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On an intermediate bivariant K-theory for C^*-algebras

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Abstract. We construct a new bivariant K-theory for C^*-algebras, that we call KE-theory. For each pair of separable graded C^*-algebras A and B, acted upon by a locally compact σ-compact group G, we define an abelian group $KE_G(A, B)$. We show that there is an associative product $KE_G(A, D) \otimes KE_G(D, B) \to KE_G(A, B)$. Various functoriality properties of the KE-theory groups and of the product are presented. The new theory is intermediate between the KK-theory of G. G. Kasparov, and the E-theory of A. Connes and N. Higson, in the sense that there are natural transformations $KK_G \to KE_G$ and $KE_G \to E_G$ preserving the products. The motivations that led to the construction of KE-theory were: (1) to give a concrete description of the map from KK-theory to E-theory, abstractly known to exist because of the universal characterization of KK-theory, (2) to construct a bivariant theory well adapted to dealing with elliptic operators, and in which the product is simpler to compute with than in KK-theory, and (3) to provide a different proof to the Baum–Connes conjecture for a-T-menable groups. This paper deals with the first two problems mentioned above; the third one will be treated somewhere else.

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1. Introduction

This paper investigates some connections between KK-theory and E-theory for C^*-algebras. Besides a wealth of functorial properties, the key feature of KK-theory (see [24, 26]) is the existence for any separable C^*-algebras A, B, and D of an associative product map $KK(A, D) \otimes KK(D, B) \longrightarrow KK(A, B)$. Following an approach indicated by J. Cuntz (see [9, 10]), N. Higson [15] gave the following description of KK-theory: it is the universal category with homotopy invariance, stability, and split-exactness. This category has separable C^*-algebras as objects, elements of KK-groups as morphisms, and the above mentioned associative Kasparov product as composition of morphisms.

In a subsequent paper [17], Higson described the universal category with homotopy invariance, stability, and *exactness*. The resulting new theory — named E-theory — has become important in C^*-algebra theory after A. Connes and

N. Higson [7] described it concretely in terms of asymptotic morphisms. (An asymptotic morphism between two C^*-algebras is a family of maps between the two, indexed by $[1, \infty)$, which satisfies the conditions of a $*$-homomorphism in the limit at ∞.) The description of KK-theory and E-theory using category theory implies, in a rather abstract way, the existence of a map $KK(A, B) \to E(A, B)$, for any two C^*-algebras A and B. This map is an isomorphism when A is nuclear [35]. Similar descriptions of the universality property for the *equivariant* theories are also known: for the equivariant KK-theory [27] under the action of a group see [36], for the equivariant E-theory under the action of a group see [14], and for both theories under the action of a groupoid see [31, 32].

Equivariant KK-theory and E-theory have become essential tools in C^*-algebra theory because of their use in solving topological and geometrical problems, notably cases of the Novikov conjecture (see [27, 33]), cases of the Baum–Connes conjecture (see [3, 4, 20]), and index theory computations (see [22]).

In this paper a new theory is constructed, that we call *KE-theory*, which is intermediate between KK-theory and E-theory. It applies to C^*-algebras that are separable, graded, and admit an action of a locally compact σ-compact Hausdorff group. For such a group G, and for any two such G-C^*-algebras A and B, the resulting abelian group is denoted by $KE_G(A, B)$. The new theory recovers the ordinary K-theory of ungraded C^*-algebras. The KE-theory groups satisfy some of the good functorial properties of the other two bivariant theories, and there exists an associative product $KE_G(A, D) \otimes KE_G(D, B) \to KE_G(A, B)$. We have also proved the existence of two natural transformations, $\Theta : KK_G(A, B) \to KE_G(A, B)$ and $\Xi : KE_G(A, B) \to E_G(A, B)$, which preserve the products. Their composition $\Xi \circ \Theta$ provides an *explicit* construction of the map $KK \to E$, abstractly known to exist because of the universality properties of the two theories (as we mentioned above). The idea of constructing a theory intermediate between KK-theory and E-theory was suggested by V. Lafforgue (private communication to N. Higson).

Intermediate theories between KK-theory and E-theory appear also in the work of J. Cuntz [11, 12]. Our construction is different in initial motivation, concrete realization, and final goal: we wanted to produce a framework for another proof to the Baum–Connes conjecture for a-T-menable groups [19, 20]. Details for this application will be given elsewhere. We also wanted a theory that works well with K-homology classes of operators on manifolds and whose product is simpler than the product in KK-theory. This last goal was only achieved in particular cases: see subsection 4.1.

The paper is structured as follows. In Section 2 we briefly review the essential definitions, theorems and constructions related to KK-theory. We also use it to set up notation. Section 3 constructs the new KE-theory. In subsection 3.1 we introduce and study its cycles, which we call *asymptotic Kasparov modules*. They may be thought of as appropriate *families* of pairs, indexed by $[1, \infty)$. Each pair consists of a Hilbert bimodule and an operator on it, that are put together in a field satisfying conditions that resemble those appearing in KK-theory, with an E-theoretical twist. An example of such a cycle in the K-homology of a C^*-algebra A, motivated by the K-homology class of the Dirac operator on a spin manifold, consists of a Hilbert space $\mathcal{H}$, a $*$–homomorphism $\varphi : A \to \mathcal{L}(\mathcal{H})$, and a family $\{F_t\}_{t\in[1,\infty)}$ of bounded linear operators on $\mathcal{H}$ satisfying:

(aKm1) $F_t = F_t^*$, for all t;

(aKm2) $\| [F_t, \varphi(a)] \| \xrightarrow{t\to\infty} 0$, for all $a \in A$;

(aKm3) $\varphi(a)\,(F_t^2 - 1)\,\varphi(a)^* \geq 0$, modulo compact operators and operators which converge in norm to zero.

Such a family $\{(\mathcal{H}, F_t)\}_{t\in[1,\infty)}$ is an asymptotic Kasparov (A,$\mathbb{C}$)-module. Note the asymptotic commutativity of (aKm2). Axiom (aKm2) encodes the pseudo-locality of first order elliptic differential operators, and axiom (aKm3) is supposed to encode the Fredholm property of elliptic operators on smooth manifolds. The definition can be also adapted to include a group action, and in subsection 3.2 we define, for a locally compact group G and two graded separable G-C^*-algebras A and B, the group $KE_G(A, B)$ of homotopy equivalence classes of asymptotic Kasparov G-(A, B)-modules. In subsection 3.3 we examine the axioms of asymptotic Kasparov modules from the perspective of two concrete examples. In subsubsection 3.3.1 we show that the KE-theory groups recover the ordinary K-theory for trivially graded C^*-algebras, and in subsubsection 3.3.2 we compute $KE_\Gamma(\mathbb{C}, \mathbb{C})$, for a discrete group Γ. Various functoriality properties of these groups are proved in the remaining part of the section.

In Section 4 the product in KE-theory is constructed using the notions of "two-dimensional" connection and quasi-central approximate unit. Let G be a locally compact group, and A_1, A_2, B_1, B_2, D be G-C^*-algebras. As in KK-theory, in its most general form, the product is a map

$$KE_G(A_1, B_1 \otimes D) \otimes KE_G(D \otimes A_2, B_2) \to KE_G(A_1 \otimes A_2, B_1 \otimes B_2),$$
$$(x, y) \mapsto x \sharp_D y\,.$$

Insight about the product in the new theory can be obtained by looking at the particular case when $B_1 = B_2 = D = \mathbb{C}$, which corresponds to the external product in K-homology. Consider two asymptotic Kasparov modules as described above: $\{(\mathcal{H}_1, F_{1,t})\}_t \in KE(A_1, \mathbb{C})$, and $\{(\mathcal{H}_2, F_{2,t})\}_t \in KE(A_2, \mathbb{C})$. Their product is

$$\{(\mathcal{H}_1 \otimes \mathcal{H}_2, F_{1,t} \otimes 1 + 1 \otimes F_{2,t})\}_t \in KE(A_1 \otimes A_2, \mathbb{C}).$$

The general case is more involved and our method is summarized in Overview 4.8. The reader familiar with KK-theory will notice that the Kasparov Technical Theorem was not used in our construction. In subsection 4.4 we analyze the algebra behind the product. We show that the product is associative and its various compatibilities with the functoriality of KE-groups are worked out. The stability of KE-theory is an easy consequence of the corresponding property of KK-theory. Subsection 4.5 plays the role of an appendix to this section. It contains the proof of Theorem 4.10 used to construct the product.

In Section 5 we define the two natural transformations $KK_G \to KE_G$ and $KE_G \to E_G$, whose composition gives an explicit characterization of the map from KK-theory into E-theory. The main open question about KE-theory is whether it coincides or not with either KK-theory or E-theory. The last subsection contains some observations about this topic. See also the paper of Ralf Meyer [29].

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2. Preliminaries: review and notation

The purpose of this section is to briefly review the essential definitions, theorems and constructions related to KK-theory. We also use it to set up notation. A standing assumption for the entire paper is: we work in the category **C*-alg**, whose objects are the *separable* and $\mathbb{Z}_2$*-graded* C^*-algebras and whose morphisms are $*$-homomorphisms that preserve the grading. Standard references for most of the constructions are [5] and [28].

2.1. C^*-algebras, Hilbert modules and tensor products. Given a graded C^*-algebra A, the *commutator* of two elements $a, b \in A$ is:

$$[a, b] = ab - (-1)^{\partial a\, \partial b} ba.$$

The C^*-algebra of complex numbers, $\mathbb{C}$, is trivially graded. As a general rule, given a locally compact space X, the C^*-algebra $C_0(X)$, of complex valued continuous functions on X vanishing at infinity, will be trivially graded. All the tensor products that we consider are graded. The minimal C^*-algebra tensor product is denoted by $\otimes$.

Let $L = [1, \infty)$ and $LL = [1, \infty) \times [1, \infty)$. For any C^*-algebra B and any locally compact space X, the C^*-algebra $B(X)$ of B-valued continuous functions on X vanishing at infinity is $B(X) = C_0(X, B) = C_0(X) \otimes B$. We further simplify and write: $BL = C_0(L, B)$, $BLL = C_0(LL, B)$, and $B[0,1] = C([0,1], B)$.

Given a Hilbert B-module $\mathcal{E}$, the C^*-algebra of *adjointable operators on* $\mathcal{E}$ is denoted by $\mathcal{L}(\mathcal{E})$. The closed ideal of *compact operators on* $\mathcal{E}$ is denoted by $\mathcal{K}(\mathcal{E})$. It is generated by the rank-one operators $\theta_{\xi,\eta}(\zeta) = \xi\langle \eta, \zeta\rangle$, for $\xi, \eta, \zeta \in \mathcal{E}$.

Let $\mathcal{E}_1$ and $\mathcal{E}_2$ be graded Hilbert modules over B_1 and B_2, respectively. The completion $\mathcal{E}_1 \otimes \mathcal{E}_2$ of the algebraic tensor product $\mathcal{E}_1 \odot \mathcal{E}_2$ with respect to the $B_1 \otimes B_2$-valued semi-inner product

$$\langle \xi_1 \otimes \eta_1, \xi_2 \otimes \eta_2\rangle = (-1)^{\partial\eta_1(\partial\xi_1 + \partial\xi_2)} \langle \xi_1, \xi_2\rangle \otimes \langle \eta_1, \eta_2\rangle$$

is a Hilbert $B_1 \otimes B_2$-module, called the *external tensor product of* $\mathcal{E}_1$ *and* $\mathcal{E}_2$. If $\varphi : B_1 \to \mathcal{L}(\mathcal{E}_2)$ is a $*$–homomorphism, one can also construct the *internal tensor product* $\mathcal{E}_1 \otimes_{B_1} \mathcal{E}_2$ *of* $\mathcal{E}_1$ *and* $\mathcal{E}_2$. (The notation $\mathcal{E}_1 \otimes_\varphi \mathcal{E}_2$ will also be used.) It is the Hilbert B_2-module obtained as completion of the algebraic tensor product $\mathcal{E}_1 \odot_{B_1} \mathcal{E}_2$ with respect to the B_2-valued semi-inner product

$$\langle \xi_1 \otimes \eta_1, \xi_2 \otimes \eta_2\rangle = \langle \eta_1, \varphi(\langle \xi_1, \xi_2\rangle)(\eta_2)\rangle.$$

In both cases the grading is $\partial(\xi \otimes \eta) = \partial\xi + \partial\eta$.

Given two Hilbert modules $\mathcal{E}_1$ and $\mathcal{E}_2$, there is an embedding

$$\mathcal{L}(\mathcal{E}_1) \otimes \mathcal{L}(\mathcal{E}_2) \to \mathcal{L}(\mathcal{E}_1 \otimes \mathcal{E}_2),$$

given by

$$(F_1 \otimes F_2)(\xi \otimes \eta) = (-1)^{\partial\xi\,\partial F_2} F_1(\xi) \otimes F_2(\eta).$$

Its restriction to compact operators gives an isomorphism

$$\mathcal{K}(\mathcal{E}_1) \otimes \mathcal{K}(\mathcal{E}_2) \simeq \mathcal{K}(\mathcal{E}_1 \otimes \mathcal{E}_2).$$

In the case of an internal tensor product of Hilbert modules, we only have a natural graded $*$–homomorphism

$$\mathcal{L}(\mathcal{E}_1) \to \mathcal{L}(\mathcal{E}_1 \otimes_{B_1} \mathcal{E}_2), \quad F \mapsto F \otimes_{B_1} 1, \quad (F \otimes_{B_1} 1)(\xi \otimes \eta) = F(\xi) \otimes \eta.$$

Given a Hilbert B-module $\mathcal{E}$ and a space X, $\mathcal{E}(X)$ is the Hilbert $B(X)$-module $C_0(X) \otimes \mathcal{E}$ (external tensor product of Hilbert modules). We shall use the notation: $\mathcal{E}L = C_0(L) \otimes \mathcal{E} = \{\mathcal{E}\}_t$ = constant family with "fiber" $\mathcal{E}$ indexed by $[1,\infty)$, $\mathcal{E}LL = C_0(LL) \otimes \mathcal{E}$ = constant family with "fiber" $\mathcal{E}$ indexed by $[1,\infty) \times [1,\infty)$.

The *multiplier algebra* $\mathcal{M}(A)$ of a C^*-algebra A is the largest C^*-algebra in which A embeds as an essential ideal. We recall the following two facts about multiplier algebras: $\mathcal{M}(\mathcal{K}(\mathcal{E})) \simeq \mathcal{L}(\mathcal{E})$, for any Hilbert B-module $\mathcal{E}$, and $\mathcal{M}\big(C_0([1,\infty), \mathcal{K}(\mathcal{E}))\big) \simeq C_b\big([1,\infty), \mathcal{L}_{str}(\mathcal{E})\big)$, where $\mathcal{L}_{str}(\mathcal{E})$ denotes the strict topology (see [1, 3.4]).

2.2. Group actions. As reference for this section see [27, Sec.1]. Besides being separable and graded, the C^*-algebras that we consider have an additional structure: the action of a group by automorphisms. *All groups are supposed to be locally compact, σ-compact and Hausdorff.* Given such a group G and a C^*-algebra A, an *action of G on A* is a group homomorphism $G \to \mathrm{Aut}(A)$, where $\mathrm{Aut}(A)$ is the group of automorphisms of A, with no topology on it. An element $a \in A$ is called *G-continuous* if the map $G \to A$, $g \mapsto g(a)$ is continuous. We denote by **G-C*-alg** the category with objects the separable graded C^*-algebras equipped with G-action compatible with the grading and having all the elements G-continuous, and with morphisms the equivariant $*$-homomorphisms. The objects of **G-C*-alg** are called *G-C^*-algebras*. The action of any group G on $\mathbb{C}$ is trivial.

Given a group G, a G-C^*-algebra B, and a Hilbert B-module $\mathcal{E}$, an *action of G on $\mathcal{E}$*, or a *G-action*, is an action of G by grading preserving linear automorphisms such that: (i) $G \times \mathcal{E} \to \mathcal{E}$, $(g, \xi) \mapsto g(\xi)$, is continuous in the norm topology of $\mathcal{E}$; (ii) $g(\xi b) = g(\xi)g(b)$; and (iii) $\langle g(\xi), g(\eta)\rangle = g(\langle \xi, \eta\rangle)$, for all $\xi, \eta \in \mathcal{E}$, $b \in B$, $g \in G$. We call such a Hilbert module $\mathcal{E}$ a *G-B-module*.

Given an action of G on $\mathcal{E}$, there is an induced action of G on $\mathcal{L}(\mathcal{E})$ as follows: $g(T)(\xi) = g(T(g^{-1}\xi))$, for all $g \in G$, $T \in \mathcal{L}(\mathcal{E})$, and $\xi \in \mathcal{E}$. In this way, for any G-C^*-algebra B, there is a canonical induced action on $\mathcal{M}(B)$. Let $\mathcal{E}_1$ be a Hilbert D-module, with a G-action, and $\mathcal{E}_2$ be a G-(D, B)-module. The action of G on the internal tensor product $\mathcal{E}_1 \otimes_D \mathcal{E}_2$ is given by $g(\xi \otimes_D \eta) = g(\xi) \otimes_D g(\eta)$, for all $\xi \in \mathcal{E}_1$, $\eta \in \mathcal{E}_2$. This implies, for $T \in \mathcal{L}(\mathcal{E}_1)$, that $g(\,T \otimes_D 1\,) = g(T) \otimes_D 1$.

The *standard Hilbert G-space* is

$$\mathcal{H}_G = L^2(G) \oplus L^2(G) \oplus \cdots ,$$

with infinitely many summands, graded alternately even and odd, and equipped with the left regular representation of G. Let $\mathcal{K} = \mathcal{K}(\mathcal{H}_G)$ be the compact operators on $\mathcal{H}_G$. For any G-C^*-algebra B, the *standard Hilbert G-B-module* is

$$\mathcal{H}_B = l^2 \otimes L^2(G) \otimes (B \oplus B^{\mathrm{op}}) \simeq \mathcal{H}_G \otimes B.$$

The following notion will be very important in our construction of the product.

Definition 2.1. Let G be a group. Consider an inclusion $I \subset B \subset A$, where A is a G-C^*-algebra, B is a separable G-C^*-subalgebra of A, and I is a σ-unital G-ideal of A. A *quasi-invariant quasi-central approximate unit for I in B* (abbreviated q.i.q.c.a.u.) is a continuous family $\{u_t\}_{t\in[1,\infty)}$ of positive, increasing, even elements of I satisfying:

(a.u.) $\| x u_t - x \| \xrightarrow{t\to\infty} 0$, for all $x \in I$;

(q.c.) $\| y u_t - u_t y \| \xrightarrow{t\to\infty} 0$, for all $y \in B$; and

(q.i.) $\| g(u_t) - u_t \| \xrightarrow{t\to\infty} 0$, uniformly on compact subsets of G.

Such quasi-invariant quasi-central approximate units always exist, for any $I \lhd A$. For a proof see [27, Lemma 1.4], or [14, 5.3]; without a group action, see [30, 3.12.14], or [2, Thm.1]. In this paper we need a *countable* approximate unit $\{u_n\}_n$ (which by interpolation gives the family $\{u_t\}_t$), and this justifies the presence of the separable subalgebra B. It is usually clear from the context what B is (the biggest subalgebra that one needs in each particular application!), and we shall usually omit mention of it.

2.3. KK-theory. Taking into account the fact that many constructions in KE-theory are motivated by KK-theory constructions and in order to have the paper self-contained, we present in this subsection a quick review of the theory of Gennadi Kasparov [27].

Definition 2.2. Consider a group G and two graded separable G-C^*-algebras A and B. A *Kasparov G-(A,B)-module*is a triple $(\mathcal{E},\varphi,F)$, where $\mathcal{E}$ is a Hilbert G-B-module, $\varphi : A \to \mathcal{L}(\mathcal{E})$ is a $*$-homomorphism, and $F \in \mathcal{L}(\mathcal{E})$ is an odd G-continuous operator such that for every $a \in A$ and $g \in G$,

$$(F - F^*)\varphi(a),\ [F,\varphi(a)],\ (F^2-1)\varphi(a),\ \text{and}\ (g(F)-F)\varphi(a)\ \text{all belong to}\ \mathcal{K}(\mathcal{E}). \tag{2.1}$$

The set of all Kasparov G-(A,B)-modules will be denoted by $kk_G(A,B)$. Note that this is not the standard notation, namely $E(A,B)$, from the literature. A Kasparov G-(A,B)-module $(\mathcal{E},\varphi,F)$ is said to be *degenerate* if for all $a \in A$ and $g \in G$: $(F-F^*)\varphi(a) = 0$, $[F,\varphi(a)] = 0$, $(F^2-1)\varphi(a) = 0$, and $(g(F)-F)\varphi(a) = 0$. Whenever there is no risk of confusion, we shall write a instead of $\varphi(a)$.

The set $KK_G(A,B)$ is defined as the quotient of $kk_G(A,B)$ by the equivalence relation generated by homotopy. Given an element $x = (\mathcal{E},\varphi,F) \in kk_G(A,B)$, its class in $KK_G(A,B)$ will be denoted with a bolded character $\boldsymbol{x}$. The *addition* of two Kasparov G-(A,B)-modules is given by direct sum. With this operation $KK_G(A,B)$ becomes an abelian group and the degenerate elements all represent the null element.

Definition 2.3. ([8, Thm. A.3], [34, Def. 10]) Let A, B, D be G-C^*-algebras,

$$x = (\mathcal{E}_1,\varphi_1,F_1) \in kk_G(A,D), \quad y = (\mathcal{E}_2,\varphi_2,F_2) \in kk_G(D,B),$$
$$\mathcal{E} = \mathcal{E}_1 \otimes_D \mathcal{E}_2.$$

Denote by $F_1 \sharp_D F_2$ the set of operators $F \in \mathcal{L}(\mathcal{E})$ satisfying:

(1) $(\mathcal{E},\varphi_1 \otimes_D 1, F) \in kk_G(A,B)$;

(2) F is an F_2-connection for $\mathcal{E}_1$; and

(3) $a\,[F_1 \otimes_D 1, F]\,a^* \geq 0$, modulo $\mathcal{K}(\mathcal{E})$, for all $a \in A$.

For any $F \in F_1 \sharp_D F_2$, the triple $z = (\mathcal{E}, \varphi_1 \otimes_D 1, F)$ will be called *the product of x and y*. We shall use the notation $\boldsymbol{z} = \boldsymbol{x} \sharp_D \boldsymbol{y}$. (The same notation $\sharp$ will also be used to designate the product in the new KE-theory. It will be clear from the context to what theory a certain product belongs. Note also that the literature uses $\otimes$ instead of $\sharp$.)

The product $\sharp_D$ exists, is unique up to homotopy, and defines a bilinear pairing:

$$KK_G(A, D) \otimes KK_G(D, B) \xrightarrow{\sharp_D} KK_G(A, B), \ (\boldsymbol{x}, \boldsymbol{y}) \mapsto \boldsymbol{x} \sharp_D \boldsymbol{y}\,. \tag{2.2}$$

Example 2.4 (External product in KK-theory). Let A_1, A_2, B_1, B_2 be G-C^*-algebras. A particular case of the Kasparov product is the *external product* given by the map:

$$KK_G(A_1, B_1) \otimes KK_G(A_2, B_2) \xrightarrow{\sharp_{\mathbb{C}}} KK_G(A_1 \otimes A_2, B_1 \otimes B_2), \tag{2.3}$$

which sends the classes of $x = (\mathcal{E}_1, \varphi_1, F_1) \in kk_G(A_1, B_1)$ and $y = (\mathcal{E}_2, \varphi_2, F_2) \in kk_G(A_2, B_2)$ to the class of

$$(\mathcal{E}, \varphi, F) = (\mathcal{E}_1 \otimes \mathcal{E}_2, \varphi_1 \otimes \varphi_2, M^{\frac{1}{2}}(F_1 \otimes 1) + N^{\frac{1}{2}}(1 \otimes F_2)),$$

with M and N given by Kasparov's Technical Theorem [27, Thm.1.5].

We make the remark that even in the external product case one cannot in general obtain the product without the "partition of unity" provided by Kasparov's Technical Theorem. The search for a theory in which the "ideal" formula for the product, $F = F_1 \otimes 1 + 1 \otimes F_2$, always holds true and which is well suited to deal with elliptic operators on manifolds motivated the new KE-theory. See Section 4 and especially subsection 4.1.

3. *KE*-theory: definitions and functorial properties

In this section we introduce the new bivariant theory.

3.1. Asymptotic Kasparov modules.

Definition 3.1. Consider a group G and two separable G-C^*-algebras A and B. A *continuous field of G-(A, B)-modules* is a countably generated G-(A, BL)-module, *i.e.* a Hilbert BL-module $\mathcal{E}$, admitting a G-action and a left action of A through an equivariant $*$–homomorphism $\varphi : A \to \mathcal{L}(\mathcal{E})$. (We recall the notation: $L = [1, \infty)$, $BL = C_0(L, B) = C_0(L) \otimes B$.) We omit G in the non-equivariant case.

A continuous field $\mathcal{E}$ of G-(A, B)-modules may be thought of as a family $\{\mathcal{E}_t\}_{t \in [1,\infty)}$ of Hilbert B-modules, each acted on the left by A and G, satisfying certain continuity conditions for the left and right actions. Indeed, for any $t \in [1, \infty)$, let $ev_t : BL \to B$ be the evaluation $*$-homomorphism at t: $ev_t(f \otimes b) = f(t)b$, for $f \in C_0(L)$ and $b \in B$. We obtain the Hilbert G-(A,B)-module $\mathcal{E}_t = \mathcal{E} \otimes_{ev_t} B$, with inner product $\langle \xi \otimes b, \xi' \otimes b' \rangle_t = b^* \, ev_t(\langle \xi, \xi' \rangle) \, b'$. The A-action on each $\mathcal{E}_t$ is $\varphi_t : A \to \mathcal{L}(\mathcal{E}_t)$, $\varphi_t(a) = \varphi(a) \otimes_{ev_t} 1$. Whenever there is no risk of confusion, we shall write a instead of $\varphi(a)$, and a_t instead of $\varphi_t(a)$. It is also the case that an operator $F \in \mathcal{L}(\mathcal{E})$ gives a family $\{F_t\}_{t\in[1,\infty)} = \{F \otimes_{ev_t} 1\}_{t\in[1,\infty)}$. When $\mathcal{E} = \mathcal{E}_\bullet L$, for a fixed Hilbert B-module $\mathcal{E}_\bullet$, the function $L \to \mathcal{L}(\mathcal{E}_\bullet)$, $t \mapsto F_t$, is "bounded and $*$-strong continuous" [16, 3.16], *i.e.* the family $\{F_t\}_t$ is norm bounded, and for each $\xi \in \mathcal{E}_\bullet$ the functions $t \mapsto F_t(\xi)$ and $t \mapsto F_t^*(\xi)$ are norm continuous. Indeed, we have:

$$\begin{aligned}\mathcal{L}(\mathcal{E}_\bullet L) = \mathcal{L}(C_0(L) \otimes \mathcal{E}_\bullet) &= \mathcal{M}(\mathcal{K}(C_0(L) \otimes \mathcal{E}_\bullet)) \\ &= \mathcal{M}(C_0(L, \mathcal{K}(\mathcal{E}_\bullet))) = C_b(L, \mathcal{L}_{str}(\mathcal{E}_\bullet)),\end{aligned}$$

and strict continuity is $*$-strong continuity. On $\mathcal{L}(\mathcal{E}_\bullet)$ $*$-strong continuity is weaker than norm continuity.

For the remaining part of this subsection we assume no group action. Given any Hilbert BL-module $\mathcal{E}$, besides the *adjointable operators* $\mathcal{L}(\mathcal{E})$ on $\mathcal{E}$ and the *compact operators* $\mathcal{K}(\mathcal{E})$, two other ideals will play an important role in our presentation:

Definition 3.2. The closed ideal of *locally compact-valued families of operators* is

$$\mathcal{C}(\mathcal{E}) = \{F \in \mathcal{L}(\mathcal{E}) \mid F f \in \mathcal{K}(\mathcal{E}), \text{ for all } f \in C_0(L)\}. \tag{3.1}$$

(Here $C_0(L)$ is viewed as a sub-C^*-algebra of $\mathcal{L}(\mathcal{E})$ as follows: let $\{b_n\}_n$ be an approximate unit for B, then for $\xi \in \mathcal{E}$, let $f(\xi) = \lim_{n\to\infty} \xi(f \otimes b_n)$.) The closed ideal of *vanishing families of operators* is

$$\mathcal{J}(\mathcal{E}) = \{F \in \mathcal{L}(\mathcal{E}) \mid \lim_{t\to\infty} \|F_t\|_{\mathcal{L}(\mathcal{E}_t)} = 0\}. \tag{3.2}$$

Lemma 3.3. $\mathcal{K}(\mathcal{E}) = \mathcal{C}(\mathcal{E}) \cap \mathcal{J}(\mathcal{E})$.

Proof. The inclusion $\mathcal{K}(\mathcal{E}) \subseteq \mathcal{C}(\mathcal{E}) \cap \mathcal{J}(\mathcal{E})$ is clear. Let $F \in \mathcal{C}(\mathcal{E}) \cap \mathcal{J}(\mathcal{E})$. From the fact that $F \in \mathcal{J}(\mathcal{E})$ it follows that for every positive integer n there exists t_n such that $\|F_t\| < 2^{-n}$, for all $t > t_n$. Consider a partition of unity for L, $\{\chi_0, \chi_1, \dots, \chi_n, \dots\}$, subordinated to the cover $[1, t_1 + 2^{-1}) \cup \bigcup_{n=1}^{\infty} (t_n, t_{n+1} + 2^{-n-1})$. Then

$$F = F \cdot 1 = \sum_{n=0}^{\infty} F \cdot \chi_n \in \mathcal{K}(\mathcal{E}),$$

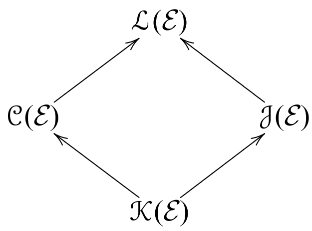
due to the fact that each term $F \cdot \chi_n$ of the sum is compact ($F \in \mathcal{C}(\mathcal{E})$), and of norm less than 2^{-n} (for $n \geq 1$). □

Lemma 3.4. *If $\mathcal{E} = \mathcal{E}_\bullet L$ is a constant family of Hilbert B-modules, then any $F \in \mathcal{C}(\mathcal{E})$ generates a norm-continuous family of operators $\{F_t\}_t$ in $\mathcal{K}(\mathcal{E}_\bullet)$, and vice versa.*

Proof. We first notice that the elements of $\mathcal{K}(\mathcal{E})$ generate norm-continuous families of operators. This is because any $\xi \in \mathcal{E}_\bullet L$ is a norm-continuous section vanishing at infinity in the constant field of Hilbert modules $\{\mathcal{E}_\bullet\}_t$. Consequently the generators $\theta_{\xi,\eta}$, $\xi, \eta \in \mathcal{E}_\bullet L$, of $\mathcal{K}(\mathcal{E})$ are norm-continuous. Now, given $F \in \mathcal{C}(\mathcal{E})$, the continuity of the family $\{F_t\}_t$ that it generates is a local property. For any t_0, choose $f \in C_c(L)$, $f \equiv 1$ in a neighborhood of t_0. The definition of $\mathcal{C}(\mathcal{E})$ says that $Ff \in \mathcal{K}(\mathcal{E})$, and consequently Ff is a norm-continuous family. This gives the norm-continuity of $\{F_t\}_t$ at t_0. □

Remark 3.5. $\mathcal{C}(\mathcal{E})$ does not coincide with $\{ F \in \mathcal{L}(\mathcal{E}) \mid F_t \in \mathcal{K}(\mathcal{E}_t), \text{ for all } t \}$. Indeed, it is not difficult to construct a $*$-strongly continuous family $\{P_t\}_{t\in[1,\infty)}$ of projections, of rank (at most) one, on an infinite dimensional Hilbert space which is not norm continuous.

We summarize the relations between these various ideals in the following diagram:



(3.3)

Definition 3.6. Let A and B be graded separable C^*-algebras (with no group action). An *asymptotic Kasparov* (A, B)*-module* is a pair $(\mathcal{E}, F)$, where $\mathcal{E}$ is a continuous field of (A,B)-modules, and $F \in \mathcal{L}(\mathcal{E})$ is odd and satisfies for any $a \in A$:

(aKm1) $(F - F^*)\varphi(a) \in \mathcal{J}(\mathcal{E})$;

(aKm2) $[F, \varphi(a)] \in \mathcal{J}(\mathcal{E})$; and

(aKm3) $\varphi(a)\,(F^2 - 1)\,\varphi(a)^* \geq 0$, modulo $\mathcal{C}(\mathcal{E}) + \mathcal{J}(\mathcal{E})$.

The set of all asymptotic Kasparov (A, B)-modules will be denoted by $ke(A, B)$.

Remark 3.7. By defining the cycles as pairs instead of triples we tried to simplify the notation. The $*$-homomorphism φ is incorporated in the definition of the continuous field.

Remark 3.8. Compare these axioms with the ones that a Kasparov module $(\mathcal{E}, \varphi, F)$ must satisfy (Definition 2.2, (2.1)). It is worth noticing that the third axiom of a Kasparov module, $(F^2 - 1)\varphi(a) \in \mathcal{K}(\mathcal{E})$, can be replaced (at least when $\|F\| \leq 1$) by $\varphi(a)(F^2-1)\varphi(a)^* \geq 0$, modulo $\mathcal{K}(\mathcal{E})$, which looks more like our (aKm3). We can also replace our (aKm3) axiom, when $\|F\| \leq 1$, with $(F^2 - 1)\varphi(a) \in \mathcal{C}(\mathcal{E}) + \mathcal{J}(\mathcal{E})$.

Remark 3.9. We introduce the following notation: given two operators $T, T' \in \mathcal{L}(\mathcal{E})$, then $T \sim T'$ if $(T - T') \in \mathcal{J}(\mathcal{E})$. With this convention (aKm1) reads $(F - F^*)\varphi(a) \sim 0$, and (aKm2) reads $[F, \varphi(a)] \sim 0$, for all $a \in A$.

Remark 3.10. In terms of families we can rephrase the conditions of Definition 3.6 as follows: $\{\mathcal{E}_t\}_{t\in[1,\infty)}$ is a family of Hilbert (A, B)-modules, $\{F_t\}_{t\in[1,\infty)}$ is a bounded $*$-strong continuous family of odd operators, meaning that for each continuous section $\xi = \{\xi_t\}_t$ the maps $t \mapsto F_t(\xi_t)$ and $t \mapsto F_t^*(\xi_t)$ are continuous sections of the field $\{\mathcal{E}_t\}_{t\in[1,\infty)}$, and for each $a \in A$

(aKm1′) $\| (F_t - F_t^*)a_t \| \xrightarrow{t\to\infty} 0$;

(aKm2′) $\| [F_t, a_t] \| \xrightarrow{t\to\infty} 0$; and

(aKm3′) $a_t\,(F_t^2 - 1)\,a_t^* = P_t^a + K_t^a$, with $P^a \geq 0$ in $\mathcal{L}(\mathcal{E})$ and $K^a \in \mathcal{C}(\mathcal{E}) + \mathcal{J}(\mathcal{E})$, both P^a and K^a depending on a. (Here a_t denotes $\varphi_t(a) = \varphi(a) \otimes_{ev_t} 1$.)

We shall use the notation $(\mathcal{E}, F) = \{(\mathcal{E}_t, F_t)\}_t$.

Example 3.11. Given a $*$-homomorphism $\psi : A \to B$, the associated asymptotic Kasparov (A,B)-module is $(BL, 0) = \{(B, 0)\}_t$, with $\varphi_t = \psi$. More generally, given a $*$-homomorphism $\psi : A \to \mathcal{K}(\mathcal{H}) \otimes B$, with $\mathcal{H}$ a separable Hilbert space, we form the asymptotic Kasparov (A,B)-module $(\mathcal{H}_B L, 0)$, with constant action of A on "fibers" as above. In this situation $P_t^a = 0$ and $K_t^a = -\psi(a)\,\psi(a)^* \in \mathcal{K}(\mathcal{H}) \otimes B \simeq \mathcal{K}(\mathcal{H}_B)$. This simple example covers the case of all Kasparov modules $(\mathcal{E}, \varphi, F)$ with $F = 0$.

Remark 3.12. In general, it is not true that a Kasparov module $(\mathcal{E}, \varphi, F)$ gives an asymptotic Kasparov module as a constant field $(\mathcal{E}L, 1 \otimes F)$. This is because (aKm2) may not be satisfied.

Example 3.13 (The K-homology class of the Dirac operator). Let M^{2n} be an even-dimensional, complete, spin^c-manifold, with spinor bundle $\mathbb{S} = \mathbb{S}_M$, and Dirac operator $D = D_M$. (D is essentially self-adjoint, and whenever functional calculus is used D actually denotes the closure $\overline{D} = D^*$.) The *fundamental asymptotic Kasparov* $(C_0(M), \mathbb{C})$*-module* is constructed as follows: $\mathcal{E} = \{L^2(M, \mathbb{S})\}_{t\in[1,\infty)}$, constant family; the action of $C_0(M)$ is the same on each 'fiber', by multiplication operators $\varphi_t(f) = M_f$; and $F = \{\chi(\frac{1}{t}D)\}_{t\in[1,\infty)}$, where χ is a *normalizing function* (*i.e.* $\chi : \mathbb{R} \to [-1, 1]$ is odd, smooth, and $\lim_{x\to\pm\infty} \chi(x) = \pm 1$; for example one could take $\chi(x) = x/(1 + x^2)^{1/2}$). We show that this is an asymptotic Kasparov module. (For a thorough exposition of elliptic operators on manifolds see [21, Chaps.10,11]. This reference also explains the terminology that we use in this example.)

- $F \in \mathcal{L}(\mathcal{E})$. Indeed, this is implied by the norm continuity of $t \mapsto \chi(\frac{1}{t}D)$.
- *F satisfies* (aKm1). As noted above, when we write D we actually mean $\overline{D} = D^*$, which is self-adjoint, and the functional calculus gives $F = F^*$.
- *F satisfies* (aKm2). For $f \in C_c^\infty(M)$ we get

$$\left[\tfrac{1}{t}D, f\right] = \tfrac{1}{t}\,(Df - fD) = \tfrac{1}{t}\,\nabla f \xrightarrow{t\to\infty} 0,$$

in norm (∇f represents Clifford multiplication by the vector field ∇f). This gives [18]:

$$\begin{aligned}\big[(\tfrac{1}{t}D \pm i)^{-1}, f\big] &= (\tfrac{1}{t}D \pm i)^{-1} f - f(\tfrac{1}{t}D \pm i)^{-1} \\ &= (\tfrac{1}{t}D \pm i)^{-1}\big(f(\tfrac{1}{t}D \pm i) - (\tfrac{1}{t}D \pm i)f\big)(\tfrac{1}{t}D \pm i)^{-1} \\ &= (\tfrac{1}{t}D \pm i)^{-1}\,(\tfrac{1}{t}\nabla f)\,(\tfrac{1}{t}D \pm i)^{-1} \xrightarrow{t\to\infty} 0.\end{aligned} \tag{3.4}$$

It follows that we obtain norm convergence $[\phi(\frac{1}{t}D), f] \xrightarrow{t\to\infty} 0$, for all $\phi \in C_0(\mathbb{R})$, $f \in C_0(M)$. The significance is that the asymptotic Kasparov module that we construct will not depend on the normalizing function, any two such having difference in $C_0(\mathbb{R})$. Moreover it suffices now to prove (aKm2) for *one* particular normalizing function χ_0. We choose it such that its distributional Fourier transform $\widehat{\chi}_0$ is compactly supported, and $s \mapsto s\,\widehat{\chi}_0(s)$ is smooth (or in $L^1(\mathbb{R})$). (Such functions exist: see [21, 10.9.3].) Some Fourier analysis next shows that:

$$\langle\, \chi_0(D)u, v\,\rangle = \int_{\mathbb{R}} \langle\, e^{isD}u, v\,\rangle\ \widehat{\chi_0}(s)\ ds, \ \text{ for all } u, v \in C_c^\infty(M, \mathbb{S}). \tag{3.5}$$

Consider for the moment a function $f \in C^\infty(M)$ which takes values in $S^1 \subset \mathbb{C}$ (*i.e.* M_f is an unitary operator), and such that ∇f is also a bounded operator. We have:

$$\begin{aligned}[\chi_0(\tfrac{1}{t}D), f] &= \chi_0(\tfrac{1}{t}D)\,f - f\,\chi_0(\tfrac{1}{t}D) = f\,\big(f^{-1}\,\chi_0(\tfrac{1}{t}D)\,f - \chi_0(\tfrac{1}{t}D)\big) \\ &= f\,\big(\chi_0(\tfrac{1}{t}f^{-1}Df) - \chi_0(\tfrac{1}{t}D)\big).\end{aligned} \tag{3.6}$$

Putting together (3.5) and (3.6), we obtain:

$$\langle\, [\chi_0(\tfrac{1}{t}D), f]u, v\,\rangle = \int_{\mathbb{R}} \langle\, \big(e^{ist^{-1}f^{-1}Df} - e^{ist^{-1}D}\big)u, \overline{f}v\,\rangle\ \widehat{\chi_0}(s)\ ds. \tag{3.7}$$

By our first computation of this paragraph, $f^{-1}Df - D = f^{-1}[D, f] = f^{-1}\nabla f$ is a bounded operator. In accordance with [21, Lemma 10.3.6], applied to $T_1 = \frac{1}{t}f^{-1}Df$ and $T_2 = \frac{1}{t}D$, we have:

$$\|e^{isT_1} - e^{isT_2}\| \le |s|\ \|T_1 - T_2\|, \ \text{ for all } s \in \mathbb{R}. \tag{3.8}$$

Because of (3.8), the inner product in the integral of (3.7) equals $|s|$ times a smooth function which is pointwise bounded by $\frac{1}{t}\,\|\nabla f\| \cdot \|u\| \cdot \|v\|$. The required norm asymptotic commutation now follows:

$$\|\,[\chi_0(\tfrac{1}{t}D), f]\,\| \le \tfrac{1}{t}\,\|\nabla f\| \int_{\mathbb{R}} |s\,\widehat{\chi_0}(s)|\ ds.$$

The computation made in the last part of the argument above is [21, Prop. 10.3.7].

Finally, any arbitrary non-identically zero $f \in C_c^\infty(M)$ can be written as a linear combination of functions on M which are S^1-valued. Indeed, $f = \mathrm{Re}(f) + i\,\mathrm{Im}(f)$, and for a real valued $f \neq 0$ one writes:

$$f = (\|f\|/2)\Big(\big(f/\|f\| + i\sqrt{1 - f^2/\|f\|^2}\big) + \big(f/\|f\| - i\sqrt{1 - f^2/\|f\|^2}\big)\Big).$$

We are through (due to the density of $C_c^\infty(M)$ in $C_0(M)$).

- *F satisfies* (aKm3). The standard theory of elliptic first order differential operators shows that $f\,\big(\chi^2(\frac{1}{t}D) - 1\big)$ is compact for $f \in C_0(M)$, so $f\,\big(F^2 - 1\big)\,\overline{f} = 0$ modulo $\mathcal{C}(\mathcal{E})$. (The norm continuity of $t \mapsto F_t$ was used again here.)

Remark 3.14. Given an asymptotic Kasparov (A, B)-module $(\mathcal{E}, F)$ then

$$(\mathcal{E}, (F + F^*)/2)$$

is another such object. Indeed, the only axiom which is not clear is (aKm3). It reduces to showing that

$$(F + F^*)^2/4 \geq (F^2 + (F^*)^2)/2,$$

which in turn is equivalent to the obvious

$$(F - F^*)\,(F - F^*)^* \geq 0.$$

We shall see in the next section that $(\mathcal{E}, F)$ and $(\mathcal{E}, (F + F^*)/2)$ are homotopic cycles and generate the same element in KE-theory.

3.2. The KE-theory groups. In this subsection we define the new bivariant theory and we study some of its functorial properties. *A group (locally compact, σ-compact, Hausdorff) is assumed to act continuously on all the objects under study.* We start with an extension of our previous Definition 3.6 to the equivariant context.

Definition 3.15. Consider a group G, and two graded separable G-C^*-algebras A and B. An *asymptotic Kasparov G-(A, B)-module*is a pair $(\mathcal{E}, F)$, where $\mathcal{E}$ is a continuous field of G-(A, B)-modules (see Definition 3.1) and $F \in \mathcal{L}(\mathcal{E})$ is an odd G-continuous operator, that satisfies (aKm1), (aKm2), (aKm3) of Definition 3.6, and the extra condition:

(aKm4) $(g(F) - F)\,\varphi(a) \in \mathcal{J}(\mathcal{E})$, for all $g \in G$, $a \in A$.

In terms of families this last condition reads:

(aKm4′) $\|\,(g_t(F_t) - F_t)\,a_t\,\| \xrightarrow{t\to\infty} 0$, for all $g \in G$, $a \in A$, and with $a_t = \varphi_t(a)$.

The set of all asymptotic Kasparov G-(A, B)-modules is denoted by $ke_G(A, B)$.

Example 3.16. Consider an equivariantly split exact sequence of G-C^*-algebras

$$0 \longrightarrow B \xrightarrow[j]{} D \underset{q}{\overset{s}{\rightleftarrows}} A \longrightarrow 0,$$

meaning that all the $*$-homomorphisms are equivariant and that $q \circ s = \mathrm{id}_A$. Let $\omega : D \to \mathcal{M}(B) = \mathcal{L}(B)$ be the canonical extension of the inclusion $B \to \mathcal{M}(B)$ (the construction of the extension given in the proof of [28, Prop. 2.1] is equivariant). Let $\{u_t\}_t$ be a quasi-invariant quasi-central approximate unit for $B \subset \omega(D) \subset \mathcal{M}(B)$. We associate to the above extension the asymptotic Kasparov G-(D, B)-module

$$\left\{\left(B \oplus B^{\mathrm{op}}, \left(\begin{smallmatrix} 0 & 1-u_t \\ 1-u_t & 0 \end{smallmatrix}\right)\right)\right\}_t,$$

the action of D being constant on fibers

$$\varphi_t : D \to \mathcal{L}(B \oplus B^{\mathrm{op}}), \quad \varphi_t(d) = \begin{pmatrix} \omega(d) & 0 \\ 0 & (-1)^{\partial d}(\omega \circ s \circ q)(d) \end{pmatrix}.$$

Its class in $KE_G(D, B)$ is the *splitting morphism* of the exact sequence (compare with [5, 17.1.2b] and [6, Sec. 5]).

Example 3.17 (The Bott element). Let V be a separable Euclidean space, and denote by $\mathcal{A}(V)$ the non-commutative C^*-algebra used by Higson–Kasparov–Trout in their proof of Bott periodicity (see [22, Def. 3.3], [19, Def. 4.1]). One considers $C_0(\mathbb{R})$ graded by even and odd functions. For a finite dimensional affine subspace V_a of V, denote by V_A^0 its linear support, and by $\mathcal{A}(V_a) = C_0(\mathbb{R}) \otimes C_0(V_a, \mathrm{Cliff}(V_a^0))$. The C^*-algebra $\mathcal{A}(V)$ is defined as the direct limit over the directed set of all finite dimensional affine subspaces $V_a \subset V$ of $\mathcal{A}(V_a)$: $\mathcal{A}(V) = \varinjlim \mathcal{A}(V_a)$. Then let $\beta : C_0(\mathbb{R}) \to \mathcal{A}(V)$ be the $*$-homomorphism given by the inclusion $(0) \subset V$, and use it to construct a family of $*$-homomorphisms

$$\{\beta_t\}_{t\in[1,\infty)} : C_0(\mathbb{R}) \to \mathcal{A}(V), \quad \beta_t(f) = \beta(f_t),$$

where $f_t(x) = f(t^{-1}x)$. For each t extend β_t to a $*$-homomorphism

$$\overline{\beta_t} : C_b(\mathbb{R}) = \mathcal{M}(C_0(\mathbb{R})) \to \mathcal{M}(\mathcal{A}(V)).$$

Consider $\lambda(x) = x/(1+x^2)^{1/2}$ and define $F_t = \overline{\beta_t}(\lambda) \in \mathcal{M}(\mathcal{A}(V))$. Further assume that a group G acts isometrically and by affine transformations on V. We associate the asymptotic Kasparov G-$(\mathbb{C}, \mathcal{A}(V))$-module $\{(\mathcal{A}(V), F_t)\}_t$, where the action of $\mathbb{C}$ is constant on fibers $\varphi_t : \mathbb{C} \to \mathcal{L}(\mathcal{A}(V))$, $\varphi_t(1) = 1$. We notice that, for each t, F_t is odd and self-adjoint (because λ has these properties), and that $\{F_t\}_t$ is actually a norm continuous family of operators. This shows that (aKm1) is satisfied, (aKm2) is trivial, and (aKm4) follows from the asymptotic equivariance of $\{\beta_t\}_t$ [19, Def. 4.3]. Finally, to see that (aKm3) holds true, note that

$$F_t^2 - 1 = -\beta_t(1/(1+x^2)) \in \mathcal{A}(V) = \mathcal{K}(\mathcal{A}(V)).$$

Consequently $F_t^2 - 1 = 0$ modulo $\mathcal{C}(\mathcal{A}(V)L)$.

Definition 3.18. An element $(\mathcal{E}, F)$ of $ke_G(A, B[0,1])$ gives, by "evaluation at s", a family

$$\{\, (\mathcal{E}_s, F_s) \in ke_G(A, B) \mid s \in [0,1] \,\},$$

with $\mathcal{E}_s = \mathcal{E} \otimes_{\mathrm{ev}_s} BL$, $F_s = F \otimes_{\mathrm{ev}_s} 1$. Such an element $(\mathcal{E}, F)$ and the family that it generates are called a *homotopy* between $(\mathcal{E}_0, F_0)$ and $(\mathcal{E}_1, F_1)$. An *operator homotopy* is a homotopy

$$\{\, (\mathcal{E}, F_s) \mid s \in [0,1] \,\},$$

with $s \mapsto F_s$ being norm continuous. Note that $\mathcal{E}$, and the action of A on it, are constant throughout an operator homotopy.

Example 3.19. Each

$$(\mathcal{E}_0, F_0) = \{(\mathcal{E}_{0,t}, F_{0,t})\}_t \in ke_G(A, B)$$

is homotopic to any of its "translates"

$$\{(\mathcal{E}_{0,t+N}, F_{0,t+N})\}_t.$$

It can also be "stretched" by a homotopy to

$$(\mathcal{E}_1, F_1) = \{(\mathcal{E}_{0,h(t)}, F_{0,h(t)})\}_t,$$

for any increasing bijective function $h : [1,\infty) \to [1,\infty)$.

Definition 3.20. An asymptotic Kasparov G-(A, B)-module $(\mathcal{E}, F)$ is said to be *degenerate* if for all $a \in A$ and $g \in G$: $(F - F^*)\varphi(a) = 0$, $[F, \varphi(a)] = 0$, $(g(F) - F)\varphi(a) = 0$, and $\varphi(a)\,(F^2 - 1)\,\varphi(a)^* \geq 0$, modulo $\mathcal{J}(\mathcal{E})$.

Remark 3.21. We want to comment on the definition of degenerate elements. The first three conditions are identical with the ones for a degenerate Kasparov module, but in (aKm3) we require positivity modulo $\mathcal{J}(\mathcal{E})$. In this way, for example, the generator of $KE(\mathbb{C}, \mathbb{C})$ will be described by $\mathcal{C}(\mathcal{H}L)/\mathcal{J}(\mathcal{H}L)$, which corresponds to the Fredholm index as invariant. This result is required by the dimension axiom that any homology theory has to satisfy.

Lemma 3.22. *If $(\mathcal{E}, F)$ is degenerate, then it is homotopic to the* 0*-module* $(0, 0)$.

Proof. The pair $(C_0([0,1)) \otimes \mathcal{E}, 1 \otimes F)$, with A acting as $1 \otimes \varphi$, is a degenerate asymptotic Kasparov (A, BI)-module, which gives a homotopy between $(\mathcal{E}, F)$ and $(0, 0)$. □

Definition 3.23. Given $(\mathcal{E}, F)$ and $(\mathcal{E}, F')$ in $ke_G(A, B)$, we say that F' is a "*small perturbation*" of F if $(F - F')\,\varphi(a) \in \mathcal{J}(\mathcal{E})$, for all $a \in A$.

Lemma 3.24. *Consider $(\mathcal{E}, F)$ in $ke_G(A, B)$, and F' a "small perturbation" of F. Then $(\mathcal{E}, F)$ and $(\mathcal{E}, F')$ are operatorially homotopic.*

Proof. Indeed, the straight line segment between F and F' is an operator homotopy:

$$\boldsymbol{F} = \{sF + (1-s)F'\}_{s\in[0,1]}.$$

We note that it is the same proof as in KK-theory for "compact perturbations" [5, Def. 17.2.4]. □

Corollary 3.25. *Any $(\mathcal{E}, F) \in ke_G(A, B)$ is homotopic to $(\mathcal{E}, (F + F^*)/2)$.*

Proof. $(F + F^*)/2$ is a "small perturbation" of F. □

From the corollary above it follows that (aKm1) can be strengthened: in Definitions 3.6 and 3.15 we could consider only self-adjoint operators F. Other changes are possible too.

A less trivial example of homotopy is provided by the next result (compare with [34, Lemma 11]). Despite the simplicity of its proof, it will be very useful when we shall analyze in depth the product in KE-theory.

Lemma 3.26. *Let $\mathcal{E}$ be a continuous field of G-(A, B)-modules. Consider two asymptotic Kasparov G-(A, B)-modules $(\mathcal{E}, F)$, $(\mathcal{E}, F') \in ke_G(A, B)$, such that $\varphi(a)\,[F, F']\,\varphi(a)^* \geq 0$, modulo $\mathfrak{C}(\mathcal{E}) + \mathfrak{J}(\mathcal{E})$, for all $a \in A$. Then $(\mathcal{E}, F)$ and $(\mathcal{E}, F')$ are (operatorially) homotopic.*

Proof. Put $F_s = \cos(s\pi/2)F + \sin(s\pi/2)F'$, for $s \in [0, 1]$. Then the family $\{(\mathcal{E}, F_s)\}_s$ realizes the required homotopy. □

Definition 3.27. The set $KE_G(A, B)$ is defined as the quotient of $ke_G(A, B)$ by the equivalence relation generated by homotopy. (We shall omit G in the non-equivariant case.) Given $x = (\mathcal{E}, F) \in ke_G(A, B)$, its class in $KE_G(A, B)$ will be denoted by $[\,x\,]$. The *addition* of two asymptotic Kasparov G-(A, B)-modules $(\mathcal{E}_1, F_1)$ and $(\mathcal{E}_2, F_2)$ is defined by $(\mathcal{E}_1, F_1) + (\mathcal{E}_2, F_2) = (\mathcal{E}_1 \oplus \mathcal{E}_2, F_1 \oplus F_2) \in ke_G(A, B)$.

Theorem 3.28. *With the notation of the previous definition, $KE_G(A, B)$ is an abelian group.*

Proof. The argument is similar to the one for KK-theory (see [34, Prop. 4]). The inverse of $(\mathcal{E}, F)$ is $(\mathcal{E}^{\mathrm{op}}, -UFU^*)$, where $\mathcal{E}^{\mathrm{op}}$ is $\mathcal{E}$ with the opposite grading, $U : \mathcal{E} \to \mathcal{E}^{\mathrm{op}}$ is the identity, and A acts on $\mathcal{E}^{\mathrm{op}}$ as $a(U\xi) = U((-1)^{\partial a} a\xi)$. □

Definition 3.29. For any group G, $1 = 1_{\mathbb{C}} \in KE_G(\mathbb{C}, \mathbb{C})$ is the class of the identity $*$-homomorphism $\psi = \mathrm{id} : \mathbb{C} \to \mathbb{C}$, *i.e.* the class of $(C_0(L), 0)$, with trivial action on $C_0(L)$. More generally, given a G-C^*-algebra A, the element $1_A \in KE_G(A, A)$ is the class of the identity $*$-homomorphism $\psi = \mathrm{id} : A \to A$ (as in Example 3.11), *i.e.* the class of $(AL, 0)$. Given an equivariant $*$–homomorphism $\psi : A \to B$ or more generally $\psi : A \to \mathcal{K} \otimes B$, its class in $KE_G(A, B)$ is denoted by $[\,\psi\,]$.

We end this subsection by defining in KE-theory (as it is the case in KK-theory and E-theory) the *higher order groups*. We recall that $\mathcal{C}_{+n}$ is the Clifford algebra of $\mathbb{R}^n$, *i.e.* the universal algebra with odd generators $\{e_1, \dots, e_n\}$ satisfying $e_i e_j + e_j e_i = +2\delta_{ij}$, for $1 \le i, j \le n$, $e_i{}^* = +e_i$, and $\|e_i\| = 1$. (The grading is the standard one, and the notation coincides with the one from [24]. The adjoint and the norm refer to the fact that $\mathcal{C}_{+n}$ can be given the structure of a C^*-algebra.)

Definition 3.30. $KE_G^n(A, B) = KE_G(A, B \otimes \mathcal{C}_{+n})$, for $n = 1, 2, \dots$.

3.3. Some examples. We further investigate, by means of examples, the significance of the axioms (aKm1)–(aKm4).

3.3.1. A non-equivariant example: *K*-theory.

Proposition 3.31. *If B is a graded separable C^*-algebra, then there are isomorphisms*

$$KE^*(\mathbb{C}, B) \simeq KK^*(\mathbb{C}, B), \quad \textit{for } * = 0, 1.$$

Consequently, when B is a trivially graded separable C^*-algebra, KE-theory recovers ordinary K-theory.

Proof. The second part follows from a well-known result in KK-theory, consequently the main point behind this proposition is the following: we shall show that the axioms of Kasparov modules can be successively modified, in the case when $A = \mathbb{C}$, to give the axioms (aKm1–3) of asymptotic Kasparov modules. This is done by constructing two intermediate abelian groups $\widetilde{KK}(\mathbb{C}, B)$ and $\widetilde{KE}(\mathbb{C}, B)$, together with group homomorphisms α, β, γ between the four groups under consideration, that can be depicted in the diagram:

$$KK(\mathbb{C}, B) \xrightarrow{\ \alpha\ } \widetilde{KK}(\mathbb{C}, B) \xrightarrow{\ \beta\ } \widetilde{KE}(\mathbb{C}, B) \xrightarrow{\ \gamma\ } KE(\mathbb{C}, B)\,. \tag{3.9}$$

(Note that $\widetilde{KK}(\mathbb{C}, B)$ has nothing to do with the group denoted by same symbol in [34, Def.2(8)].) The claimed isomorphism between the KK-theory group and the KE-theory group is deduced from the fact that α, β, and γ are proven to be isomorphisms.

$\widetilde{KK}(\mathbb{C}, B)$ is the abelian group (under direct sum) of homotopy classes of triples $(\mathcal{E}, \varphi, F)$, where $\mathcal{E}$ is a Hilbert B-module, admitting an action of $\mathbb{C}$ via a $*$-homomorphism $\varphi : \mathbb{C} \to \mathcal{L}(\mathcal{E})$, and $F \in \mathcal{L}(\mathcal{E})$ is an odd operator such that:

$$\varphi(1) = \mathrm{id},\ F = F^*,\ \text{and } (F^2 - 1/2) \ge 0,\ \text{modulo } \mathcal{K}(\mathcal{E}). \tag{3.10}$$

To construct the group homomorphism $\alpha : KK(\mathbb{C}, B) \to \widetilde{KK}(\mathbb{C}, B)$ we recall some of the standard simplifications of the axioms for a Kasparov module. Let $(\mathcal{E}, \varphi, F) \in kk(\mathbb{C}, B)$ be an arbitrary Kasparov module. By replacing F with

$F' = (F + F^*)/2$ we find a homotopic module $(\mathcal{E}, \varphi, F')$ with the operator self-adjoint. Next, consider the projection $\varphi(1) = P \in \mathcal{L}(\mathcal{E})$. The triple $(\mathcal{E}, \varphi, F')$ is operator homotopic to

$$(\mathcal{E}, \varphi, PF'P) = (P\mathcal{E}, \varphi, PF'P) + ((1-P)\mathcal{E}, \varphi, 0),$$

with the second summand being degenerate. Consequently, in the homotopy class of the initial Kasparov module we find a representative $(\widetilde{\mathcal{E}}, \widetilde{\varphi}, \widetilde{F}) = (P\mathcal{E}, \mathrm{id}, PF'P)$, with $1 \in \mathbb{C}$ acting as identity, $\widetilde{F}$ self-adjoint, and $\widetilde{F}^2 = 1 \geq 1/2$, modulo $\mathcal{K}(\widetilde{\mathcal{E}})$. This defines the group homomorphism α (all the changes above preserve homotopies and direct sums): $\alpha([(\mathcal{E}, \varphi, F)]) = [(\widetilde{\mathcal{E}}, \widetilde{\varphi}, \widetilde{F})]$. For the inverse map, let $\psi : \mathbb{R} \to \mathbb{R}$ be $\psi(x) = -1$ for $x \leq -1/\sqrt{2}$, $\psi(x) = \sqrt{2}\,x$ for $x \in (-1/\sqrt{2}, 1/\sqrt{2})$, and $\psi(x) = 1$ for $x \geq 1/\sqrt{2}$. Define

$$\alpha' : \widetilde{KK}(\mathbb{C}, B) \to KK(\mathbb{C}, B), \quad \alpha'([(\widetilde{\mathcal{E}}, \widetilde{\varphi}, \widetilde{F})]) = [(\widetilde{\mathcal{E}}, \widetilde{\varphi}, \psi(\widetilde{F}))].$$

The only non-trivial checking is $\psi(\widetilde{F})^2 - 1 = 2\widetilde{F}^2 - 1 \geq 0$ modulo $\mathcal{K}(\widetilde{\mathcal{E}})$. We observe that $[\psi(\widetilde{F}), \widetilde{F}] \geq 0$ and consequently both compositions $\alpha' \circ \alpha$ and $\alpha \circ \alpha'$ give results homotopic with the initial module. It follows that α is an isomorphism, with $\alpha^{-1} = \alpha'$.

Define next $\widetilde{KE}(\mathbb{C}, B)$ to be the abelian group (under direct sum) of homotopy classes of *asymptotic* Kasparov $(\mathbb{C}, B)$-modules $(\widehat{\mathcal{E}}, \widehat{F})$ satisfying the *extra conditions*:

$$\varphi(1) = \mathrm{id}, \; \widehat{F} = \widehat{F}^*, \; \text{ and } (\widehat{F}^2 - 1/2) \geq 0, \; \text{ modulo } \mathcal{C}(\widehat{\mathcal{E}}). \tag{3.11}$$

The map $\gamma : \widetilde{KE}(\mathbb{C}, B) \to KE(\mathbb{C}, B)$ is the forgetting map at the level of asymptotic Kasparov modules. To define the inverse γ', let $(\widehat{\mathcal{E}}, \widehat{F})$ be an arbitrary asymptotic Kasparov module. We can make the action of $\mathbb{C}$ unital as in KK-theory: there is a homotopy followed by a "small perturbation" connecting $(\widehat{\mathcal{E}}, \widehat{F})$ with $(\widehat{\mathcal{E}}', \widehat{F}'') = (P\widehat{\mathcal{E}}, P\widehat{F}P)$, where $P = \varphi(1)$. As we have already observed in Corollary 3.25, there is a homotopy from this last pair to another one $(\widehat{\mathcal{E}}', \widehat{F}')$, with $\widehat{F}'$ self-adjoint. Finally, (aKm3) implies that $(\widehat{F}_t{}')^2 - 1 \geq U_t + V_t$, with $U = \{U_t\}_t \in \mathcal{C}(\widehat{\mathcal{E}}')$ and $V = \{V_t\}_t \in \mathcal{J}(\widehat{\mathcal{E}}')$. Let T be such that $\|V_t\| < 1/2$, for all $t > T$. It follows that $(\widehat{F}_t{}')^2 - 1/2 \geq U_t$, for $t > T$. We define γ' via a "translation" (see Example 3.19):

$$\gamma' : KE(\mathbb{C}, B) \to \widetilde{KE}(\mathbb{C}, B), \quad \gamma' : \{(\widehat{\mathcal{E}}_t, \widehat{F}_t)\}_t \mapsto \{(\widehat{\mathcal{E}}'{}_{t+T}, \widehat{F}'{}_{t+T})\}_t.$$

All the operations used to define γ' preserve homotopies and direct sums, and consequently both $\gamma' \circ \gamma$ and $\gamma \circ \gamma'$ are the identity, and $\gamma^{-1} = \gamma'$.

Finally, define

$$\beta : \widetilde{KK}(\mathbb{C}, B) \to \widetilde{KE}(\mathbb{C}, B), \; (\widetilde{\mathcal{E}}, \widetilde{\varphi}, \widetilde{F}) \mapsto \{(\widetilde{\mathcal{E}}, \widetilde{F})\}_t \;\; \text{(constant family)}, \tag{3.12}$$

and

$$\beta' : \widetilde{KE}(\mathbb{C}, B) \to \widetilde{KK}(\mathbb{C}, B), \ (\widehat{\mathcal{E}}, \widehat{F}) = \{ (\widehat{\mathcal{E}_t}, \widehat{F_t}) \}_t \mapsto (\widehat{\mathcal{E}_1}, \widehat{\varphi}_1, \widehat{F_1})$$

(the "fiber" at $t = 1$).

The composition $\beta' \circ \beta = \mathrm{id}$ is obvious. Let now $(\widehat{\mathcal{E}}, \widehat{F}) = \{ (\widehat{\mathcal{E}_t}, \widehat{F_t}) \}_t$ be an element of $\widetilde{KE}(\mathbb{C}, B)$. There exists a homotopy $(\boldsymbol{\mathcal{E}}, \boldsymbol{F})$ between $(\widehat{\mathcal{E}}, \widehat{F})$ and $(\beta \circ \beta')((\widehat{\mathcal{E}}, \widehat{F})) = \{ (\widehat{\mathcal{E}_1}, \widehat{F_1}) \}_t$ given by explicit formulas:

$$\boldsymbol{\mathcal{E}}_{t,s} = \widehat{\mathcal{E}}_{s+(1-s)t}, \ \boldsymbol{F}_{t,s} = \widehat{F}_{s+(1-s)t}, \quad \text{for } s \in [0,1], t \in [1, \infty).$$

This proves that β is also an isomorphism, with $\beta^{-1} = \beta'$.

The claimed isomorphism is $\gamma \circ \beta \circ \alpha : KK(\mathbb{C}, B) \to KE(\mathbb{C}, B)$. Finally we get:

$$KK^1(\mathbb{C}, B) \overset{\text{def}}{\simeq} KK(\mathbb{C}, B \otimes \mathcal{C}_{+1}) \overset{\text{as above}}{\simeq} KE(\mathbb{C}, B \otimes \mathcal{C}_{+1}) \overset{\text{def}}{\simeq} KE^1(\mathbb{C}, B). \quad \square$$

Remark 3.32. The proof above implies the following KE-description of K-theory: homotopy classes of *constant families* $\{(\mathcal{E}, F)\}_t$, with 1 acting as identity on $\mathcal{E} = \mathcal{E}^{(0)} \oplus \mathcal{E}^{(1)}$, and $F = \begin{pmatrix} 0 & T^* \\ T & 0 \end{pmatrix}$ satisfying $T^*T - 1 \in \mathcal{K}(\mathcal{E}^{(0)})$ and $TT^* - 1 \in \mathcal{K}(\mathcal{E}^{(1)})$. Compare with [5, 17.5.4] and with the construction of the general map between KK- and KE-theory in subsection 5.1.

3.3.2. An equivariant example: $KE_\Gamma(\mathbb{C}, \mathbb{C})$, for Γ discrete. The next result is similar with Remark 2, after [27, 2.15], namely the dual of the Green–Julg theorem in KK-theory. The proof makes use of some results that are fully justified in Section 4.

Proposition 3.33. *Let Γ be a discrete group and A a separable Γ-C^*-algebra, then $KE^*_\Gamma(A, \mathbb{C}) = KE^*(C^*(\Gamma, A), \mathbb{C})$.*

Proof. We start by choosing a cycle $(\mathcal{E}, F) \in ke_\Gamma(A, \mathbb{C})$. Using the Stability Theorem 4.21 and the Stabilization Theorem [25, Thm. 2], we can assume that $\mathcal{E} = HL$, for a fixed Hilbert space H (see also 5.7). The field of Hilbert spaces $\mathcal{E}$ is endowed with an action $U : \Gamma \to \mathcal{L}(\mathcal{E})$ and an equivariant $*$-representation $\varphi : A \to \mathcal{L}(\mathcal{E})$. Thus the structure of $\mathcal{E}$ guarantees the existence of families of unitary representations $\{ U_t : \Gamma \to \mathcal{U}(H) \}_t$ and equivariant $*$-homorphisms $\{ \varphi_t : A \to \mathcal{L}(H) \}_t$. We denote $U_t(g)$ by $g_t \in \mathcal{U}(H)$, for $t \in [1, \infty)$. The equivariance of each φ_t implies that we actually have a family of covariant representations of the dynamical system (A, Γ). Consequently we can construct actions $\widetilde{\varphi_t} : C_c(\Gamma, A) \to \mathcal{L}(H)$ in the usual way: $\widetilde{\varphi_t}(f) = \sum_{g \in \Gamma} \varphi_t(a_g) \, g_t$, for $f = \sum_{g \in \Gamma} a_g \, \delta_g \in C_c(\Gamma, A)$. Note that $\{ \widetilde{\varphi_t}(f) \}_t$ is bounded and $*$-strong continuous for each $f \in C_c(\Gamma, A)$, because the families $\{\varphi_t(a_g)\}_t$ and $\{g_t\}_t$ have this property, for all $g \in \Gamma$ and $a_g \in A$. Using the norm density of $C_c(\Gamma, A)$ in $C^*(\Gamma, A)$ we obtain a representation $\widetilde{\varphi} : C^*(\Gamma, A) \to \mathcal{L}(HL)$. If we let $\widetilde{\mathcal{E}}$ denote the Hilbert module $\mathcal{E}$ endowed with the representation $\widetilde{\varphi}$ and if we let $\widetilde{F} = F$, then we claim that $(\widetilde{\mathcal{E}}, \widetilde{F})$ is in $ke(C^*(\Gamma, A), \mathbb{C})$. It is enough to check the axioms for $f = a_g\,\delta_g \in C_c(\Gamma, A)$.

- $\widetilde{F}$ *satisfies* (aKm1). $(\widetilde{F} - \widetilde{F}^*)\widetilde{\varphi}(f) = (F - F^*)\varphi(a_g)g \sim 0$, by (aKm1) for F.
- $\widetilde{F}$ *satisfies* (aKm2). Indeed:

$$\begin{aligned}[\widetilde{F}, \widetilde{\varphi}(f)] &= F\varphi(a_g)g - (-1)^{\partial a_g}\varphi(a_g)gF\\ &= [F, \varphi(a_g)]g + (-1)^{\partial a_g}\varphi(a_g)(F - g(F))g \sim 0,\\ &\qquad\qquad\qquad \text{by (aKm2) and (aKm4) for } F.\end{aligned}$$

- $\widetilde{F}$ *satisfies* (aKm3).

$$\begin{aligned}\widetilde{\varphi}(f)\,(\widetilde{F}^2 - 1)\,\widetilde{\varphi}(f)^* &= \varphi(a_g)\,g\,(F^2 - 1)\,g^{-1}\,\varphi(a_g^*)\\ &\sim \varphi(a_g)\,(F^2 - 1)\,\varphi(a_g^*) && \text{by (aKm4) for } F\\ &\geq 0,\ \text{modulo } \mathfrak{C}(\mathcal{E}) + \mathfrak{J}(\mathcal{E}), && \text{by (aKm3) for } F.\end{aligned}$$

The computation above shows also that a homotopy in $ke_\Gamma(A, C([0, 1]))$ is sent to a homotopy in $ke(C^*(\Gamma, A), C([0, 1]))$. We obtain in this way a group homorphism

$$KE_\Gamma(A, \mathbb{C}) \to KE(C^*(\Gamma, A), \mathbb{C}), \quad \boldsymbol{[}\,(\mathcal{E}, F)\,\boldsymbol{]} \mapsto \boldsymbol{[}\,(\widetilde{\mathcal{E}}, \widetilde{F})\,\boldsymbol{]}.$$

For the inverse group homomorphism, consider an asymptotic Kasparov module $(\widetilde{\mathcal{E}}, \widetilde{F}) \in ke(C^*(\Gamma, A), \mathbb{C})$, where $\widetilde{\mathcal{E}} = HL$ for a fixed Hilbert space H, with representation $\widetilde{\varphi} : C^*(\Gamma, A) \to \mathcal{L}(\widetilde{\mathcal{E}})$. The goal is to define an action of Γ on $\widetilde{\mathcal{E}}$. If A is unital and $\widetilde{\varphi}(1_A\delta_e)$ acts identically on $\widetilde{\mathcal{E}}$ we obtain immediately a representation of A on $\widetilde{\mathcal{E}}$ and a group action of Γ on $\widetilde{\mathcal{E}}$ by defining $\varphi(a) := \widetilde{\varphi}(a\delta_e)$, for $a \in A$, and $U_g := \widetilde{\varphi}(1_A\delta_g)$, for $g \in \Gamma$. Denote by $(\mathcal{E}, F)$ the pair $(\widetilde{\mathcal{E}}, \widetilde{F})$ with the actions of Γ and A obtained in this way. We claim that $(\mathcal{E}, F)$ belongs to $ke_\Gamma(A, \mathbb{C})$ and the only non-trivial axiom to be checked is (aKm4). We have:

$$\begin{aligned}(\,g(F) - F\,)\ \varphi(a) &= \big(U_g F U_{g^{-1}} - F\big)\,\varphi(a)\\ &= \big(\widetilde{\varphi}(1_A\delta_g)\,\widetilde{F}\widetilde{\varphi}(1_A\delta_{g^{-1}}) - \widetilde{F}\big)\,\widetilde{\varphi}(a\delta_e)\\ &= [\widetilde{\varphi}(1_A\delta_g), \widetilde{F}]\,\widetilde{\varphi}(a\delta_e) + \big(\widetilde{F}\widetilde{\varphi}(1_A\delta_e) - \widetilde{F}\big)\,\widetilde{\varphi}(a\delta_e)\\ &\sim 0,\ \text{by (aKm2) for } \widetilde{F} \text{ and identical action of } \widetilde{\varphi}(1_A\delta_e).\end{aligned}$$

To deal with the general case, apply Proposition 4.7 to first obtain a cycle $(\widetilde{\mathcal{E}}', \widetilde{F}')$ homotopic to $(\widetilde{\mathcal{E}}, \widetilde{F})$, with an essential action $\widetilde{\varphi}'$ of $C^*(\Gamma, A)$ on $\widetilde{\mathcal{E}}'$. Now extend the action $\widetilde{\varphi}'$ to $C^*(\Gamma, A^\sim)$ and reduce to the case discussed above. Finally we notice that the homotopies in $ke(C^*(\Gamma, A), C([0, 1]))$ are sent by the above constructions to homotopies in $ke_\Gamma(A, C([0,1]))$ and it is clear that we obtain in this way the inverse group homomorphism. Finally, using Bott periodicity, it follows that

$$\begin{aligned} KE^1_\Gamma(A,\mathbb{C}) &\overset{\text{Bott}}{\simeq} KE_\Gamma(A\otimes \mathcal{C}_{+1},\mathbb{C}) \\ &\overset{\text{above}}{\simeq} KE(C^*(\Gamma, A\otimes\mathcal{C}_{+1}),\mathbb{C}) \simeq KE(C^*(\Gamma,A)\otimes\mathcal{C}_{+1},\mathbb{C}) \\ &\overset{\text{Bott}}{\simeq} KE^1(C^*(\Gamma,A),\mathbb{C}). \end{aligned}$$ □

3.4. Functoriality properties. We discuss next some of the functoriality properties of the KE-groups. They are similar to the ones that the KK-theory groups satisfy.

(a) Given a $*$-homomorphism $\psi : A_1 \to A$, we obtain a map:

$$\psi^* : ke_G(A,B) \to ke_G(A_1,B)\,,\ (\mathcal{E},F)\mapsto(\psi^*\mathcal{E},F).$$

Here $\psi^*\mathcal{E}$ denotes the same Hilbert module $\mathcal{E}$, but with left action by A_1 given by the composition $\varphi\circ\psi : A_1 \to \mathcal{L}(\mathcal{E})$. We observe that ψ^* respects direct sums, and homotopy of asymptotic Kasparov modules. Consequently we get a well-defined map, denoted by the same symbol, at the level of groups:

$$\psi^* : KE_G(A,B) \to KE_G(A_1,B).$$

It is clear that for $*$-homomorphisms $A_2 \overset{\omega}{\longrightarrow} A_1 \overset{\psi}{\longrightarrow} A$ we have $(\psi\circ\omega)^* = \omega^*\circ\psi^*$.

(b) Let $\psi : B \to B_1$ be a $*$-homomorphism. Using $1\otimes\psi : BL \to B_1L$, we obtain a map:

$$\psi_* : ke_G(A,B) \to ke_G(A,B_1)\,,\quad (\mathcal{E},F)\mapsto(\mathcal{E}\otimes_{1\otimes\psi} B_1L, F\otimes_{1\otimes\psi} 1).$$

This map also respects direct sums, and homotopy of asymptotic Kasparov modules, and so gives a well-defined map: $\psi_* : KE_G(A,B) \to KE_G(A,B_1)$.

(c) For any G-C^*-algebra D there is a map:

$$\sigma_D : ke_G(A,B) \to ke_G(A\otimes D, B\otimes D),\quad (\mathcal{E},F)\mapsto(\mathcal{E}\otimes D, F\otimes 1). \tag{3.13}$$

It passes to quotients and gives a map $\sigma_D : KE_G(A,B) \to KE_G(A\otimes D, B\otimes D)$. Indeed, we verify first that the axioms for asymptotic Kasparov modules are satisfied.

- $F\otimes 1$ *satisfies* (aKm1).

$$(F\otimes 1 - (F\otimes 1)^*)\,(a\otimes d) = (F-F^*)a\otimes d \in \mathfrak{J}(\mathcal{E})\otimes D \subseteq \mathfrak{J}(\mathcal{E}\otimes D).$$

- $F\otimes 1$ *satisfies* (aKm2).

$$(F\otimes 1)\,(a\otimes d) - (-1)^{\partial a+\partial d}(a\otimes d)\,(F\otimes 1) = [F,a]\otimes d \in \mathfrak{J}(\mathcal{E})\otimes D \subseteq \mathfrak{J}(\mathcal{E}\otimes D).$$

- $F \otimes 1$ *satisfies* (aKm3).

$$(a \otimes d)\,(F^2 \otimes 1)(a^* \otimes d^*) = aF^2a^* \otimes dd^* \geq aa^* \otimes dd^*,$$
$$\text{modulo } \mathcal{C}(\mathcal{E}) \otimes D + \mathcal{J}(\mathcal{E} \otimes D) \subseteq \mathcal{C}(\mathcal{E} \otimes D) + \mathcal{J}(\mathcal{E} \otimes D).$$

The last inclusion follows from the isomorphism $\mathcal{K}(\mathcal{F} \otimes D) \simeq \mathcal{K}(\mathcal{F}) \otimes D$, where $\mathcal{F}$ is any Hilbert module.

- $F \otimes 1$ *satisfies* (aKm4).

$$(g(F \otimes 1) - F \otimes 1)\,(a \otimes d) = (g(F) - F)a \otimes d \in \mathcal{J}(\mathcal{E} \otimes D).$$

Finally, σ_D sends homotopic asymptotic Kasparov modules to homotopic asymptotic Kasparov modules, and this shows that σ_D is well defined at the level of groups.

Proposition 3.34 (Homotopy invariance). *The bifunctor $KE_G(A, B)$ is homotopy invariant in both variables:*

(a) *let $\psi_0, \psi_1 : A_1 \to A$ be homotopic $*$-homomorphisms; then, for any B, $\psi_0^* = \psi_1^* : KE_G(A_1, B) \to KE_G(A, B)$;*

(b) *let $\psi_0, \psi_1 : B \to B_1$ be homotopic $*$-homomorphisms; then, for any A, $\psi_{0*} = \psi_{1*} : KE_G(A, B) \to KE_G(A, B_1)$.*

Proof. Once again we may follow the same proof as in KK-theory.

(a) Let $\boldsymbol{\psi} : A_1 \to A[0,1]$ be a homotopy between ψ_0 and ψ_1. If $(\mathcal{E}, F) \in ke_G(A, B)$, then $\boldsymbol{\psi}^*\big(\sigma_{C([0,1])}((\mathcal{E}, F))\big) \in ke_G(A_1, B[0,1])$ gives a homotopy between $\psi_0^*((\mathcal{E}, F))$ and $\psi_1^*((\mathcal{E}, F))$.

(b) Let $\boldsymbol{\psi} : B \to B_1[0,1]$ be a homotopy between ψ_0 and ψ_1. Because ev_0 and ev_1 are essential $*$-homomorphisms, it follows that $\psi_{i*} = ev_{i*} \circ \boldsymbol{\psi}_*$, for $i = 0, 1$. Consequently, given $(\mathcal{E}, F) \in ke_G(A, B)$, $\boldsymbol{\psi}_*((\mathcal{E}, F))$ gives a homotopy between $\psi_{0*}((\mathcal{E}, F))$ and $\psi_{1*}((\mathcal{E}, F))$. □

3.5. Some technical results. We conclude this section with a technical result (namely Lemma 3.35), three definitions, and a "diagonalization" process, that will be used in the definition of the product in Section 4. Recall that any self-adjoint element x of a C^*-algebra can be written as a difference of two positive elements $x = x_+ - x_-$, with $x_+ x_- = x_- x_+ = 0$. The element x_- is called the negative part of x.

Lemma 3.35. *Let A and B be separable G-C^*-algebras. Given $(\mathcal{E}, F) \in ke_G(A, B)$, there exists a self-adjoint element $u \in \mathcal{C}(\mathcal{E})^{(0)}$ satisfying:*

(i) $[u, F] \in \mathcal{J}(\mathcal{E})$;

(ii) $[u, a] \in \mathcal{J}(\mathcal{E})$, *for all $a \in A$;*

(iii) $(1 - u^2)\big(a\,(F^2 - 1)\,a^*\big)_- \in \mathcal{J}(\mathcal{E})$, *for all $a \in A$; and*

(iv) $(g(u) - u) \in \mathcal{J}(\mathcal{E})$, *for all $g \in G$.*

Proof. Consider a dense subset $\{a_n\}_{n=1}^{\infty}$ in A, and an appropriate (see below) cover of $[1,\infty)$ by closed intervals $\{I_n\}_{n=0}^{\infty}$, of the form $I_n = [t_n, t_{n+2}]$, with $t_0 = 1$, and $\{t_n\}_n$ being a strictly increasing sequence with $\lim_{n\to\infty} t_n = \infty$. Choose a partition of unity $\{\mu_n\}_{n=0}^{\infty}$ in $C_0([1,\infty))$ subordinated to this cover. For each positive integer n, let $r_n : BL \to B(I_n)$ be the restriction $*$-homomorphism, and use it to define the restriction of $\mathcal{E}$ and F to I_n: $\mathcal{E}|_{I_n} = (r_n)_*(\mathcal{E})$, $F|_{I_n} = (r_n)_*(F)$.

Let $u_{0,0}$ be an arbitrary even self-adjoint element of $\mathcal{K}(\mathcal{E}|_{I_0})$. For each $n \geq 1$, construct a quasi-invariant approximate unit $\{u_{n,k}\}_{k=1}^{\infty}$ for $\mathcal{K}(\mathcal{E}|_{I_n})$, which is quasi-central for $F|_{I_n}$ and $A|_{I_n}$. There exists an index k_n such that

$$\| [u_{n,k_n}, F] \| < 1/n, \quad \| [u_{n,k_n}, a_m] \| < 1/n,$$
$$\| (1 - u_{n,k_n}^2) \left(a_m(F^2 - 1)a_m^*\right)_- \| < 1/n,$$

for $m = 1, 2, \ldots, n$. (For the third inequality, recall that (aKm3) implies

$$(a_m(F^2 - 1)a_m^*)_- \in \mathcal{C}(\mathcal{E}) + \mathcal{J}(\mathcal{E}),$$

with the $\mathcal{C}(\mathcal{E})$ part restricting to an element of $\mathcal{K}(\mathcal{E}|_{I_n})$, and the $\mathcal{J}(\mathcal{E})$ part having norm $< 1/2n$ by our initial choice of the partition $\{I_n\}_n$.)

Define: $u = \sum_{n=0}^{\infty} \mu_n \, u_{n,k_n} \in \mathcal{C}(\mathcal{E})$. We observe that (i) is satisfied, and that (ii) and (iii) hold true for all the elements of the dense subset $\{a_n\}_n$ of A. A density argument finishes the proof. To have (iv) satisfied, one uses quasi-invariance, and a similar argument after choosing a dense subset $\{g_n\}_{n=1}^{\infty}$ of G. □

Remark. The diagram (3.3) shows that the operators that appear in (i), (ii), and (iv) of the lemma above actually belong to $\mathcal{K}(\mathcal{E})$.

Definition 3.36. A *section of* $[1,\infty) \times [1,\infty)$ is any increasing continuous function $h : [1,\infty) \to [1,\infty)$, with $h(1) = 1$, $\lim_{t\to\infty} h(t) = \infty$, differentiable on $[1,\infty)$, except maybe for a countable set of points where it has finite one-sided derivatives. (The differentiability assumption is just a convenience.)

The result below will be used in the next section. Its elementary proof is left to the reader.

Lemma 3.37. *Given a countable family* $\{h_n\}_n$ *of sections of* $[1,\infty) \times [1,\infty)$*, one can find a suitable strictly increasing sequence of numbers* $\{1 = x_0, x_1, x_2, \ldots, x_n, \ldots\}$*, with* $\lim_{n\to\infty} x_n = \infty$*, and a section* h *satisfying the following condition: for each* n*,* $h \geq h_i$*, for all* $i = 1, 2, \ldots, n$*, over the closed interval* $[x_{n-1}, x_n]$.

Definition 3.38. Consider a Hilbert BLL-module $\mathcal{E}$. Given a section h of $[1,\infty) \times [1,\infty)$ as in Definition 3.36, consider the restriction $*$-homomorphism:

$$\mathrm{Res}_h : BLL \to BL, \quad f \mapsto f|_{\mathrm{graph}(h)}.$$

(The parameter $t \in L$ in BL is such that $(t, h(t)) \in \mathrm{graph}(h) \subset [1,\infty) \times [1,\infty)$.) The *restriction of* $\mathcal{E}$ *to the graph of* h is the Hilbert BL-module

$$\mathcal{E}_h := (\mathrm{Res}_h)_* \, (\mathcal{E}) = \mathcal{E} \otimes_{\mathrm{Res}_h} BL.$$

Consider now any operator $F \in \mathcal{L}(\mathcal{E})$. The *restriction of* F *to the graph of* h is the operator $F_h := (\mathrm{Res}_h)_*(F) = F \otimes_{\mathrm{Res}_h} 1 \in \mathcal{L}(\mathcal{E}_h)$.

Definition 3.39. Given a Hilbert BLL-module $\mathcal{E}$, we abuse notation and define

$$\mathcal{J}(\mathcal{E}) = \{\ F \in \mathcal{L}(\mathcal{E}) \mid \lim_{t_1,t_2\to\infty} \|F_{(t_1,t_2)}\| = 0\ \}.$$

Here (t_1, t_2) designates a point in $LL = [1,\infty)\times[1,\infty)$, and the limit is taken when both t_1 and t_2 approach infinity. Note that if $F \in \mathcal{J}(\mathcal{E})$ then $F_h \in \mathcal{J}(\mathcal{E}_h)$ for any section h of $[1,\infty)\times[1,\infty)$. We also define

$$\mathcal{C}(\mathcal{E}) = \{F \in \mathcal{L}(\mathcal{E}) \mid F\ f \in \mathcal{K}(\mathcal{E}),\ \text{for all } f \in C_0(LL)\}.$$

4. *KE*-theory: construction of the product

In this section the product is defined and various properties, including its associativity, are proved.

4.1. A motivational example. Let G be a locally compact σ-compact Hausdorff group, and A_1, A_2, B_1, B_2, D be separable G-C^*-algebras. The aim is to construct a certain bilinear map

$$KE_G(A_1, B_1 \otimes D) \otimes KE_G(D \otimes A_2, B_2) \to KE_G(A_1 \otimes A_2, B_1 \otimes B_2). \quad (4.1)$$

This will be the *product in* KE*-theory* (compare with the product in KK-theory and in E-theory), and its construction is based on the particular case when $B_1 = A_2 = \mathbb{C}$. The intuition, based on examples coming from K-homology and K-theory, is that the product should have the form:

$$\big(\,(\mathcal{E}_1, F_1)\,,\,(\mathcal{E}_2, F_2)\,\big) \mapsto \big(\,\mathcal{E}_1 \boxtimes \mathcal{E}_2\,,\ F_1 \boxtimes 1 + 1 \boxtimes F_2\,\big), \quad (4.2)$$

where $\boxtimes$ is a certain "tensor product". Kasparov [24, 26] succeeded to overcome the serious technical difficulties that arise in making sense of (4.2). We start our approach by providing a construction of the product (4.1) in the case when $D = \mathbb{C}$, known as *external product*. By doing so, we shall present a case when the formula (4.2) is actually correct, using ordinary tensor products. We shall also see the axioms (aKm1)–(aKm4) at work, and understand some of the difficulties involved in the general construction.

Example 4.1 (External product). Consider elements $(\mathcal{E}_1, F_1) \in ke_G(A_1, B_1)$ and $(\mathcal{E}_2, F_2) \in ke_G(A_2, B_2)$. Construct the G-$(A_1 \otimes A_2, B_1L \otimes B_2L)$-module $\mathcal{E} = \mathcal{E}_1 \otimes \mathcal{E}_2$ (external tensor product of Hilbert modules), and $F = F_1 \otimes 1 + 1 \otimes F_2 \in \mathcal{L}(\mathcal{E})$. The claim is that the restriction $(\mathrm{Res}_h)_*\,((\mathcal{E}, F))$ to the graph of any section h satisfies (aKm1)–(aKm4). Indeed, due to the inclusions $\mathcal{J}(\mathcal{E}_1) \otimes \mathcal{L}(\mathcal{E}_2) \subset \mathcal{J}(\mathcal{E})$ and

$\mathcal{L}(\mathcal{E}_1) \otimes \mathcal{J}(\mathcal{E}_2) \subset \mathcal{J}(\mathcal{E})$, it is easy to see that $(F - F^*)a$, $[F, a]$, $(g(F) - F)a \in \mathcal{J}(\mathcal{E})$, for all $a = a_1 \otimes a_2 \in A_1 \otimes A_2$ (recall Definition 3.39 for the meaning of $\mathcal{J}(\mathcal{E})$). We also have:

$$
\begin{aligned}
&(a_1 \otimes a_2)\big(F^2 - 1\big)(a_1 \otimes a_2)^* \\
&\qquad = (a_1 \otimes a_2)\big(F_1^2 \otimes 1 + 1 \otimes F_2^2 - 1\big)(a_1 \otimes a_2)^* \\
&\qquad = \begin{cases} a_1(F_1^2 - 1)a_1^* \otimes a_2 a_2^* + a_1 a_1^* \otimes a_2 F_2^2 a_2^* \ \geq 0, \\ \qquad\qquad \text{modulo } J_1 = \big(\mathcal{C}(\mathcal{E}_1) + \mathcal{J}(\mathcal{E}_1)\big) \otimes \mathcal{L}(\mathcal{E}_2), \text{ and} \\ a_1 F_1^2 a_1^* \otimes a_2 a_2^* + a_1 a_1^* \otimes a_2 (F_2^2 - 1) a_2^* \ \geq 0, \\ \qquad\qquad \text{modulo } J_2 = \mathcal{L}(\mathcal{E}_1) \otimes \big(\mathcal{C}(\mathcal{E}_2) + \mathcal{J}(\mathcal{E}_2)\big). \end{cases}
\end{aligned}
$$

Apply Lemma 4.2, with J_1, J_2 ideals in $\mathcal{L}(\mathcal{E}_1) \otimes \mathcal{L}(\mathcal{E}_2)$, to see that

$$(a_1 \otimes a_2)(F^2 - 1)(a_1 \otimes a_2)^* \geq 0, \quad \text{modulo } J_1 J_2 \subseteq \mathcal{C}(\mathcal{E}) + \mathcal{J}(\mathcal{E}).$$

There is only one thing left: in order to obtain a right $(B_1 \otimes B_2)L$-module (and not a $(B_1 \otimes B_2)LL$-module as $\mathcal{E}$ is) we restrict $\mathcal{E}$ and F to the graph of $h(t) = t$. It is clear that F_h satisfies (aKm1)–(aKm4). The class of $(\mathrm{Res}_h)_*\,\big((\mathcal{E}, F)\big)$ in $KE_G(A_1 \otimes A_2, B_1 \otimes B_2)$ is called the *external product* of $(\mathcal{E}_1, F_1)$ and $(\mathcal{E}_2, F_2)$. Compare with Example 2.4.

Conclusion. The external product of two asymptotic Kasparov G-modules $\{(\mathcal{E}_{1,t}, F_{1,t})\}_t$ and $\{(\mathcal{E}_{2,t}, F_{2,t})\}_t$ will be the asymptotic Kasparov G-module

$$\{(\mathcal{E}_{1,t} \otimes \mathcal{E}_{2,t}, F_{1,t} \otimes 1 + 1 \otimes F_{2,t})\}_t.$$

In the above example we used:

Lemma 4.2. *Let J_1 and J_2 be closed ideals of the C^*-algebra A. If $a \geq 0 \mod J_1$, and $a \geq 0 \mod J_2$, then $a \geq 0 \mod J_1 J_2 = J_1 \cap J_2$.*

4.2. Two-dimensional connections. As in Kasparov's KK-theory, the general product will involve internal tensor products of Hilbert modules. Given a Hilbert DL-module $\mathcal{E}_1$ and a Hilbert BL-module $\mathcal{E}_2$, their tensor product (internal or external) will be a continuous field of modules over $[1, \infty) \times [1, \infty)$ (to be precise, it will be a module over the algebra BLL or $(D \otimes B)LL$). We shall call such modules over $[1, \infty) \times [1, \infty)$, and corresponding families of operators, "two-dimensional". The ones indexed by $[1, \infty)$ are "one-dimensional". Our construction of the product will be based on an appropriate notion of connection, which is going to be a "two-dimensional" operator. The original definition of connection, on which ours is modelled, appears in [8, Def. A.1] and [34, Def. 8].

Definition 4.3. Assume that the following elements are given: a Hilbert DL-module $\mathcal{E}_1$, a Hilbert (D, BL)-module $\mathcal{E}_2$, and $F_2 \in \mathcal{L}(\mathcal{E}_2)$. Consider the Hilbert BLL-module $\mathcal{E} = \mathcal{E}_1 \otimes_{DL} \mathcal{E}_2 L$, with $\mathcal{E}_2 L = C_0(L) \otimes \mathcal{E}_2$. An operator $\underline{F} \in \mathcal{L}(\mathcal{E})$ is called an *F_2-connection for $\mathcal{E}_1$* if it has the same degree as F_2 and if it satisfies, for every *compactly supported* ξ in $\mathcal{E}_1$,

$$\left(T_\xi \, (1 \otimes F_2) - (-1)^{\partial\xi \cdot \partial F_2} \underline{F} \; T_\xi \right) \in \mathcal{J}(\mathcal{E}_2 L, \mathcal{E}),$$

and
$$\left((1 \otimes F_2) \; T_\xi^* - (-1)^{\partial\xi \cdot \partial F_2} T_\xi^* \; \underline{F} \right) \in \mathcal{J}(\mathcal{E}, \mathcal{E}_2 L).$$

Here $T_\xi \in \mathcal{L}(\mathcal{E}_2 L, \mathcal{E})$ is defined by $T_\xi(g \otimes \eta) = \xi \otimes_{DL} (g \otimes \eta)$, for $g \in C_0(L)$, and $\eta \in \mathcal{E}_2$. Moreover $\mathcal{J}(\mathcal{E}_2 L, \mathcal{E}) = \{T \in \mathcal{L}(\mathcal{E}_2 L, \mathcal{E}) \mid \lim_{t_1,t_2 \to \infty} \|T_{(t_1,t_2)}\| = 0\}$, and $\mathcal{J}(\mathcal{E}, \mathcal{E}_2 L)$ is defined similarly.

Remark 1. The above two conditions that a connection must satisfy are better remembered through the gradedly commutative modulo $\mathcal{J}$ diagrams

$$\begin{array}{ccc} \mathcal{E}_2 L & \xrightarrow{1 \otimes F_2} & \mathcal{E}_2 L \\ {\scriptstyle T_\xi}\downarrow & & \downarrow{\scriptstyle T_\xi} \\ \mathcal{E} & \xrightarrow[\underline{F}]{} & \mathcal{E} \end{array} \quad \text{and} \quad \begin{array}{ccc} \mathcal{E}_2 L & \xrightarrow{1 \otimes F_2} & \mathcal{E}_2 L \\ {\scriptstyle T_\xi^*}\uparrow & & \uparrow{\scriptstyle T_\xi^*} \\ \mathcal{E} & \xrightarrow[\underline{F}]{} & \mathcal{E} \end{array} \;. \tag{4.3}$$

Remark 2. Note that the graded commutativity from (4.3) is modulo $\mathcal{J}$, and not modulo $\mathcal{K}$, as in KK-theory. This is due to condition (aKm2). The role played by a connection is nevertheless the same as in KK-theory, namely to give a good replacement for the operator $1 \otimes F_2$ on $\mathcal{E} = \mathcal{E}_1 \otimes_{DL} \mathcal{E}_2 L$. And $\mathcal{E}$ being a BLL-module forces the connection to be a “two-dimensional” family. There are also situations where one can construct “one-dimensional” connections. For example, if $\mathcal{E}_1$ is a D-module, then for $\mathcal{E} = \mathcal{E}_1 \otimes_D \mathcal{E}_2$ we define a connection to be an operator $\underline{F}$ on $\mathcal{E}$ such that

$$\left(T_\xi \, F_2 - (-1)^{\partial\xi \cdot \partial F_2} \underline{F} \; T_\xi \right) \in \mathcal{J}(\mathcal{E}_2, \mathcal{E}) \;\text{ and }\; \left(F_2 \; T_\xi^* - (-1)^{\partial\xi \cdot \partial F_2} T_\xi^* \; \underline{F} \right) \in \mathcal{J}(\mathcal{E}, \mathcal{E}_2).$$

A connection like this is used, for example, in Proposition 4.7. The existence of such connections is justified as in the proof of the next result.

Proposition 4.4. *Consider the notation of the previous definition, with $\varphi_2 : D \to \mathcal{L}(\mathcal{E}_2)$ denoting the left action of D on $\mathcal{E}_2$. If F_2 satisfies, for all $d \in D$, $[F_2, \varphi_2(d)] \in \mathcal{J}(\mathcal{E}_2)$, then an F_2-connection exists for any countably generated $\mathcal{E}_1$.*

Proof. According with the Stabilization Theorem [25, Thm. 2], there exists an element $V \in \mathcal{L}(\mathcal{E}_1, \mathcal{H}_{(DL)^\sim})$ of degree 0 such that $V^* V = 1$. (This follows from the isomorphism $\mathcal{E}_1 \oplus \mathcal{H}_{(DL)^\sim} \simeq \mathcal{H}_{(DL)^\sim}$.) By construction, the unit of $(DL)^\sim$ acts as identity operator on $\mathcal{E}_2 L$. There is then an obvious isomorphism

$$W : \mathcal{H}_{(DL)^\sim} \otimes_{(DL)^\sim} \mathcal{E}_2 L \to \mathcal{H} \otimes \mathcal{E}_2 L,$$

given on elementary tensors by $W((v \otimes f) \otimes_{(DL)^\sim} \eta) = v \otimes f\eta$, for $v \in \mathcal{H}$, $f \in (DL)^\sim$, $\eta \in \mathcal{E}_2 L$. (In $\mathcal{H} \otimes \mathcal{E}_2 L$ the tensor product is an external one.) We obtain an F_2-connection $\underline{F}$ by imposing the commutativity of the diagram below:

$$\begin{array}{ccc}
\mathcal{E}_1 \otimes_{DL} \mathcal{E}_2 L & \xrightarrow{\ \underline{F}\ } & \mathcal{E}_1 \otimes_{DL} \mathcal{E}_2 L \\
{\scriptstyle V \otimes_{(DL)^\sim} 1}\big\downarrow & & \big\uparrow{\scriptstyle V^* \otimes_{(DL)^\sim} 1} \\
\mathcal{H}_{(DL)^\sim} \otimes_{(DL)^\sim} \mathcal{E}_2 L & & \mathcal{H}_{(DL)^\sim} \otimes_{(DL)^\sim} \mathcal{E}_2 L \\
{\scriptstyle W}\big\downarrow & & \big\uparrow{\scriptstyle W^{-1}} \\
\mathcal{H} \otimes \mathcal{E}_2 L & \xrightarrow[1 \otimes (1 \otimes F_2)]{} & \mathcal{H} \otimes \mathcal{E}_2 L
\end{array} ,$$

i.e.

$$\underline{F} = (V^* \otimes 1)\, W^{-1}\, (1 \otimes (1 \otimes F_2))\, W\, (V \otimes 1). \tag{4.4}$$

We shall verify only one of the conditions for an F_2-connection (the other one being similar). Let ξ be a compactly supported homogeneous section of $\mathcal{E}_1$, and $V(\xi) = \sum_{i=1}^{\infty} e_i \otimes f_i$, where $\{e_i\}_{i=1}^{\infty}$ is an orthonormal basis in $\mathcal{H}$, and $\sum_{i=1}^{\infty} f_i^* f_i < \infty$ in DL. We have of course $\partial\xi = \partial e_i + \partial f_i$, and $\text{supp}(f_i) \subseteq \text{supp}(\xi)$. A direct computation gives for any $\eta \in \mathcal{E}_2 L$:

$$\begin{aligned}
&W(V \otimes 1)\big(T_\xi\,(1 \otimes F_2) - (-1)^{\partial\xi\cdot\partial F_2}\underline{F}\,T_\xi\big)(\eta) \\
&= W(\,V(\xi) \otimes_{(DL)^\sim} (1 \otimes F_2)(\eta)\,) - (-1)^{\partial\xi\cdot\partial F_2}\,(1 \otimes (1 \otimes F_2))\,W\,(V(\xi) \otimes_{(DL)^\sim} \eta\,) \\
&= \sum_{i=1}^{\infty} e_i \otimes f_i(1 \otimes F_2)(\eta) - (-1)^{\partial f_i\cdot\partial F_2} \sum_{i=1}^{\infty} e_i \otimes (1 \otimes F_2)(f_i\eta) \\
&= \sum_{i=1}^{\infty} e_i \otimes [f_i, 1 \otimes F_2](\eta).
\end{aligned}$$

Consequently, it remains to show the convergence of the last infinite sum and that it belongs to $\mathfrak{J}(\mathcal{E}_2 L, \mathcal{E})$. This is accomplished by proving the convergence in *operator norm* of the partial sums $S_I = \sum_{i=1}^{I} e_i \otimes [f_i, 1 \otimes F_2]$, using the expression given after the second equal sign in the above computation. The desired result follows because the partial sums belong to $\mathfrak{J}(\mathcal{E}_2 L, \mathcal{H} \otimes \mathcal{E}_2 L)$. (The last observation uses the hypothesis on F_2 and on ξ.)

Fix $\varepsilon > 0$. We have:

$$
\begin{aligned}
&\left\|(S_{I+k}-S_I)(\eta)\right\| \\
&\qquad = \left\| \sum_{i=I+1}^{I+k} e_i \otimes f_i(1\otimes F_2)(\eta) - (-1)^{\partial f_i \cdot \partial F_2} \sum_{i=I+1}^{I+k} e_i \otimes (1\otimes F_2) f_i(\eta) \right\| \\
&\qquad \leq \underbrace{\left\| \sum_{i=I+1}^{I+k} e_i \otimes f_i(1\otimes F_2)(\eta) \right\|}_{\alpha} + \underbrace{\left\| \sum_{i=I+1}^{I+k} e_i \otimes (1\otimes F_2) f_i(\eta) \right\|}_{\beta}.
\end{aligned}
$$

Now:

$$
\alpha^2 = \left\| \langle (1\otimes F_2)(\eta), \Big(\sum_{i=I+1}^{I+k} f_i^* f_i \Big)(1\otimes F_2)(\eta) \rangle \right\| \leq \left\| \sum_{i=I+1}^{I+k} f_i^* f_i \right\| \cdot \|F_2\|^2 \cdot \|\eta\|^2 .
$$

Choose I such that $\left\| \sum_{i\in\Omega} f_i^* f_i \right\| \leq \varepsilon^2/(4\|F_2\|^2)$, for every finite set Ω which does not intersect $\{1, 2, \ldots, I\}$. Next:

$$
\begin{aligned}
\beta^2 &= \left\| \langle \eta, \sum_{i=I+1}^{I+k} f_i^* \, (1\otimes F_2)^* \, (1\otimes F_2) \, f_i \, (\eta) \rangle \right\| \\
&\leq \|F_2^* F_2\| \cdot \left\| \sum_{i=I+1}^{I+k} f_i^* f_i \right\| \cdot \|\eta\|^2 = \|F_2\|^2 \cdot \left\| \sum_{i=I+1}^{I+k} f_i^* f_i \right\| \cdot \|\eta\|^2 .
\end{aligned}
$$

For the chosen I, we obtain: $\alpha + \beta \leq (\varepsilon/2 + \varepsilon/2)\|\eta\|$. Consequently, $\|S_{I+k} - S_I\| \leq \varepsilon$, for all positive integers k. This proves the norm convergence of the double sum and the proposition. □

The next result gathers some useful properties of connections (compare with [34, Prop. 9]). The same notation as in Definition 4.3 is used.

Proposition 4.5. (i) *Let $\underline{F}$ be an F_2-connection for $\mathcal{E}_1$, and $\underline{F}'$ be an F_2'-connection for $\mathcal{E}_1$. Then $(\underline{F} + \underline{F}')$ is an $(F_2 + F_2')$-connection for $\mathcal{E}_1$, and $(\underline{F}\,\underline{F}')$ is an $(F_2\, F_2')$-connection for $\mathcal{E}_1$.*

(ii) *The linear space of 0-connections for $\mathcal{E}_1$ is*

$$
\left\{ \underline{F} \in \mathcal{L}(\mathcal{E}) \mid (K \otimes_{DL} 1)\underline{F},\ \underline{F}(K \otimes_{DL} 1) \in \mathcal{J}(\mathcal{E}),\ \textit{for all } K \in \mathcal{K}(\mathcal{E}_1) \right\}.
$$

Proof. Both (i) and (ii) follow immediately from the definition of connection. □

Lemma 4.6. *Consider the notation of Definition 4.3 and assume that a separable set $K \subset \mathcal{C}(\mathcal{E}_1)$ is given. Then there exists a section h_{00} of $[1, \infty) \times [1, \infty)$ such that for any other section $h \geq h_{00}$ the following holds:*

$$
(\mathrm{Res}_h)_* \left([\, k \otimes_{DL} 1, \underline{F}\,] \right) \in \mathcal{J}\left((\mathrm{Res}_h)_* \left(\mathcal{E} \right) \right), \ \textit{for all } k \in K.
$$

Proof. Choose a dense subset $\{k_n\}_{n=1}^{\infty}$ of K. Assume that one is able to find for each k_n a section h_n such that $(\mathrm{Res}_h)_*([k_n \otimes_{DL} 1, \underline{F}]) \in \mathcal{J}((\mathrm{Res}_h)_*(\mathcal{E}))$, for any $h \geq h_n$. Apply the diagonalization process described in Lemma 3.37 to obtain a section h_{00} which makes the conclusion true for all k_n's. A density argument shows that the result holds for all $k \in K$.

Consequently it is enough to construct a section that works for a single element $k \in K$. As in the proof of (4.12) in the Technical Theorem (subsection 4.5), one uses a partition of unity for L, an approximation of $k \otimes_{DL} 1$ by finite sums $\sum_i T_{\xi_i} T^*_{\eta_i}$, with $\xi_i, \eta_i \in \mathcal{E}_1$, and the properties of connections that $\underline{F}$ satisfies. □

We conclude this section about connections with a result that corresponds to [27, Lemma 2.8]. Its justification is almost identical to the one in [27] (see also [5, 18.3.6]), with the observation that in our case the connection $\widetilde{F}$ used on $\widetilde{\mathcal{E}} = Z(A, A^{\sim}) \otimes_{A^{\sim}[0,1]} \mathcal{E}[0,1]$ is of the type mentioned in Remark 2 after Definition 4.3.

Proposition 4.7. *Let A be σ-unital. For any asymptotic Kasparov module $(\mathcal{E}, F) \in ke_G(A, B)$, there exists a homotopy equivalent asymptotic Kasparov module $(\mathcal{E}', F')$, where $\mathcal{E}' = A \otimes_{\varphi} \mathcal{E} = \overline{\varphi(A)\mathcal{E}}$.*

4.3. Construction of the product. We are now ready to give the construction of the product (4.1) in the case when $B_1 = A_2 = \mathbb{C}$. Before stating the main theorem we present an overview of the proof.

Overview 4.8. Consider two asymptotic Kasparov modules $(\mathcal{E}_1, F_1) \in ke_G(A, D)$ and $(\mathcal{E}_2, F_2) \in ke_G(D, B)$. Their product, which is an element in $ke_G(A, B)$, is obtained by performing the following sequence of steps.

Step 1. Find a self-adjoint $u \in \mathcal{C}(\mathcal{E}_1)^{(0)}$ such that:

(1) $[u, F_1] \in \mathcal{J}(\mathcal{E}_1)$,

(2) $[u, a] \in \mathcal{J}(\mathcal{E}_1)$, for all $a \in A$,

(3) $(1 - u^2)\big(a\,(F_1^2 - 1)\,a^*\big)_- \in \mathcal{J}(\mathcal{E}_1)$, for all $a \in A$,

(4) $(g(u) - u) \in \mathcal{J}(\mathcal{E}_1)$, for all $g \in G$.

Step 2. Define $\mathcal{E} = \mathcal{E}_1 \otimes_{DL} \mathcal{E}_2 L$. Find $\underline{F} = \underline{F}^*$ an F_2-connection for $\mathcal{E}_1$, and define $F = F_1 \otimes_{DL} 1 + (u \otimes_{DL} 1)\,\underline{F}$. (The self-adjointness of $\underline{F}$ is just a convenience.)

Step 3. Choose a section h_{00} of $[1,\infty) \times [1,\infty)$ such that the restrictions of the following operators to the graph of any other section $h \geq h_{00}$ are in $\mathcal{J}((\mathrm{Res}_h)_*(\mathcal{E}))$:

(5) $[u \otimes_{DL} 1, \underline{F}]$,

(6) $[u F_1 \otimes_{DL} 1, \underline{F}]$,

(7) $[u\,a \otimes_{DL} 1, \underline{F}]$, for all $a \in A$.

Step 4. Find $h_0 \geq h_{00}$ such that the restriction to the graph of any $h \geq h_0$ of

(8) $(u \otimes_{DL} 1)\big(\underline{F}^2 - 1\big)(u \otimes_{DL} 1)$ is positive modulo $\mathcal{C}((\mathrm{Res}_h)_*(\mathcal{E})) + \mathcal{J}((\mathrm{Res}_h)_*(\mathcal{E}))$, and of

(9) $(u \otimes_{DL} 1)\big(g(\underline{F}) - \underline{F}\big)$ is in $\mathcal{J}((\mathrm{Res}_h)_*(\mathcal{E}))$, for all $g \in G$.

Once a triple $(u, \underline{F}, h_0)$ satisfying (1)–(9) is constructed, the conclusion is that the restriction of $(\mathcal{E}, F)$ to the graph of any $h \geq h_0$ gives an asymptotic Kasparov G-(A, B)-module $(\mathcal{E}_h, F_h)$, that we call a *product of* $(\mathcal{E}_1, F_1)$ *by* $(\mathcal{E}_2, F_2)$:

$$\begin{aligned} \mathcal{E}_h &= (\mathrm{Res}_h)_*(\mathcal{E}) = (\mathrm{Res}_h)_*(\mathcal{E}_1 \otimes_{DL} \mathcal{E}_2 L), \\ F_h &= (\mathrm{Res}_h)_*(F) = (\mathrm{Res}_h)_*(F_1 \otimes_{DL} 1 + (u \otimes_{DL} 1)\underline{F}) \\ &= \widetilde{F_1 \otimes_{D,h} 1} + \widetilde{1 \otimes_{D,h} F_2}. \end{aligned} \tag{4.5}$$

The notation

$$\widetilde{F_1 \otimes_{D,h} 1} = (\mathrm{Res}_h)_*(F_1 \otimes_{DL} 1), \quad \text{and} \quad \widetilde{1 \otimes_{D,h} F_2} = (\mathrm{Res}_h)_*((u \otimes_{DL} 1)\underline{F})$$

is suggested by the form of the product in the external product case. Note that in terms of families (4.5) reads:

$$(\mathcal{E}_h, F_h) = \big\{ \big(\mathcal{E}_{1,t} \otimes_D \mathcal{E}_{2,h(t)}, F_{1,t} \otimes_D 1 + (u_t \otimes_D 1)\underline{F}_{(t,h(t))} \big) \big\}_{t \in [1,\infty)}. \tag{4.6}$$

Remark 4.9. We do not have an axiomatic definition of the product as in [34, Def. 10], [8, Thm. A.3], so the situation is more like in E-theory.

The following theorem guarantees that Steps 1–4 of Overview 4.8 can be performed. Its proof will be given in subsection 4.5.

Theorem 4.10 (Technical Theorem). *Let G be a locally compact σ-compact Hausdorff group, and let A, B, and D be separable graded G-C^*-algebras. Consider two asymptotic Kasparov modules $(\mathcal{E}_1, F_1) \in ke_G(A, D)$ and $(\mathcal{E}_2, F_2) \in ke_G(D, B)$. There exists a triple $(u, \underline{F}, h_0)$, with u a self-adjoint element of $\mathcal{C}^{(0)}(\mathcal{E}_1)$, $\underline{F}$ an F_2-connection for $\mathcal{E}_1$, and h_0 a section of $[1,\infty) \times [1,\infty)$, as in Overview 4.8, such that for any other section $h \geq h_0$*

$$(\mathcal{E}_h, F_h) = (\mathrm{Res}_h)_*\big(\mathcal{E}_1 \otimes_{DL} \mathcal{E}_2 L, F_1 \otimes_{DL} 1 + (u \otimes_{DL} 1)\underline{F} \big)$$

is an asymptotic Kasparov G-(A, B)-module.

We can now give the definition of the *product map in KE-theory* in the form of:

Theorem 4.11. *With the notation of the above theorem, the map*

$$((\mathcal{E}_1, F_1), (\mathcal{E}_2, F_2)) \mapsto (\mathcal{E}_{h_0}, F_{h_0})$$

passes to quotients and defines the product map*:*

$$KE_G(A, D) \otimes KE_G(D, B) \xrightarrow{\sharp_D} KE_G(A, B), \quad (x, y) \mapsto x \sharp_D y. \tag{4.7}$$

Proof. The notation is that of Overview 4.8.

(I) *Independence of h.* For any two $h_1, h_2 \geq h_0$ we have a homotopy between $(\mathcal{E}_{h_1}, F_{h_1})$ and $(\mathcal{E}_{h_2}, F_{h_2})$ given by the explicit formula:

$$\left\{ \left((\mathrm{Res}_{sh_1+(1-s)h_2})_* \left(\mathcal{E} \right), (\mathrm{Res}_{sh_1+(1-s)h_2})_* \left(F \right) \right) \right\}_{s\in[0,1]}.$$

(II) *Independence of the triple* $(u, \underline{F}, h_0)$.

(a) As above, one can construct a homotopy between two asymptotic Kasparov modules corresponding to different h_0's satisfying Step 4. This proves the independence of h_0.

(b) In order to show independence of $\underline{F}$, consider two F_2-connections $\underline{F}$ and $\underline{F}'$ and the same u. Now $(\underline{F} - \underline{F}')$ is a 0-connection, and Proposition 4.5(ii) implies that there exists a section h such that $(\mathrm{Res}_h)_* \left((u \otimes_{DL} 1)\, \underline{F} - (u \otimes_{DL} 1)\, \underline{F}' \right) \in \mathfrak{J}(\mathcal{E}_h)$. Further modify h such that both F_h and F'_h give elements in $ke_G(A, B)$. Lemma 3.24 applies and gives a homotopy between F_h and F'_h.

(c) To show independence of u, choose two different such elements u and u', both satisfying the requirements of Step 1, same $\underline{F}$, and an h that works for both choices. We obtain a homotopy by the formula:

$$\left\{(\mathrm{Res}_h)_* \left(F_1 \otimes_{DL} 1 + \left(s(u \otimes_{DL} 1) + (1-s)(u' \otimes_{DL} 1)\right) \underline{F} \right) \right\}_{s\in[0,1]}.$$

Combining (a), (b), and (c) above we get that the homotopy class of the element $(\mathcal{E}_h, F_h)$ constructed in Theorem 4.10 does not depend on the triple $(u, \underline{F}, h_0)$.

(III) *Passage to quotients.* Our goal is to show that the homotopy class of the product does not depend on the representatives in the class of $(\mathcal{E}_1, F_1)$ and $(\mathcal{E}_2, F_2)$, respectively. Consider $(\boldsymbol{\mathcal{E}_1}, \boldsymbol{F_1}) \in ke_G(A, D[0,1])$ a homotopy between $(\mathcal{E}_{1,0}, F_{1,0})$ and $(\mathcal{E}_{1,1}, F_{1,1})$. A product $(\boldsymbol{\mathcal{E}}, \boldsymbol{F})$ of $(\boldsymbol{\mathcal{E}_1}, \boldsymbol{F_1})$ by $\sigma_{C[0,1]}((\mathcal{E}_2, F_2))$ represents a homotopy between the product of $(\mathcal{E}_{1,0}, F_{1,0})$ by $(\mathcal{E}_2, F_2)$ and a product of $(\mathcal{E}_{1,1}, F_{1,1})$ by $(\mathcal{E}_2, F_2)$. Consider now $(\boldsymbol{\mathcal{E}_2}, \boldsymbol{F_2}) \in ke_G(D, B[0,1])$. A product $(\boldsymbol{\mathcal{E}}, \boldsymbol{F})$ of $(\mathcal{E}_1, F_1)$ by $(\boldsymbol{\mathcal{E}_2}, \boldsymbol{F_2})$ represents a homotopy between the product of $(\mathcal{E}_1, F_1)$ by $(\mathcal{E}_{2,0}, F_{2,0})$ and a product of $(\mathcal{E}_1, F_1)$ by $(\mathcal{E}_{2,1}, F_{2,1})$ We obtain that the map from the statement does pass to a well-defined map at the level of KE-theory groups. □

Using Theorem 4.11 and the map σ, we are now in position to construct the general product (4.1) mentioned at the very beginning of this section (compare with the definition in KK-theory [27, Def. 2.12]).

Definition 4.12. Let G be a group, and let A_1, A_2, B_1, B_2, D be G-C^*-algebras. The *general product in KE-theory* is the map

$$KE_G(A_1, B_1 \otimes D) \otimes KE_G(D \otimes A_2, B_2) \to KE_G(A_1 \otimes A_2, B_1 \otimes B_2), \quad (4.1)$$

defined by:

$$x \sharp_D y = \sigma_{A_2}(x) \sharp_{B_1 \otimes D \otimes A_2} \sigma_{B_1}(y) . \tag{4.8}$$

The *external product* corresponds to $D = \mathbb{C}$.

This subsection is concluded by showing that, in the case of the external product, the asymptotic Kasparov module constructed in Example 4.1 is homotopic with the one given by the general product of Definition 4.12. This will show that Example 4.1 really represents the construction of a product, and not merely of some other asymptotic Kasparov module. Let $x \in KE_G(A_1, B_1)$ be represented by $(\mathcal{E}_1, F_1)$, and $y \in KE_G(A_2, B_2)$ be represented by $(\mathcal{E}_2, F_2)$. (One may need to apply Proposition 4.7 to make sure that $\mathcal{E}_2 = A_2\mathcal{E}_2$.) According with Definition 4.12, $x \sharp_{\mathbb{C}} y = \sigma_{A_2}(x) \sharp_{B_1 \otimes A_2} \sigma_{B_1}(y)$. Now, $\sigma_{A_2}(x)$ is represented by $(\mathcal{E}_1 \otimes A_2, F_1 \otimes 1)$, and $\sigma_{B_1}(y)$ is represented by $(B_1 \otimes \mathcal{E}_2, 1 \otimes F_2)$. To obtain a module that represents the product we follow the steps given in Overview 4.8. The element u of *Step 1* can be chosen of the form $\{\widetilde{u_t} \otimes \alpha_{h(t)}\}_t$, with $\{\widetilde{u_t}\}_t$ a q.i.q.c.a.u. for $\mathcal{K}(\mathcal{E}_1)$, $\{\alpha_t\}_t$ an a.u. for A_2, and h an arbitrary section. In *Step 2* we identify $\mathcal{E}$ with $\mathcal{E}_1 \otimes A_2\mathcal{E}_2 = \mathcal{E}_1 \otimes \mathcal{E}_2$, which is a Hilbert $(B_1L \otimes B_2L)$-module, acted on the left by $A_1 \otimes A_2$. As two-dimensional connection we can take the constant field $\{1 \otimes F_{2,t_2}\}_{(t_1,t_2) \in LL}$. With the choices and identifications made so far, any section h_{00} will do in *Step 3*. In *Step 4* choose a section h that makes the restriction to its graph an asymptotic Kasparov module:

$$\left(\mathcal{E}_h, F_h\right) = \left\{ \left(\mathcal{E}_{1,t} \otimes \mathcal{E}_{2,h(t)}, F_{1,t} \otimes 1 + \widetilde{u_t} \otimes \alpha_{h(t)} F_{2,h(t)}\right) \right\}_t \in ke_G(A_1 \otimes A_2, B_1 \otimes B_2).$$

Lemma 3.26 applies and gives a homotopy between $(\mathcal{E}_h, F_h)$ and

$$\left(\mathcal{E}'_h, F'_h\right) = \left\{ \left(\mathcal{E}_{1,t} \otimes \mathcal{E}_{2,h(t)}, F_{1,t} \otimes 1 + 1 \otimes F_{2,h(t)}\right) \right\}_t \in ke_G(A_1 \otimes A_2, B_1 \otimes B_2).$$

Finally we notice that $\{(\mathcal{E}_{2,h(t)}, F_{2,h(t)})\}_t$ is just another representative of y, obtained by "stretching" (Example 3.19) the initial representative $(\mathcal{E}_2, F_2) = \{(\mathcal{E}_{2,t}, F_{2,t})\}_t)$. Consequently, using two homotopies, we succeeded to show that the product $\sharp_{\mathbb{C}}$ of Definition 4.12 is what we called external tensor product in Example 4.1.

4.4. Properties of the product. The properties of the product in KE-theory are very similar to the ones that the Kasparov product satisfies in KK-theory. For our first result compare with [27, Thm. 2.14].

Theorem 4.13. *The product $\sharp$ satisfies the following functoriality properties:*

(i) *it is bilinear;*

(ii) *it is contravariant in A,* i.e. $f^*(x) \sharp_D y = f^*(x \sharp_D y)$, *for any $*$-homomorphism $f : A_1 \to A$, $x \in KE_G(A, D)$, and $y \in KE_G(D, B)$;*

(iii) *it is covariant in B,* i.e. $g_*(x \sharp_D y) = x \sharp_D g_*(y)$, *for any $*$-homomorphism $g : B \to B_1$, $x \in KE_G(A, D)$, and $y \in KE_G(D, B)$;*

(iv) *it is functorial in* D, i.e. $f_*(x) \sharp_{D_2} y = x \sharp_{D_1} f^*(y)$, *for any* $*$-*homomorphism* $f : D_1 \to D_2$, $x \in KE_G(A, D_1)$, *and* $y \in KE_G(D_2, B)$;

(v) $\sigma_{D_1}(x \sharp_D y) = \sigma_{D_1}(x) \sharp_{D\otimes D_1} \sigma_{D_1}(y)$, *for* $x \in KE_G(A,D)$ *and* $y \in KE_G(D,B)$.

Proof. (i) Let

$$x = [(\mathcal{E}_1, F_1)] \in KE_G(A, D),$$
$$y_1 = [(\mathcal{E}_2, F_2)], \quad y_2 = [(\mathcal{E}_2', F_2')] \in KE_G(D, B).$$

Then:

$$\begin{aligned} x \sharp_D y_1 &= [(\mathrm{Res}_{h_1})_*\big((\mathcal{E}_1 \otimes_{DL} \mathcal{E}_2 L, F_1 \otimes_{DL} 1 + (u \otimes_{DL} 1)\underline{F}\,)\big)],\\ x \sharp_D y_2 &= [(\mathrm{Res}_{h_2})_*\big((\mathcal{E}_1 \otimes_{DL} \mathcal{E}_2' L, F_1 \otimes_{DL} 1 + (u \otimes_{DL} 1)\underline{F}'\,)\big)],\\ y_1 + y_2 &= [(\mathcal{E}_2 \oplus \mathcal{E}_2', F_2 \oplus F_2')]. \end{aligned}$$

Let $h = \sup\{h_1, h_2\}$. Using $\mathcal{E}_1 \otimes_{DL} (\mathcal{E}_2 \oplus \mathcal{E}_2')L \simeq (\mathcal{E}_1 \otimes_{DL} \mathcal{E}_2 L) \oplus (\mathcal{E}_1 \otimes_{DL} \mathcal{E}_2' L)$, the definition of connection shows that $(\underline{F} \oplus \underline{F}')$ is an $(F_2 \oplus F_2')$-connection for $\mathcal{E}_1$. It is clear that:

$$\begin{aligned} &x \sharp_D y_1 + x \sharp_D y_2\\ &= [(\mathrm{Res}_h)_*\big((\mathcal{E}_1 \otimes_{DL} (\mathcal{E}_2 \oplus \mathcal{E}_2')L, F_1 \otimes_{DL} (1 \oplus 1) + (u \otimes_{DL} (1 \oplus 1))(\underline{F} \oplus \underline{F}'\,)\big)]\\ &= x \sharp_D (y_1 + y_2). \end{aligned}$$

The linearity in the first variable is simpler.

(ii–iv) A proof using the definition of the product can be given as for (i) above, but these properties are also a direct consequence of the associativity of the product (see Theorem 4.15 below) and of the following remark: $f^*(x) = [f] \sharp_A x$ and $g_*(y) = y \sharp_B [g]$.

(v) With $x = [(\mathcal{E}_1, F_1)]$ and $y = [(\mathcal{E}_2, F_2)]$, $\sigma_{D_1}(x \sharp_D y)$ is represented by the restriction of

$$\big((\mathcal{E}_1 \otimes_{DL} \mathcal{E}_2 L) \otimes D_1, (F_1 \otimes_{DL} 1) \otimes 1 + ((u \otimes_{DL} 1)\underline{F}) \otimes 1\big)$$

to the graph of a section h. Let $\mathcal{E}_1' = \mathcal{E}_1 \otimes D_1$, $\mathcal{E}_2' = \mathcal{E}_2 \otimes D_1$, $D' = D \otimes D_1$. The product $\sigma_{D_1}(x) \sharp_{D\otimes D_1} \sigma_{D_1}(y)$ is represented by the restriction of

$$\big(\mathcal{E}_1' \otimes_{D'L} \mathcal{E}_2' L, (F_1 \otimes 1) \otimes_{D'L} 1 + (\widetilde{u} \otimes_{D'L} 1)\underline{F}'\big).$$

Under the identification $\mathcal{E}_1' \otimes_{D'L} \mathcal{E}_2' L \simeq (\mathcal{E}_1 \otimes_{DL} \mathcal{E}_2 L) \otimes D_1$, we can take $\underline{F}' = \underline{F} \otimes 1$. Given any quasi-invariant approximate unit $\widetilde{d} = \{d_t\}_t$ for D_1, we can choose $\widetilde{u} = u \otimes \widetilde{d} \in \mathcal{C}^{(0)}(\mathcal{E}_1 \otimes D_1)$. Finally, after considering a common section for both products, Lemma 3.26 applies and gives a homotopy between the two representatives. □

Remark. In the proof of the next theorem the language of elementary calculus will be used again in order to "visualize" the construction of a double product in KE-theory. A "3D-cartesian coordinate system" is assumed, with LLL viewed as "octant" in this system. The quotation marks required by such imprecise, but suggestive we hope, terminology will be dropped.

Definition 4.14. A *3D-section* is a function $h : L \to LL$, $t \mapsto (h_2(t), h_3(t))$, with h_2 and h_3 ordinary sections.

Theorem 4.15 (Associativity of the product). *Let A, B, D, and E be G-C^*-algebras. Then, for any $x_1 \in KE_G(A, D)$, $x_2 \in KE_G(D, E)$, and $x_3 \in KE_G(E, B)$,*

$$(x_1 \sharp_D x_2) \sharp_E x_3 \;=\; x_1 \sharp_D (x_2 \sharp_E x_3) \,.$$

Proof. Assume that x_1, x_2, x_3 are represented by

$$(\mathcal{E}_1, F_1) \in ke_G(A, D), \quad (\mathcal{E}_2, F_2) \in ke_G(D, E), \quad (\mathcal{E}_3, F_3) \in ke_G(E, B),$$

respectively. We shall use the notation:

$$\mathcal{E}_{12} = \mathcal{E}_1 \otimes_{DL} \mathcal{E}_2 L, \quad \mathcal{E}_{23} = \mathcal{E}_2 \otimes_{EL} \mathcal{E}_3 L, \quad \mathcal{E} = \mathcal{E}_1 \otimes_{DL} \mathcal{E}_2 L \otimes_{ELL} \mathcal{E}_3 LL,$$
$$x_{12,3} = (x_1 \sharp_D x_2) \sharp_E x_3 \,, \quad x_{1,23} = x_1 \sharp_D (x_2 \sharp_E x_3) \,.$$

An inner product $(\xi \otimes_{DL} \eta \otimes_{ELL} \zeta) \in \mathcal{E}$ is abbreviated as $(\xi \otimes_D \eta \otimes_E \zeta)$, and similarly for operators on $\mathcal{E}$. In LLL, the first copy of L and the first coordinate t_1 correspond to $\mathcal{E}_1$, the second copy of L and the second coordinate t_2 correspond to $\mathcal{E}_2$, and the third copy of L and the third coordinate t_3 correspond to $\mathcal{E}_3$.

We first describe the product $x_{12,3}$. As explained in the previous subsection, $x_1 \sharp_D x_2$ is constructed from a triple $(u_1, \underline{F_{12}}, h_{12})$, with $u_1 \in \mathcal{C}(\mathcal{E}_1)$, $\underline{F_{12}}$ an F_2-connection for $\mathcal{E}_1$, and h_{12} a section in the (t_1, t_2)-plane. It is represented by

$$(\mathcal{E}_{12,h_{12}}, F_{12,h_{12}}) = (\mathrm{Res}_{h_{12}})_* \left((\mathcal{E}_{12}, F_{12}) \right),$$

where

$$F_{12} = F_1 \otimes_{DL} 1 + (u_1 \otimes_{DL} 1)\underline{F_{12}}.$$

The product $x_{12,3}$ is constructed from a triple $(u_{12,h_{12}}, \underline{F_{12,3}}, h_3)$, with $u_{12,h_{12}} \in \mathcal{C}(\mathcal{E}_{12,h_{12}})$, $\underline{F_{12,3}}$ an F_3-connection for $\mathcal{E}_{12,h_{12}}$, and h_3 a section in the "surface" $\Sigma_1 = \{(t_1, t_2, t_3) \in LLL \,|\, t_2 = h_{12}(t_1)\}$. It is represented by the restriction to the graph of h_3 of $\big(\mathcal{E}_{12,h_{12}} \otimes_{EL} \mathcal{E}_3 L, F_{12,h_{12}} \otimes_{EL} 1 + (u_{12,h_{12}} \otimes_{EL} 1)\underline{F_{12,3}}\big)$. There is a simpler way of describing a representative. Define the 3D-section $h(t) = (h_{12}(t), h_3(t))$. Consider the three-dimensional objects $\mathcal{E}$ and

$$F = F_1 \otimes_D 1 \otimes_E 1 + (u_1 \otimes_D 1 \otimes_E 1)(\underline{F_{12}} \otimes_E 1) + (u_{12} \otimes_E 1)\underline{F},$$

with u_1, $\underline{F_{12}}$ as before, $u_{12} \in \mathcal{C}(\mathcal{E}_{12})$, and $\underline{F}$ a three-dimensional F_3-connection for $\mathcal{E}_{12}$. (Such a three-dimensional connection is a straightforward generalization of our definition for two-dimensional connections. See (4.10) for one of the defining, commutative up to $\mathcal{J}$, diagrams.) The product is represented by the restriction of $(\mathcal{E}, F)$ to the graph of h.

Similarly, $x_2 \sharp_E x_3$ is constructed from a triple $(u_2, \underline{F_{23}}, h_{23})$, with $u_2 \in \mathcal{C}(\mathcal{E}_2)$, $\underline{F_{23}}$ an F_3-connection for $\mathcal{E}_2$, and h_{23} a section in the (t_2, t_3)-plane. It is represented by $(\mathcal{E}_{23,h_{23}}, F_{23,h_{23}}) = (\mathrm{Res}_{h_{23}})_* \big((\mathcal{E}_{23}, F_{23}) \big)$, where

$$F_{23} = F_2 \otimes_{EL} 1 + (u_2 \otimes_{EL} 1)\underline{F_{23}}.$$

The product $x_{1,23}$ is constructed from a triple $(u_1, \underline{F_{1,23}}, h'_3)$, with the same u_1 as before, $\underline{F_{1,23}}$ an $F_{23,h_{23}}$-connection for $\mathcal{E}_1$, and h'_3 a section in the "surface" $\Sigma_2 = \{(t_1, t_2, t_3) \in LLL \,|\, t_3 = h_{23}(t_2)\}$. Let h' be the 3D-section whose graph is given by the graph of h'_3. We can describe a representative for $x_{1,23}$ by the restriction to the graph of h' of

$$F' = F_1 \otimes_{DL} 1 + (u_1 \otimes_{DL} 1)\underline{F_{1,23}},$$

with $\underline{F_{1,23}}$ an F_{23}-connection for $\mathcal{E}_1$. The properties of connections given in Proposition 4.5 imply that we can take $\underline{F_{1,23}} = \underline{F_{12}} \otimes_E 1 + \underline{U_2}\,\underline{F'}$, where $\underline{U_2}$ is an $(u_2 \otimes_{EL} 1)$-connection for $\mathcal{E}_1$, and $\underline{F'}$ is an $\underline{F_{23}}$-connection for $\mathcal{E}_1$. The best way to see this choice for $\underline{F_{1,23}}$ is through the diagram below, which represents the first of the two diagrams (4.3) for the connections under discussion (the other one being constructed in a similar way):

$$\begin{array}{ccccc}
(\mathcal{E}_3 L)L & \xrightarrow{1\otimes(1\otimes F_3)} & (\mathcal{E}_3 L)L & & \\
{\scriptstyle f_1\otimes T_\eta}\downarrow & & \downarrow{\scriptstyle f_1\otimes T_\eta} & & \\
(\mathcal{E}_2 \otimes_{EL} \mathcal{E}_3 L)L & \xrightarrow{1\otimes \underline{F_{23}}} & (\mathcal{E}_2 \otimes_{EL} \mathcal{E}_3 L)L & \xrightarrow{1\otimes(u_2\otimes_{EL}1)} & (\mathcal{E}_2 \otimes_{EL} \mathcal{E}_3 L)L \\
{\scriptstyle T_\xi}\downarrow & & \downarrow{\scriptstyle T_\xi} & & \downarrow{\scriptstyle T_\xi} \\
\mathcal{E} & \xrightarrow[\underline{F'}]{} & \mathcal{E} & \xrightarrow[\underline{U_2}]{} & \mathcal{E}
\end{array} \tag{4.9}$$

(In the diagram: $f_1 \in C_0(L)$, $\eta \in \mathcal{E}_2$, $\xi \in \mathcal{E}_1$. We also have made the identification: $\mathcal{E}_1 \otimes_{DL} (\mathcal{E}_2 L \otimes_{ELL} \mathcal{E}_3 LL) \simeq \mathcal{E} \simeq (\mathcal{E}_1 \otimes_{DL} \mathcal{E}_2 L) \otimes_{ELL} \mathcal{E}_3 LL$.) The bottom squares of (4.9) show that $\underline{U_2}\,\underline{F'}$ is indeed a $(u_2 \otimes_{EL} 1)\underline{F_{23}}$-connection for $\mathcal{E}_1$. The left squares of (4.9) are nothing but an F_3-connection for $\mathcal{E}_{12}$:

$$\begin{array}{ccc}
(\mathcal{E}_3)LL & \xrightarrow{(1\otimes 1)\otimes F_3} & (\mathcal{E}_3)LL \\
{\scriptstyle T_{\xi\otimes_{DL}(f_1\otimes\eta)}}\downarrow & & \downarrow{\scriptstyle T_{\xi\otimes_{DL}(f_1\otimes\eta)}} \\
\mathcal{E} & \xrightarrow[\underline{F'}=\underline{F}]{} & \mathcal{E}
\end{array} \tag{4.10}$$

The outcome of all the above is the following: $x_{12,3}$ and $x_{1,23}$ can be represented by the restriction of three dimensional pairs $(\mathcal{E}, F)$ and $(\mathcal{E}, F')$, where

$$\begin{aligned} F &= F_1 \otimes_D 1 \otimes_E 1 + (u_1 \otimes_D 1 \otimes_E 1)(\underline{F_{12}} \otimes_E 1) + (u_{12} \otimes_E 1)\,\underline{F}, \\ F' &= F_1 \otimes_D 1 \otimes_E 1 + (u_1 \otimes_D 1 \otimes_E 1)(\underline{F_{12}} \otimes_E 1) + (u_1 \otimes_D 1 \otimes_E 1)\,\underline{U_2}\,\underline{F}, \end{aligned} \tag{4.11}$$

to the graphs of appropriate sections h and h', respectively. We complete the proof by showing that h and h' can be chosen the same, and that F and F' are homotopic.

The proof of the Technical Theorem given in subsection 4.5 (see also the remark that follows that proof) shows that, while the section h_0 that appears in the triple $(u, \underline{F}, h_0)$ used to define the product of two KE-modules is an important element, the "right decay conditions" actually hold true on a two dimensional object, namely over $\cup_{n=0}^{\infty}[T_{1,n}, T_{1,n+1}] \times [T_{2,n}, \infty)$, or over $\{\,(t_1, t_2) \in LL \mid t_2 \geq h_0(t_1)\,\}$. (Notation as in the proof of the Technical Theorem.) This implies that in the computation of a product the section is important only through the fact that it captures the behavior when both $t_1 \to \infty$ and $t_2 \to \infty$. This observation is summarized as:

Lemma 4.16. *The products* $(\,x_1 \sharp_D x_2\,) \sharp_E x_3$ *and* $x_1 \sharp_D (\,x_2 \sharp_E x_3\,)$ *can be computed by restricting the operators of (4.11) to a* common *3D-section* h.

We need one more result:

Lemma 4.17. *Define:* $\mathcal{J}_0(\mathcal{E}) = \{F \in \mathcal{L}(\mathcal{E}) \mid \lim_{t_1,t_2,t_3 \to \infty} \|F_{(t_1,t_2,t_3)}\| = 0\}$. *(Here* $t_1, t_2, t_3 \to \infty$ *means* $t_i \to \infty$, *for* $i = 1, 2, 3$.*) Then* $[\,u_1 \otimes_D 1 \otimes_E 1, \underline{U_2}\,] \in \mathcal{J}_0(\mathcal{E})$, *and* u_{12} *can be chosen such that* $[\,u_{12} \otimes_E 1, (u_1 \otimes_D 1 \otimes_E 1)\,\underline{U_2}\,] \in \mathcal{J}_0(\mathcal{E})$.

Proof. Modulo an element in $\mathcal{J}(\mathcal{E}_1) \otimes_D 1 \otimes_E 1 \subset \mathcal{J}_0(\mathcal{E})$, $u_1 \otimes_D 1 \otimes_E 1$ can be approximated on compact intervals in the t_1-variable by finite sums $\sum_i (T_{\xi_i} T_{\eta_i}^* \otimes_E 1)$, with $\xi_i, \eta_i \in \mathcal{E}_1$ compactly supported. (See the proof of the Technical Theorem in subsection 4.5.) This implies:

$$\begin{aligned} &[u_1 \otimes_D 1 \otimes_E 1, \underline{U_2}] \\ &\sim \sum_i \big((T_{\xi_i} T_{\eta_i}^* \otimes_E 1)\underline{U_2} - \underline{U_2}(T_{\xi_i} T_{\eta_i}^* \otimes_E 1)\big) && \bmod \mathcal{J}_0(\mathcal{E}) \\ &\sim (-1)^{\partial \eta_i} \sum_i \big(T_{\xi_i}(1 \otimes (u_2 \otimes_{EL} 1))T_{\eta_i}^* - T_{\xi_i}(1 \otimes (u_2 \otimes_{EL} 1))T_{\eta_i}^*\big) && \bmod \mathcal{J}_0(\mathcal{E}) \\ &= 0. \end{aligned}$$

This proves the first inclusion. For the second one, use the same approximation for $u_1 \otimes_D 1 \otimes_E 1$ as above to see that, modulo $\mathcal{J}_0(\mathcal{E})$, $(u_1 \otimes_D 1 \otimes_E 1)\,\underline{U_2}$ is an element of $\mathcal{L}(\mathcal{E}_{12}) \otimes_E 1$. The claimed asymptotic commutativity follows by actually imposing it as an *extra requirement* for u_{12} (besides the conditions that appear in Step 1, Overview 4.8). □

This last lemma implies that $a\,[F, F']\,a^* \geq 0$, modulo $\mathcal{J}(\mathcal{E}_h)$, for any section h, and consequently Lemma 3.26 gives the required homotopy. We have showed that $x_{12,3} = x_{1,23}$ in $KE_G(A, B)$, and this completes the proof of Theorem 4.15. □

Remark. There is another way to see the homotopy between the operators from (4.11). It uses the following result, whose proof is left to the reader:

Lemma 4.18. $(u_1 \otimes_D 1 \otimes_E 1)\,\underline{U_2}$ *satisfies the (properly modified) conditions of Step 1, Overview 4.8, that* $(u_{12} \otimes_E 1)$ *satisfies.*

Consequently, the straight line homotopy

$$\{\,(1-s)(u_{12} \otimes_E 1) + s(u_1 \otimes_D 1 \otimes_E 1)\,\underline{U_2}\,\}_{s\in[0,1]}$$

can be used to give a homotopy between F and F'.

Recall that $1 = 1_{\mathbb{C}} \in KE_G(\mathbb{C}, \mathbb{C})$ is the class of the identity homomorphism $\psi = \mathrm{id} : \mathbb{C} \to \mathbb{C}$. For the next result compare with [26, Thm. 4.5] and [34, Prop. 17]. The proof is left to the reader. Note that Theorem 4.15 and Proposition 4.19 imply that $KE_G(A, A)$ is a ring with unit, for any G-C^*-algebra A.

Proposition 4.19. *Let A and B be separable G-C^*-algebras, then*

$$1_{\mathbb{C}} \sharp_{\mathbb{C}}\, x \;=\; x \sharp_{\mathbb{C}}\, 1_{\mathbb{C}} \;= x, \; \textit{for any } x \in KE_G(A, B).$$

The following notion is important in further studying the properties of KE-theory and in applications.

Definition 4.20. Let D_1 and D_2 be G-C^*-algebras. An element $\alpha \in KE_G(D_1, D_2)$ is called a *KE-equivalence* (or *invertible*) if there exists an element $\beta \in KE_G(D_2, D_1)$ such that $\alpha \sharp_{D_2} \beta \;= 1_{D_1}$ and $\beta \sharp_{D_1} \alpha \;= 1_{D_2}$. If such an element α exists then D_1 and D_2 are called *KE-equivalent*.

We use KE-equivalence to state a result that bears considerable theoretical significance:

Theorem 4.21 (Stability in KE-theory). *For any G-C^*-algebra A, A and $A \otimes \mathcal{K}(\mathcal{H}_G)$ are KE-equivalent.*

The proof follows from the corresponding result in KK-theory, as explained in Corollary 5.3. Another proof can be given by rephrasing [27, 2.18] in terms of KE-theory groups.

Corollary 4.22. *For any separable G-C^*-algebras A and B, we have*

$$\begin{aligned} KE_G(A, B) \simeq KE_G(A, B \otimes \mathcal{K}(\mathcal{H}_G)) &\simeq KE_G(A \otimes \mathcal{K}(\mathcal{H}_G), B) \\ &\simeq KE_G(A \otimes \mathcal{K}(\mathcal{H}_G), B \otimes \mathcal{K}(\mathcal{H}_G)). \end{aligned}$$

4.5. The proof of the technical theorem. In this subsection the following is proved:

Technical Theorem (Theorem 4.10). *Let G be a locally compact σ-compact Hausdorff group, and let A, B, and D be separable graded G-C^*-algebras. Consider two asymptotic Kasparov modules $(\mathcal{E}_1, F_1) \in ke_G(A, D)$ and $(\mathcal{E}_2, F_2) \in ke_G(D, B)$. There exists a triple $(u, \underline{F}, h_0)$, with u a self-adjoint element of $\mathcal{C}^{(0)}(\mathcal{E}_1)$, $\underline{F}$ an F_2-connection for $\mathcal{E}_1$, and h_0 a section of $[1,\infty)\times[1,\infty)$, as in Overview 4.8, such that for any other section $h \geq h_0$*

$$(\mathcal{E}_h, F_h) = (\mathrm{Res}_h)_*\big(\,\mathcal{E}_1 \otimes_{DL} \mathcal{E}_2 L,\, F_1 \otimes_{DL} 1 + (u \otimes_{DL} 1)\,\underline{F}\,\big)$$

is an asymptotic Kasparov G-(A, B)-module.

Proof. We shall justify Steps 1–4 of the Overview 4.8.

Step 1. Step 1, in which u is constructed, is nothing but Lemma 3.35 applied to $(\mathcal{E}_1, F_1)$.

Step 2. The existence of the connection $\underline{F} = \underline{F}^*$ in Step 2 follows from Proposition 4.4 (and after choosing $F_2 = F_2^*$). As it will become clear from the proof, the self-adjointness of $\underline{F}$ is just a convenience. It enables us to reduce some of the computations to the unified requirements of Step 3. So far we succeeded to create the pair of "two-dimensional" objects $(\mathcal{E}, F) = (\mathcal{E}_1 \otimes_{DL} \mathcal{E}_2 L, F_1 \otimes_{DL} 1 + (u \otimes_{DL} 1)\,\underline{F})$.

Step 3. For Step 3, we obtain h_{00} by applying Lemma 4.6 for the set $K = \{u, uF_1, ua_1, ua_2, \ldots, ua_n, \ldots\}$, where $\{a_n\}_{n=1}^{\infty}$ is a dense subset of A.

Step 4. The essential Step 4 is concerned with finding an appropriate section h_0 such that $(\mathcal{E}_{h_0}, F_{h_0}) = (\mathrm{Res}_{h_0})_*\,((\mathcal{E}, F))$ will be the asymptotic Kasparov G-(A, B)-module which represents the product. For this to happen, the axioms (aKm1)–(aKm4) must be satisfied. The tensor products that appear below are all inner (over DL), but the C^*-algebra will be omitted in order to simplify the writing.

- The simple computation:

$$\begin{aligned}(F - F^*)(a \otimes 1) &= \big(F_1 \otimes 1 + (u \otimes 1)\,\underline{F} - F_1^* \otimes 1 - \underline{F}^*\,(u \otimes 1)\big)(a \otimes 1)\\ &= (F_1 - F_1^*)a \otimes 1 + [u \otimes 1, \underline{F}](a \otimes 1),\end{aligned}$$

  shows that (aKm1) for F_h is satisfied for *any* $h \geq h_{00}$, due to (aKm1) for F_1, and (5) of Step 3.

- Next, given $a \in A$, we have

$$\begin{aligned}&[F, a \otimes 1]\\ &= F_1 a \otimes 1 + (u \otimes 1)\,\underline{F}\,(a \otimes 1) - (-1)^{\partial a} a F_1 \otimes 1 - (-1)^{\partial a}(au \otimes 1)\,\underline{F}\\ &= [F_1, a] \otimes 1 - (-1)^{\partial a}[ua \otimes 1, \underline{F}] + (-1)^{\partial a}([u, a] \otimes 1)\,\underline{F} + [u \otimes 1, \underline{F}]\,(a \otimes 1).\end{aligned}$$

Consequently (aKm2) for F_h is also satisfied for *any* $h \geq h_{00}$, because of (aKm2) for F_1, and (2), (5), and (7).

- For (aKm3), it is noted that:

$$\begin{aligned} a&\left(F^2-1\right)a^* \\ &= (a\otimes 1)\Big(\, F_1^2\otimes 1 + (u\otimes 1)\underline{F}(F_1\otimes 1) + (F_1\otimes 1)(u\otimes 1)\underline{F} \\ &\qquad - (u\otimes 1)\underline{F}\,[\underline{F}, u\otimes 1] + (u\otimes 1)\underline{F}^2(u\otimes 1) - 1\,\Big)\,(a^*\otimes 1) \\ &\sim \left((au)\,F_1^2\,(au)^*\right)\otimes 1 + (1-u^2)\left(a(F_1^2-1)a^*\right)\otimes 1 \\ &\quad - (a\otimes 1)(u\otimes 1)\underline{F}\,[\underline{F}, u\otimes 1](a^*\otimes 1) \\ &\quad + (a\otimes 1)\big(\,([F_1,u]\otimes 1)\underline{F} + [uF_1\otimes 1, \underline{F}] + [u\otimes 1, \underline{F}](F_1\otimes 1)\,\big)(a^*\otimes 1) \\ &\quad + (a\otimes 1)\,(u\otimes 1)\left(\underline{F}^2-1\right)(u\otimes 1)\,(a\otimes 1)^*, \text{ modulo } \mathcal{J}(\mathcal{E}_1)\otimes_{DL} 1. \end{aligned}$$

(For the second equality $\sim$ above, we used (1) and (2) of Step 1, and the self-adjointness of u.) The restriction of the first six terms to *any* $h \geq h_{00}$ will give a positive element modulo $\mathcal{J}(\mathcal{E}_h)$, because of (3), (5) and (6). So we shall have (aKm3) satisfied provided that

$$(u\otimes 1)\left(\underline{F}^2-1\right)(u\otimes 1) \text{ restricts to a positive element modulo } \mathcal{C}(\mathcal{E}_h)+\mathcal{J}(\mathcal{E}_h). \tag{4.12}$$

Showing (4.12) is a critical point in the construction. Let $\{I_n\}_{n=0}^\infty$ be a cover of $[1,\infty)$ by closed intervals of the form $I_n = [t_n, t_{n+2}]$, with $t_0 = 1$, and $\{t_n\}_n$ being a strictly increasing sequence with $\lim_{n\to\infty} t_n = \infty$. Let $T_{1,n} = t_n$, for $n \geq 0$, and $T_{2,0} = 1$. If $\{\mu_n\}_{n=0}^\infty$ is a partition of unity subordinate to this cover, then $u\otimes 1 = \sum_{n=0}^\infty (\mu_n u\otimes 1)$. For each $n\geq 1$, we can approximate $(\mu_n u\otimes 1)$ by a *self-adjoint* finite rank operator

$$K_n = \sum_{i=1}^{N_n} T_{\xi_i}\,T_{\eta_i}^* = \sum_{i=1}^{N_n} T_{\eta_i}\,T_{\xi_i}^*, \text{ with } \xi_i,\eta_i\in\mathcal{E}_1|_{I_n}, \text{ for } i=1,2,\dots,N_n, \tag{4.13}$$

and such that $\|(\mu_n u\otimes 1) - K_n\| < 1/(24n(\|F_2\|^2+1))$. Note that:

$$\begin{aligned} K_n\left(\underline{F}^2-1\right)K_n^* &= \sum_{i,j=1}^{N_n} T_{\xi_i}\,T_{\eta_i}^*\left(\underline{F}^2-1\right)T_{\eta_j}\,T_{\xi_j}^* \\ &\sim \sum_{i,j=1}^{N_n} T_{\xi_i}\,T_{\eta_i}^* T_{\eta_j}\left((1\otimes F_2^2)-1\right)T_{\xi_j}^*, \qquad \text{modulo } \mathcal{J}(\mathcal{E}) \\ &= \sum_{i,j=1}^{N_n} T_{\xi_i}\,\langle \eta_i,\eta_j\rangle\left((1\otimes F_2^2)-1\right)T_{\xi_j}^*. \end{aligned} \tag{4.14}$$

There exists $\tau_{n,1}$ such that $\left\|(\underline{F}^2 T_{\eta_j} - T_{\eta_j}(1 \otimes F_2^2))_{(t_1,t_2)}\right\| < 1/(12nN_n^2)$, for all η_j, all $t_1 \in I_n$, and all $t_2 > \tau_{n,1}$. This implies that the error of the commutation that was used for the second line of equation (4.14) is smaller than $1/(12n)$, in norm and when restricted to the graph of any section h whose values on I_n are bigger than $\tau_{n,1}$. Using the characterization of positive operators on Hilbert modules [28, 4.1] that generalizes the familiar one from Hilbert space theory, we see that the matrix $P = \big(\langle \eta_i, \eta_j\rangle\big) \in M_{N_n}(DL)$ is positive. Consequently $P = QQ^*$, with $Q = \big(d_{ij}\big)$, and we get:

$$\begin{aligned}\sum_{i,j=1}^{N_n} & T_{\xi_i}\,\langle \eta_i, \eta_j\rangle\,\big((1\otimes F_2^2)-1\big)\,T_{\xi_j}^* \\ &= \sum_{i,j=1}^{N_n} T_{\xi_i}\,\Big(\sum_{k=1}^{N_n} d_{ik}\,d_{jk}^*\Big)\,\big((1\otimes F_2^2)-1\big)\,T_{\xi_j}^* \\ &\sim \sum_{k=1}^{N_n}\Big(\Big(\sum_{i=1}^{N_n} T_{\xi_i}\,d_{ik}\Big)\,\big((1\otimes F_2^2)-1\big)\,\Big(\sum_{j=1}^{N_n} T_{\xi_j}\,d_{jk}\Big)^*\Big), \quad \text{modulo } \mathcal{J}(\mathcal{E}).\end{aligned} \tag{4.15}$$

There exists $\tau_{n,2}$ such that $\left\|[d_{jk}, 1\otimes F_2^2]_{(t_1,t_2)}\right\| < 1/(12nN_n^3)$, for all d_{jk}, all $t_1 \in I_n$, and all $t_2 > \tau_{n,2}$. This implies that the error due to asymptotic commutativity ((aKm2) for F_2, used to obtain the second line of equation (4.15)) is smaller than $1/(12n)$, in norm and when restricted to the graph of any section h whose values on I_n are bigger than $\tau_{n,2}$. Let $\{\delta_m\}_m$ be an approximate unit in D. Because of (aKm3) for F_2,

$$\sum_{k=1}^{N_n}\Big(\Big(\sum_{i=1}^{N_n} T_{\xi_i}\,d_{ik}\Big)\,\big(\,1\otimes \delta_m(F_2^2-1)\delta_m\,\big)\,\Big(\sum_{j=1}^{N_n} T_{\xi_j}\,d_{jk}\Big)^*\Big) \tag{4.16}$$

is positive modulo $\mathcal{C}(\mathcal{E}|_{I_n}) + \mathcal{J}(\mathcal{E}|_{I_n})$. Choose m_0 such that the entire sum from (4.16) approximates the one from the second line of (4.15) by $1/(12n)$.

Let $T_{2,n} = \max\{\tau_{n,1}, \tau_{n,2}, T_{2,(n-1)} + 1\}$. (To be precise, there is also a $\tau_{n,3}$ coming from (aKm4) to be taken into account, but we ignore it for the moment.) Once the sequence $\{T_{2,n}\}_n$ has been constructed, we define h_0 on $[T_{1,n}, T_{1,(n+1)}]$ as the linear function satisfying $h_0(T_{1,n}) = T_{2,n}$ and $h_0(T_{1,(n+1)}) = T_{2,(n+1)}$. The estimates above show that the restriction to the graph of $h_0|_{I_n}$ of

$$(\mu_n u \otimes 1)\,(\underline{F}^2 - 1)\,(\mu_n u \otimes 1)^*$$

is positive modulo $\mathcal{C}(\mathcal{E}_{h_0})$, with an error which is smaller than $1/(3n)$, in norm. At most three such terms are non-zero over I_n, this proves (4.12) for any $h \geq h_0$, and consequently F_h satisfies (aKm3).

- Finally, for any $g \in G$, we have:

$$\begin{aligned}
&\big(g(F) - F\big)(a \otimes 1) \\
&\quad = \big(g\,(F_1 \otimes 1) + g\,(u \otimes 1)\; g\,(\underline{F}) - (F_1 \otimes 1) - (u \otimes 1)\,\underline{F}\big)(a \otimes 1) \\
&\quad = \big(g(F_1) - F_1\big)a \otimes 1 + \big((g(u) - u) \otimes 1\big)\, g(\underline{F})(a \otimes 1) \\
&\qquad + (u \otimes 1)\big(g(\underline{F}) - \underline{F}\big)(a \otimes 1).
\end{aligned}$$

Due to (aKm4) for F_1 and (4) of Step 1, the first two terms put no extra constraints on h_0. For the third one, $u \otimes 1$ can be approximated, as in the proof of (aKm3), on each interval I_n, by a finite sum $\sum_i T_{\xi_i} T^*_{\eta_i}$. A simple computation shows that $g\,T^*_{\eta_i} = T^*_{g(\eta_i)}$. Consequently:

$$\begin{aligned}
T_{\xi_i} T^*_{\eta_i}\big(g(\underline{F}) - \underline{F}\big) &= T_{\xi_i}\, g\big(g^{-1}(T^*_{\eta_i})\,\underline{F}\big) - T_{\xi_i} T^*_{\eta_i} \underline{F} \\
&\sim (-1)^{\partial\eta_i}\, T_{\xi_i}\, g\big(F_2 T^*_{g^{-1}(\eta_i)}\big) - (-1)^{\partial\eta_i}\, T_{\xi_i}\, F_2 T^*_{\eta_i}, \quad \text{modulo } \mathcal{J}(\mathcal{E}) \\
&= (-1)^{\partial\eta_i}\, T_{\xi_i}\, \big(g(F_2) - F_2\big)\, T^*_{\eta_i}.
\end{aligned}$$

Further modification (increase) of h_0, using (aKm4) for F_2, will make the above errors go to zero when restricted to the graph of h_0. (This is the place where the $\tau_{n,3}$ mentioned when we defined $T_{2,n}$ makes its appearance.) This shows that (aKm4) holds for F_h, for any $h \geq$ (new h_0), and the proof of the Technical Theorem is complete. □

Remark. The only important fact that h_0 encodes in the construction of the product is a certain behavior that occurs when $t_1 \to \infty$ and $t_2 \to \infty$, with h_0 correlating t_1 and t_2. We have noticed that certain decay properties hold true on entire "stripes" $[T_{1,n}, T_{1,n+1}] \times [T_{2,n}, \infty)$, and not only on the graph of h_0. This observation is used in the proof of the associativity of the product (see Lemma 4.16), where it allows us to focus on the analysis of the operators that appear in the construction rather than on the sections.

5. *KE*-theory: comparison with *KK*-theory and *E*-theory

Assume that a group G (locally compact, σ-compact, Hausdorff) is given. In this final section we construct two functors: $\Theta : \mathbf{KK_G} \to \mathbf{KE_G}$ and $\Xi : \mathbf{KE_G} \to \mathbf{E_G}$. The three categories have all the same objects: the separable and graded G-C^*-algebras. The morphisms of $\mathbf{KK_G}$ (see [15, 17, 36]) are the KK-theory groups, with composition given by the Kasparov product. The morphisms of $\mathbf{KE_G}$ are the KE-theory groups, with composition given by the product defined in Section 4. The morphisms of $\mathbf{E_G}$ (see [14, 19]) are the E-theory groups, with the corresponding composition product. Both functors are the identity on objects.

One consequence of the existence of these two functors is the construction of an *explicit* natural transformation, namely the composition $\Xi \circ \Theta$, between KK-theory and E-theory. This transformation *preserves the product structures* of the two theories. This connecting functor is roughly:

$$\begin{array}{ccccc} KK_G(A,B) & \xrightarrow{\Theta} & KE_G(A,B) & \xrightarrow{\Xi} & E_G(A,B) \\ (\mathcal{E},\varphi,F) & \mapsto & \big\{\big(\mathcal{E},(1-u_t)F(1-u_t)\big)\big\}_t & \mapsto & \big\{f\otimes a \overset{\varphi_t}{\mapsto} f\big((1-u_t)F(1-u_t)\big)\,a\big\}_t \end{array} \tag{5.1}$$

Here $\{u_t\}_t$ is a quasi-invariant quasi-central approximate unit for $\mathcal{K}(\mathcal{E})$, and $\{\varphi_t\}_t : C_0((-1,1)) \otimes A \dashrightarrow B \otimes \mathcal{K}$ is an asymptotic family constructed using functional calculus. The suggestive but somehow imprecise (see subsection 5.2) formula of the composition in (5.1), namely $\Xi \circ \Theta : KK_G(A,B) \to E_G(A,B)$, $(\mathcal{E},\varphi,F) \mapsto \{f \otimes a \overset{\varphi_t}{\mapsto} f\,((1-u_t)F(1-u_t))\; a\}_t$ appears also in [31, 4.5.1], in the context of groupoid actions.

5.1. The map $KK_G \to KE_G$. Let A and B be G-C^*-algebras. Consider $(\mathcal{E},\varphi,F) \in kk_G(A,B)$. Denote by $C^*(\mathcal{K}(\mathcal{E}),A,F)$ the smallest C^*-subalgebra of $\mathcal{L}(\mathcal{E})$ that contains $\mathcal{K}(\mathcal{E})$, $\varphi(A)$, and F, and let $u=\{u_t\}_{t\in[1,\infty)}$ be a quasi-invariant quasi-central approximate unit for $\mathcal{K}(\mathcal{E}) \subset C^*(\mathcal{K}(\mathcal{E}),A,F) \subset \mathcal{L}(\mathcal{E})$. It will be convenient, at least for notational purposes, to regard u as an element of $\mathcal{C}(\mathcal{E}L)$. We make the notation: $\widehat{\mathcal{E}} = \mathcal{E}L$ (constant family of modules), and

$$\widehat{F} = \{(1-u_t)F(1-u_t)\}_t = (1-u)F(1-u).$$

Then

$$\big\{\big(\mathcal{E},(1-u_t)F(1-u_t)\big)\big\}_t = \big(\widehat{\mathcal{E}},\widehat{F}\big)$$

is an asymptotic Kasparov G-(A,B)-module. The connection between the KK-theory and KE-theory groups is given by the following two results, both proved in [13].

Theorem 5.1. *With the above notation, the map*

$$\Theta : kk_G(A,B) \to ke_G(A,B),\ \big(\mathcal{E},\varphi,F\big) \mapsto \big\{\big(\mathcal{E},(1-u_t)F(1-u_t)\big)\big\}_{t\in[1,\infty)}, \tag{5.2}$$

passes to quotients and gives a group homomorphism

$$\Theta : KK_G(A,B) \to KE_G(A,B).$$

Theorem 5.2. $\Theta : \mathbf{KK_G} \longrightarrow \mathbf{KE_G}$ *is a functor,* i.e. *preserves the products.*

One consequence is worth noticing:

Corollary 5.3. *A KK-equivalence is sent by Θ into a KE-equivalence. In particular we obtain that A and $A\otimes\mathcal{K}(\mathcal{H}_G)$ are KE-theory equivalent, for any G-C^*-algebra A, and that the KE-theory groups satisfy Bott periodicity.*

5.2. The map $KE_G \to E_G$. The E-theory groups were introduced and studied in [6, 7], the equivariant ones under the action of a group in [14], and under the action of a groupoid in [32]. We use here the approach taken in [19]. Let $\mathcal{S}$ be the C^*-algebra $C_0(\mathbb{R})$ graded by even and odd functions.

Definition 5.4. ([19, Def.2.2]) We denote by $E_G(A, B)$the set of all homotopy equivalence classes of asymptotic families from $\mathcal{S}A \otimes \mathcal{K}(\mathcal{H}_G) = \mathcal{S} \otimes A \otimes \mathcal{K}(\mathcal{H}_G)$ to $B \otimes \mathcal{K}(\mathcal{H}_G)$: $E_G(A, B) = [\![\mathcal{S}A \otimes \mathcal{K}(\mathcal{H}_G), B \otimes \mathcal{K}(\mathcal{H}_G)]\!]$.

Our construction of the connecting map between KE-theory and E-theory is performed via a description of the E-theory groups which involves $C_0((-1,1))$ instead of $\mathcal{S}$. Such a modification seems more appropriate when working with bounded operators. As with $\mathcal{S} = C_0(\mathbb{R})$, the C^*-algebra $C_0((-1,1))$ will be graded by even and odd functions.

Let A and B be G-C^*-algebras. We consider first a particular case of asymptotic Kasparov (A, B)-modules: $(\mathcal{E}, F) = \{(\mathcal{E}_\bullet, F_t)\}_t \in ke_G(A, B)$, where $\mathcal{E}_\bullet$ is a *fixed* Hilbert G-B-module acted upon by A through a family of $*$-homomorphisms $\varphi_t : A \to \mathcal{L}(\mathcal{E}_\bullet)$. This means that $F_t = F_t^* \in \mathcal{L}(\mathcal{E}_\bullet)$ is an odd self-adjoint operator, for every t, such that $[F_t, a_t]$, $(g(F_t) - F_t)a_t$ converge in norm to 0 as $t \to \infty$, for all $a \in A$, $g \in G$, and that $a_t(F_t^2 - 1)a_t^* \geq 0$, modulo compacts, with an error that converges in norm to 0 as $t \to \infty$.

Proposition 5.5. *The family of maps*

$$\phi_F = \{\phi_{F,t}\}_{t \in [1,\infty)} : C_0((-1,1)) \otimes A \to \mathcal{K}(\mathcal{E}_\bullet),\ f \otimes a \overset{\phi_{F,t}}{\mapsto} f(F_t)\, a_t, \quad (5.3)$$

for $f \in C_0((-1,1))$, $a \in A$, is an asymptotic family, in the sense of E-theory [14, Def. 1.3].

Proof. The varying family of functional calculus $*$-homomorphisms

$$\{\chi_t\}_t : C_0((-1,1)) \to \mathcal{L}(\mathcal{E}_\bullet),\ \chi_t : f \mapsto f(F_t),$$

and the family of $*$-homomorphisms $\{\varphi_t\}_t$, are used to obtain the asymptotic family

$$\phi_F = \{\chi_t \otimes \varphi\}_t : C_0((-1,1)) \otimes A \to \mathcal{L}(\mathcal{E}_\bullet),\ f \otimes a \mapsto f(F_t)\, a_t.$$

The asymptotic commutativity $[\chi_t(f), \varphi_t(a)] = [f(F_t), a_t] \xrightarrow{t\to\infty} 0$ follows from (aKm2) (and the mechanics behind the functional calculus). Finally, in order to compute the range of φ_t's we shall show that

$$a\, f(F) \in \mathcal{C}(\mathcal{E}) + \mathcal{J}(\mathcal{E}),\ \text{for any } f \in C_c((-1,1)). \quad (5.4)$$

Granted this, (aKm2) implies that $f(F)\,a \in \mathcal{C}(\mathcal{E}) + \mathcal{J}(\mathcal{E})$, which makes ϕ_F asymptotically equivalent to a $\mathcal{K}(\mathcal{E}_\bullet)$-valued asymptotic family. A density argument shows that the desired inclusion holds for every $f \in C_0((-1,1))$.

To prove (5.4), let $a(F^2-1)a^* = p_+ - p_- \in \mathcal{L}(\mathcal{E})$ be the decomposition into positive and negative parts. The axiom (aKm3) implies that $p_- \in \mathcal{C}(\mathcal{E}) + \mathcal{J}(\mathcal{E})$. There exists $g \in C_c((-1,1))$ (concretely $g(x) = f(x)/\sqrt{1-x^2}$, for $x \in \operatorname{supp}(f)$) such that $f(x)\overline{f}(x) = g(x)(1-x^2)\overline{g}(x)$. Then:

$$\begin{aligned}
0 &\le a\, f(F)\, \overline{f}(F)\, a^* \\
&= a\, g(F)\,(1-F^2)\,\overline{g}(F)\, a^* \\
&= g(F)\, a\,(1-F^2)\, a^*\, \overline{g}(F) && \text{(modulo } \mathcal{J}(\mathcal{E})) \\
&= -g(F)\, a\,(F^2-1)\, a^*\, \overline{g}(F) \\
&= g(F)\, p_-\, \overline{g}(F) && \text{(because of initial positivity)} \\
&\in \mathcal{C}(\mathcal{E}) + \mathcal{J}(\mathcal{E}) && \text{(because of (aKm3)).}
\end{aligned}$$

Consequently, by polar decomposition, $a\, f(F) \in \mathcal{C}(\mathcal{E}) + \mathcal{J}(\mathcal{E})$. □

The asymptotic family constructed above indicates that a "$C_0((-1,1))$-picture" of E-theory is in order. The next lemma is the first step towards such a characterization.

Lemma 5.6. *Let A and D be G-C^*-algebras, and consider an equivariant asymptotic family $\phi_F = \{\phi_{F,t}\}_t : C_0((-1,1)) \otimes A \dashrightarrow D$. Then there exists a unique, up to homotopy, equivariant asymptotic family $\psi_F = \{\psi_{F,t}\}_t : \mathcal{S}A \dashrightarrow D$ such that the diagram*

$$\begin{array}{ccc}
C_0((-1,1)) \otimes A & \xrightarrow{\ \phi_F\ } & D \\
{\scriptstyle \text{inclusion}}\Big\downarrow{\scriptstyle \iota} & & \Big\| \\
\mathcal{S}A & \xrightarrow[\ \psi_F\ ]{} & D
\end{array}$$

commutes up to homotopy.

To discuss the general case we mention the following possible simplification in the definition of asymptotic Kasparov modules:

Proposition 5.7. *Given two G-C^*-algebras A and B, let $A' = A \otimes \mathcal{K}(\mathcal{H}_G)$ and $B' = B \otimes \mathcal{K}(\mathcal{H}_G)$. Then, in the definition of $KE_G(A', B')$ it is enough to consider modules of the form $(\mathcal{H}_{B'L}, F)$.*

Remark 5.8. In general, it may *not* be true that each element of $KE_G(A,B)$ can be represented by an asymptotic Kasparov module with trivial field $\mathcal{E}$.

The proposition implies that the previous construction of the asymptotic morphism associated to an asymptotic Kasparov module with constant "fibers" can be carried over to the general case. Consider an arbitrary Kasparov module $(\mathcal{E}, F) \in ke_G(A,B)$. We can construct an asymptotic morphism

$$\phi : C_0((-1,1)) \otimes A \dashrightarrow \mathcal{C}(\mathcal{E})/\mathcal{K}(\mathcal{E}).$$

This in turn gives an asymptotic morphism:

$$\phi \otimes 1 : C_0((-1,1)) \otimes A \otimes \mathcal{K}(L^2(G)) \dashrightarrow \mathcal{C}(\mathcal{E} \otimes L^2(G))/\mathcal{K}(\mathcal{E} \otimes L^2(G)). \quad (5.5)$$

By ignoring the action of G, apply the Stabilization Theorem (with $G = \{e\}$) to get a non-equivariant isometry $V : \mathcal{E} \to \mathcal{H}_{BL}$. Apply next the Fell's trick to construct an equivariant BL-linear isometry $W : \mathcal{E} \otimes L^2(G) \to \mathcal{H}_{BL} \otimes L^2(G)$. Use it, and the fact that now we have a *constant field* $\mathcal{H}_{BL}$ of modules, to transform the asymptotic morphism $\phi \otimes 1$ of (5.5) into an asymptotic family:

$$\phi_F : C_0((-1,1)) \otimes A \otimes \mathcal{K}(L^2(G)) \dashrightarrow \mathcal{K}(\mathcal{H}_B) \otimes \mathcal{K}(L^2(G)). \quad (5.6)$$

After tensoring with $\mathcal{K}$, we can use Lemma 5.6 to obtain an asymptotic morphism $\psi_F : \mathcal{S}A \otimes \mathcal{K} \dashrightarrow B \otimes \mathcal{K}$. The connection between KE-theory and E-theory is given by:

Theorem 5.9. *For any group G, and any two G-C^*-algebras A and B, the map $\Xi : (\mathcal{E}, F) \mapsto \psi_F$, from asymptotic Kasparov G-(A,B)-modules to asymptotic families from $\mathcal{S}A \otimes \mathcal{K}$ to $B \otimes \mathcal{K}$, passes to quotients and gives a natural group homomorphism*

$$\Xi : KE_G(A,B) \to E_G(A,B), \ \Xi((\mathcal{E},F)) = [\![\psi_F]\!]. \quad (5.7)$$

Moreover we have:

Theorem 5.10. $\Xi : \mathbf{KE_G} \longrightarrow \mathbf{E_G}$ *is a functor.*

5.3. An important open question. The most important open question about KE-theory is whether it coincides or not with either KK-theory or E-theory. Our initial investigations did not provide a conclusive answer. Ralf Meyer made a suggestion which may indicate that, at least in the non-equivariant case, KE-theory coincides with KK-theory. We succinctly present below the argument and refer the reader to [29] for more details. Recall the following description of KK-theory from [23]: $KK(A,B)$ is the group of classes of asymptotic morphisms whose individual maps, φ_t's, are all completely positive linear contractions,

$$[\![C_0(\mathbb{R}) \otimes A \otimes \mathcal{K}, C_0(\mathbb{R}) \otimes B \otimes \mathcal{K}]\!]_{\mathrm{cp}}.$$

In this approach the map from KK-theory to E-theory is the forgetful functor. (As a side remark, in this characterization the C^*-algebra $C_0(\mathbb{R})$ is not graded and the definition of asymptotic morphisms is the original one, as in [7], and not as in Definition 5.4, as we used it in our paper. To the best of our knowledge the equivalence of the two definitions of the E-theory groups is not worked out in detail in the literature.) The reviewer's key observation is that the $*$-homomorphism

$$\phi_F : C_0((-1,1)) \otimes A \to (\mathcal{C}(\mathcal{E}) + \mathcal{J}(\mathcal{E}))/\mathcal{J}(\mathcal{E}),$$

obtained as in the proof of Proposition 5.5 and under the assumption that $\mathcal{E}$ is *trivial*, can be lifted in the extension

$$0 \to \mathfrak{J}(\mathcal{E}) \to \mathfrak{C}(\mathcal{E}) + \mathfrak{J}(\mathcal{E}) \to (\mathfrak{C}(\mathcal{E}) + \mathfrak{J}(\mathcal{E}))/\mathfrak{J}(\mathcal{E}) \to 0$$

to a completely positive contraction $\alpha : C_0((-1,1)) \otimes A \to \mathfrak{C}(\mathcal{E})$. This construction uses an argument similar to [23, Lemma 4.1], some density arguments, and the closure property of the maps with completely positive lifting from [2]. The point now is that, because the asymptotic morphisms in the range of Ξ are completely positive contractions, the map Ξ factors through KK-theory. To finish the proof along this lines, one will have to deal next with the possibility that $\mathcal{E}$ is not trivial, as noticed in Remark 5.8, and with proving that Ξ is the inverse of Θ. The equivariant case is a lot more complicated, because of the various group actions involved. We notice though that there is a description of $KK_G(A, B)$ as completely positive and equivariant contractions [37], so it would be interesting to find if the argument sketched above can be made to work in the equivariant setting as well.

References

[1] C. Akemann, G. Pedersen and J. Tomiyama, Multipliers of C^*-algebras, *J. Functional Analysis*, **13** (1973), 277–301. Zbl 0258.46052 MR 0470685

[2] W. Arveson, Notes on extensions of C^*-algebras, *Duke Math. J.*, **44** (1977), no. 2, 329–355. Zbl 0368.46052 MR 0438137

[3] P. Baum and A. Connes, Geometric K-theory for Lie groups and foliations, *Enseign. Math. (2)*, **46** (2000), no. 1-2, 3–42. Zbl 0985.46042 MR 1769535

[4] P. Baum, A. Connes and N. Higson, Classifying space for proper actions and K-theory of group C^*-algebras, in *C^*-algebras: 1943–1993 (San Antonio, TX, 1993)*, 240–291, Contemp. Math., 167, Amer. Math. Soc., Providence, RI, 1994. Zbl 0830.46061 MR 1292018

[5] B. Blackadar, *K-theory for operator algebras*, Second edition, Cambridge Univ. Press, Cambridge, 1998. Zbl 0913.46054 MR 1656031

[6] A. Connes and N. Higson, *Almost homomorphisms and KK-theory*, unpublished manuscript, 1989. Available at: `http://www.math.psu.edu/higson/research.html`

[7] A. Connes and N. Higson, Déformations, morphismes asymptotiques et K-théorie bivariante, *C. R. Acad. Sci. Paris Sér. I Math.*, **311** (1990), no. 2, 101–106. Zbl 0717.46062 MR 1065438

[8] A. Connes and G. Skandalis, The longitudinal index theorem for foliations, *Publ. Res. Inst. Math. Sci.*, **20** (1984), no. 6, 1139–1183. Zbl 0575.58030 MR 0775126

[9] J. Cuntz, Generalized homomorphisms between C^*-algebras and KK-theory, in *Dynamics and processes (Bielefeld, 1981)*, 31–45, Lecture Notes in Math., 1031, Springer, Berlin, 1983. Zbl 0561.46034 MR 0733641

[10] J. Cuntz, K-theory and C^*-algebras, in *Algebraic K-theory, number theory, geometry and analysis (Bielefeld, 1982)*, 55–79, Lecture Notes in Math., 1046, Springer, Berlin, 1984. Zbl 0548.46056 MR 750677

[11] J. Cuntz, Bivariante K-Theorie für lokalkonvexe Algebren und der Chern-Connes-Charakter, *Doc. Math.*, **2** (1997), 139–182 (electronic). Zbl 0920.19004 MR 1456322

[12] J. Cuntz, A general construction of bivariant K-theories on the category of C^*-algebras, in *Operator algebras and operator theory (Shanghai, 1997)*, 31–43, Contemp. Math., 228, Amer. Math. Soc., Providence, RI, 1998. Zbl 0930.46055 MR 1667652

[13] D. Dumitraşcu, *A new approach to bivariant K-theory*, Ph.D. thesis, Penn State University, 2001.

[14] E. Guentner, N. Higson and J. Trout, Equivariant E-theory for C^*-algebras, *Mem. Amer. Math. Soc.*, **148** (2000), no. 703, viii+86 pp. Zbl 0983.19003 MR 1711324

[15] N. Higson, A characterization of KK-theory, *Pacific J. Math.*, **126** (1987), no. 2, 253–276. Zbl 0645.46054 MR 0869779

[16] N. Higson, A primer on KK-theory, in *Operator theory: operator algebras and applications, Part 1 (Durham, NH, 1988)*, 239–283, Proc. Sympos. Pure Math., 51, Amer. Math. Soc., Providence, RI, 1990. Zbl 0718.46052 MR 1077390

[17] N. Higson, Categories of fractions and excision in KK-theory, *J. Pure Appl. Algebra*, **65** (1990), no. 2, 119–138. Zbl 0718.46053 MR 1068250

[18] N. Higson, A note on the cobordism invariance of the index, *Topology*, **30** (1991), no. 3, 439–443. Zbl 0731.58065 MR 1113688

[19] N. Higson and G. G. Kasparov, Operator K-theory for groups which act properly and isometrically on Hilbert space, *Electron. Res. Announc. Amer. Math. Soc.*, **3** (1997), 131–142 (electronic). Zbl 0888.46046 MR 1487204

[20] N. Higson and G. G. Kasparov, E-theory and KK-theory for groups which act properly and isometrically on Hilbert space, *Invent. Math.*, **144** (2001), no. 1, 23–74. Zbl 0988.19003 MR 1821144

[21] N. Higson and J. Roe, *Analytic K-homology*, Oxford Univ. Press, Oxford, 2000. Zbl 0968.46058 MR 1817560

[22] N. Higson, G. Kasparov and J. Trout, A Bott periodicity theorem for infinite dimensional Euclidean space, *Adv. Math.*, **135** (1998), no. 1, 1–40. Zbl 0911.46040 MR 1617411

[23] T. G. Houghton-Larsen and K. Thomsen, Universal (co)homology theories, *K-Theory*, **16** (1999), no. 1, 1–27. Zbl 0918.19005 MR 1673935

[24] G. G. Kasparov, Topological invariants of elliptic operators, I: K-homology, *Math. USSR-Izv.*, **9** (1975), no. 4, 751–792; translated from *Izv. Akad. Nauk SSSR Ser. Mat.*, **39** (1975), no. 4, 796–838. Zbl 0328.58016 MR 0488027

[25] G. G. Kasparov, Hilbert C^*-modules: theorems of Stinespring and Voiculescu, *J. Operator Theory*, **4** (1980), no. 1, 133–150. Zbl 0456.46059 MR 0587371

[26] G. G. Kasparov, The operator K-functor and extensions of C^*-algebras, *Math. USSR Izvestiya*, **16** (1981), 513–572; translated from *Izv. Akad. Nauk SSSR Ser. Mat.*, **44** (1980), no. 3, 571–636. Zbl 0448.46051 MR 0582160

[27] G. G. Kasparov, Equivariant KK-theory and the Novikov conjecture, *Invent. Math.*, **91** (1988), no. 1, 147–201. Zbl 0647.46053 MR 0918241

[28] E. C. Lance, *Hilbert C^*-modules: a toolkit for operator algebraists*, Cambridge Univ. Press, Cambridge, 1995. Zbl 0822.46080 MR 1325694

[29] R. Meyer, Comparison of KE-theory and KK-theory, *J. Noncommut. Geom.*, **10** (2016), no. 3, 1131–1142.

[30] G. K. Pedersen, *C^*-algebras and their automorphism groups*, Academic Press, London, 1979. Zbl 0416.46043 MR 0548006

[31] R. Popescu, *E-théorie équivariante et groupoïdes*, thèse de doctorat, Université Claude Bernard - Lyon I, 2000.

[32] R. Popescu, Equivariant E-theory for groupoids acting on C^*-algebras, *J. Funct. Anal.*, **209** (2004), no. 2, 247–292. Zbl 1059.46054 MR 2044224

[33] J. Rosenberg, Group C^*-algebras and topological invariants, in *Operator algebras and group representations, Vol. II (Neptun, 1980)*, 95–115, Pitman, Boston, MA, 1984. Zbl 0524.22009 MR 0733308

[34] G. Skandalis, Some remarks on Kasparov theory, *J. Funct. Anal.*, **56** (1984), no. 3, 337–347. Zbl 0561.46035 MR 0743845

[35] G. Skandalis, Une notion de nuclearité en K-théorie (d'aprés J. Cuntz), *K-Theory*, **1** (1988), no. 6, 549–573. Zbl 0653.46065 MR 0953916

[36] K. Thomsen, The universal property of equivariant KK-theory, *J. Reine Angew. Math.*, **504** (1998), 55–71. Zbl 0918.19004 MR 1656818

[37] K. Thomsen, Asymptotic homomorphisms and equivariant KK-theory, *J. Funct. Anal.*, **163** (1999), no. 2, 324–343. Zbl 0929.46059 MR 1680467

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