigoplus_{n \geq 0} \operatorname{Ext}_{A^e}^n(A, A),$$

which is called the *Hochschild cohomology ring* of A. Moreover, $HH^*(A)$ is graded commutative, that is, $\alpha \cup \beta = (-1)^{mn} \beta \cup \alpha$ for $\alpha \in HH^n(A)$ and $\beta \in HH^m(A)$. As usual, we call an element $\alpha \in HH^n(A)$ homogeneous of degree n, and its degree will be denoted by $|\alpha|$.

The Lie bracket is defined as follows. Let $f \in C^n(A, M)$, $g \in C^m(A)$. If $n, m \geq 1$, then for $1 \leq i \leq n$, the so-called *brace operation* $f \circ_i g \in C^{n+m-1}(A, M)$ is defined by

$$f \circ_i g(a_1 \otimes \cdots a_{n+m-1})$$
$$= f(a_1 \otimes \cdots \otimes a_{i-1} \otimes g(a_i \otimes \cdots \otimes a_{i+m-1}) \otimes a_{i+m} \otimes \cdots \otimes a_{n+m-1});$$

if $n \geq 1$ and $m = 0$, then $g \in A$ and for $1 \leq i \leq n$, set

$$f \circ_i g(a_1 \otimes \cdots a_{n-1}) = f(a_1 \otimes \cdots \otimes a_{i-1} \otimes g \otimes a_i \otimes \cdots \otimes a_{n-1});$$

for any other case, set $f \circ_i g$ to be zero. Define

$$f \circ g = \sum_{i=1}^{n} (-1)^{(m-1)(i-1)} f \circ_i g \in C^{n+m-1}(A, M)$$

and for $f \in C^n(A)$, $g \in C^m(A)$, define

$$[f, g] = f \circ g - (-1)^{(n-1)(m-1)} g \circ f \in C^{n+m-1}(A).$$

The above $[\,,\,]$ induces a well defined *(graded) Lie bracket* in Hochschild cohomology

$$[\,,\,] : HH^n(A) \times HH^m(A) \longrightarrow HH^{n+m-1}(A)$$

such that $(HH^*(A), \cup, [\,,\,])$ is a *Gerstenhaber algebra*, that is, for homogeneous

elements α, β, γ in $HH^*(A)$, the following three conditions hold:

- $(HH^*(A), \cup)$ is an associative algebra and it is graded commutative, that is, the cup product $\cup$ is an associative multiplication and satisfies $\alpha \cup \beta = (-1)^{|\alpha||\beta|}\beta \cup \alpha$;
- $(HH^*(A), [\,,\,])$ is a graded Lie algebra, that is, the Lie bracket $[\,,\,]$ satisfies $[\alpha, \beta] = -(-1)^{(|\alpha|-1)(|\beta|-1)}[\beta, \alpha]$ and the graded Jacobi identity;
- Poisson rule: $[\alpha \cup \beta, \gamma] = [\alpha, \gamma] \cup \beta + (-1)^{|\alpha|(|\gamma|-1)}\alpha \cup [\beta, \gamma]$.

We now assume that A is a *symmetric k-algebra*, that is, A is isomorphic to its dual $D(A) = \operatorname{Hom}_k(A, k)$ as A^e-modules, or equivalently, if there exists a symmetric associative non-degenerate bilinear form $\langle\,,\,\rangle : A \times A \longrightarrow k$. This bilinear form induces a duality between the Hochschild cohomology and the Hochschild homology. In fact, for any $n \geq 0$ there is an isomorphism between $HH^n(A)$ and $HH_n(A)$ induced by the following canonical isomorphisms

$$\operatorname{Hom}_k(A \otimes_{A^e} \operatorname{Bar}_n(A), k) \simeq \operatorname{Hom}_{A^e}(\operatorname{Bar}_n(A), D(A)) \simeq \operatorname{Hom}_{A^e}(\operatorname{Bar}_n(A), A).$$

Via this duality, we have, for $n \geq 1$, an operator $\triangle : C^n(A) \longrightarrow C^{n-1}(A)$ which corresponds to the *Connes' B-operator* (denoted by B) on the Hochschild homology complex. More precisely, for any $f \in C^n(A)$, $\triangle(f) \in C^{n-1}(A)$ is given by the equation

$$\langle \triangle(f)(a_1 \otimes \cdots \otimes a_{n-1}), a_n \rangle$$
$$= \sum_{i=1}^{n}(-1)^{i(n-1)}\langle f(a_i \otimes \cdots \otimes a_{n-1} \otimes a_n \otimes a_1 \otimes \cdots \otimes a_{i-1}), 1 \rangle.$$

From the well known properties of the Connes' B-operator B (cf. [13, Chapter 2]), it is easy to see that the operator $\triangle$ is a chain map such that the induced operation $\triangle$ on Hochschild cohomology $HH^*(A)$ squares to zero (in fact, $\triangle^2 = 0$ holds on normalized Hochschild cochain complex level). It turns out that the Gerstenhaber algebra $(HH^*(A), \cup, [\,,\,])$ together with the operator $\triangle$ is a *Batalin–Vilkovsky algebra* (BV-algebra), that is, in addition to be a Gerstenhaber algebra, $(HH^*(A), \triangle)$ is a complex and

$$[\alpha, \beta] = -(-1)^{(|\alpha|-1)|\beta|}(\triangle(\alpha \cup \beta) - \triangle(\alpha) \cup \beta - (-1)^{|\alpha|}\alpha \cup \triangle(\beta))$$

for all homogeneous elements $\alpha, \beta \in HH^*(A)$.

Remark 3.1. The sign in the definition of a BV-algebra depends on the choice of the definitions of cup product and Lie bracket. If we define $\alpha \cup' \beta = (-1)^{|\alpha||\beta|}\alpha \cup \beta$ and $\triangle'(\alpha) = (-1)^{(|\alpha|-1)} \triangle(\alpha)$, then we get

$$[\alpha, \beta] = (-1)^{|\alpha|}(\triangle'(\alpha \cup' \beta) - \triangle'(\alpha) \cup' \beta - (-1)^{|\alpha|}\alpha \cup' \triangle'(\beta)),$$

which is the equality in the usual definition of a BV-algebra (see, for example [9, 15]). We choose the sign convention from [19] because of our convention of the definitions of cup product and Connes' B-operator in the Hochschild (co)homology theory.

4. Reminder on group cohomology

Let G be a finite group and U a left kG-module. The group cohomology of G with coefficient in U is defined to be $H^n(G, U) = \mathrm{Ext}^n_{kG}(k, U)$. The complex $\mathrm{Bar}_*(kG) \otimes_{kG} k$ is the standard resolution of the trivial module k. In fact, as the setwise self-homotopy s_n over $\mathrm{Bar}_*(kG)$ are right module homomorphisms, $\mathrm{Bar}_*(kG) \otimes_{kG} k$ is exact and thus a projective resolution of $kG \otimes_{kG} k \simeq k$. We write the complex $C^*(G, U) = \mathrm{Hom}_{kG}(\mathrm{Bar}_*(kG) \otimes_{kG} k, U)$. Therefore, for $n \geq 0$,

$$C^n(G, U) \simeq \mathrm{Hom}_{kG}((kG \otimes \overline{kG}^{\otimes n} \otimes kG) \otimes_{kG} k, U)$$

$$\simeq \mathrm{Hom}_{kG}(kG \otimes \overline{kG}^{\otimes n}, U)$$

$$\simeq \mathrm{Hom}_k(\overline{kG}^{\otimes n}, U) \simeq \mathrm{Map}(\overline{G}^{\times n}, U),$$

and the differential is given by

$$\delta_0(x)(g) = gx - x,$$

for $x \in U$ and $g \in \overline{G}$, and

$$\delta_n(\varphi)(g_1, \ldots, g_{n+1}) = g_1 \varphi(g_2, \ldots, g_{n+1})$$
$$+ \sum_{i=1}^n (-1)^i \varphi(g_1, \ldots, g_i g_{i+1}, \ldots, g_{n+1}) + (-1)^{n+1} \varphi(g_1, \ldots, g_n),$$

for $\varphi : \overline{G}^{\times n} \longrightarrow U$ and $g_1, \ldots, g_{n+1} \in \overline{G}$.

Of particular interest to us are the following two cases which relate group cohomology to Hochschild cohomology and in fact which underly our two realisations of the additive decomposition of the Hochschild cohomology of a group algebra.

Note that we have an algebra isomorphism $(kG)^e \simeq k(G \times G)$ given by $g_1 \otimes g_2 \longmapsto (g_1, g_2^{-1})$, for $g_1, g_2 \in G$. Thus we can also identify each kG-kG-bimodule M as a left $k(G \times G)$-module by $(g_1, g_2) \cdot x = g_1 x g_2^{-1}$. In the sequel, we shall write the Hochschild cohomology complex for the group algebra kG in terms of $k(G \times G)$-modules.

Case 1: $M = kG$. The module kG with the obvious kG-kG-bimodule, or equivalently, the $k(G \times G)$-module kG with action: $(g_1, g_2) \cdot x = g_1 x g_2^{-1}$ for $g_1, g_2 \in G$. Consider G as a subgroup of $G \times G$ via the diagonal embedding $G \to G \times G, g \mapsto (g, g)$, and it is easy to verify that there is a $k(G \times G)$-module isomorphism $\mathrm{Ind}_G^{G \times G} k = k(G \times G) \otimes_{kG} k \simeq kG, (g_1, g_2) \otimes 1 \longmapsto g_1 g_2^{-1}$. So we have

$$HH^n(kG, kG) \simeq \mathrm{Ext}^n_{k(G \times G)}(kG, kG) \simeq \mathrm{Ext}^n_{k(G \times G)}(\mathrm{Ind}_G^{G \times G} k, kG)$$

$$\simeq \mathrm{Ext}^n_{kG}(k, \mathrm{Res}_G^{G \times G} kG) = \mathrm{Ext}^n_{kG}(k, {}_c kG)$$

$$= H^n(G, {}_c kG),$$

where the third isomorphism is given by the adjoint equivalence and $_c kG$ is considered as a left kG-module by conjugation: $g \cdot x = gxg^{-1}$ for $g, x \in G$. This verifies a well known fact observed by Eilenberg and MacLane [5]: the Hochschild cohomology $HH^n(kG, kG)$ of kG with coefficients in kG is isomorphic to the ordinary group cohomology $H^n(G, kG)$ of G with coefficients in kG under the conjugation.

Case 2: $M = k$. The trivial kG-kG-bimodule, or equivalently, the $k(G \times G)$-module k with action: $(g_1, g_2) \cdot 1 = 1$ for $g_1, g_2 \in G$. Since we have

$$HH^n(kG, k) \simeq \mathrm{Ext}^n_{k(G \times G)}(kG, k) \simeq \mathrm{Ext}^n_{k(G \times G)}(k(G \times G) \otimes_{kG} k, k)$$
$$\simeq \mathrm{Ext}^n_{kG}(k, k) = H^n(G, k),$$

the Hochschild cohomology $HH^n(kG, k)$ of kG with coefficients in k is isomorphic to the ordinary group cohomology $H^n(G, k)$. Another way to see this lies in the fact that the two complexes $C^*(kG, k)$ and $C^*(G, k)$ coincide.

We can deduce the second case from the first one. In fact, the subspace $k(\sum_{g \in G} g) \subseteq kG$ is a sub-$(G \times G)$-module of kG (and also sub-G-module of $_c kG$), which is isomorphic to the trivial module. Via the isomorphisms in Case 1, $HH^*(kG, k(\sum_{g \in G} g))$ corresponds to $H^*(G, k(\sum_{g \in G} g))$.

We can in fact define a cup product and Lie bracket over

$$H^*(G, k) = \bigoplus_{n \geq 0} H^n(G, k)$$

such that it becomes a Gerstenhaber algebra. One sees that the cup product and the Lie bracket over $HH^*(kG)$ restrict to $H^*(G, k)$ by [7, Corollary 2.2], so $H^*(G, k)$ is a Gerstenhaber subalgebra of $HH^*(kG)$. In fact, as in [7, Proof of Theorem 1.8], there is a chain map at the cohomology complex level:

$$\mathrm{Hom}_{kG}(\mathrm{Bar}_n(kG) \otimes_{kG} k, k) = C^n(kG, k) \hookrightarrow C^n(kG)$$
$$= \mathrm{Hom}_{k(G \times G)}(\mathrm{Bar}_n(kG), kG),$$

$$(\varphi : \overline{G}^{\times n} \longrightarrow k) \longmapsto (\psi : \overline{G}^{\times n} \longrightarrow kG), \quad \psi(g_1, \ldots, g_n) = \varphi(g_1, \ldots, g_n) g_1 \cdots g_n.$$

This inclusion map preserves the brace operations in the following sense.

Let $\varphi_1 \in C^n(kG, k) \simeq \mathrm{Map}(\overline{G}^{\times n}, k)$, $\varphi_2 \in C^m(kG, k)$, and let $\widehat{\varphi}_1 \in C^n(kG)$, $\widehat{\varphi}_2 \in C^m(kG)$ be the corresponding elements under the above inclusion map. Then $\widehat{\varphi}_1 \circ_i \widehat{\varphi}_2 = \widehat{\varphi_1 \circ_i \varphi_2} \in C^{m+n-1}(kG)$.

Recall that kG is a symmetric algebra with the bilinear form

$$\langle \, , \, \rangle : kG \times kG \longrightarrow k,$$

$$\langle g, h \rangle = \begin{cases} 1 & \text{if } g = h^{-1}, \\ 0 & \text{otherwise}, \end{cases}$$

for $g, h \in G$. So there is a well-defined BV-algebra structure on $HH^*(kG)$. We shall see later that $H^*(G, k)$ is furthermore a sub-BV-algebra of $HH^*(kG)$.

5. The first realization of the additive decomposition

Let k be a field and G a finite group. Then the Hochschild cohomology ring of the group algebra kG admits an additive decomposition:

$$HH^*(kG) \simeq \bigoplus_{x \in X} H^*(C_G(x), k),$$

where X is a set of representatives of conjugacy classes of elements of G and $C_G(x) = \{g \in G \mid gx = xg\}$ is the centralizer subgroup of G. In this section, we give an explicit construction of the additive decomposition. The main technique we used here is to construct comparison maps based on some setwise self-homotopies.

The following is a proof of the additive decomposition which consists of a series of isomorphisms. Our first realization of the additive decomposition will follow this series of isomorphisms.

$$HH^*(kG, kG) = \operatorname{Ext}^*_{(kG)^e}(kG, kG) \simeq \operatorname{Ext}^*_{k(G \times G)}(kG, kG)$$

$\overset{(1)}{\simeq} \operatorname{Ext}^*_{k(G \times G)}(\operatorname{Ind}_G^{G \times G} k, kG)$

because $_{k(G \times G)}kG \simeq \operatorname{Ind}_G^{G \times G} k = {}_{k(G \times G)}k(G \times G) \otimes_{kG} k$ where $k(G \times G)$ is endowed with the right kG-module structure via the diagonal map

$G \to G \times G, g \mapsto (g, g)$

$\overset{(2)}{\simeq} \operatorname{Ext}^*_{kG}(k, \operatorname{Res}_G^{G \times G} kG) = \operatorname{Ext}^*_{kG}(k, \operatorname{Hom}_{k(G \times G)}(k(G \times G), kG))$
$$= \operatorname{Ext}^*_{kG}(k, {}_ckG) = H^*(G, {}_ckG)$$

by the adjoint pair

$(_{k(G \times G)}k(G \times G) \otimes_{kG} -, \operatorname{Hom}_{k(G \times G)}(_{k(G \times G)}k(G \times G)_{kG}, -))$

$\overset{(3)}{\simeq} \oplus_{x \in X} \operatorname{Ext}^*_{kG}(k, {}_ckC_x)$

because $_ckG = \oplus_{x \in X} {}_ckC_x$ where $_ckC_x$ is the kG-module generated by the elements in the conjugacy class $C_x = \{gxg^{-1} | g \in G\}$

$\overset{(4)}{\simeq} \oplus_{x \in X} \operatorname{Ext}^*_{kG}(k, \operatorname{Coind}_{C_G(x)}^G k)$

because as left kG-modules, $_ckC_x \simeq \operatorname{Coind}_{C_G(x)}^G k = \operatorname{Hom}_{kC_G(x)}(kG, k)$

$\overset{(5)}{\simeq} \oplus_{x \in X} \operatorname{Ext}^*_{kC_G(x)}(\operatorname{Res}_{C_G(x)}^G k, k)$

by the adjoint pair $(_{kC_G(x)}kG \otimes_{kG} -, \operatorname{Hom}_{kC_G(x)}(_{kC_G(x)}kG_{kG}, -))$

$\overset{(6)}{\simeq} \oplus_{x \in X} \operatorname{Ext}^*_{kC_G(x)}(k, k)) = \oplus_{x \in X} H^*(C_G(x), k)$

We shall express explicitly these isomorphisms step by step using the bar resolution.

The first step. By definition, the Hochschild cohomology groups $HH^*(kG, kG)$ can be computed using the bar resolution $\mathrm{Bar}_*(kG)$. On the other hand, $\mathrm{Bar}_*(kG) \otimes_{kG} k$ is a free resolution of k as left kG-module, and therefore

$$k(G \times G) \otimes_{kG} \mathrm{Bar}_*(kG) \otimes_{kG} k$$

is also a free resolution of the $k(G \times G)$-module $k(G \times G) \otimes_{kG} k \simeq kG$. Notice that the terms in $\mathrm{Bar}_*(kG)$ are still viewed as the usual kG-kG-bimodules when we do the above tensor products.

Let us write explicitly the resolution $k(G \times G) \otimes_{kG} \mathrm{Bar}_*(kG) \otimes_{kG} k$. Under the identification

$$k(G \times G) \otimes_{kG} \mathrm{Bar}_n(kG) \otimes_{kG} k \simeq k(G \times G) \otimes_{kG} (kG \otimes \overline{kG}^{\otimes n} \otimes kG) \otimes_{kG} k$$
$$\simeq k(G \times G) \otimes \overline{kG}^{\otimes n}$$
$$\simeq kG \otimes kG \otimes \overline{kG}^{\otimes n},$$

and the differential is as follows (we only write down the maps on base elements here and later):

$$kG \otimes kG \longrightarrow kG, \quad x \otimes y \longmapsto xy^{-1};$$
$$kG \otimes kG \otimes \overline{kG} \longrightarrow kG \otimes kG, \quad x \otimes y \otimes g_1 \longmapsto xg_1 \otimes yg_1 - x \otimes y;$$
$$\vdots$$
$$kG \otimes kG \otimes \overline{kG}^{\otimes n} \longrightarrow kG \otimes kG \otimes \overline{kG}^{\otimes n-1}, x \otimes y \otimes g_1 \otimes \cdots \otimes g_n \longmapsto$$
$$xg_1 \otimes yg_1 \otimes g_2 \otimes \cdots \otimes g_n + \sum_{i=1}^{n-1} (-1)^i x \otimes y \otimes g_1 \otimes \cdots \otimes g_i g_{i+1} \otimes \cdots \otimes g_n$$
$$+(-1)^n x \otimes y \otimes g_1 \otimes \cdots \otimes g_{n-1}.$$

We also have

$$\mathrm{Hom}_{k(G \times G)}(k(G \times G) \otimes_{kG} \mathrm{Bar}_n(kG) \otimes_{kG} k, kG) \simeq \mathrm{Hom}_k(\overline{kG}^{\otimes n}, kG)$$
$$\simeq \mathrm{Map}(\overline{G}^{\times n}, kG).$$

Using this identification, $H^*(\mathrm{Hom}_{k(G \times G)}(k(G \times G) \otimes_{kG} \mathrm{Bar}_*(kG) \otimes_{kG} k, kG))$ is given by the following cochain complex:

$$0 \longrightarrow kG \xrightarrow{\delta_0} \mathrm{Map}(\overline{G}, kG) \xrightarrow{\delta_1} \cdots \longrightarrow \mathrm{Map}(\overline{G}^{\times n}, kG) \xrightarrow{\delta_n} \cdots,$$

where the differential is given by

$$\delta_0(x)(g) = gxg^{-1} - x \quad \text{for } x \in kG \text{ and } g \in \overline{G},$$

and for $\varphi : \overline{G}^{\times n} \longrightarrow kG$ and $g_1, \ldots, g_{n+1} \in \overline{G}$,

$$\delta_n(\varphi)(g_1, \ldots, g_{n+1}) = g_1\varphi(g_2, \ldots, g_{n+1})g_1^{-1}$$
$$+ \sum_{i=1}^{n}(-1)^i\varphi(g_1, \ldots, g_ig_{i+1}, \ldots, g_{n+1}) + (-1)^{n+1}\varphi(g_1, \ldots, g_n).$$

We will show that the two complexes $k(G \times G) \otimes_{kG} \mathrm{Bar}_*(kG) \otimes_{kG} k$ and $\mathrm{Bar}_*(kG)$ are isomorphic and therefore there is an isomorphism

$$H^*(\mathrm{Hom}_{k(G\times G)}(\mathrm{Bar}_*(kG), kG))$$
$$\simeq H^*(\mathrm{Hom}_{k(G\times G)}(k(G \times G) \otimes_{kG} \mathrm{Bar}_*(kG) \otimes_{kG} k, kG)) \quad (1)$$

To do this, we need to construct the comparison maps between the two free resolutions $\mathrm{Bar}_*(kG)$ and $k(G \times G) \otimes_{kG} \mathrm{Bar}_*(kG) \otimes_{kG} k$ of the above $k(G \times G)$-module kG. As explained in Section 2, this is reduced to construct setwise self-homotopys over these resolutions. Our principle here is to choose those setwise self-homotopys so that the computations and results are as simple as possible.

We choose a setwise self-homotopy over $\mathrm{Bar}_*(kG)$ as follows:

$$u_{-1} : kG \to kG \otimes kG, \quad g \mapsto g \otimes 1,$$

and for $n \geq 0$,

$$u_n : kG \otimes \overline{kG}^{\otimes n} \otimes kG \longrightarrow kG \otimes \overline{kG}^{\otimes n} \otimes kG,$$
$$g_0 \otimes g_1 \otimes \cdots \otimes g_{n+1} \longmapsto (-1)^{n+1}g_0 \otimes g_1 \otimes \cdots \otimes g_{n+1} \otimes 1.$$

Using $\{u_n\}$ we can construct a comparison map

$$\alpha_* : k(G \times G) \otimes_{kG} \mathrm{Bar}_*(kG) \otimes_{kG} k = kG \otimes kG \otimes k\overline{G}^{\otimes *} \longrightarrow \mathrm{Bar}_*(kG)$$
$$= kG \otimes kG^{\otimes *} \otimes kG$$

as follows (as before we only write down the maps on base elements):

$$\alpha_{-1} : kG \longrightarrow kG, \quad x \longmapsto x,$$
$$\alpha_0 : kG \otimes kG \longrightarrow kG \otimes kG, \quad x \otimes y \longmapsto x \otimes y^{-1},$$
$$\alpha_1 : kG \otimes kG \otimes \overline{kG} \longrightarrow kG \otimes \overline{kG} \otimes kG, \quad x \otimes y \otimes g_1 \longmapsto -xg_1 \otimes g_1^{-1} \otimes y^{-1},$$
$$\vdots$$
$$\alpha_n : kG \otimes kG \otimes \overline{kG}^{\otimes n} \longrightarrow kG \otimes \overline{kG}^{\otimes n} \otimes kG,$$
$$x \otimes y \otimes g_1 \otimes \cdots \otimes g_n \longmapsto (-1)^{\frac{n(n+1)}{2}}xg_1 \cdots g_n \otimes g_n^{-1} \otimes \cdots \otimes g_1^{-1} \otimes y^{-1}.$$

Similarly, we choose a setwise self-homotopy over $k(G \times G) \otimes_{kG} \mathrm{Bar}_*(kG) \otimes_{kG} k$ as follows:

$$v_{-1} : kG \to kG \otimes kG, \quad g \mapsto g \otimes 1,$$

and for $n \geq 0$,

$$v_n : kG \otimes kG \otimes \overline{kG}^{\otimes n} \longrightarrow kG \otimes kG \otimes \overline{kG}^{\otimes n},$$
$$x \otimes y \otimes g_1 \otimes \cdots \otimes g_n \longmapsto xy^{-1} \otimes 1 \otimes y \otimes g_1 \otimes \cdots \otimes g_n.$$

Using $\{v_n\}$ we can construct a comparison map

$$\beta_* : \mathrm{Bar}_*(kG) = kG \otimes kG^{\otimes *} \otimes kG \longrightarrow k(G \times G) \otimes_{kG} \mathrm{Bar}_*(kG) \otimes_{kG} k$$
$$= kG \otimes kG \otimes kG^{\otimes *}$$

as follows:

$$\beta_{-1} : kG \longrightarrow kG, \quad x \longmapsto x,$$
$$\beta_0 : kG \otimes kG \longrightarrow kG \otimes kG, \quad x \otimes y \longmapsto x \otimes y^{-1},$$
$$\beta_1 : kG \otimes \overline{kG} \otimes kG \longrightarrow kG \otimes kG \otimes \overline{kG}, \quad x \otimes g_1 \otimes y \longmapsto -xg_1 \otimes y^{-1} \otimes g_1^{-1},$$
$$\vdots$$
$$\beta_n : kG \otimes \overline{kG}^{\otimes n} \otimes kG \longrightarrow kG \otimes kG \otimes \overline{kG}^{\otimes n},$$
$$x \otimes g_1 \otimes \cdots \otimes g_n \otimes y \longmapsto (-1)^{\frac{n(n+1)}{2}} xg_1 \cdots g_n \otimes y^{-1} \otimes g_n^{-1} \otimes \cdots \otimes g_1^{-1}.$$

It is easy to check that the chain maps $\{\alpha_n\}$ and $\{\beta_n\}$ are inverse to each other, and therefore we get an isomorphism

$$\mathrm{Hom}_{k(G \times G)}(\mathrm{Bar}_*(kG), kG) \longrightarrow \mathrm{Hom}_{k(G \times G)}(k(G \times G) \otimes_{kG} \mathrm{Bar}_*(kG) \otimes_{kG} k, kG),$$
$$(\varphi : \overline{G}^{\times n} \longrightarrow kG) \longmapsto (\varphi_1 : \overline{G}^{\times n} \longrightarrow kG),$$
$$\varphi_1(g_1, \ldots, g_n) = (-1)^{\frac{n(n+1)}{2}} g_1 \cdots g_n \varphi(g_n^{-1}, \cdots, g_1^{-1}).$$

Its inverse is given by

$$\mathrm{Hom}_{k(G \times G)}(k(G \times G) \otimes_{kG} \mathrm{Bar}_*(kG) \otimes_{kG} k, kG) \longrightarrow \mathrm{Hom}_{k(G \times G)}(\mathrm{Bar}_*(kG), kG),$$
$$(\varphi_1 : \overline{G}^{\times n} \longrightarrow kG) \longmapsto (\varphi : \overline{G}^{\times n} \longrightarrow kG),$$
$$\varphi(g_1, \ldots, g_n) = (-1)^{\frac{n(n+1)}{2}} g_1 \cdots g_n \varphi_1(g_n^{-1}, \ldots, g_1^{-1}).$$

Passing to the cohomology, we realize an isomorphism in (1) and its inverse.

The second step. Since

$$({}_{k(G\times G)}k(G \times G) \otimes_{kG} -, \mathrm{Hom}_{k(G\times G)}({}_{k(G\times G)}k(G \times G)_{kG}, -))$$

is an adjoint pair, we have an isomorphism (here $k(G \times G)$ is viewed as a right kG-module by diagonal action)

$$\mathrm{Hom}_{k(G\times G)}(k(G\times G)\otimes_{kG}\mathrm{Bar}_*(kG)\otimes_{kG}k, kG) \simeq \mathrm{Hom}_{kG}(\mathrm{Bar}_*(kG)\otimes_{kG}k, {}_ckG).$$

Passing to the cohomology, we get an isomorphism

$$H^*(\mathrm{Hom}_{k(G\times G)}(k(G \times G) \otimes_{kG} \mathrm{Bar}_*(kG) \otimes_{kG} k, kG))$$
$$\simeq H^*(\mathrm{Hom}_{kG}(\mathrm{Bar}_*(kG) \otimes_{kG} k, {}_ckG)). \quad (2)$$

Recall that the right hand side is just the ordinary group cohomology $H^*(G, kG)$ of G with coefficients in ${}_ckG$. We also have

$$\mathrm{Hom}_{kG}(\mathrm{Bar}_n(kG) \otimes_{kG} k, kG) \simeq \mathrm{Hom}_{kG}(kG \otimes \overline{kG}^{\otimes n}, kG)$$
$$\simeq \mathrm{Hom}_k(\overline{kG}^{\otimes n}, kG) \simeq \mathrm{Map}(\overline{G}^{\times n}, kG).$$

Using this identification, $H^*(G, kG) = H^*(\mathrm{Hom}_{kG}(\mathrm{Bar}_*(kG) \otimes_{kG} k, kG))$ is given by the following cochain complex:

$$0 \longrightarrow kG \xrightarrow{\delta_0} \mathrm{Map}(\overline{G}, kG) \xrightarrow{\delta_1} \cdots \longrightarrow \mathrm{Map}(\overline{G}^{\times n}, kG) \xrightarrow{\delta_n} \cdots,$$

where the differential is given by

$$\delta_0(x)(g) = gxg^{-1} - x$$

for $x \in kG$ and $g \in \overline{G}$, and

$$\delta_n(\varphi)(g_1, \ldots, g_{n+1}) = g_1\varphi(g_2, \ldots, g_{n+1})g_1^{-1}$$
$$+ \sum_{i=1}^n (-1)^i \varphi(g_1, \ldots, g_i g_{i+1}, \ldots, g_{n+1}) + (-1)^{n+1}\varphi(g_1, \ldots, g_n).$$

for $\varphi : \overline{G}^{\times n} \longrightarrow kG$ and $g_1, \ldots, g_{n+1} \in \overline{G}$. So formally the left hand side and the right hand side in (2) are identical, though they have different meaning. It is also easy to check that under the above identifications, the adjoint isomorphisms are identity maps:

$$\mathrm{Hom}_{k(G\times G)}(k(G \times G) \otimes_{kG} \mathrm{Bar}_*(kG) \otimes_{kG} k, kG)$$
$$\longrightarrow \mathrm{Hom}_{kG}(\mathrm{Bar}_*(kG) \otimes_{kG} k, {}_ckG),$$
$$(\varphi_1 : \overline{G}^{\times n} \longrightarrow kG) \longmapsto (\varphi_2 : \overline{G}^{\times n} \longrightarrow kG), \quad \varphi_2(g_1, \cdots, g_n) = \varphi_1(g_1, \cdots, g_n).$$

Its inverse is given by

$$\mathrm{Hom}_{kG}(\mathrm{Bar}_*(kG) \otimes_{kG} k, {}_ckG)$$
$$\longrightarrow \mathrm{Hom}_{k(G \times G)}(k(G \times G) \otimes_{kG} \mathrm{Bar}_*(kG) \otimes_{kG} k, kG),$$
$$(\varphi_2 : \overline{G}^{\times n} \longrightarrow kG) \longmapsto (\varphi_1 : \overline{G}^{\times n} \longrightarrow kG), \quad \varphi_1(g_1, \ldots, g_n) = \varphi_2(g_1, \ldots, g_n).$$

Passing to the cohomology, we realize an isomorphism in (2) and its inverse.

The third step. We choose a complete set X of representatives of the conjugacy classes in the finite group G. Take $x \in X$. Then $C_x = \{gxg^{-1} | g \in G\}$ is the conjugacy class corresponding to x and $C_G(x) = \{g \in G | gxg^{-1} = x\}$ is the centralizer subgroup. Clearly the k-space kC_x generated by the elements in C_x is a left kG-module under the conjugation action. We choose a right coset decomposition of $C_G(x)$ in G: $G = C_G(x)\gamma_{1,x} \cup C_G(x)\gamma_{2,x} \cup \cdots \cup C_G(x)\gamma_{n_x,x}$ (equivalently, $G = \gamma_{1,x}^{-1}C_G(x) \cup \gamma_{2,x}^{-1}C_G(x) \cup \cdots \cup \gamma_{n_x,x}^{-1}C_G(x)$ is a left coset decomposition of $C_G(x)$ in G), and such that $C_x = \{x = \gamma_{1,x}^{-1}x\gamma_{1,x}, \gamma_{2,x}^{-1}x\gamma_{2,x}, \ldots, \gamma_{n_x,x}^{-1}x\gamma_{n_x,x}\}$. (We will always take $\gamma_{1,x} = 1$, and we write x_i for $\gamma_{i,x}^{-1}x\gamma_{i,x}$.) Then we have the following kG-module isomorphisms:

$$_ckC_x \simeq \mathrm{Ind}_{C_G(x)}^G k = {}_{kG}kG \otimes_{kC_G(x)} k, \quad x_i \longmapsto \gamma_{i,x}^{-1} \otimes 1,$$
$$_ckC_x \simeq \mathrm{Coind}_{C_G(x)}^G k = \mathrm{Hom}_{kC_G(x)}({}_{kC_G(x)}kG_{kG}, {}_{kC_G(x)}k),$$
$$x_i \longmapsto \gamma_i : kG \longrightarrow k, \gamma_i(\gamma_{j,x}) = \delta_{ij},$$

where in the first isomorphism, the left kG-module structure on kG is the usual left multiplication and the right $kC_G(x)$-module structure on kG is given by restriction, and k is the trivial $kC_G(x)$-module, and the same as in the second isomorphism.

In the second step, we have arrived at the ordinary group cohomology $H^*(G, kG)$ of G with coefficients in $_ckG$. This $_ckG$ has a kG-module decomposition:

$$_ckG = \bigoplus_{x \in X} {}_ckC_x.$$

Denote by $\pi_x : kG \longrightarrow kC_x$ and $i_x : kC_x \longrightarrow kG$ the canonical projection and the canonical injection, respectively. Then we have the following isomorphism

$$\mathrm{Hom}_{kG}(\mathrm{Bar}_*(kG) \otimes_{kG} k, {}_ckG) \longrightarrow \bigoplus_{x \in X} \mathrm{Hom}_{kG}(\mathrm{Bar}_*(kG) \otimes_{kG} k, {}_ckC_x),$$

$$(\varphi_2 : \overline{G}^{\times n} \longrightarrow kG) \longmapsto \varphi_3 = \{\varphi_{3,x} | x \in X\}, \text{ where } \varphi_{3,x} = \pi_x\varphi_2 : \overline{G}^{\times n} \longrightarrow kC_x.$$

Its inverse is given by

$$\bigoplus_{x \in X} \mathrm{Hom}_{kG}(\mathrm{Bar}_*(kG) \otimes_{kG} k, kC_x) \longrightarrow \mathrm{Hom}_{kG}(\mathrm{Bar}_*(kG) \otimes_{kG} k, kG),$$

$$\varphi_3 = \{\varphi_{3,x} : \overline{G}^{\times n} \longrightarrow kC_x | x \in X\} \longmapsto \left(\varphi_2 = \sum_{x \in X} i_x \varphi_{3,x} : \overline{G}^{\times n} \longrightarrow kG\right).$$

Passing to the cohomology, we realize an isomorphism:

$$H^*(G, {}_ckG) \simeq \bigoplus_{x \in X} H^*(G, {}_ckC_x). \tag{3}$$

The fourth step. We have stated in the third step the following kG-module isomorphism

$${}_ckC_x \simeq \mathrm{Hom}_{kC_G(x)}(kG, k), \quad x_i \longmapsto \gamma_i : kG \longrightarrow k, \gamma_i(\gamma_{j,x}) = \delta_{ij}.$$

Therefore we have the following isomorphism

$$\mathrm{Hom}_{kG}(\mathrm{Bar}_*(kG) \otimes_{kG} k, {}_ckC_x)$$
$$\longrightarrow \mathrm{Hom}_{kG}(\mathrm{Bar}_*(kG) \otimes_{kG} k, \mathrm{Hom}_{kC_G(x)}(kG, k)),$$

$$(\varphi_{3,x} : \overline{G}^{\times n} \longrightarrow kC_x) \longmapsto (\varphi_{4,x} : \overline{G}^{\times n} \longrightarrow \mathrm{Hom}_{kC_G(x)}(kG, k)),$$

where if we write $\varphi_{3,x}(g_1, g_2, \ldots, g_n) = \sum_{i=1}^{n_x} a_{i,x} x_i$, then $\varphi_{4,x}(g_1, g_2, \ldots, g_n)$ maps $\gamma_{i,x}$ to $a_{i,x}$ for any i. The inverse isomorphism is given by

$$\mathrm{Hom}_{kG}(\mathrm{Bar}_*(kG) \otimes_{kG} k, \mathrm{Hom}_{kC_G(x)}(kG, k))$$
$$\longrightarrow \mathrm{Hom}_{kG}(\mathrm{Bar}_*(kG) \otimes_{kG} k, {}_ckC_x),$$

$$(\varphi_{4,x} : \overline{G}^{\times n} \longrightarrow \mathrm{Hom}_{kC_G(x)}(kG, k)) \longmapsto (\varphi_{3,x} : \overline{G}^{\times n} \longrightarrow kC_x),$$

where if $\varphi_{4,x}(g_1, g_2, \ldots, g_n)$ maps $\gamma_{i,x}$ to $a_{i,x}$ for any i, then $\varphi_{3,x}(g_1, g_2, \ldots, g_n) = \sum_{i=1}^{n_x} a_{i,x} x_i$. Passing to the cohomology, we realize an isomorphism:

$$H^*(G, kC_x) \simeq H^*(\mathrm{Hom}_{kG}(\mathrm{Bar}_*(kG) \otimes_{kG} k, \mathrm{Hom}_{kC_G(x)}(kG, k))). \tag{4}$$

The fifth step. Since $(kG \otimes_{kG} -, \mathrm{Hom}_{kC_G(x)}(kG, -))$ is an adjoint pair (restriction and coinduction), we have the following isomorphism

$$\mathrm{Hom}_{kG}(\mathrm{Bar}_*(kG) \otimes_{kG} k, \mathrm{Hom}_{kC_G(x)}(kG, k))$$
$$\longrightarrow \mathrm{Hom}_{kC_G(x)}(\mathrm{Bar}_*(kG) \otimes_{kG} k, k).$$

Passing to the cohomology, we get an isomorphism

$$H^*(\mathrm{Hom}_{kG}(\mathrm{Bar}_*(kG) \otimes_{kG} k, \mathrm{Hom}_{kC_G(x)}(kG,k)))$$
$$\simeq H^*(\mathrm{Hom}_{kC_G(x)}(\mathrm{Bar}_*(kG) \otimes_{kG} k, k)), \quad (5)$$

where the right hand side is isomorphic to the ordinary group cohomology $H^*(C_G(x), k)$ of $C_G(x)$ with coefficients in the trivial module k. Since there are $kC_G(x)$-module isomorphisms

$$\mathrm{Bar}_*(kG) \otimes_{kG} k \simeq \bigoplus_{i=1}^{n_x} kC_G(x)\gamma_{i,x} \otimes \overline{kG}^{\otimes *},$$

we have

$$\mathrm{Hom}_{kC_G(x)}(\mathrm{Bar}_*(kG) \otimes_{kG} k, k) \simeq \mathrm{Hom}_k \left(\bigoplus_{i=1}^{n_x} k\gamma_{i,x} \otimes \overline{kG}^{\otimes n}, k \right)$$
$$\simeq \mathrm{Map}(S_x \times \overline{G}^{\times n}, k),$$

where $S_x = \{\gamma_{1,x}, \ldots, \gamma_{n_x,x}\}$ (cf. the third step). Using this identification, the adjoint isomorphism is given by

$$\mathrm{Hom}_{kG}(\mathrm{Bar}_*(kG) \otimes_{kG} k, \mathrm{Hom}_{kC_G(x)}(kG,k))$$
$$\longrightarrow \mathrm{Hom}_{kC_G(x)}(\mathrm{Bar}_*(kG) \otimes_{kG} k, k),$$

$$(\varphi_{4,x} : \overline{G}^{\times n} \longrightarrow \mathrm{Hom}_{kC_G(x)}(kG,k)) \longmapsto (\varphi_{5,x} : S_x \times \overline{G}^{\times n} \longrightarrow k),$$

where if $\varphi_{4,x}(g_1, g_2, \ldots, g_n)$ maps $\gamma_{i,x}$ to $a_{i,x}$ for any i, then $\varphi_{5,x}(\gamma_{i,x}, g_1, g_2, \ldots, g_n) = a_{i,x}$ for any i. The inverse isomorphism is given by

$$\mathrm{Hom}_{kC_G(x)}(\mathrm{Bar}_*(kG) \otimes_{kG} k, k)$$
$$\longrightarrow \mathrm{Hom}_{kG}(\mathrm{Bar}_*(kG) \otimes_{kG} k, \mathrm{Hom}_{kC_G(x)}(kG,k)), s$$

$$(\varphi_{5,x} : S_x \times \overline{G}^{\times n} \longrightarrow k) \longmapsto (\varphi_{4,x} : \overline{G}^{\times n} \longrightarrow \mathrm{Hom}_{kC_G(x)}(kG,k)),$$

where if $\varphi_{5,x}(\gamma_{i,x}, g_1, g_2, \ldots, g_n) = a_{i,x}$ for any i, then $\varphi_{4,x}(g_1, g_2, \ldots, g_n)$ maps $\gamma_{i,x}$ to $a_{i,x}$ for any i. Passing to the cohomology, we realize an isomorphism in (5) and its inverse.

The sixth step. In the fifth step, we have arrived at the ordinary group cohomology $H^*(C_G(x), k)$ of $C_G(x)$ with coefficients in the trivial module k, where $H^*(C_G(x), k)$ is computed by the cochain complex

$$\mathrm{Hom}_{kC_G(x)}(\mathrm{Bar}_*(kG) \otimes_{kG} k, k).$$

By the identification in the fifth step, this is given by the following cochain complex:

$$0 \longrightarrow k^{\times n_x} \xrightarrow{\delta_0} \mathrm{Map}(S_x \times \overline{G}, k) \xrightarrow{\delta_1} \cdots \longrightarrow \mathrm{Map}(S_x \times \overline{G}^{\times n}, k) \xrightarrow{\delta_n} \cdots,$$

where the differential is given by $\delta_0(\{a_{i,x}\})((\gamma_{j,x}, g_1)) = a_{s_j,x} - a_{j,x}$, such that $a_{s_j,x}$ is determined as follows: for $\{a_{i,x}\} \in k^{\times n_x}, \gamma_{j,x} \in S_x, g_1 \in \overline{G}$, we have

$$\gamma_{j,x} g_1 = h_{j,1} \gamma_{s_j,x}$$

for some $h_{j,1} \in C_G(x)$ and for some $1 \le s_j \le n_x$, and

$$\delta_n(\varphi)(\gamma_{j,x}, g_1, \ldots, g_{n+1}) = \varphi(\gamma_{s_j,x}, g_2, \ldots, g_{n+1})$$
$$+ \sum_{i=1}^{n} (-1)^i \varphi(\gamma_{j,x}, g_1, \ldots, g_i g_{i+1}, \ldots, g_{n+1}) + (-1)^{n+1} \varphi(\gamma_{j,x}, g_1, \ldots, g_n),$$

for $\varphi : S_x \times \overline{G}^{\times n} \longrightarrow k, \gamma_{j,x} \in S_x, g_1, \ldots, g_{n+1} \in \overline{G}$ such that $\gamma_{j,x} g_1 = h_{j,1} \gamma_{s_j,x}$. (Remark that for a fixed $g_1 \in \overline{G}, \{s_1, s_2, \ldots, s_{n_x}\}$ is a permutation of $\{1, 2, \ldots, n_x\}$.)

The above computation for $H^*(C_G(x), k)$ uses the projective resolution $\mathrm{Bar}_*(kG) \otimes_{kG} k$ of the trivial $kC_G(x)$-module k, which is identified as the following complex. (It is in fact a projective resolution of the trivial kG-module k, but we view it as a complex of $kC_G(x)$-modules by restriction.)

$$\cdots \longrightarrow kG \otimes \overline{kG}^{\otimes n} \xrightarrow{d_n} \cdots \longrightarrow kG \otimes \overline{kG} \xrightarrow{d_1} kG \xrightarrow{d_0} k \longrightarrow 0,$$

where the differential is given by

$$d_0(g_0) = 1,$$

for $g_0 \in G$, and

$$d_n(g_0, g_1, \ldots, g_n) = g_0 g_1 \otimes g_2 \otimes \cdots \otimes g_n$$
$$+ \sum_{i=1}^{n-1} (-1)^i g_0 \otimes \cdots \otimes g_i g_{i+1} \otimes \cdots \otimes g_n + (-1)^n g_0 \otimes g_1 \otimes \cdots \otimes g_{n-1},$$

for $g_0 \in G, g_1, \ldots, g_n \in \overline{G}$. We now use another projective resolution $\mathrm{Bar}_*(kC_G(x)) \otimes_{kC_G(x)} k$ of the trivial $kC_G(x)$-module k, which is identified as the following complex

$$\cdots \longrightarrow kC_G(x) \otimes \overline{kC_G(x)}^{\otimes n} \xrightarrow{d_n} \cdots$$
$$\cdots \longrightarrow kC_G(x) \otimes \overline{kC_G(x)} \xrightarrow{d_1} kC_G(x) \xrightarrow{d_0} k \longrightarrow 0,$$

where the differential is given by

$$d_0(h_0) = 1,$$

for $h_0 \in C_G(x)$, and

$$d_n(h_0, h_1, \ldots, h_n) = h_0 h_1 \otimes h_2 \otimes \cdots \otimes h_n$$
$$+ \sum_{i=1}^{n-1} (-1)^i h_0 \otimes \cdots \otimes h_i h_{i+1} \otimes \cdots \otimes h_n + (-1)^n h_0 \otimes h_1 \otimes \cdots \otimes h_{n-1},$$

for $h_0 \in C_G(x), h_1, \ldots, h_n \in \overline{C_G(x)}$. We have

$$\mathrm{Hom}_{kC_G(x)}(\mathrm{Bar}_*(kC_G(x)) \otimes_{kC_G(x)} k, k) \simeq \mathrm{Map}(\overline{C_G(x)}^{\times n}, k),$$

so $H^*(C_G(x), k)$ can also be computed by the following cochain complex

$$0 \longrightarrow k \xrightarrow{\delta_0} \mathrm{Map}(\overline{C_G(x)}, k) \xrightarrow{\delta_1} \cdots \longrightarrow \mathrm{Map}(\overline{C_G(x)}^{\times n}, k) \xrightarrow{\delta_n} \cdots,$$

where the differential is given by

$$\delta_0(a)(h_1) = 0,$$

for $a \in k, h_1 \in \overline{C_G(x)}$, and

$$\delta_n(\varphi)(h_1, \ldots, h_{n+1}) = \varphi(h_2, \ldots, h_{n+1})$$
$$+ \sum_{i=1}^{n} (-1)^i \varphi(h_1, \ldots, h_i h_{i+1}, \ldots, h_{n+1}) + (-1)^{n+1} \varphi(h_1, \ldots, h_n),$$

for $\varphi : \overline{C_G(x)}^{\times n} \longrightarrow k, h_1, \ldots, h_{n+1} \in \overline{C_G(x)}$. Clearly, we have

$$H^*(\mathrm{Hom}_{kC_G(x)}(\mathrm{Bar}_*(kG) \otimes_{kG} k, k))$$
$$\simeq H^*(\mathrm{Hom}_{kC_G(x)}(\mathrm{Bar}_*(kC_G(x)) \otimes_{kC_G(x)} k, k)). \quad (6)$$

To give an explicit isomorphism in (6), we need to construct the comparison maps between two projective resolutions $\mathrm{Bar}_*(kG) \otimes_{kG} k$ and $\mathrm{Bar}_*(kC_G(x)) \otimes_{kC_G(x)} k$ of the trivial $kC_G(x)$-module k.

The comparison map from $\mathrm{Bar}_*(kC_G(x)) \otimes_{kC_G(x)} k$ to $\mathrm{Bar}_*(kG) \otimes_{kG} k$ is just the inclusion map

$$\iota : kC_G(x) \otimes \overline{kC_G(x)}^{\otimes n} \hookrightarrow kG \otimes \overline{kG}^{\otimes n}.$$

This is obvious or can be obtained using a setwise self-homotopy on $\mathrm{Bar}_*(kG) \otimes_{kG} k$ (see below for its explicit form).

To construct the comparison map on the reverse direction, we use a setwise self-homotopy over $kC_G(x) \otimes \overline{kC_G(x)}^{\otimes *}$ as follows:

$$kC_G(x) \otimes \overline{kC_G(x)}^{\otimes n} \longrightarrow kC_G(x) \otimes \overline{kC_G(x)}^{\otimes n+1},$$
$$h_0 \otimes h_1 \otimes \cdots \otimes h_n \longmapsto 1 \otimes h_0 \otimes h_1 \otimes \cdots \otimes h_n,$$

for $h_0 \in C_G(x), h_1, \ldots, h_n \in \overline{C_G(x)}$. Then we get a comparison map

$$\rho : \mathrm{Bar}_*(kG) \otimes_{kG} k \longrightarrow \mathrm{Bar}_*(kC_G(x)) \otimes_{kC_G(x)} k$$

as follows:

$$\rho_{-1} : k \longrightarrow k, \quad 1 \longmapsto 1,$$
$$\rho_0 : kG \longrightarrow kC_G(x), \quad h\gamma_{i,x} \longmapsto h, \text{ for } h \in C_G(x),$$
$$\rho_1 : kG \otimes \overline{kG} \longrightarrow kC_G(x) \otimes \overline{kC_G(x)}, \quad h\gamma_{i,x} \otimes g_1 \longmapsto h \otimes h_{i,1},$$

where $\gamma_{i,x} g_1 = h_{i,1}\gamma_{s_i,x}$ for $h_{i,1} \in \overline{C_G(x)}$,

$$\vdots$$

$$\rho_n : kG \otimes \overline{kG}^{\otimes n} \longrightarrow kC_G(x) \otimes \overline{kC_G(x)}^{\otimes n},$$
$$h\gamma_{i,x} \otimes g_1 \otimes \cdots \otimes g_n \longmapsto h \otimes h_{i,1} \otimes \cdots \otimes h_{i,n},$$

where $h_{i,1}, \ldots, h_{i,n} \in \overline{C_G(x)}$ are determined by the sequence $\{g_1, \ldots, g_n\}$ as follows:

$$\gamma_{i,x} g_1 = h_{i,1}\gamma_{s_i^1,x}, \quad \gamma_{s_i^1,x} g_2 = h_{i,2}\gamma_{s_i^2,x}, \quad \cdots \quad \gamma_{s_i^{n-1},x} g_n = h_{i,n}\gamma_{s_i^n,x}.$$

Notice that $\rho \circ \iota = \mathrm{Id}$ and $\iota \circ \rho \neq \mathrm{Id}$. It follows that we have two homomorphisms:

$$\mathrm{Hom}_{kC_G(x)}(\mathrm{Bar}_*(kG) \otimes_{kG} k, k) \longrightarrow \mathrm{Hom}_{kC_G(x)}(\mathrm{Bar}_*(kC_G(x)) \otimes_{kC_G(x)} k, k),$$
$$(\varphi_{5,x} : S_x \times \overline{G}^{\times n} \longrightarrow k) \longmapsto (\varphi_{6,x} : \overline{C_G(x)}^{\times n} \longrightarrow k),$$
$$\varphi_{6,x}(h_1, \ldots, h_n) = \varphi_{5,x}(1, h_1, \ldots, h_n) = a_{1,x},$$

where $a_{1,x}$ is the coefficient of x in $\varphi_{3,x}(h_1, \ldots, h_n) = \sum_{i=1}^{n_x} a_{i,x} x_i$; and

$$\mathrm{Hom}_{kC_G(x)}(\mathrm{Bar}_*(kC_G(x)) \otimes_{kC_G(x)} k, k) \longrightarrow \mathrm{Hom}_{kC_G(x)}(\mathrm{Bar}_*(kG) \otimes_{kG} k, k),$$
$$(\varphi_{6,x} : \overline{C_G(x)}^{\times n} \longrightarrow k) \longmapsto (\varphi_{5,x} : S_x \times \overline{G}^{\times n} \longrightarrow k),$$
$$\varphi_{5,x}(\gamma_{i,x}, g_1, \ldots, g_n) = \varphi_{6,x}(h_{i,1}, \ldots, h_{i,n}),$$

where for $h_{i,1}, \ldots, h_{i,n} \in \overline{C_G(x)}$ are determined by the sequence $\{g_1, \ldots, g_n\}$ as follows:

$$\gamma_{i,x} g_1 = h_{i,1}\gamma_{s_i^1,x}, \quad \gamma_{s_i^1,x} g_2 = h_{i,2}\gamma_{s_i^2,x}, \quad \cdots \quad \gamma_{s_i^{n-1},x} g_n = h_{i,n}\gamma_{s_i^n,x}.$$

Since both ι and ρ induce the identity map $1 : k \longrightarrow k$, by Lemma 2.3, we have inverse isomorphisms between $H^*(\mathrm{Hom}_{kC_G(x)}(\mathrm{Bar}_*(kG) \otimes_{kG} k, k))$ and

$$H^*(\mathrm{Hom}_{kC_G(x)}(\mathrm{Bar}_*(kC_G(x)) \otimes_{kC_G(x)} k, k)).$$

The correspondence is induced by $\varphi_{5,x} \longleftrightarrow \varphi_{6,x}$, as we stated above. So we realize an isomorphism in (6) and its inverse.

Summarizing the above six steps, we get the following main result in this section.

Theorem 5.1. *Let k be a field and G a finite group. Consider the additive decomposition of Hochschild cohomology ring of the group algebra kG:*

$$HH^*(kG) \simeq \bigoplus_{x \in X} H^*(C_G(x), k),$$

where X is a set of representatives of conjugacy classes of elements of G and $C_G(x)$ is the centralizer subgroup of G. Recall that we choose a right coset decomposition of $C_G(x)$ in G:

$$G = C_G(x)\gamma_{1,x} \cup C_G(x)\gamma_{2,x} \cup \cdots \cup C_G(x)\gamma_{n_x,x}$$

equivalently,

$$G = \gamma_{1,x}^{-1}C_G(x) \cup \gamma_{2,x}^{-1}C_G(x) \cup \cdots \cup \gamma_{n_x,x}^{-1}C_G(x)$$

is a left coset decomposition of $C_G(x)$ in G, and such that

$$C_x = \{x = \gamma_{1,x}^{-1}x\gamma_{1,x}, \gamma_{2,x}^{-1}x\gamma_{2,x}, \ldots, \gamma_{n_x,x}^{-1}x\gamma_{n_x,x}\}.$$

We will always take $\gamma_{1,x} = 1$, and we write x_i for $\gamma_{i,x}^{-1}x\gamma_{i,x}$.

We compute the Hochschild cohomology

$$HH^*(kG) = H^*(\mathrm{Hom}_{k(G \times G)}(\mathrm{Bar}_*(kG), kG))$$

by the classical normalized bar resolution, and we compute the group cohomology $H^*(C_G(x), k)$ by

$$H^*(\mathrm{Hom}_{kC_G(x)}(\mathrm{Bar}_*(kC_G(x)) \otimes_{kC_G(x)} k, k)).$$

Then, we can realize an isomorphism in additive decomposition as follows:

$$HH^*(kG) \xrightarrow{\sim} \bigoplus_{x \in X} H^*(C_G(x), k),$$

$$[\varphi : \overline{G}^{\times n} \longrightarrow kG] \longmapsto [\widehat{\varphi}] = \bigoplus_{x \in X}[\widehat{\varphi}_x], \quad \widehat{\varphi}_x : \overline{C_G(x)}^{\times n} \longrightarrow k,$$

$$\widehat{\varphi}_x(h_1, \ldots, h_n) = a_{1,x},$$

where $\pi_x((-1)^{\frac{n(n+1)}{2}} h_1 \cdots h_n \varphi(h_n^{-1}, \ldots, h_1^{-1})) = \sum_{i=1}^{n_x} a_{i,x}x_i$.

In other words, $\widehat{\varphi}_x(h_1, \ldots, h_n)$ is just the coefficient of x in

$$(-1)^{\frac{n(n+1)}{2}} h_1 \cdots h_n \varphi(h_n^{-1}, \ldots, h_1^{-1}) \in kG.$$

The inverse of the above isomorphism is given as follows:

$$\bigoplus_{x \in X} H^*(C_G(x), k) \xrightarrow{\sim} HH^*(kG),$$

$$[\widehat{\varphi}] = \bigoplus_{x \in X} [\widehat{\varphi}_x], \quad \widehat{\varphi}_x : \overline{C_G(x)}^{\times n} \longrightarrow k \longmapsto [\varphi : \overline{G}^{\times n} \longrightarrow kG],$$

$$\varphi(g_1, \ldots, g_n) = (-1)^{\frac{n(n+1)}{2}} g_1 \cdots g_n \sum_{x \in X} \sum_{i=1}^{n_x} \widehat{\varphi}_x(h'_{i,1}, \ldots, h'_{i,n}) x_i,$$

where for $x \in X, h'_{i,1}, \ldots, h'_{i,n} \in \overline{C_G(x)}$ are determined by the sequence $\{g_n^{-1}, \ldots, g_1^{-1}\}$ as follows:

$$\gamma_{i,x} g_n^{-1} = h'_{i,1} \gamma_{s_i^1, x}, \quad \gamma_{s_i^1, x} g_{n-1}^{-1} = h'_{i,2} \gamma_{s_i^2, x}, \quad \cdots \quad \gamma_{s_i^{n-1}, x} g_1^{-1} = h'_{i,n} \gamma_{s_i^n, x}.$$

Proof. This is a direct consequence by applying the above isomorphisms from (1) to (6) and their inverses. For an element $\varphi : \overline{G}^{\times n} \longrightarrow kG$ in the nth term $C^n(kG) \simeq \mathrm{Map}(\overline{G}^{\times n}, kG)$ of the Hochschild cohomology complex, $[\varphi]$ denotes the corresponding element in the Hochschild cohomology group $HH^n(kG)$. Note that the elements $h'_{i,1}, \ldots, h'_{i,n}$ depend on $x \in X$ and the sequence $\{g_n^{-1}, \ldots, g_1^{-1}\}$. For the simplicity of notations, we avoid to write them down explicitly. $\square$

Remark 5.2. (a) The correspondence in Theorem 5.1 makes use of the same line employed by Siegel and Witherspoon in [17]. The difference is: they realize each step between cohomology groups using standard operations like restriction, induction, conjugation, etc., while we construct maps directly in each step on the cohomology complex level.

(b) In [17], as the authors proved that $HH^*(kG) \simeq H^*(G, _ckG)$ as graded algebras, they concentrated on $H^*(G, _ckG)$ instead of $HH^*(kG)$ in most part of their paper. If we only consider the isomorphisms (2)–(5), then the correspondence in Theorem 5.1 become simpler:

$$H^*(G, _ckG) \xrightarrow{\sim} \bigoplus_{x \in X} H^*(C_G(x), k),$$

$$[\varphi : \overline{G}^{\times n} \longrightarrow kG] \longmapsto [\widehat{\varphi}] = \bigoplus_{x \in X} [\widehat{\varphi}_x], \quad \widehat{\varphi}_x : \overline{C_G(x)}^{\times n} \longrightarrow k,$$

$$\widehat{\varphi}_x(h_1, \ldots, h_n) = a_{1,x}, \text{ the coefficient of } x \text{ in } \varphi(h_1, \ldots, h_n) \in kG;$$

$$\bigoplus_{x \in X} H^*(C_G(x), k) \xrightarrow{\sim} H^*(G, _ckG),$$

$$[\widehat{\varphi}] = \bigoplus_{x \in X} [\widehat{\varphi}_x], \quad \widehat{\varphi}_x : \overline{C_G(x)}^{\times n} \longrightarrow k \longmapsto [\varphi : \overline{G}^{\times n} \longrightarrow kG],$$

$$\varphi(g_1, \ldots, g_n) = \sum_{x \in X} \sum_{i=1}^{n_x} \widehat{\varphi}_x(h_{i,1}, \ldots, h_{i,n}) x_i,$$

where for $x \in X, h_{i,1}, \ldots, h_{i,n} \in \overline{C_G(x)}$ are determined by the sequence $\{g_1, \ldots, g_n\}$ as follows:

$$\gamma_{i,x} g_1 = h_{i,1} \gamma_{s_i^1, x}, \quad \gamma_{s_i^1, x} g_2 = h_{i,2} \gamma_{s_i^2, x}, \quad \cdots \quad \gamma_{s_i^{n-1}, x} g_n = h_{i,n} \gamma_{s_i^n, x}.$$

6. Another realization of the additive decomposition

In [4], Cibils and Solotar constructed a subcomplex of the Hochschild cohomology complex for each conjugacy class, and then they showed that for a finite abelian group, the subcomplex is isomorphic to the complex computing group cohomology. We will generalize this to any finite group: for each conjugacy class, this complex computes the cohomology of the corresponding centralizer subgroup. As a result, we give a second way to realize the additive decomposition.

As before, let k be a field and G a finite group. Recall that the Hochschild cohomology $HH^*(kG)$ of the group algebra kG can be computed by the following (cochain) complex:

$$0 \longrightarrow kG \xrightarrow{\delta_0} \mathrm{Map}(\overline{G}, kG) \xrightarrow{\delta_1} \cdots \longrightarrow \mathrm{Map}(\overline{G}^{\times n}, kG) \xrightarrow{\delta_n} \cdots, \qquad (\mathcal{H}^*)$$

where the differential is given by

$$\delta_0(x)(g) = gx - xg$$

for $x \in kG$ and $g \in \overline{G}$, and

$$\delta_n(\varphi)(g_1, \ldots, g_{n+1}) = g_1 \varphi(g_2, \ldots, g_{n+1})$$
$$+ \sum_{i=1}^{n} (-1)^i \varphi(g_1, \ldots, g_i g_{i+1}, \ldots, g_{n+1}) + (-1)^{n+1} \varphi(g_1, \ldots, g_n) g_{n+1}$$

for $\varphi : \overline{G}^{\times n} \longrightarrow kG$ and $g_1, \ldots, g_{n+1} \in \overline{G}$. We keep the following notations in Section 3: X is a complete set of representatives of the conjugacy classes in the finite group G. For $x \in X$, $C_x = \{gxg^{-1} | g \in G\}$ is the conjugacy class corresponding to x and $C_G(x) = \{g \in G | gxg^{-1} = x\}$ is the centralizer subgroup. Now take a conjugacy class C_x and define

$$\mathcal{H}_x^0 = kC_x,$$

and for $n \geq 1$,

$$\mathcal{H}_x^n = \{\varphi : \overline{G}^{\times n} \longrightarrow kG \mid \varphi(g_1, \ldots, g_n) \in k[g_1 \cdots g_n C_x] \subset kG, \forall g_1, \ldots, g_n \in \overline{G}\},$$

where $g_1 \cdots g_n C_x$ denotes the subset of G by multiplying $g_1 \cdots g_n$ on C_x and $k[g_1 \cdots g_n C_x]$ is the k-subspace of kG generated by this set. Note that we have

$$g_1 \cdots g_n C_x = C_x g_1 \cdots g_n \quad \text{and} \quad k[g_1 \cdots g_n C_x] = k[C_x g_1 \cdots g_n].$$

Let $\mathcal{H}_x^* = \bigoplus_{n \geq 0} \mathcal{H}_x^n$. Cibils and Solotar [4, p. 20, Proof of the theorem] observed that $\mathcal{H}_x^*$ is a subcomplex of $\mathcal{H}^*$ and $\mathcal{H}^* = \bigoplus_{x \in X} \mathcal{H}_x^*$.

Lemma 6.1. *$\mathcal{H}_x^*$ is canonically isomorphic to the complex*

$$\mathrm{Hom}_{kG}(\mathrm{Bar}_*(kG) \otimes_{kG} k, kC_x),$$

which computes the group cohomology $H^*(G, kC_x)$ of G with coefficients in kC_x, where kC_x is a left kG-module under conjugation.

Proof. We know from Section 3 that the complex $\mathrm{Hom}_{kG}(\mathrm{Bar}_*(kG) \otimes_{kG} k, kC_x)$ is identified as the following complex:

$$0 \longrightarrow kC_x \xrightarrow{\delta_0} \mathrm{Map}(\overline{G}, kC_x) \xrightarrow{\delta_1} \cdots \longrightarrow \mathrm{Map}(\overline{G}^{\times n}, kC_x) \xrightarrow{\delta_n} \cdots,$$

where the differential is given by

$$\delta_0(x)(g) = gxg^{-1} - x$$

for $x \in kC_x$ and $g \in \overline{G}$, and

$$\delta_n(\varphi)(g_1, \ldots, g_{n+1}) = g_1 \varphi(g_2, \ldots, g_{n+1}) g_1^{-1}$$
$$+ \sum_{i=1}^{n} (-1)^i \varphi(g_1, \ldots, g_i g_{i+1}, \ldots, g_{n+1}) + (-1)^{n+1} \varphi(g_1, \ldots, g_n)$$

for $\varphi : \overline{G}^{\times n} \longrightarrow kC_x$ and $g_1, \ldots, g_{n+1} \in \overline{G}$. A direct computation shows that the following map is an isomorphism of complexes:

$$\mathcal{H}_x^* \longrightarrow \mathrm{Hom}_{kG}(\mathrm{Bar}_*(kG) \otimes_{kG} k, kC_x),$$
$$(\varphi_1 : \overline{G}^{\times n} \longrightarrow kG) \longmapsto (\varphi_2 : \overline{G}^{\times n} \longrightarrow kC_x),$$
$$\varphi_2(g_1, \ldots, g_n) = \varphi_1(g_1, \ldots, g_n) g_n^{-1} \cdots g_1^{-1}.$$

Its inverse is given by

$$\mathrm{Hom}_{kG}(\mathrm{Bar}_*(kG) \otimes_{kG} k, kC_x) \longrightarrow \mathcal{H}_x^*,$$
$$(\varphi_2 : \overline{G}^{\times n} \longrightarrow kC_x) \longmapsto (\varphi_1 : \overline{G}^{\times n} \longrightarrow kG),$$
$$\varphi_1(g_1, \ldots, g_n) = \varphi_2(g_1, \ldots, g_n) g_1 \cdots g_n.$$

Passing to the cohomology, we have $H^*(\mathcal{H}_x^*) \simeq H^*(G, kC_x)$. $\qquad \square$

Remark 6.2. Since the first three steps of the previous section realize

$$HH^*(kG) \simeq H^*(G, {}_ckG) \simeq \oplus_{x \in X} H^*(C_G(x), k),$$

these isomorphisms also give a decomposition of the complex $\mathcal{H}^*$, which computes $HH^*(kG)$. In fact, during these three steps, we establish the following isomorphisms of complexes

$$\mathcal{H}^* = \mathrm{Hom}_{k(G \times G)}(\mathrm{Bar}_*(kG), kG)$$

$$\overset{(1)}{\simeq} \mathrm{Hom}_{k(G \times G)}(k(G \times G) \otimes_{kG} \mathrm{Bar}_*(kG) \otimes_{kG} k, kG)$$

$$\overset{(2)}{\simeq} \mathrm{Hom}_{kG}(\mathrm{Bar}_*(kG) \otimes_{kG} k, {}_ckG)$$

$$\overset{(3)}{=} \bigoplus_{x \in X} \mathrm{Hom}_{kG}(\mathrm{Bar}_*(kG) \otimes_{kG} k, {}_ckC_x).$$

So the complex $\mathrm{Hom}_{kG}(\mathrm{Bar}_*(kG) \otimes_{kG} k, {}_ckC_x)$ is isomorphic to a subcomplex of $\mathcal{H}^*$ and we verify easily that this subcomplex is just the above defined $\mathcal{H}_x^*$. However, the isomorphism between these two complexes is as follows:

$$\mathcal{H}_x^* \longrightarrow \mathrm{Hom}_{kG}(\mathrm{Bar}_*(kG) \otimes_{kG} k, kC_x),$$

$$(\varphi_1 : \overline{G}^{\times n} \longrightarrow kG) \longmapsto (\varphi_2 : \overline{G}^{\times n} \longrightarrow kC_x),$$

$$\varphi_2(g_1, \ldots, g_n) = (-1)^{\frac{n(n+1)}{2}} g_1 \cdots g_n \varphi_1(g_n^{-1}, \ldots, g_1^{-1}).$$

Its inverse is given by

$$\mathrm{Hom}_{kG}(\mathrm{Bar}_*(kG) \otimes_{kG} k, kC_x) \longrightarrow \mathcal{H}_x^*,$$

$$(\varphi_2 : \overline{G}^{\times n} \longrightarrow kC_x) \longmapsto (\varphi_1 : \overline{G}^{\times n} \longrightarrow kG),$$

$$\varphi_1(g_1, \ldots, g_n) = (-1)^{\frac{n(n+1)}{2}} g_1 \cdots g_n \varphi_1(g_n^{-1}, \ldots, g_1^{-1}).$$

Note that this isomorphism differs from the one in Lemma 6.1 by an automorphism of the complex $\mathrm{Hom}_{kG}(\mathrm{Bar}_*(kG) \otimes_{kG} k, {}_ckC_x)$, which sends $\varphi : \overline{G}^{\times n} \to kC_x$ to $\varphi' : \overline{G}^{\times n} \to kC_x$ with

$$\varphi'(g_1, \ldots, g_n) = (-1)^{\frac{n(n+1)}{2}} g_1 \cdots g_n \varphi(g_n^{-1}, \ldots, g_1^{-1}) g_n^{-1} \cdots g_1^{-1}.$$

On the other hand, we have shown that the complex

$$\mathrm{Hom}_{kG}(\mathrm{Bar}_*(kG) \otimes_{kG} k, kC_x)$$

is isomorphic to the complex

$$\mathrm{Hom}_{kC_G(x)}(\mathrm{Bar}_*(kC_G(x)) \otimes_{kC_G(x)} k, k),$$

which computes the group cohomology $H^*(C_G(x), k)$ of the centralizer subgroup $C_G(x)$ with coefficients in the trivial module k. (cf. Section 5, from the fourth step to the sixth step.) Therefore we get another realization to the additive decomposition.

Theorem 6.3. *Let k be a field and G a finite group. Consider the additive decomposition of Hochschild cohomology ring of the group algebra kG:*

$$HH^*(kG) \simeq \bigoplus_{x \in X} H^*(C_G(x), k)$$

where X is a set of representatives of conjugacy classes of elements of G and $C_G(x)$ is the centralizer subgroup of G. We compute the Hochschild cohomology $HH^*(kG) = H^*(\mathrm{Hom}_{k(G \times G)}(\mathrm{Bar}_*(kG), kG))$ by the classical normalized bar resolution, and we compute the group cohomology $H^*(C_G(x), k)$ by

$$H^*(\mathrm{Hom}_{kC_G(x)}(\mathrm{Bar}_*(kC_G(x)) \otimes_{kC_G(x)} k, k)).$$

Then, we can realize an isomorphism in additive decomposition as follows:

$$HH^*(kG) \xrightarrow{\sim} \bigoplus_{x \in X} H^*(C_G(x), k),$$

$$[\varphi_x : \overline{G}^{\times n} \longrightarrow kG], \quad \varphi_x \in \mathcal{H}_x^n \longmapsto [\widehat{\varphi}_x : \overline{C_G(x)}^{\times n} \longrightarrow k],$$

$$\widehat{\varphi}_x(h_1, \ldots, h_n) = a_{1,x},$$

where

$$\varphi_x(h_1, \ldots, h_n) h_n^{-1} \cdots h_1^{-1} = \sum_{i=1}^{n_x} a_{i,x} x_i \in kC_x.$$

In other words, $\widehat{\varphi}_x(h_1, \ldots, h_n)$ is just the coefficient of x in $\varphi_x(h_1, \ldots, h_n) h_n^{-1} \cdots h_1^{-1} \in kC_x$. The inverse of the above isomorphism is given as follows:

$$\bigoplus_{x \in X} H^*(C_G(x), k) \xrightarrow{\sim} HH^*(kG),$$

$$[\widehat{\varphi}_x : \overline{C_G(x)}^{\times n} \longrightarrow k] \longmapsto [\varphi_x : \overline{G}^{\times n} \longrightarrow kG], \quad \varphi_x \in \mathcal{H}_x^n,$$

$$\varphi_x(g_1, \ldots, g_n) = \sum_{i=1}^{n_x} \widehat{\varphi}_x(h_{i,1}, \ldots, h_{i,n}) x_i g_1 \cdots g_n,$$

where $h_{i,1}, \ldots, h_{i,n} \in \overline{C_G(x)}$ are determined by the sequence $\{g_1, \ldots, g_n\}$ as follows:

$$\gamma_{i,x} g_1 = h_{i,1} \gamma_{s_i^1, x}, \quad \gamma_{s_i^1, x} g_2 = h_{i,2} \gamma_{s_i^2, x}, \quad \cdots \quad \gamma_{s_i^{n-1}, x} g_n = h_{i,n} \gamma_{s_i^n, x}.$$

Proof. This is a combination of Lemma 6.1 and the correspondence from the fourth step to the sixth step in Section 3. $\qquad \square$

By Remark 6.2, the two realizations of the additive decomposition in Theorem 5.1 and Theorem 6.3 are essentially the same on the cohomology group level. In the sequel, we prefer to the second realization since it is simpler.

7. The cup product formula

We keep the notations of the previous sections: k is a field, and G is a finite group, and so on. We describe the cup product formula for the Hochschild cohomology ring $HH^*(kG)$ in terms of the additive decomposition.

We shall define a product over $\sum_{x \in X} H^*(C_G(x), k)$ such that the isomorphism

$$HH^*(kG) \simeq \bigoplus_{x \in X} H^*(C_G(x), k)$$

realized in Theorem 6.3 becomes an isomorphism of graded algebras.

Let $[\widehat{\varphi_x}] \in H^*(C_G(x), k)$ (respectively $[\widehat{\varphi_y}] \in H^m(C_G(y), k)$) represented by the map $\widehat{\varphi_x} : \overline{C_G(x)}^{\times n} \longrightarrow k$ (respectively by $\widehat{\varphi_y} : \overline{C_G(y)}^{\times m} \longrightarrow k$). Define

$$[\widehat{\varphi_x} \cup \widehat{\varphi_y}] = \sum_{z \in X} [(\widehat{\varphi_x} \cup \widehat{\varphi_y})_z] \in \bigoplus_{z \in X} H^*(C_G(z), k)$$

with $(\widehat{\varphi_x} \cup \widehat{\varphi_y})_z : \overline{C_G(z)}^{\times n+m} \to k$ as follows:

$$(\widehat{\varphi_x} \cup \widehat{\varphi_y})_z(h_1, \ldots, h_n, h_{n+1}, \ldots, h_{n+m})$$
$$= \sum_{(i,j) \in I_1'} \widehat{\varphi_x}(h_{i,1}, \ldots, h_{i,n}) \widehat{\varphi_y}(h_{j,1}, \ldots, h_{j,m}),$$

where

- I_1' is the set of pairs (i, j) such that $x_i h_1 \cdots h_n y_j (h_1 \cdots h_n)^{-1} = z$;
- $\gamma_{i,x} h_1 = h_{i,1} \gamma_{s_i^1, x}, \quad \gamma_{s_i^1, x} h_2 = h_{i,2} \gamma_{s_i^2, x}, \quad \cdots \quad \gamma_{s_i^{n-1}, x} h_n = h_{i,n} \gamma_{s_i^n, x};$
- $\gamma_{j,y} h_{n+1} = h_{j,1} \gamma_{s_j^1, y}, \quad \gamma_{s_j^1, y} h_{n+2} = h_{j,2} \gamma_{s_j^2, y}, \quad \cdots$
$$\cdots \quad \gamma_{s_j^{m-1}, j} h_{n+m} = h_{j,m} \gamma_{s_j^m, y}.$$

Theorem 7.1. *With the product defined above, the isomorphism in Theorem 6.3 is an isomorphism of graded algebras.*

Proof. We shall show that with respect to the isomorphism in Theorem 6.3, the product defined above coincide with the cup product on the cohomology complex level.

Let

$$[\widehat{\varphi_x}](\widehat{\varphi_x} : \overline{C_G(x)}^{\times n} \longrightarrow k) \quad \text{and} \quad [\widehat{\varphi_y}](\widehat{\varphi_y} : \overline{C_G(y)}^{\times m} \longrightarrow k)$$

be two elements in $H^n(C_G(x), k)$ and in $H^m(C_G(y), k)$, respectively. Denote by

$$[\varphi_x : \overline{G}^{\times n} \longrightarrow kG](\varphi_x \in \mathcal{H}_x^n) \quad \text{and} \quad [\varphi_y : \overline{G}^{\times m} \longrightarrow kG](\varphi_y \in \mathcal{H}_y^m)$$

the corresponding elements in $HH^*(kG)$. By Theorem 6.3,

$$\varphi_x : \overline{G}^{\times n} \longrightarrow kG, \quad (g_1, \ldots, g_n) \mapsto \sum_{i=1}^{n_x} \widehat{\varphi_x}(h_{i,1}, \ldots, h_{i,n})x_i g_1 \cdots g_n,$$

with

$$\gamma_{i,x} g_1 = h_{i,1} \gamma_{s_i^1, x}, \quad \gamma_{s_i^1, x} g_2 = h_{i,2} \gamma_{s_i^2, x}, \quad \cdots \quad \gamma_{s_i^{n-1}, x} g_n = h_{i,n} \gamma_{s_i^n, x}.$$

A similar formula for φ_y works as well.

Now denote by $\varphi_x \cup \varphi_y : \overline{G}^{\times(n+m)} \longrightarrow kG$ the cup product. By the definition of the cup product, for any $z \in X$, we obtain $(\varphi_x \cup \varphi_y)_z \in \mathcal{H}_z^{n+m}$ given by

$$(\varphi_x \cup \varphi_y)_z : \overline{G}^{\times(n+m)} \longrightarrow kG,$$

$$(g_1, \ldots, g_n, \ldots, g_{n+m})$$
$$\mapsto \sum_{k=1}^{n_z} \sum_{(i,j) \in I_k} \widehat{\varphi_x}(h_{i,1}, \ldots, h_{i,n})\widehat{\varphi_y}(h_{j,1}, \ldots, h_{j,m})z_k g_1 \cdots g_{n+m},$$

where

- I_k is the set of pairs (i, j) such that $x_i g_1 \cdots g_n y_j (g_1 \cdots g_n)^{-1} = z_k$;
- $\gamma_{i,x} g_1 = h_{i,1} \gamma_{s_i^1, x}, \quad \gamma_{s_i^1, x} g_2 = h_{i,2} \gamma_{s_i^2, x}, \quad \cdots \quad \gamma_{s_i^{n-1}, x} g_n = h_{i,n} \gamma_{s_i^n, x}$;
- $\gamma_{j,y} g_{n+1} = h_{j,1} \gamma_{s_j^1, y}, \quad \gamma_{s_j^1, y} g_{n+2} = h_{j,2} \gamma_{s_j^2, y}, \quad \cdots$
$$\cdots \quad \gamma_{s_j^{m-1}, j} g_{n+m} = h_{j,m} \gamma_{s_j^m, y}.$$

Note that I_k depends on the elements $g_1, \ldots, g_n$. Again by Theorem 6.3, we obtain an element in $H^*(C_G(z), k)$ of the following form:

$$\overline{C_G(z)}^{\times(n+m)} \longrightarrow k$$

$$(h_1, \ldots, h_n, h_{n+1}, \ldots, h_{n+m}) \longmapsto \sum_{(i,j) \in I_1} \widehat{\varphi_x}(h_{i,1}, \ldots, h_{i,n})\widehat{\varphi_y}(h_{j,1}, \ldots, h_{j,m}),$$

which is just $(\widehat{\varphi_x} \cup \widehat{\varphi_y})_z$ defined before. $\square$

Similarly we can prove the following result.

Theorem 7.2. *The isomorphism in Theorem 5.1 is an isomorphism of graded algebras with respect to the following product defined on $\bigoplus_{x \in X} H^m(C_G(x), k)$.*

Let

$$[\widehat{\varphi_x}](\widehat{\varphi_x} : \overline{C_G(x)}^{\times n} \longrightarrow k) \quad and \quad [\widehat{\varphi_y}](\widehat{\varphi_y} : \overline{C_G(y)}^{\times m} \longrightarrow k)$$

be two elements in $H^n(C_G(x), k)$ and in $H^m(C_G(y), k)$, respectively. Define

$$[\widehat{\varphi_x} \cup \widehat{\varphi_y}] = \sum_{z \in X} [(\widehat{\varphi_x} \cup \widehat{\varphi_y})_z] \in \bigoplus_{z \in X} H^*(C_G(z), k)$$

with $(\widehat{\varphi_x} \cup \widehat{\varphi_y})_z : \overline{C_G(z)}^{\times n+m} \longrightarrow k$ *as follows:*

$$(\widehat{\varphi_x} \cup \widehat{\varphi_y})_z (h_1, \ldots, h_n, h_{n+1}, \ldots, h_{n+m})$$
$$= (-1)^{nm} \sum_{(i,j) \in I_1} \widehat{\varphi_x}(h'_{i,1}, \ldots, h'_{i,n}) \widehat{\varphi_y}(h'_{j,1}, \ldots, h'_{j,m}),$$

where

- I_1 *is the set of pairs* (i, j) *such that* $h_1 \cdots h_m x_i (h_1 \cdots h_m)^{-1} y_j = z_1 = z;$
- $\gamma_{i,x} h_{m+1} = h'_{i,1} \gamma_{s_i^1, x}, \quad \gamma_{s_i^1, x} h_{m+2} = h'_{i,2} \gamma_{s_i^2, x}, \quad \cdots$
$$\cdots \quad \gamma_{s_i^{n-1}, x} h_{n+m} = h'_{i,n} \gamma_{s_i^n, x};$$
- $\gamma_{j,y} h_1 = h'_{j,1} \gamma_{s_j^1, y}, \quad \gamma_{s_j^1, y} h_2 = h'_{j,2} \gamma_{s_j^2, y}, \quad \cdots \quad \gamma_{s_j^{m-1}, j} h_m = h'_{j,m} \gamma_{s_j^m, y}.$

Remark 7.3. (1) By Remark 5.2 (*a*), our cup product formulae in Theorems 7.1 and 7.2 are consistent with Siegel and Witherspoon's formula in [17, Theorem 5.1] up to an isomorphism.

(2) From our realization of the graded algebra isomorphism

$$HH^*(kG) \simeq \bigoplus_{x \in X} H^*(C_G(x), k) = H^*(G, k) \oplus \left(\bigoplus_{x \in X - \{1\}} H^*(C_G(x), k) \right),$$

it is clear that $H^*(G, k)$ can be seen as a graded subalgebra of $HH^*(kG)$ and each $H^*(C_G(x), k)$ is a graded $H^*(G, k)$-submodule of $HH^*(kG)$. Therefore, the additive decomposition gives an isomorphism of graded $H^*(G, k)$-modules.

8. The $\triangle$ operator formula

Let k be a field and G a finite group. Recall that the group algebra kG is a symmetric algebra with the bilinear form

$$\langle \, , \, \rangle : kG \times kG \longrightarrow k,$$
$$\langle g, h \rangle = \begin{cases} 1 & \text{if } g = h^{-1} \\ 0 & \text{otherwise} \end{cases}$$

for $g, h \in G$. For $n \geq 1$, the operator $\triangle : HH^n(kG) \longrightarrow HH^{n-1}(kG)$ on the Hochschild cohomology is defined by the equation

$$\langle \triangle(\varphi)(g_1, \ldots, g_{n-1}), g_n \rangle = \sum_{i=1}^{n} (-1)^{i(n-1)} \langle \varphi(g_i, \ldots, g_{n-1}, g_n, g_1, \ldots, g_{i-1}), 1 \rangle,$$

where $\varphi \in C^n(kG) \simeq \text{Map}(\overline{G}^{\times n}, kG)$, $\triangle(\varphi) \in C^{n-1}(kG) \simeq \text{Map}(\overline{G}^{\times n-1}, kG)$.

Equivalently,

$$
\begin{aligned}
\triangle (\varphi)(g_1, \ldots, g_{n-1}) \\
= \sum_{g_n \in G} \sum_{i=1}^{n} (-1)^{i(n-1)} \langle \varphi(g_i, \ldots, g_{n-1}, g_n, g_1, \ldots, g_{i-1}), 1 \rangle g_n^{-1}.
\end{aligned}
$$

For example, when $n = 1$, $\triangle : HH^1(kG) \longrightarrow HH^0(kG)$ is given by $\triangle(\varphi) = \sum_{g \in G} \langle \varphi(g), 1 \rangle g^{-1}$. This operator together with the cup product $\cup$ and the Lie bracket $[\,,\,]$ defines a BV algebra structure on $HH^*(kG)$.

We know from Section 4 that, for a conjugacy class C_x of G, $\mathcal{H}_x^* = \bigoplus_{n \geq 0} \mathcal{H}_x^n$ is a subcomplex of the Hochschild cohomology complex $\mathcal{H}^*$, where

$$
\mathcal{H}_x^n = \{\varphi : \overline{G}^{\times n} \longrightarrow kG \mid \varphi(g_1, \ldots, g_n) \in k[g_1 \cdots g_n C_x] \subset kG, \forall g_1, \ldots, g_n \in \overline{G}\}.
$$

Lemma 8.1. *The operator $\triangle : \mathcal{H}^n \longrightarrow \mathcal{H}^{n-1}$ restricts to $\triangle_x : \mathcal{H}_x^n \longrightarrow \mathcal{H}_x^{n-1}$ for each conjugacy class C_x.*

Proof. We need to show that $\triangle(\varphi) \in \mathcal{H}_x^{n-1}$ for each $\varphi \in \mathcal{H}_x^n$. Let $g_1, \ldots, g_{n-1} \in \overline{G}$. Since

$$
\begin{aligned}
\triangle (\varphi)(g_1, \ldots, g_{n-1}) \\
= \sum_{g_n \in G} \sum_{i=1}^{n} (-1)^{i(n-1)} \langle \varphi(g_i, \ldots, g_{n-1}, g_n, g_1, \ldots, g_{i-1}), 1 \rangle g_n^{-1},
\end{aligned}
$$

it suffices to prove the following statement: if

$$
\langle \varphi(g_i, \ldots, g_{n-1}, g_n, g_1, \ldots, g_{i-1}), 1 \rangle \neq 0
$$

for some i, then $g_n^{-1} \in g_1 \cdots g_{n-1} C_x$. Indeed,

$$
\langle \varphi(g_i, \ldots, g_{n-1}, g_n, g_1, \ldots, g_{i-1}), 1 \rangle \neq 0
$$

implies that $1 \in g_i \cdots g_{n-1} g_n g_1 \cdots g_{i-1} C_x$, or equivalently

$$
g_n^{-1} \in g_1 \cdots g_{i-1} C_x g_i \cdots g_{n-1} = g_1 \cdots g_{n-1} C_x. \qquad \square
$$

Now we can determine the behavior of the operator $\triangle$ under the additive decomposition.

Theorem 8.2. *Let $\widehat{\triangle}_x : H^n(C_G(x), k) \longrightarrow H^{n-1}(C_G(x), k)$ be the map induced by the operator $\triangle_x : HH^n(kG) \longrightarrow HH^{n-1}(kG)$ via the isomorphism established in Lemma 6.1. Then $\widehat{\triangle}_x$ is defined as follows:*

$$\widehat{\triangle}_x(\psi)(h_1, \ldots, h_{n-1})$$
$$= \sum_{i=1}^{n}(-1)^{i(n-1)}\psi(h_i, \ldots, h_{n-1}, h_{n-1}^{-1} \cdots h_1^{-1}x^{-1}, h_1, \ldots, h_{i-1})$$

for $\psi : \overline{C_G(x)}^{\times n} \longrightarrow k$ and for $h_1, \ldots, h_{n-1} \in \overline{C_G(x)}$. For example, when $n = 1$, $\widehat{\triangle}_x : H^1(C_G(x), k) \longrightarrow H^0(C_G(x), k)$ sends $\psi : \overline{C_G(x)} \longrightarrow k$ to $\psi(x^{-1})$.

Proof. We shall prove that the following diagram

$$
\begin{array}{ccc}
H^n(\mathcal{H}_x^*) & \xrightarrow{\triangle_x} & H^{n-1}(\mathcal{H}_x^*) \\
\wr \downarrow & & \wr \downarrow \\
H^n(C_G(x), k) & \xrightarrow{\widehat{\triangle}_x} & H^{n-1}(C_G(x), k)
\end{array}
$$

is commutative, where the vertical isomorphisms are given in Lemma 6.1.

Take an element $\psi : \overline{C_G(x)}^{\times n} \longrightarrow k$ in

$$\operatorname{Hom}_{kC_G(x)}(\operatorname{Bar}_n(kC_G(x)) \otimes_{kC_G(x)} k, k)$$

and denote by $\varphi : \overline{G}^{\times n} \longrightarrow kG$ the corresponding element in $\mathcal{H}_x^n$. By Theorem 6.3, for any $h_1, \ldots, h_n \in \overline{C_G(x)}$, $\psi(h_1, \ldots, h_n)$ is equal to the coefficient of x in $\varphi(h_1, \ldots, h_n)h_n^{-1} \cdots h_1^{-1} \in kC_x$. We should prove that $\widehat{\triangle}_x(\psi)$ corresponds to $\triangle_x(\varphi)$ via the isomorphism in Lemma 6.1.

Now

$$\triangle_x(\varphi)(g_1, \ldots, g_{n-1})$$
$$= \sum_{g_n \in G}\sum_{i=1}^{n}(-1)^{i(n-1)}\langle\varphi(g_i, \ldots, g_{n-1}, g_n, g_1, \ldots, g_{i-1}), 1\rangle g_n^{-1}.$$

For any $h_1, \ldots, h_{n-1} \in \overline{C_G(x)}$, consider the coefficient of x in

$$\triangle_x(\varphi)(h_1, \ldots, h_{n-1})h_{n-1}^{-1} \cdots h_1^{-1} \in kC_x,$$

or equivalently the coefficient of $xh_1 \cdots h_{n-1}$ in

$$\triangle_x(\varphi)(h_1, \ldots, h_{n-1}) \in k[h_1 \cdots h_{n-1}C_x].$$

This coefficient is equal to

$$\langle \triangle_x(\varphi)(h_1,\ldots,h_{n-1}), h_{n-1}^{-1}\cdots h_1^{-1}x^{-1}\rangle$$
$$= \sum_{i=1}^{n}(-1)^{i(n-1)}\langle\varphi(h_i,\ldots,h_{n-1}, h_{n-1}^{-1}\cdots h_1^{-1}x^{-1}, h_1,\ldots,h_{i-1}), 1\rangle.$$

On the other hand, we also know that

$$\psi(h_i,\ldots,h_{n-1}, h_{n-1}^{-1}\cdots h_1^{-1}x^{-1}, h_1,\ldots,h_{i-1})$$

is equal to the coefficient of x in

$$\varphi(h_i,\ldots,h_{n-1}, h_{n-1}^{-1}\cdots h_1^{-1}x^{-1}, h_1,\ldots,h_{i-1})h_{i-1}^{-1}\cdots h_1^{-1}xh_1\cdots h_{n-1}h_{n-1}^{-1}\cdots h_i^{-1}$$
$$= \varphi(h_i,\ldots,h_{n-1}, h_{n-1}^{-1}\cdots h_1^{-1}x^{-1}, h_1,\ldots,h_{i-1})x \in kC_x,$$

which is again equal to $\langle\varphi(h_i,\ldots,h_{n-1}, h_{n-1}^{-1}\cdots h_1^{-1}x^{-1}, h_1,\ldots,h_{i-1}), 1\rangle$. It follows that

$$\widehat{\triangle}_x(\psi)(h_1,\ldots,h_{n-1})$$
$$= \sum_{i=1}^{n}(-1)^{i(n-1)}\psi(h_i,\ldots,h_{n-1}, h_{n-1}^{-1}\cdots h_1^{-1}x^{-1}, h_1,\ldots,h_{i-1}).$$

We have proved that $\widehat{\triangle}_x(\psi)$ corresponds to $\triangle_x(\varphi)$ via the isomorphism in Lemma 6.1 and the diagram is commutative (even at the cohomology complex level). $\square$

Remark 8.3. By [7, Corollary 2.2], we know that $H^*(G,k)$ is a Gerstenhaber subalgebra of $HH^*(kG)$ under the inclusion map:

$$\mathrm{Hom}_{kG}(\mathrm{Bar}_*(kG)\otimes_{kG}k, k) \hookrightarrow \mathrm{Hom}_{k(G\times G)}(\mathrm{Bar}_*(kG), kG),$$
$$(\varphi : \overline{G}^{\times n} \longrightarrow k) \longmapsto (\psi : \overline{G}^{\times n} \longrightarrow kG),$$
$$\psi(g_1,\ldots,g_n) = \varphi(g_1,\ldots,g_n)g_1\cdots g_n,$$

which is in fact induced by the isomorphism in Lemma 6.1 corresponding to $x = 1$.

Notice that by notations in Section 6, $\psi \in \mathcal{H}_1^n$. So motivated by Theorem 8.2, we can similarly define an operator $\triangle_1 : H^n(G,k) \longrightarrow H^{n-1}(G,k)$ in the group cohomology $H^*(G,k)$ as follows:

$$\triangle_1(\varphi)(g_1,\ldots,g_{n-1}) = \sum_{i=1}^{n}(-1)^{i(n-1)}\varphi(g_i,\ldots,g_{n-1}, g_{n-1}^{-1}\cdots g_1^{-1}, g_1,\ldots,g_{i-1})$$

for $\varphi : \overline{G}^{\times n} \longrightarrow k$ and for $g_1,\ldots,g_{n-1} \in \overline{G}$. In particular,

$$\triangle_1 : H^1(G,k) \longrightarrow H^0(G,k)$$

is given by $\varphi \mapsto \varphi(1)$.

We prove that $H^*(G, k)$ is in fact a BV subalgebra of $HH^*(kG)$.

Corollary 8.4. *Let k be a field and G a finite group. Then $H^*(G, k) \hookrightarrow HH^*(kG)$ is a BV subalgebra.*

Proof. The inclusion in known to preserve the product structure. In fact this is a direct consequence of Theorem 7.1. In that result, taking $x = 1 = y$, we always has $z = 1$ if the set I_1' is not empty.

As the inclusion $H^n(G, k) \hookrightarrow HH^*(kG)$ induced by the isomorphism in Lemma 6.1 corresponding to $x = 1$, Theorem 8.2 shows that it preserves the $\triangle$-operator. Since this operator together with the cup product $\cup$ and the Lie bracket $[\,,\,]$ define a BV algebra structure on $HH^*(kG)$, via the isomorphism in Lemma 6.1, we deduce that the Lie bracket $[\,,\,]$ restricts to $H^*(G, k) = H^*(C_G(1), k)$. $\square$

Now we specialize to the case of abelian groups. Let G be an abelian group. In this case, the Hochschild cohomology ring $HH^*(kG)$ of the group algebra kG is isomorphic to the tensor product algebra of kG and the group cohomology ring $H^*(G, k)$: $HH^*(kG) \simeq kG \otimes_k H^*(G, k)$. According to [4], this isomorphism is given as follows. For G an abelian group, conjugacy classes are elements of G, hence a cochain φ_x of $\mathcal{H}_x^n$ for $x \in G$ attributes a scalar multiple of $g_1 \cdots g_n x$ for each $(g_1, \ldots, g_n) \in \overline{G}^{\times n}$ and we obtain in this way a map $\overline{\varphi_x} : \overline{G}^{\times n} \longrightarrow k$. It is easy to see that the map $\widehat{\varphi_x}$ in Theorem 6.3 is just this scalar.

Now Theorem 7.1 shows that the map $\varphi \longmapsto \Sigma_{x \in G}(x \otimes \overline{\varphi_x})$ defines a ring isomorphism $C^*(kG) \longrightarrow kG \otimes C^*(kG, k)$ compatible with the differentials, and therefore it induces the above isomorphism. Theorem 8.2 specializes to the following statement.

Proposition 8.5. *Let k be a field and G a finite abelian group. Under the above isomorphism $HH^*(kG) \simeq kG \otimes_k H^*(G, k)$, the operator*

$$\triangle : HH^n(kG) \longrightarrow HH^{n-1}(kG)$$

corresponds to the sum of operators

$$x \otimes \overline{\triangle_x} : x \otimes H^n(G, k) \longrightarrow x \otimes H^{n-1}(G, k),$$

where $x \in G$ and $\overline{\triangle_x} : H^n(G, k) \longrightarrow H^{n-1}(G, k)$ is defined as follows:

$$\overline{\triangle_x}(\overline{\varphi})(g_1, \ldots, g_{n-1})$$
$$= \sum_{i=1}^{n}(-1)^{i(n-1)}\overline{\varphi}(g_i, \ldots, g_{n-1}, g_{n-1}^{-1} \cdots g_1^{-1} x^{-1}, g_1, \ldots, g_{i-1})$$

for $\overline{\varphi} : \overline{G}^{\times n} \longrightarrow k$ and for $g_1, \ldots, g_{n-1} \in \overline{G}$. When $n = 1$,

$$\overline{\triangle_x} : H^1(G, k) \longrightarrow H^0(G, k)$$

is given by $\overline{\varphi} \mapsto \overline{\varphi}(x^{-1})$.

Remark 8.6. We could also use the first realization to deduce a formula of the $\triangle$ operator. However, this formula is much more complicated than that of Theorem 8.2. We refrain from giving it here.

In a BV-algebra, we have the following equation (see [9]; Here we have changed the original equation according to the sign convention in Remark 3.1 and we omit the sign $\cup$ in the equation):

$$\triangle (\alpha\beta\gamma) = (-1)^{|\alpha||\beta||\gamma|}[(-1)^{|\gamma|} \triangle (\alpha\beta)\gamma + \alpha \triangle (\beta\gamma) + (-1)^{|\alpha||\beta|}\beta \triangle (\alpha\gamma)$$
$$- (-1)^{|\alpha|} \triangle (\alpha)\beta\gamma - (-1)^{|\alpha|+|\beta|-|\alpha||\gamma|}\alpha(\triangle(\beta))\gamma$$
$$- (-1)^{|\alpha|+|\beta|+|\gamma|}\alpha\beta \triangle (\gamma)],$$

where $\alpha, \beta, \gamma \in HH^*(A)$ are homogeneous elements. So in order to compute the $\triangle$ operator in $HH^*(A)$, it suffices to find the value of $\triangle$ on each generator and on the cup product of every two generators. Finally, let us mention that we can use the cup product formula, the $\triangle$ operator formula and the following formula to compute the Lie bracket:

$$[\alpha, \beta] = -(-1)^{(|\alpha|-1)|\beta|}(\triangle(\alpha \cup \beta) - \triangle(\alpha) \cup \beta - (-1)^{|\alpha|}\alpha \cup \triangle(\beta)).$$

9. The symmetric group of degree 3

There are a few computations in literature on the BV structures of the Hochschild cohomology rings of some commutative algebras, see for example, [20]. As far as we know, there is no concrete computation in non-commutative case. In this section, we use our method to compute the BV structure of the Hochschild cohomology rings of the group algebra $\mathbb{F}_3 S_3$. The associative ring structure has been determined by Siegel and Witherspoon [17] using their cup product formula. So we only need to compute the $\triangle$ operator and the Lie bracket.

Let $G = S_3 = \langle a, b \mid a^3 = 1 = b^2, bab = a^{-1} \rangle$. Choose the conjugacy class representatives as $1, a, b$. The corresponding centralizers are $H_1 = G, H_2 = \langle a \rangle$ and $H_3 = \langle b \rangle$. So $HH^*(\mathbb{F}_3 S_3) \simeq H^*(S_3) \oplus H^*(\langle a \rangle) \oplus H^*(\langle b \rangle)$. The ring structures of $H^*(S_3)$, of $H^*(\langle a \rangle)$, and of $H^*(\langle b \rangle)$ are well-known (see for example, [6]). $H^*(S_3) = \mathbb{F}_3[u, v]/(u^2)$, where u and v have degrees of 3 and 4, respectively. $H^*(\langle a \rangle) = \mathbb{F}_3[w_1, w_2]/(w_1^2)$, where w_1 and w_2 have degrees of 1 and 2, respectively. $H^*(\langle b \rangle) = \mathbb{F}_3$, since $\mathbb{F}_3\langle b \rangle$ is semisimple. Identify the elements u, v with their images in $HH^*(\mathbb{F}_3 S_3)$ and denote by X_1, X_2 the images of the elements (resp.) w_1, w_2 under the additive decomposition. Then Siegel and Witherspoon proved in [17] the following presentation for the Hochschild cohomology ring $HH^*(\mathbb{F}_3 S_3)$: $HH^*(\mathbb{F}_3 S_3)$ is generated as an algebra by elements $u, v, C_1 = 1 + a + a^2, C_2 = b(1 + a + a^2), X_1, X_2$ of degrees (resp.) $3, 4, 0, 0, 1$

and 2, subject to the relations

$$uX_1 = 0, \quad vX_1 = uX_2, \quad uC_2 = 0 = vC_2,$$
$$C_i X_j = 0 = C_i C_j (i, j \in \{1, 2\}), \quad X_1 X_2 = uC_1, \quad X_2^2 = vC_1$$

in addition to the graded commutative relations.

Our formula in Theorem 8.2 for $\triangle$ operator is based on the normalized bar resolution. However, the real computations of the Hochschild cohomology or the group cohomology are based on the minimal projective resolutions. So we need to construct comparison maps between the minimal projective resolution and normalized bar resolution (by the same technique introduced in Section 2), and then we can transfer our formula in Theorem 8.2 to the minimal Hochschild cohomology level. By Theorem 8.2, the operator $\triangle : HH^n(\mathbb{F}_3 S_3) \longrightarrow HH^{n-1}(\mathbb{F}_3 S_3)$ restricts to the operators $\widehat{\triangle}_b : H^n(\langle b \rangle) \longrightarrow H^{n-1}(\langle b \rangle)$, $\widehat{\triangle}_a : H^n(\langle a \rangle) \longrightarrow H^{n-1}(\langle a \rangle)$, and $\widehat{\triangle}_1 : H^n(S_3) \longrightarrow H^{n-1}(S_3)$. Since $\mathbb{F}_3 \langle b \rangle$ is semisimple and $H^*(\langle b \rangle)$ is concentrated in degree zero, $\widehat{\triangle}_b$ is trivial.

To compute $\widehat{\triangle}_a$, we first recall the minimal projective resolution P_a^* of the trivial $\mathbb{F}_3 \langle a \rangle$-module $\mathbb{F}_3$:

$$\cdots \longrightarrow \mathbb{F}_3 \langle a \rangle \xrightarrow{a-1} \mathbb{F}_3 \langle a \rangle \xrightarrow{1+a+a^2} \mathbb{F}_3 \langle a \rangle \xrightarrow{a-1} \mathbb{F}_3 \langle a \rangle \xrightarrow{\epsilon} \mathbb{F}_3 \longrightarrow 0,$$

where the differential ϵ is given by $\epsilon(\lambda_1 + \lambda_2 a + \lambda_3 a^2) = \lambda_1 + \lambda_2 + \lambda_3$, and the differential $a - 1$ means multiplying by $a - 1$, etc.. There is a setwise self-homotopy over P_a^* as follows:

$$t_{-1} : \mathbb{F}_3 \to \mathbb{F}_3 \langle a \rangle, \quad 1 \mapsto 1,$$
$$t_0 : \mathbb{F}_3 \langle a \rangle \to \mathbb{F}_3 \langle a \rangle, \quad 1 \mapsto 0, a \mapsto 1, a^2 \mapsto 1 + a,$$
$$t_1 : \mathbb{F}_3 \langle a \rangle \to \mathbb{F}_3 \langle a \rangle, \quad 1 \mapsto 0, a \mapsto 0, a^2 \mapsto 1,$$
$$t_2 : \mathbb{F}_3 \langle a \rangle \to \mathbb{F}_3 \langle a \rangle, \quad 1 \mapsto 0, a \mapsto 1, a^2 \mapsto 1 + a,$$
$$t_3 : \mathbb{F}_3 \langle a \rangle \to \mathbb{F}_3 \langle a \rangle, \quad 1 \mapsto 0, a \mapsto 0, a^2 \mapsto 1,$$
$$\vdots$$

We also have the normalized bar resolution $\mathrm{Bar}_*(\mathbb{F}_3 \langle a \rangle) \otimes_{\mathbb{F}_3 \langle a \rangle} \mathbb{F}_3$ of the trivial $\mathbb{F}_3 \langle a \rangle$-module $\mathbb{F}_3$, which is identified as the following complex

$$\cdots \longrightarrow \mathbb{F}_3 \langle a \rangle \otimes \overline{\mathbb{F}_3 \langle a \rangle}^{\otimes n} \xrightarrow{d_n} \cdots \longrightarrow \mathbb{F}_3 \langle a \rangle \otimes \overline{\mathbb{F}_3 \langle a \rangle} \xrightarrow{d_1} \mathbb{F}_3 \langle a \rangle \xrightarrow{d_0} \mathbb{F}_3 \longrightarrow 0,$$

where the differential is given by

$$d_0(g_0) = 1$$

for $g_0 \in \langle a \rangle$, and

$$d_n(g_0, g_1, \ldots, g_n) = g_0 g_1 \otimes g_2 \otimes \cdots \otimes g_n$$
$$+ \sum_{i=1}^{n-1} (-1)^i g_0 \otimes \cdots \otimes g_i g_{i+1} \otimes \cdots \otimes g_n + (-1)^n g_0 \otimes g_1 \otimes \cdots \otimes g_{n-1}$$

for $g_0 \in \langle a \rangle, g_1, \ldots, g_n \in \overline{\langle a \rangle}$. There is a setwise self-homotopy over $\mathrm{Bar}_*(\mathbb{F}_3\langle a \rangle) \otimes_{\mathbb{F}_3\langle a \rangle} \mathbb{F}_3$ as follows:

$$s_n : \mathbb{F}_3\langle a \rangle \otimes \overline{\mathbb{F}_3\langle a \rangle}^{\otimes n} \longrightarrow \mathbb{F}_3\langle a \rangle \otimes \overline{\mathbb{F}_3\langle a \rangle}^{\otimes n+1},$$
$$g_0 \otimes g_1 \otimes \cdots \otimes g_n \longmapsto 1 \otimes g_0 \otimes g_1 \otimes \cdots \otimes g_n,$$

where $g_0 \in \langle a \rangle, g_1, \ldots, g_n \in \overline{\langle a \rangle}$. Using $\{s_n\}$ and $\{t_n\}$ we get comparison maps $\Phi : P_a^* \longrightarrow \mathrm{Bar}_*(\mathbb{F}_3\langle a \rangle) \otimes_{\mathbb{F}_3\langle a \rangle} \mathbb{F}_3$ and $\Psi : \mathrm{Bar}_*(\mathbb{F}_3\langle a \rangle) \otimes_{\mathbb{F}_3\langle a \rangle} \mathbb{F}_3 \longrightarrow P_a^*$. We write down the maps up to degree 4 explicitly:

$$\Phi : P_a^* \longrightarrow \mathrm{Bar}_*(\mathbb{F}_3\langle a \rangle) \otimes_{\mathbb{F}_3\langle a \rangle} \mathbb{F}_3$$
$$\Phi_{-1} = \mathrm{id} : \mathbb{F}_3 \longrightarrow \mathbb{F}_3,$$
$$\Phi_0 = \mathrm{id} : \mathbb{F}_3\langle a \rangle \longrightarrow \mathbb{F}_3\langle a \rangle,$$
$$\Phi_1 : \mathbb{F}_3\langle a \rangle \longrightarrow \mathbb{F}_3\langle a \rangle \otimes \overline{\mathbb{F}_3\langle a \rangle}, \quad g \mapsto g \otimes a,$$

for $g = 1, a, a^2$,

$$\Phi_2 : \mathbb{F}_3\langle a \rangle \longrightarrow \mathbb{F}_3\langle a \rangle \otimes \overline{\mathbb{F}_3\langle a \rangle}^{\otimes 2}, \quad g \mapsto g \otimes a \otimes a + g \otimes a^2 \otimes a,$$

for $g = 1, a, a^2$,

$$\Phi_3 : \mathbb{F}_3\langle a \rangle \longrightarrow \mathbb{F}_3\langle a \rangle \otimes \overline{\mathbb{F}_3\langle a \rangle}^{\otimes 3}, \quad g \mapsto g \otimes a \otimes a \otimes a + g \otimes a \otimes a^2 \otimes a,$$

for $g = 1, a, a^2$,

$$\Phi_4 : \mathbb{F}_3\langle a \rangle \longrightarrow \mathbb{F}_3\langle a \rangle \otimes \overline{\mathbb{F}_3\langle a \rangle}^{\otimes 4},$$
$$g \mapsto g \otimes a \otimes a \otimes a \otimes a + g \otimes a \otimes a \otimes a^2 \otimes a + g \otimes a^2 \otimes a \otimes a \otimes a$$
$$+ g \otimes a^2 \otimes a \otimes a^2 \otimes a,$$

for $g = 1, a, a^2$;

$$\Psi : \mathrm{Bar}_*(\mathbb{F}_3\langle a \rangle) \otimes_{\mathbb{F}_3\langle a \rangle} \mathbb{F}_3 \longrightarrow P_a^*$$
$$\Psi_{-1} = \mathrm{id} : \mathbb{F}_3 \longrightarrow \mathbb{F}_3,$$
$$\Psi_0 = \mathrm{id} : \mathbb{F}_3\langle a \rangle \longrightarrow \mathbb{F}_3\langle a \rangle,$$
$$\Psi_1 : \mathbb{F}_3\langle a \rangle \otimes \overline{\mathbb{F}_3\langle a \rangle} \longrightarrow \mathbb{F}_3\langle a \rangle, \quad g \otimes a \mapsto g, \quad g \otimes a^2 \mapsto g(1 + a),$$

for $g = 1, a, a^2$,

$$\Psi_2 : \mathbb{F}_3\langle a\rangle \otimes \overline{\mathbb{F}_3\langle a\rangle}^{\otimes 2} \longrightarrow \mathbb{F}_3\langle a\rangle, \quad g \otimes a \otimes a \mapsto 0,$$

$$g \otimes a \otimes a^2 \mapsto g, \quad g \otimes a^2 \otimes a \mapsto g, \quad g \otimes a^2 \otimes a^2 \mapsto g,$$

for $g = 1, a, a^2$,

$$\Psi_3 : \mathbb{F}_3\langle a\rangle \otimes \overline{\mathbb{F}_3\langle a\rangle}^{\otimes 3} \longrightarrow \mathbb{F}_3\langle a\rangle, \quad g \otimes a \otimes a \otimes a \mapsto 0, \quad g \otimes a \otimes a \otimes a^2 \mapsto 0,$$

$$g \otimes a \otimes a^2 \otimes a \mapsto 0, \quad g \otimes a \otimes a^2 \otimes a^2 \mapsto 0, \quad g \otimes a^2 \otimes a \otimes a \mapsto 0,$$

$$g \otimes a^2 \otimes a \otimes a^2 \mapsto g(1+a), \quad g \otimes a^2 \otimes a^2 \otimes a \mapsto g(1+a),$$

$$g \otimes a^2 \otimes a^2 \otimes a^2 \mapsto ga,$$

for $g = 1, a, a^2$,

$$\Psi_4 : \mathbb{F}_3\langle a\rangle \otimes \overline{\mathbb{F}_3\langle a\rangle}^{\otimes 4} \longrightarrow \mathbb{F}_3\langle a\rangle, \quad g \otimes a \otimes a \otimes a \otimes a \mapsto 0,$$

$$g \otimes a \otimes a \otimes a \otimes a^2 \mapsto 0, g \otimes a \otimes a \otimes a^2 \otimes a \mapsto 0,$$

$$g \otimes a \otimes a \otimes a^2 \otimes a^2 \mapsto 0, \quad g \otimes a \otimes a^2 \otimes a \otimes a \mapsto 0,$$

$$g \otimes a \otimes a^2 \otimes a \otimes a^2 \mapsto g, \quad g \otimes a \otimes a^2 \otimes a^2 \otimes a \mapsto g,$$

$$g \otimes a \otimes a^2 \otimes a^2 \otimes a^2 \mapsto g, g \otimes a^2 \otimes a \otimes a \otimes a \mapsto 0,$$

$$g \otimes a^2 \otimes a \otimes a \otimes a^2 \mapsto 0, \quad g \otimes a^2 \otimes a \otimes a^2 \otimes a \mapsto 0,$$

$$g \otimes a^2 \otimes a \otimes a^2 \otimes a^2 \mapsto 0, \quad g \otimes a^2 \otimes a^2 \otimes a \otimes a \mapsto 0,$$

$$g \otimes a^2 \otimes a^2 \otimes a \otimes a^2 \mapsto g, g \otimes a^2 \otimes a^2 \otimes a^2 \otimes a \mapsto g,$$

$$g \otimes a^2 \otimes a^2 \otimes a^2 \otimes a^2 \mapsto 0,$$

for $g = 1, a, a^2$.

We have the following commutative diagram:

Clearly both the representatives of w_1 and w_2 in the group cohomology

$$H^*(\langle a\rangle) = \mathbb{F}_3[w_1, w_2]/(w_1{}^2)$$

can be chosen as

$$\epsilon : \mathbb{F}_3\langle a \rangle \longrightarrow \mathbb{F}_3, \lambda_1 + \lambda_2 a + \lambda_3 a^2 \mapsto \lambda_1 + \lambda_2 + \lambda_3.$$

By abuse of notation, we have

$$\widehat{\triangle}_a(w_1) = \widehat{\triangle}_a(w_1\Psi_1) \circ \Phi_0,$$
$$\widehat{\triangle}_a(w_2) = \widehat{\triangle}_a(w_2\Psi_2) \circ \Phi_1.$$

A straightforward calculation shows that $\widehat{\triangle}_a(w_1) = -1$ and $\widehat{\triangle}_a(w_2) = 0$. Similarly, we can get that

$$\widehat{\triangle}_a(w_1 w_2) = \widehat{\triangle}_a((w_1\Psi_1)(w_2\Psi_2)) \circ \Phi_2,$$
$$\widehat{\triangle}_a(w_2^2) = \widehat{\triangle}_a((w_2\Psi_2)^2) \circ \Phi_3.$$

By direct computation, we have $(w_1\Psi_1)(w_2\Psi_2)\Phi_3(1) = 1$ and $(w_2\Psi_2)^2)\Phi_4(1) = 1$, which imply that both the representatives of $w_1 w_2$ and w_2^2 in the group cohomology $H^*(\langle a \rangle) = \mathbb{F}_3[w_1, w_2]/(w_1{}^2)$ are given by ϵ. So again a straightforward calculation shows that $\widehat{\triangle}_a(w_1 w_2) = -w_2$ and $\widehat{\triangle}_a(w_2^2) = 0$.

Next we compute $\widehat{\triangle}_1$. First of all, we need to construct a minimal projective resolution P_1^* of the trivial $\mathbb{F}_3 S_3$-module $\mathbb{F}_3$. Recall that the group algebra $\mathbb{F}_3 S_3$ can be identified as the $\mathbb{F}_3$-algebra A given by the following quiver and relations:

$$\overset{1}{\circ} \underset{\beta}{\overset{\alpha}{\rightleftarrows}} \overset{2}{\circ}, \quad \alpha\beta\alpha = \beta\alpha\beta = 0.$$

Let $_A A = Ae_1 \oplus Ae_2 = \langle e_1, \alpha, \beta\alpha \rangle \oplus \langle e_2, \beta, \alpha\beta \rangle$ be the decomposition of the regular module into the indecomposable projective modules. Then we have the following (recall that all the computations take place over $\mathbb{F}_3$):

$$e_1 = -(1 + b), e_2 = -(1 - b),$$
$$\alpha = -a(1 - a)(1 + b) = -(1 - b)a(1 - a) = -(a - a^2 + ab - a^2 b),$$
$$\beta = -a(1 - a)(1 - b) = -(1 + b)a(1 - a) = -(a - a^2 - ab + a^2 b),$$
$$\beta\alpha = (1 + b)(1 + a + a^2)(1 + b) = -(1 + a + b + a^2 + ab + a^2 b),$$
$$\alpha\beta = (1 - b)(1 + a + a^2)(1 - b) = -(1 + a - b + a^2 - ab - a^2 b),$$
$$1 = e_1 + e_2, \quad a = e_1 + e_2 - \alpha - \beta - \alpha\beta - \beta\alpha, \quad b = e_1 - e_2,$$
$$a^2 = e_1 + e_2 + \alpha + \beta - \alpha\beta - \beta\alpha, \quad ab = e_1 - e_2 - \alpha + \beta + \alpha\beta - \beta\alpha,$$
$$a^2 b = e_1 - e_2 + \alpha - \beta + \alpha\beta - \beta\alpha.$$

Ae_1 is the projective cover of the trivial module $\mathbb{F}_3$ since a and b act trivially on $Ae_1/\mathrm{rad}(Ae_1)$; Ae_2 is the projective cover of the sign module sgn since a (resp., b)

acts trivially (resp., by multiplying -1) on $Ae_2/\operatorname{rad}(Ae_2)$. Now it is easy to write down the minimal projective resolution P_1^* of the trivial $\mathbb{F}_3 S_3$-module $\mathbb{F}_3$ is as follows:

$$\cdots \longrightarrow Ae_1 \xrightarrow{\partial_4} Ae_1 \xrightarrow{\partial_3} Ae_2 \xrightarrow{\partial_2} Ae_2 \xrightarrow{\partial_1} Ae_1 \xrightarrow{\partial_0} \mathbb{F}_3 \longrightarrow 0,$$

where the differential is given as follows:

$$\partial_0 : e_1 \mapsto 1, \alpha \mapsto 0, \beta\alpha \mapsto 0,$$
$$\partial_1 : e_2 \mapsto \alpha, \beta \mapsto \beta\alpha, \alpha\beta \mapsto 0,$$
$$\partial_2 : e_2 \mapsto \alpha\beta, \beta \mapsto 0, \alpha\beta \mapsto 0,$$
$$\partial_3 : e_1 \mapsto \beta, \alpha \mapsto \alpha\beta, \beta\alpha \mapsto 0,$$
$$\partial_4 : e_1 \mapsto \beta\alpha, \alpha \mapsto 0, \beta\alpha \mapsto 0,$$
$$\partial_5 : e_2 \mapsto \alpha, \beta \mapsto \beta\alpha, \alpha\beta \mapsto 0,$$
$$\partial_6 : e_2 \mapsto \alpha\beta, \beta \mapsto 0, \alpha\beta \mapsto 0,$$
$$\partial_7 : e_1 \mapsto \beta, \alpha \mapsto \alpha\beta, \beta\alpha \mapsto 0,$$
$$\partial_8 : e_1 \mapsto \beta\alpha, \alpha \mapsto 0, \beta\alpha \mapsto 0,$$
$$\vdots$$

Using the Lowey diagram structures of Ae_1 and Ae_2, we can easily construct a setwise self-homotopy over P_1^* as follows:

$$t_{-1} : \mathbb{F}_3 \to Ae_1, \quad 1 \mapsto e_1,$$
$$t_0 : Ae_1 \to Ae_2, \quad e_1 \mapsto 0, \alpha \mapsto e_2, \beta\alpha \mapsto \beta,$$
$$t_1 : Ae_2 \to Ae_2, \quad e_2 \mapsto 0, \beta \mapsto 0, \alpha\beta \mapsto e_2,$$
$$t_2 : Ae_2 \to Ae_1, \quad e_2 \mapsto 0, \beta \mapsto e_1, \alpha\beta \mapsto \alpha,$$
$$t_3 : Ae_1 \to Ae_1, \quad e_1 \mapsto 0, \alpha \mapsto 0, \beta\alpha \mapsto e_1,$$
$$t_4 : Ae_1 \to Ae_2, \quad e_1 \mapsto 0, \alpha \mapsto e_2, \beta\alpha \mapsto \beta,$$
$$t_5 : Ae_2 \to Ae_2, \quad e_2 \mapsto 0, \beta \mapsto 0, \alpha\beta \mapsto e_2,$$
$$t_6 : Ae_2 \to Ae_1, \quad e_2 \mapsto 0, \beta \mapsto e_1, \alpha\beta \mapsto \alpha,$$
$$t_7 : Ae_1 \to Ae_1, \quad e_1 \mapsto 0, \alpha \mapsto 0, \beta\alpha \mapsto e_1,$$
$$t_8 : Ae_1 \to Ae_2, \quad e_1 \mapsto 0, \alpha \mapsto e_2, \beta\alpha \mapsto \beta,$$
$$\vdots$$

We also have the normalized bar resolution $\operatorname{Bar}_*(A) \otimes_A \mathbb{F}_3$ of the trivial A-module $\mathbb{F}_3$, which is identified as the following complex

$$\cdots \longrightarrow A \otimes \overline{A}^{\otimes n} \xrightarrow{d_n} \cdots \longrightarrow A \otimes \overline{A} \xrightarrow{d_1} A \xrightarrow{d_0} \mathbb{F}_3 \longrightarrow 0,$$

where the differential is given by

$$d_0(g_0) = 1$$

for $g_0 \in S_3$, and

$$d_n(g_0, g_1, \ldots, g_n) = g_0 g_1 \otimes g_2 \otimes \cdots \otimes g_n$$
$$+ \sum_{i=1}^{n-1} (-1)^i g_0 \otimes \cdots \otimes g_i g_{i+1} \otimes \cdots \otimes g_n + (-1)^n g_0 \otimes g_1 \otimes \cdots \otimes g_{n-1}$$

for $g_0 \in S_3, g_1, \ldots, g_n \in \overline{S_3}$. There is a setwise self-homotopy over $\mathrm{Bar}_*(A) \otimes_A \mathbb{F}_3$ as follows:

$$s_n : A \otimes \overline{A}^{\otimes n} \longrightarrow A \otimes \overline{A}^{\otimes n+1},$$
$$g_0 \otimes g_1 \otimes \cdots \otimes g_n \longmapsto 1 \otimes g_0 \otimes g_1 \otimes \cdots \otimes g_n,$$

where $g_0 \in S_3, g_1, \ldots, g_n \in \overline{S_3}$. As before, we want to use $\{s_n\}$ and $\{t_n\}$ to get comparison maps $\Lambda : P_1^* \longrightarrow \mathrm{Bar}_*(A) \otimes_A \mathbb{F}_3$ and $\Theta : \mathrm{Bar}_*(A) \otimes_A \mathbb{F}_3 \longrightarrow P_1^*$. Here the situation is a bit different, since P_1^* is not a free resolution. However, if we replace $s_n(x)$ by $\widetilde{s_n}(x) = e_1 s_n(e_1 x) + e_2 s_n(e_2 x)$, then the method introduced in Section 2 still works. We write down the comparison maps up to degree 8 explicitly:

$$\Lambda : P_1^* \longrightarrow \mathrm{Bar}_*(A) \otimes_A \mathbb{F}_3$$
$$\Lambda_{-1} = \mathrm{id} : \mathbb{F}_3 \longrightarrow \mathbb{F}_3,$$
$$\Lambda_0 : A e_1 \longrightarrow A,$$
$$a e_1 \mapsto a e_1,$$
$$\Lambda_1 : A e_2 \longrightarrow A \otimes \overline{A},$$
$$e_2 \mapsto e_2 \otimes \alpha, \beta \mapsto \beta \otimes \alpha, \alpha\beta \mapsto \alpha\beta \otimes \alpha,$$
$$\Lambda_2 : A e_2 \longrightarrow A \otimes \overline{A}^{\otimes 2},$$
$$e_2 \mapsto e_2 \otimes \alpha\beta \otimes \alpha, \beta \mapsto \beta \otimes \alpha\beta \otimes \alpha, \alpha\beta \mapsto \alpha\beta \otimes \alpha\beta \otimes \alpha,$$
$$\Lambda_3 : A e_1 \longrightarrow A \otimes \overline{A}^{\otimes 3},$$
$$e_1 \mapsto e_1 \otimes \beta \otimes \alpha\beta \otimes \alpha, \alpha \mapsto \alpha \otimes \beta \otimes \alpha\beta \otimes \alpha, \beta\alpha \mapsto \beta\alpha \otimes \beta \otimes \alpha\beta \otimes \alpha,$$
$$\Lambda_4 : A e_1 \longrightarrow A \otimes \overline{A}^{\otimes 4},$$
$$e_1 \mapsto e_1 \otimes \beta\alpha \otimes \beta \otimes \alpha\beta \otimes \alpha, \alpha \mapsto \alpha \otimes \beta\alpha \otimes \beta \otimes \alpha\beta \otimes \alpha,$$
$$\beta\alpha \mapsto \beta\alpha \otimes \beta\alpha \otimes \beta \otimes \alpha\beta \otimes \alpha,$$
$$\Lambda_5 : A e_2 \longrightarrow A \otimes \overline{A}^{\otimes 5},$$
$$e_2 \mapsto e_2 \otimes \alpha \otimes \beta\alpha \otimes \beta \otimes \alpha\beta \otimes \alpha, \beta \mapsto \beta \otimes \alpha \otimes \beta\alpha \otimes \beta \otimes \alpha\beta \otimes \alpha,$$
$$\alpha\beta \mapsto \alpha\beta \otimes \alpha \otimes \beta\alpha \otimes \beta \otimes \alpha\beta \otimes \alpha,$$

$$\Lambda_6 : Ae_2 \longrightarrow A \otimes \overline{A}^{\otimes 6},$$

$$e_2 \mapsto e_2 \otimes \alpha\beta \otimes \alpha \otimes \beta\alpha \otimes \beta \otimes \alpha\beta \otimes \alpha, \beta \mapsto \beta \otimes \alpha\beta \otimes \alpha \otimes \beta\alpha \otimes \beta \otimes \alpha\beta \otimes \alpha,$$

$$\alpha\beta \mapsto \alpha\beta \otimes \alpha\beta \otimes \alpha \otimes \beta\alpha \otimes \beta \otimes \alpha\beta \otimes \alpha,$$

$$\Lambda_7 : Ae_1 \longrightarrow A \otimes \overline{A}^{\otimes 7},$$

$$e_1 \mapsto e_1 \otimes \beta \otimes \alpha\beta \otimes \alpha \otimes \beta\alpha \otimes \beta \otimes \alpha\beta \otimes \alpha,$$

$$\alpha \mapsto \alpha \otimes \beta \otimes \alpha\beta \otimes \alpha \otimes \beta\alpha \otimes \beta \otimes \alpha\beta \otimes \alpha,$$

$$\beta\alpha \mapsto \beta\alpha \otimes \beta \otimes \alpha\beta \otimes \alpha \otimes \beta\alpha \otimes \beta \otimes \alpha\beta \otimes \alpha,$$

$$\Lambda_8 : Ae_1 \longrightarrow A \otimes \overline{A}^{\otimes 8},$$

$$e_1 \mapsto e_1 \otimes \beta\alpha \otimes \beta \otimes \alpha\beta \otimes \alpha \otimes \beta\alpha \otimes \beta \otimes \alpha\beta \otimes \alpha,$$

$$\alpha \mapsto \alpha \otimes \beta\alpha \otimes \beta \otimes \alpha\beta \otimes \alpha \otimes \beta\alpha \otimes \beta \otimes \alpha\beta \otimes \alpha,$$

$$\beta\alpha \mapsto \beta\alpha \otimes \beta\alpha \otimes \beta \otimes \alpha\beta \otimes \alpha \otimes \beta\alpha \otimes \beta \otimes \alpha\beta \otimes \alpha;$$

$$\Theta : \mathrm{Bar}_*(A) \otimes_A \mathbb{F}_3 \longrightarrow P_1^*$$

$$\Theta_{-1} = \mathrm{id} : \mathbb{F}_3 \longrightarrow \mathbb{F}_3,$$

$$\Theta_0 : A \longrightarrow Ae_1,$$

$$a \mapsto ae_1,$$

for $a \in A$,

$$\Theta_1 : A \otimes \overline{A} \longrightarrow Ae_2,$$

$$1 \otimes g_1 \mapsto -e_2 - \beta, 0, e_2 - \beta, -e_2 - \beta, e_2 - \beta,$$

for $g_1 = a, b, a^2, ab, a^2 b$,

$$\Theta_2 : A \otimes \overline{A}^{\otimes 2} \longrightarrow Ae_2,$$

$$1 \otimes g_1 \otimes g_2 \mapsto -e_2, 0, 0, 0, e_2$$

for $g_1 = a, b, a^2, ab, a^2 b$, and where

$$\Theta_1(1 \otimes g_2) = -e_2 - \beta,$$

$$1 \otimes g_1 \otimes g_2 \mapsto 0, 0, e_2, -e_2, 0$$

for $g_1 = a, b, a^2, ab, a^2 b$, and where

$$\Theta_1(1 \otimes g_2) = e_2 - \beta,$$

$$1 \otimes g_1 \otimes g_2 \mapsto 0,$$

for any other case,

$$\Theta_3 : A \otimes \overline{A}^{\otimes 3} \longrightarrow Ae_1,$$

$$1 \otimes g_1 \otimes g_2 \otimes g_3 \mapsto -e_1 - \alpha, 0, e_1 - \alpha, e_1 + \alpha, -e_1 + \alpha,$$

for $g_1 = a, b, a^2, ab, a^2b$, and where

$$\Theta_2(1 \otimes g_2 \otimes g_3) = e_2,$$
$$1 \otimes g_1 \otimes g_2 \otimes g_3 \mapsto e_1 + \alpha, 0, -e_1 + \alpha, -e_1 - \alpha, e_1 - \alpha,$$

for $g_1 = a, b, a^2, ab, a^2b$, and where

$$\Theta_2(1 \otimes g_2 \otimes g_3) = -e_2,$$
$$1 \otimes g_1 \otimes g_2 \otimes g_3 \mapsto 0,$$

for any other case,

$$\Theta_4 : A \otimes \overline{A}^{\otimes 4} \longrightarrow Ae_1,$$
$$1 \otimes g_1 \otimes g_2 \otimes g_3 \otimes g_4 \mapsto e_1, 0, 0, 0, e_1$$

for $g_1 = a, b, a^2, ab, a^2b$, and where

$$\Theta_3(1 \otimes g_2 \otimes g_3 \otimes g_4) = e_1 + \alpha,$$
$$1 \otimes g_1 \otimes g_2 \otimes g_3 \otimes g_4 \mapsto 0, 0, e_1, e_1, 0$$

for $g_1 = a, b, a^2, ab, a^2b$, and where

$$\Theta_3(1 \otimes g_2 \otimes g_3 \otimes g_4) = e_1 - \alpha,$$
$$1 \otimes g_1 \otimes g_2 \otimes g_3 \otimes g_4 \mapsto -e_1, 0, 0, 0, -e_1$$

for $g_1 = a, b, a^2, ab, a^2b$, and where

$$\Theta_3(1 \otimes g_2 \otimes g_3 \otimes g_4) = -e_1 - \alpha,$$
$$1 \otimes g_1 \otimes g_2 \otimes g_3 \otimes g_4 \mapsto 0, 0, -e_1, -e_1, 0$$

for $g_1 = a, b, a^2, ab, a^2b$, and where

$$\Theta_3(1 \otimes g_2 \otimes g_3 \otimes g_4) = -e_1 + \alpha,$$
$$1 \otimes g_1 \otimes g_2 \otimes g_3 \otimes g_4 \mapsto 0,$$

for any other case,

$$\Theta_5 : A \otimes \overline{A}^{\otimes 5} \longrightarrow Ae_2,$$
$$1 \otimes g_1 \otimes g_2 \otimes g_3 \otimes g_4 \otimes g_5 \mapsto -e_2 - \beta, 0, e_2 - \beta, -e_2 - \beta, e_2 - \beta,$$

for $g_1 = a, b, a^2, ab, a^2b$, and where

$$\Theta_4(1 \otimes g_2 \otimes g_3 \otimes g_4 \otimes g_5) = e_1,$$

$$1 \otimes g_1 \otimes g_2 \otimes g_3 \otimes g_4 \otimes g_5 \mapsto e_2 + \beta, 0, -e_2 + \beta, e_2 + \beta, -e_2 + \beta,$$

for $g_1 = a, b, a^2, ab, a^2b$, and where

$$\Theta_4(1 \otimes g_2 \otimes g_3 \otimes g_4 \otimes g_5) = -e_1,$$
$$1 \otimes g_1 \otimes g_2 \otimes g_3 \otimes g_4 \otimes g_5 \mapsto 0,$$

for any other case,

$$\Theta_6 : A \otimes \overline{A}^{\otimes 6} \longrightarrow Ae_2,$$
$$1 \otimes g_1 \otimes g_2 \otimes g_3 \otimes g_4 \otimes g_5 \otimes g_6 \mapsto e_2, 0, 0, 0, -e_2$$

for $g_1 = a, b, a^2, ab, a^2b$, and where

$$\Theta_5(1 \otimes g_2 \otimes g_3 \otimes g_4 \otimes g_5 \otimes g_6) = e_2 + \beta,$$
$$1 \otimes g_1 \otimes g_2 \otimes g_3 \otimes g_4 \otimes g_5 \otimes g_6 \mapsto 0, 0, e_2, -e_2, 0$$

for $g_1 = a, b, a^2, ab, a^2b$, and where

$$\Theta_5(1 \otimes g_2 \otimes g_3 \otimes g_4 \otimes g_5 \otimes g_6) = e_2 - \beta,$$
$$1 \otimes g_1 \otimes g_2 \otimes g_3 \otimes g_4 \otimes g_5 \otimes g_6 \mapsto -e_2, 0, 0, 0, e_2$$

for $g_1 = a, b, a^2, ab, a^2b$, and where

$$\Theta_5(1 \otimes g_2 \otimes g_3 \otimes g_4 \otimes g_5 \otimes g_6) = -e_2 - \beta,$$
$$1 \otimes g_1 \otimes g_2 \otimes g_3 \otimes g_4 \otimes g_5 \otimes g_6 \mapsto 0, 0, -e_2, e_2, 0$$

for $g_1 = a, b, a^2, ab, a^2b$, and where

$$\Theta_5(1 \otimes g_2 \otimes g_3 \otimes g_4 \otimes g_5 \otimes g_6) = -e_2 + \beta,$$
$$1 \otimes g_1 \otimes g_2 \otimes g_3 \otimes g_4 \otimes g_5 \otimes g_6 \mapsto 0,$$

for any other case,

$$\Theta_7 : A \otimes \overline{A}^{\otimes 7} \longrightarrow Ae_1,$$
$$1 \otimes g_1 \otimes g_2 \otimes g_3 \otimes g_4 \otimes g_5 \otimes g_6 \otimes g_7 \mapsto -e_1 - \alpha, 0, e_1 - \alpha, e_1 + \alpha, -e_1 + \alpha$$

for $g_1 = a, b, a^2, ab, a^2b$, and where

$$\Theta_6(1 \otimes g_2 \otimes g_3 \otimes g_4 \otimes g_5 \otimes g_6 \otimes g_7) = e_2,$$
$$1 \otimes g_1 \otimes g_2 \otimes g_3 \otimes g_4 \otimes g_5 \otimes g_6 \otimes g_7 \mapsto e_1 + \alpha, 0, -e_1 + \alpha, -e_1 - \alpha, e_1 - \alpha$$

for $g_1 = a, b, a^2, ab, a^2b$, and where

$$\Theta_6(1 \otimes g_2 \otimes g_3 \otimes g_4 \otimes g_5 \otimes g_6 \otimes g_7) = -e_2,$$

$$1 \otimes g_1 \otimes g_2 \otimes g_3 \otimes g_4 \otimes g_5 \otimes g_6 \otimes g_7 \mapsto 0,$$

for any other case,

$$\Theta_8 : A \otimes \overline{A}^{\otimes 8} \longrightarrow Ae_1,$$
$$1 \otimes g_1 \otimes g_2 \otimes g_3 \otimes g_4 \otimes g_5 \otimes g_6 \otimes g_7 \otimes g_8 \mapsto e_1, 0, 0, 0, e_1$$

for $g_1 = a, b, a^2, ab, a^2b$, and where

$$\Theta_7(1 \otimes g_2 \otimes g_3 \otimes g_4 \otimes g_5 \otimes g_6 \otimes g_7 \otimes g_8) = e_1 + \alpha,$$
$$1 \otimes g_1 \otimes g_2 \otimes g_3 \otimes g_4 \otimes g_5 \otimes g_6 \otimes g_7 \otimes g_8 \mapsto 0, 0, e_1, e_1, 0$$

for $g_1 = a, b, a^2, ab, a^2b$, and where

$$\Theta_7(1 \otimes g_2 \otimes g_3 \otimes g_4 \otimes g_5 \otimes g_6 \otimes g_7 \otimes g_8) = e_1 - \alpha,$$
$$1 \otimes g_1 \otimes g_2 \otimes g_3 \otimes g_4 \otimes g_5 \otimes g_6 \otimes g_7 \otimes g_8 \mapsto -e_1, 0, 0, 0, -e_1$$

for $g_1 = a, b, a^2, ab, a^2b$, and where

$$\Theta_7(1 \otimes g_2 \otimes g_3 \otimes g_4 \otimes g_5 \otimes g_6 \otimes g_7 \otimes g_8) = -e_1 - \alpha,$$
$$1 \otimes g_1 \otimes g_2 \otimes g_3 \otimes g_4 \otimes g_5 \otimes g_6 \otimes g_7 \otimes g_8 \mapsto 0, 0, -e_1, -e_1, 0$$

for $g_1 = a, b, a^2, ab, a^2b$, and where

$$\Theta_7(1 \otimes g_2 \otimes g_3 \otimes g_4 \otimes g_5 \otimes g_6 \otimes g_7 \otimes g_8) = -e_1 + \alpha,$$
$$1 \otimes g_1 \otimes g_2 \otimes g_3 \otimes g_4 \otimes g_5 \otimes g_6 \otimes g_7 \otimes g_8 \mapsto 0,$$

for any other case. Note that both the representatives of u and v in the group cohomology $H^*(S_3) = \mathbb{F}_3[u, v]/(u^2)$ can be chosen as $Ae_1 \longrightarrow \mathbb{F}_3$, $e_1 \mapsto 1$, $\alpha \mapsto 0$, $\beta\alpha \mapsto 0$. Since $|u| = 3$ and $|v| = 4$, we have $\widehat{\triangle}_1(u) = 0$ and $\widehat{\triangle}_1(v) = \lambda u$ for some $\lambda \in \mathbb{F}_3$. We have

$$\widehat{\triangle}_1(v) = \widehat{\triangle}_1(v\Theta_4) \circ \Lambda_3,$$

and $\widehat{\triangle}_1(v\Theta_4)$ can be computed by our formula in Theorem 8.2. Since

$$\begin{aligned}
\widehat{\triangle}_1(v\Theta_4) \circ \Lambda_3(e_1) &= e_1\widehat{\triangle}_1(v\Theta_4)(\beta \otimes \alpha\beta \otimes \alpha) \\
&= -\widehat{\triangle}_1(v\Theta_4)(\beta \otimes \alpha\beta \otimes \alpha) - b\widehat{\triangle}_1(v\Theta_4)(\beta \otimes \alpha\beta \otimes \alpha) \\
&= -2\widehat{\triangle}_1(v\Theta_4)(\beta \otimes \alpha\beta \otimes \alpha) \\
&= \widehat{\triangle}_1(v\Theta_4)(\beta \otimes \alpha\beta \otimes \alpha).
\end{aligned}$$

By a MAPLE calculation (see [14]: A MAPLE program for computing $\widehat{\triangle}_1$.), we obtain that $\widehat{\triangle}_1(v\Theta_4)(\beta \otimes \alpha\beta \otimes \alpha) = 0$, and therefore $\widehat{\triangle}_1(v) = 0$. Since $|uv| = 7$ and

$|v^2| = 8$, we have $\widehat{\triangle}_1(uv) = \mu u^2$ and $\widehat{\triangle}_1(v^2) = \mu' uv$ for some $\mu, \mu' \in \mathbb{F}_3$. Since $u^2 = 0$, $\widehat{\triangle}_1(uv) = 0$, and we only need to compute $\widehat{\triangle}_1(v^2)$. The representative of v^2 in the group cohomology $H^*(S_3) = \mathbb{F}_3[u, v]/(u^2)$ can also be chosen (up to a sign) as $Ae_1 \longrightarrow \mathbb{F}_3, e_1 \mapsto 1, \alpha \mapsto 0, \beta\alpha \mapsto 0$. We have

$$\widehat{\triangle}_1(v^2) = \widehat{\triangle}_1(v^2\Theta_8) \circ \Lambda_7,$$

$$\widehat{\triangle}_1(v^2\Theta_8) \circ \Lambda_7(e_1) = e_1\widehat{\triangle}_1(v^2\Theta_7)(\beta \otimes \alpha\beta \otimes \alpha \otimes \beta\alpha \otimes \beta \otimes \alpha\beta \otimes \alpha)$$

$$= \widehat{\triangle}_1(v^2\Theta_7)(\beta \otimes \alpha\beta \otimes \alpha \otimes \beta\alpha \otimes \beta \otimes \alpha\beta \otimes \alpha).$$

Similarly by a MAPLE calculation (see [14]: A MAPLE program for computing $\widehat{\triangle}_1$), we obtain that $\widehat{\triangle}_1(v^2\Theta_8) \circ \Lambda_7(e_1) = 0$, and therefore $\widehat{\triangle}_1(v^2) = 0$.

Finally, based on the above computations, we deal with the Lie brackets. Since we have the following Possion rule: $[\alpha \cup \beta, \gamma] = [\alpha, \gamma] \cup \beta + (-1)^{|\alpha|(|\gamma|-1)}\alpha \cup [\beta, \gamma]$, it suffices to write down the Lie brackets between generators in $HH^*(\mathbb{F}_3 S_3)$. Recall that $HH^*(\mathbb{F}_3 S_3)$ is generated as an algebra by elements u, v, $C_1 = 1 + a + a^2$, $C_2 = b(1 + a + a^2)$, X_1, X_2 of degrees (resp.) $3, 4, 0, 0, 1$, and 2, subject to the relations

$$uX_1 = 0, \quad vX_1 = uX_2, \quad uC_2 = 0 = vC_2,$$

$$C_iX_j = 0 = C_iC_j (i, j \in \{1, 2\}), \quad X_1X_2 = uC_1, \quad X_2^2 = vC_1$$

in addition to the graded commutative relations. Using the formulas (Here we omit the sign $\cup$ in the equation)

$$[\alpha, \beta] = -(-1)^{(|\alpha|-1)|\beta|}(\triangle(\alpha\beta) - \triangle(\alpha)\beta - (-1)^{|\alpha|}\alpha \triangle (\beta))$$

and

$$[\alpha, \beta] = -(-1)^{(|\alpha|-1)(|\beta|-1)}[\beta, \alpha],$$

we do the concrete computations as follows:

$$[u, u] = 0, [u, v] = \widehat{\triangle}_1(uv) - \widehat{\triangle}_1(u)v + u\widehat{\triangle}_1(v) = 0, [v, u] = 0,$$

$$[u, C_1] = -(\triangle(uC_1) - \triangle(u)C_1 + u \triangle (C_1)) = -\widehat{\triangle}_a(X_1X_2) = X_2, [C_1, u] = -X_2,$$

$$[u, C_2] = -(\triangle(uC_2) - \triangle(u)C_2 + u \triangle (C_2)) = 0, [C_2, u] = 0,$$

$$[u, X_1] = -(\triangle(uX_1) - \triangle(u)X_1 + u \triangle (X_1)) = u, [X_1, u] = -u,$$

$$[u, X_2] = -(\triangle(uX_2) - \triangle(u)X_2 + u \triangle (X_2)) = -\widehat{\triangle}(uX_2) = 0, [X_2, u] = 0,$$

$$[v, v] = -(\triangle(v^2) - \triangle(v)v - v \triangle (v)) = 0,$$

$$[v, C_1] = -(\triangle(vC_1) - \triangle(v)C_1 + v \triangle (C_1)) = -\widehat{\triangle}_a(X_2^2) = 0, [C_1, v] = 0,$$

$$[v, C_2] = -(\triangle(vC_2) - \triangle(v)C_2 + v \triangle (C_2)) = 0, [C_2, v] = 0,$$

$$[v, X_1] = \triangle(vX_1) - \triangle(v)X_1 - v \triangle (X_1) = -v, [X_1, v] = v,$$

$$[v, X_2] = -(\triangle(vX_2) - \triangle(v)X_2 - v \triangle(X_2)) = \pm\widehat{\triangle}_a(X_2^3) = 0, [X_2, v] = 0,$$
$$[C_1, C_1] = [C_1, C_2] = [C_2, C_1] = [C_2, C_2] = 0,$$
$$[C_1, X_1] = \triangle(C_1 X_1) - \triangle(C_1)X_1 - C_1 \triangle(X_1) = C_1, [X_1, C_1] = -C_1,$$
$$[C_1, X_2] = -(\triangle(C_1 X_2) - \triangle(C_1)X_2 - C_1 \triangle(X_2)) = 0, [X_2, C_1] = 0,$$
$$[C_2, X_1] = \triangle(C_2 X_1) - \triangle(C_2)X_1 - C_2 \triangle(X_1) = C_2, [X_1, C_2] = -C_2,$$
$$[C_2, X_2] = -(\triangle(C_2 X_2) - \triangle(C_2)X_2 - C_2 \triangle(X_2)) = 0, [X_2, C_2] = 0,$$
$$[X_1, X_1] = 0, [X_1, X_2] = -(\triangle(X_1 X_2) - \triangle(X_1)X_2 + X_1 \triangle(X_2)) = 0, [X_1, X_2] = 0,$$
$$[X_2, X_2] = -(\triangle(X_2^2) - \triangle(X_2)X_2 - X_2 \triangle(X_2)) = 0.$$

In the fifth line of the above computation, we use the fact that $\widehat{\triangle}(uX_2) = 0$. The reason is as follows: uX_2 is an element of degree 5, under the additive decompositon, it corresponds to an element in $H^*(\langle a \rangle)$ and has the form $\pm w_1 w_2{}^2$. It follows from the formula in the last paragragh of Section 8 that $\widehat{\triangle}(uX_2) = \widehat{\triangle}_a(\pm w_1 w_2{}^2) = 0$.

Remark 9.1. By a recent result of Menichi (see [16, p. 321]), the Lie bracket of the group cohomology $H^*(G)$ for a finite group G must be trivial. The above computation shows that this is indeed the case for $H^*(S_3) = \mathbb{F}_3[u, v]/(u^2)$. Note that to verify $[v, v] = 0$, we have used the MAPLE program in [14].

Remark 9.2. Observe in the above example that the generators of $HH^*(\mathbb{F}_3 S_3)$ are "multiplicative closed" under Lie bracket: the Lie bracket $[\alpha, \beta]$ of two generators α and β is a scalar multiple of another generator. Also if $[\alpha, \beta] \neq 0$, then $[\alpha, \beta]$ is equal to $-[\beta, \alpha]$.

References

[1] N. Bian, P. Zhang and G. Zhang, Setwise homotopy category, *Appl. Categ. Structures*, **17** (2009), no. 6, 561–565. Zbl 1210.18014 MR 2564122

[2] D. Benson, *Representations and cohomology, II: Cohomology of groups and modules*, Cambridge University Press, 1991. Zbl 0731.20001 MR 1156302

[3] C. Cibils, Tensor product of Hopf bimodules over a group, *Proc. Amer. Math. Soc.*, **125** (1997), 1315–1321. Zbl 0865.18005 MR 1371118

[4] C. Cibils and A. Solotar, Hochschild cohomology algebra of abelian groups, *Arch. Math.*, **68** (1997), 17–21. Zbl 0872.16004 MR 1421841

[5] S. Eilenberg and S. MacLane, Cohomology theory in abstract groups. I, *Ann. Math.*, **48** (1947), 51–78. Zbl 0029.34001 MR 0019092

[6] L. Evens, *The cohomology of groups*, Clarendon Press, 1991. Zbl 0742.20050 MR 1144017

[7]   M. A. Farinati and A. L. Solotar, G-structure on the cohomology of Hopf algebras, *Proc. Amer. Math. Soc.*, **132** (2004), 2859–2865. Zbl 1063.16010 MR 2063104

[8]   M. Gerstenhaber, The cohomology structure of an associative ring, *Ann. Math.*, **78** (1963), no. 2, 267–288. Zbl 0131.27302 MR 0161898

[9]   E. Getzler, Batalin–Vilkovisky algebras and two-dimensional topological field theories, *Comm. Math. Phys.*, **159** (1994), no. 2, 265–285. Zbl 0807.17026 MR 1256989

[10] T. Holm, The Hochschild cohomology ring of a modular group algebra: the commutative case, *Comm. Algebra*, **24** (1996), 1957–1969. Zbl 0853.16016 MR 1386022

[11] A. Ivanov, S. O. Ivanov, Y. Volkov and G. D. Zhou, On the Hochschild cohomology ring of the quaternion group of order eight in characteristic two, *J. Algebra*, **435** (2015), 174–203. Zbl 06435591 MR 3343216

[12] J. Le and G. D. Zhou, On the Hochschild cohomology of tensor product of algebras, *J. Pure Appl. Algebra*, **218** (2014), no. 8, 1463–1477. Zbl 1328.16002 MR 3175033

[13] J. L. Loday, *Cyclic Homology*, Grundlehren der mathematischen Wissenschaften, 301, Springer, 1998. Zbl 0885.18007 MR 1600246

[14] Y. M. Liu and G. D. Zhou, A Maple program, available at: http://math.bnu.edu.cn/~liuym/.

[15] L. Menichi, Batalin–Vilkovisky algebras and cyclic cohomology of Hopf algebras, *K-Theory*, **32** (2004), no. 3, 231–251. Zbl 1101.19003 MR 2114167

[16] L. Menichi, Connes-Moscovici characteristic map is a Lie algebra morphism, *J. Algebra*, **331** (2011), 311–337. Zbl 1256.16010 MR 2774661

[17] S. S. Siegel and S. J. Witherspoon, The Hochschild cohomology ring of a group algebra, *Proc. London Math. Soc.*, **79** (1999), no. 3, 131–157. Zbl 1044.16005 MR 1687539

[18] R. Taillefer, Injective Hopf bimodules, cohomologies of infinite dimensional Hopf algebras and graded-commutativity of the Yoneda product, *J. Algebra*, **276** (2004), 259–279. Zbl 1063.16048 MR 2054397

[19] T. Tradler, The Batalin–Vilkovisky algebra on Hochschild cohomology induced by infinitely inner products, *Ann. Inst. Fourier (Grenoble)*, **58** (2008), no. 7, 2351–2379. Zbl 1218.16004 MR 2498354

[20] T. Yang, A Batalin–Vilkovisky Algebra Structure on the Hochschild Cohomology of Truncated Polynomials, *Topology and its Applications*, **160** (2013), no. 13, 1633–1651. Zbl 1282.55013 MR 3091339

Y. Liu, School of Mathematical Sciences, Beijing Normal University, Beijing 100875, China;
and BCMIIS, Beijing 100048, China
E-mail: ymliu@bnu.edu.cn

G. Zhou, Department of Mathematics, Shanghai Key laboratory of PMMP,
East China Normal University, Dong Chuan Road 500, Shanghai 200241, China
E-mail: gdzhou@math.ecnu.edu.cn