

## Matrix factorizations and motivic measures

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**Abstract.** This article is the continuation of [17]. We use categories of matrix factorizations to define a morphism of rings (= a Landau–Ginzburg motivic measure) from the (motivic) Grothendieck ring of varieties over  $\mathbb{A}^1$  to the Grothendieck ring of saturated dg categories (with relations coming from semi-orthogonal decompositions into admissible subcategories). Our Landau–Ginzburg motivic measure is the analog for matrix factorizations of the motivic measure in [5] whose definition involved bounded derived categories of coherent sheaves. On the way we prove smoothness and a Thom–Sebastiani theorem for enhancements of categories of matrix factorizations.

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### 1. Introduction

This article is the partner of [17]. The mutual goal of these two articles is the construction of an interesting Landau–Ginzburg motivic measure: a ring morphism from the (motivic) Grothendieck ring of varieties over  $\mathbb{A}^1$  to another ring. The terminology Landau–Ginzburg comes from physics where a morphism  $W: X \rightarrow \mathbb{A}^1$  is considered as a superpotential on a variety  $X$ .

Let  $k$  be an algebraically closed field of characteristic zero. Let  $X$  be a smooth variety (over  $k$ ) and  $W: X \rightarrow \mathbb{A}^1 = \mathbb{A}_k^1$  a morphism (also viewed as an element of  $\Gamma(X, \mathcal{O}_X)$ ). We denote the category of matrix factorizations of  $W$  by  $\mathbf{MF}(X, W)$  (see [17]). Taking the product over all the categories  $\mathbf{MF}(X, W - a)$ , for  $a \in k$ , defines the singularity category  $\mathbf{MF}(W)$  of  $W$ ,

$$\mathbf{MF}(W) := \prod_{a \in k} \mathbf{MF}(X, W - a).$$

Only finitely many factors of this product are non-zero, and  $\mathbf{MF}(W)$  vanishes if and only if  $W$  is a smooth morphism (see Lemma 4.13). Let  $\mathbf{MF}(W)^{\text{dg}}$  be a suitable enhancement (in the differential  $\mathbb{Z}_2$ -graded setting, where  $\mathbb{Z}_2 = \mathbb{Z}/2\mathbb{Z}$ ) of  $\mathbf{MF}(W)$ ,

and let  $\mathbf{MF}(W)^{\mathrm{dg},\natural}$  be the corresponding enhancement of the Karoubi envelope of  $\mathbf{MF}(W)$ .

The (motivic) Grothendieck group  $K_0(\mathrm{Var}_{\mathbb{A}^1})$  of varieties over  $\mathbb{A}^1$  is defined as the free abelian group on isomorphism classes  $[X]_{\mathbb{A}^1} = [X, W]$  of varieties  $W: X \rightarrow \mathbb{A}^1$  over  $\mathbb{A}^1$  subject to the relations  $[X]_{\mathbb{A}^1} = [X - Y]_{\mathbb{A}^1} + [Y]_{\mathbb{A}^1}$  whenever  $Y \subset X$  is a closed subvariety. Given  $W: X \rightarrow \mathbb{A}^1$  and  $V: Y \rightarrow \mathbb{A}^1$  we define  $W * V: X \times Y \rightarrow \mathbb{A}^1$  by  $(W * V)(x, y) = W(x) + V(y)$ . This operation turns  $K_0(\mathrm{Var}_{\mathbb{A}^1})$  into a commutative ring.

We denote by  $K_0(\mathrm{sat}_k^{\mathbb{Z}_2})$  the Grothendieck group of saturated dg categories (see Definition 2.23), i.e. the free abelian group on quasi-equivalence classes of saturated (= proper, smooth and triangulated) dg (= differential  $\mathbb{Z}_2$ -graded) categories with relations coming from semi-orthogonal decompositions into admissible subcategories on the level of homotopy categories. The tensor product of dg categories gives rise to a ring structure on  $K_0(\mathrm{sat}_k^{\mathbb{Z}_2})$ . One may think of  $K_0(\mathrm{sat}_k^{\mathbb{Z}_2})$  as a Grothendieck ring of suitable pretriangulated dg categories. Now we can state our main result.

**Theorem 1.1** (see Theorem 5.2). *There is a unique morphism*

$$\mu: K_0(\mathrm{Var}_{\mathbb{A}^1}) \rightarrow K_0(\mathrm{sat}_k^{\mathbb{Z}_2}) \tag{1.1}$$

*of rings (= a Landau–Ginzburg motivic measure) that maps  $[X, W]$  to the class of  $\mathbf{MF}(W)^{\mathrm{dg},\natural}$  whenever  $X$  is a smooth variety and  $W: X \rightarrow \mathbb{A}^1$  is a proper morphism.*

*In particular,  $\mu$  is a morphism of abelian groups and maps  $[X, W]$  to the class of  $\mathbf{MF}(W)^{\mathrm{dg},\natural}$  whenever  $X$  is a smooth (connected) variety and  $W: X \rightarrow \mathbb{A}^1$  is a projective morphism. These two properties determine  $\mu$  uniquely.*

Let us sketch the main steps of the proof of this theorem.

We first show that  $\mathbf{MF}(W)^{\mathrm{dg}}$  and  $\mathbf{MF}(W)^{\mathrm{dg},\natural}$  are smooth dg categories (Theorem 4.24), and proper if  $W$  is proper (Proposition 4.26). Hence, for proper  $W$ ,  $\mathbf{MF}(W)^{\mathrm{dg},\natural}$  is a saturated dg category and defines an element of  $K_0(\mathrm{sat}_k^{\mathbb{Z}_2})$ . The proof of smoothness takes advantage of good properties of object oriented Čech enhancements of matrix factorization categories; for example, the standard duality and external tensor products admit natural lifts to these enhancements. On the way we show a Thom–Sebastiani Theorem (Theorem 4.23); it says that given smooth varieties  $X$  and  $Y$  with morphisms  $W: X \rightarrow \mathbb{A}^1$  and  $V: Y \rightarrow \mathbb{A}^1$ , the two dg categories  $\mathbf{MF}(W)^{\mathrm{dg}} \otimes \mathbf{MF}(V)^{\mathrm{dg}}$  and  $\mathbf{MF}(W * V)^{\mathrm{dg}}$  are Morita equivalent. If  $W$  is proper, properness follows essentially from [23, Cor. 1.24].

According to [1, Theorem 5.1]),  $K_0(\mathrm{Var}_{\mathbb{A}^1})$  has a presentation with generators the isomorphism classes  $[X, W]$ , where  $X$  is a smooth variety and  $W$  is a proper (or projective) morphism, and relations coming from blowing-ups. Using this, the semi-orthogonal decompositions for projective space bundles and blowing-ups we established in [17, Theorems 3.2 and 3.5] imply that there is a morphism of abelian groups  $\mu: K_0(\mathrm{Var}_{\mathbb{A}^1}) \rightarrow K_0(\mathrm{sat}_k^{\mathbb{Z}_2})$  sending  $[X, W]$  to the class of  $\mathbf{MF}(W)^{\mathrm{dg},\natural}$  for

smooth  $X$  and proper  $W$ , and that this morphism is already uniquely determined by its values on  $[X, W]$  for smooth  $X$  and projective  $W$ .

It remains to show multiplicativity of  $\mu$ . If  $W: X \rightarrow \mathbb{A}^1$  and  $V: Y \rightarrow \mathbb{A}^1$  are proper, the product of the classes of  $\mathbf{MF}(W)^{\mathrm{dg}, \mathfrak{h}}$  and  $\mathbf{MF}(V)^{\mathrm{dg}, \mathfrak{h}}$  in  $K_0(\mathrm{sat}_k^{\mathbb{Z}_2})$  is isomorphic to the class of  $\mathbf{MF}(W * V)^{\mathrm{dg}, \mathfrak{h}}$ , by the Thom–Sebastiani Theorem 4.23. However,  $W * V$  is not proper in general, so it is a priori not clear that  $\mu$  maps  $[X \times Y, W * V]$  to the class of  $\mathbf{MF}(W * V)^{\mathrm{dg}, \mathfrak{h}}$ . To ensure this we furthermore have to compactify the morphism  $W * V$  in a nice way (Proposition 6.1) in order to obtain multiplicativity. This finishes the sketch of proof of Theorem 1.1.

The Landau–Ginzburg measure (1.1) sends  $\mathbb{L}_{(\mathbb{A}^1, 0)} := [\mathbb{A}^1, 0]$  to 1. It is the analog of the motivic measure constructed in [5] using bounded derived categories of coherent sheaves. We refer the reader to the introduction of [17] for more details. There we discuss also our related work.

For the convenience of readers who are more familiar with smooth and proper dg algebras than with saturated dg categories let us explain that the Grothendieck group  $K_0(\mathrm{sat}_k^{\mathbb{Z}_2})$  can also be described using proper and smooth dg algebras. We call two proper and smooth dg algebras “dg Morita equivalent” if their derived categories are connected by a zig-zag of tensor equivalences (cf. Remark 2.39). Define the Grothendieck group  $K_0(\mathrm{prsmalg}_k^{\mathbb{Z}_2})$  of proper and smooth dg algebras as the quotient of the free abelian group on dg Morita equivalence classes  $\underline{A}$  of proper and smooth dg algebras  $A$  by the subgroup generated by the elements  $\underline{R} - (\underline{A} + \underline{B})$  whenever  $R$  is a proper and smooth dg algebra such that there are dg algebras  $A$  and  $B$  together with a dg  $A \otimes B^{\mathrm{op}}$ -module  $N = {}_B N_A$  such that  $R = \begin{pmatrix} A & 0 \\ N & B \end{pmatrix}$  (see Def. 2.42). The tensor product of dg algebras turns  $K_0(\mathrm{prsmalg}_k^{\mathbb{Z}_2})$  into a ring. Under our assumption that  $k$  is a field we show that mapping a proper and smooth dg algebra  $A$  to its triangulated envelope  $\mathrm{Perf}(A)$  induces an isomorphism

$$K_0(\mathrm{prsmalg}_k^{\mathbb{Z}_2}) \xrightarrow{\sim} K_0(\mathrm{sat}_k^{\mathbb{Z}_2})$$

of rings (Proposition 2.26 and Remark 2.45). Using this isomorphism, the Landau–Ginzburg motivic measure (1.1) may be viewed as a morphism of rings

$$\mu: K_0(\mathrm{Var}_{\mathbb{A}^1}) \rightarrow K_0(\mathrm{prsmalg}_k^{\mathbb{Z}_2}).$$

In this interpretation,  $\mu$  can be described more concretely as follows. Given a smooth variety  $X$  and a proper morphism  $W: X \rightarrow \mathbb{A}^1$ , choose a classical generator in each category  $\mathbf{MF}(X, W - a)$  and let  $A_a$  be its endomorphism dg algebra (computed in a suitable enhancement). Then the image of  $[X, W]$  under  $\mu$  is the class of  $\prod_{a \in k} A_a$ .

Section 2 contains the definitions of various Grothendieck rings of dg categories. We even work over an arbitrary commutative ground ring  $k$  there. Besides the Grothendieck rings  $K_0(\mathrm{sat}_k^{\mathbb{Z}_2})$  and  $K_0(\mathrm{prsmalg}_k^{\mathbb{Z}_2})$  explained above we also introduce the modified Grothendieck ring  $K'_0(\mathrm{sat}_k^{\mathbb{Z}_2})$  of saturated dg categories and the

Grothendieck ring  $K_0(\text{prsm}_k^{\mathbb{Z}_2})$  of proper and smooth dg categories. There are canonical morphisms

$$K_0(\text{sat}_k^{\mathbb{Z}_2}) \twoheadrightarrow K'_0(\text{sat}_k^{\mathbb{Z}_2}) \xleftarrow{\sim} K_0(\text{prsm}_k^{\mathbb{Z}_2}) \xleftarrow{\sim} K_0(\text{prsmalg}_k^{\mathbb{Z}_2})$$

of rings. The first morphism is the obvious surjection: the definition of  $K'_0(\text{sat}_k^{\mathbb{Z}_2})$  is obtained from that of  $K_0(\text{sat}_k^{\mathbb{Z}_2})$  by dropping the words “into admissible subcategories”. It is an isomorphism if  $k$  is a field (Proposition 2.26). The other two morphisms are isomorphisms (see Remark 2.45).

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## 2. Grothendieck rings of dg categories

Our aim is to define and compare some Grothendieck rings of saturated dg  $k$ -categories (or proper and smooth dg  $k$ -algebras) where  $k$  is a commutative ring. Initially we follow [5] but pay more attention to finiteness conditions and work over an arbitrary commutative ring. We use notions and results from [11, 30, 32]. For the convenience of the reader we repeat some proofs. In this section dg stands for “differential  $\mathbb{Z}$ -graded”.

**Remark 2.1.** All results of this section (and the results we cite) are also true in the differential  $\mathbb{Z}_n$ -graded setting (where  $\mathbb{Z}_n := \mathbb{Z}/n\mathbb{Z}$ ), for any  $n \in \mathbb{Z}$ , unless said otherwise (the standard notions we use have their obvious counterpart in this setting). The proofs are easily adapted. We could even work over a graded commutative differential  $\mathbb{Z}_n$ -graded  $k$ -algebra  $K$  as in [19].

In fact, in the rest of this article, we only need the differential  $\mathbb{Z}_2$ -graded setting for  $k$  a field. To exclude misunderstandings, a dg module in this setting is a  $\mathbb{Z}_2$ -graded  $k$ -module  $V = V_0 \oplus V_1$  together with a differential  $d: V \rightarrow V$  of degree 1, i. e.  $k$ -linear maps  $d_i: V_i \rightarrow V_{i+1}$  for  $i \in \mathbb{Z}_2$  such that  $d_{i+1}d_i = 0$ .

We choose to explain the case of an arbitrary commutative ring  $k$  since it is only slightly more difficult than that of a field.

**2.1. Dg categories and their module categories.** Our notation coincides with that of [19, sections 2 and 3] (if one puts  $K = k$  there).

Let  $k$  be a commutative ring. We denote by  $C(k)$  the category whose objects are dg  $(k)$ -modules. Morphisms in  $C(k)$  are degree zero morphisms that commute

with the respective differentials. Note that the category  $C(k)$  is abelian and closed symmetric monoidal with the obvious tensor product  $\otimes := \otimes_k$ .

If  $\mathcal{A}$  is a dg category (= a category enriched in  $C(k)$ ) we denote the category with the same objects and closed degree zero morphism (resp. closed degree zero morphisms up to homotopy) by  $Z_0(\mathcal{A})$  (resp.  $[\mathcal{A}]$ .) For example, there is an obvious dg category  $Mod(k)$  such that  $Z_0(Mod(k)) = C(k)$ . We denote the category of small dg categories by  $dgcat_k$ .

Let  $\mathcal{A}$  be a small dg category. We denote by  $Mod(\mathcal{A})$  the dg category of (right) dg  $\mathcal{A}$ -modules (= dg functors  $\mathcal{A}^{op} \rightarrow Mod(k)$ ). The dg functor  $Y: \mathcal{A} \rightarrow Mod(\mathcal{A})$ ,  $A \mapsto Y(A) := \widehat{A} := \mathcal{A}(-, A)$ , is full and faithful and called Yoneda embedding. We write  $C(\mathcal{A}) := Z_0(Mod(\mathcal{A}))$  and  $\mathcal{H}(\mathcal{A}) := [Mod(\mathcal{A})]$  (resp.  $D(\mathcal{A})$ ) for the homotopy (resp. derived) category of dg  $\mathcal{A}$ -modules. We equip  $C(\mathcal{A})$  with the (cofibrantly generated) projective model structure (cf. [19, Thm. 2.2]). Its weak equivalences are the quasi-isomorphisms, and its fibrations are the epimorphisms.

Let  $Mod(\mathcal{A})_{cf} \subset Mod(\mathcal{A})$  be the full dg subcategory of all cofibrant objects in  $C(\mathcal{A})$  (all objects are fibrant). We denote by  $\overline{\mathcal{A}} \subset Mod(\mathcal{A})$  the smallest strict full dg subcategory which contains the zero module, all  $\widehat{A}$ , for  $A \in \mathcal{A}$ , and is closed under cones of closed degree zero morphisms (and then also under all shifts). Any object of  $\overline{\mathcal{A}}$  is a semi-free dg  $\mathcal{A}$ -module and hence cofibrant. The situation is illustrated by the diagram

$$\mathcal{A} \xrightarrow{Y} \overline{\mathcal{A}} \subset Mod(\mathcal{A})_{cf} \subset Mod(\mathcal{A}).$$

The three categories on the right are (strongly) pretriangulated. The canonical full and faithful dg functor  $\mathcal{A}^{pre-tr} \rightarrow Mod(\mathcal{A})$  from [4, §1, §4] (an extended version of the Yoneda embedding) has precisely  $\overline{\mathcal{A}}$  as its essential image, so  $\mathcal{A}^{pre-tr}$  and  $\overline{\mathcal{A}}$  are dg equivalent. So  $\overline{\mathcal{A}}$  is a pretriangulated envelope of  $\mathcal{A}$ . We pass to the respective homotopy categories and obtain the first row in the commutative diagram

$$\begin{array}{ccccc}
 [\mathcal{A}] & \xrightarrow{[Y]} & [\overline{\mathcal{A}}] & \subset & [Mod(\mathcal{A})_{cf}] & \subset & \mathcal{H}(\mathcal{A}) \\
 & & \downarrow \sim & & \cup & \searrow \sim & \downarrow \\
 & & tria(\mathcal{A}) & \subset & thick(\mathcal{A}) & & D(\mathcal{A}) \\
 & & & & & \searrow \sim & \cup \\
 & & & & & & per(\mathcal{A}).
 \end{array}$$

Here  $tria(\mathcal{A})$  is defined to be the smallest strict full triangulated subcategory of  $[Mod(\mathcal{A})_{cf}]$  that contains all  $\widehat{A}$ , for  $A \in \mathcal{A}$ ,  $thick(\mathcal{A})$  is in addition required to be closed under direct summands in  $[Mod(\mathcal{A})_{cf}]$ , and  $per(\mathcal{A})$  is defined to be the thick envelope of  $\{\widehat{A} \mid A \in \mathcal{A}\}$  in  $D(\mathcal{A})$ . The three indicated triangulated equivalences are well-known (or obvious). They show that  $\overline{\mathcal{A}}$  (together with equivalence  $[\overline{\mathcal{A}}] \rightarrow tria(\mathcal{A})$ ) is an enhancement of  $tria(\mathcal{A})$ , and that  $Mod(\mathcal{A})_{cf}$

is an enhancement of  $[Mod(\mathcal{A})_{cf}]$  and  $D(\mathcal{A})$ . We define  $Perf(\mathcal{A})$  to be the full subcategory of  $Mod(\mathcal{A})_{cf}$  whose objects coincide with those of  $thick(\mathcal{A})$ . Then  $Perf(\mathcal{A})$  is (strongly) pretriangulated and an enhancement of  $thick(\mathcal{A})$  and  $per(\mathcal{A})$ .

The categories  $[Mod(\mathcal{A})_{cf}]$  and  $D(\mathcal{A})$  have arbitrary (in particular countable) coproducts. Hence they are Karoubian (= idempotent complete), and so are  $thick(\mathcal{A})$  and  $per(\mathcal{A})$ . In particular  $thick(\mathcal{A})$  can be viewed as the Karoubi envelope (= idempotent completion) of  $tria(\mathcal{A})$ . Note also that  $D(\mathcal{A})$  is compactly generated and that the subcategory  $D(\mathcal{A})^c$  of compact objects in  $D(\mathcal{A})$  is precisely  $per(\mathcal{A})$ , i. e.  $D(\mathcal{A})^c = per(\mathcal{A})$  (cf. the discussion around equation (2.4) in [19]).

**2.2. Triangulated dg categories.** Recall that a dg functor  $F: \mathcal{A} \rightarrow \mathcal{B}$  is a **quasi-equivalence** if

- (qe1) for all objects  $a_1, a_2 \in \mathcal{A}$ , the morphism  $F: \mathcal{A}(a_1, a_2) \rightarrow \mathcal{B}(Fa_1, Fa_2)$  is a quasi-isomorphism, and
- (qe2) the induced functor  $[F]: [\mathcal{A}] \rightarrow [\mathcal{B}]$  on homotopy categories is essentially surjective.

If (qe1) holds, then (qe2) is equivalent to the condition that  $[F]: [\mathcal{A}] \rightarrow [\mathcal{B}]$  is an equivalence.

**Definition 2.2** ([32, Def. 2.4.5]). A dg category  $\mathcal{A}$  is **triangulated** if the Yoneda embedding induces a quasi-equivalence  $\mathcal{A} \rightarrow Perf(\mathcal{A})$ . (It is enough to require that  $[\mathcal{A}] \rightarrow [Perf(\mathcal{A})]$  is essentially surjective.)

**Lemma 2.3.** *A dg category  $\mathcal{A}$  is triangulated if and only if it is pretriangulated and  $[\mathcal{A}]$  is Karoubian.*

*Proof.* Note that  $\mathcal{A} \rightarrow Perf(\mathcal{A})$  factors as  $\mathcal{A} \hookrightarrow \overline{\mathcal{A}} \subset Perf(\mathcal{A})$ .

Assume that  $\mathcal{A}$  is triangulated. Then it is clear that  $\mathcal{A} \rightarrow \overline{\mathcal{A}}$  is a quasi-equivalence which precisely means that  $\mathcal{A}$  is pretriangulated. Since  $[\mathcal{A}] \rightarrow [Perf(\mathcal{A})]$  is an equivalence and  $[Perf(\mathcal{A})] = thick(\mathcal{A})$  is Karoubian, the same is true for  $[\mathcal{A}]$ .

Conversely, if  $\mathcal{A}$  is pretriangulated and  $[\mathcal{A}]$  is Karoubian, then

$$[\mathcal{A}] \xrightarrow{\sim} [\overline{\mathcal{A}}] \xrightarrow{\sim} tria(\mathcal{A}),$$

so  $tria(\mathcal{A})$  is Karoubian and coincides with its Karoubi envelope  $thick(\mathcal{A}) = [Perf(\mathcal{A})]$ . This implies that  $[\mathcal{A}] \rightarrow [Perf(\mathcal{A})]$  is an equivalence, so  $\mathcal{A}$  is triangulated. □

**Corollary 2.4** ([32, Lemma. 2.6]). *Let  $\mathcal{A}$  be a dg category. Then  $Perf(\mathcal{A})$  is a triangulated dg category, i. e. the morphism  $Perf(\mathcal{A}) \rightarrow Perf(Perf(\mathcal{A}))$  induced by the Yoneda functor is a quasi-equivalence.*

*Proof.* We have observed above that  $Perf(\mathcal{A})$  is pretriangulated, and that  $[Perf(\mathcal{A})] = thick(\mathcal{A})$  is Karoubian. □

So passing from a dg category  $\mathcal{A}$  to  $Perf(\mathcal{A})$  means taking a triangulated envelope.

**2.3. Some general results.**

**Lemma 2.5.** *Let  $F: \mathcal{A} \rightarrow \mathcal{B}$  be a dg functor and assume that the following condition on  $\mathcal{A}$  holds: For all  $X \in \mathcal{A}$  and  $r \in \mathbb{Z}$  there is an object  $Z \in \mathcal{A}$  and morphisms  $f \in \mathcal{A}(X, Z)^{-r}$  and  $g \in \mathcal{A}(Z, X)^r$  such that  $df = 0$ ,  $dg = 0$ , and  $fg$  is homotopic to  $\text{id}_Z$  and  $gf$  is homotopic to  $\text{id}_X$ . In other words, the essential image of  $[\mathcal{A}]$  in  $[\text{Mod}(\mathcal{A})]$  is closed under all shifts. (This condition is for example satisfied if  $\mathcal{A}$  is pretriangulated or closed under all shifts.)*

*Then  $F$  is a quasi-equivalence if and only if  $[F]: [\mathcal{A}] \rightarrow [\mathcal{B}]$  is an equivalence.*

*Proof.* We prove the non-trivial implication. Assume that  $[F]$  is an equivalence. Then obviously (qe2) is satisfied. Let  $A, X \in \mathcal{A}$ . Let  $r \in \mathbb{Z}$  and let  $Z, f, g$  be as above. Consider the commutative diagram

$$\begin{array}{ccc} [r]\mathcal{A}(A, X) & \xrightarrow{f \circ ?} & \mathcal{A}(A, Z) \\ \downarrow [r]F & & \downarrow F \\ [r]\mathcal{B}(FA, FX) & \xrightarrow{F(f) \circ ?} & \mathcal{B}(FA, FZ) \end{array}$$

in  $C(k)$ . If we apply  $H^0$  to this diagram, the horizontal arrows and the vertical arrow on the right become isomorphisms. Hence the same is true for the vertical arrow on the left, i. e.  $H^r(F): H^r(\mathcal{A}(A, X)) \rightarrow H^r(\mathcal{A}(FA, FX))$  is an isomorphism. This proves (qe1). □

Any dg functor  $f: \mathcal{A} \rightarrow \mathcal{B}$  gives rise to the dg functor  $f^* = \text{prod}_{\mathcal{A}}^{\mathcal{B}} = (- \otimes_{\mathcal{A}} \mathcal{B}): \text{Mod}(\mathcal{A}) \rightarrow \text{Mod}(\mathcal{B})$  called extension of scalars functor, and we have a commutative diagram

$$\begin{array}{ccc} \mathcal{A} & \xrightarrow{Y} & \text{Mod}(\mathcal{A}) \\ \downarrow f & & \downarrow f^* \\ \mathcal{B} & \xrightarrow{Y} & \text{Mod}(\mathcal{B}) \end{array} \tag{2.1}$$

since  $f^*(\widehat{A}) = \widehat{f(A)}$  for all  $A \in \mathcal{A}$ .

**Lemma 2.6.** *Let  $f: \mathcal{A} \rightarrow \mathcal{B}$  be a morphism in  $\text{dgc}at_k$ . Then:*

(a) *The extension of scalars functor  $f^*$  induces dg functors*

$$\begin{aligned} f^*: \text{Mod}(\mathcal{A})_{cf} &\rightarrow \text{Mod}(\mathcal{B})_{cf}, \\ f^*: \text{Perf}(\mathcal{A}) &\rightarrow \text{Perf}(\mathcal{B}), \quad \text{and} \quad f^*: \overline{\mathcal{A}} \rightarrow \overline{\mathcal{B}}. \end{aligned}$$

(b) *If  $f$  is full and faithful, then all these functors  $f^*$  are full and faithful.*

(c) If  $f$  is a quasi-equivalence, then

$$f^*: \text{Mod}(\mathcal{A})_{cf} \rightarrow \text{Mod}(\mathcal{B})_{cf},$$

$$f^*: \text{Perf}(\mathcal{A}) \rightarrow \text{Perf}(\mathcal{B}), \quad \text{and} \quad f^*: \overline{\mathcal{A}} \rightarrow \overline{\mathcal{B}}$$

are quasi-equivalences.

*Proof.* We prove (a). Note first that  $f^*$  (viewed as a functor  $C(\mathcal{A}) \rightarrow C(\mathcal{B})$  where both categories are viewed as model categories with the projective model structure) maps cofibrations to cofibrations since its right adjoint  $f_* = \text{res}_{\mathcal{A}}^{\mathcal{B}}$  maps trivial fibrations (= epimorphic quasi-isomorphisms) to trivial fibrations. In particular  $f^*$  induces a dg functor  $f^*: \text{Mod}(\mathcal{A})_{cf} \rightarrow \text{Mod}(\mathcal{B})_{cf}$ . For the remaining statements of (a) use  $f^*(\widehat{A}) = \widehat{f(A)}$  for all  $A \in \mathcal{A}$  and the fact that a dg functor preserves shifts and cones of closed degree zero morphisms (see [5, 4.3]).

In order to prove (b) assume that  $f$  is full and faithful. The right adjoint of

$$f^* = \text{prod}_{\mathcal{A}}^{\mathcal{B}}: \text{Mod}(\mathcal{A}) \rightarrow \text{Mod}(\mathcal{B}) \tag{2.2}$$

is restriction  $f_* = \text{res}_{\mathcal{A}}^{\mathcal{B}}$ . Hence  $f^*$  is full and faithful if and only if the unit  $X \rightarrow \text{res}_{\mathcal{A}}^{\mathcal{B}}(\text{prod}_{\mathcal{A}}^{\mathcal{B}}(X))$  of this adjunction is an isomorphism for all  $X \in \text{Mod}(\mathcal{A})$ . But

$$\begin{aligned} &(\text{res}_{\mathcal{A}}^{\mathcal{B}}(\text{prod}_{\mathcal{A}}^{\mathcal{B}}(X)))(A) = (\text{prod}_{\mathcal{A}}^{\mathcal{B}}(X))(f(A)) \\ &= \text{cok} \left( \bigoplus_{A''', A'' \in \mathcal{A}} X(A''') \otimes \mathcal{A}(A'', A''') \otimes \mathcal{B}(f(A), f(A'')) \right) \\ &\hspace{15em} \rightarrow \bigoplus_{A' \in \mathcal{A}} X(A') \otimes \mathcal{B}(f(A), f(A')) \\ &\xleftarrow{\sim} \text{cok} \left( \bigoplus_{A''', A'' \in \mathcal{A}} X(A''') \otimes \mathcal{A}(A'', A''') \otimes \mathcal{A}(A, A'') \right) \\ &\hspace{15em} \rightarrow \bigoplus_{A' \in \mathcal{A}} X(A') \otimes \mathcal{A}(A, A') \\ &\xrightarrow{\sim} X(A), \end{aligned}$$

where the first arrow is an isomorphism since  $f$  is full and faithful, and the second arrow is the obvious evaluation morphism. Under this identification the unit becomes the identity which shows that the functor  $f^*$  in (2.2) is full and faithful. Then  $f^*$  is obviously also full and faithful on all full subcategories of  $\text{Mod}(\mathcal{A})$ .

Let us prove (c). Assume that  $f$  is a quasi-equivalence. View  $\mathcal{A} \subset \text{Mod}(\mathcal{A})$  and  $\mathcal{B} \subset \text{Mod}(\mathcal{B})$  as full dg subcategories via the Yoneda embedding. It is easy to prove that  $f^*: \overline{\mathcal{A}} \rightarrow \overline{\mathcal{B}}$  is a quasi-equivalence (this statement corresponds to [6, Prop. 2.5] under the dg equivalences  $\mathcal{A}^{\text{pre-tr}} \xrightarrow{\sim} \overline{\mathcal{A}}$  and  $\mathcal{B}^{\text{pre-tr}} \xrightarrow{\sim} \overline{\mathcal{B}}$ ). In

particular,  $[f^*]: [\overline{\mathcal{A}}] \rightarrow [\overline{\mathcal{B}}]$  and hence  $[f^*]: \text{tria}(\mathcal{A}) \rightarrow \text{tria}(\mathcal{B})$  are equivalences. It extends to an equivalence between the corresponding Karoubi envelopes given by  $[f^*]: \text{thick}(\mathcal{A}) \rightarrow \text{thick}(\mathcal{B})$ . This also implies that

$$L f^* = (- \otimes_{\mathcal{A}}^L \mathcal{B}): D(\mathcal{A}) \rightarrow D(\mathcal{B})$$

is an equivalence [10, 4.2, Lemma]. Now Lemma 2.5 shows that

$$f^*: \text{Perf}(\mathcal{A}) \rightarrow \text{Perf}(\mathcal{B}) \quad \text{and} \quad f^*: \text{Mod}(\mathcal{A})_{cf} \rightarrow \text{Mod}(\mathcal{B})_{cf}$$

are quasi-equivalences. □

**Lemma 2.7** ([5, Lemma 4.16]). *Let  $\mathcal{A}$  be a full dg subcategory of a dg category  $\mathcal{B}$ . Assume that  $[\mathcal{A}] \subset [\mathcal{B}]$  is dense in the sense that any object of  $[\mathcal{B}]$  is a direct summand of an object of  $[\mathcal{A}]$ . Then  $[\overline{\mathcal{A}}]$  is dense in  $[\overline{\mathcal{B}}]$  where we view  $\overline{\mathcal{A}}$  as a full dg subcategory of  $\overline{\mathcal{B}}$  via Lemma 2.6.(b).*

*Proof.* We view  $\mathcal{A} \subset \overline{\mathcal{A}}$  and  $\mathcal{B} \subset \overline{\mathcal{B}}$  as full dg subcategories via the Yoneda embedding. Any object of  $[\mathcal{B}]$  is a direct summand of an object of  $[\mathcal{A}]$  and hence also of an object of  $[\overline{\mathcal{A}}]$ . If an object of  $[\overline{\mathcal{B}}]$  is a direct summand of an object of  $[\overline{\mathcal{A}}]$ , then all its shifts have the same property.

Assume that  $f: X \rightarrow Y$  is a closed degree zero morphism in  $\overline{\mathcal{B}}$ , and that  $X \oplus X' \cong M$  and  $Y \oplus Y' \cong N$  in  $[\overline{\mathcal{B}}]$  with  $M \in [\overline{\mathcal{A}}]$  and  $N \in [\overline{\mathcal{A}}]$ . Consider  $f \oplus 0: X \oplus X' \rightarrow Y \oplus Y'$  and let  $g: M \rightarrow N$  be a closed degree zero morphism in  $\overline{\mathcal{A}}$  corresponding to  $f \oplus 0$  in  $[\overline{\mathcal{B}}]$ . Then

$$\text{Cone}(g) \cong \text{Cone}(f \oplus 0) \cong \text{Cone}(f) \oplus Y' \oplus [1]X'$$

in  $[\overline{\mathcal{B}}]$ . Hence  $\text{Cone}(f)$  is a direct summand of the object  $\text{Cone}(g) \in [\overline{\mathcal{A}}]$ .

Now use that any object of  $[\overline{\mathcal{B}}]$  is built up from the objects of  $\mathcal{B}$  using shifts and cones of closed degree zero morphisms. □

**2.4. Perfection of tensor products.**

**Proposition 2.8** (cf. [5, Prop. 4.17]). *Let  $\mathcal{A}$  and  $\mathcal{B}$  be dg categories. Then the dg functor  $f: \mathcal{A} \otimes \mathcal{B} \rightarrow \text{Perf}(\mathcal{A}) \otimes \mathcal{B}$  obtained from  $\mathcal{A} \rightarrow \text{Perf}(\mathcal{A})$  is full and faithful and induces (by extension of scalars along  $f$ ) a quasi-equivalence*

$$f^*: \text{Perf}(\mathcal{A} \otimes \mathcal{B}) \rightarrow \text{Perf}(\text{Perf}(\mathcal{A}) \otimes \mathcal{B})$$

of dg categories.

*Proof.* The sequence  $\mathcal{A} \hookrightarrow \overline{\mathcal{A}} \subset \text{Perf}(\mathcal{A})$  of full and faithful dg functors yields a sequence

$$\mathcal{A} \otimes \mathcal{B} \hookrightarrow \overline{\mathcal{A}} \otimes \mathcal{B} \subset \text{Perf}(\mathcal{A}) \otimes \mathcal{B}$$

of full and faithful dg functors whose composition is  $f$ . By Lemma 2.6.(b) we obtain full and faithful dg functors

$$\overline{\mathcal{A} \otimes \mathcal{B}} \hookrightarrow \overline{\overline{\mathcal{A}} \otimes \mathcal{B}} \subset \overline{\text{Perf}(\mathcal{A}) \otimes \mathcal{B}}.$$

The functor on the left is an equivalence of dg categories (and in particular a quasi-equivalence) since both categories are built up from the objects of  $\mathcal{A} \otimes \mathcal{B}$  using shifts and cones of closed degree zero morphisms. The first row of the commutative diagram

$$\begin{array}{ccccc} \overline{\mathcal{A} \otimes \mathcal{B}} & \xrightarrow{\sim} & \overline{\overline{\mathcal{A}} \otimes \mathcal{B}} & \subset & \overline{\text{Perf}(\mathcal{A}) \otimes \mathcal{B}} \\ \downarrow \sim & & \downarrow \sim & & \downarrow \sim \\ \text{tria}(\mathcal{A} \otimes \mathcal{B}) & \xrightarrow{\sim} & \text{tria}(\overline{\mathcal{A}} \otimes \mathcal{B}) & \subset & \text{tria}(\text{Perf}(\mathcal{A}) \otimes \mathcal{B}) \\ \downarrow \wr & & \downarrow \wr & & \downarrow \wr \\ \text{thick}(\mathcal{A} \otimes \mathcal{B}) & \xrightarrow{\sim} & \text{thick}(\overline{\mathcal{A}} \otimes \mathcal{B}) & = & \text{thick}(\text{Perf}(\mathcal{A}) \otimes \mathcal{B}) \\ \parallel & & \parallel & & \parallel \\ \text{Perf}(\mathcal{A} \otimes \mathcal{B}) & \xrightarrow{\sim} & \text{Perf}(\overline{\mathcal{A}} \otimes \mathcal{B}) & = & \text{Perf}(\text{Perf}(\mathcal{A}) \otimes \mathcal{B}) \end{array}$$

is obtained by passing to the respective homotopy categories. Its left arrow is an equivalence, and we claim that its inclusion is dense: Since  $[\overline{\mathcal{A}}] \xrightarrow{\sim} \text{tria}(\mathcal{A})$ , the inclusion  $[\overline{\mathcal{A}}] \subset [\text{Perf}(\mathcal{A})] = \text{thick}(\mathcal{A})$  is dense; then  $[\overline{\mathcal{A}} \otimes \mathcal{B}] \subset [\text{Perf}(\mathcal{A}) \otimes \mathcal{B}]$  is dense, too, and Lemma 2.7 shows our claim. The second row is in the obvious way equivalent to the first one, and passing to the third row means taking the respective Karoubi envelopes; in particular, the dense inclusion becomes an equality. The fourth row is equal to the third row, and Lemma 2.5 proves that both arrows in  $\text{Perf}(\mathcal{A} \otimes \mathcal{B}) \rightarrow \text{Perf}(\overline{\mathcal{A}} \otimes \mathcal{B}) \rightarrow \text{Perf}(\text{Perf}(\mathcal{A}) \otimes \mathcal{B})$  are quasi-equivalences. The composition of these arrows is  $f^*$ .  $\square$

We equip  $\text{dgc}at_k$  with the (cofibrantly generated) model structure<sup>1</sup> from [29] (cf. [19, section 2.7]). Its weak equivalences are the quasi-equivalences, and the cofibrant dg categories are precisely the retracts of semi-free dg categories. We denote the homotopy category of  $\text{dgc}at_k$  with respect to these weak equivalences by  $\text{Heq}_k$ . We fix a cofibrant replacement functor  $Q$ . If  $\mathcal{A}$  and  $\mathcal{B}$  are dg categories we define  $\mathcal{A} \otimes^L \mathcal{B} := Q(\mathcal{A}) \otimes Q(\mathcal{B})$  and

$$\mathcal{A} \odot \mathcal{B} := \text{Perf}(\mathcal{A} \otimes^L \mathcal{B}). \tag{2.3}$$

One may consider  $\mathcal{A} \odot \mathcal{B}$  as a triangulated envelope of  $\mathcal{A} \otimes^L \mathcal{B}$  (cf. Corollary 2.4).

<sup>1</sup> In case that  $k$  is a field (which is all we need in this article), the rest of this section can be simplified: we don't need this model structure and can assume that  $Q(\mathcal{A}) = \mathcal{A}$  for any dg category  $\mathcal{A}$  (since all we need is that  $Q(\mathcal{A})$  is h-flat; but any dg module over a field  $k$  is k-h-flat).

**Lemma 2.9.** *Quasi-equivalences  $\mathcal{A} \rightarrow \mathcal{A}'$  and  $\mathcal{B} \rightarrow \mathcal{B}'$  give rise to a quasi-equivalence  $\mathcal{A} \odot \mathcal{B} \rightarrow \mathcal{A}' \odot \mathcal{B}'$ .*

*Proof.* Observe that  $\mathcal{A} \otimes^{\mathbf{L}} \mathcal{B} = Q(\mathcal{A}) \otimes Q(\mathcal{B}) \rightarrow Q(\mathcal{A}') \otimes Q(\mathcal{B}) = \mathcal{A}' \otimes^{\mathbf{L}} \mathcal{B}$  is a quasi-equivalence since the cofibrant dg category  $Q(\mathcal{B})$  is k-h-flat (by [19, Lemmata 2.14 and 2.15]). Hence we obtain a quasi-equivalence  $\mathcal{A} \odot \mathcal{B} \rightarrow \mathcal{A}' \odot \mathcal{B}$  by Lemma 2.6.(c).  $\square$

**Proposition 2.10.** *Let  $\mathcal{A}$  and  $\mathcal{B}$  be dg categories. Then the natural morphism*

$$\mathcal{A} \odot \mathcal{B} = \text{Perf}(\mathcal{A} \otimes^{\mathbf{L}} \mathcal{B}) \rightarrow \text{Perf}((\text{Perf}(\mathcal{A})) \otimes^{\mathbf{L}} (\text{Perf}(\mathcal{B}))) = \text{Perf}(\mathcal{A}) \odot \text{Perf}(\mathcal{B})$$

*in  $\text{dgcat}_k$  is a quasi-equivalence (and becomes an isomorphism in  $\text{Heq}_k$ ).*

*Proof.* Let  $Y: \mathcal{A} \rightarrow \text{Perf}(\mathcal{A})$  be the full and faithful Yoneda dg functor. The cofibrant replacement functor  $Q$  comes with a natural transformation  $Q \rightarrow \text{id}$  and yields the commutative square

$$\begin{array}{ccc} Q(\mathcal{A}) & \xrightarrow{Q(Y)} & Q(\text{Perf}(\mathcal{A})) \\ \downarrow & & \downarrow \\ \mathcal{A} & \xrightarrow{Y} & \text{Perf}(\mathcal{A}) \end{array}$$

whose vertical arrows are trivial fibrations. We tensor this diagram with  $Q(\mathcal{B})$  and obtain the commutative square

$$\begin{array}{ccc} Q(\mathcal{A}) \otimes Q(\mathcal{B}) & \xrightarrow{Q(Y) \otimes \text{id}} & Q(\text{Perf}(\mathcal{A})) \otimes Q(\mathcal{B}) \\ \downarrow & & \downarrow \\ \mathcal{A} \otimes Q(\mathcal{B}) & \xrightarrow{Y \otimes \text{id}} & \text{Perf}(\mathcal{A}) \otimes Q(\mathcal{B}) \end{array}$$

whose vertical arrows are still quasi-equivalences since  $Q(\mathcal{B})$  is k-h-flat (they are even trivial fibrations by the characterization of the trivial fibrations, see [19, after Thm. 2.11]).

These morphisms of dg categories induce by extension of scalars a commutative diagram

$$\begin{array}{ccc} \text{Perf}(Q(\mathcal{A}) \otimes Q(\mathcal{B})) & \xrightarrow{(Q(Y) \otimes \text{id})^*} & \text{Perf}(Q(\text{Perf}(\mathcal{A})) \otimes Q(\mathcal{B})) \\ \downarrow & & \downarrow \\ \text{Perf}(\mathcal{A} \otimes Q(\mathcal{B})) & \xrightarrow{(Y \otimes \text{id})^*} & \text{Perf}(\text{Perf}(\mathcal{A}) \otimes Q(\mathcal{B})) \end{array} \tag{2.4}$$

whose vertical arrows are quasi-equivalences (Lemma 2.6.(c)). The lower horizontal arrow is a quasi-equivalence by Proposition 2.8. This implies that the upper horizontal arrow

$$\text{Perf}(\mathcal{A} \otimes^{\mathbf{L}} \mathcal{B}) \xrightarrow{(Y \otimes^{\mathbf{L}} \text{id})^*} \text{Perf}(\text{Perf}(\mathcal{A}) \otimes^{\mathbf{L}} \mathcal{B})$$

is a quasi-equivalence. The same reasoning shows that the morphism  $(\text{id} \otimes^{\mathbf{L}} Y)^*$  from the right-hand side to  $\text{Perf}((\text{Perf}(\mathcal{A})) \otimes^{\mathbf{L}} \text{Perf}(\mathcal{B}))$  is a quasi-equivalence.  $\square$

**2.5. Proper, smooth, and saturated dg categories.**

**Definition 2.11** (cf. [32, Def. 2.4], [30, Def. 2.3]). Let  $\mathcal{A}$  be a dg category.

- (a)  $\mathcal{A}$  is **locally (k-)perfect** (or **locally (k-)proper**) if  $\mathcal{A}(A, A')$  is a perfect dg  $k$ -module (i. e. in  $\text{per}(k)$ ) for all  $A, A' \in \mathcal{A}$ .
- (b)  $\mathcal{A}$  **has a compact generator** if the triangulated category  $D(\mathcal{A})$  has a compact generator. An equivalent condition is that  $\text{per}(\mathcal{A})$  has a classical generator (use [2, Thm. 2.1.2]).
- (c)  $\mathcal{A}$  is **(k-)proper** if it is locally perfect and has a compact generator.
- (d)  $\mathcal{A}$  is **(k-)smooth** if  $\mathcal{A}$  considered as a dg  $Q(\mathcal{A}) \otimes Q(\mathcal{A})^{\text{op}}$ -module, is in  $\text{per}(Q(\mathcal{A}) \otimes Q(\mathcal{A})^{\text{op}})$ .
- (e)  $\mathcal{A}$  is **(k-)saturated** if it is (k-)proper, (k-)smooth and triangulated (see Def. 2.2).

If  $A$  is a dg algebra, then  $\widehat{A}$  is a compact generator of  $D(A)$ , hence  $A$  has a compact generator. Hence  $A$  is proper if and only if it is locally perfect, i. e. if  $A$  is a perfect dg  $k$ -module. The same statement is true for  $\mathcal{A}$  a dg category with finitely many isoclasses of objects in  $[\mathcal{A}]$ :  $\bigoplus_{A \in [\mathcal{A}]/\cong} \widehat{A}$  is a compact generator of  $D(\mathcal{A})$ .

**Lemma 2.12.** *The notions introduced in Definitions 2.11 and 2.2 are all invariant under quasi-equivalences.*

*Proof.* Let  $F: \mathcal{A} \rightarrow \mathcal{B}$  be a quasi-equivalence.

Locally perfect: If all  $\mathcal{B}(B, B')$  are perfect dg  $k$ -modules, the same is true for all  $\mathcal{A}(A, A')$ . If all  $\mathcal{A}(A, A')$  are in  $\text{per}(k)$ , then all  $\mathcal{B}(F(A), F(A'))$  are in  $\text{per}(k)$ . In order to show that all  $\mathcal{B}(B, B')$  are perfect use that  $[F]$  is an equivalence.

Has a compact generator: It is well-known (cf. proof of Lemma 2.6.(c)) that  $F$  induces an equivalence  $\mathbf{L}F^*: D(\mathcal{A}) \rightarrow D(\mathcal{B})$  of triangulated categories.

Proper: Clear from above.

Smooth: See [19, Lemma 3.12].

Triangulated: By Lemma 2.6 we have a commutative diagram

$$\begin{array}{ccc} \mathcal{A} & \xrightarrow{Y} & \text{Perf}(\mathcal{A}) \\ \downarrow f & & \downarrow f^* \\ \mathcal{B} & \xrightarrow{Y} & \text{Perf}(\mathcal{B}) \end{array}$$

whose vertical arrows are quasi-equivalences. Hence the upper horizontal arrow is a quasi-equivalence if and only if the lower horizontal arrow is a quasi-equivalence.

Saturated: Clear from above.  $\square$

**Lemma 2.13** ([32, Lemma 2.6]). *Let  $\mathcal{A}$  be a dg category. Then  $\mathcal{A}$  is locally perfect (resp. has a compact generator resp. is proper resp. is smooth) if and only if  $\text{Perf}(\mathcal{A})$  has the corresponding property.*

*Proof.* Locally perfect: The Yoneda functor  $Y: \mathcal{A} \rightarrow \text{Perf}(\mathcal{A})$  is full and faithful. Hence  $\mathcal{A}$  is certainly locally perfect if  $\text{Perf}(\mathcal{A})$  is locally perfect. Conversely assume that  $\mathcal{A}$  is locally perfect. It is easy to see that  $\overline{\mathcal{A}}(A, A')$  is a perfect dg module for all  $A, A' \in \overline{\mathcal{A}}$ . If  $U$  is in  $\text{Perf}(\mathcal{A})$ , then there is an object  $U' \in \text{Perf}(\mathcal{A})$  and an object  $X \in \overline{\mathcal{A}}$  such that  $U \oplus U' \cong X$  in  $[\text{Perf}(\mathcal{A})]$ . Let  $Y \in \overline{\mathcal{A}}$ . Then  $\text{Perf}(\mathcal{A})(U, Y)$  is a direct summand of  $\text{Perf}(\mathcal{A})(U \oplus U', Y)$  which is in  $D(\mathbf{k})$  (even in  $[\text{Mod}(\mathbf{k})]$ ) isomorphic to  $\text{Perf}(\mathcal{A})(X, Y) = \overline{\mathcal{A}}(X, Y)$ . Hence  $\text{Perf}(\mathcal{A})(U, Y)$  is a perfect dg  $\mathbf{k}$ -module. Similarly we show that  $\text{Perf}(\mathcal{A})(U, V)$  is a perfect dg  $\mathbf{k}$ -module for  $V$  in  $\text{Perf}(\mathcal{A})$ . This implies that  $\text{Perf}(\mathcal{A})$  is locally perfect.

Let us prove the remaining claims. The Yoneda functor  $Y: \mathcal{A} \rightarrow \text{Perf}(\mathcal{A})$  gives rise to the commutative diagram

$$\begin{array}{ccc} \mathcal{A} & \xrightarrow{Y} & \text{Perf}(\mathcal{A}) \\ \downarrow Y & & \downarrow Y^* \\ \text{Perf}(\mathcal{A}) & \xrightarrow{Y} & \text{Perf}(\text{Perf}(\mathcal{A})), \end{array}$$

by Lemma 2.6.(a). Since  $\text{Perf}(\mathcal{A})$  is triangulated (Corollary 2.4), the lower horizontal arrow is a quasi-equivalence. Note that  $[\text{Perf}(\mathcal{A})] \xrightarrow{\sim} \text{per}(\mathcal{A})$  is classically generated by the objects in the image of the Yoneda functor  $[Y]: [\mathcal{A}] \rightarrow [\text{Perf}(\mathcal{A})]$ . These statements imply that  $Y^*$  induces an equivalence on homotopy categories and hence is a quasi-equivalence by Lemma 2.5. In particular

$$\mathbf{L}Y^* = (- \otimes_{\mathcal{A}}^{\mathbf{L}} \text{Perf}(\mathcal{A})): \text{per}(\mathcal{A}) \rightarrow \text{per}(\text{Perf}(\mathcal{A}))$$

is an equivalence, and hence also

$$\mathbf{L}Y^* = (- \otimes_{\mathcal{A}}^{\mathbf{L}} \text{Perf}(\mathcal{A})): D(\mathcal{A}) \rightarrow D(\text{Perf}(\mathcal{A}))$$

(use [10, 4.2, Lemma]).

This immediately implies the claims concerning compact generators and properness, and also the claim concerning smoothness (by [19, Thm. 3.17]).  $\square$

**2.6. Smoothness and properness of tensor products.** We start with some observations. Let  $\mathcal{R}$  and  $\mathcal{S}$  be dg categories, and assume that  $\mathcal{R}$  is  $\mathbf{k}$ -h-flat (i. e. all morphism spaces  $\mathcal{R}(R, R')$  are  $\mathbf{k}$ -h-flat). Then the obvious dg bifunctor  $\text{Mod}(\mathcal{R}) \times \text{Mod}(\mathcal{S}) \rightarrow \text{Mod}(\mathcal{R} \otimes \mathcal{S}), (X, Y) \mapsto X \otimes Y$ , induces the left derived functor

$$\begin{aligned} D(\mathcal{R}) \times D(\mathcal{S}) &\rightarrow D(\mathcal{R} \otimes \mathcal{S}), \\ (X, Y) &\mapsto X \otimes^{\mathbf{L}} Y := c(X) \otimes Y, \end{aligned} \tag{2.5}$$

where  $c(X) \rightarrow X$  is a cofibrant resolution; note for this that a cofibrant dg  $\mathcal{R}$ -module is a retract of a semi-free dg  $\mathcal{R}$ -module [19, Lemma 2.7] and hence k-h-flat by our assumption on  $\mathcal{R}$ . If  $Y$  is k-h-flat, then the obvious morphism  $X \otimes^{\mathbf{L}} Y \rightarrow X \otimes Y$  is an isomorphism. It is easy to see that the bifunctor (2.5) induces a bifunctor

$$\text{per}(\mathcal{R}) \times \text{per}(\mathcal{S}) \rightarrow \text{per}(\mathcal{R} \otimes \mathcal{S}).$$

In particular (for  $\mathcal{R} = \mathcal{S} = \mathbf{k}$ ), if  $X$  and  $Y$  are perfect dg modules, then

$$X \otimes^{\mathbf{L}} Y \in \text{per}(\mathbf{k});$$

if they are perfect and  $Y$  is k-h-flat, then  $X \otimes Y$  is in  $\text{per}(\mathbf{k})$ .

**Lemma 2.14.** *Let  $\mathcal{A}$  and  $\mathcal{B}$  be smooth dg categories. Then  $\mathcal{A} \otimes^{\mathbf{L}} \mathcal{B}$  is smooth.*

*Proof.* Let  $Q(\mathcal{A}) \rightarrow \mathcal{A}$  and  $Q(\mathcal{B}) \rightarrow \mathcal{B}$  be cofibrant resolutions. Then

$$Q(\mathcal{A}) \in \text{per}(Q(\mathcal{A}) \otimes Q(\mathcal{A})^{\text{op}}) \quad \text{and} \quad Q(\mathcal{B}) \in \text{per}(Q(\mathcal{B}) \otimes Q(\mathcal{B})^{\text{op}})$$

by assumption.

Note that both diagonal dg bimodules  $Q(\mathcal{A})$  and  $Q(\mathcal{B})$  are k-h-flat, and that both dg categories  $\mathcal{R} = Q(\mathcal{A}) \otimes Q(\mathcal{A})^{\text{op}}$  and  $\mathcal{S} = Q(\mathcal{B}) \otimes Q(\mathcal{B})^{\text{op}}$  are k-h-flat, by [19, Lemma 2.14]. Then, using the obvious isomorphism

$$\mathcal{R} \otimes \mathcal{S} \xrightarrow{\sim} (Q(\mathcal{A}) \otimes Q(\mathcal{B})) \otimes (Q(\mathcal{A}) \otimes Q(\mathcal{B}))^{\text{op}},$$

the above discussion shows that

$$Q(\mathcal{A}) \otimes Q(\mathcal{B}) \in \text{per}((Q(\mathcal{A}) \otimes Q(\mathcal{B})) \otimes (Q(\mathcal{A}) \otimes Q(\mathcal{B}))^{\text{op}}),$$

and this dg bimodule is the diagonal bimodule. Since  $Q(\mathcal{A}) \otimes Q(\mathcal{B})$  is k-h-flat this implies that  $\mathcal{A} \otimes^{\mathbf{L}} \mathcal{B} = Q(\mathcal{A}) \otimes Q(\mathcal{B})$  is smooth (since smoothness can be checked using a k-h-flat resolution, by [19, Lemma 3.6]).  $\square$

**Lemma 2.15.** *Let  $\mathcal{A}$  and  $\mathcal{B}$  be locally perfect dg categories. Then  $\mathcal{A} \otimes^{\mathbf{L}} \mathcal{B}$  is a locally perfect dg category. In particular, if  $\mathcal{A}$  and  $\mathcal{B}$  are proper dg algebras, then  $\mathcal{A} \otimes^{\mathbf{L}} \mathcal{B}$  is a proper dg algebra.*

*Proof.* If  $Q(\mathcal{A}) \rightarrow \mathcal{A}$  and  $Q(\mathcal{B}) \rightarrow \mathcal{B}$  are cofibrant resolutions, both  $Q(\mathcal{A})$  and  $Q(\mathcal{B})$  are locally perfect (Lemma 2.12). Since both  $Q(\mathcal{A})$  and  $Q(\mathcal{B})$  are k-h-flat [19, Lemma 2.14], the above discussion shows that  $\mathcal{A} \otimes^{\mathbf{L}} \mathcal{B} = Q(\mathcal{A}) \otimes Q(\mathcal{B})$  is locally perfect. The last claim is immediate since a dg algebra is proper if and only if it is locally perfect.  $\square$

**2.7. Back to saturated dg categories.** Recall from Definition 2.11 that a dg category  $\mathcal{T}$  is saturated if it is triangulated, smooth and proper.

**Proposition 2.16** (cf. [31, Prop. 13], and [32, Lemma 2.6]). *A dg category  $\mathcal{T}$  has a compact generator if and only if there is a dg algebra  $A$  such that  $\text{Perf}(\mathcal{T})$  and  $\text{Perf}(A)$  are isomorphic in  $\text{Heq}_k$ . If such an  $A$  is given, then  $\text{Perf}(\mathcal{T})$  is smooth (resp. proper) if and only if  $A$  is smooth (resp. proper).*

*In particular, a dg category  $\mathcal{T}$  is saturated if and only if there is a smooth and proper dg algebra  $A$  such that  $\mathcal{T}$  and  $\text{Perf}(A)$  are isomorphic in  $\text{Heq}_k$ .*

*Proof.* If  $A$  is a dg algebra, then  $\widehat{A}$  is a classical generator of  $[\text{Perf}(A)] \xrightarrow{\sim} \text{per}(A)$ . If  $\text{Perf}(\mathcal{T})$  and  $\text{Perf}(A)$  are isomorphic in  $\text{Heq}_k$  then  $[\text{Perf}(\mathcal{T})] \cong [\text{Perf}(A)]$ , so  $\mathcal{T}$  has a compact generator.

Conversely, assume that  $\mathcal{T}$  has a compact generator. Let  $E \in \text{Perf}(\mathcal{T})$  be such that  $E$  is a classical generator of  $[\text{Perf}(\mathcal{T})] \xrightarrow{\sim} \text{per}(\mathcal{T})$ . Let  $A := (\text{Perf}(\mathcal{T}))(E, E)$ . We consider the dg algebra  $A$  also as a dg category with one object  $\star$ . The obvious inclusion  $i: A \rightarrow \text{Perf}(\mathcal{T}), \star \mapsto E$ , gives by Lemma 2.6.(b) rise to the commutative diagram

$$\begin{array}{ccc} A & \xrightarrow{Y} & \text{Perf}(A) \\ \downarrow i & & \downarrow i^* \\ \text{Perf}(\mathcal{T}) & \xrightarrow{Y} & \text{Perf}(\text{Perf}(\mathcal{T})) \end{array}$$

whose vertical arrows are full and faithful. The lower horizontal arrow is a quasi-isomorphism (Corollary 2.4); in particular, it induces a triangulated equivalence

$$[\text{Perf}(\mathcal{T})] \xrightarrow[\sim]{[Y]} [\text{Perf}(\text{Perf}(\mathcal{T}))].$$

This implies that  $[\text{Perf}(\text{Perf}(\mathcal{T}))]$  is the thick envelope of  $\widehat{E} = Y(E)$ . Note that  $[\text{Perf}(A)] = \text{thick}(A)$  is the thick envelope of  $\widehat{\star}$  and that  $i^*(\widehat{\star}) = i^*(Y(\star)) = Y(i(\star)) = \widehat{E}$ . Since  $[\text{Perf}(A)]$  is Karoubian this implies that  $[i^*]: [\text{Perf}(A)] \rightarrow [\text{Perf}(\text{Perf}(\mathcal{T}))]$  is a triangulated equivalence. Then Lemma 2.5 shows that the vertical arrow  $i^*$  in the above commutative square is a quasi-equivalence. This shows that  $\text{Perf}(A)$  and  $\text{Perf}(\mathcal{T})$  are connected by a zig-zag of quasi-equivalences.

Lemmata 2.12 and 2.13 then yield the second claim, and for the last claim use additionally Corollary 2.4. □

**Proposition 2.17.** *Let  $\mathcal{S}, \mathcal{T}$  be saturated dg categories. Then  $\mathcal{S} \odot \mathcal{T}$  (defined in (2.3)) is a saturated dg category.*

Note that Lemmata 2.13, 2.14, 2.15 and Corollary 2.4 show that  $\mathcal{S} \odot \mathcal{T}$  is locally perfect, smooth, and triangulated.

*Proof.* By Proposition 2.16 there are smooth and proper dg algebras  $A$  and  $B$  such that  $\mathcal{S} \cong \text{Perf}(A)$  and  $\mathcal{T} \cong \text{Perf}(B)$  in  $\text{Heq}_k$ . Then we have isomorphisms

$$\mathcal{S} \odot \mathcal{T} \cong (\text{Perf}(A)) \odot (\text{Perf}(B)) \xrightarrow{\sim} \text{Perf}(A \otimes^L B) = A \odot B$$

in  $\text{Heq}_k$  by Lemma 2.9 and Proposition 2.10. Lemmata 2.14 and 2.15 show that  $A \otimes^L B$  is smooth and proper, so Proposition 2.16 again proves the claim.  $\square$

For later use we include the following result which is similar to Proposition 2.16.

**Proposition 2.18.** *Let  $\mathcal{E}$  be a pretriangulated dg category and let  $E \in \mathcal{E}$  be an object that becomes a classical generator of  $[\mathcal{E}]$ . Consider the dg algebra  $A := \mathcal{E}(E, E)$ . Then there is a quasi-equivalence  $\text{Perf}(A) \rightarrow \text{Perf}(\mathcal{E})$  of dg categories, and the dg functor  $\mathcal{E}(E, -): \mathcal{E} \rightarrow \text{Mod}(A)$  induces a full and faithful triangulated functor  $\mathcal{E}(E, -): [\mathcal{E}] \rightarrow \text{per}(A)$  that extends to an equivalence between the Karoubi envelope of  $[\mathcal{E}]$  and  $\text{per}(A)$ .*

*Moreover,  $A$  is smooth (resp. proper) if and only if  $\mathcal{E}$  has this property if and only if  $\text{Perf}(\mathcal{E})$  has this property. In particular,  $A$  is smooth and proper if and only if  $\text{Perf}(\mathcal{E})$  is saturated.*

*Proof.* We consider  $A$  as a dg category. Mapping its unique object to  $E$  defines a dg functor  $i: A \rightarrow \mathcal{E}$ . Lemma 2.6.(b) shows that the induced extension of scalars functor  $i^*: \text{Perf}(A) \rightarrow \text{Perf}(\mathcal{E})$  is full and faithful. It maps  $A$  to  $\widehat{E}$ . The induced functor  $i^*: \text{thick}(A) \rightarrow \text{thick}(\mathcal{E})$  on homotopy categories is full and faithful, and moreover essentially surjective since  $\widehat{E}$  is a classical generator of  $\text{thick}(\mathcal{E})$  and  $\text{thick}(A)$  is idempotent complete. Now Lemma 2.5 shows that  $i^*: \text{Perf}(A) \rightarrow \text{Perf}(\mathcal{E})$  is a quasi-equivalence.

A quasi-inverse of  $i^*: \text{thick}(A) \rightarrow \text{thick}(\mathcal{E})$  is given by restriction along  $i$ . This restriction composed with  $[\mathcal{E}] \rightarrow \text{thick}(\mathcal{E})$  is given by  $\mathcal{E}(E, -)$ . Moreover,  $\text{thick}(\mathcal{E})$  is the Karoubi envelope of  $[\mathcal{E}]$ ,  $[\mathcal{E}] \rightarrow [\widehat{\mathcal{E}}]$  is an equivalence since  $\mathcal{E}$  is pretriangulated, and  $\text{thick}(A) \xrightarrow{\sim} \text{per}(A)$ .

The remaining claims follow from the Lemmata 2.13 and 2.12 and Corollary 2.4.  $\square$

**2.8. Semi-orthogonal decompositions.** We refer the reader to [17, appendix A] for the definition and elementary properties of semi-orthogonal decompositions.

The first part of the following result says that a semi-orthogonal decomposition of the homotopy category  $[\mathcal{T}]$  of a pretriangulated dg category  $\mathcal{T}$  induces a semi-orthogonal decomposition of  $[\text{Perf}(\mathcal{T})]$ . This may be viewed as a dg lift of [17, Cor. A.12]. Its formulation is a bit technical since the components of a semi-orthogonal decomposition are required to be strict subcategories. A related result is given in Lemma 2.34 below.

**Proposition 2.19.** *Assume that  $\mathcal{T}$  is a pretriangulated dg category with full dg subcategories  $\mathcal{U}$  and  $\mathcal{V}$  such that  $[\mathcal{T}] = \langle [\mathcal{U}], [\mathcal{V}] \rangle$  is a semi-orthogonal decomposition (resp. a semi-orthogonal decomposition into admissible subcategories).*

*Then there is an induced semi-orthogonal decomposition*

$$[\text{Perf}(\mathcal{T})] = \langle [\text{Perf}(\mathcal{U})'], [\text{Perf}(\mathcal{V})'] \rangle$$

*(into admissible subcategories). Here  $\text{Perf}(\mathcal{U})'$  is the full dg subcategory of  $\text{Perf}(\mathcal{T})$  such that  $[\text{Perf}(\mathcal{U})']$  is the strict closure of  $[\text{Perf}(\mathcal{U})]$  in  $[\text{Perf}(\mathcal{T})]$ . In particular, there is an obvious quasi-equivalence  $\text{Perf}(\mathcal{U}) \rightarrow \text{Perf}(\mathcal{U})'$ . The dg category  $\text{Perf}(\mathcal{V})'$  is defined similarly.*

*More generally, let  $\mathcal{R}$  be a  $k$ -h-flat dg category. Then there is a semi-orthogonal decomposition*

$$[\text{Perf}(\mathcal{R} \otimes \mathcal{T})] = \langle [\text{Perf}(\mathcal{R} \otimes \mathcal{U})'], [\text{Perf}(\mathcal{R} \otimes \mathcal{V})'] \rangle$$

*(into admissible subcategories) where the involved dg subcategories are defined in the obvious way.*

*Proof.* The first claim is the special case  $\mathcal{R} = k$  of the second claim which we prove now. Assume that  $[\mathcal{T}] = \langle [\mathcal{U}], [\mathcal{V}] \rangle$  is a semi-orthogonal decomposition. The inclusions  $\mathcal{U} \subset \mathcal{T}$  and  $\mathcal{V} \subset \mathcal{T}$  give rise to full and faithful dg functors  $\mathcal{R} \otimes \mathcal{U} \rightarrow \mathcal{R} \otimes \mathcal{T}$ ,  $\mathcal{R} \otimes \mathcal{V} \rightarrow \mathcal{R} \otimes \mathcal{T}$ . Lemma 2.6.(b) shows that the induced dg functors

$$\begin{aligned} \widetilde{\mathcal{U}} &:= \text{Perf}(\mathcal{R} \otimes \mathcal{U}) \rightarrow \widetilde{\mathcal{T}} := \text{Perf}(\mathcal{R} \otimes \mathcal{T}), \\ \widetilde{\mathcal{V}} &:= \text{Perf}(\mathcal{R} \otimes \mathcal{V}) \rightarrow \widetilde{\mathcal{T}} = \text{Perf}(\mathcal{R} \otimes \mathcal{T}) \end{aligned}$$

are full and faithful. We view  $\widetilde{\mathcal{U}}$  and  $\widetilde{\mathcal{V}}$  as full dg subcategories of  $\widetilde{\mathcal{T}}$ . From  $[\mathcal{T}]([\mathcal{V}], [\mathcal{U}]) = 0$  we see that  $\mathcal{T}(v, u)$  is acyclic for all  $v \in \mathcal{V}$  and  $u \in \mathcal{U}$ .

Let  $r, r' \in \mathcal{R}$ ,  $u \in \mathcal{U}$  and  $v \in \mathcal{V}$ . Since  $\mathcal{R}(r, r')$  is  $k$ -h-flat,  $\mathcal{R}(r, r') \otimes \mathcal{T}(v, u)$  is acyclic. This implies that

$$[\widetilde{\mathcal{T}}](\widehat{(r, v)}, \widehat{(r', u)}) = [\mathcal{R} \otimes \mathcal{T}](r, v), (r', u) = H^0(\mathcal{R}(r, r') \otimes \mathcal{T}(v, u)) = 0.$$

Since  $[\widetilde{\mathcal{U}}] = \text{thick}(\mathcal{R} \otimes \mathcal{U})$  is classically generated by the objects  $\widehat{(r, u)}$ , for  $r \in \mathcal{R}$  and  $u \in \mathcal{U}$ , and similarly for  $[\widetilde{\mathcal{V}}]$ , we see that  $[\widetilde{\mathcal{T}}](\widehat{[\mathcal{V}]}, \widehat{[\mathcal{U}]}) = 0$ .

Let  $r \in \mathcal{R}$  and  $t \in \mathcal{T}$ . There are  $v \in \mathcal{V}$  and  $u \in \mathcal{U}$  such that there is a triangle  $v \rightarrow t \rightarrow u \rightarrow [1]v$  in  $[\mathcal{T}]$ . Consider the dg functor  $i_r: \mathcal{T} \rightarrow \mathcal{R} \otimes \mathcal{T}$ ,  $t' \mapsto (r, t')$ . It induces a commutative diagram

$$\begin{array}{ccc} \mathcal{T} & \xrightarrow{Y} & \text{Perf}(\mathcal{T}) \\ \downarrow i_r & & \downarrow i_r^* \\ \mathcal{R} \otimes \mathcal{T} & \xrightarrow{Y} & \widetilde{\mathcal{T}} = \text{Perf}(\mathcal{R} \otimes \mathcal{T}) \end{array}$$

(see Lemma 2.6.(a)). If we pass to homotopy categories, the upper horizontal and the right vertical functor are triangulated functors; they map the above triangle to the triangle

$$\widehat{(r, v)} \rightarrow \widehat{(r, t)} \rightarrow \widehat{(r, u)} \rightarrow [1]\widehat{(r, v)}$$

in  $[\widetilde{\mathcal{T}}]$ . Note that  $[\widetilde{\mathcal{T}}]$  is classically generated by the objects  $\widehat{(r, t)}$ , for  $r \in \mathcal{R}$  and  $t \in \mathcal{T}$ , and that both  $[\widetilde{\mathcal{V}}]$  and  $[\widetilde{\mathcal{U}}]$  are Karoubian (by Lemma 2.3 and Corollary 2.4). Define  $\widetilde{\mathcal{V}}'$  to be the full dg subcategory of  $\widetilde{\mathcal{T}}$  such that  $[\widetilde{\mathcal{V}}']$  is the closure under isomorphisms of  $[\widetilde{\mathcal{V}}]$  in  $[\widetilde{\mathcal{T}}]$ ; define  $\widetilde{\mathcal{U}}'$  similarly. (Note that  $\widetilde{\mathcal{V}} \rightarrow \widetilde{\mathcal{V}}'$  is a quasi-equivalence by Lemma 2.5.)

From [17, Lemma A.6.(b)] we see that  $[\widetilde{\mathcal{T}}] = \langle [\widetilde{\mathcal{U}}'], [\widetilde{\mathcal{V}}'] \rangle$  is a semi-orthogonal decomposition. In particular,  $[\widetilde{\mathcal{V}}']$  is right admissible and  $[\widetilde{\mathcal{U}}']$  is left admissible in  $[\widetilde{\mathcal{T}}]$ .

Assume now in addition that  $[\mathcal{U}]$  is right admissible in  $[\mathcal{T}]$ . Then [17, Lemma A.11.(a)] says that  $[\mathcal{T}] = \langle [\mathcal{U}]^\perp, [\mathcal{U}] \rangle$  is a semi-orthogonal decomposition. Let  $\mathcal{U}^\dagger$  be the full dg subcategory of  $\mathcal{T}$  that has the same objects as  $[\mathcal{U}]^\perp$ , so  $[\mathcal{U}^\dagger] = [\mathcal{U}]^\perp$ . Then the above argument shows that  $[\widetilde{\mathcal{U}}']$  is right admissible. Similarly, left admissibility of  $[\mathcal{V}]$  implies that  $[\widetilde{\mathcal{V}}']$  is left admissible.  $\square$

**Proposition 2.20.** *Let  $\mathcal{T}$  be a pretriangulated dg category with full dg subcategories  $\mathcal{U}$  and  $\mathcal{V}$  such that  $[\mathcal{T}] = \langle [\mathcal{U}], [\mathcal{V}] \rangle$  is a semi-orthogonal decomposition. Then  $\mathcal{U}$  and  $\mathcal{V}$  are pretriangulated as well. Moreover, if  $\mathcal{T}$  is triangulated (resp. is locally perfect resp. has a compact generator resp. is smooth resp. is proper resp. is saturated) then  $\mathcal{U}$  and  $\mathcal{V}$  have the same property.*

*Proof.* It is clear that  $\mathcal{U}$  and  $\mathcal{V}$  are pretriangulated.

Triangulated: If  $[\mathcal{T}]$  is Karoubian, so are  $[\mathcal{U}]$  and  $[\mathcal{V}]$  since  $[\mathcal{U}] = [\mathcal{V}]^\perp$  and  $[\mathcal{V}] = {}^\perp[\mathcal{U}]$ , and we can apply Lemma 2.3.

Locally perfect: this is obviously passed to any full dg subcategory.

Has a compact generator. The first claim of Proposition 2.19 (together with Lemmata 2.12 and 2.13) shows that we can assume that  $\mathcal{T}$  is triangulated. Then by assumption  $[\mathcal{T}] \xrightarrow{\sim} [\text{Perf}(\mathcal{T})] \xrightarrow{\sim} \text{per}(\mathcal{T})$  has a classical generator. The obvious functors  $[\mathcal{U}] \rightarrow [\mathcal{T}]/[\mathcal{V}]$  and  $[\mathcal{V}] \rightarrow [\mathcal{T}]/[\mathcal{U}]$  are equivalences of triangulated categories. This implies that  $[\mathcal{U}] \xrightarrow{\sim} \text{per}(\mathcal{U})$  and  $[\mathcal{V}] \xrightarrow{\sim} \text{per}(\mathcal{V})$  have classical generators, i. e.  $\mathcal{U}$  and  $\mathcal{V}$  have compact generators.

Smooth: Let  $\mathcal{E} \subset \mathcal{T}$  be the full dg subcategory of  $\mathcal{T}$  whose objects are the union of the objects of  $\mathcal{U}$  and  $\mathcal{V}$ . Let  $\mathcal{V}'$  be the full dg subcategory of  $\mathcal{V}$  obtained by ignoring all objects that are also in  $\mathcal{U}$ . Let  $\mathcal{E}' \subset \mathcal{E}$  be the (in general non-full) dg subcategory with the same objects and morphism spaces as  $\mathcal{E}$  except that we set  $\mathcal{E}'(V', U) = 0$  for all  $V' \in \mathcal{V}'$  and  $U \in \mathcal{U}$ . Then  $\mathcal{E}' \rightarrow \mathcal{E}$  is a quasi-equivalence since all  $\mathcal{T}(V', U)$  are acyclic. Symbolically, this inclusion may be written as

$\mathcal{E}' = \begin{bmatrix} \mathcal{U} & 0 \\ \mathcal{V}'\tau_{\mathcal{U}} & \mathcal{V}' \end{bmatrix} \subset \mathcal{E} = \begin{bmatrix} \mathcal{U} & \mathcal{U}\tau_{\mathcal{V}'} \\ \mathcal{V}'\tau_{\mathcal{U}} & \mathcal{V}' \end{bmatrix}$ . Lemma 2.6 implies that  $\text{Perf}(\mathcal{E}') \rightarrow \text{Perf}(\mathcal{E})$  is a quasi-equivalence and that  $\text{Perf}(\mathcal{E}) \rightarrow \text{Perf}(\mathcal{T})$  is full and faithful; but in fact this last arrow is also a quasi-equivalence: on homotopy categories it induces an equivalence since each object of  $[\text{Perf}(\mathcal{T})]$  is an extension of an object of  $[\text{Perf}(\mathcal{U})]$  by an object of  $[\text{Perf}(\mathcal{V})]$  (by Proposition 2.19), so we can use Lemma 2.5. Hence smoothness of  $\mathcal{T}$  implies smoothness of  $\mathcal{E}'$  (using Lemmata 2.12 and 2.13 again), and then [19, Thm. 3.24] implies smoothness of both  $\mathcal{U}$  and  $\mathcal{V}$ .

Proper, saturated: Clear from above. □

**Corollary 2.21.** *Let  $\mathcal{D}$  be a triangulated category with a semi-orthogonal decomposition  $\mathcal{D} = \langle \mathcal{A}, \mathcal{B} \rangle$ . Then the following properties of an enhancement of  $\mathcal{D}$  are passed on to the induced enhancements of  $\mathcal{A}$  and  $\mathcal{B}$ : being triangulated, being locally perfect, having a compact generator, smoothness, properness, being saturated.*

**2.9. Grothendieck ring of saturated dg categories.** Let  $\text{sat}_k$  be the full subcategory of  $\text{dgc}at_k$  consisting of saturated dg categories. By  $\underline{\text{sat}}_k$  we denote the set of isomorphism classes in  $\text{Heq}_k$  of these categories (cf. Lemma 2.12). Given a saturated dg category  $\mathcal{T}$ , we write  $\underline{\mathcal{T}}$  for its class in  $\underline{\text{sat}}_k$ .

**Proposition 2.22.** *The map  $(\mathcal{S}, \mathcal{T}) \mapsto \mathcal{S} \odot \mathcal{T}$  induces a multiplication  $\bullet$  on  $\underline{\text{sat}}_k$  that turns  $\underline{\text{sat}}_k$  into a commutative monoid with unit  $\underline{\text{Perf}(k)}$ .*

*Proof.* Lemma 2.9 and Proposition 2.17 show that  $\bullet$  is well defined. Let  $\mathcal{S}_1, \mathcal{S}_2,$  and  $\mathcal{S}_3$  be saturated dg categories. We know that  $\mathcal{S}_i \rightarrow \text{Perf}(\mathcal{S}_i)$  is a quasi-equivalence. Hence to obtain associativity of  $\bullet$  it is enough to prove that  $(\text{Perf}(\mathcal{S}_1) \odot \text{Perf}(\mathcal{S}_2)) \odot \text{Perf}(\mathcal{S}_3)$  and  $(\text{Perf}(\mathcal{S}_1) \odot \text{Perf}(\mathcal{S}_2)) \odot \text{Perf}(\mathcal{S}_3)$  are isomorphic in  $\text{Heq}_k$ . Proposition 2.10 and Lemma 2.6.(c) reduce this to showing that  $(\mathcal{S}_1 \otimes^L \mathcal{S}_2) \otimes^L \mathcal{S}_3$  and  $\mathcal{S}_1 \otimes^L (\mathcal{S}_2 \otimes^L \mathcal{S}_3)$  are isomorphic in  $\text{Heq}_k$ . But this is easy to see since cofibrant dg categories are  $k$ -h-flat. Similarly, commutativity follows from  $\mathcal{S}_1 \otimes^L \mathcal{S}_2 \cong \mathcal{S}_2 \otimes^L \mathcal{S}_1$ , and  $\mathcal{S}_1 \otimes^L k \cong \mathcal{S}_1$  proves that  $\underline{\text{Perf}(k)}$  is the unit. □

Denote by  $\mathbb{Z}\underline{\text{sat}}_k$  the (commutative associative unital) monoid ring of  $\underline{\text{sat}}_k$ , i. e. the free abelian group on  $\underline{\text{sat}}_k$  with  $\mathbb{Z}$ -bilinear multiplication induced by  $\bullet$ .

**Definition 2.23** (cf. [5, Def. 5.1, 8.1]). The **Grothendieck group  $K_0(\text{sat}_k)$  of saturated dg categories** is defined to be the quotient of  $\mathbb{Z}\underline{\text{sat}}_k$  by the subgroup generated by the elements (the “semi-orthogonal relations”)  $\underline{\mathcal{T}} - (\underline{\mathcal{U}} + \underline{\mathcal{V}})$  whenever there is a saturated dg category  $\mathcal{T}$  with full dg subcategories  $\mathcal{U}$  and  $\mathcal{V}$  such that  $[\mathcal{T}] = \langle [\mathcal{U}], [\mathcal{V}] \rangle$  is a semi-orthogonal decomposition into admissible subcategories. (We do not require that  $\mathcal{U}$  and  $\mathcal{V}$  are saturated; this is automatic by Proposition 2.20.)

If  $0$  is the trivial dg algebra (considered as a dg category) and if  $\emptyset$  is the empty dg category, then  $0 = \text{Perf}(\emptyset)$ , and we have  $\underline{0} = \underline{\text{Perf}(\emptyset)} = 0$  in  $K_0(\text{sat}_k)$ .

**Proposition 2.24.** *The multiplication  $\bullet$  on  $\mathbb{Z}\text{sat}_k$  induces a multiplication on  $K_0(\text{sat}_k)$  such that  $\mathbb{Z}\text{sat}_k \rightarrow K_0(\text{sat}_k)$  is a ring morphism. Equipped with this multiplication, we call  $K_0(\text{sat}_k)$  the **Grothendieck ring of saturated dg categories**.*

*Proof.* Let  $I \subset \mathbb{Z}\text{sat}_k$  be the subgroup generated by the “semi-orthogonal relations”. We need to show that  $I$  is an ideal in  $\mathbb{Z}\text{sat}_k$ . Assume that  $\mathcal{T}$  is a saturated dg category with (saturated) dg subcategories  $\mathcal{U}$  and  $\mathcal{V}$  such that  $[\mathcal{T}] = \langle [\mathcal{U}], [\mathcal{V}] \rangle$  is a semi-orthogonal decomposition into admissible subcategories.

Let  $\mathcal{S}$  be any saturated dg category. We need to prove that

$$\underline{\mathcal{S}} \bullet \underline{\mathcal{T}} - (\underline{\mathcal{S}} \bullet \underline{\mathcal{U}} + \underline{\mathcal{S}} \bullet \underline{\mathcal{V}}) = \underline{\text{Perf}(\mathcal{S} \otimes^{\mathbf{L}} \mathcal{T})} - (\underline{\text{Perf}(\mathcal{S} \otimes^{\mathbf{L}} \mathcal{U})} + \underline{\text{Perf}(\mathcal{S} \otimes^{\mathbf{L}} \mathcal{V})})$$

is an element of  $I$ . Observe that  $\mathcal{S} \otimes^{\mathbf{L}} \mathcal{A} = Q(\mathcal{S}) \otimes Q(\mathcal{A}) \rightarrow Q(\mathcal{S}) \otimes \mathcal{A}$  is a quasi-equivalence (for  $\mathcal{A}$  an arbitrary dg category) since  $Q(\mathcal{S})$  is  $k$ -h-flat. Lemma 2.6.(c) shows that the above element is equal to

$$\underline{\text{Perf}(Q(\mathcal{S}) \otimes \mathcal{T})} - \underline{\text{Perf}(Q(\mathcal{S}) \otimes \mathcal{U})} + \underline{\text{Perf}(Q(\mathcal{S}) \otimes \mathcal{V})}.$$

But this element lies in  $I$  by Proposition 2.19. □

**Remark 2.25.** Gonçalo Tabuada shows in [28, section 7] that for  $k$  a field (and in the differential  $\mathbb{Z}$ -graded situation) there is a surjective morphism  $K_0(\text{sat}_k) \rightarrow K_0(\text{Hmo}_0^{cl})$  of commutative rings. We refer the reader to [28] for the definition of  $K_0(\text{Hmo}_0^{cl})$ . This ring is non-zero, so the same is true for  $K_0(\text{sat}_k)$ .

The results of the following Subsections 2.10, 2.11 and 2.12 are dispensable for Sections 3, 4, 5 and 6.

**2.10. Modified Grothendieck ring of saturated dg categories.** By omitting the words “into admissible subcategories” in Definition 2.23 we define the **modified Grothendieck group  $K'_0(\text{sat}_k)$  of saturated dg categories**. The proof of Proposition 2.24 shows that  $K'_0(\text{sat}_k)$  becomes a ring with multiplication induced by  $\bullet$ . There is an obvious surjective morphism

$$K_0(\text{sat}_k) \rightarrow K'_0(\text{sat}_k) \tag{2.6}$$

of rings.

**Proposition 2.26.** *If  $k$  is a field, then the map (2.6) is an isomorphism.*

The proof of this result requires some additional care in the differential  $\mathbb{Z}_n$ -graded setting.

*Proof of Prop. 2.26 in the differential  $\mathbb{Z}$ -graded setting.* Let  $\mathcal{T}$  be a saturated dg category  $\mathcal{T}$  with full dg subcategories  $\mathcal{U}$  and  $\mathcal{V}$  such that  $[\mathcal{T}] = \langle [\mathcal{U}], [\mathcal{V}] \rangle$  is a semi-orthogonal decomposition. We have seen that this already implies that both  $\mathcal{U}$  and  $\mathcal{V}$  are saturated dg categories. Then [27, Thm. 3.1] (and Proposition 2.16) show that

$[\mathcal{U}]$  and  $[\mathcal{V}]$  are “saturated” in the sense that they are Ext-finite and all covariant and contravariant cohomological functors  $[\mathcal{T}] \rightarrow \text{Vect}(\mathbf{k})$  of finite type are representable. Here  $\text{Vect}(\mathbf{k})$  denotes the category of vector spaces over  $\mathbf{k}$ . Then [3, Prop. 2.6] tells us that  $[\mathcal{U}]$  and  $[\mathcal{V}]$  are both admissible in  $[\mathcal{T}]$ .  $\square$

The following proposition is a variant of [2, Thm. 1.3]. We denote the category of finite dimensional vector spaces over  $\mathbf{k}$  by  $\text{Vect}_{\text{fd}}(\mathbf{k})$ .

**Proposition 2.27.** *Let  $\mathbf{k}$  be a field and  $\mathcal{D}$  a triangulated (in the usual Verdier sense)  $\mathbf{k}$ -linear category. Assume that*

- (a)  $\dim_{\mathbf{k}} \mathcal{D}(A, B) < \infty$  for all objects  $A, B$  of  $\mathcal{D}$ ;
- (b)  $\mathcal{D}$  has a strong generator  $E$  such that there is some  $n > 0$  such that  $E \cong [n]E$ ; and
- (c)  $\mathcal{D}$  is Karoubian.

*Then every  $\mathbf{k}$ -linear cohomological functor  $\mathcal{D}^{\text{op}} \rightarrow \text{Vect}_{\text{fd}}(\mathbf{k})$  is representable.*

If  $\mathcal{D}$  is  $n$ -periodic (for some  $n \in \mathbb{Z}$ ) in the sense that each object  $A \in \mathcal{D}$  is isomorphic to  $[n]A$ , then condition (b) is satisfied if and only if  $\mathcal{D}$  has a strong generator.

*Proof.* Let  $Z = \{0, 1, \dots, n - 1\}$ . Then for each  $m \in \mathbb{Z}$  there is an  $z \in Z$  such that  $[m]E \cong [z]E$ . Now observe that the proof of [2, Thm. 1.3] contains all the ideas needed to prove this proposition. Only Subsection 2.4 “Construction of resolutions” there needs to be modified: one essentially replaces all direct sums indexed by  $\mathbb{Z}$  by direct sums indexed by  $Z$ .  $\square$

We deduce a variant of [27, Thm. 3.1].

**Proposition 2.28.** *Let  $\mathbf{k}$  be a field and  $n \in \mathbb{Z}$ . Let  $\mathcal{T}$  be a saturated  $d\mathbb{Z}_n\mathbf{g}$  (= differential  $\mathbb{Z}_n$ -graded) category. Then all covariant and contravariant  $\mathbf{k}$ -linear cohomological functors  $[\mathcal{T}] \rightarrow \text{Vect}_{\text{fd}}(\mathbf{k})$  are representable.*

*Proof.* We follow the proof of [27, Thm. 3.1] but use Proposition 2.27 instead of [2, Thm. 1.3]. By Proposition 2.16 we can assume that  $\mathcal{T} = \text{Perf}(A)$  for a smooth and proper  $d\mathbb{Z}_n\mathbf{g}$  algebra  $A$ . The argument from the proof of [27, Thm. 3.1] shows that  $[\mathcal{T}]$  has a strong generator. Hence we can apply Proposition 2.27 and obtain that every  $\mathbf{k}$ -linear cohomological functor  $[\mathcal{T}]^{\text{op}} \rightarrow \text{Vect}_{\text{fd}}(\mathbf{k})$  is representable.

We claim that  $\mathcal{T}^{\text{op}}$  is also saturated. It is certainly pretriangulated, locally proper, and smooth (see for example [19, Remark 3.11]). Observe that

$$[\mathcal{T}^{\text{op}}] = [\mathcal{T}]^{\text{op}} \cong \text{per}(A)^{\text{op}} \cong \text{per}(A^{\text{op}})$$

where the last equivalence comes from the proof of [27, Thm. 3.1]. This shows that  $[\mathcal{T}^{\text{op}}]$  is Karoubian, so  $\mathcal{T}^{\text{op}}$  is triangulated, and that  $\mathcal{T}^{\text{op}}$  has a compact generator. Hence the above argument applied to the saturated  $d\mathbb{Z}_n\mathbf{g}$  category  $\mathcal{T}^{\text{op}}$  shows that any  $\mathbf{k}$ -linear cohomological functor  $[\mathcal{T}] = [\mathcal{T}^{\text{op}}]^{\text{op}} \rightarrow \text{Vect}_{\text{fd}}(\mathbf{k})$  is representable.  $\square$

*Proof of Prop. 2.26 in the differential  $\mathbb{Z}_n$ -graded setting (for some  $n \in \mathbb{Z}$ ).* Let  $\mathcal{T}$  be a saturated  $d\mathbb{Z}_n\text{g}$  category  $\mathcal{T}$  with full  $d\mathbb{Z}_n\text{g}$  subcategories  $\mathcal{U}$  and  $\mathcal{V}$  such that  $[\mathcal{T}] = \langle [\mathcal{U}], [\mathcal{V}] \rangle$  is a semi-orthogonal decomposition. We already know that both  $\mathcal{U}$  and  $\mathcal{V}$  are saturated  $d\mathbb{Z}_n\text{g}$  categories, so we can apply Proposition 2.28 to  $\mathcal{U}$  and  $\mathcal{V}$ . The proof of [3, Prop. 2.6] then tells us that  $[\mathcal{U}]$  and  $[\mathcal{V}]$  are both admissible in  $[\mathcal{T}]$ .  $\square$

**2.11. Grothendieck ring of proper and smooth dg categories.** The aim of this section is to give an alternative description of the modified Grothendieck ring  $K'_0(\text{sat}_k)$  of saturated dg categories using smooth and proper dg categories (which are not necessarily triangulated).

Recall from [29] and [28] (and corrections) that there are three model category structures on  $\text{dgc}(\mathbb{k})$ . They are all cofibrantly generated by the same set of generating cofibrations. In particular they have the same cofibrations, cofibrant objects and trivial fibrations, and hence we can use the same cofibrant replacement functor.

Above we have used the model category structure whose weak equivalences are the quasi-equivalences and have denoted the corresponding homotopy category by  $\text{Heq}_k$ . Now we will work with the model structure whose weak equivalences are the Morita equivalences (= dg foncteurs de Morita). The corresponding homotopy category will be denoted  $\text{Hmo}_k$ .

Recall that a dg functor  $f: \mathcal{A} \rightarrow \mathcal{B}$  is a Morita equivalence if the restriction of scalars functor  $D(\mathcal{B}) \rightarrow D(\mathcal{A})$  is an equivalence of triangulated categories. It is easy to see that  $f$  is a Morita equivalence if and only if  $f^*: \text{Perf}(\mathcal{A}) \rightarrow \text{Perf}(\mathcal{B})$  is a quasi-equivalence (use Lemmata 2.5 and 2.6, and [16, Lemma 2.12]). For example, if  $\mathcal{A}$  is any dg category, the Yoneda morphism  $\mathcal{A} \rightarrow \text{Perf}(\mathcal{A})$  is a Morita equivalence by Proposition 2.8.

**Example 2.29.** Let  $\mathcal{T}$  be a dg category with a compact generator. Let  $E \in \text{Perf}(\mathcal{T})$  be a classical generator of  $[\text{Perf}(\mathcal{T})]$ , and let  $A := (\text{Perf}(\mathcal{T}))(E, E)$  be its endomorphism dg algebra. Then the proof of Proposition 2.16 shows that the obvious dg functor  $A \rightarrow \text{Perf}(\mathcal{T})$  is a Morita equivalence. Moreover, Lemma 2.30 below shows that  $\mathcal{T}$  is proper (resp. smooth) if and only if  $A$  is proper (resp. smooth).

**Lemma 2.30.** *The following properties of dg categories are invariant under Morita equivalences: being locally perfect, having a compact generator, properness, smoothness.*

*Proof.* This follows from the above and Lemmata 2.12 and 2.13.  $\square$

**Lemma 2.31.** *Morita equivalences  $\mathcal{A} \rightarrow \mathcal{A}'$  and  $\mathcal{B} \rightarrow \mathcal{B}'$  give rise to a Morita equivalence  $\mathcal{A} \otimes^{\mathbb{L}} \mathcal{B} \rightarrow \mathcal{A}' \otimes^{\mathbb{L}} \mathcal{B}'$ .*

*Proof.* Clearly  $Q(\mathcal{A}) \rightarrow Q(\mathcal{A}')$  is a Morita equivalence, so

$$\text{Perf}(Q(\mathcal{A})) \rightarrow \text{Perf}(Q(\mathcal{A}')) \quad \text{and} \quad \text{Perf}(Q(\mathcal{A})) \otimes Q(\mathcal{B}) \rightarrow \text{Perf}(Q(\mathcal{A}')) \otimes Q(\mathcal{B})$$

(by [19, Lemma 2.15]), and

$$\text{Perf}(\text{Perf}(Q(\mathcal{A})) \otimes Q(\mathcal{B})) \rightarrow \text{Perf}(\text{Perf}(Q(\mathcal{A}')) \otimes Q(\mathcal{B}))$$

(by Lemma 2.6.(c)) are quasi-equivalences. Then Proposition 2.8 shows that

$$\text{Perf}(Q(\mathcal{A}) \otimes Q(\mathcal{B})) \rightarrow \text{Perf}(Q(\mathcal{A}') \otimes Q(\mathcal{B}))$$

is a quasi-equivalence, so

$$Q(\mathcal{A}) \otimes Q(\mathcal{B}) \rightarrow Q(\mathcal{A}') \otimes Q(\mathcal{B})$$

is a Morita equivalence. □

**Lemma 2.32.** *Let  $\mathcal{A}$  and  $\mathcal{B}$  be proper dg categories. Then  $\mathcal{A} \otimes^L \mathcal{B}$  is a proper dg category.*

*Proof.* Example 2.29 shows that there are Morita equivalences  $\mathcal{A} \rightarrow \text{Perf}(\mathcal{A})$  and  $\mathcal{B} \rightarrow \text{Perf}(\mathcal{B})$  for proper dg algebras  $\mathcal{A}$  and  $\mathcal{B}$ . Lemma 2.15 implies that  $\mathcal{A} \otimes^L \mathcal{B}$  is a proper dg algebra/category. Lemma 2.31 shows that

$$\mathcal{A} \otimes^L \mathcal{B} \rightarrow \text{Perf}(\mathcal{A}) \otimes^L \text{Perf}(\mathcal{B}) \leftarrow \mathcal{A} \otimes^L \mathcal{B}$$

consists of Morita equivalences, and hence Lemma 2.30 shows that  $\mathcal{A} \otimes^L \mathcal{B}$  is proper. □

Let  $\text{prsm}_k$  be the full subcategory of  $\text{dgcats}_k$  consisting of proper and smooth dg categories. By  $\underline{\text{prsm}}_k$  we denote the set of isomorphism classes in  $\text{Hmo}_k$  of these categories (cf. Lemma 2.30). Given a proper and smooth dg category  $\mathcal{T}$ , we write  $\underline{\mathcal{T}}$  for its class in  $\underline{\text{prsm}}_k$ .

**Proposition 2.33.** *The map  $(\mathcal{S}, \mathcal{T}) \mapsto \mathcal{S} \otimes^L \mathcal{T}$  induces a multiplication  $\bullet$  on  $\underline{\text{prsm}}_k$  that turns  $\underline{\text{prsm}}_k$  into a commutative monoid with unit  $\underline{k}$ .*

*Proof.* Lemmata 2.31, 2.32 and 2.14 show that  $\bullet$  is well defined. We leave the easy proofs of associativity, commutativity, and of  $\underline{k}$  being the unit to the reader. □

Denote by  $\mathbb{Z}\underline{\text{prsm}}_k$  the (commutative associative unital) monoid ring of  $\underline{\text{prsm}}_k$ , i. e. the free abelian group on  $\underline{\text{prsm}}_k$  with  $\mathbb{Z}$ -bilinear multiplication induced by  $\bullet$ .

Let  $\mathcal{T}$  be a dg category. Following [19, section 3.2] we write  $\mathcal{T} = \begin{bmatrix} \mathcal{U} & 0 \\ * & \mathcal{V} \end{bmatrix}$  if  $\mathcal{U}$  and  $\mathcal{V}$  are full dg subcategories of  $\mathcal{T}$  such that  $\mathcal{T}(\mathcal{V}, \mathcal{U}) = 0$  and such that the set of objects of  $\mathcal{T}$  is the disjoint union of the sets of objects of  $\mathcal{U}$  and of  $\mathcal{V}$ . (Conversely, any two dg categories  $\mathcal{U}$  and  $\mathcal{V}$  together with a dg  $\mathcal{U} \otimes \mathcal{V}^{\text{op}}$ -module  $N = {}_{\mathcal{V}}N_{\mathcal{U}}$  give rise to such a “directed” or “lower triangular” dg category  $\begin{bmatrix} \mathcal{U} & 0 \\ N & \mathcal{V} \end{bmatrix}$ .)

**Lemma 2.34.** *Assume that  $\mathcal{U}$  and  $\mathcal{V}$  are full dg subcategories of a dg category  $\mathcal{T}$  such that  $\mathcal{T} = \begin{bmatrix} \mathcal{U} & 0 \\ * & \mathcal{V} \end{bmatrix}$ . Then there is an induced semi-orthogonal decomposition*

$$[\text{Perf}(\mathcal{T})] = \langle [\text{Perf}(\mathcal{U})'], [\text{Perf}(\mathcal{V})'] \rangle$$

where  $\text{Perf}(\mathcal{U})'$  is the full dg subcategory of  $\text{Perf}(\mathcal{T})$  such that  $[\text{Perf}(\mathcal{U})']$  is the strict closure of  $[\text{Perf}(\mathcal{U})]$  in  $[\text{Perf}(\mathcal{T})]$ , and  $\text{Perf}(\mathcal{V})'$  is defined similarly. In particular, there are obvious quasi-equivalences  $\text{Perf}(\mathcal{U}) \rightarrow \text{Perf}(\mathcal{U})'$  and  $\text{Perf}(\mathcal{V}) \rightarrow \text{Perf}(\mathcal{V})'$ .

*Proof.* The proof is similar to (but easier than) the proof of Proposition 2.19 and also based on [17, Lemma A.6.(b)] □

**Definition 2.35.** The **Grothendieck group  $K_0(\text{prsm}_k)$  of proper and smooth dg categories** is defined to be the quotient of  $\mathbb{Z}\text{prsm}_k$  by the subgroup generated by the elements (the “directed relations”)  $\underline{\mathcal{T}} - (\underline{\mathcal{U}} + \underline{\mathcal{V}})$  whenever there is a smooth and proper dg category  $\mathcal{T}$  with full dg subcategories  $\mathcal{U}$  and  $\mathcal{V}$  such that  $\mathcal{T} = \begin{bmatrix} \mathcal{U} & 0 \\ * & \mathcal{V} \end{bmatrix}$ . (We do not require that  $\mathcal{U}$  and  $\mathcal{V}$  are smooth and proper since this is automatic: use Lemma 2.34, Proposition 2.20, and Lemma 2.30 (applied to  $\mathcal{U} \rightarrow \text{Perf}(\mathcal{U})$  and  $\mathcal{V} \rightarrow \text{Perf}(\mathcal{V})$ .)

If  $0$  is the trivial dg algebra (considered as a dg category) and if  $\emptyset$  is the empty dg category, then  $\emptyset \rightarrow 0$  is a Morita equivalence and we have  $\underline{\emptyset} = \underline{0} = 0$  in  $K_0(\text{prsm}_k)$ .

**Proposition 2.36.** *The multiplication  $\bullet$  on  $\mathbb{Z}\text{prsm}_k$  induces a multiplication on  $K_0(\text{prsm}_k)$  such that  $\mathbb{Z}\text{prsm}_k \rightarrow K_0(\text{prsm}_k)$  is a ring morphism. Equipped with this multiplication, we call  $\overline{K_0}(\text{prsm}_k)$  the **Grothendieck ring of proper and smooth dg categories**.*

*Proof.* Let  $I \subset \mathbb{Z}\text{prsm}_k$  be the subgroup generated by the “directed relations”. We need to show that  $I$  is an ideal in  $\mathbb{Z}\text{prsm}_k$ . Assume that  $\mathcal{T}$  is a smooth and proper dg category with full dg subcategories  $\mathcal{U}$  and  $\mathcal{V}$  such that  $\mathcal{T} = \begin{bmatrix} \mathcal{U} & 0 \\ * & \mathcal{V} \end{bmatrix}$ . Let  $\mathcal{S}$  be any saturated dg category. Then

$$\begin{aligned} \underline{\mathcal{S}} \bullet \underline{\mathcal{T}} - (\underline{\mathcal{S}} \bullet \underline{\mathcal{U}} + \underline{\mathcal{S}} \bullet \underline{\mathcal{V}}) &= \underline{\mathcal{S}} \otimes^{\mathbf{L}} \underline{\mathcal{T}} - (\underline{\mathcal{S}} \otimes^{\mathbf{L}} \underline{\mathcal{U}} + \underline{\mathcal{S}} \otimes^{\mathbf{L}} \underline{\mathcal{V}}) \\ &= \underline{Q(\mathcal{S})} \otimes \underline{\mathcal{T}} - (\underline{Q(\mathcal{S})} \otimes \underline{\mathcal{U}} + \underline{Q(\mathcal{S})} \otimes \underline{\mathcal{V}}) \end{aligned}$$

is an element of  $I$  since  $\underline{Q(\mathcal{S})} \otimes \underline{\mathcal{T}} = \begin{bmatrix} \underline{Q(\mathcal{S})} \otimes^{\mathbf{L}} \mathcal{U} & 0 \\ * & \underline{Q(\mathcal{S})} \otimes^{\mathbf{L}} \mathcal{V} \end{bmatrix}$ . □

**Proposition 2.37.** *The map  $\mathcal{T} \mapsto \text{Perf}(\mathcal{T})$  (for proper and smooth  $\mathcal{T}$ ) induces an isomorphism*

$$K_0(\text{prsm}_k) \xrightarrow{\sim} K'_0(\text{sat}_k)$$

of rings with inverse morphism induced by  $\mathcal{S} \mapsto \underline{\mathcal{S}}$  (for saturated  $\mathcal{S}$ ).

*Proof.* Morita equivalences  $\mathcal{T} \rightarrow \mathcal{T}'$  between proper and smooth dg categories induce quasi-equivalences  $\text{Perf}(\mathcal{T}) \rightarrow \text{Perf}(\mathcal{T}')$  between saturated dg categories, and  $\mathcal{T} \rightarrow \text{Perf}(\mathcal{T})$  is a Morita equivalence. Quasi-equivalences are certainly Morita equivalences, and  $\mathcal{S} \rightarrow \text{Perf}(\mathcal{S})$  is a quasi-equivalence for saturated  $\mathcal{S}$ . Hence we get isomorphisms  $\underline{\text{prsm}}_k \rightarrow \underline{\text{sat}}_k$  of monoids (multiplicativity is obvious for the inverse  $\underline{\mathcal{S}} \mapsto \underline{\mathcal{S}}$ ) and  $\underline{\mathbb{Z}\text{prsm}}_k \rightarrow \underline{\mathbb{Z}\text{sat}}_k$  of unital rings. Lemma 2.34 shows that the “directed relations” in  $\underline{\mathbb{Z}\text{prsm}}_k$  go to zero in  $K'_0(\text{sat}_k)$ .

We claim that the “semi-orthogonal relations” in  $\underline{\mathbb{Z}\text{sat}}_k$  go to zero in  $K_0(\text{prsm}_k)$  under  $\underline{\mathcal{S}} \mapsto \underline{\mathcal{S}}$ . Namely, let  $\mathcal{S}$  be a saturated dg category with full dg subcategories  $\mathcal{U}$  and  $\mathcal{V}$  such that  $[\mathcal{S}] = \langle [\mathcal{U}], [\mathcal{V}] \rangle$  is a semi-orthogonal decomposition. Let  $\mathcal{V}'$  be the full dg subcategory of  $\mathcal{V}$  consisting of all objects that are not in  $\mathcal{U}$ . From the proof of Proposition 2.20 we see that the obvious dg functor  $\left[ \begin{smallmatrix} \mathcal{U} & 0 \\ \mathcal{V}' & \mathcal{V}' \end{smallmatrix} \right] \rightarrow \mathcal{S}$  is a Morita equivalence. Obviously  $\mathcal{V}' \rightarrow \mathcal{V}$  is a Morita equivalence ( $\mathcal{V}'$  may be empty). We obtain  $\underline{\mathcal{S}} = \left[ \begin{smallmatrix} \mathcal{U} & 0 \\ \mathcal{V}' & \mathcal{V}' \end{smallmatrix} \right] = \underline{\mathcal{U}} + \underline{\mathcal{V}'} = \underline{\mathcal{U}} + \underline{\mathcal{V}}$  in  $K_0(\text{prsm}_k)$ . This shows the claim and proves the proposition.  $\square$

**2.12. Grothendieck ring of proper and smooth dg algebras.** The aim of this section is to give an alternative description of the ring  $K_0(\text{prsm}_k)$  using smooth and proper dg algebras (instead of categories).

**Lemma 2.38.** *Let  $\mathcal{A}, \mathcal{B}$  be dg categories, and let  $X = {}_{\mathcal{B}}X_{\mathcal{A}}$  be a dg  $\mathcal{A} \otimes \mathcal{B}^{\text{op}}$ -module. Assume that  $(-\otimes_{\mathcal{B}}^L X): D(\mathcal{B}) \rightarrow D(\mathcal{A})$  is an equivalence of triangulated categories. Then  $\mathcal{A}$  and  $\mathcal{B}$  are isomorphic in  $\text{Hmo}_k$ .*

**Remark 2.39.** Lemma 2.38 shows that two dg categories  $\mathcal{A}$  and  $\mathcal{B}$  are “dg Morita equivalent” in the sense of [19, Def. 3.13] if and only if they are isomorphic in  $\text{Hmo}_k$ .

*Proof.* Let  $\mathcal{A}' := Q(\mathcal{A}) \rightarrow \mathcal{A}$  and  $\mathcal{B}' := Q(\mathcal{B}) \rightarrow \mathcal{B}$  be cofibrant resolutions. Consider  $X$  by restriction of scalars as a dg  $\mathcal{A}' \otimes \mathcal{B}'^{\text{op}}$ -module, and let  $X' \rightarrow X$  be a cofibrant resolution in  $C(\mathcal{A}' \otimes \mathcal{B}'^{\text{op}})$ . Corollary 3.15 and the beginning of the proof of Proposition 3.16 in [19] show that the dg functor

$$T_{X'} := (- \otimes_{\mathcal{B}'} X'): \text{Mod}(\mathcal{B}') \rightarrow \text{Mod}(\mathcal{A}')$$

directly descends to an equivalence  $T_{X'}: D(\mathcal{B}') \rightarrow D(\mathcal{A}')$  of triangulated categories. On compact objects we obtain an equivalence  $T_{X'}: \text{per}(\mathcal{B}') \rightarrow \text{per}(\mathcal{A}')$ .

For  $B' \in \mathcal{B}'$  note that  $T_{X'}(\widehat{B}') = X'(-, B')$  is a cofibrant dg  $\mathcal{A}'$ -module by [19, Prop. 2.10.(b) and Lemma 2.14]. Hence the dg functor  $T_{X'}$  maps cofibrant dg  $\mathcal{B}'$ -modules to cofibrant dg  $\mathcal{A}'$ -modules (use [19, Lemma 2.7]).

This shows that the dg functor  $T_{X'}: \text{Perf}(\mathcal{B}') \rightarrow \text{Perf}(\mathcal{A}')$  induced by  $T_{X'}$  is a quasi-equivalence. Hence

$$\mathcal{B} \leftarrow \mathcal{B}' \rightarrow \text{Perf}(\mathcal{B}') \xrightarrow{T_{X'}} \text{Perf}(\mathcal{A}') \leftarrow \mathcal{A}' \rightarrow \mathcal{A}$$

consists of Morita equivalences, so  $\mathcal{A}$  and  $\mathcal{B}$  are isomorphic in  $\text{Hmo}_k$ .  $\square$

**Lemma 2.40.** *Let  $f: \mathcal{S} \rightarrow \mathcal{T}$  be a Morita equivalence between dg categories having a compact generator. Let  $A$  and  $B$  be endomorphism dg algebras of objects  $E \in \text{Perf}(\mathcal{S})$  and  $F \in \text{Perf}(\mathcal{T})$ , respectively, that become classical generators of  $[\text{Perf}(\mathcal{S})]$  and  $[\text{Perf}(\mathcal{T})]$ , respectively. If  $\mathcal{S}$  and  $\mathcal{T}$  are proper and smooth, the same is true for  $A$  and  $B$ . Then there is a dg  $B \otimes A^{\text{op}}$ -module  $X = {}_A X_B$  such that  $(-\otimes_A^L X): D(A) \rightarrow D(B)$  is an equivalence of triangulated categories.*

*Proof.* Example 2.29 shows that the obvious dg functors  $A \rightarrow \text{Perf}(\mathcal{S})$  and  $B \rightarrow \text{Perf}(\mathcal{T})$  are Morita equivalences and that properness and smoothness is passed on from  $\mathcal{S}$  and  $\mathcal{T}$  to  $A$  and  $B$ . The Morita equivalences  $A \rightarrow \text{Perf}(\mathcal{S}) \xrightarrow{f^*} \text{Perf}(\mathcal{T}) \leftarrow B$  (where  $f^*$  is even a quasi-equivalence) define equivalences

$$\begin{aligned} [\text{Mod}(A)_{cf}] &\xrightarrow{\sim} [\text{Mod}(\text{Perf}(\mathcal{S}))_{cf}] \xrightarrow[\sim]{[(f^*)^*]} [\text{Mod}(\text{Perf}(\mathcal{T}))_{cf}] \\ &\xrightarrow{\sim} D(\text{Perf}(\mathcal{T})) \xrightarrow[\sim]{\text{res}_B^{\text{Perf}(\mathcal{T})}} D(B) \end{aligned}$$

mapping  $A$  to  $(\text{Perf}(\mathcal{S}))(-, E)$  to  $(\text{Perf}(\mathcal{T}))(-, f^*(E))$  to  $(\text{Perf}(\mathcal{T}))(F, f^*(E))$ . Hence we can take  $X$  to be the dg  $B \otimes A^{\text{op}}$ -module  $(\text{Perf}(\mathcal{T}))(F, f^*(E))$ .  $\square$

Let  $\text{prsmalg}_k$  be the full subcategory of  $\text{dgcats}_k$  consisting of proper and smooth dg algebras (= dg categories with one object). We consider the equivalence relation on the objects of  $\text{prsmalg}_k$  generated by  $A \sim B$  if there is a dg  $B \otimes A^{\text{op}}$ -module  $X = {}_A X_B$  such that  $(-\otimes_A^L X): D(A) \rightarrow D(B)$  is an equivalence of triangulated categories. Let  $\underline{\text{prsmalg}}_k$  be the set of equivalence classes. Given a proper and smooth dg algebra  $A$  we denote its class by  $\underline{A}$ .

**Lemma 2.41.** *The inclusion  $\text{prsmalg}_k \rightarrow \text{prsm}_k$  induces an isomorphism*

$$\underline{\text{prsmalg}}_k \xrightarrow{\sim} \underline{\text{prsm}}_k$$

*of sets and then an isomorphism*

$$\underline{\mathbb{Z}\text{prsmalg}}_k \xrightarrow{\sim} \underline{\mathbb{Z}\text{prsm}}_k$$

*of abelian groups.*

*Proof.* The first isomorphism trivially yields the second one which we prove now. Lemma 2.38 shows that the map  $\underline{\text{prsmalg}}_k \rightarrow \underline{\text{prsm}}_k$  is well-defined. If  $\mathcal{T}$  is any proper and smooth dg category, take any  $E \in \underline{\text{Perf}}(\mathcal{T})$  that is a classical generator of  $[\text{Perf}(\mathcal{T})]$ , and let  $A = \text{Perf}(\mathcal{T})(E, E)$ . Then  $A \rightarrow \text{Perf}(\mathcal{T})$  is a Morita equivalence and  $A$  is proper and smooth (see Example 2.29). This shows surjectivity. Injectivity follows from Lemma 2.40.  $\square$

Let  $A$  and  $B$  be dg algebras, and let  $N = {}_B N_A$  be a dg  $A \otimes B^{\text{op}}$ -module. Then we can form the dg algebra  $\begin{pmatrix} A & 0 \\ N & B \end{pmatrix}$ . We use round brackets in order to distinguish  $\begin{pmatrix} A & 0 \\ N & B \end{pmatrix}$  from the dg category  $\left[ \begin{smallmatrix} A & 0 \\ N & B \end{smallmatrix} \right]$  with two objects.

**Definition 2.42.** The **Grothendieck group  $K_0(\text{prsmalg}_k)$  of proper and smooth dg algebras** is defined to be the quotient of the abelian group  $\mathbb{Z}\text{prsmalg}_k$  by the subgroup generated by the elements (the “lower-triangular matrix algebra relations”)  $\underline{R} - (\underline{A} + \underline{B})$  whenever  $R$  is a proper and smooth dg algebra such that there are dg algebras  $A$  and  $B$  together with a dg  $A \otimes B^{\text{op}}$ -module  $N = {}_B N_A$  such that  $R = \begin{pmatrix} A & 0 \\ N & B \end{pmatrix}$ . (We do not require that  $A$  and  $B$  are smooth and proper since this is automatic: properness of  $A$  and  $B$  is obvious, and smoothness follows from [19, Thm. 3.24 and Rem. 3.25].)

**Proposition 2.43.** *The isomorphism  $\mathbb{Z}\text{prsmalg}_k \xrightarrow{\sim} \mathbb{Z}\text{prsm}_k$  of abelian groups induces an isomorphism*

$$K_0(\text{prsmalg}_k) \xrightarrow{\sim} K_0(\text{prsm}_k)$$

of abelian groups.

*Proof.* It is easy to see that the dg algebra  $\begin{pmatrix} A & 0 \\ N & B \end{pmatrix}$  and the dg category  $\left[ \begin{smallmatrix} A & 0 \\ N & B \end{smallmatrix} \right]$  are Morita equivalent, cf. [19, Rem. 3.25]. Hence the “lower-triangular matrix algebra relations” go to zero in  $K_0(\text{prsm}_k)$  and we obtain a morphism

$$K_0(\text{prsmalg}_k) \rightarrow K_0(\text{prsm}_k)$$

of groups.

Let  $\mathcal{T}$  be a smooth and proper dg category with full dg subcategories  $\mathcal{U}$  and  $\mathcal{V}$  such that  $\mathcal{T} = \left[ \begin{smallmatrix} \mathcal{U} & 0 \\ * & \mathcal{V} \end{smallmatrix} \right]$ . Then we have a semi-orthogonal decomposition

$$[\text{Perf}(\mathcal{T})] = \langle [\text{Perf}(\mathcal{U})], [\text{Perf}(\mathcal{V})] \rangle$$

by Lemma 2.34. Choose  $u \in \text{Perf}(\mathcal{U})$  and  $v \in \text{Perf}(\mathcal{V})$  that become classical generators of  $[\text{Perf}(\mathcal{U})]$  and  $[\text{Perf}(\mathcal{V})]$  respectively. Then  $u \oplus v$  is a classical generator of  $[\text{Perf}(\mathcal{T})]$ . Let  $A, B, R$  be the endomorphism dg algebras of  $u, v, u \oplus v$ , respectively. Then the “directed relation”  $\underline{\mathcal{T}} - (\underline{\mathcal{U}} + \underline{\mathcal{V}})$  in  $\mathbb{Z}\text{prsm}_k$  is mapped to  $\underline{R} - (\underline{A} + \underline{B})$  in  $\mathbb{Z}\text{prsmalg}_k$  under the inverse of the isomorphism of Lemma 2.41.

Note that  $R = \begin{pmatrix} A & (\text{Perf}(\mathcal{T}))(v,u) \\ (\text{Perf}(\mathcal{T}))(u,v) & B \end{pmatrix}$ . But since  $(\text{Perf}(\mathcal{T}))(v,u)$  is acyclic the obvious morphism  $R' := \begin{pmatrix} A & 0 \\ (\text{Perf}(\mathcal{T}))(u,v) & B \end{pmatrix} \rightarrow R$  is a quasi-isomorphism of dg algebras. Hence  $\underline{R} - (\underline{A} + \underline{B}) = \underline{R}' - (\underline{A} + \underline{B})$  goes to zero in  $K_0(\text{prsmalg}_k)$ , and this implies the proposition.  $\square$

Note that up to now  $\text{prsmalg}_k$  is only a set and  $\mathbb{Z}\text{prsmalg}_k$  and  $K_0(\text{prsmalg}_k)$  are only abelian groups. However the isomorphisms from Lemma 2.41 and Proposition 2.43 enable us to equip these structures with multiplication maps  $\bullet$

which are obviously induced by  $(A, B) \mapsto A \otimes^L B$ . So  $\text{prsmalg}_k$  is a monoid with unit  $k$ , and  $\mathbb{Z}\text{prsmalg}_k$  and  $K_0(\text{prsmalg}_k)$  are commutative rings with unit  $k$ .

**Definition 2.44.** We call  $K_0(\text{prsmalg}_k)$  with the multiplication  $\bullet$  induced by  $(A, B) \mapsto A \otimes^L B$ , the **Grothendieck ring of proper and smooth dg algebras**.

**Remark 2.45.** If we combine Propositions 2.43 and 2.37 we obtain ring isomorphisms

$$K_0(\text{prsmalg}_k) \xrightarrow{\sim} K_0(\text{prsm}_k) \xrightarrow{\sim} K'_0(\text{sat}_k)$$

induced by  $A \mapsto A$  (for  $A$  a proper and smooth dg algebra) and  $\mathcal{T} \mapsto \text{Perf}(\mathcal{T})$  (for  $\mathcal{T}$  a proper and smooth dg category). The inverse map  $K'_0(\text{sat}_k) \xrightarrow{\sim} K_0(\text{prsmalg}_k)$  is induced by mapping a saturated dg category  $\mathcal{S}$  to the endomorphism dg algebra of an arbitrary classical generator of  $[\mathcal{S}]$ .

### 3. Grothendieck ring of varieties over $\mathbb{A}^1$

A variety is a reduced separated scheme of finite type over a field  $k$  (not necessarily irreducible). In this section we assume that  $k$  has characteristic zero. If  $X$  and  $Y$  are schemes over  $k$  we abbreviate  $X \times Y := X \times_{\text{Spec } k} Y$ . Denote by  $\mathbb{A}^1 = \mathbb{A}_k^1$  the affine line over  $k$ . An  $\mathbb{A}^1$ -variety is a variety  $X$  together with a morphism  $X \rightarrow \mathbb{A}^1$ .

**Definition 3.1** ([1]). The **(motivic) Grothendieck group  $K_0(\text{Var}_{\mathbb{A}^1})$  of varieties over  $\mathbb{A}^1$**  is the free abelian group on isomorphism classes  $[X]_{\mathbb{A}^1}$  of varieties  $X \rightarrow \mathbb{A}^1$  over  $\mathbb{A}^1$  subject to the relations  $[X]_{\mathbb{A}^1} = [X - Y]_{\mathbb{A}^1} + [Y]_{\mathbb{A}^1}$  whenever  $Y \subset X$  is a closed subvariety.

Sometimes we write  $[X, W]$  instead of  $[X]_{\mathbb{A}^1}$  if we want to emphasize the morphism  $W: X \rightarrow \mathbb{A}^1$ . The following theorem describes two alternative presentations of the Grothendieck group  $K_0(\text{Var}_{\mathbb{A}^1})$  of varieties over  $\mathbb{A}^1$ .

**Theorem 3.2** ([1, Thm. 5.1]). *The obvious morphisms from the following two abelian groups to  $K_0(\text{Var}_{\mathbb{A}^1})$  are isomorphisms.*

- (sm)  $K_0^{\text{sm}}(\text{Var}_{\mathbb{A}^1})$ , the free abelian group on isomorphism classes  $[X]_{\mathbb{A}^1}$  of  $\mathbb{A}^1$ -varieties which are smooth over  $k$ , subject to the relations

$$[X]_{\mathbb{A}^1} = [X - Y]_{\mathbb{A}^1} + [Y]_{\mathbb{A}^1},$$

where  $X$  is smooth over  $k$ , and  $Y \subset X$  is a  $k$ -smooth closed subvariety.

- (bl)  $K_0^{\text{bl}}(\text{Var}_{\mathbb{A}^1})$ , the free abelian group on isomorphism classes  $[X]_{\mathbb{A}^1}$  of  $\mathbb{A}^1$ -varieties which are smooth over  $k$  and proper over  $\mathbb{A}^1$  subject to relations

$$[\emptyset]_{\mathbb{A}^1} = 0 \quad \text{and} \quad [\text{Bl}_Y(X)]_{\mathbb{A}^1} - [E]_{\mathbb{A}^1} = [X]_{\mathbb{A}^1} - [Y]_{\mathbb{A}^1},$$

where  $X$  is smooth over  $k$  and proper over  $\mathbb{A}^1$ ,  $Y \subset X$  is a  $k$ -smooth closed subvariety,  $\text{Bl}_Y(X)$  is the blowing-up of  $X$  along  $Y$ , and  $E$  is the exceptional divisor of this blowing-up.

In case (sm) we can restrict to varieties which are in addition quasi-projective over  $\mathbb{A}^1$  (and hence quasi-projective over  $k$ ), and in case (bl) to varieties which are projective over  $\mathbb{A}^1$  (and hence quasi-projective over  $k$ ). In both cases we can restrict to connected varieties.

The presentation (bl) of  $K_0(\text{Var}_{\mathbb{A}^1})$  is very important for us whereas the presentation (sm) is not used in the rest of this article.

Using that  $\mathbb{A}^1$  is an abelian algebraic group we now turn  $K_0(\text{Var}_{\mathbb{A}^1})$  into a commutative ring with unit. Given varieties  $W: X \rightarrow \mathbb{A}^1$  and  $V: Y \rightarrow \mathbb{A}^1$  define  $W * V$  to be the composition

$$W * V: X \times Y \xrightarrow{W \times V} \mathbb{A}^1 \times \mathbb{A}^1 \xrightarrow{+} \mathbb{A}^1.$$

From Definition 3.1 it is clear that  $[X, W] \cdot [Y, V] := [X \times Y, W * V]$  turns the abelian group  $K_0(\text{Var}_{\mathbb{A}^1})$  into a commutative ring with unit  $[\text{Spec } k, 0]$ , the class of the zero function  $\text{Spec } k \xrightarrow{0} \mathbb{A}^1$ .

The same recipe turns  $K_0^{\text{sm}}(\text{Var}_{\mathbb{A}^1})$  into a ring such that  $K_0^{\text{sm}}(\text{Var}_{\mathbb{A}^1}) \rightarrow K_0(\text{Var}_{\mathbb{A}^1})$  is an isomorphism of rings. Note however that this recipe does not work for  $K_0^{\text{bl}}(\text{Var}_{\mathbb{A}^1})$ : if  $W: X \rightarrow \mathbb{A}^1$  and  $V: Y \rightarrow \mathbb{A}^1$  are projective,  $W * V$  is not projective in general.

**Remark 3.3.** We denote the class of the zero morphism  $\mathbb{A}^1 \xrightarrow{0} \mathbb{A}^1$  by

$$\mathbb{L}_{(\mathbb{A}^1, 0)} := [\mathbb{A}^1, 0].$$

Let us justify this. Similar as above one defines the Grothendieck ring  $K_0(\text{Var}_k)$  of varieties over  $k$ , with multiplication given by  $[X] \cdot [Y] := [X \times Y]$ . The map  $K_0(\text{Var}_k) \rightarrow K_0(\text{Var}_{\mathbb{A}^1})$  given by  $[X] \mapsto [X, 0]$  is then a morphism of unital rings. It maps the class  $\mathbb{L}_k$  of  $\mathbb{A}^1 \rightarrow \text{Spec } k$  to  $\mathbb{L}_{(\mathbb{A}^1, 0)}$ .

**Definition 3.4.** A **Landau–Ginzburg (LG) motivic measure** is a morphism of unital rings from  $K_0(\text{Var}_{\mathbb{A}^1})$  to another ring.

#### 4. Thom–Sebastiani Theorem and smoothness

We now start to consider categories of matrix factorizations. Our notation and many results are explained in [17]. Our aim in this section is to prove the Thom–Sebastiani Theorem 4.23 and the smoothness result of Theorem 4.24.

We fix a field  $k$  which can be arbitrary in section 4.1 and is assumed to be algebraically closed and of characteristic zero starting from section 4.2. By a scheme we mean a scheme over  $k$ , and by a variety a reduced separated scheme of finite type over  $k$  (as in Section 3).

In this and the following section dg means “differential  $\mathbb{Z}_2$ -graded”. When we refer to results from Section 2 we always mean the differential  $\mathbb{Z}_2$ -graded version (see Remark 2.1) for  $k = k$ .

**4.1. Object oriented Čech enhancements for matrix factorizations.** This section runs parallel to [18, section C]; we will therefore often refer to results there and assume that the reader is familiar with the notation and arguments there.

We say that a scheme  $X$  satisfies condition (srNfKd) if

(srNfKd)  $X$  is a separated regular Noetherian scheme of finite Krull dimension.

This is the condition we have worked with in [17]. From the discussion there it is clear that this condition implies condition (GSP+) in [18].

**Remark 4.1.** If schemes  $X$  and  $Y$  satisfy condition (srNfKd) it is in general not true that so does  $X \times Y$ . Hence as soon as we work on products we need to require condition (srNfKd) there. To avoid this annoyance one may work with smooth varieties, i. e. separated smooth schemes of finite type (over the field  $k$ ). Every smooth variety satisfies condition (srNfKd), and products of smooth varieties are again smooth varieties.

Let  $X$  be a scheme satisfying condition (srNfKd) and let  $\mathcal{U} = (U_s)_{s \in S}$  be a finite affine open covering of  $X$ . Given a vector bundle  $P$  on  $X$  we can consider its (finite) ordered Čech resolution

$$\mathcal{C}_{\text{ord}}^\bullet(P) := \left( \prod_{s_0 \in S} U_{\{s_0\}} P \rightarrow \prod_{\substack{s_0, s_1 \in S, \\ s_0 < s_1}} U_{\{s_0, s_1\}} P \rightarrow \cdots \right)$$

with the usual differentials where we abbreviate  $U_I := \bigcap_{i \in I} U_i$  for a subset  $I \subset S$  and use the notation  $j_* P := j_* j^*(P)$  if  $j: V \hookrightarrow X$  is the inclusion of an open subscheme; note that  $\mathcal{C}_{\text{ord}}^\bullet(P)$  depends on  $\mathcal{U}$  and also on the choice of a total order  $<$  on  $S$ . However, we can and will neglect the choice of  $<$  since different choices lead to isomorphic resolutions.

Let  $W: X \rightarrow \mathbb{A}^1$  be a morphism. Consider the functor that maps a vector bundle  $P$  on  $X$  to  $\mathcal{C}_{\text{ord}}^\bullet(P)$ . If we apply it to an object  $E \in \mathbf{MF}(X, W)$  we obtain a (bounded) complex in  $Z_0(\text{Qcoh}(X, W))$  that we denote by  $\mathcal{C}_{\text{ord}}^\bullet(E)$ . We denote its totalization by  $\mathcal{C}_{\text{ord}}(E) := \text{Tot}(\mathcal{C}_{\text{ord}}^\bullet(E)) \in \text{Qcoh}(X, W)$ .

Let  $\mathbf{MF}_{\check{\text{Cob}}}(X, W)$  (omitting  $\mathcal{U}$  from the notation) be the smallest full dg subcategory of  $\text{Qcoh}(X, W)$  that contains all objects  $\mathcal{C}_{\text{ord}}(E)$  for  $E \in \mathbf{MF}(X, W)$ , is closed under shifts, under cones of closed degree zero morphisms and under taking homotopy equivalent objects (i. e. objects that are isomorphic in  $[\text{Qcoh}(X, W)]$ ). It is strongly pretriangulated.

**Proposition 4.2.** *The dg category  $\mathbf{MF}_{\check{\text{Cob}}}(X, W)$  is naturally an enhancement of  $\mathbf{MF}(X, W)$ . More precisely, the natural functor*

$$\varepsilon: [\mathbf{MF}_{\check{\text{Cob}}}(X, W)] \rightarrow \text{DQcoh}(X, W)$$

*is full and faithful and its essential image coincides with the closure under isomorphisms of  $\mathbf{MF}(X, W) \subset \text{DQcoh}(X, W)$  (see [17, Thm. 2.9]). We call  $\mathbf{MF}_{\check{\text{Cob}}}(X, W)$  the **object oriented Čech enhancement** of  $\mathbf{MF}(X, W)$ .*

*Proof.* It is clear that the essential image of  $\varepsilon$  is as claimed: given  $E \in \mathbf{MF}(X, W)$ , the obvious morphism  $E \rightarrow \mathcal{C}_{\text{ord}}(E)$  becomes an isomorphism in  $\text{DQcoh}(X, W)$ .

Let  $E, F \in \mathbf{MF}(X, W)$ . In order to prove that  $\varepsilon$  is full and faithful it is enough to show that

$$\text{Hom}_{[\text{Qcoh}(X, W)]}(\mathcal{C}_{\text{ord}}(E), [m]\mathcal{C}_{\text{ord}}(F)) \rightarrow \text{Hom}_{\text{DQcoh}(X, W)}(\mathcal{C}_{\text{ord}}(E), [m]\mathcal{C}_{\text{ord}}(F))$$

is an isomorphism, for any  $m \in \mathbb{Z}_2$ . Note that  $\mathcal{C}_{\text{ord}}(F)$  is constructed as an iterated cone from shifts of objects  ${}_{\nu}F := j_*j^*(F)$ , where  $I \subset S$  and

$$j: V := U_I := \bigcap_{i \in I} U_i \rightarrow X$$

is the corresponding open embedding. Hence, as in the proof of [18, Lemma C.2], we need to show the following two claims.

- (a)  $\text{Hom}_{[\text{Qcoh}(X, W)]}(E, [n]{}_{\nu}F) \rightarrow \text{Hom}_{\text{DQcoh}(X, W)}(E, [n]{}_{\nu}F)$  is an isomorphism, for any  $n \in \mathbb{Z}_2$ .
- (b)  $\text{Hom}_{\text{Qcoh}(X, W)}(\text{Tot}(\mathcal{C}_{\text{ord}}^{\bullet}(E)), {}_{\nu}F) \rightarrow \text{Hom}_{\text{Qcoh}(X, W)}(E, {}_{\nu}F)$  is a quasi-isomorphism.

Proof of (a): Note that  $\mathbf{R}j_* = j_*$  and  $\mathbf{L}j^* = j^*$ , by [17, Lemma 2.38], since  $j$  is open and affine. Hence by the adjunctions  $(j^*, j_*)$  it is enough to show that

$$\text{Hom}_{[\mathbf{MF}(V, W)]}(j^*(E), [n]j^*(F)) \rightarrow \text{Hom}_{\mathbf{MF}(V, W)}(j^*(E), [n]j^*(F))$$

is an isomorphism (we use that  $\mathbf{MF}(V, W) \rightarrow \text{DQcoh}(V, W)$  is full and faithful, by [17, Thm. 2.9]). But  $[\mathbf{MF}(V, W)] \xrightarrow{\sim} \mathbf{MF}(V, W)$  since  $V$  is affine, by [17, Lemma 2.17].

Proof of (b): The domain of the given morphism is the totalization of the bounded complex

$$\cdots \rightarrow \text{Hom}_{\text{Qcoh}(X, W)}(\mathcal{C}_{\text{ord}}^1(E), {}_{\nu}F) \rightarrow \text{Hom}_{\text{Qcoh}(X, W)}(\mathcal{C}_{\text{ord}}^0(E), {}_{\nu}F) \rightarrow 0$$

in  $Z_0(\text{Sh}(\text{Spec } k, 0))$ . We can also view this complex as a  $\mathbb{Z}_2 \times \mathbb{Z}$ -graded double complex. Hence the given morphism is the totalization of a morphism of double complexes. Then [17, Lemma 2.46.(a)] shows that it is enough to show that

$$\text{Hom}_{\mathcal{C}(\text{Qcoh}(X))}(\mathcal{C}_{\text{ord}}^{\bullet}(E_s), {}_{\nu}F_t) \rightarrow \text{Hom}_{\mathcal{C}(\text{Qcoh}(X))}(E_s, {}_{\nu}F_t)$$

is a quasi-isomorphism for all  $s, t \in \mathbb{Z}_2$ . But this is true by the argument that shows that the morphism in [18, Formula (C.2)] is a quasi-isomorphism ( $\mathcal{C}_*$  there is denoted  $\mathcal{C}_{\text{ord}}^{\bullet}$  here; implicitly we replace  $V$  by one of its connected components).  $\square$

**Remark 4.3** (cf. [18, Rem. C.3]). The objects of  $\mathbf{MF}_{\check{\text{Cob}}}(X, W)$  are precisely the objects of  $\text{Qcoh}(X, W)$  that are homotopy equivalent to an object of the form  $\mathcal{C}_{\text{ord}}(E)$ , for  $E \in \mathbf{MF}(X, W)$ .

Let  $Y$  be another scheme and assume that  $Y$  and  $X \times Y$  satisfy condition (srNfKd) (cf. Remark 4.1). We fix a morphism  $V: Y \rightarrow \mathbb{A}^1$  and a finite affine open covering  $\mathcal{V}$  of  $Y$ . We consider the product covering  $\mathcal{U} \times \mathcal{V}$  on  $X \times Y$ . In order to prove the analog of [18, Prop. C.11] we let  $\mathbf{MF}_{\check{\text{Cob}}\boxtimes}(X \times Y, W * V)$  be the smallest full dg subcategory of  $\text{Qcoh}(X \times Y, W * V)$  that contains all objects  $\mathcal{C}_{\text{ord}}(E) \boxtimes \mathcal{C}_{\text{ord}}(F)$  for  $E \in \mathbf{MF}(X, W)$  and  $F \in \mathbf{MF}(Y, V)$ , all objects  $\mathcal{C}_{\text{ord}}(G)$  for  $G \in \mathbf{MF}(X \times Y, W * V)$ , is closed under shifts, cones of closed degree zero morphisms and under taking homotopy equivalent objects. It is strongly pretriangulated.

**Proposition 4.4.** *The dg category  $\mathbf{MF}_{\check{\text{Cob}}\boxtimes}(X \times Y, W * V)$  is naturally an enhancement of  $\mathbf{MF}(X \times Y, W * V)$ . In fact, it is equal to the enhancement  $\mathbf{MF}_{\check{\text{Cob}}}(X \times Y, W * V)$ .*

*Proof.* Use the techniques of proof from Proposition 4.2 and [18, Prop. C.11, Cor. C.12].  $\square$

Consider now  $X \times X$  with the morphism  $W * (-W): X \times X \rightarrow \mathbb{A}^1$  and with the product covering  $\mathcal{U} \times \mathcal{U}$ , and assume that  $X \times X$  satisfies condition (srNfKd). Let  $\Delta: X \rightarrow X \times X$  be the diagonal inclusion. Note that  $\Delta^*(W * (-W)) = 0$  so that the dg functor  $\Delta_*: \text{Qcoh}(X, 0) \rightarrow \text{Qcoh}(X \times X, W * (-W))$  is well-defined.

**Lemma 4.5.** *Let  $E \in \mathbf{MF}(X, W)$ ,  $F \in \mathbf{MF}(X, -W)$ ,  $G \in \mathbf{MF}(X, 0)$ , and let  $m \in \mathbb{Z}_2$ . Then the canonical map*

$$\begin{aligned} & \text{Hom}_{[\text{Qcoh}(X \times X, W * (-W))]}(\mathcal{C}_{\text{ord}}(E) \boxtimes \mathcal{C}_{\text{ord}}(F), [m]\Delta_*(\mathcal{C}_{\text{ord}}(G))) \\ & \rightarrow \text{Hom}_{\text{DQcoh}(X \times X, W * (-W))}(\mathcal{C}_{\text{ord}}(E) \boxtimes \mathcal{C}_{\text{ord}}(F), [m]\Delta_*(\mathcal{C}_{\text{ord}}(G))) \end{aligned}$$

*is an isomorphism.*

*Proof.* Again use the above techniques and the proof of [18, Lemma C.13] (note that  $\mathbf{R}\Delta_* = \Delta_*$  by [17, Remark 2.39]).  $\square$

We come back to the product situation  $X \times Y$  with morphism  $W * V$  and covering  $\mathcal{U} \times \mathcal{V}$ .

**Lemma 4.6.** *The dg functor*

$$\boxtimes: \mathbf{MF}_{\check{\text{Cob}}}(X, W) \otimes \mathbf{MF}_{\check{\text{Cob}}}(Y, V) \rightarrow \mathbf{MF}_{\check{\text{Cob}}}(X \times Y, W * V) \quad (4.1)$$

*induced from  $(- \boxtimes -): \text{Qcoh}(X, W) \times \text{Qcoh}(Y, V) \rightarrow \text{Qcoh}(X \times Y, W * V)$  is quasi-fully faithful, i. e. induces quasi-isomorphisms between morphisms spaces.*

*Proof.* This is an easy generalization of [18, Lemma C.14] since we can consider the graded components separately.  $\square$

The dg bifunctor (4.1) lifts the dg bifunctor

$$\boxtimes: \mathbf{MF}(X, W) \otimes \mathbf{MF}(Y, V) \rightarrow \mathbf{MF}(X \times Y, W * V)$$

of triangulated categories (cf. [18, Rem. C.15]).

**4.1.1. Equivalence of enhancements.**

**Lemma 4.7.** *The enhancements  $\text{InjQcoh}_{\mathbf{MF}}(X, W)$  (defined in [17, section 2.6.1]) and  $\text{MF}'_{\check{\text{Cob}}}(X, W)$  of  $\mathbf{MF}(X, W)$  are equivalent.*

*Proof.* For the first statement use the method of proof of [5, Lemma 6.2] or [17, Prop. 2.50]. □

**4.1.2. Version for arbitrary curved sheaves.** In the following section 4.1.3 we need a small generalization of the previous constructions and results.

Recall from [17, Thm. 2.25] that the functor  $\text{DQcoh}(X, W) \rightarrow \text{DSh}^{\text{co}}(X, W)$  is full and faithful and that  $\text{InjSh}(X, W)$  is naturally an enhancement of  $\text{DSh}^{\text{co}}(X, W)$ . Let  $\mathbf{MF}'(X, W)$  be the essential image of  $\mathbf{MF}(X, W)$  under the full and faithful functor  $\mathbf{MF}(X, W) \rightarrow \text{DSh}^{\text{co}}(X, W)$  (see [17, Thm. 2.9]); so  $\mathbf{MF}(X, W) \rightarrow \mathbf{MF}'(X, W)$  is an equivalence.

Denote by  $\text{MF}'_{\check{\text{Cob}}}(X, W)$  the smallest full dg subcategory of  $\text{Sh}(X, W)$  that contains all objects of  $\text{MF}_{\check{\text{Cob}}}(X, W)$  and is closed under taking homotopy equivalent objects. Then the inclusion  $\text{MF}_{\check{\text{Cob}}}(X, W) \rightarrow \text{MF}'_{\check{\text{Cob}}}(X, W)$  is a quasi-equivalence. If we define  $\text{MF}'_{\check{\text{Cob}}\boxtimes}(X \times Y, W * V)$  similarly it is clear that all propositions, lemmata and remarks of section 4.1 remain true if we replace  $\text{MF}_{\check{\text{Cob}}}$  by  $\text{MF}'_{\check{\text{Cob}}}$ ,  $\text{MF}_{\check{\text{Cob}}\boxtimes}$  by  $\text{MF}'_{\check{\text{Cob}}\boxtimes}$ ,  $\mathbf{MF}$  by  $\mathbf{MF}'$ , and  $\text{DQcoh}(-, ?)$  by  $\text{DSh}^{\text{co}}(-, ?)$ . The full dg subcategory  $\text{InjSh}_{\mathbf{MF}'}(X, W)$  of  $\text{InjSh}(X, W)$  consisting of objects of  $\mathbf{MF}'(X, W)$  is naturally an enhancement of  $\mathbf{MF}'(X, W)$ , and the obvious variation of Lemma 4.7 is true; in fact, all the enhancements of  $\mathbf{MF}'(X, W)$  we have defined are equivalent.

**4.1.3. Lifting the duality.** Recall the duality

$$D = D_X = (-)^\vee = \mathcal{H}om(-, \mathcal{D}): \mathbf{MF}(X, W)^{\text{op}} \rightarrow \mathbf{MF}(X, -W)$$

from [17, section 2.5.5] where  $\mathcal{D} = \mathcal{D}_X = (0 \rightleftarrows \mathcal{O}_X) \in \mathbf{MF}(X, 0)$ . Our aim is to lift its extension

$$D: \mathbf{MF}'(X, W)^{\text{op}} \rightarrow \mathbf{MF}'(X, -W) \tag{4.2}$$

to a dg functor  $\text{MF}'_{\check{\text{Cob}}}(X, W) \rightarrow \text{MF}'_{\check{\text{Cob}}}(X, -W)$  between the respective enhancements. Consider the dg functor

$$\tilde{D} := \mathcal{H}om(-, \mathcal{C}_{\text{ord}}(\mathcal{D})): \text{Sh}(X, W)^{\text{op}} \rightarrow \text{Sh}(X, -W).$$

**Lemma 4.8.** *Let  $E \in \mathbf{MF}(X, W)$  and consider the canonical morphism*

$$\alpha: E \rightarrow \mathcal{C}_{\text{ord}}(E)$$

*in  $Z_0(\text{Sh}(X, W))$ . Then the induced morphism*

$$\begin{aligned} \tilde{D}(\alpha): \tilde{D}(\mathcal{C}_{\text{ord}}(E)) &= \mathcal{H}om(\mathcal{C}_{\text{ord}}(E), \mathcal{C}_{\text{ord}}(\mathcal{D})) \rightarrow \tilde{D}(E) = \mathcal{H}om(E, \mathcal{C}_{\text{ord}}(\mathcal{D})) \\ &= \mathcal{C}_{\text{ord}}(E^\vee), \end{aligned}$$

is a homotopy equivalence, i. e. an isomorphism in  $[\text{Sh}(X, -W)]$ . See [18, Rem. C.6] for the last identification.

*Proof.* Write  $\alpha^* := [1]D(\alpha)$ . We have to show that

$$\text{Cone}(\alpha^*) = \mathcal{H}om(\text{Cone}(\alpha), \mathcal{C}_{\text{ord}}(\mathcal{D}))$$

is contractible. Using the method of proof of [18, Lemma C.7] (we can assume that  $X$  is irreducible) we see that  $\text{Cone}(\alpha^*)$  has a filtration with subquotients  $\text{Cone}(\alpha^*)_I^K$  labeled by pairs  $(I, K)$  where  $I \subset S$  is a non-empty subset and  $K \subset S \setminus I$  a (possibly empty) subset, such that  $\text{Cone}(\alpha^*)_I^K$  consists (if we forget some differentials) of all summands  $\mathcal{H}om_{(U_J E, U_I \mathcal{D})}$  for  $K \subset J \subset (I \cup K)$ . Moreover, for fixed  $(I, K)$ , all these summands are isomorphic to  $\mathcal{H}_I^K := \mathcal{H}om(E_{U_{I \cup K}}, U_I \mathcal{D})$ , and  $\text{Cone}(\alpha^*)_I^K$  is isomorphic to the totalization of the augmented chain complex of a (non-empty) simplex  $\Sigma$  with coefficients in  $\mathcal{H}_I^K$ . By the latter we mean the complex in  $Z_0(\text{Sh}(X, -W))$  that arises from tensoring the augmented chain complex of  $\Sigma$  with the object  $\mathcal{H}_I^K \in \text{Sh}(X, -W)$ . Since the augmented chain complex is homotopy equivalent to zero, the same is true for this complex, and then for its totalization.  $\square$

**Corollary 4.9.** *The dg functor  $\tilde{D}$  induces a dg functor*

$$\tilde{D} = \mathcal{H}om(-, \mathcal{C}_{\text{ord}}(\mathcal{D})) : \text{MF}'_{\text{Cob}}(X, W)^{\text{op}} \rightarrow \text{MF}'_{\text{Cob}}(X, -W)$$

which lifts the duality  $D$  in (4.2).

*Proof.* Adapt the proof of [18, Cor. C.8].  $\square$

The canonical morphism

$$\begin{aligned} \theta_F : F \rightarrow \tilde{D}^2(F) &= \mathcal{H}om(\mathcal{H}om(F, \mathcal{C}_{\text{ord}}(\mathcal{D})), \mathcal{C}_{\text{ord}}(\mathcal{D})), \\ f &\mapsto (\lambda \mapsto \lambda(f)) \end{aligned} \tag{4.3}$$

(for  $F \in \text{Sh}(X, W)$ ) defines a morphism  $\theta : \text{id} \rightarrow \tilde{D}^2$  of dg functors

$$\text{Sh}(X, W) \rightarrow \text{Sh}(X, W)$$

and, by Corollary 4.9, also of dg functors

$$\text{MF}'_{\text{Cob}}(X, W) \rightarrow \text{MF}'_{\text{Cob}}(X, W).$$

**Lemma 4.10.** *For each  $F \in \text{MF}'_{\text{Cob}}(X, W)$ , the morphism  $\theta_F$  in (4.3) is a homotopy equivalence.*

*Proof.* Adapt the proof of [18, Lemma C.9]. Instead of quasi-isomorphisms we need to speak about morphisms in  $Z_0(\text{Sh}(X, \pm W))$  that become isomorphisms in  $\text{DSh}(X, \pm W)$ .  $\square$

**Corollary 4.11.** *The dg functor*

$$\widetilde{D} = \mathcal{H}om(-, \mathcal{C}_{\text{ord}}(\mathcal{D})): \mathbf{MF}'_{\text{Cob}}(X, W)^{\text{op}} \rightarrow \mathbf{MF}'_{\text{Cob}}(X, -W)$$

*is a quasi-equivalence. The induced functor  $[\widetilde{D}]$  on homotopy categories is an equivalence and a duality in the sense that the natural morphism  $\theta: \text{id} \rightarrow [\widetilde{D}]^2$  is an isomorphism.*

*Proof.* Lemma 4.10 shows that  $\theta: \text{id} \rightarrow [\widetilde{D}]^2$  is an isomorphism. In particular,  $[\widetilde{D}]$  is an equivalence, and  $\widetilde{D}$  is a quasi-equivalence.  $\square$

**4.2. The singularity category of a function.** We assume now and for the rest of Section 4 that our field  $k$  is algebraically closed and of characteristic zero. Let  $X$  be a smooth variety, i. e. a separated smooth scheme of finite type (over  $k$ ), cf. Remark 4.1. Let  $W: X \rightarrow \mathbb{A}^1$  be a morphism. We identify  $k = \mathbb{A}^1(k)$  with the set of closed points of  $\mathbb{A}^1$ .

**Definition 4.12.** We define the **singularity category of  $W$**  as the product

$$\mathbf{MF}(W) := \prod_{a \in k} \mathbf{MF}(X, W - a).$$

Note that only finitely many factors of this product are non-zero. To show this we can assume that  $X$  is connected (see [17, Rem. 2.6]). If  $W$  is constant, then  $W = b$  for some  $b \in k$  and  $\mathbf{MF}(W) = \mathbf{MF}(X, 0)$  by [17, Lemma 2.28]. Otherwise  $W$  is flat and Orlov’s theorem says that  $\text{cok}: \mathbf{MF}(X, W - a) \rightarrow D_{\text{Sg}}(X_a)$  is an equivalence [17, Thm. 2.8] where  $X_a$  is the scheme theoretic fiber over  $a \in k$ . By generic smoothness on the target [9, Cor. III.10.7]  $X_a$  is smooth for all but finitely many values  $a \in k$ . If  $X_a$  is smooth, then  $D_{\text{Sg}}(X_a) = 0$ .

**Lemma 4.13.** *We have  $\mathbf{MF}(W) = 0$  if and only if  $W$  is a smooth morphism.*

*Proof.* If  $W: X \rightarrow \mathbb{A}^1$  is smooth, then it is in particular flat, so Orlov’s equivalence  $\text{cok}: \mathbf{MF}(X, W - a) \xrightarrow{\sim} D_{\text{Sg}}(X_a)$  and the fact that all  $X_a$  are regular show that  $\mathbf{MF}(W) = 0$ .

Conversely, assume that  $\mathbf{MF}(W) = 0$ . We can in addition assume that  $X$  is connected and non-empty. Then  $W$  is either constant or flat. If  $W$  is constant, we obtain  $\mathbf{MF}(X, 0) = \mathbf{MF}(W) = 0$ . This is a contradiction since  $\mathbf{MF}(X, 0)$  obviously has non-zero objects (use [17, Prop. 2.30]). So assume that  $W$  is flat. Then  $D_{\text{Sg}}(X_a) = 0$  for all  $a \in k$ , so all fibers  $X_a$  are (regular and) smooth. This together with flatness of  $W$  already implies that  $W$  is smooth (by [15, Def. 4.3.35]).  $\square$

**Remark 4.14.** As made precise by Lemma 4.13, one may think of  $\mathbf{MF}(W)$  as measuring the singularity of  $W$ . The above discussion implies that  $\mathbf{MF}(W)$  is nonzero for a constant function  $W$  (if  $X \neq \emptyset$ ), hence a constant function is considered to be singular. This would not be the case if we had defined  $\mathbf{MF}(W)$  as the product of the categories  $D_{\text{Sg}}(X_a)$ .

Let  $\text{Sing}(W) \subset X$  be the closed subscheme defined by the vanishing of the section  $dW \in \Gamma(X, \Omega_{X/k}^1)$  of the cotangent bundle. Its closed points are the critical points of  $W$ . Let  $\text{Crit}(W) = W(\text{Sing}(W)(k)) \subset \mathbb{A}^1(k) = k$  be the (finite) set of critical values of  $W$ . The above discussion shows that

$$\mathbf{MF}(W) = \prod_{a \in \text{Crit}(W)} \mathbf{MF}(X, W - a).$$

and we emphasize again that this product is finite.

Recall that we defined in [17, section 2.6] and in section 4.1 the enhancements

$$\begin{aligned} & \text{InjQcoh}_{\mathbf{MF}}(X, W - a), \\ & \mathbf{MF}_{\check{\text{C}}\text{mor}}(X, W - a), \quad \mathbf{MF}(X, W - a) / \text{AcyclMF}(X, W - a), \\ & \mathbf{MF}_{\check{\text{C}}\text{ob}}(X, W - a) \quad \text{and} \quad \mathbf{MF}'_{\check{\text{C}}\text{ob}}(X, W - a) \end{aligned}$$

of  $\mathbf{MF}(X, W - a)$  and showed that they are equivalent (three of these enhancements depend on the choice of a (finite) affine open covering of  $X$ ). Fix one of these enhancements and denote it by  $\mathbf{MF}(X, W - a)^{\text{dg}}$ . Then

$$\mathbf{MF}(W)^{\text{dg}} := \prod_{a \in \text{Crit}(W)} \mathbf{MF}(X, W - a)^{\text{dg}}$$

is an enhancement of  $\mathbf{MF}(W)$ . Since the pretriangulated dg category  $\mathbf{MF}(W)$  might not be triangulated (cf. Lemma 2.3) we will mainly work with its “triangulated dg envelope”<sup>2</sup>

$$\mathbf{MF}(W)^{\text{dg}, \natural} := \text{Perf}(\mathbf{MF}(W)^{\text{dg}}) = \prod_{a \in \text{Crit}(W)} \text{Perf}(\mathbf{MF}(X, W - a)^{\text{dg}}). \quad (4.4)$$

Then  $\mathbf{MF}(W)^{\text{dg}, \natural}$  is an enhancement of the Karoubi envelope of  $\mathbf{MF}(W)$ . Note that the quasi-equivalence class of  $\mathbf{MF}(W)^{\text{dg}, \natural}$  does not depend on the above choices of enhancements, by Lemma 2.6.(c).

**Remark 4.15.** Let us give a more concrete description of  $\mathbf{MF}(W)^{\text{dg}, \natural}$  that we will mainly use later on: For each  $a \in \text{Crit}(W)$  choose an object  $E(a) \in \mathbf{MF}(X, W - a)^{\text{dg}}$  that becomes a classical generator in  $[\mathbf{MF}(X, W - a)^{\text{dg}}] \cong \mathbf{MF}(X, W - a)$  (use [17, Prop. 2.53]). Let  $A(a)$  be the endomorphism dg algebra of  $E(a)$  in  $\mathbf{MF}(X, W - a)^{\text{dg}}$ , i. e.

$$A(a) = \text{End}_{\mathbf{MF}(X, W - a)^{\text{dg}}}(E(a)).$$

Then  $A = \prod_{a \in \text{Crit}(W)} A(a)$  is the endomorphism dg algebra of  $E = (E(a))$  in  $\mathbf{MF}(W)^{\text{dg}}$ . Proposition 2.18 yields a quasi-equivalence

$$\text{Perf}(A) = \prod_{a \in \text{Crit}(W)} \text{Perf}(A(a)) \rightarrow \mathbf{MF}(W)^{\text{dg}, \natural}$$

<sup>2</sup> If  $\mathcal{A}$  and  $\mathcal{B}$  are non-empty dg categories, scalar extension along the two projections  $\mathcal{A} \times \mathcal{B} \rightarrow \mathcal{A}$  and  $\mathcal{A} \times \mathcal{B} \rightarrow \mathcal{B}$  defines an equivalence  $\text{Perf}(\mathcal{A} \times \mathcal{B}) \rightarrow \text{Perf}(\mathcal{A}) \times \text{Perf}(\mathcal{B})$  of dg categories. This explains the second equality.

and also provides a triangulated equivalence

$$\overline{\mathbf{MF}(W)} = \prod_{a \in \text{Crit}(W)} \overline{\mathbf{MF}(X, W - a)} \xrightarrow{\sim} \prod_{a \in \text{Crit}(W)} \text{per}(A(a)) = \text{per}(A) \quad (4.5)$$

where  $\overline{\mathcal{T}}$  denotes the Karoubi envelope of a triangulated category  $\mathcal{T}$ . Moreover, it says that smoothness and properness of  $\mathbf{MF}(W)^{\text{dg}}$  (resp.  $\mathbf{MF}(W)^{\text{dg}, \natural}$ ) can be tested on  $A$ , and that  $\mathbf{MF}(W)^{\text{dg}, \natural}$  is saturated if and only if  $A$  is smooth and proper.

**4.3. Products and generators.** Let  $X$  and  $Y$  (and hence  $X \times Y$ ) be smooth varieties and let  $W: X \rightarrow \mathbb{A}^1$  and  $V: Y \rightarrow \mathbb{A}^1$  be morphisms. Our aim is to prove Proposition 4.22 below. We start with some preparations.

An object  $E \in \text{Coh}(X_0)$  can be considered as an object

$$\mu(E) := (0 \rightrightarrows E) \in \text{Coh}(X, W).$$

For flat  $W$  recall the equivalence  $\text{cok}: \mathbf{MF}(X, W) \rightarrow D_{\text{Sg}}(X_0)$  from [17, Thm. 2.8].

**Lemma 4.16** ([14, Lemma 2.18]). *Assume that  $W: X \rightarrow \mathbb{A}^1$  is flat, and let  $E \in \text{Coh}(X_0)$ . Suppose that  $P \rightarrow \mu(E)$  is a morphism in  $Z_0(\text{Coh}(X, W))$  with  $P \in \mathbf{MF}(X, W)$  and cone in  $\text{Acycl}[\text{Coh}(X, W)]$  (such a morphism exists by [17, Thm. 2.10.(b)]). Then there is an isomorphism  $\text{cok}(P) := \text{cok}(p_1) \cong E$  in  $D_{\text{Sg}}(X_0)$ .*

*Proof.* We elaborate on the proof of [14, Lemma 2.18]. From the proof of [23, Prop. 1.23] we see that there is an exact sequence (for any  $l \gg 0$ )

$$0 \rightarrow E' \rightarrow L^{-l+1} \rightarrow \dots \rightarrow L^0 \rightarrow E \rightarrow 0 \quad (4.6)$$

in  $\text{Coh}(X_0)$  where all  $L^l$  are locally free coherent sheaves and  $E'$  is a Cohen–Macaulay sheaf (as defined in [22, Lemma-Def. 1]).

The proof of [22, Thm. 3.5] shows that there is an object

$$Q := (Q_1 \xrightarrow[q_0]{q_1} Q_0) \in \mathbf{MF}(X, W)$$

such that  $\text{cok}(q_1) = E'$  as coherent sheaves. Let

$$K := (Q_1 \xrightarrow[W]{1} Q_1)$$

and note that

$$0 \rightarrow K \xrightarrow{(1, q_1)} Q \rightarrow \mu(E') \rightarrow 0$$

is a short exact sequence in  $Z_0(\text{Coh}(X, W))$ . It gives rise to a triangle in  $\text{DCoh}(X, W)$ . Since  $K = 0$  in  $[\mathbf{MF}(X, W)]$  we obtain an isomorphism  $Q \xrightarrow{\sim} \mu(E')$  in  $\text{DCoh}(X, W)$ .

If  $L \in \text{Coh}(X_0)$  is locally free we claim that  $\mu(L)$  vanishes in  $\text{DCoh}(X, W)$ . Indeed,  $L$  is Cohen–Macaulay, so the above argument shows that there is an object  $M \in \mathbf{MF}(X, W)$  such that  $\text{cok}(m_1) = L$  in  $\text{Coh}(X_0)$  and  $M \xrightarrow{\sim} \mu(L)$  in  $\text{DCoh}(X, W)$ . Since  $L$  vanishes in  $D_{\text{Sg}}(X_0)$  we see that  $M$  vanishes in  $\mathbf{MF}(X, W)$  and a fortiori in  $\text{DCoh}(X, W)$ .

If we apply  $\mu$  to (4.6) and use this claim for the  $\mu(L')$  we see that  $[l]\mu(E') \cong \mu(E)$  in  $\text{DCoh}(X, W)$ .

By assumption we have  $P \cong \mu(E)$  in  $\text{DCoh}(X, W)$ . Combined with the above isomorphisms this shows that  $P \cong [l]Q$  in  $\text{DCoh}(X, W)$ . Since both  $P$  and  $Q$  are in  $\mathbf{MF}(X, W)$  and  $\mathbf{MF}(X, W) \rightarrow \text{DCoh}(X, W)$  is an equivalence, we have  $P \cong [l]Q$  in  $\mathbf{MF}(X, W)$ . This shows that  $\text{cok}(p_1) \cong [l]\text{cok}(q_1) = [l]E' \cong E$  in  $D_{\text{Sg}}(X_0)$  where we use (4.6) for the last isomorphism.  $\square$

**Corollary 4.17.** *Assume that  $W: X \rightarrow \mathbb{A}^1$  and  $V: Y \rightarrow \mathbb{A}^1$  are flat. Let  $E \in \text{Coh}(X_0)$  and  $F \in \text{Coh}(Y_0)$ . Let  $P \rightarrow \mu(E)$  be a morphism in  $Z_0(\text{Coh}(X, W))$  with  $P \in \mathbf{MF}(X, W)$  and cone in  $\text{Acycl}[\text{Coh}(X, W)]$ , and let  $Q \rightarrow \mu(F)$  be a morphism in  $Z_0(\text{Coh}(Y, V))$  with  $Q \in \mathbf{MF}(Y, V)$  and cone in  $\text{Acycl}[\text{Coh}(Y, V)]$ . Then*

$$\text{cok}(P \boxtimes Q) \cong E \boxtimes F$$

in  $D_{\text{Sg}}((X \times Y)_0)$ . Here we use the closed embedding  $X_0 \times Y_0 \subset (X \times Y)_0$  in order to consider  $E \boxtimes F \in \text{Coh}(X_0 \times Y_0)$  as an object of  $\text{Coh}((X \times Y)_0)$ .

*Proof.* The morphism  $P \boxtimes Q \rightarrow \mu(E) \boxtimes \mu(F) = \mu(E \boxtimes F)$  has cone in  $\text{Acycl}[\text{Coh}(X \times Y, W * V)]$ : it factors as  $P \boxtimes Q \rightarrow \mu(E) \boxtimes Q \rightarrow \mu(E) \boxtimes \mu(F)$ , both morphisms have cone in  $\text{Acycl}[\text{Coh}(X \times Y, W * V)]$ , and we can use the octahedral axiom. Since  $W * V$  is flat we can apply Lemma 4.16.  $\square$

We need that certain categories have classical generators.

**Theorem 4.18.** *For  $Z, Z_1, Z_2$  separated schemes of finite type, we have:*

- (a) *The category  $D^b(\text{Coh}(Z))$  has a classical generator.*
- (b) *If  $T_1$  and  $T_2$  are classical generators of  $D^b(\text{Coh}(Z_1))$  and  $D^b(\text{Coh}(Z_2))$ , respectively, then  $T_1 \boxtimes T_2$  is a classical generator of  $D^b(\text{Coh}(Z_1 \times Z_2))$ .*

*Proof.* See [26, Thm. 7.38] or [16, Thm. 6.3] for the first statement. The proof of [16, Thm. 6.3] shows that there are classical generators  $S_1$  and  $S_2$  of  $D^b(\text{Coh}(Z_1))$  and  $D^b(\text{Coh}(Z_2))$ , respectively, such that  $S_1 \boxtimes S_2$  is a classical generator of  $D^b(\text{Coh}(Z_1 \times Z_2))$ . From  $S_1 \in \text{thick}(T_1)$  we obtain  $S_1 \boxtimes S_2 \in \text{thick}(T_1 \boxtimes S_2)$ , so  $T_1 \boxtimes S_2$  is a classical generator of  $D^b(\text{Coh}(Z_1 \times Z_2))$ . Similarly, we see that  $T_1 \boxtimes T_2$  is a classical generator of  $D^b(\text{Coh}(Z_1 \times Z_2))$ .  $\square$

If  $Z$  is a locally Noetherian scheme (over our  $k$ ) its regular locus is open [7, Rem. 6.25(4)]. We equip its closed complement  $Z^{\text{sing}} \subset Z$  of singular points with the unique structure of a reduced closed subscheme of  $Z$ .

**Proposition 4.19.** *Let  $Z$  be a scheme satisfying condition (ELF) in [21], and let  $i: Z^{\text{sing}} \hookrightarrow Z$  be the inclusion of the singular locus. Let  $T \in D^b(\text{Coh}(Z^{\text{sing}}))$  be a classical generator. Then the image of  $i_*(T)$  in  $D_{\text{Sg}}(Z)$  is a classical generator of  $D_{\text{Sg}}(Z)$ .*

*Proof.* We use the notation of [21]. The object  $i_*(T)$  is a classical generator of  $D_{Z^{\text{sing}}}^b(\text{Coh}(Z))$  (by [16, Lemma 6.9]) and the obvious functor

$$D_{Z^{\text{sing}}}^b(\text{Coh}(Z))/\mathfrak{Pctf}_{Z^{\text{sing}}}(Z) \rightarrow D_{\text{Sg}}(Z)$$

is full and faithful, and dense in the sense that any object of  $D_{\text{Sg}}(Z)$  is a direct summand of an object of  $D_{Z^{\text{sing}}}^b(\text{Coh}(Z))/\mathfrak{Pctf}_{Z^{\text{sing}}}(Z)$  [21, Lemma 2.6 and Prop. 2.7]. These statements obviously imply that  $i_*(T)$  becomes a classical generator of  $D_{\text{Sg}}(Z)$ .  $\square$

We come back to our setting with  $W: X \rightarrow \mathbb{A}^1$  and  $V: Y \rightarrow \mathbb{A}^1$ . Recall that  $\text{Sing}(W) \subset X$  is the closed subscheme defined by the vanishing of  $dW$ . If  $Z$  is a scheme we denote by  $|Z|$  the corresponding reduced closed subscheme.

**Remark 4.20.** Assume that  $X$  is connected, and let  $a \in k$ . If  $W = a$  then  $\text{Sing}(W) \cap X_a = X$  and  $(X_a)^{\text{sing}} = \emptyset$ . Otherwise the singular points of  $X_a$  are precisely the elements of the scheme-theoretic intersection  $\text{Sing}(W) \cap X_a$ , i. e. we have the equality

$$|\text{Sing}(W) \cap X_a| = (X_a)^{\text{sing}} \tag{4.7}$$

of varieties. This is trivial if  $W$  is constant  $\neq a$ , and otherwise it follows from the Jacobian criterion applied to  $W - a$  (see e. g. the proof of [20, Thm. III.§4.4]).

We obviously have

$$\text{Sing}(W * V) = \text{Sing}(W) \times \text{Sing}(V). \tag{4.8}$$

This implies that

$$\text{Crit}(W * V) = \text{Crit}(W) + \text{Crit}(V) := \{a + b \mid a \in \text{Crit}(W), b \in \text{Crit}(V)\}.$$

**Lemma 4.21.** *Let  $c \in k$ . Then*

$$|\text{Sing}(W * V) \cap (X \times Y)_c| = \coprod_{\substack{a \in \text{Crit}(W), b \in \text{Crit}(V), \\ a+b=c}} |\text{Sing}(W) \cap X_a| \times |\text{Sing}(V) \cap Y_b|$$

*as subvarieties of  $X \times Y$ . If  $c \notin \text{Crit}(W * V)$  then  $|\text{Sing}(W * V) \cap (X \times Y)_c| = \emptyset$ .*

*Proof.* The set  $\text{Crit}(W) \subset k$  of critical values of  $W$  is finite, by generic smoothness on the target. Hence  $|\text{Sing}(W)| = \coprod_{a \in \text{Crit}(W)} |\text{Sing}(W) \cap X_a|$ , and similarly for  $V$  and  $W * V$ . Hence we can rewrite both sides of (4.8) and obtain

$$\coprod_{c \in \text{Crit}(W * V)} |\text{Sing}(W * V) \cap (X \times Y)_c| = \coprod_{\substack{a \in \text{Crit}(W), \\ b \in \text{Crit}(V)}} |\text{Sing}(W) \cap X_a| \times |\text{Sing}(V) \cap Y_b|$$

These statements imply the lemma.  $\square$

The functor  $\boxtimes: \mathbf{MF}(X, W - a) \times \mathbf{MF}(Y, V - b) \rightarrow \mathbf{MF}(X \times Y, W * V - a - b)$  gives rise to the functor  $\boxtimes: \mathbf{MF}(W) \times \mathbf{MF}(V) \rightarrow \mathbf{MF}(W * V)$  defined by

$$E \boxtimes F := \left( \bigoplus_{a+b=c} E(a) \boxtimes F(b) \right)_{c \in k} \tag{4.9}$$

for  $E = (E(a))_{a \in k}$  and  $F = (F(b))_{b \in k}$ . Note that only finitely many of the objects  $E(a) \boxtimes F(b)$  are non-zero.

Obviously, an object  $E = (E(a))_{a \in k} \in \mathbf{MF}(W)$  is a classical generator if and only if  $E(a) \in \mathbf{MF}(X, W - a)$  is a classical generator for the finitely many critical values  $a$  of  $W$ .

**Proposition 4.22.** *Let  $E \in \mathbf{MF}(W)$  and  $F \in \mathbf{MF}(V)$  be classical generators. Then  $E \boxtimes F$  is a classical generator of  $\mathbf{MF}(W * V)$ .*

*Proof.* Observe first that it is enough to prove the result for suitably chosen classical generators  $E$  and  $F$ , see the end of the proof of Theorem 4.18.

It is certainly enough to prove the proposition under the additional assumption that both  $X$  and  $Y$  are connected (cf. [17, Rem. 2.6]). Then  $W$  is either constant or flat, and similarly for  $V$ .

**Case 1.** Both  $W$  and  $V$  are flat.

**Step 1:** Fix a critical value  $a \in \text{Crit}(W)$  of  $W$ . Let  $S_a \in D^b(\text{Coh}((X_a)^{\text{sing}}))$  be a classical generator (which exists by Theorem 4.18.(a)). By replacing  $S_a$  by the direct sum of its cohomologies we can and will assume that  $S_a \in \text{Coh}((X_a)^{\text{sing}})$ . Let  $s_a: (X_a)^{\text{sing}} \hookrightarrow X_a$  be the closed embedding. By [17, Thm. 2.10.(b)] there is an object  $E(a) \in \mathbf{MF}(X, W - a)$  together with a morphism  $E(a) \rightarrow \mu(s_{a*}(S_a))$  in  $Z_0(\text{Coh}(X, W - a))$  whose cone is in  $\text{Acycl}[\text{Coh}(X, W - a)]$ . By Lemma 4.16 we have  $\text{cok}(E(a)) \cong s_{a*}(S_a)$  in  $D_{\text{Sg}}(X_a)$ . Proposition 4.19 then shows that  $E(a)$  is a classical generator of  $\mathbf{MF}(X, W - a)$ . Letting  $a$  vary we see that  $E := (E(a))_{a \in \text{Crit}(W)}$  is a classical generator of  $\mathbf{MF}(W)$ .

**Step 2:** Similarly we find for each  $b \in \text{Crit}(V)$  an object  $T_b \in \text{Coh}((Y_b)^{\text{sing}})$  that is a classical generator of  $D^b(\text{Coh}((Y_b)^{\text{sing}}))$  and then  $F(b) \in \mathbf{MF}(Y, V - b)$  together with a morphism  $F(b) \rightarrow \mu(t_{b*}(T_b))$  in  $Z_0(\text{Coh}(Y, V - b))$  whose cone is in  $\text{Acycl}[\text{Coh}(Y, V - b)]$  such that  $\text{cok}(F(b)) \cong t_{b*}(T_b)$  in  $D_{\text{Sg}}(Y_b)$  where  $t_b: (Y_b)^{\text{sing}} \hookrightarrow Y_b$ . Then  $F := (F(b))_{b \in \text{Crit}(V)}$  is the classical generator of  $\mathbf{MF}(V)$  we will consider.

**Step 3:** Fix  $c \in \text{Crit}(W * V)$  a critical value of  $W * V$ . Theorem 4.18.(b), Lemma 4.21 and equation (4.7) in Remark 4.20, and Proposition 4.19 imply that the image of

$$\bigoplus_{\substack{a \in \text{Crit}(W), b \in \text{Crit}(V), \\ a+b=c}} s_{a*}(S_a) \boxtimes t_{b*}(T_b)$$

in  $D_{\text{Sg}}((X \times Y)_c)$  is a classical generator of  $D_{\text{Sg}}((X \times Y)_c)$ . But Corollary 4.17 shows that this object is isomorphic to  $\text{cok}((E \boxtimes F)(c))$  in  $D_{\text{Sg}}((X \times Y)_c)$ .

Hence  $E \boxtimes F$  is a classical generator of  $\mathbf{MF}(W * V)$ . This proves the proposition if both  $W$  and  $V$  are flat.

**Case 2.** Precisely one of  $W, V$  is flat.

Without loss of generality assume that  $W$  is flat and that  $b_0 := V \in k$ . Then  $\mathbf{MF}(V) = \mathbf{MF}(Y, 0)$  by [17, Lemma 2.28], and [17, Rem. 2.54] shows that there is a vector bundle  $Q$  on  $Y$  that is a classical generator of  $D^b(\text{Coh}(Y))$  such that  $\mu(Q) = (0 \rightrightarrows Q)$  is a classical generator of  $\mathbf{MF}(Y, 0)$ . Now define  $F \in \mathbf{MF}(V)$  by  $F(b_0) := \mu(Q)$  and  $F(b) = 0$  for all  $b \neq b_0$ . Define the classical generator  $E = (E(a))$  of  $\mathbf{MF}(W)$  as in Step 1. We have  $\text{Sing}(W * V) = \text{Sing}(W) \times Y = \coprod_{a \in \text{Crit}(W)} (X_a)^{\text{sing}} \times Y$  and  $((X \times Y)_c)^{\text{sing}} = (X_{c-b_0})^{\text{sing}} \times Y$ . Adjusting the above method it is easy to see that  $E \boxtimes F$  is a classical generator of  $\mathbf{MF}(W * V)$ .

**Case 3.** Both  $W$  and  $V$  are constant.

Then

$$\mathbf{MF}(W) = \mathbf{MF}(X, 0), \quad \mathbf{MF}(V) = \mathbf{MF}(Y, 0)$$

and

$$\mathbf{MF}(W * V) = \mathbf{MF}(X \times Y, 0).$$

Let  $Q$  be a vector bundle on  $Y$  as in the previous case, and let  $P$  be a vector bundle on  $X$  that generates  $D^b(\text{Coh}(X))$  classically and such that  $\mu(P)$  is a classical generator of  $\mathbf{MF}(X, 0)$ . Theorem 4.18.(b) (or [2, Lemma 3.4.1, 3.1, 2.1] since  $X$  and  $Y$  are smooth) shows that  $P \boxtimes Q$  is a classical generator of  $D^b(\text{Coh}(X \times Y))$ . Then  $\mu(P \boxtimes Q) = \mu(P) \boxtimes \mu(Q)$  is a classical generator of  $\mathbf{MF}(X \times Y, 0)$ , by [17, Rem. 2.54]. This shows what we need.  $\square$

**4.4. Thom–Sebastiani Theorem.** Note that the definition of  $\odot$  in (2.3) simplifies since we work over the field  $k$ . We can and will assume that  $\mathcal{A} \otimes^{\mathbf{L}} \mathcal{B} = \mathcal{A} \otimes \mathcal{B}$  and  $\mathcal{A} \odot \mathcal{B} = \text{Perf}(\mathcal{A} \otimes \mathcal{B})$ .

**Theorem 4.23** (Thom–Sebastiani Theorem). *Let  $X$  and  $Y$  be smooth varieties with morphisms  $W: X \rightarrow \mathbb{A}^1$  and  $V: Y \rightarrow \mathbb{A}^1$ . Then the two dg categories*

$$\mathbf{MF}(W)^{\text{dg}, \natural} \odot \mathbf{MF}(V)^{\text{dg}, \natural} \quad \text{and} \quad \mathbf{MF}(W * V)^{\text{dg}, \natural}$$

*are quasi-equivalent. An equivalent statement is that the two dg categories*

$$\mathbf{MF}(W)^{\text{dg}} \otimes \mathbf{MF}(V)^{\text{dg}} \quad \text{and} \quad \mathbf{MF}(W * V)^{\text{dg}}$$

*are Morita equivalent, i. e. isomorphic in  $\text{Hmo}_k$ .*

The assertion of this theorem is not new. A proof is contained in the preprint [25] using higher techniques of derived algebraic geometry. A different proof is claimed in [14].

*Proof.* The equivalence of the two statements follows from Proposition 2.8 and Lemma 2.31.

Fix finite affine open coverings of  $X$  and  $Y$  and consider the product covering of  $X \times Y$ . We can assume that we have used the object oriented Čech enhancements (see Proposition 4.2) when defining  $\mathbf{MF}(W)^{\text{dg}}$ , i. e.

$$\begin{aligned} \mathbf{MF}(W)^{\text{dg}} &= \mathbf{MF}(W)_{\check{\text{Cob}}} := \prod_{a \in \text{Crit}(W)} \mathbf{MF}_{\check{\text{Cob}}}(X, W - a), \\ \mathbf{MF}(W)^{\text{dg}, \natural} &= \mathbf{MF}(W)_{\check{\text{Cob}}}^{\natural} := \prod_{a \in \text{Crit}(W)} \text{Perf}(\mathbf{MF}_{\check{\text{Cob}}}(X, W - a)). \end{aligned}$$

Similarly we consider and define the dg categories  $\mathbf{MF}_{\check{\text{Cob}}}(Y, V - b)$ ,  $\mathbf{MF}(V)^{\text{dg}} = \mathbf{MF}(V)_{\check{\text{Cob}}}$  and  $\mathbf{MF}(V)^{\text{dg}, \natural} = \mathbf{MF}(V)_{\check{\text{Cob}}}^{\natural}$ . On  $X \times Y$  we consider the object oriented Čech enhancement  $\mathbf{MF}_{\check{\text{Cob}}}(X \times Y, W * V)$  of  $\mathbf{MF}(X \times Y, W * V)$  (see Proposition 4.4) and then define  $\mathbf{MF}(W * V)^{\text{dg}} = \mathbf{MF}(W * V)_{\check{\text{Cob}}}$  and  $\mathbf{MF}(W * V)^{\text{dg}, \natural} = \mathbf{MF}(W * V)_{\check{\text{Cob}}}^{\natural}$  accordingly. To ease the notation we abbreviate  $\text{Hom}_{\check{\text{Cob}}} = \text{Hom}_{\mathbf{MF}_{\check{\text{Cob}}}(X, W - a)}$ , and similarly for the other dg categories just mentioned.

Let  $E = (E(a))_{a \in k} \in \mathbf{MF}(W)$  be a classical generator. Its canonical lift to the enhancement  $\mathbf{MF}(W)_{\check{\text{Cob}}}$  is the object  $\mathcal{C}_{\text{ord}}(E) := (\mathcal{C}_{\text{ord}}(E(a)))_{a \in k}$ . As explained in Remark 4.15 we obtain the dg algebra  $A = \prod A(a) = \text{End}_{\check{\text{Cob}}}(\mathcal{C}_{\text{ord}}(E))$  and a quasi-equivalence  $\text{Perf}(A) \rightarrow \mathbf{MF}(W)_{\check{\text{Cob}}}^{\natural}$ . Similarly, starting from a classical generator  $F \in \mathbf{MF}(V)$ , we obtain a dg algebra  $B = \prod B(b) = \text{End}_{\check{\text{Cob}}}(\mathcal{C}_{\text{ord}}(F))$  and a quasi-equivalence  $\text{Perf}(B) \rightarrow \mathbf{MF}(V)_{\check{\text{Cob}}}^{\natural}$ . Proposition 2.10 and Lemma 2.9 then provide quasi-equivalences

$$A \odot B = \text{Perf}(A \otimes B) \rightarrow \text{Perf}(A) \odot \text{Perf}(B) \rightarrow \mathbf{MF}(W)_{\check{\text{Cob}}}^{\natural} \odot \mathbf{MF}(V)_{\check{\text{Cob}}}^{\natural}.$$

On the other hand  $E \boxtimes F$  is a classical generator of  $\mathbf{MF}(W * V)$  by Proposition 4.22. As its lift to the enhancement  $\mathbf{MF}(W * V)_{\check{\text{Cob}}}$  we use the object  $\mathcal{C}_{\text{ord}}(E) \boxtimes \mathcal{C}_{\text{ord}}(F)$  defined in the obvious manner, cf. (4.9). Let

$$M = \prod M(c) = \text{End}_{\check{\text{Cob}}}(\mathcal{C}_{\text{ord}}(E) \boxtimes \mathcal{C}_{\text{ord}}(F)).$$

Remark 4.15 again provides a quasi-equivalence

$$\text{Perf}(M) \rightarrow \mathbf{MF}(W * V)_{\check{\text{Cob}}}^{\natural}.$$

By Lemma 2.6.(c) it is hence sufficient to show that there is a quasi-isomorphism

$$A \otimes B \rightarrow M \tag{4.10}$$

of dg algebras. For  $c \in \text{Crit}(W * V)$  the dg algebra  $M(c)$  is a matrix algebra in the sense that

$$\begin{aligned} M(c) &= \text{End}_{\check{\text{Cob}}}((\mathcal{C}_{\text{ord}}(E) \boxtimes \mathcal{C}_{\text{ord}}(F))(c)) \\ &= \bigoplus_{\substack{a+b=c, \\ a'+b'=c}} \text{Hom}_{\check{\text{Cob}}}(\mathcal{C}_{\text{ord}}(E(a)) \boxtimes \mathcal{C}_{\text{ord}}(F(b)), \mathcal{C}_{\text{ord}}(E(a')) \boxtimes \mathcal{C}_{\text{ord}}(F(b'))), \end{aligned}$$

where the (finite) direct sum is taken over all  $a, a' \in \text{Crit}(W)$  and  $b, b' \in \text{Crit}(V)$  satisfying the given condition. Note that  $A \otimes B = \prod_{c \in \text{Crit}(W * V)} (A \otimes B)(c)$  where  $(A \otimes B)(c)$  is defined by

$$(A \otimes B)(c) := \prod_{a+b=c} A(a) \otimes B(b) = \prod_{a+b=c} \text{End}_{\check{\text{Cob}}}^{\times}(\mathcal{C}_{\text{ord}}(E(a))) \otimes \text{End}_{\check{\text{Cob}}}^{\times}(\mathcal{C}_{\text{ord}}(F(b))).$$

We define the morphism (4.10) of dg algebras using Lemma 4.6. This lemma then says that  $(A \otimes B)(c)$  goes quasi-isomorphically (even isomorphically, by inspection of the proof) onto the diagonal subalgebra of  $M(c)$ . Hence we need to show that the off diagonal part of each  $M(c)$  is acyclic.

Let  $a, a' \in \text{Crit}(W)$  and  $b, b' \in \text{Crit}(V)$  and assume that  $a + b = c = a' + b'$  but  $a \neq a'$  (and hence  $b \neq b'$ ). We need to prove that both cohomologies of  $\text{Hom}_{\check{\text{Cob}}}(\mathcal{C}_{\text{ord}}(E(a)) \boxtimes \mathcal{C}_{\text{ord}}(F(b)), \mathcal{C}_{\text{ord}}(E(a')) \boxtimes \mathcal{C}_{\text{ord}}(F(b')))$  are zero. Equivalently we need to show that

$$\text{Hom}_{\text{MF}(X \times Y, W * V - c)}(E(a) \boxtimes F(b), [p]E(a') \boxtimes F(b'))$$

is zero for  $p \in \mathbb{Z}_2$ . We can use the morphism oriented Čech enhancement  $\text{MF}_{\check{\text{Cmor}}}(X \times Y, W * V - c)$  (see [17, Prop. 2.50]) for this and need to show that both cohomologies of

$$\begin{aligned} \text{Hom}_{\text{MF}_{\check{\text{Cmor}}}(X \times Y, W * V - c)}(E(a) \boxtimes F(b), E(a') \boxtimes F(b')) \\ = C(\mathcal{U} \times \mathcal{V}, \mathcal{H}om(E(a) \boxtimes F(b), E(a') \boxtimes F(b'))) \end{aligned}$$

vanish. It is certainly sufficient to show that the object

$$\mathcal{H}om(E(a) \boxtimes F(b), E(a') \boxtimes F(b'))$$

of  $\text{MF}(X \times Y, 0)$  is zero in  $[\text{MF}(X \times Y, 0)]$  (use for example [17, Lemma 2.48]).

We have  $\mathcal{H}om(E(a), E(a')) \in \text{MF}(X, a - a')$  and  $\mathcal{H}om(F(b), F(b')) \in \text{MF}(Y, b - b')$ , cf. [17, section 2.5.3]. The  $\boxtimes$ -product of these two objects is then in  $\text{MF}(X \times Y, 0)$  and the obvious closed degree zero morphism

$$\boxtimes: \mathcal{H}om(E(a), E(a')) \boxtimes \mathcal{H}om(F(b), F(b')) \rightarrow \mathcal{H}om(E(a) \boxtimes F(b), E(a') \boxtimes F(b')) \tag{4.11}$$

is an isomorphism: this can be checked componentwise and locally on  $\text{Spec } R \subset X$  and  $\text{Spec } S \subset Y$  and boils down to the fact that the obvious map

$$\text{Hom}_R(M, M') \otimes \text{Hom}_S(N, N') \rightarrow \text{Hom}_{R \otimes S}(M \otimes N, M' \otimes N')$$

is an isomorphism for  $M, M' \in \text{Mod}(R)$  and  $N, N' \in \text{Mod}(S)$  with  $M$  and  $N$  finitely generated projective.

Since we assume that  $a \neq a'$ , [17, Lemma 2.28] shows that

$$\mathcal{H}om(E(a), E(a')) = 0$$

in  $[\mathbf{MF}(X, a - a')]$ . We then see from (4.11) that

$$\mathcal{H}om(E(a) \boxtimes F(b), E(a') \boxtimes F(b')) = 0$$

in  $[\mathbf{MF}(X \times Y, 0)]$ . □

**4.5. Smoothness.** Theorem 4.24 below is the analog of [18, Cor. 4.4] for matrix factorizations, and we use the same strategy of proof.

**Theorem 4.24.** *Let  $X$  be a smooth variety with a morphism  $W: X \rightarrow \mathbb{A}^1$ . Then the dg categories  $\mathbf{MF}(W)^{\text{dg}}$  and  $\mathbf{MF}(W)^{\text{dg}, \natural}$  are smooth over  $k$ .*

*Proof.* Recall that smoothness is invariant under quasi-equivalence. We proceed as in the beginning of the proof of the Thom–Sebastiani Theorem 4.23, but we use the enhancements  $\mathbf{MF}'_{\text{Cob}}(X, W - a)$  (see section 4.1.2). The reason is that the duality  $D = (-)^\vee: \mathbf{MF}'_{\text{Cob}}(X, W - a)^{\text{op}} \rightarrow \mathbf{MF}'_{\text{Cob}}(X, -W + a)$  can then be lifted to the dg functor  $\tilde{D}: \mathbf{MF}'_{\text{Cob}}(X, W - a)^{\text{op}} \rightarrow \mathbf{MF}'_{\text{Cob}}(X, -W + a)$ , see Corollary 4.9. So we assume that  $\mathbf{MF}(W)^{\text{dg}} = \mathbf{MF}(W)'_{\text{Cob}} := \prod_{a \in k} \mathbf{MF}'_{\text{Cob}}(X, W - a)$  and  $\mathbf{MF}(W)^{\text{dg}, \natural} = \mathbf{MF}(W)^{\natural}_{\text{Cob}} := \prod \text{Perf}(\mathbf{MF}'_{\text{Cob}}(X, W - a))$ . It is clear how to extend the duality  $D$  and its lift  $\tilde{D}$  to  $D: \mathbf{MF}(W)^{\text{op}} \rightarrow \mathbf{MF}(-W)$  and  $\tilde{D}: (\mathbf{MF}(W)'_{\text{Cob}})^{\text{op}} \rightarrow \mathbf{MF}(-W)'_{\text{Cob}}$ , respectively.

Let  $E = (E(a))_{a \in k}$  be a classical generator of  $\mathbf{MF}(W)$  and consider the dg algebra  $A = \prod A(a) = \text{End}_{\text{Cob}}(\mathcal{C}_{\text{ord}}(E))$ . Here we abbreviate  $\text{End}_{\text{Cob}} = \text{End}_{\mathbf{MF}(W)'_{\text{Cob}}}$  and use similar notation in the following. By Remark 4.15 it is enough to prove that  $A$  is smooth, i. e. that  $A \in \text{per}(A \otimes A^{\text{op}})$ .

Since  $E^\vee$  is a classical generator of  $\mathbf{MF}(-W)$ , Proposition 4.22 says that  $E \boxtimes E^\vee$  is a classical generator of  $\mathbf{MF}(W * (-W))$ . We will use the lift

$$P := \mathcal{C}_{\text{ord}}(E) \boxtimes \tilde{D}(\mathcal{C}_{\text{ord}}(E))$$

of this generator to the enhancement  $\mathbf{MF}(W * (-W))'_{\text{Cob}}$ , cf. Lemma 4.8. Here  $\mathbf{MF}(W * (-W))'_{\text{Cob}}$  is defined in the obvious way using the product covering of  $X \times X$ . Note that

$$\tilde{D}: A^{\text{op}} = \text{End}_{\text{Cob}}(\mathcal{C}_{\text{ord}}(E))^{\text{op}} \rightarrow \text{End}_{\text{Cob}}(\tilde{D}(\mathcal{C}_{\text{ord}}(E)))$$

is a quasi-isomorphism of dg algebras by Corollary 4.11. Recall that we showed in the proof of the Thom–Sebastiani Theorem 4.23 that the natural morphism (4.10) is a quasi-isomorphism. Transferred to our setting this means that the morphism

$$\boxtimes: \text{End}_{\check{\text{Cob}}}(\mathcal{C}_{\text{ord}}(E)) \otimes \text{End}_{\check{\text{Cob}}}(\mathcal{C}_{\text{ord}}(E^\vee)) \rightarrow \text{End}_{\check{\text{Cob}}}(\mathcal{C}_{\text{ord}}(E) \boxtimes \mathcal{C}_{\text{ord}}(E^\vee))$$

is a quasi-isomorphism of dg algebras. We also have the isomorphism

$$\widetilde{D}(\mathcal{C}_{\text{ord}}(E)) \xrightarrow{\sim} \mathcal{C}_{\text{ord}}(E^\vee)$$

in  $\prod_{a \in k} [\text{Sh}(X, -W + a)]$  from Lemma 4.8. These three facts (and the fact that  $\boxtimes$  preserves quasi-isomorphisms) show that both arrows in

$$A \otimes A^{\text{op}} \xrightarrow{\text{id} \otimes \widetilde{D}} \text{End}_{\check{\text{Cob}}}(\mathcal{C}_{\text{ord}}(E)) \otimes \text{End}_{\check{\text{Cob}}}(\widetilde{D}(\mathcal{C}_{\text{ord}}(E))) \xrightarrow{\boxtimes} \text{End}_{\check{\text{Cob}}}(P)$$

are quasi-isomorphism of dg algebras. Restriction of dg modules along their composition defines an equivalence of the corresponding perfect derived categories; combined with Proposition 2.18 we obtain a full and faithful functor

$$F := \text{Hom}_{\check{\text{Cob}}}(P, -): [\mathbf{MF}(W * (-W))'_{\check{\text{Cob}}}] \rightarrow \text{per}(A \otimes A^{\text{op}})$$

of triangulated categories. Note that  $A \otimes A^{\text{op}} = \prod_{a, a' \in k} A(a) \otimes A(a')^{\text{op}}$  and  $\text{per}(A \otimes A^{\text{op}}) = \prod_{a, a' \in k} \text{per}(A(a) \otimes A(a')^{\text{op}})$ . Under this identification, the  $(a, a')$ -component of  $F$  is given by

$$F_{a, a'} = \text{Hom}_{\check{\text{Cob}}}(P_{a, a'}, -): [\mathbf{MF}'_{\check{\text{Cob}}}(X \times X, W * (-W) - a + a')] \rightarrow \text{per}(A(a) \otimes A(a')^{\text{op}})$$

where  $P_{a, a'} := \mathcal{C}_{\text{ord}}(E(a)) \boxtimes \widetilde{D}(\mathcal{C}_{\text{ord}}(E(a')))$ . We also see that smoothness of  $A$  is equivalent to smoothness of all  $A(a)$ , for  $a \in k$ .

Let  $\Delta: X \rightarrow X \times X$  be the diagonal embedding. Note that  $\Delta^*(W * (-W)) = 0$ . Consider the object  $\mathcal{D} = \mathcal{D}_X = (0 \rightrightarrows \mathcal{O}_X) \in \mathbf{MF}(X, 0)$  and the canonical morphism  $\mathcal{D} \rightarrow \mathcal{C}_{\text{ord}}(\mathcal{D})$  in  $\text{Qcoh}(X, 0)$  that becomes an isomorphism in  $\text{DQcoh}(X, 0)$ . Since  $\Delta$  is affine and proper,

$$\Delta_*(\mathcal{D}) \rightarrow \Delta_*(\mathcal{C}_{\text{ord}}(\mathcal{D}))$$

in  $\text{Qcoh}(X \times X, W * (-W))$  becomes an isomorphism in  $\text{DQcoh}(X \times X, W * (-W))$ , and  $\Delta_*(\mathcal{D})$  is in  $\mathbf{MF}'(X \times X, W * (-W))$  (cf. [17, Rem. 2.39 and Lemma 2.37]).

Find  $I \in \text{InjQcoh}(X \times X, W * (-W))$  and  $T \in \mathbf{MF}(X \times X, W * (-W))$  together with morphisms  $\Delta_*(\mathcal{C}_{\text{ord}}(\mathcal{D})) \rightarrow I$  and  $T \rightarrow I$  in  $Z_0(\text{Qcoh}(X \times X, W * (-W)))$  that become isomorphisms in  $\text{DQcoh}(X \times X, W * (-W))$ . These morphisms induce quasi-morphisms

$$\begin{aligned} F_{a, a}(\mathcal{C}_{\text{ord}}(T)) &\rightarrow \text{Hom}_{\text{Sh}(X \times X, W * (-W))}(P_{a, a}, I) \\ &\leftarrow \text{Hom}_{\text{Sh}(X \times X, W * (-W))}(P_{a, a}, \Delta_*(\mathcal{C}_{\text{ord}}(\mathcal{D}))) \end{aligned}$$

of dg  $A(a) \otimes A(a)^{\text{op}}$ -modules (for  $a \in k$ ): this is proved using the version of Proposition 4.4 explained in section 4.1.2, Lemma 4.8 and Lemma 4.5; and [17, Thm. 2.25 and Remark 2.14]). These three dg modules are in  $\text{per}(A(a) \otimes A(a)^{\text{op}})$  since  $F(\mathcal{C}_{\text{ord}}(T)) \in \text{per}(A \otimes A^{\text{op}})$  and hence  $F_{a,a}(\mathcal{C}_{\text{ord}}(T)) \in \text{per}(A(a) \otimes A(a)^{\text{op}})$ .

Observe that the obvious adjunctions provide isomorphisms of dg  $A(a) \otimes A(a)^{\text{op}}$ -modules

$$\begin{aligned} & \text{Hom}_{\text{Sh}(X \times X, W^*(-W))}(P_{a,a}, \Delta_*(\mathcal{C}_{\text{ord}}(\mathcal{D}))) \\ & \xrightarrow{\sim} \text{Hom}_{\text{Sh}(X,0)}(\Delta^*(\mathcal{C}_{\text{ord}}(E(a)) \boxtimes \tilde{D}(\mathcal{C}_{\text{ord}}(E(a)))) , \mathcal{C}_{\text{ord}}(\mathcal{D})) \\ & = \text{Hom}_{\text{Sh}(X,0)}(\mathcal{C}_{\text{ord}}(E(a)) \otimes \tilde{D}(\mathcal{C}_{\text{ord}}(E(a))), \mathcal{C}_{\text{ord}}(\mathcal{D})) \\ & \xrightarrow{\sim} \text{Hom}_{\text{Sh}(X,0)}(\mathcal{C}_{\text{ord}}(E(a)), \mathcal{H}om(\tilde{D}(\mathcal{C}_{\text{ord}}(E(a))), \mathcal{C}_{\text{ord}}(\mathcal{D}))) \\ & = \text{Hom}_{\text{Sh}(X,0)}(\mathcal{C}_{\text{ord}}(E(a)), \tilde{D}^2(\mathcal{C}_{\text{ord}}(E(a)))) . \end{aligned}$$

Now use Lemma 4.10. The canonical morphism

$$\theta = \theta_{\mathcal{C}_{\text{ord}}(E(a))}: \mathcal{C}_{\text{ord}}(E(a)) \rightarrow \tilde{D}^2(\mathcal{C}_{\text{ord}}(E(a)))$$

is a homotopy equivalence, so

$$\begin{aligned} \theta_*: \text{Hom}_{\text{Sh}(X,0)}(\mathcal{C}_{\text{ord}}(E(a)), \mathcal{C}_{\text{ord}}(E(a))) \\ \rightarrow \text{Hom}_{\text{Sh}(X,0)}(\mathcal{C}_{\text{ord}}(E(a)), \tilde{D}^2(\mathcal{C}_{\text{ord}}(E(a)))) \end{aligned}$$

is a homotopy equivalence; moreover, it is a morphism of dg  $A(a) \otimes A(a)^{\text{op}}$ -modules. The object on the left is the diagonal dg  $A(a) \otimes A(a)^{\text{op}}$ -module  $A(a)$  which is hence in  $\text{per}(A(a) \otimes A(a)^{\text{op}})$ . This proves smoothness of  $A(a)$ , for any  $a \in k$ . As observed above this just means that  $A$  is smooth.  $\square$

**Corollary 4.25.** *Let  $X$  be a smooth variety with a morphism  $W: X \rightarrow \mathbb{A}^1$ . Then the dg category  $\mathbf{MF}(X, W)^{\text{dg}}$  is smooth over  $k$ .*

*Proof.* We can assume that  $\mathbf{MF}(X, W)^{\text{dg}} = \mathbf{MF}'_{\check{\text{C}}_{\text{ob}}}(X, W)$ . In the proof of Theorem 4.24 we have seen that  $A(0) = \text{End}_{\check{\text{C}}_{\text{ob}}}(\mathcal{C}_{\text{ord}}(E(0)))$  is smooth. This implies the claim.  $\square$

**4.6. Properness.**

**Proposition 4.26.** *Let  $X$  be a smooth variety with a morphism  $W: X \rightarrow \mathbb{A}^1$ , and assume that  $W|_{\text{Sing}(W)}: \text{Sing}(W) \rightarrow \mathbb{A}^1$  is proper (for example  $W$  could be proper), or equivalently, that  $\text{Sing}(W)$  is complete. Then the dg categories  $\mathbf{MF}(X, W)^{\text{dg}}$ ,  $\text{Perf}(\mathbf{MF}(X, W)^{\text{dg}})$ ,  $\mathbf{MF}(W)^{\text{dg}}$ , and  $\mathbf{MF}(W)^{\text{dg}, \natural}$  are proper over  $k$ .*

*Proof.* We know that  $|\text{Sing}(W)| = \coprod_{a \in \text{Crit}(W)} |\text{Sing}(W) \cap X_a|$ . Hence  $|\text{Sing}(W)| \rightarrow \mathbb{A}^1$  factors as  $|\text{Sing}(W)| \rightarrow \text{Crit}(W) \subset \mathbb{A}^1$ . This implies that  $\text{Sing}(W) \rightarrow \mathbb{A}^1$  is proper if and only if  $\text{Sing}(W) \rightarrow \text{Spec } k$  is proper.

Let  $E$  be a classical generator of  $\mathbf{MF}(X, W)$  and  $A$  the dg algebra of its endomorphisms in  $\mathbf{MF}(X, W)^{\text{dg}}$ . It is certainly enough to show  $A$  is proper (cf. Remark 4.15), i.e. that  $A \in \text{per}(k)$ . Since  $k$  is a field this just means  $H_l(A) = \text{Hom}_{\mathbf{MF}(X, W)}(E, [l]E)$  is finite dimensional for  $l \in \mathbb{Z}_2$ . We can assume that  $X$  is connected, so that  $W$  is either flat or constant.

Assume that  $W$  is flat. Then we have the equivalence

$$\text{cok: } \mathbf{MF}(X, W) \xrightarrow{\sim} D_{\text{Sg}}(X_0)$$

and  $\dim_k \text{Hom}_{D_{\text{Sg}}(X_0)}(M, N) < \infty$  for all  $M, N \in D_{\text{Sg}}(X_0)$  by [23, Cor. 1.24]: note that  $X_0$  is Gorenstein and that  $(X_0)^{\text{sing}} = |\text{Sing}(W) \cap X_0|$  (see equation (4.7) in Remark 4.20) is complete. This implies that  $A$  is proper over  $k$ .

Now assume that  $W$  is constant. In case  $W \neq 0$  we have  $\mathbf{MF}(X, W) = 0$  by [17, Lemma 2.28] and the claim is trivial. So assume  $W = 0$ . We can assume that  $E = (0 \rightrightarrows P)$  with  $P$  a vector bundle on  $X$  (see [17, Rem. 2.54]) and that  $\mathbf{MF}(X, 0)^{\text{dg}} = \mathbf{MF}_{\text{Cmor}}(X, 0)$ . Then  $A = C(\mathcal{U}, \text{Hom}(E, E)) = C(\mathcal{U}, (0 \rightrightarrows \text{Hom}(P, P)))$ , and hence  $H_l(A) = \bigoplus H^n(X, \text{Hom}(P, P))$  where the direct sum is over all  $n \in \mathbb{Z}$  with  $n = l$  in  $\mathbb{Z}_2$ . We have  $\dim_k H_l(A) < \infty$  by [9, Thm. III.2.7] and [8, Thm. 3.2.1] since  $X$  is Noetherian of finite dimension and  $X = \text{Sing}(W)$  is complete.  $\square$

**4.7. Conclusion.** Recall the Grothendieck ring of saturated dg categories from Proposition 2.24 and Definition 2.23. Since we work here in the differential  $\mathbb{Z}_2$ -graded setting and over the field  $k$  (cf. Remark 2.1) we denote it by  $K_0(\text{sat}_k^{\mathbb{Z}_2})$ . Similarly we denote the monoid from Proposition 2.22 by  $\text{sat}_k^{\mathbb{Z}_2}$ .

**Theorem 4.27.** *Let  $X$  be a smooth variety with a morphism  $W: X \rightarrow \mathbb{A}^1$  such that  $\text{Sing}(W)$  is complete (for example  $W$  could be proper). Then  $\text{Perf}(\mathbf{MF}(X, W)^{\text{dg}})$  and  $\mathbf{MF}(W)^{\text{dg}, \natural}$  are saturated dg categories and hence define elements  $\overline{\text{Perf}(\mathbf{MF}(X, W)^{\text{dg}})}$  and  $\overline{\mathbf{MF}(W)^{\text{dg}, \natural}}$  of  $K_0(\text{sat}_k^{\mathbb{Z}_2})$ . If  $Y$  is another smooth variety with a morphism  $V: Y \rightarrow \mathbb{A}^1$  such that  $\text{Sing}(V)$  is complete, then*

$$\overline{\mathbf{MF}(W)^{\text{dg}, \natural}} \bullet \overline{\mathbf{MF}(V)^{\text{dg}, \natural}} = \overline{\mathbf{MF}(W * V)^{\text{dg}, \natural}} \tag{4.12}$$

in the monoid  $\text{sat}_k^{\mathbb{Z}_2}$  and hence in the ring  $K_0(\text{sat}_k^{\mathbb{Z}_2})$ .

*Proof.* The dg categories  $\text{Perf}(\mathbf{MF}(X, W)^{\text{dg}})$  and  $\mathbf{MF}(W)^{\text{dg}, \natural}$  are smooth, proper and triangulated, i.e. saturated, by Theorem 4.24, Corollary 4.25, Proposition 4.26, Corollary 2.4, and Lemma 2.13. Equality (4.12) is then a direct consequence of the Thom–Sebastiani Theorem 4.23.  $\square$

**Remark 4.28.** Consider the set  $M$  of isomorphism classes  $[X]_{\mathbb{A}^1}$  of  $\mathbb{A}^1$ -varieties  $W: X \rightarrow \mathbb{A}^1$  with  $X$  smooth over  $k$  and  $\text{Sing}(W)$  complete. If  $W: X \rightarrow \mathbb{A}^1$  and  $V: Y \rightarrow \mathbb{A}^1$  are  $\mathbb{A}^1$ -varieties satisfying these conditions, so does  $W * V: X \times Y \rightarrow \mathbb{A}^1$

(by equation (4.8)). Hence  $[X]_{\mathbb{A}^1} \cdot [Y]_{\mathbb{A}^1} := [X \times Y]_{\mathbb{A}^1}^1$  turns  $M$  into a commutative monoid with unit the class of  $\text{Spec } k \xrightarrow{0} \mathbb{A}^1$ . One may view  $M$  as a “Grothendieck monoid” of certain varieties over  $\mathbb{A}^1$ . Then Theorem 4.27 says that mapping the class of  $W: X \rightarrow \mathbb{A}^1$  as above to  $\underline{\mathbf{MF}}(W)^{\text{dg}, \natural}$  defines a (unital) morphism  $M \rightarrow \underline{\text{sat}}_k^{\mathbb{Z}_2}$  of monoids.

**5. Landau–Ginzburg motivic measure**

Let  $k$  be an algebraically closed field of characteristic zero. We continue to work in the differential  $\mathbb{Z}_2$ -graded setting. Our aim in this section is to prove Theorem 5.2.

Recall the Grothendieck ring  $K_0(\text{Var}_{\mathbb{A}^1})$  of varieties over  $\mathbb{A}^1$  from Section 3 and the Grothendieck ring  $K_0(\text{sat}_k^{\mathbb{Z}_2})$  of saturated dg categories from Proposition 2.24. We first state an additive precursor of Theorem 5.2 which only uses the additive structures on  $K_0(\text{Var}_{\mathbb{A}^1})$  and  $K_0(\text{sat}_k^{\mathbb{Z}_2})$ .

**Proposition 5.1.** *There is a unique morphisms*

$$K_0(\text{Var}_{\mathbb{A}^1}) \rightarrow K_0(\text{sat}_k^{\mathbb{Z}_2})$$

*of abelian groups that maps  $[X]_{\mathbb{A}^1} = [X, W]$  to  $\underline{\text{Perf}}(\underline{\mathbf{MF}}(X, W)^{\text{dg}})$  whenever  $X$  is a smooth variety and  $W: X \rightarrow \mathbb{A}^1$  is a proper morphism. This morphism of abelian groups is uniquely determined by its values on  $[X, W]$  for smooth (connected)  $X$  and projective  $W$ .*

*Proof.* Recall the isomorphism  $K_0^{\text{bl}}(\text{Var}_{\mathbb{A}^1}) \xrightarrow{\sim} K_0(\text{Var}_{\mathbb{A}^1})$  of abelian groups from Theorem 3.2 (and that one may restrict to connected varieties or projective morphisms in (bl)). This shows uniqueness.

If  $X$  and  $W$  are as above, then  $\underline{\text{Perf}}(\underline{\mathbf{MF}}(X, W)^{\text{dg}})$  is saturated by Theorem 4.27. Hence to show existence we only need to see that the relation  $[\emptyset]_{\mathbb{A}^1} = 0$  and the blowing-up relations go to zero under  $[X, W] \mapsto \underline{\text{Perf}}(\underline{\mathbf{MF}}(X, W)^{\text{dg}})$ . It is trivial that  $[\emptyset]_{\mathbb{A}^1}$  goes to  $\underline{\text{Perf}}(\emptyset) = 0$ . It is enough to consider the blowing-up relations when blowing-up a connected smooth subvariety, and in this case we can use [17, Cor. 3.3 and 3.16] and Proposition 2.19. □

Let us formulate the main result of this article.

**Theorem 5.2.** *Let  $k$  be an algebraically closed field of characteristic zero. There is a unique morphism*

$$\mu: K_0(\text{Var}_{\mathbb{A}^1}) \rightarrow K_0(\text{sat}_k^{\mathbb{Z}_2})$$

*of rings (= a Landau–Ginzburg motivic measure) that maps  $[X, W]$  to  $\underline{\mathbf{MF}}(W)^{\text{dg}, \natural}$  whenever  $X$  is a smooth variety and  $W: X \rightarrow \mathbb{A}^1$  is a proper morphism.*

In particular,  $\mu$  is a morphism of abelian groups and maps  $[X, W]$  to  $\underline{\mathbf{MF}}(W)^{\text{dg}, \mathfrak{h}}$  whenever  $X$  is a smooth (connected) variety and  $W: X \rightarrow \mathbb{A}^1$  is a projective morphism. These two properties determine  $\mu$  uniquely.

*Proof.* If  $\mathcal{A}$  and  $\mathcal{B}$  are saturated dg categories, then  $\mathcal{A} \times \mathcal{B}$  is saturated and  $\underline{\mathcal{A}} \times \underline{\mathcal{B}} = \underline{\mathcal{A}} + \underline{\mathcal{B}}$  in  $K_0(\text{sat}_k^{\mathbb{Z}^2})$  since there are semi-orthogonal decompositions  $[\mathcal{A} \times \mathcal{B}] = \langle [\mathcal{A}], [\mathcal{B}] \rangle = \langle [\mathcal{B}], [\mathcal{A}] \rangle$ . If we use the isomorphism  $K_0^{\text{bl}}(\text{Var}_{\mathbb{A}^1}) \xrightarrow{\sim} K_0(\text{Var}_{\mathbb{A}^1})$  of abelian groups and proceed as in the proof of Proposition 5.1 (using the defining equation (4.4)) we see that there is a unique morphism  $\mu: K_0(\text{Var}_{\mathbb{A}^1}) \rightarrow K_0(\text{sat}_k^{\mathbb{Z}^2})$  of abelian groups mapping  $[X, W]$  to  $\underline{\mathbf{MF}}(W)^{\text{dg}, \mathfrak{h}}$  whenever  $X$  is smooth and  $W$  is proper, and that it is uniquely determined by its values on  $[X, W]$  for  $X$  smooth (connected) and  $W$  projective. It is clear that  $\mu$  sends the unit  $[\text{Spec } k, 0]$  of  $K_0(\text{Var}_{\mathbb{A}^1})$  to the unit  $\underline{\text{Perf}}(k)$  of  $K_0(\text{sat}_k^{\mathbb{Z}^2})$ .

We need to prove that  $\mu$  is compatible with multiplication. Recall that the multiplication is easy to define on  $K^0(\text{Var}_{\mathbb{A}^1})$  but not on  $K_0^{\text{bl}}(\text{Var}_{\mathbb{A}^1})$ . Let  $X$  and  $Y$  be smooth connected varieties with projective morphisms  $W: X \rightarrow \mathbb{A}^1$  and  $V: Y \rightarrow \mathbb{A}^1$ . By definition of  $\mu$  and Theorem 4.27 we have

$$\mu([X, W]) \bullet \mu([Y, V]) = \underline{\mathbf{MF}}(W)^{\text{dg}, \mathfrak{h}} \bullet \underline{\mathbf{MF}}(V)^{\text{dg}, \mathfrak{h}} = \underline{\mathbf{MF}}(W * V)^{\text{dg}, \mathfrak{h}}$$

in  $K_0(\text{sat}_k^{\mathbb{Z}^2})$ . If  $W$  or  $V$  is constant, then  $W * V$  is projective and hence  $\underline{\mathbf{MF}}(W * V)^{\text{dg}, \mathfrak{h}} = \mu([X \times Y, W * V])$ , so  $\mu$  is multiplicative.

Hence we can assume that both  $W$  and  $V$  are flat. Since  $W * V$  might not be projective, it is not clear that  $\mu$  maps  $[X \times Y, W * V]$  to  $\underline{\mathbf{MF}}(W * V)^{\text{dg}, \mathfrak{h}}$ . In order to prove this it is enough to find smooth varieties  $Z_i$  together with projective morphisms  $W_i: Z_i \rightarrow \mathbb{A}^1$  and integers  $n_i$  such that

$$[X \times Y, W * V] = \sum_i n_i [Z_i, W_i] \quad \text{in } K_0(\text{Var}_{\mathbb{A}^1}),$$

and 
$$\underline{\mathbf{MF}}(W * V)^{\text{dg}, \mathfrak{h}} = \sum_i n_i \underline{\mathbf{MF}}(W_i)^{\text{dg}, \mathfrak{h}} \quad \text{in } K_0(\text{sat}_k^{\mathbb{Z}^2}).$$

This can be done using Proposition 6.1 below which shows that the morphism  $W * V$  can be “compactified” in a nice way. Using the notation introduced there, it is easy to see that

$$[X \times Y, W * V] = [Z, h] - \sum_i [D_i, h_i] + \sum_{i < j} [D_{ij}, h_{ij}] - \dots + (-1)^s [D_{12\dots s}, h_{12\dots s}]$$

in  $K_0(\text{Var}_{\mathbb{A}^1})$ . On the right-hand side,  $Z$  and all  $D_{i_1 \dots i_p}$  are smooth quasi-projective varieties, and  $h$  and all  $h_{i_1 \dots i_p}$  are projective morphisms, by Proposition 6.1.(iv). Hence we obtain

$$\mu([X \times Y, W * V]) = \underline{\mathbf{MF}}(h)^{\text{dg}, \mathfrak{h}} - \sum_i \underline{\mathbf{MF}}(h_i)^{\text{dg}, \mathfrak{h}} + \dots + (-1)^s \underline{\mathbf{MF}}(h_{12\dots s})^{\text{dg}, \mathfrak{h}}.$$

Lemma 4.13 and Proposition 6.1.(iv) again show that  $\underline{\mathbf{MF}}(h_{i_1 \dots i_p})^{\text{dg}, \mathfrak{h}} = 0$  for all tuples  $(i_1, \dots, i_p)$  with  $p \geq 1$ . Hence it is enough to show that

$$\underline{\mathbf{MF}}(h)^{\text{dg}, \mathfrak{h}} = \underline{\mathbf{MF}}(W * V)^{\text{dg}, \mathfrak{h}}.$$

Let  $j: X \times Y \rightarrow Z$  be the open inclusion, and let  $a \in k$ . The functor

$$j^*: \mathbf{MF}(Z, h - a) \rightarrow \mathbf{MF}(X \times Y, W * V - a)$$

lifts to a dg functor

$$j^*: \mathbf{MF}(Z, h - a)^{\text{dg}} \rightarrow \mathbf{MF}(X \times Y, W * V - a)^{\text{dg}}$$

if we work for example with the enhancements using injective quasi-coherent sheaves. From the defining equation (4.4) it is clearly enough to show that this functor is a quasi-equivalence, or equivalently, that

$$j^*: \mathbf{MF}(Z, h - a) \rightarrow \mathbf{MF}(X \times Y, W * V - a)$$

is an equivalence. Note that  $W * V$  and hence  $h$  are flat, so we can use Orlov’s equivalence [17, Thm. 2.8] and have to prove that

$$j^*: D_{\text{Sg}}(Z_a) \rightarrow D_{\text{Sg}}((X \times Y)_a)$$

is an equivalence. But equation (4.7) in Remark 4.20 and Proposition 6.1.(ii) imply that

$$(Z_a)^{\text{sing}} = |\text{Sing}(h) \cap Z_a| = |\text{Sing}(W * V) \cap Z_a| = ((X \times Y)_a)^{\text{sing}} \subset (X \times Y)_a,$$

so we can use [24, Prop. 1.3]. □

**Remark 5.3.** Let  $X$  and  $Y$  be smooth varieties with proper morphisms  $W: X \rightarrow \mathbb{A}^1$  and  $V: Y \rightarrow \mathbb{A}^1$ . Then we see from Theorems 5.2 and 4.27 that

$$\begin{aligned} \mu([X \times Y, W * V]) &= \mu([X, W]) \bullet \mu([Y, V]) \\ &= \underline{\mathbf{MF}}(W)^{\text{dg}, \mathfrak{h}} \bullet \underline{\mathbf{MF}}(V)^{\text{dg}, \mathfrak{h}} = \underline{\mathbf{MF}}(W * V)^{\text{dg}, \mathfrak{h}}. \end{aligned}$$

This shows that the Landau–Ginzburg motivic measure  $\mu$  sends  $W * V: X \times Y \rightarrow \mathbb{A}^1$  to  $\underline{\mathbf{MF}}(W * V)^{\text{dg}, \mathfrak{h}}$  even though  $W * V$  might not be proper. This statement is slightly more general than what we showed in the proof of Theorem 5.2.

**Remark 5.4.** From [17, Corollary 3.3] we see that

$$\mu([\mathbb{P}_k^n, 0]) = (n + 1)\underline{\text{Perf}}(k) = n + 1.$$

Recall the element  $\mathbb{L}_{(\mathbb{A}^1, 0)} := [\mathbb{A}^1, 0] \in K_0(\text{Var}_{\mathbb{A}}^1)$  from Remark 3.3. Then we obtain

$$\mu(\mathbb{L}_{(\mathbb{A}^1, 0)}) = \mu([\mathbb{P}_k^1, 0]) - \mu([\text{Spec } k, 0]) = 2 - 1 = 1.$$

This implies that  $\mu$  factors to a morphism

$$\mu: K_0(\text{Var}_{\mathbb{A}^1})/(\mathbb{L}_{(\mathbb{A}^1,0)} - 1) \rightarrow K_0(\text{sat}_k^{\mathbb{Z}_2})$$

of rings, cf. [5, sect. 8.2].

If  $W: X \rightarrow \mathbb{A}^1$  is a proper and smooth morphism, then certainly  $\mathbf{MF}(W) = 0$  by Lemma 4.13 and hence  $\mu([X, W]) = 0$ . This yields many other elements of the kernel of  $\mu$ . For example  $[\mathbb{A}^1, \text{id}_{\mathbb{A}^1}]$  lies in the kernel of  $\mu$ .

### 6. Compactification

Let  $k$  be an algebraically closed field of characteristic zero.

**Proposition 6.1.** *Let  $X$  and  $Y$  be smooth varieties and let  $W: X \rightarrow \mathbb{A}^1$  and  $V: Y \rightarrow \mathbb{A}^1$  be projective morphisms (hence  $X$  and  $Y$  are quasi-projective varieties). Consider the convolution*

$$W * V: X \times Y \xrightarrow{W \times V} \mathbb{A}^1 \times \mathbb{A}^1 \xrightarrow{+} \mathbb{A}^1.$$

*Then there exists a smooth quasi-projective variety  $Z$  with an open embedding  $X \times Y \hookrightarrow Z$  and a projective morphism  $h: Z \rightarrow \mathbb{A}^1$  such that the following conditions are satisfied.*

(i) *The diagram*

$$\begin{array}{ccc} X \times Y & \hookrightarrow & Z \\ \downarrow W * V & & \downarrow h \\ \mathbb{A}^1 & = & \mathbb{A}^1 \end{array}$$

*commutes.*

- (ii) *All critical points of  $h$  are contained in  $X \times Y$ , i. e.  $\text{Sing}(W * V) = \text{Sing}(h)$ .*
- (iii) *We have  $Z \setminus X \times Y = \bigcup_{i=1}^s D_i$  where the  $D_i$  are pairwise distinct smooth prime divisors. More precisely,  $Z \setminus X \times Y$  is the support of a snc (= simple normal crossing) divisor.*
- (iv) *For every  $p$ -tuple  $(i_1, \dots, i_p)$  of indices (with  $p \geq 1$ ) the morphism*

$$h_{i_1 \dots i_p}: D_{i_1 \dots i_p} := D_{i_1} \cap \dots \cap D_{i_p} \rightarrow \mathbb{A}^1$$

*induced by  $h$  is projective and smooth. In particular, all  $D_{i_1 \dots i_p}$  are smooth quasi-projective varieties.*

**Remark 6.2.** To prove Proposition 6.1 one may assume that both  $X$  and  $Y$  are connected. If the map  $W$  is not flat then its image is one point  $W(X) = a \in \mathbb{A}^1$ ,  $X$  is projective, and the map  $W * V: X \times Y \rightarrow \mathbb{A}^1$  is already projective. So we can take  $Z = X \times Y$  and  $h = W * V$ . This shows that Proposition 6.1 is interesting only in case both  $W$  and  $V$  are flat. The proof given below works in general.

We need some preparations for the proof of this proposition. Let  $U$  be a scheme and  $I \subset \mathcal{O}_U$  an ideal sheaf. We say that the pair  $(U, I)$  satisfies condition **(K)** if

- (K)  $U$  is a reduced scheme of finite type (over  $k$ ),  $I$  is not zero on any irreducible component of  $U$ , and the closed subscheme  $\mathcal{V}(I)$  defined by  $I$  contains the singular locus  $U^{\text{sing}}$  of  $U$ .

**Remark 6.3.** We recall some results on resolution of singularities and monomialization (principalization) from [13]. Assume that  $(U, I)$  as above satisfies condition **(K)**. Let  $\tilde{U} \rightarrow U$  be the resolution of singularities from [13, Thm. 3.36] (it seems preferable to start with a reduced scheme there). Then  $\tilde{U}$  together with the inverse image ideal sheaf  $\tilde{I}$  of  $I$  under  $\tilde{U} \rightarrow U$  also satisfies condition **(K)**:  $\tilde{U}$  is again reduced [15, Lemma 8.1.2] of finite type,  $\tilde{I}$  is not zero on any irreducible component of  $\tilde{U}$  since  $\tilde{U} \rightarrow U$  is birational (as confirmed to us by János Kollár), and  $\tilde{U}^{\text{sing}} = \emptyset$ . So we can apply monomialization (principalization) [13, Thm. 3.35] to this inverse image ideal sheaf (and the empty snc divisor) and obtain a morphism  $c(U) = c_I(U) \rightarrow \tilde{U}$ . Let  $\gamma = \gamma_U = \gamma_{U,I}$  be the composition  $c(U) \rightarrow \tilde{U} \rightarrow U$ . Then  $c(U)$  is a smooth scheme of finite type over  $k$ , the inverse image ideal sheaf  $\gamma^{-1}(I) \cdot \mathcal{O}_{c(U)} \subset \mathcal{O}_{c(U)}$  is the ideal sheaf of a snc divisor, and  $\gamma$  is an isomorphism over  $U \setminus \mathcal{V}(I)$ . Moreover,  $\gamma$  is a composition of blowing-up morphisms and in particular a proper morphism. If  $U$  is quasi-projective (resp. projective), so is  $c(U)$ , and  $\gamma$  is a projective morphism. As described in [13, 3.34.1], the association

$$(U, I) \mapsto (c_I(U) \xrightarrow{\gamma} U) \tag{6.1}$$

commutes with smooth (and in particular étale) morphisms. This means that any smooth or étale morphism  $f: U' \rightarrow U$ , gives rise to a pullback diagram

$$\begin{array}{ccc} c_{f^{-1}(I) \cdot \mathcal{O}_{U'}}(U') & \longrightarrow & c_I(U) \\ \downarrow \gamma_{U'} & & \downarrow \gamma_U \\ U' & \xrightarrow{f} & U. \end{array} \tag{6.2}$$

The following proposition provides useful compactifications and describes them “étale locally”. We view  $\mathbb{A}^1 \subset \mathbb{P}^1$ ,  $z \mapsto [1, z]$ , as an open subvariety, and let  $\infty = [0, 1] \in \mathbb{P}^1$ . We write  $\mathbb{A}^1_\infty$  for  $\mathbb{A}^1$  viewed as an open neighborhood of  $\infty$  via  $z \mapsto [z, 1]$ .

**Proposition 6.4.** *Let  $X$  be a smooth (quasi-projective) variety and let  $W: X \rightarrow \mathbb{A}^1$  be a projective morphism. Let  $I_\infty$  be the ideal sheaf of the closed subvariety  $\{\infty\} \subset \mathbb{P}^1$ . Then there is a smooth projective variety  $\bar{X}$  with an open embedding  $X \hookrightarrow \bar{X}$  and a projective morphism  $\bar{W}: \bar{X} \rightarrow \mathbb{P}^1$  such that the diagram*

$$\begin{array}{ccc} X & \hookrightarrow & \bar{X} \\ \downarrow W & & \downarrow \bar{W} \\ \mathbb{A}^1 & \hookrightarrow & \mathbb{P}^1 \end{array} \tag{6.3}$$

is a pullback diagram and such that the inverse image ideal sheaf

$$\overline{W}^{-1}(I_\infty) \cdot \mathcal{O}_{\overline{X}} \subset \mathcal{O}_{\overline{X}}$$

is a locally monomial ideal, i. e. the ideal sheaf of a snc divisor.

In particular, for any (closed) point  $p$  in the fiber  $\overline{X}_\infty := \overline{W}^{-1}(\infty)$  at infinity, there are an étale morphism  $u: U \rightarrow \overline{X}$  with  $p$  in its image, uniformizing parameters  $\underline{x} = (x_1, \dots, x_m)$  on  $U$  and a tuple  $\mu = (\mu_1, \dots, \mu_s)$  of positive integers, for some  $1 \leq s \leq m$ , such that the diagram

$$\begin{array}{ccc} \overline{X} & \xleftarrow{u} & U & \xrightarrow{\underline{x}} & \mathbb{A}^m \\ \downarrow \overline{W} & & & & \downarrow \mu \\ \mathbb{P}^1 & \supset & & & \mathbb{A}_\infty^1 \end{array}$$

commutes, where  $\underline{x}$  is the morphism given by the uniformizing parameters and  $\mu$  is the morphism mapping  $(t_1, \dots, t_m)$  to  $t^\mu := t_1^{\mu_1} \dots t_s^{\mu_s}$ .

*Proof.* By assumption on  $W$  we have a commutative diagram, for some  $N \in \mathbb{N}$ ,

$$\begin{array}{ccccc} X & \hookrightarrow & \mathbb{P}_{\mathbb{A}^1}^N & \hookrightarrow & \mathbb{P}_{\mathbb{P}^1}^N \\ \downarrow W & & \downarrow & & \downarrow \\ \mathbb{A}^1 & = & \mathbb{A}^1 & \hookrightarrow & \mathbb{P}^1 \end{array}$$

where the first arrow in the first row is a closed embedding. Let  $K$  be the closure of  $X$  in  $\mathbb{P}_{\mathbb{P}^1}^N = \mathbb{P}^N \times \mathbb{P}^1$ . Then  $K$  is a projective variety with an open embedding  $X \subset K$  and a projective morphism  $\kappa: K \rightarrow \mathbb{P}^1$  such that the diagram

$$\begin{array}{ccc} X & \hookrightarrow & K \\ \downarrow W & & \downarrow \kappa \\ \mathbb{A}^1 & \hookrightarrow & \mathbb{P}^1 \end{array}$$

is a pullback diagram. This compactifies  $W: X \rightarrow \mathbb{A}^1$  at infinity. The singular points of  $K$  are all contained in the fiber of  $\kappa$  over  $\infty$ . So clearly  $(K, \kappa^{-1}(I_\infty) \cdot \mathcal{O}_K)$  satisfies condition (K), and we obtain a morphism  $\gamma: c_{\kappa^{-1}(I_\infty) \cdot \mathcal{O}_K}(K) \rightarrow K$  as explained in Remark 6.3. Define  $\overline{X} := c_{\kappa^{-1}(I_\infty) \cdot \mathcal{O}_K}(K)$  and  $\overline{W} := \kappa \circ \gamma$ . Then  $\overline{X}$  is smooth projective and  $\overline{W}$  is projective, and from the construction we obtain the pullback diagram (6.3).

It remains to provide the local description of  $\overline{W}$  around  $p \in \overline{X}_\infty$ . We can assume that  $p$  is a closed point. Let  $\overline{X}' := \overline{W}^{-1}(\mathbb{A}_\infty^1)$  and view the restriction  $\overline{W}: \overline{X}' \rightarrow \mathbb{A}_\infty^1$  as a regular function on  $\overline{X}'$ . It generates the inverse image ideal

sheaf  $\overline{W}^{-1}(I_\infty) \cdot \mathcal{O}_{\overline{X}}$ , so its divisor is the snc divisor of this ideal sheaf. Hence there is an open neighborhood  $U'$  of  $p$  in  $\overline{X}$  with uniformizing parameters  $(\underline{y}_1, \dots, \underline{y}_m)$  centered at  $p$  and a tuple  $\mu = (\mu_1, \dots, \mu_s)$  of positive integers, for some  $1 \leq s \leq m$ , such that  $\overline{W} = v \underline{y}_1^{\mu_1} \dots \underline{y}_s^{\mu_s}$  for some unit  $v$  in  $\mathcal{O}_{\overline{X}}(U')$ . Let  $u: U \rightarrow U'$  be the étale morphism extracting a  $\mu_1$ -th root of  $v$ . Then

$$\underline{x}_1 := \underline{y}_1 v^{1/\mu_1}, \underline{x}_2 := \underline{y}_2, \dots, \underline{x}_m := \underline{y}_m$$

defines uniformizing parameters on  $U$  which satisfy  $\underline{x}_1^{\mu_1} \dots \underline{x}_s^{\mu_s} = \overline{W} \circ u$ . □

We introduce another condition needed in the proof of Proposition 6.1. Let  $(U, I)$  satisfy condition (K) and let  $\gamma: c(U) \rightarrow U$  be as in Remark 6.3. Write the snc divisor corresponding to  $\gamma^{-1}(I) \cdot \mathcal{O}_{c(U)}$  as  $\sum_{i=1}^s n_i E_i$  with pairwise distinct prime divisors  $E_1, \dots, E_s$  and all  $n_i > 0$ . Let  $f: U \rightarrow \mathbb{A}^1$  be a regular function. We say that the triple  $(U, I, f)$  satisfies condition (NoCrit-Sm) if

(NoCrit-Sm) No critical point of the morphism  $f \circ \gamma: c_I(U) \rightarrow \mathbb{A}^1$  is contained in  $E_1 \cup \dots \cup E_s$ , and for every tuple  $(i_1, \dots, i_p)$  of indices (with  $p \geq 1$ ) the morphism  $E_{i_1} \cap \dots \cap E_{i_p} \rightarrow \mathbb{A}^1$  induced by  $f \circ \gamma$  is smooth.

*Proof of Proposition 6.1.* Consider the morphism

$$\begin{aligned} \sigma: \mathbb{P}^1 \times \mathbb{A}^1 &\rightarrow \mathbb{P}^1 \times \mathbb{P}^1, \\ ([z'_0, z'_1], v') &\mapsto ([z'_0, z'_1], [z'_0, z'_0 v' - z'_1]). \end{aligned}$$

The image of  $\sigma$  is  $\mathbb{A}^1 \times \mathbb{A}^1 \cup \{(\infty, \infty)\}$ . The fiber of  $\sigma$  over  $(\infty, \infty)$  is  $E := \{\infty\} \times \mathbb{A}^1$ , and  $\sigma$  induces an isomorphism  $\mathbb{A}^1 \times \mathbb{A}^1 \xrightarrow{\sim} \mathbb{A}^1 \times \mathbb{A}^1$ . Note moreover that the diagram

$$\begin{array}{ccccc} \mathbb{P}^1 \times \mathbb{P}^1 & \supset & \mathbb{A}^1 \times \mathbb{A}^1 & \xleftarrow{\sim \sigma} & \mathbb{A}^1 \times \mathbb{A}^1 & \subset & \mathbb{P}^1 \times \mathbb{A}^1 & (6.4) \\ & & \downarrow + & & \downarrow \text{pr}_2 & & \downarrow \text{pr}_2 \\ & & \mathbb{A}^1 & = & \mathbb{A}^1 & = & \mathbb{A}^1 \end{array}$$

commutes. It says that addition  $\mathbb{A}^1 \times \mathbb{A}^1 \xrightarrow{+} \mathbb{A}^1$  corresponds under  $\sigma$  to the second projection; this projection can easily be extended to the projective morphism on the right. This little construction already does the job in case  $X = Y = \mathbb{A}^1$  and  $W = V = \text{id}_{\mathbb{A}^1}$ .

Now let  $X, Y$  be smooth (quasi-projective) varieties and let  $W: X \rightarrow \mathbb{A}^1$  and  $V: Y \rightarrow \mathbb{A}^1$  be projective morphisms. We can and will assume that  $X$  and  $Y$  are irreducible.

We choose  $X \hookrightarrow \overline{X} \xrightarrow{\overline{W}} \mathbb{P}^1$  and  $Y \hookrightarrow \overline{Y} \xrightarrow{\overline{V}} \mathbb{P}^1$  having the properties described in Proposition 6.4. Consider the pullback diagram

$$\begin{CD} T @>\widehat{\sigma}>> \overline{X} \times \overline{Y} \\ @V\theta VV @VV\overline{W} \times \overline{V} V \\ \mathbb{P}^1 \times \mathbb{A}^1 @>\sigma>> \mathbb{P}^1 \times \mathbb{P}^1. \end{CD} \tag{6.5}$$

Note that the upper horizontal arrow  $\widehat{\sigma}$  in this diagram induces an isomorphism

$$\widehat{\sigma}: T' := \theta^{-1}(\mathbb{A}^1 \times \mathbb{A}^1) \xrightarrow{\sim} X \times Y.$$

From (6.4) it is obvious that under this isomorphism the morphism  $\text{pr}_2 \circ \theta|_{T'}: T' \rightarrow \mathbb{A}^1$  corresponds to  $W * V: X \times Y \rightarrow \mathbb{A}^1$ .

We need to analyze  $T$  “étale locally” around an arbitrary point of  $\theta^{-1}(E)$ . Let  $I_E \subset \mathcal{O}_{\mathbb{P}^1 \times \mathbb{A}^1}$  be the ideal sheaf of  $E$ . Our analysis will in particular show that the pair  $(T, \theta^{-1}(I_E) \cdot \mathcal{O}_T)$  satisfies condition (K), so that the morphism

$$\gamma: Z := c_{\theta^{-1}(I_E) \cdot \mathcal{O}_T}(T) \rightarrow T,$$

is available. We will then see that  $Z$  together with the composition

$$h: Z \xrightarrow{\gamma} T \xrightarrow{\theta} \mathbb{P}^1 \times \mathbb{A}^1 \xrightarrow{\text{pr}_2} \mathbb{A}^1$$

does the job.

Define

$$B := \text{Spec } k[z_0, v, (z_0v - 1)^{-1}]$$

and embed this as an open subvariety of  $\mathbb{P}^1 \times \mathbb{A}^1$  via  $(z'_0, v') \mapsto ([z'_0, 1], v')$ . So  $B$  is contained in  $\mathbb{A}^1_\infty \times \mathbb{A}^1$ . We have  $B = \sigma^{-1}(\mathbb{A}^1_\infty \times \mathbb{A}^1_\infty)$ . Note that  $B$  contains  $E = \{\infty\} \times \mathbb{A}^1 = \{z'_0 = 0\}$  and that  $\sigma$  induces the morphism

$$\begin{aligned} \sigma: B &\rightarrow \mathbb{A}^1_\infty \times \mathbb{A}^1_\infty, \\ (z'_0, v') &\mapsto ([z'_0, 1], [z'_0(z'_0v' - 1)^{-1}, 1]). \end{aligned} \tag{6.6}$$

Let  $t \in \theta^{-1}(E)$  be the closed point around which we will analyze  $T$  “étale locally”. Define  $(x_\infty, y_\infty) := \widehat{\sigma}(t) \in \overline{X}_\infty \times \overline{Y}_\infty$ . We use the local description of  $\overline{W}$  around  $x_\infty \in \overline{X}_\infty$  given by Proposition 6.4. There is an étale morphism  $u: U \rightarrow \overline{X}$  whose image contains  $x_\infty$  such that  $\overline{W} \circ u$  can be factorized as  $U \xrightarrow{\underline{x}} \mathbb{A}^m \xrightarrow{\underline{\mu}} \mathbb{A}^1_\infty \subset \mathbb{P}^1$  for uniformizing parameters  $\underline{x}$  and a suitable  $\underline{\mu}$ . Similarly we describe  $\overline{V}$  locally around  $y_\infty \in \overline{Y}_\infty$  by an étale morphism  $u': U' \rightarrow \overline{Y}$  such that  $\overline{V} \circ u'$  is given by  $U' \xrightarrow{\underline{y}} \mathbb{A}^n \xrightarrow{\underline{v}} \mathbb{A}^1_\infty \subset \mathbb{P}^1$  for suitable  $\underline{y}$  and  $\underline{v}$ . Then  $(\overline{W} \times \overline{V}) \circ (u \times u')$  is equal to the composition

$$U \times U' \xrightarrow{\underline{x} \times \underline{y}} \mathbb{A}^m \times \mathbb{A}^n \xrightarrow{\underline{\mu} \times \underline{v}} \mathbb{A}^1_\infty \times \mathbb{A}^1_\infty \subset \mathbb{P}^1 \times \mathbb{P}^1.$$

Consider the pullback diagram

$$\begin{array}{ccc} S & \longrightarrow & \mathbb{A}^m \times \mathbb{A}^n \\ \downarrow \theta' & & \downarrow \mu \times v \\ B & \xrightarrow{\sigma} & \mathbb{A}^1_\infty \times \mathbb{A}^1_\infty. \end{array}$$

Note that the pullback of  $T$  along the étale morphism  $u \times u'$  coincides with the pullback of  $S$  along the étale morphism  $\underline{x} \times \underline{y}$ . Let us denote this pullback by  $\widehat{S}$ . The étale morphism  $\widehat{S} \rightarrow T$  contains  $t$  in its image.

From (6.6) we see that  $S$  can be described explicitly as

$$S = \text{Spec } k[v, x, y, (x^\mu v - 1)^{-1}] / (x^\mu - (x^\mu v - 1)y^v)$$

where  $x = x_1, \dots, x_m$  and  $y = y_1, \dots, y_n$  and  $x^\mu = x_1^{\mu_1} \dots x_s^{\mu_s}$  and similarly for  $y^v$ . Note that  $\theta'^*(z_0) = x^\mu$  and  $\text{pr}_2 \circ \theta' = v$ . Let  $S' \rightarrow S$  be the (surjective) étale morphism extracting the  $v_1$ -th root of the invertible element  $(x^\mu v - 1)$ . Define new coordinates  $y'_1 := y_1(x^\mu v - 1)^{1/v_1}$ ,  $y'_2 := y_2, \dots, y'_n := y_n$ . Then  $S'$  is given by

$$S' = \text{Spec } k[v, x, y', (x^\mu v - 1)^{-1/v_1}] / (x^\mu - y'^v).$$

There is an obvious étale morphism from  $S'$  to the open subscheme

$$S'' := \text{Spec } k[v, x, y', (x^\mu v - 1)^{-1}] / (x^\mu - y'^v)$$

of

$$S''' = \text{Spec } k[v] \times L$$

where  $L := \text{Spec } k[x, y'] / (x^\mu - y'^v)$ .

Up to now we have constructed a zig-zag of étale morphisms

$$T \leftarrow \widehat{S} \rightarrow S \leftarrow S' \rightarrow S'' \rightarrow S'''.$$

Let  $\widehat{T}$  be the pullback of  $\widehat{S} \rightarrow S$  and  $S' \rightarrow S$ . Hence we have étale morphisms

$$T \xleftarrow{\alpha} \widehat{T} \xrightarrow{\beta} S''' \tag{6.7}$$

and  $t$  is in the image of  $\alpha$  (since  $S' \rightarrow S$  is surjective). The ideal sheaves  $\theta^{-1}(I_E) \cdot \mathcal{O}_T$  on  $T$  and  $(x^\mu)$  on  $S'''$  have the same inverse image ideal sheaf on  $\widehat{T}$  (which also comes from the ideal sheaf  $(x^\mu)$  on  $S$ ). Correspondingly, we have  $\alpha^{-1}(T') = \beta^{-1}(S''' \setminus \mathcal{V}(x^\mu))$  (recall that  $\widehat{\sigma}: T' \xrightarrow{\sim} X \times Y$ ). Note also that  $\alpha^*(\text{pr}_2 \circ \theta) = \beta^*(v)$  as functions  $\widehat{T} \rightarrow \mathbb{A}^1$ .

Lemma 6.5 tells us that  $L$  is a reduced scheme of finite type, that the ideal  $(x^\mu)$  does not vanish on any irreducible component of  $L$  and that  $\mathcal{V}(x^\mu) \supset L^{\text{sing}}$  (the singular locus of each component is contained in  $\mathcal{V}(x^\mu)$ , and different components

do not intersect outside  $\mathcal{V}(x^\mu)$ . This just means that  $(L, (x^\mu))$  satisfies condition (K). Hence the same is true for  $(S''', (x^\mu))$ .

From [12, Prop. I.4.9] we see that  $\widehat{T}$  and  $\alpha(\widehat{T})$  are reduced schemes. This implies that  $\widehat{T}$  and  $T$  (let  $t$  vary) are quasi-projective varieties.

We claim that  $T$  is irreducible. Obviously,  $L \setminus \mathcal{V}(x^\mu)$  is open and dense in  $L$ . By Lemma 6.6 below, applied to  $\widehat{T} \xrightarrow{\beta} S''' \rightarrow L$ , we see that  $\beta^{-1}(S''' \setminus \mathcal{V}(x^\mu))$  is open and dense in  $\widehat{T}$ . Recall that  $\alpha^{-1}(T') = \beta^{-1}(S''' \setminus \mathcal{V}(x^\mu))$ . We obtain that  $\alpha(\alpha^{-1}(T'))$  is open and dense in  $\alpha(\widehat{T})$ . In particular,  $t$  is in the closure of  $T'$  in  $T$ . Since  $t \in \theta^{-1}(E) = T \setminus T'$  was arbitrary and  $T' \xrightarrow{\sim} X \times Y$  is irreducible this proves that  $T$  is irreducible.

Now it is clear that  $\theta^{-1}(I_E) \cdot \mathcal{O}_T$  does not vanish  $T$ , and certainly we have

$$T^{\text{sing}} \subset \theta^{-1}(E) = \mathcal{V}(\theta^{-1}(I_E) \cdot \mathcal{O}_T).$$

This proves that  $(T, \theta^{-1}(I_E) \cdot \mathcal{O}_T)$  satisfies condition (K).

Hence we can apply Remark 6.3 to  $(T, \theta^{-1}(I_E) \cdot \mathcal{O}_T)$  and obtain a morphism  $\gamma: c_{\theta^{-1}(I_E) \cdot \mathcal{O}_T}(T) \rightarrow T$ . Define  $Z := c_{\theta^{-1}(I_E) \cdot \mathcal{O}_T}(T)$  and

$$h: Z \xrightarrow{\gamma} T \xrightarrow{\theta} \mathbb{P}^1 \times \mathbb{A}^1 \xrightarrow{\text{pr}_2} \mathbb{A}^1.$$

Then  $h: Z \rightarrow \mathbb{A}^1$  is a projective morphism and  $Z$  is a smooth quasi-projective variety. By construction  $\gamma$  induces an isomorphism  $\gamma^{-1}(T') \xrightarrow{\sim} T'$ . Using the isomorphism  $\widehat{\sigma}: T' \xrightarrow{\sim} X \times Y$  we hence find an open embedding  $X \times Y \hookrightarrow Z$ . We claim that this datum satisfies conditions (i)–(iv). Indeed, (i) and (iii) hold by construction. All  $D_{i_1 \dots i_p} := D_{i_1} \cap \dots \cap D_{i_p}$  are smooth quasi-projective varieties since the  $D_i$  are the irreducible components of the support of a snc divisor, and all morphisms  $h_{i_1 \dots i_p}: D_{i_1 \dots i_p} := D_{i_1} \cap \dots \cap D_{i_p} \rightarrow \mathbb{A}^1$  induced by  $h$  are projective since  $h: Z \rightarrow \mathbb{A}^1$  is projective.

Hence we need to show condition (ii) and the smoothness part of condition (iv), or, equivalently, that condition (NoCrit-Sm) holds for the triple

$$(T, \theta^{-1}(I_E) \cdot \mathcal{O}_T, \text{pr}_2 \circ \theta).$$

From Remark 6.3 we see that this can be checked locally on  $T$ , and even on an étale covering of  $T$  (we use that the map (6.1) commutes with étale morphisms). The zig-zag (6.7) of étale maps and the fact that we already know that  $(S''', (x^\mu))$  satisfies condition (K) shows that it is enough to show condition (NoCrit-Sm) for  $(S''', (x^\mu), v)$ .

Consider  $L$  with the ideal sheaf  $(x^\mu)$  and the structure morphism

$$\text{can}: L \rightarrow \text{Spec } k.$$

Let  $\gamma_L: c(L) = c_{(x^\mu)}(L) \rightarrow L$  be the morphism from Remark 6.3. Let  $\sum_{j=1}^r m_j F_j$  (with pairwise distinct  $F_j$ 's and all  $m_j > 0$ ) be the snc divisor corresponding

to  $(\gamma_L^*(x^\mu))$ . Then for every tuple  $(j_1, \dots, j_q)$  of indices (with  $q \geq 1$ ) the intersection  $F_{j_1} \cap \dots \cap F_{j_q}$  is regular, i. e.  $\text{can} \circ \gamma_L: F_{j_1} \cap \dots \cap F_{j_q} \rightarrow \text{Spec } k$  is smooth.

Note that  $(S''', (x^\mu), v)$  is obtained from  $(L, (x^\mu), \text{can})$  via base change along the smooth morphism  $\text{Spec } k[v] \rightarrow \text{Spec } k$ . So the two squares in the diagram

$$\begin{array}{ccc}
 c_{(x^\mu)}(S''') & \longrightarrow & c_{(x^\mu)}(L) \\
 \downarrow \gamma_{S'''} & & \downarrow \gamma_L \\
 S''' & \longrightarrow & L \\
 \downarrow v & & \downarrow \text{can} \\
 \mathbb{A}^1 = \text{Spec } k[v] & \longrightarrow & \text{Spec } k.
 \end{array}$$

are pullback diagrams (we use that the map (6.1) commutes with smooth morphisms, cf. diagram (6.2)). Let  $F'_j = \text{Spec } k[v] \times F_j$ . Then  $\sum_{j=1}^r m_j F'_j$  is the snc divisor corresponding to  $(\gamma_{S'''}^*(x^\mu))$ , and it is obvious that condition (NoCrit-Sm) holds for  $(S''', (x^\mu), v)$ : the morphism  $v \circ \gamma_{S'''}$  is smooth (since  $\text{can} \circ \gamma_L$  is smooth) and hence has no critical point at all, and all morphisms  $v \circ \gamma_L: F'_{j_1} \cap \dots \cap F'_{j_q} \rightarrow \text{Spec } k[v]$  are smooth since they are obtained from  $\text{can} \circ \gamma_L: F_{j_1} \cap \dots \cap F_{j_q} \rightarrow \text{Spec } k$  by base change.  $\square$

**Lemma 6.5.** *Let*

$$p := x^\mu - y^v := x_1^{\mu_1} x_2^{\mu_2} \dots x_s^{\mu_s} - y_1^{v_1} \dots y_t^{v_t}$$

be a polynomial in  $k[x, y] := k[x_1, \dots, x_s, y_1, \dots, y_t]$  with  $s, t > 0$  and all  $\mu_i > 0$  and all  $v_j > 0$ . Let  $d = \text{gcd}(\mu_1, \mu_2, \dots, \mu_s, v_1, \dots, v_t)$ . Then

$$p = \prod_{\zeta \in \sqrt[d]{1}} (x^{\mu/d} - \zeta y^{v/d}) \tag{6.8}$$

is the factorization of  $p$  into irreducibles in  $k[x, y]$  (where  $\sqrt[d]{1}$  denotes the set of all  $d$ -th roots of unity in  $k$ ); obviously, all factors are distinct and appear with multiplicity one. Here we use the shorthand notation  $x^{\mu/d} = x_1^{\mu_1/d} \dots x_s^{\mu_s/d}$ , and similarly for  $y^{v/d}$ . In particular,  $p$  is irreducible in  $k[x, y]$  if  $d = 1$ . (If  $s, t \leq n$  then the above factorization into irreducibles obviously is also a factorization into irreducibles in  $k[x_1, \dots, x_n, y_1, \dots, y_n]$ .)

The proof of this lemma was motivated by a proof of its special case  $s = t = 1$  on Stackexchange by Qiaochu Yuan. We thank Jan Büthe for a discussion of the general case.

*Proof.* From  $T^d - 1 = \prod_{\zeta \in \sqrt[d]{1}} (T - \zeta)$  we obtain by substituting  $T = \frac{U}{V}$  that

$$U^d - V^d = \prod_{\zeta \in \sqrt[d]{1}} (U - \zeta V).$$

From  $p = (x^{\mu/d})^d - (y^{v/d})^d$  we hence obtain formula (6.8), and it is enough to show that each factor  $(x^{\mu/d} - \zeta y^{v/d})$  is irreducible in  $k[x, y]$ .

For this it is enough to show that  $p$  is irreducible if  $d = 1$  (since then also any polynomial  $x^\mu - \lambda y^v$  will be irreducible for  $\lambda \in k^\times$ : put  $y'_1 := \sqrt[v]{\lambda} y_1$ ; alternatively, adapt the following proof so that it works directly for  $x^\mu - \lambda y^v$ ).

Let  $f$  be an irreducible factor of  $p$  in  $k[x, y]$ . The group  $\mathbb{Z}_{\mu_1}$  acts on  $k[x, y]$  by algebra automorphisms such that the generator 1 of  $\mathbb{Z}_{\mu_1}$  maps  $x_1$  to  $\zeta_{\mu_1} x_1$  where  $\zeta_{\mu_1}$  is a fixed primitive  $\mu_1$ -th root of unity. By combining the analog commuting actions on the other variables we obtain an action of

$$Z := \mathbb{Z}_\mu \times \mathbb{Z}_v := \mathbb{Z}_{\mu_1} \times \cdots \times \mathbb{Z}_{\mu_s} \times \mathbb{Z}_{v_1} \times \cdots \times \mathbb{Z}_{v_t}$$

on  $k[x, y]$ .

Note that  $p \in k[x, y]^Z = k[x_1^{\mu_1}, \dots, y_t^{v_t}]$ . Any element of the  $Z$ -orbit  $Z \cdot f$  of  $f$  also is an irreducible factor of  $p$ . Some of these irreducible factors might be associated. Let  $F$  be the product of all these irreducible factors up to  $k^\times$ -multiples. (More precisely we mean the following: the group  $Z$  acts on  $\mathbb{P}(k[x, y])$ , and the multiplication of  $k[x, y]$  induces a multiplication on  $\mathbb{P}(k[x, y])$  which is compatible with the  $Z$ -action. Let  $F \in k[x, y]$  be an element such that  $[F] = \prod_{g \in Z \cdot [f]} g$  in  $\mathbb{P}(k[x, y])$ .) Then  $F|p$ .

It is clear that  $z \cdot F \in k^\times F$  for all  $z \in Z$ . We claim that in fact  $F \in k[x, y]^Z$ .

Let  $\rho: Z \rightarrow k^\times$  be the morphism of groups such that  $z \cdot F = \rho(z)F$  for all  $z \in Z$ . If we apply the element  $z_1 := (1, 0, 0, \dots, 0) \in Z$  to the monomial  $x^\alpha y^\beta$  we obtain  $\zeta_{\mu_1}^{\alpha_1} x^\alpha y^\beta$ . If this monomial  $x^\alpha y^\beta$  appears with non-zero coefficient in  $F$  we must have  $\rho(z_1) = \zeta_{\mu_1}^{\alpha_1}$ . Hence if another monomial  $x^{\alpha'} y^{\beta'}$  also appears with non-zero coefficient in  $F$ , then  $\zeta_{\mu_1}^{\alpha_1} = \zeta_{\mu_1}^{\alpha'_1}$ , or equivalently,  $\alpha_1 - \alpha'_1 \in \mathbb{Z}_{\mu_1}$ . This implies that we can write  $F = x_1^\gamma G$  with  $G \in k[x_1^{\mu_1}, x_2, \dots, x_n, y_1, \dots, y_n]$ , for some  $\gamma \in \mathbb{N}$  (for example the smallest exponent of  $x_1$  that appears in a monomial that appears in  $F$  with non-zero coefficient). Since  $F|p$  this implies that  $x_1^\gamma|p$  which is obviously only possible if  $\gamma = 0$ . We can iterate this argument and eventually see that  $F \in k[x_1^{\mu_1}, x_2^{\mu_2}, \dots, x_s^{\mu_s}, y_1^{v_1}, \dots, y_t^{v_t}] = k[x, y]^Z$ , proving our claim.

Hence we have  $F|p$  in  $k[x, y]^Z$ . Write  $a_1 := x_1^{\mu_1}, \dots, a_s := x_s^{\mu_s}$ , and  $b_1 := y_1^{v_1}, \dots, b_t := y_t^{v_t}$ . Then  $p = a - b := a_1 \dots a_s - b_1 \dots b_t$  and this element is irreducible in  $k[x, y]^Z = k[a, b]$  (it is linear in  $a_1$  and the coefficient  $a_2 \dots a_s$  of  $a_1$  and the constant coefficient  $b_1 \dots b_t$  have greatest common divisor 1). Since  $F$  is not a unit this implies that  $F = p$  up to a multiple in  $k^\times$ .

Denote by  $\deg_{x_i}(g)$  the degree of an element  $g \in k[x, y]$  in  $x_i$ . Let  $l$  be the cardinality of the orbit of  $[f]$  in  $\mathbb{P}(k[x, y])$ , i. e.  $F$  is the product of  $l$  irreducible elements obtained from  $f$ . Then

$$\mu_i = \deg_{x_i}(p) = \deg_{x_i}(F) = l \deg_{x_i}(f).$$

This and the same argument for the degrees in the  $y_j$ 's show that  $l$  is a common divisor of all the  $\mu_i$  and  $v_j$ .

If  $d = 1$  we obtain  $l = 1$ , i. e.  $F = f$  up to a multiple in  $k^\times$ . Hence  $F$  and  $p$  are irreducible in  $k[x, y]$ .  $\square$

**Lemma 6.6.** *Let  $f: X \rightarrow Y$  be an open morphism of Noetherian schemes. If  $V \subset Y$  is open and dense, then  $f^{-1}(V)$  is open and dense in  $X$ .*

*Proof.* Let  $C$  be an irreducible component of  $X$ . Let  $C^\circ$  be obtained from  $C$  by removing all points that lie in an irreducible component distinct from  $C$ . Then  $C^\circ$  is open in  $X$  and non-empty, so  $f(C^\circ)$  is open and non-empty and hence contains a point of  $V$ . Then  $f^{-1}(V) \cap C^\circ$  is open in  $C$  and non-empty, and hence dense in  $C$ . This implies that  $C \subset f^{-1}(V)$ .  $\square$

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