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Measured quantum transformation groupoids

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Abstract. In this article, when $\mathbb{G}$ is a locally compact quantum group, we associate, to a braided-commutative $\mathbb{G}$-Yetter–Drinfel'd algebra $(N, \mathfrak{a}, \widehat{\mathfrak{a}})$ equipped with a normal faithful semi-finite weight verifying some appropriate condition (in particular if it is invariant with respect to $\mathfrak{a}$, or to $\widehat{\mathfrak{a}}$), a structure of a measured quantum groupoid. The dual structure is then given by $(N, \widehat{\mathfrak{a}}, \mathfrak{a})$. Examples are given, especially the situation of a quotient type co-ideal of a compact quantum group. This construction generalizes the standard construction of a transformation groupoid. Most of the results were announced by the second author in 2011, at a conference in Warsaw.

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1. Introduction

1.1. Locally compact quantum groups. The theory of locally compact quantum groups, developed by J. Kustermans and S. Vaes [22, 23], provides a comprehensive framework for the study of quantum groups in the setting of C^*-algebras and von Neumann algebras. It includes a far reaching generalization of the classical Pontrjagin duality of locally compact abelian groups, that covers all locally compact groups. Namely, if G is a locally compact group, its dual $\widehat{G}$ will be the von Neumann algebra $\mathcal{L}(G)$ generated by the left regular representation λ_G of G, equipped with a coproduct Γ_G from $\mathcal{L}(G)$ on $\mathcal{L}(G) \otimes \mathcal{L}(G)$ defined, for all $s \in G$, by $\Gamma_G(\lambda_G(s)) = \lambda_G(s) \otimes \lambda_G(s)$, and with a normal semi-finite faithful weight, called the Plancherel weight φ_G, associated via the Tomita–Takesaki construction, to the left Hilbert algebra defined by the algebra $\mathcal{K}(G)$ of continuous functions with compact support (with convolution as product), this weight φ_G being left- and right-invariant with respect to Γ_G [38, VII, 3].

This theory builds on many preceding works, by G. Kac, G. Kac and L. Vainerman, the first author and J.-M. Schwartz [18, 19], S. Baaj and G. Skandalis [4], A. Van Daele,

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S. Woronowicz [47, 50, 51] and many others. See the monography written by the second author for a survey of that theory [39], and the introduction of [19] for a sketch of the historical background. It seems to have reached now a stable situation, because it fits the needs of operator algebraists for many reasons:

First, the axioms of this theory are very simple and elegant: they can be given in both C^*-algebras and von Neumann algebras, and these two points of view are equivalent, as A. Weil had shown it was the fact for groups (namely any measurable group equipped with a left-invariant positive measure bears a topology which makes it locally compact, and this measure is then the Haar measure [46]). In a von Neumann setting, a locally compact quantum group is just a von Neumann algebra, equipped with a co-associative coproduct, and two normal faithful semi-finite weights, one left-invariant with respect to that coproduct, and one right-invariant. Then, many other data are constructed, in particular a multiplicative unitary (as defined in [4]) which is manageable (as defined in [51]).

Second, all preceeding attemps [19, 50] appear as particular cases of locally compact quantum groups; and many interesting examples were constructed [43, 48, 49].

Third, many constructions of harmonic analysis, or concerning group actions on C^*-algebras and von Neumann algebras, were generalized up to locally compact quantum groups [41].

Finally, many constructions made by algebraists at the level of Hopf $*$-algebras, or multipliers Hopf $*$-algebras can be generalized for locally compact quantum groups. This is the case, for instance, for Drinfel'd double of a quantum group [10], and for Yetter–Drinfel'd algebras which were well-known in an algebraic approach in [26].

1.2. Measured Quantum Groupoids. In two articles [44, 45], J.-M. Vallin has introduced two notions (pseudo-multiplicative unitary, Hopf bimodule), in order to generalize, to the groupoid case, the classical notions of multiplicative unitary [4] and of a co-associative coproduct on a von Neumann algebra. Then, F. Lesieur [24], starting from a Hopf bimodule, when there exist a left-invariant operator-valued weight and a right-invariant operator-valued weight, mimicking in that wider setting what was done in [22, 23], obtained a pseudo-multiplicative unitary, and called "measured quantum groupoids" these objects. A new set of axioms had been given in an appendix of [13]. In [13] and [14], most of the results given in [41] were generalized up to measured quantum groupoids.

This theory, up to now, bears two defects:

First, it is only a theory in a von Neumann algebra setting. The second author had made many attemps in order to provide a C^*-algebra version of it (see [39] for a survey); these attemps were fruitful, but not sufficient to complete a theory equivalent to the von Neumann one.

Second, there is a lack of interesting examples. For instance, the transformation groupoid (i.e. the groupoid given by a locally compact group right acting on a locally compact space), which is the first non-trivial example of a groupoid [32, 1.2.a], had no quantum analog up to this article.

1.3. Measured quantum transformation groupoid. This article is devoted to the construction of a family of examples of measured quantum groupoids. Most of the results were announced in [40]. The key point, is, when looking at a transformation groupoid given by a locally compact group G having a right action $\mathfrak{a}$ on a locally compact space X, to add the fact that the dual $\widehat{G}$ is trivially right acting also on $L^\infty(X)$, and that the triple $(L^\infty(X), \mathfrak{a}, \mathrm{id})$ is a G-Yetter–Drinfel'd algebra, and, more precisely, a braided-commutative G-Yetter–Drinfel'd algebra.

The aim of this article is to generalize the construction of transformation groupoids, using this remark which shows that this generalization is not to be found for any action of a locally compact quantum group, but for a braided-commutative $\mathbb{G}$-Yetter–Drinfel'd algebra.

Then, for any locally compact quantum group $\mathbb{G}$, looking at any braided-commutative Yetter–Drinfel'd algebra $(N, \mathfrak{a}, \widehat{\mathfrak{a}})$, it is possible to put a structure of Hopf bimodule on the crossed product $\mathbb{G} \ltimes_{\mathfrak{a}} N$, equipped with a left-invariant operator-valued weight, and with a right-invariant operator-valued weight. In order to get a measured quantum groupoid, one has to choose on N (which is the basis of the measured quantum groupoid) a normal faithful semi-finite weight ν that satisfies some condition with respect to the action $\mathfrak{a}$; for example, ν could be invariant with respect to $\mathfrak{a}$. It appears then that the dual measured quantum groupoid is the structure associated to the braided-commutative Yetter–Drinfel'd algebra $(N, \widehat{\mathfrak{a}}, \mathfrak{a})$.

In an algebraic framework, similar results were obtained in [25] and [3]. It is also interesting to notice that, as for locally compact quantum groups, the framework of measured quantum groupoids appears to be a good structure in which the algebraic results can be generalized.

The article is organized as follows:

In Section 2 are recalled all the necessary results needed: namely locally compact quantum groups (2.1), actions of locally compact quantum groups on a von Neumann algebra (2.2), Drinfel'd double of a locally compact quantum group (2.3), Yetter–Drinfel'd algebras (2.4), and braided-commutative Yetter–Drinfel'd algebras (2.5).

In Section 3, we study relatively invariant weights with respect to an action, and then invariant weights for a Yetter–Drinfel'd algebra, and prove that such a weight exists when the von Neumann algebra N is properly infinite.

In Section 4, we construct the Hopf–von Neumann structure associated to a braided-commutative $\mathbb{G}$-Yetter–Drinfel'd algebra. The precise definition of such a structure is given in 4.1 and 4.2. We construct also a co-inverse of this Hopf–von Neumann structure.

In Section 5, we study the conditions to put on the weight ν to construct a measured quantum groupoid associated to a braided-commutative $\mathbb{G}$-Yetter–Drinfel'd algebra. These conditions hold, in particular, if the weight ν is invariant with respect to $\mathfrak{a}$. The precise definition and properties of measured quantum groupoids are given in 5.1, 5.2, 5.3.

In Section 6, we obtain the dual of this measured quantum groupoid, which is the measured quantum groupoid obtained when permuting the actions $\mathfrak{a}$ and $\widehat{\mathfrak{a}}$.

Finally, in Section 7, we give several examples of measured quantum groupoids which can be constructed this way, and in Section 8, we study more carefully the case of a quotient type co-ideal of a compact quantum group: in that situation, one of the measured quantum groupoids constructed in 7.4.4 is Morita equivalent to the quantum subgroup.

2. Preliminaries

2.1. Locally compact quantum groups. A quadruplet $\mathbb{G} = (M, \Gamma, \varphi, \psi)$ is a *locally compact quantum group* if:

(i) M is a von Neumann algebra,

(ii) Γ is an injective unital $*$-homomorphism from M into the von Neumann tensor product $M \otimes M$, called a *coproduct*, satisfying $(\Gamma \otimes \mathrm{id})\Gamma = (\mathrm{id} \otimes \Gamma)\Gamma$ (the coproduct is called *co-associative*),

(iii) φ is a normal faithful semi-finite weight on M^+ which is *left-invariant*, i.e.,

$$(\mathrm{id} \otimes \varphi)\Gamma(x) = \varphi(x)1_M \quad \text{for all } x \in \mathfrak{M}_\varphi^+;$$

(iv) ψ is a normal faithful semi-finite weight on M^+ which is *right-invariant*, i.e.,

$$(\psi \otimes \mathrm{id})\Gamma(x) = \psi(x)1_M \quad \text{for all } x \in \mathfrak{M}_\psi^+.$$

In this definition (and in the following), $\otimes$ means the von Neumann tensor product, $(\mathrm{id} \otimes \varphi)$ (resp. $(\psi \otimes \mathrm{id})$) is an operator-valued weight from $M \otimes M$ to $M \otimes \mathbb{C}$ (resp. $\mathbb{C} \otimes M$). This is the definition of the von Neumann version of a locally compact quantum group [23]. See also, of course [22].

We shall use the usual data H_φ, J_φ, Δ_φ of Tomita–Takesaki theory associated to the weight φ (see [38, Chap. 6–9], [36, Chap. 10], [35, Chap. 1–2]), which, for simplification, we write as H, J, Δ. We regard M as a von Neumann algebra on H_φ and identify the opposite von Neumann algebra M^{o} with the commutant M'.

On the Hilbert tensor product $H \otimes H$, Kustermanns and Vaes constructed a unitary W, called the *fundamental unitary*, which satisfies the *pentagonal equation*

$$W_{23}W_{12} = W_{12}W_{13}W_{23},$$

where, we use, as usual, the leg-numbering notation. This unitary contains all the data of $\mathbb{G}$: M is the weak closure of the vector space (which is an algebra)

$\{(\mathrm{id} \otimes \omega)(W) : \omega \in B(H)_*\}$ and Γ is given by [22, 3.17]

$$\Gamma(x) = W^*(1 \otimes x)W \quad \text{for all } x \in M,$$

and

$$(\mathrm{id} \otimes \omega_{J_\varphi \Lambda_\varphi(y_1^* y_2), \Lambda_\varphi(x)})(W) = (\mathrm{id} \otimes \omega_{J_\varphi \Lambda_\varphi(y_2), J_\varphi(y_1)})\Gamma(x^*)$$

for all x, y_1, y_2 in $\mathfrak{N}_\varphi$. It is then possible to construct an unital anti-$*$-automorphism R of M which is involutive ($R^2 = \mathrm{id}$), defined by

$$R[(\mathrm{id} \otimes \omega_{\xi,\eta})(W)] = (\mathrm{id} \otimes \omega_{J\eta, J\xi})(W) \quad \text{for all } \xi, \eta \in H.$$

This map is a *co-inverse* (often called the *unitary antipode*), which means that

$$\Gamma \circ R = \varsigma \circ (R \otimes R) \circ \Gamma,$$

where ς is the flip of $M \otimes M$ [22, 5.26]. It is straightforward to get that $\varphi \circ R$ is a right-invariant normal semi-finite faithful weight and, thanks to a unicity theorem, is therefore proportional to ψ. We shall always suppose that $\psi = \varphi \circ R$.

Associated to (M, Γ) is a *dual* locally compact quantum group $(\widehat{M}, \widehat{\Gamma})$, where $\widehat{M}$ is the weak closure of the vector space (which is an algebra) $\{(\omega \otimes \mathrm{id})(W) : \omega \in B(H)_*\}$, and $\widehat{\Gamma}$ is given by

$$\widehat{\Gamma}(y) = \sigma W(y \otimes 1)W^*\sigma \quad \text{for all } y \in \widehat{M}.$$

Here, σ denotes the flip of $H \otimes H$. Let

$$\|\omega\|_\varphi = \sup\{|\omega(x^*)| : x \in \mathfrak{N}_\varphi, \varphi(x^*x) \leq 1\}, \quad I_\varphi = \{\omega \in M_* : \|\omega\|_\varphi < \infty\}.$$

Then, it is possible to define a normal semi-finite faithful weight $\widehat{\varphi}$ on $\widehat{M}$ such that $\widehat{\varphi}((\omega \otimes \mathrm{id})(W)^*(\omega \otimes \mathrm{id})(W)) = \|\omega\|_\varphi^2$ [22, 8.13], and it is possible to prove that $\widehat{\varphi}$ is left-invariant with respect to $\widehat{\Gamma}$ [22, 8.15]. Moreover, the application $y \mapsto Jy^*J$ is a unital anti-$*$-automorphism $\widehat{R}$ of $\widehat{M}$, which is involutive ($\widehat{R}^2 = \mathrm{id}$) and is a co-inverse. Therefore, $\widehat{\varphi} \circ \widehat{R}$ is right-invariant with respect to $\widehat{\Gamma}$.

Therefore $\widehat{\mathbb{G}} = (\widehat{M}, \widehat{\Gamma}, \widehat{\varphi}, \widehat{\varphi} \circ \widehat{R})$ is a locally compact quantum group, called the *dual* of $\mathbb{G}$. Its multiplicative unitary $\widehat{W}$ is equal to $\sigma W^* \sigma$. The bidual locally compact quantum group $\widehat{\widehat{\mathbb{G}}}$ is equal to $\mathbb{G}$. In particular, the construction of the dual weight, when applied to $\widehat{\mathbb{G}}$ gives that, for any ω in $\widehat{M}_*$, $(\mathrm{id} \otimes \omega)(W^*)$ belongs to $\mathfrak{N}_\varphi$ if and only if ω belongs to $I_{\widehat{\varphi}}$, and we have then $\|\Lambda_\varphi((\mathrm{id} \otimes \omega)(W^*))\| = \|\omega\|_{\widehat{\varphi}}$.

The Hilbert space $H_{\widehat{\varphi}}$ is isomorphic to (and will be identified with) H. For simplification, we write $\widehat{J}$ for $J_{\widehat{\varphi}}$ and $\widehat{\Delta}$ for $\Delta_{\widehat{\varphi}}$; we have, for all $x \in M$, $R(x) = \widehat{J}x^*\widehat{J}$ [23, 2.1]. The operator W satisfies

$$(\widehat{\Delta}^{it} \otimes \Delta^{it})W(\widehat{\Delta}^{-it} \otimes \Delta^{-it}) = W$$

and $(\widehat{J} \otimes J)W(\widehat{J} \otimes J) = W^*$.

Associated to (M,Γ) is *a scaling group*, which is a one-parameter group τ_t of automorphisms of M, such that [23, 2.1], for all $x\in M$, $t\in\mathbb{R}$, we have $\tau_t(x)=\widehat{\Delta}^{it}x\widehat{\Delta}^{-it}$, satisfying $\Gamma\circ\tau_t=(\tau_t\otimes\tau_t)\Gamma$ [22, 5.12], $R\circ\tau_t=\tau_t\circ R$ [22, 5.21], and $\Gamma\circ\sigma_t^{\varphi}=(\tau_t\otimes\sigma_t^{\varphi})\Gamma$ [22, 5.38] (and, therefore, $\Gamma\circ\sigma_t^{\varphi\circ R}=(\sigma_t^{\varphi\circ R}\otimes\tau_{-t})\Gamma$ [22, 5.17]).

The application $S=R\circ\tau_{-i/2}$ is called the *antipode* of $\mathbb{G}$.

The modular groups of the weights φ and $\varphi\circ R$ commute, which leads to the definition of the *scaling constant* $\lambda\in\mathbb{R}$ and the *modulus*, which is a positive self-adjoint operator δ affiliated to M, such that $(D\varphi\circ R:D\varphi)_t=\lambda^{it^2/2}\delta^{it}$.

We have $\varphi\circ\tau_t=\lambda^t\varphi$, and the canonical implementation of τ_t is given by a positive non-singular operator P defined by $P^{it}\Lambda_\varphi(x)=\lambda^{t/2}\Lambda_\varphi(\tau_t(x))$. Moreover, the operator $\widehat{\Delta}$ is equal to the closure of $PJ\delta^{-1}J$, and the operator $\widehat{\delta}$ is equal to the closure of $P^{-1}J\delta J\delta^{-1}\Delta^{-1}$ ([23, 2.1] and [54, 2.5]).

We have $\widehat{J}J=\lambda^{i/4}J\widehat{J}$ [23, 2.12]. The operator $\widehat{P}$ is equal to P, the scaling constant $\widehat{\lambda}$ is equal to λ^{-1}. Moreover, we have [54, 3.4]

$$W(\widehat{\Delta}^{it}\otimes\widehat{\Delta}^{it})W^*=\delta^{it}\widehat{\Delta}^{it}\otimes\widehat{\Delta}^{it}.$$

A *representation* of $\mathbb{G}$ on a Hilbert space K is a unitary $U\in M\otimes B(K)$, satisfying $(\Gamma\otimes\mathrm{id})(U)=U_{23}U_{13}$. It is well known that such a representation satisfies that, for any ξ,η in K, the operator $(\mathrm{id}\otimes\omega_{\xi,\eta})(U)$ belongs to $\mathcal{D}(S)$ and that

$$S[(\mathrm{id}\otimes\omega_{\xi,\eta})(U)]=(\mathrm{id}\otimes\omega_{\xi,\eta})(U^*)$$

(a proof for measured quantum groupoids can be found in [13, 5.10]).

Other locally compact quantum groups are $\mathbb{G}^{\mathrm{o}}=(M,\varsigma\circ\Gamma,\varphi\circ R,\varphi)$ (the *opposite* locally compact quantum group) and $\mathbb{G}^{\mathrm{c}}=(M',(j\otimes j)\circ\Gamma\circ j,\varphi\circ j,\varphi\circ R\circ j)$ (the *commutant* locally compact quantum group) where $j(x)=J_\varphi x^*J_\varphi$ is the canonical anti-$*$-isomorphism between M and M' given by Tomita–Takesaki theory. It is easy to get that $\widehat{\mathbb{G}^{\mathrm{o}}}=(\widehat{\mathbb{G}})^{\mathrm{c}}$ and $\widehat{\mathbb{G}^{\mathrm{c}}}=(\widehat{\mathbb{G}})^{\mathrm{o}}$ [23, 4.2]. We have $M\cap\widehat{M}=M'\cap\widehat{M}=M\cap\widehat{M}'=M'\cap\widehat{M}'=\mathbb{C}$. The multiplicative unitary W^{o} of $\mathbb{G}^{\mathrm{o}}$ is equal to $(\widehat{J}\otimes\widehat{J})W(\widehat{J}\otimes\widehat{J})$, and the multiplicative unitary W^{c} of $\mathbb{G}^{\mathrm{c}}$ is equal to $(J\otimes J)W(J\otimes J)$.

Moreover, the norm closure of the space $\{(\mathrm{id}\otimes\omega)(W):\omega\in B(H)_*\}$ is a C^*-algebra denoted $C_0^{\mathrm{r}}(\mathbb{G})$, which is invariant under R, and, together with the restrictions of Γ, φ and $\varphi\circ R$ will give the *reduced C^*-algebraic locally compact quantum group* [22,23]. In [21] was defined also a *universal* version $C_0^{\mathrm{u}}(\mathbb{G})$, which is equipped with a coproduct Γ_u. There exists a canonical surjective $*$-homomorphism $\pi_{\mathbb{G}}$ from $C_0^{\mathrm{u}}(\mathbb{G})$ to $C_0^{\mathrm{r}}(\mathbb{G})$, such that $(\pi_{\mathbb{G}}\otimes\pi_{\mathbb{G}})\Gamma_u=\Gamma\circ\pi_{\mathbb{G}}$. Then, $\varphi\circ\pi_{\mathbb{G}}$ (resp. $\varphi\circ R\circ\pi_{\mathbb{G}}$) is a (non-faithful) weight on $C_0^{\mathrm{u}}(\mathbb{G})$ which is left-invariant (resp. right-invariant).

If G is a locally compact group equipped with a left Haar measure ds, then, by duality of the Banach algebra structure of $L^1(G,ds)$, it is possible to define a co-associative coproduct Γ_G^a on $L^\infty(G,ds)$ and to give to $(L^\infty(G,ds),\Gamma_G^a,ds,ds^{-1})$

a structure of locally compact quantum group, called G again; any locally compact quantum group whose underlying von Neumann algebra is abelian is of that type. Then, its dual locally compact quantum group $\widehat{G}$ is $(\mathcal{L}(G), \Gamma_G^s, \varphi_G, \varphi_G))$, where $\mathcal{L}(G)$ is the von Neumann algebra generated by the left regular representation λ_G of G on $L^2(G, ds)$, Γ_G^s is defined, for all $s \in G$, by $\Gamma_G^s(\lambda_G(s)) = \lambda_G(s) \otimes \lambda_G(s)$, and φ_G is defined, for any f in the algebra $\mathcal{K}(G)$ of continuous functions with compact support, by $\varphi_G(\int_G f(s)\lambda_G(s)ds) = f(e)$, where e is the neutral element of G. Any locally compact quantum group which is symmetric (i.e. such that $\varsigma \circ \Gamma = \Gamma$) is of that type.

Let (A, Γ) be a *compact quantum group*, that is, A is a unital C^*-algebra and Γ is a coassociative coproduct from A to $A \otimes_{\min} A$ satisfying the cancellation property, i.e., $(A \otimes_{\min} 1)\Gamma(A)$ and $(1 \otimes_{\min} A)\Gamma(A)$ are dense in $A \otimes_{\min} A$ [50]. Then, there exists a left- and right-invariant state ω on A, and we can always restrict to the case when ω is faithful. Moreover, Γ extends to a normal $*$-homomorphism from $\pi_\omega(A)''$ to the (von Neumann) tensor product $\pi_\omega(A)'' \otimes \pi_\omega(A)''$, which we shall still denote by Γ, for simplification, and ω can be extended to a normal faithful state on $\pi_\omega(A)''$, we shall still denote ω for simplification. Then, $(\pi_\omega(A)'', \Gamma, \omega, \omega)$ is a locally compact quantum group, which we shall call the von Neumann version of (A, Γ). Its dual is called a discrete quantum group.

2.2. Left actions of a locally compact quantum group. A *left action* of a locally compact quantum group $\mathbb{G}$ on a von Neumann algebra N is an injective unital $*$-homomorphism $\mathfrak{a}$ from N into the von Neumann tensor product $M \otimes N$ such that

$$(\mathrm{id} \otimes \mathfrak{a})\mathfrak{a} = (\Gamma \otimes \mathrm{id})\mathfrak{a},$$

where id means the identity on M or on N as well [41, 1.1].

We shall denote by $N^{\mathfrak{a}}$ the sub-algebra of N such that $x \in N^{\mathfrak{a}}$ if and only if $\mathfrak{a}(x) = 1 \otimes x$ [41, .2]. If $N^{\mathfrak{a}} = \mathbb{C}$, the action $\mathfrak{a}$ is called *ergodic*. The formula $T_{\mathfrak{a}} = (\varphi \circ R \otimes \mathrm{id})\mathfrak{a}$ defines a normal faithful operator-valued weight from N onto $N^{\mathfrak{a}}$. We shall say that $\mathfrak{a}$ is *integrable* if and only if this operator-valued weight is semi-finite [41, 1.3, 1.4].

To any left action is associated [41, 2.1] a *crossed product* $\mathbb{G} \ltimes_{\mathfrak{a}} N = (\mathfrak{a}(N) \cup \widehat{M} \otimes \mathbb{C})''$ on which $\widehat{\mathbb{G}}^{\mathrm{o}}$ acts canonically by a left action $\tilde{\mathfrak{a}}$, called the *dual action* [41, 2.2], as follows:

$$\tilde{\mathfrak{a}}(X) = (\widehat{W}^{\mathrm{o}*} \otimes 1)(1 \otimes X)(\widehat{W}^{\mathrm{o}} \otimes 1) \quad \text{for all } X \in \mathbb{G} \ltimes_{\mathfrak{a}} N;$$

in particular, for any $x \in N$ and $y \in \widehat{M}$,

$$\tilde{\mathfrak{a}}(\mathfrak{a}(x)) = 1 \otimes \mathfrak{a}(x), \quad \tilde{\mathfrak{a}}(y \otimes 1) = \widehat{\Gamma}^{\mathrm{o}}(y) \otimes 1.$$

Moreover, we have $(\mathbb{G} \ltimes_{\mathfrak{a}} N)^{\tilde{\mathfrak{a}}} = \mathfrak{a}(N)$ [41, 2.7].

The operator-valued weight $T_{\tilde{\mathfrak{a}}} = (\widehat{\varphi} \otimes \mathrm{id}) \circ \tilde{\mathfrak{a}}$ is semi-finite [41, 2.5], which allows, for any normal faithful semi-finite weight ν on N, to define a lifted or *dual* normal faithful semi-finite weight $\tilde{\nu}$ on $\mathbb{G} \ltimes_{\mathfrak{a}} N$ by $\tilde{\nu} = \nu \circ \mathfrak{a}^{-1} \circ T_{\tilde{\mathfrak{a}}}$ [41, 3.1]. The Hilbert space $H_{\tilde{\nu}}$ is canonically isomorphic to (and will be identified with) the Hilbert tensor product $H \otimes H_\nu$ [41, 3.4 and 3.10], and this isomorphism identifies, for $x \in \mathfrak{N}_\nu$ and $y \in \mathfrak{N}_{\widehat{\varphi}}$, the vector $\Lambda_{\tilde{\nu}}((y \otimes 1)\mathfrak{a}(x))$ with $\Lambda_{\widehat{\varphi}}(y) \otimes \Lambda_\nu(x)$. Moreover, for any $X \in \mathfrak{N}_{\tilde{\nu}}$, there exists a family of operators X_i of the form $X_i = \Sigma_j (y_{i,j} \otimes 1)\mathfrak{a}(x_{i,j})$, such that X_i is weakly converging to X and $\Lambda_{\tilde{\nu}}(X_i)$ is converging to $\Lambda_{\tilde{\nu}}(X)$ [41, 3.4 and 3.10].

Then

$$U^{\mathfrak{a}}_\nu = J_{\tilde{\nu}}(\widehat{J} \otimes J_\nu)$$

is a unitary which belongs to $M \otimes B(H_\nu)$, satisfies $(\Gamma \otimes \mathrm{id})(U^{\mathfrak{a}}_\nu) = (U^{\mathfrak{a}}_\nu)_{23}(U^{\mathfrak{a}}_\nu)_{13}$ and implements $\mathfrak{a}$ in the sense that $\mathfrak{a}(x) = U^{\mathfrak{a}}_\nu(1 \otimes x)(U^{\mathfrak{a}}_\nu)^*$ for all $x \in N$ [41, 3.6, 3.7 and 4.4]. The operator $U^{\mathfrak{a}}_\nu$ is called *the canonical implementation* of $\mathfrak{a}$ on H_ν. Moreover, we have, trivially, $(U^{\mathfrak{a}}_\nu)^* = (\widehat{J} \otimes J_\nu)J_{\tilde{\nu}} = (\widehat{J} \otimes J_\nu)U^{\mathfrak{a}}_\nu(\widehat{J} \otimes J_\nu)$, and we get that

$$J_{\tilde{\nu}}\Lambda_{\tilde{\nu}}((y \otimes 1)\mathfrak{a}(x)) = U^{\mathfrak{a}}_\nu(\widehat{J}\Lambda_{\widehat{\varphi}}(y) \otimes J_\nu\Lambda_\nu(x)).$$

If we take another normal faithful semi-finite weight ψ on N, there exists a unitary u from H_ν onto H_ψ which intertwines the standard representations π_ν and π_ψ, and we have then $U^{\mathfrak{a}}_\psi = (1 \otimes u)U^{\mathfrak{a}}_\nu(1 \otimes u^*)$ [41, 4.1].

The application $(\varsigma \otimes \mathrm{id})(\mathrm{id} \otimes \mathfrak{a})$ is a left action of $\mathbb{G}$ on $B(H) \otimes N$. Moreover, in the proof of [41, 4.4], we find that $(\sigma \otimes \mathrm{id})U^{(\varsigma\otimes\mathrm{id})(\mathrm{id}\otimes\mathfrak{a})}_{Tr\otimes\nu}(\sigma \otimes \mathrm{id}) = 1 \otimes U^{\mathfrak{a}}_\nu$, where σ is the flip from $H \otimes H_\nu$ to $H_\nu \otimes H$, or vice versa.

A *right action* of a locally compact quantum $\mathbb{G}$ on a von Neumann algebra N is an injective unital $*$-homomorphism $\mathfrak{a}$ from N into the von Neumann tensor product $N \otimes M$ such that

$$(\mathfrak{a} \otimes \mathrm{id})\mathfrak{a} = (\mathrm{id} \otimes \Gamma)\mathfrak{a}.$$

Then, $\varsigma\mathfrak{a}$ is a left action of $\mathbb{G}^{\mathrm{o}}$ on N (where ς is the flip from $N \otimes M$ onto $M \otimes N$).

In [54, 2.4] and [2, Appendix] is defined, for any normal faithful semi-finite weight ν on N and $t \in \mathbb{R}$, the *Radon–Nykodym derivative*

$$(D\nu \circ \mathfrak{a} : D\nu)_t = \Delta^{it}_{\tilde{\nu}}(\widehat{\Delta}^{-it} \otimes \Delta^{-it}_\nu).$$

This unitary, denoted D_t for simplification, belongs to $M \otimes N$ and

$$(\Gamma \otimes \mathrm{id})(D_t) = (\mathrm{id} \otimes \mathfrak{a})(D_t)(1 \otimes D_t),$$

([2, 10.3] or [53, 3.4] and [54, 3.7]). Moreover, it is straightforward to get

$$D_{t+s} = D_t(\tau_t \otimes \sigma^\nu_t)(D_s) = D_s(\tau_s \otimes \sigma^\nu_s)(D_t).$$

2.3. Drinfel'd double of a locally compact quantum group. Let $\mathbb{G} = (M, \Gamma, \varphi, \varphi \circ R)$ be a locally compact quantum group, $\widehat{\mathbb{G}} = (\widehat{M}, \widehat{\Gamma}, \widehat{\varphi}, \widehat{\varphi} \circ \widehat{R})$ its dual, then it is possible to construct [2, 27, 52] another locally compact quantum group

$$D(\mathbb{G}) = (M \otimes \widehat{M}, \Gamma_D, \varphi \otimes \widehat{\varphi} \circ \widehat{R}, \varphi \otimes \widehat{\varphi} \circ \widehat{R}),$$

called the *Drinfel'd double* of $\mathbb{G}$, where Γ_D is defined by

$$\Gamma_D(x \otimes y) = \mathrm{Ad}(1 \otimes \sigma W \otimes 1)(\Gamma(x) \otimes \widehat{\Gamma}(y))$$

for all $x \in M$, $y \in \widehat{M}$. Here and throughout this paper, given a unitary U on a Hilbert space $\mathfrak{H}$, we denote by $\mathrm{Ad}(U)$ the automorphism of $B(\mathfrak{H})$ defined as usual by $x \mapsto UxU^*$ for all $x \in B(\mathfrak{H})$.

The co-inverse R_D of $D(\mathbb{G})$ is given by

$$R_D(x \otimes y) = \mathrm{Ad}(W^*)(R(x) \otimes \widehat{R}(y)).$$

This locally compact quantum group is always unimodular, which means that the left-invariant weight is also right-invariant. In the sense of [42, 2.9], $\widehat{\mathbb{G}}$ and $\mathbb{G}$ are closed quantum subgroups of $\widehat{D(\mathbb{G})}$, which means that the injection of $\widehat{M}$ (resp. M) into the underlying von Neumann algebra of its dual $\widehat{D(\mathbb{G})}$ preserve the coproduct. (See 7.4.1 for more details about this definition.)

2.4. Yetter–Drinfel'd algebras. Let $\mathbb{G} = (M, \Gamma, \varphi, \varphi \circ R)$ be a locally compact quantum group and $\widehat{\mathbb{G}} = (\widehat{M}, \widehat{\Gamma}, \widehat{\varphi}, \widehat{\varphi} \circ \widehat{R})$ its dual. A $\mathbb{G}$-*Yetter–Drinfel'd algebra* [28] is a von Neumann algebra N with a left action $\mathfrak{a}$ of $\mathbb{G}$ and a left action $\widehat{\mathfrak{a}}$ of $\widehat{\mathbb{G}}$ such that

$$(\mathrm{id} \otimes \mathfrak{a})\widehat{\mathfrak{a}}(x) = \mathrm{Ad}(\sigma W \otimes 1)(\mathrm{id} \otimes \widehat{\mathfrak{a}})\mathfrak{a}(x) \quad \text{for all } x \in N.$$

One should remark that if $(N, \mathfrak{a}, \widehat{\mathfrak{a}})$ is a $\mathbb{G}$-Yetter–Drinfel'd algebra, then $(N, \widehat{\mathfrak{a}}, \mathfrak{a})$ is a $\widehat{\mathbb{G}}$-Yetter–Drinfel'd algebra.

If B is a von Neumann sub-algebra of N such that $\mathfrak{a}(B) \subset M \otimes B$ and $\widehat{\mathfrak{a}}(B) \subset \widehat{M} \otimes B$, then, it is clear that the restriction $\mathfrak{a}_{|B}$ (resp. $\widehat{\mathfrak{a}}_{|B}$) is a left action of $\mathbb{G}$ (resp. $\widehat{\mathbb{G}}$) on B, and that $(B, \mathfrak{a}_{|B}, \widehat{\mathfrak{a}}_{|B})$ is a Yetter–Drinfel'd algebra, which we shall call a sub-$\mathbb{G}$-Yetter–Drinfel'd algebra of $(N, \mathfrak{a}, \widehat{\mathfrak{a}})$.

2.4.1 Theorem ([28, 3.2]). *Let $\mathbb{G} = (M, \Gamma, \varphi, \varphi \circ R)$ be a locally compact quantum group, $\widehat{\mathbb{G}} = (\widehat{M}, \widehat{\Gamma}, \widehat{\varphi}, \widehat{\varphi} \circ \widehat{R})$ its dual, $D(\mathbb{G})$ its Drinfel'd double and N a von Neumann algebra equipped with a left action $\mathfrak{a}$ of $\mathbb{G}$ and a left action $\widehat{\mathfrak{a}}$ of $\widehat{\mathbb{G}}$. Then the following conditions are equivalent:*

(i) $(N, \mathfrak{a}, \widehat{\mathfrak{a}})$ *is a $\mathbb{G}$-Yetter–Drinfel'd algebra;*

(ii) $(\mathrm{id} \otimes \widehat{\mathfrak{a}})\mathfrak{a}$ *is a left action of $D(\mathbb{G})$ on N.*

2.4.2 Theorem ([28, 3.2]). *Let $\mathbb{G} = (M, \Gamma, \varphi, \varphi \circ R)$ be a locally compact quantum group, $\widehat{\mathbb{G}} = (\widehat{M}, \widehat{\Gamma}, \widehat{\varphi}, \widehat{\varphi} \circ \widehat{R})$ its dual, $D(\mathbb{G})$ its Drinfeld's double and $\mathfrak{a}_D$ a left action of $D(\mathbb{G})$ on a von Neumann algebra N. Then there exist a left action $\mathfrak{a}$ of $\mathbb{G}$ on N and a left action $\widehat{\mathfrak{a}}$ of $\widehat{\mathbb{G}}$ on N such that $\mathfrak{a}_D = (\mathrm{id} \otimes \widehat{\mathfrak{a}})\mathfrak{a}$. These actions are determined by the conditions*

$$(\mathrm{id} \otimes \mathrm{id} \otimes \mathfrak{a})\mathfrak{a}_D = \mathrm{Ad}(1 \otimes \sigma W \otimes 1)(\Gamma \otimes \mathrm{id} \otimes \mathrm{id})\mathfrak{a}_D,$$
$$(\mathrm{id} \otimes \mathrm{id} \otimes \widehat{\mathfrak{a}})\mathfrak{a}_D = (\mathrm{id} \otimes \widehat{\Gamma} \otimes \mathrm{id})\mathfrak{a}_D,$$

and $(N, \mathfrak{a}, \widehat{\mathfrak{a}})$ is a $\mathbb{G}$-Yetter–Drinfel'd algebra.

2.4.3 Proposition. *With the notation of 2.4.2, we have $N^{\mathfrak{a}_D} = N^{\mathfrak{a}} \cap N^{\widehat{\mathfrak{a}}}$.*

Proof. As $\mathfrak{a}_D = (\mathrm{id} \otimes \widehat{\mathfrak{a}})\mathfrak{a}$, we get that $N^{\mathfrak{a}} \cap N^{\widehat{\mathfrak{a}}} \subset N^{\mathfrak{a}_D}$. On the other hand, using the formula $(\mathrm{id} \otimes \mathrm{id} \otimes \widehat{\mathfrak{a}})\mathfrak{a}_D = (\mathrm{id} \otimes \widehat{\Gamma} \otimes \mathrm{id})\mathfrak{a}_D$, we get that every $x \in N^{\mathfrak{a}_D}$ belongs to $N^{\widehat{\mathfrak{a}}}$. Moreover, using the formula $(\mathrm{id} \otimes \mathrm{id} \otimes \mathfrak{a})\mathfrak{a}_D = \mathrm{Ad}(1 \otimes \sigma W \otimes 1)(\Gamma \otimes \mathrm{id} \otimes \mathrm{id})\mathfrak{a}_D$, we then get that every $x \in N^{\mathfrak{a}_D}$ also belongs to $N^{\mathfrak{a}}$. □

2.4.4 Proposition. *Let $\mathbb{G} = (M, \Gamma, \varphi, \varphi \circ R)$ be a locally compact quantum group, $\widehat{\mathbb{G}} = (\widehat{M}, \widehat{\Gamma}, \widehat{\varphi}, \widehat{\varphi} \circ \widehat{R})$ its dual, $(N, \mathfrak{a}, \widehat{\mathfrak{a}})$ a $\mathbb{G}$-Yetter–Drinfel'd algebra and ν a normal faithful semi-finite weight on N. Let $t \in \mathbb{R}$, $D_t = (D\nu \circ \mathfrak{a} : D\nu)_t$ and $\widehat{D}_t = (D\nu \circ \widehat{\mathfrak{a}} : D\nu)_t$. Then*

$$\mathrm{Ad}(\sigma W \otimes 1)[(\mathrm{id} \otimes \widehat{\mathfrak{a}})(D_t)(1 \otimes \widehat{D}_t)] = (\mathrm{id} \otimes \mathfrak{a})(\widehat{D}_t)(1 \otimes D_t),$$

and if $\tilde{\nu}$ and $\widetilde{\widehat{\nu}}$ denote the weights on $\mathbb{G} \ltimes_{\mathfrak{a}} N$ and $\widehat{\mathbb{G}} \ltimes_{\widehat{\mathfrak{a}}} N$, respectively, dual to ν, then

$$\mathrm{Ad}(\sigma W \otimes 1)[(\mathrm{id} \otimes \widehat{\mathfrak{a}})(D_t)(\widehat{\Delta}^{it} \otimes \Delta_{\widetilde{\widehat{\nu}}}^{it})] = (\mathrm{id} \otimes \mathfrak{a})(\widehat{D}_t)(\Delta^{it} \otimes \Delta_{\tilde{\nu}}^{it}).$$

Proof. As $(\tau_t \otimes \widehat{\tau}_t)(W) = W$ for all $t \in \mathbb{R}$, the first equation is a straightforward application of [2, 10.4]. The second one follows easily using the relations

$$\begin{aligned}(\widehat{\Delta}^{it} \otimes \Delta_{\widetilde{\widehat{\nu}}}^{it})(W^*\sigma \otimes 1) &= (1 \otimes \widehat{D}_t)(\widehat{\Delta}^{it} \otimes \Delta^{it} \otimes \Delta_{\nu}^{it})(W^*\sigma \otimes 1) \\ &= (1 \otimes \widehat{D}_t)(W^*\sigma \otimes 1)(\Delta^{it} \otimes \widehat{\Delta}^{it} \otimes \Delta_{\nu}^{it})\end{aligned}$$

and $D_t(\widehat{\Delta}^{it} \otimes \Delta_{\nu}^{it}) = \Delta_{\tilde{\nu}}^{it}$. □

2.4.5. Basic example and De Commer's construction [7]. We can consider the coproduct Γ_D of $D(\mathbb{G})$ as a left action of $D(\mathbb{G})$ on $M \otimes \widehat{M}$. Using 2.4.1, we get that there exist a left action $\mathfrak{b}$ of $\mathbb{G}$ on $M \otimes \widehat{M}$ and a left action $\widehat{\mathfrak{b}}$ of $\widehat{\mathbb{G}}$ on $M \otimes \widehat{M}$ such that $\Gamma_D = (\mathrm{id} \otimes \widehat{\mathfrak{b}})\mathfrak{b}$. We easily obtain that for all $X \in M \otimes \widehat{M}$,

$$\mathfrak{b}(X) = (\Gamma \otimes \mathrm{id})(X), \quad \widehat{\mathfrak{b}}(X) = \mathrm{Ad}(\sigma W \otimes 1)[(\mathrm{id} \otimes \widehat{\Gamma})(X)].$$

Therefore, $\mathfrak{b}$ and $\widehat{\mathfrak{b}}$ appear as the actions associated by [7, 6.5.2] to the closed quantum subgroups $\mathbb{G}$ and $\widehat{\mathbb{G}}$ of $\widehat{D(\mathbb{G})}$.

De Commer's construction allows us to make a link between this basic example and any Yetter–Drinfel'd algebra; namely, if $(N, \mathfrak{a}, \widehat{\mathfrak{a}})$ is a Yetter–Drinfel'd algebra, let us define $\mathfrak{a}_D = (\mathrm{id} \otimes \widehat{\mathfrak{a}})\mathfrak{a}$ the left action of $D(\mathbb{G})$ on N, and, given a normal, semi-finite faithful weight ν on N, let $U_\nu^{\mathfrak{a}_D}$, $U_\nu^{\mathfrak{a}}$, $U_\nu^{\widehat{\mathfrak{a}}}$ be the canonical implementation of $\mathfrak{a}_D$, $\mathfrak{a}$, $\widehat{\mathfrak{a}}$. In the sense of De Commer, $\mathfrak{a}$ and $\widehat{\mathfrak{a}}$ are "restrictions" (to $\mathbb{G}$ and $\widehat{\mathbb{G}}$) of $\mathfrak{a}_D$ and, using [7, 6.5.3 and 6.5.4], we get that

$$(\mathfrak{b} \otimes \mathrm{id})(U_\nu^{\mathfrak{a}_D}) = (U_\nu^{\mathfrak{a}})_{14}(U_\nu^{\mathfrak{a}_D})_{234}, \quad (\widehat{\mathfrak{b}} \otimes \mathrm{id})(U_\nu^{\mathfrak{a}_D}) = (U_\nu^{\widehat{\mathfrak{a}}})_{14}(U_\nu^{\mathfrak{a}_D})_{234}.$$

In particular,

$$\begin{aligned}(U_\nu^{\mathfrak{a}_D})_{125}(U_\nu^{\mathfrak{a}_D})_{345} &= (\Gamma_D \otimes \mathrm{id})(U_\nu^{\mathfrak{a}_D}) \\ &= (\mathrm{id} \otimes \widehat{\mathfrak{b}} \otimes \mathrm{id})(\mathfrak{b} \otimes \mathrm{id})(U_\nu^{\mathfrak{a}_D}) \\ &= (\mathrm{id} \otimes \widehat{\mathfrak{b}} \otimes \mathrm{id})[(U_\nu^{\mathfrak{a}})_{14}(U_\nu^{\mathfrak{a}_D})_{234}] = (U_\nu^{\mathfrak{a}})_{15}(U_\nu^{\widehat{\mathfrak{a}}})_{25}(U_\nu^{\mathfrak{a}_D})_{345},\end{aligned}$$

whence $U_\nu^{\mathfrak{a}_D} = (U_\nu^{\mathfrak{a}})_{23}(U_\nu^{\widehat{\mathfrak{a}}})_{13}$. As this result depends on an unpublished part of [7], we shall give a different proof of this formula in 3.8, using the techniques of invariant weights, and then give several technical corollaries of this fact which will be used throughout this paper.

2.5. Braided-commutativity of Yetter–Drinfel'd algebras.

2.5.1 Definition. Let $\mathbb{G}$ be a locally compact quantum group and $\mathfrak{a}$ a left action of $\mathbb{G}$ on a von Neumann algebra N. For any $x \in N$, let us define

$$\begin{aligned}\mathfrak{a}^{\mathrm{c}}(x^{\mathrm{o}}) &= (j \otimes .^{\mathrm{o}})\mathfrak{a}(x) = \mathrm{Ad}(J \otimes J_\nu)[\mathfrak{a}(x)^*], \\ \mathfrak{a}^{\mathrm{o}}(x^{\mathrm{o}}) &= (R \otimes .^{\mathrm{o}})\mathfrak{a}(x) = \mathrm{Ad}(\widehat{J} \otimes J_\nu)[\mathfrak{a}(x)^*].\end{aligned}$$

Then $\mathfrak{a}^{\mathrm{c}}$ is a left action of $\mathbb{G}^{\mathrm{c}}$ on N^{o}, and $\mathfrak{a}^{\mathrm{o}}$ is a left action of $\mathbb{G}^{\mathrm{o}}$ on N^{o}.

Let ν be a normal semi-finite faithful weight on N and ν^{o} the normal semi-finite faithful weight on N^{o} defined by $\nu^{\mathrm{o}}(x^{\mathrm{o}}) = \nu(x)$ for any $x \in N^+$. Let $D_t = (D\nu \circ \mathfrak{a} : D\nu)_t$, $D_t^{\mathrm{o}} = (D\nu^{\mathrm{o}} \circ \mathfrak{a}^{\mathrm{o}} : D\nu^{\mathrm{o}})_t$, which belongs to $M \otimes N^{\mathrm{o}}$, and $D_t^{\mathrm{c}} = D(\nu^{\mathrm{o}} \circ \mathfrak{a}^{\mathrm{c}} : D\nu^{\mathrm{o}})_t$, which belongs to $M' \otimes N^{\mathrm{o}}$. Then for all $t \in \mathbb{R}$,

$$D_{-t}^{\mathrm{o}} = \mathrm{Ad}(\widehat{J} \otimes J_\nu)[D_t], \quad D_{-t}^{\mathrm{c}} = \mathrm{Ad}(J \otimes J_\nu)[D_t].$$

2.5.2 Lemma. *Let $\mathbb{G}$ be a locally compact quantum group, $\mathfrak{a}$ a left action of $\mathbb{G}$ on a von Neumann algebra N, ν a normal faithful semi-finite weight on N, and $U_\nu^{\mathfrak{a}}$ the standard implementation of $\mathfrak{a}$. Then:*

(i) $(\mathbb{G} \ltimes_{\mathfrak{a}} N)' = U_\nu^{\mathfrak{a}}(\mathbb{G}^{\mathrm{o}} \ltimes_{\mathfrak{a}^{\mathrm{o}}} N^{\mathrm{o}})(U_\nu^{\mathfrak{a}})^*$;

(ii) $(U_\nu^{\mathfrak{a}})^*$ *is the standard implementation of the left action* $\mathfrak{a}^{\mathrm{o}}$ *on* N^{o} *with respect to the opposite weight* ν^{o}. *In particular,* $(U_\nu^{\mathfrak{a}})^*$ *is a representation of* $\mathbb{G}^{\mathrm{o}}$ *and* $\mathfrak{a}^{\mathrm{o}}(x^{\mathrm{o}}) = (U_\nu^{\mathfrak{a}})^*(1 \otimes x^{\mathrm{o}})U_\nu^{\mathfrak{a}}$ *for all* $x \in N$.

(iii) $\Delta_{\tilde\nu}^{it} U_\nu^{\mathfrak{a}} = U_\nu^{\mathfrak{a}} D_{-t}^{\mathrm{o}}(\widehat{\Delta}^{it} \otimes \Delta_\nu^{it})$ *and* $\mathrm{Ad}(\widehat{\Delta}^{it} \otimes \Delta_\nu^{it})[(U_\nu^{\mathfrak{a}})^*] = (D_{-t}^{\mathrm{o}})^*(U_\nu^{\mathfrak{a}})^* D_t$ *for all* $t \in \mathbb{R}$.

Proof. (i) The relation $U_\nu^{\mathfrak{a}} = J_{\tilde\nu}(\widehat{J} \otimes J_\nu)$ and the definition of the crossed products imply

$$\begin{aligned} U_\nu^{\mathfrak{a}}(\mathbb{G}^{\mathrm{o}} \ltimes_{\mathfrak{a}^{\mathrm{o}}} N^{\mathrm{o}})(U_\nu^{\mathfrak{a}})^* &= J_{\tilde\nu}(\widehat{J} \otimes J_\nu)((\widehat{J}\widehat{M}\widehat{J} \otimes 1_{H_\nu}) \cup \mathfrak{a}^{\mathrm{o}}(N^{\mathrm{o}}))''(\widehat{J} \otimes J_\nu)J_{\tilde\nu} \\ &= J_{\tilde\nu}(\mathbb{G} \ltimes_{\mathfrak{a}} N)J_{\tilde\nu} \\ &= (\mathbb{G} \ltimes_{\mathfrak{a}} N)'. \end{aligned}$$

(ii) Denote by μ the weight on $\mathbb{G}^{\mathrm{o}} \ltimes_{\mathfrak{a}^{\mathrm{o}}} N^{\mathrm{o}}$ dual to ν^{o}. By §3 in [41], there exists a GNS-map $\Lambda_\mu \colon \mathfrak{N}_\mu \to H \otimes H_\nu$ determined by

$$\Lambda_\mu((\widehat{J} y \widehat{J} \otimes 1_{H_\nu})\mathfrak{a}^{\mathrm{o}}(x^{\mathrm{o}})^*) = \widehat{J}\widehat{\Lambda}(y) \otimes J_\nu \Lambda_\nu(x) \tag{1}$$

for all $y \in \mathfrak{N}_{\widehat{\phi}}$ and $x \in \mathfrak{N}_\nu$, and the standard implementation $U_{\nu^{\mathrm{o}}}^{\mathfrak{a}^{\mathrm{o}}}$ of $\mathfrak{a}^{\mathrm{o}}$ with respect to ν^{o} is given by $U_{\nu^{\mathrm{o}}}^{\mathfrak{a}^{\mathrm{o}}} = J_\mu(\widehat{J} \otimes J_\nu)$.

On the other hand, the GNS-map $\Lambda_{\tilde\nu}$ for the dual weight $\tilde\nu$ yields a GNS-map $\Lambda_{\tilde\nu^{\mathrm{o}}}$ for the opposite $\tilde\nu^{\mathrm{o}}$ on the commutant $J_{\tilde\nu}(M \ltimes_{\mathfrak{a}} N)J_{\tilde\nu}$, determined by

$$\Lambda_{\tilde\nu^{\mathrm{o}}}(J_{\tilde\nu}(y \otimes 1)\mathfrak{a}(x)J_{\tilde\nu}) = J_{\tilde\nu}\Lambda_{\tilde\nu}((y \otimes 1)\mathfrak{a}(x)) = J_{\tilde\nu}(\widehat{\Lambda}(y) \otimes \Lambda_\nu(x)) \tag{2}$$

for $y \in \mathfrak{N}_{\widehat{\phi}}$ and $x \in \mathfrak{N}_\nu$.

Comparing (1) with (2) and using the relation $U_\nu^{\mathfrak{a}} = J_{\tilde\nu}(\widehat{J} \otimes J_\nu)$, we can conclude that

$$\Lambda_\mu((U_\nu^{\mathfrak{a}})^* a U_\nu^{\mathfrak{a}}) = (U_\nu^{\mathfrak{a}})^* \Lambda_{\tilde\nu^{\mathrm{o}}}(a)$$

for all $a \in \mathfrak{N}_{\tilde\nu^{\mathrm{o}}}$. Consequently, $J_\mu = (U_\nu^{\mathfrak{a}})^* J_{\tilde\nu} U_\nu^{\mathfrak{a}}$ and $U_{\nu^{\mathrm{o}}}^{\mathfrak{a}^{\mathrm{o}}} = J_\mu(\widehat{J} \otimes J_\nu) = (U_\nu^{\mathfrak{a}})^*$.

(iii) Using 2.2, we have:

$$\begin{aligned} \Delta_{\tilde\nu}^{it} U_\nu^{\mathfrak{a}}(\widehat{\Delta}^{-it} \otimes \Delta_\nu^{-it}) &= \Delta_{\tilde\nu}^{it} J_{\tilde\nu}(\widehat{J} \otimes J_\nu)(\widehat{\Delta}^{-it} \otimes \Delta_\nu^{-it}) \\ &= J_{\tilde\nu}\Delta_{\tilde\nu}^{it}(\widehat{J} \otimes J_\nu)(\widehat{\Delta}^{-it} \otimes \Delta_\nu^{-it}) \\ &= J_{\tilde\nu} D_t(\widehat{\Delta}^{it} \otimes \Delta_\nu^{it})(\widehat{J} \otimes J_\nu)(\widehat{\Delta}^{-it} \otimes \Delta_\nu^{-it}) \\ &= J_{\tilde\nu}(\widehat{J} \otimes J_\nu)D_{-t}^{\mathrm{o}} \\ &= U_\nu^{\mathfrak{a}} D_{-t}^{\mathrm{o}} \end{aligned}$$

from which we get the first formula, and then the second one by taking the adjoints. □

2.5.3 Definition. Let $\mathbb{G}$ be a locally compact quantum group and $(N, \mathfrak{a}, \widehat{\mathfrak{a}})$ a $\mathbb{G}$-Yetter–Drinfel'd algebra. Since $\mathrm{Ad}(\widehat{J}J) = \mathrm{Ad}(J\widehat{J})$, the following two properties are equivalent:

(i) $\mathfrak{a}^{\mathrm{c}}(N^{\mathrm{o}})$ and $\widehat{\mathfrak{a}}^{\mathrm{c}}(N^{\mathrm{o}})$ commute;

(ii) $\mathfrak{a}^{\mathrm{o}}(N^{\mathrm{o}})$ and $\widehat{\mathfrak{a}}^{\mathrm{o}}(N^{\mathrm{o}})$ commute;

We shall say that $(N, \mathfrak{a}, \widehat{\mathfrak{a}})$ is *braided-commutative* if these conditions are fulfilled.

It is clear that any sub-$\mathbb{G}$-Yetter–Drinfel'd algebra of a braided-commutative $\mathbb{G}$-Yetter–Drinfel'd algebra is also braided-commutative.

2.5.4 Theorem ([40]). *Let $\mathbb{G}$ be a locally compact quantum group, $(N, \mathfrak{a}, \widehat{\mathfrak{a}})$ a $\mathbb{G}$-Yetter–Drinfel'd algebra, ν a normal faithful semi-finite weight on N, and $U_\nu^{\mathfrak{a}}$ the standard implementation of $\mathfrak{a}$. Define an injective anti-$*$-homomorphism β by*

$$\beta(x) = U_\nu^{\mathfrak{a}}\widehat{\mathfrak{a}}^{\mathrm{o}}(x^{\mathrm{o}})(U_\nu^{\mathfrak{a}})^* = \mathrm{Ad}(U_\nu^{\mathfrak{a}}(U_\nu^{\widehat{\mathfrak{a}}})^*)[1 \otimes J_\nu x^* J_\nu] \quad \textit{for all } x \in N.$$

Then:

(i) *$\beta(N)$ commutes with $\mathfrak{a}(N)$.*

(ii) *$(N, \mathfrak{a}, \widehat{\mathfrak{a}})$ is braided-commutative if and only if $\beta(N) \subset \mathbb{G} \ltimes_{\mathfrak{a}} N$.*

Proof. (i) The two formulas for $\beta(x)$ coincide by Lemma 2.5.2 (ii), and clearly, $\beta(N) \subseteq U_\nu^{\mathfrak{a}}(\widehat{M} \otimes N^{\mathrm{o}})(U_\nu^{\mathfrak{a}})^*$ commutes with $\mathfrak{a}(N) = U_\nu^{\mathfrak{a}}(1 \otimes N)(U_\nu^{\mathfrak{a}})^*$.

(ii) Using Lemma 2.5.2 (i), we see that $\beta(N) = U_\nu^{\mathfrak{a}}\widehat{\mathfrak{a}}^{\mathrm{o}}(N^{\mathrm{o}})(U_\nu^{\mathfrak{a}})^*$ lies in $\mathbb{G} \ltimes_{\mathfrak{a}} N$ if and only if it commutes with $(\mathbb{G} \ltimes_{\mathfrak{a}} N)' = U_\nu^{\mathfrak{a}}(\mathbb{G}^{\mathrm{o}} \ltimes_{\mathfrak{a}^{\mathrm{o}}} N^{\mathrm{o}})(U_\nu^{\mathfrak{a}})^*$, that is, if and only if $\widehat{\mathfrak{a}}^{\mathrm{o}}(N^{\mathrm{o}})$ commutes with $\widehat{J}\widehat{M}\widehat{J} \otimes 1_{H_\nu}$ and with $\mathfrak{a}^{\mathrm{o}}(N^{\mathrm{o}})$. But since $\widehat{\mathfrak{a}}^{\mathrm{o}}(N^{\mathrm{o}}) \subseteq \widehat{M} \otimes N^{\mathrm{o}}$, the first condition is always satisfied. □

2.5.5 Proposition. *Let $\mathbb{G}$ be a locally compact quantum group and $(N, \mathfrak{a}, \widehat{\mathfrak{a}})$ a braided-commutative $\mathbb{G}$-Yetter–Drinfel'd algebra. Then $N^{\mathfrak{a}} \subseteq Z(N)$ and $N^{\widehat{\mathfrak{a}}} \subseteq Z(N)$.*

Proof. Using 2.5.1, we get that the algebra $1 \otimes (N^{\mathfrak{a}})^{\mathrm{o}}$ commutes with $\widehat{\mathfrak{a}}^{\mathrm{o}}(N^{\mathrm{o}})$, and, therefore, that $1 \otimes N^{\mathfrak{a}}$ commutes with $\widehat{\mathfrak{a}}(N)$. As it commutes with $B(H) \otimes 1$, it will commute with $B(H) \otimes N$, by [41, Th. 2.6]. This is the first result. Applying it to the braided-commutative $\widehat{\mathbb{G}}$-Yetter–Drinfel'd algebra $(N, \widehat{\mathfrak{a}}, \mathfrak{a})$, we get the second result. □

3. Invariant weights on Yetter–Drinfel'd algebras

In this chapter, we recall the definition (3.1) and basic properties (3.2), (3.3) of a normal semi-finite faithful weight on a von Neumann algebra N, relatively invariant with respect to a left action $\mathfrak{a}$ of a locally compact quantum group $\mathbb{G}$ on N. Then, we study the case of an invariant weight on a Yetter–Drinfel'd algebra $(N, \mathfrak{a}, \widehat{\mathfrak{a}})$ (3.4), (3.5), and we prove that if N is properly infinite, there exists such a weight (3.10).

3.1 Definition. Let $\mathbb{G}$ be a locally compact quantum group and $\mathfrak{a}$ a left action of $\mathbb{G}$ on a von Neumann algebra N. Let k be a positive invertible operator affiliated to M. A normal faithful semi-finite weight ν on N is said to be k-*invariant* under $\mathfrak{a}$ if for all $x \in N^+$,

$$(\mathrm{id} \otimes \nu)\mathfrak{a}(x) = \nu(x)k.$$

Applying Γ to this formula, one gets $\Gamma(k) = k \otimes k$, whence k^{it} is a (one-dimensional) representation of $\mathbb{G}$ for all $t \in \mathbb{R}$. So, k^{it} belongs to the von Neumann subalgebra $I(M)$ of M generated by all unitaries u of M such that $\Gamma(u) = u \otimes u$. As $I(M)$ is globally invariant by τ_t and R, using [2, 10.5], we get that it is a locally compact quantum group, whose scaling group will be the restriction of τ_t to $I(M)$. Since this locally compact quantum group is cocommutative, we therefore get that the restriction of τ_t to $I(M)$ is trivial, from which we get that $\tau_t(k) = k$ for all $t \in \mathbb{R}$.

This property implies that P and k (resp. $\widehat{\Delta}$ and k) strongly commute. Therefore their product kP (resp. $k\widehat{\Delta}$) is closable, and its closure will be denoted again kP (resp. $k\widehat{\Delta}$).

It is proved in [54, 4.1] that ν is k-invariant if and only if, for all $t \in \mathbb{R}$, we have $(D\nu \circ \mathfrak{a} : D\nu)_t = k^{-it} \otimes 1$ (or, equivalently, $\Delta_{\tilde{\nu}}^{it} = k^{-it}\widehat{\Delta}^{it} \otimes \Delta_\nu^{it}$).

If $k = 1$, we shall say that ν is *invariant* under $\mathfrak{a}$.

3.2 Proposition. *Let $\mathbb{G}$ be a locally compact quantum group, $\mathfrak{a}$ a left action of $\mathbb{G}$ on a von Neumann algebra N, and ν_1 and ν_2 two k-invariant normal faithful semi-finite weights on N. Then $(D\nu_1 : D\nu_2)_t$ belongs to $N^{\mathfrak{a}}$ for all $t \in \mathbb{R}$.*

Proof. For $k = 1$, this result had been proved in [14, 7.8] for right actions of measured quantum groupoids. To get it for left actions of locally compact quantum groups is just a translation. The generalization for any k is left to the reader (see [41, 3.9]). □

3.3 Proposition. *Let $\mathbb{G}$ be a locally compact quantum group, $\mathfrak{a}$ a left action of $\mathbb{G}$ on a von Neumann algebra N, and ν a k-invariant faithful normal semi-finite weight on N. Then:*

(i) $\mathfrak{a}(\sigma_t^\nu(x)) = (\mathrm{Ad}\, k^{-it} \circ \tau_t \otimes \sigma_t^\nu)\mathfrak{a}(x)$ *for all $x \in N$ and $t \in \mathbb{R}$;*

(ii) *for all $x \in \mathfrak{N}_\nu$, $\xi \in \mathcal{D}(k^{-1/2})$ and $\eta \in H$, $(\omega_{k^{-1/2}\xi,\eta} \otimes \mathrm{id})\mathfrak{a}(x)$ belongs to $\mathfrak{N}_\nu$, and the canonical implementation $U_\nu^{\mathfrak{a}}$ is given by*

$$(\omega_{\xi,\eta} \otimes \mathrm{id})(U_\nu^{\mathfrak{a}})\Lambda_\nu(x) = \Lambda_\nu[(\omega_{k^{-1/2}\xi,\eta} \otimes \mathrm{id})\mathfrak{a}(x)].$$

Proof. (i) Since $\Delta_{\tilde{\nu}}^{it} = k^{-it}\widehat{\Delta}^{it} \otimes \Delta_\nu^{it}$,

$$\mathfrak{a}(\sigma_t^\nu(x)) = \sigma_t^{\tilde{\nu}}(\mathfrak{a}(x)) = (k^{-it}\widehat{\Delta}^{it} \otimes \Delta_\nu^{it})\mathfrak{a}(x)(\widehat{\Delta}^{-it}k^{it} \otimes \Delta_\nu^{-it})$$

for all $t \in \mathbb{R}$.

(ii) The first result of (ii) is proved (for $k = \delta^{-1}$) in [41, 2.4], and the general case can be proved the same way. □

3.4 Theorem. *Let* $\mathbb{G}$ *be a locally compact quantum group,* $(N, \mathfrak{a}, \widehat{\mathfrak{a}})$ *a* $\mathbb{G}$*-Yetter–Drinfel'd algebra,* $\mathfrak{a}_D = (\mathrm{id} \otimes \widehat{\mathfrak{a}})\mathfrak{a}$ *the action of* $D(\mathbb{G})$ *introduced in 2.4.1, and* ν *a faithful normal semi-finite weight on* N*. Then the following conditions are equivalent:*

(i) *the weight* ν *is invariant under* $\mathfrak{a}$ *and invariant under* $\widehat{\mathfrak{a}}$*.*

(ii) *the weight* ν *is invariant under* $\mathfrak{a}_D$*.*

Proof. The fact that (i) implies (ii) is trivial. Suppose that (ii) holds. Choose a state ω in $\widehat{M}_*$ and define $\nu' = (\omega \otimes \nu)\widehat{\mathfrak{a}}$. As $(\mathrm{id} \otimes \mathrm{id} \otimes \nu)\mathfrak{a}_D = \nu$, we get that $(\mathrm{id} \otimes \nu')\mathfrak{a} = \nu$.

But

$$\begin{aligned}(\mathrm{id} \otimes \mathrm{id} \otimes \nu')\mathfrak{a}_D &= (\mathrm{id} \otimes \mathrm{id} \otimes (\omega \otimes \nu)\widehat{\mathfrak{a}})(\mathrm{id} \otimes \widehat{\mathfrak{a}})\mathfrak{a} \\ &= (\mathrm{id} \otimes \mathrm{id} \otimes \omega \otimes \nu)(\mathrm{id} \otimes \widehat{\Gamma} \otimes \mathrm{id})(\mathrm{id} \otimes \widehat{\mathfrak{a}})\mathfrak{a},\end{aligned}$$

and, for any state ω' in $\widehat{M}_*$,

$$(\mathrm{id} \otimes \omega' \otimes \nu')\mathfrak{a}_D = (\mathrm{id} \otimes (\omega' \otimes \omega) \circ \widehat{\Gamma} \otimes \nu)\mathfrak{a}_D = \nu.$$

Therefore, by linearity, we get that $(\mathrm{id} \otimes \mathrm{id} \otimes \nu')\mathfrak{a}_D = \nu$. On the other hand,

$$\begin{aligned}(\mathrm{id} \otimes \mathrm{id} \otimes \nu')\mathfrak{a}_D &= \mathrm{Ad}(W^*\sigma)(\mathrm{id} \otimes \mathrm{id} \otimes \nu')(\mathrm{id} \otimes \mathfrak{a})\widehat{\mathfrak{a}} \\ &= \mathrm{Ad}(W^*\sigma)(\mathrm{id} \otimes (\mathrm{id} \otimes \nu')\mathfrak{a})\widehat{\mathfrak{a}} \\ &= \mathrm{Ad}(W^*\sigma)(\mathrm{id} \otimes \nu)\widehat{\mathfrak{a}}\end{aligned}$$

But, as $(\mathrm{id} \otimes \mathrm{id} \otimes \nu')\mathfrak{a}_D = \nu$, we get that $\nu = (\mathrm{id} \otimes \nu)\widehat{\mathfrak{a}}$, and, therefore, ν is invariant under $\widehat{\mathfrak{a}}$. So, we get that $\nu' = \nu$, and ν is invariant under $\mathfrak{a}$. □

3.5 Definition. Let $\mathbb{G}$ be a locally compact quantum group and $(N, \mathfrak{a}, \widehat{\mathfrak{a}})$ a $\mathbb{G}$-Yetter–Drinfel'd algebra. A normal faithful semi-finite weight on N will be called *Yetter–Drinfel'd invariant* if it satisfies one of the equivalent conditions of 3.4.

3.6 Theorem. *Let* $\mathbb{G}$ *be a locally compact quantum group,* $(N, \mathfrak{a}, \widehat{\mathfrak{a}})$ *a* $\mathbb{G}$*-Yetter–Drinfel'd algebra and* $\mathfrak{a}_D = (\mathrm{id} \otimes \widehat{\mathfrak{a}})\mathfrak{a}$ *the action of* $D(\mathbb{G})$ *introduced in 2.4.1. If* $\mathfrak{a}_D$ *is integrable, then there exists a Yetter–Drinfel'd invariant normal faithful semi-finite weight on* N*.*

Proof. Clear by [41, 2.5], using the fact that the locally compact quantum group $D(\mathbb{G})$ is unimodular. □

3.7 Corollary. *Let $\mathbb{G} = (M, \Gamma, \varphi, \psi)$ be a locally compact quantum group and $(N, \mathfrak{a}, \widehat{\mathfrak{a}})$ a $\mathbb{G}$-Yetter–Drinfel'd algebra. Denote by H the Hilbert space $L^2(M) = L^2(\widehat{M})$. Then $(B(H) \otimes N, (\varsigma \otimes \mathrm{id})(\mathrm{id} \otimes \alpha), (\varsigma \otimes \mathrm{id})(\mathrm{id} \otimes \widehat{\alpha}))$ is a $\mathbb{G}$-Yetter–Drinfel'd algebra which has a normal semi-finite faithful Yetter–Drinfel'd invariant weight.*

Proof. Let $\mathfrak{a}_D = (\mathrm{id} \otimes \widehat{\mathfrak{a}})\mathfrak{a}$ be the action of $D(\mathbb{G})$ introduced in 2.4.1. Using [41, 2.6], we know that the action $(\varsigma \otimes \mathrm{id})(\mathrm{id} \otimes \mathfrak{a}_D)$ is a left action of $D(\mathbb{G})$ which is cocycle-equivalent to the bidual action of $\mathfrak{a}_D$. As this bidual action is integrable [41, 2.5], it has a Yetter–Drinfel'd invariant semi-finite faithful weight by 3.6. Using [41, 2.6.3], one gets that this weight is invariant as well under $(\varsigma \otimes \mathrm{id})(\mathrm{id} \otimes \mathfrak{a}_D)$. □

3.8 Corollary. *Let $\mathbb{G}$ be a locally compact quantum group, $(N, \mathfrak{a}, \widehat{\mathfrak{a}})$ a $\mathbb{G}$-Yetter–Drinfel'd algebra, ν a normal semi-finite faithful weight on N, $U_\nu^{\mathfrak{a}}$ and $U_\nu^{\widehat{\mathfrak{a}}}$ the canonical implementations of the actions $\mathfrak{a}$ and $\widehat{\mathfrak{a}}$, and β the anti-$*$-homomorphism introduced in 2.5.4. Then:*

(i) *the unitary implementations of the actions $\mathfrak{a}$, $\widehat{\mathfrak{a}}$ and $\mathfrak{a}_D$ are linked by the relation*
$$U_\nu^{\mathfrak{a}_D} = (U_\nu^{\mathfrak{a}})_{23}(U_\nu^{\widehat{\mathfrak{a}}})_{13};$$

(ii) $(U_\nu^{\mathfrak{a}})_{13}(U_\nu^{\widehat{\mathfrak{a}}})_{23} = W_{12}(U_\nu^{\widehat{\mathfrak{a}}})_{23}(U_\nu^{\mathfrak{a}})_{13}W_{12}^*$;

(iii) $\mathrm{Ad}(1 \otimes U_\nu^{\widehat{\mathfrak{a}}}(U_\nu^{\mathfrak{a}})^*)[W \otimes 1] = (U_\nu^{\mathfrak{a}})_{13}^* W_{12} = (U_\nu^{\mathfrak{a}})_{23}^* W_{12}^* (U_\nu^{\mathfrak{a}})_{23}$;

(iv) *writing $\beta^\dagger$ for the map $x^{\mathrm{o}} \mapsto \beta(x)$, we have*
$$\mathrm{Ad}(W \otimes 1)[1 \otimes \beta(x)] = (\mathrm{id} \otimes \beta^\dagger)(\mathfrak{a}^{\mathrm{o}}(x^{\mathrm{o}})) \quad \textit{for all } x \in N.$$

Proof. (i) Suppose first that there is a faithful semi-finite Yetter–Drinfel'd invariant weight ν' for $(N, \mathfrak{a}, \widehat{\mathfrak{a}})$. Then, for ξ_1, ξ_2, η_1, η_2 in H, $x \in \mathfrak{N}_\nu$, we get, using 3.3,
$$\begin{aligned}(\omega_{\xi_1\otimes\xi_2,\eta_1\otimes\eta_2} \otimes \mathrm{id})(U_{\nu'}^{\mathfrak{a}_D})\Lambda_{\nu'}(x) &= \Lambda_{\nu'}[(\omega_{\xi_1\otimes\xi_2,\eta_1\otimes\eta_2} \otimes \mathrm{id})\mathfrak{a}_D(x)] \\ &= \Lambda_{\nu'}[(\omega_{\xi_2,\eta_2} \otimes \mathrm{id})\widehat{\mathfrak{a}}(\omega_{\xi_1,\eta_1} \otimes \mathrm{id})\mathfrak{a}(x)] \\ &= (\omega_{\xi_2,\eta_2} \otimes \mathrm{id})(U_{\nu'}^{\widehat{\mathfrak{a}}})(\omega_{\xi_1,\eta_1} \otimes \mathrm{id})(U_{\nu'}^{\mathfrak{a}})\Lambda_{\nu'}(x)m,\end{aligned}$$
from which we get (i) for such a weight ν'. Applying that result to 3.7, we get that there exists a normal semi-finite faithful weight ψ on $B(H) \otimes N$ such that
$$U_\psi^{(\varsigma\otimes\mathrm{id})(\mathrm{id}\otimes\mathfrak{a}_D)} = (U_\psi^{(\varsigma\otimes\mathrm{id})(\mathrm{id}\otimes\widehat{\mathfrak{a}})})_{234}(U_\psi^{(\varsigma\otimes\mathrm{id})(\mathrm{id}\otimes\mathfrak{a})})_{134}.$$
Using now [41, 4.1], we get that for every normal semi-finite faithful weight on N,
$$U_{Tr\otimes\nu}^{(\varsigma\otimes\mathrm{id})(\mathrm{id}\otimes\mathfrak{a}_D)} = (U_{Tr\otimes\nu}^{(\varsigma\otimes\mathrm{id})(\mathrm{id}\otimes\widehat{\mathfrak{a}})})_{234}(U_{Tr\otimes\nu}^{(\varsigma\otimes\mathrm{id})(\mathrm{id}\otimes\mathfrak{a})})_{134}$$
which by [41, 4.4] implies (i).

(ii) From (i) we get that $(U^{\mathfrak{a}}_{\nu})_{23}(\widehat{U^{\mathfrak{a}}_{\nu}})_{13}$ is a representation of $D(\mathbb{G})$. Therefore,

$$\begin{aligned}(\widehat{U^{\mathfrak{a}}_{\nu}})_{45}(U^{\mathfrak{a}}_{\nu})_{35}(\widehat{U^{\mathfrak{a}}_{\nu}})_{25}(U^{\mathfrak{a}}_{\nu})_{15}&\\ &= \mathrm{Ad}(1\otimes\sigma W\otimes 1\otimes 1)[(\Gamma\otimes\widehat{\Gamma}\otimes\mathrm{id})((\widehat{U^{\mathfrak{a}}_{\nu}})_{23}(U^{\mathfrak{a}}_{\nu})_{13})]\\ &= \mathrm{Ad}(1\otimes\sigma W\otimes 1\otimes 1)[(\widehat{U^{\mathfrak{a}}_{\nu}})_{45}(\widehat{U^{\mathfrak{a}}_{\nu}})_{35}(U^{\mathfrak{a}}_{\nu})_{25}(U^{\mathfrak{a}}_{\nu})_{15}]\\ &= (\widehat{U^{\mathfrak{a}}_{\nu}})_{45}\,\mathrm{Ad}(1\otimes\sigma W\otimes 1\otimes 1)[(\widehat{U^{\mathfrak{a}}_{\nu}})_{35}(U^{\mathfrak{a}}_{\nu})_{25}](U^{\mathfrak{a}}_{\nu})_{15},\end{aligned}$$

from which we infer that

$$(U^{\mathfrak{a}}_{\nu})_{35}(\widehat{U^{\mathfrak{a}}_{\nu}})_{25} = \mathrm{Ad}(1\otimes\sigma W\otimes 1\otimes 1)[(\widehat{U^{\mathfrak{a}}_{\nu}})_{35}(U^{\mathfrak{a}}_{\nu})_{25}].$$

After renumbering the legs, we obtain (ii).

(iii) The relation $W^*_{12}(U^{\mathfrak{a}}_{\nu})^*_{23}W_{12} = (\Gamma\otimes\mathrm{id})(U^{\mathfrak{a}}_{\nu})^* = (U^{\mathfrak{a}}_{\nu})^*_{13}(U^{\mathfrak{a}}_{\nu})^*_{23}$ implies

$$(U^{\mathfrak{a}}_{\nu})^*_{23}W_{12}(U^{\mathfrak{a}}_{\nu})_{23} = W_{12}(U^{\mathfrak{a}}_{\nu})^*_{13}.$$

Using (ii), we get

$$(U^{\mathfrak{a}}_{\nu})_{13}(\widehat{U^{\mathfrak{a}}_{\nu}})_{23} = W_{12}(\widehat{U^{\mathfrak{a}}_{\nu}})_{23}(U^{\mathfrak{a}}_{\nu})^*_{23}W^*_{12}(U^{\mathfrak{a}}_{\nu})_{23}$$

and, therefore,

$$W^*_{12}(U^{\mathfrak{a}}_{\nu})_{13} = (\widehat{U^{\mathfrak{a}}_{\nu}})_{23}(U^{\mathfrak{a}}_{\nu})^*_{23}W^*_{12}(U^{\mathfrak{a}}_{\nu})_{23}(\widehat{U^{\mathfrak{a}}_{\nu}})^*_{23}$$

which implies (iii).

(iv) Relation (iii) and 2.5.2 imply

$$\begin{aligned}\mathrm{Ad}(W_{12})[\beta(x)_{23}] &= \mathrm{Ad}(W_{12}(U^{\mathfrak{a}}_{\nu})_{23}(\widehat{U^{\mathfrak{a}}_{\nu}})_{23})[1\otimes 1\otimes x^{\mathrm{o}}]\\ &= \mathrm{Ad}((U^{\mathfrak{a}}_{\nu})_{23}(\widehat{U^{\mathfrak{a}}_{\nu}})^*_{23}(U^{\mathfrak{a}}_{\nu})^*_{13}W_{12})[1\otimes 1\otimes x^{\mathrm{o}}]\\ &= \mathrm{Ad}((U^{\mathfrak{a}}_{\nu})_{23}(\widehat{U^{\mathfrak{a}}_{\nu}})^*_{23})[\mathfrak{a}^{\mathrm{o}}(x^{\mathrm{o}})_{13}]\\ &= (\mathrm{id}\otimes\beta^{\dagger})(\mathfrak{a}^{\mathrm{o}}(x^{\mathrm{o}})).\end{aligned}$$ □

3.8.1 Remark. We have quickly shown in 2.4.5 that (i) can also be deduced from a particular case of [7, 6,5], which remains unpublished.

3.9 Lemma. *Let N be a properly infinite von Neumann algebra.*

(i) *Let $(e_n)_{n\in\mathbb{N}}$ be a sequence of pairwise orthogonal projections in N, equivalent to 1 and whose sum is 1, and let $(v_n)_{n\in\mathbb{N}}$ be a sequence of isometries in N such that $v_n^*v_n = 1$ and $v_nv_n^* = e_n$ for all $n\in\mathbb{N}$, (and, therefore $v_i^*v_j = 0$*

if $i \neq j$). Let H be a separable Hilbert space and $u_{i,j}$ a set of matrix units of $B(H)$ acting on an orthonormal basis $(\xi_i)_i$. For any $x \in N$, let

$$\Phi(x) = \sum_{i,j} u_{i,j} \otimes v_i^* x v_j$$

Then Φ is an isomorphism of N onto $B(H) \otimes N$, and $\Phi^{-1}(1 \otimes x) = \sum_i v_i x v_i^*$.

(ii) *Let $\mathfrak{a}$ be a left action of a locally compact quantum group $\mathbb{G} = (M, \Gamma, \varphi, \psi)$ with separable predual M_* on N. Then the operator $V = \sum_n (1 \otimes v_n)\mathfrak{a}(v_n^*)$ exists, is a unitary in $M \otimes N$ and a cocycle for $\mathfrak{a}$, that is, $(\Gamma \otimes \mathrm{id})(V) = (1 \otimes V)(\mathrm{id} \otimes \mathfrak{a})(V)$. Moreover, the actions $(\varsigma \otimes \mathrm{id})(\mathrm{id} \otimes \mathfrak{a})$ and $(\mathrm{id} \otimes \Phi)\mathfrak{a}\Phi^{-1}$ are linked by the relation*

$$(\varsigma \otimes \mathrm{id})(\mathrm{id} \otimes \mathfrak{a})(X) = \mathrm{Ad}((\mathrm{id} \otimes \Phi)(V))[(\mathrm{id} \otimes \Phi)\mathfrak{a}\Phi^{-1}(X)].$$

(iii) *Let ϕ be a normal semi-finite faithful weight on N. Then for each $n \in \mathbb{N}$, the weight ϕ_n on N defined by $\phi_n(x) = \phi(v_n x v_n^*)$ for all $x \in N^+$ is faithful, normal and semi-finite, and $\phi \circ \Phi^{-1} = \sum_n (\omega_{\xi_n} \otimes \phi_n)$.*

(iv) *Let ψ be a normal semi-finite faithful weight on $B(H) \otimes N$. Then, with the notations of* (iii) *$(\psi \circ \Phi)_n(x) = \psi(u_{n,n} \otimes x)$ for all $x \in N^+$. If ψ is invariant under $(\varsigma \otimes \mathrm{id})(\mathrm{id} \otimes \mathfrak{a})$, then each $(\psi \circ \Phi)_n$ is a normal semi-finite faithful weight on N, invariant under $\mathfrak{a}$.*

Proof. (i) This result is taken from [37, Th. 4.6].

(ii) This assertion is proved in [12, Th. IV.3] for right actions of Kac algebras, but remains true for left actions of any locally compact quantum group.

(iii) Let $(\xi_i)_{i \in \mathbb{N}}$ be the orthonormal basis of H defined by the matrix units $u_{i,j}$. Then we can define an isometry I from $L^2(N)$ into $H \otimes L^2(N)$ by $I\eta = \sum_n \xi_n \otimes v_n^* \eta$ for all $\eta \in L^2(N)$. It is then straightforward to get that, for all sequences $(\eta_n)_{n \in \mathbb{N}}$ such that $\sum_n \|\eta_n\|^2 < \infty$, we have $I^*(\sum_n \xi_n \otimes \eta_n) = \sum_n v_n \eta_n$. Therefore, I is unitary and $\Phi(x) = I x I^*$ and for all $x \in N$. So, for any $\zeta \in L^2(N)$, $\omega_\zeta \circ \Phi^{-1}$ is equal to the normal weight $\sum_n \omega_{\xi_n} \otimes \omega_{v_n^* \zeta}$. Hence, $\phi \circ \Phi^{-1}$ is the weight $\sum_n \omega_{\xi_n} \otimes \phi_n$.

Let now $x \in N$ such that $\phi_n(x^* x) = 0$. By definition, we get that $x v_n^* = 0$ and therefore $x = 0$. So, the weight ϕ_n is faithful. As ϕ is semi-finite, there exists in $\mathfrak{M}_\phi^+$ an increasing family $x_k \uparrow 1$. For all $n \in \mathbb{N}$, we get $y_k = (\omega_{\xi_n} \otimes \mathrm{id})\Phi(x_k) \uparrow 1$ and $\phi_n(y_k) = (\omega_{\xi_n} \otimes \phi_n)\Phi(x_k) \leq \phi(x_k) < \infty$, which gives that ϕ_n is semi-finite.

(iv) First,

$$(\psi \circ \Phi)_n(x) = (\psi \circ \Phi)(v_n^* x v_n) = \psi\left(\sum_{i,j} u_{i,j} \otimes v_i^* v_n x v_n^* v_j\right) = \psi(u_{n,n} \otimes x).$$

If ψ is invariant under $(\varsigma \otimes \mathrm{id})(\mathrm{id} \otimes \mathfrak{a})$, then it is clear that all $(\psi \circ \Phi)_n$ are normal semi-finite faithful weights on N, invariant under $\mathfrak{a}$. □

3.10 Corollary. *Let $\mathbb{G} = (M, \Gamma, \varphi, \psi)$ be a locally compact quantum group such that the predual M_* is separable, and $(N, \mathfrak{a}, \widehat{\mathfrak{a}})$ a $\mathbb{G}$-Yetter–Drinfel'd algebra, where N is a properly infinite von Neumann algebra. Then this $\mathbb{G}$-Yetter–Drinfel'd algebra has a normal faithful semi-finite invariant weight.*

Proof. Use the left action $\mathfrak{a}_D = (\mathrm{id} \otimes \widehat{\mathfrak{a}})\mathfrak{a}$ of $D(\mathbb{G})$ on N and apply 3.7 and 3.9 (iv). □

4. The Hopf bimodule associated to a braided-commutative Yetter–Drinfel'd algebra

In this chapter, we recall the definition of the relative tensor product of Hilbert spaces, and of the fiber product of von Neumann algebras (4.1). Then, we recall the definition of a Hopf bimodule (4.2) and a co-inverse. Starting then from a braided-commutative Yetter–Drinfel'd algebra $(N, \mathfrak{a}, \widehat{\mathfrak{a}})$, and any normal semi-finite faithful weight ν on N, we first construct an isomorphism of the Hilbert spaces $H \otimes H \otimes H_\nu$ and $(H \otimes H_\nu) {}_\beta\underset{\nu}{\otimes}{}_{\mathfrak{a}} (H \otimes H_\nu)$ (4.3) and then show that the dual action $\tilde{\mathfrak{a}}$ of $\widehat{\mathbb{G}}^{\mathrm{o}}$ on the crossed product $\mathbb{G} \ltimes_{\mathfrak{a}} N$, modulo this isomorphism, can be interpreted as a coproduct on $\mathbb{G} \ltimes_{\mathfrak{a}} N$ (4.4). Finally, we construct an involutive anti-$*$-automorphism of $\mathbb{G} \ltimes_{\mathfrak{a}} N$ which turns out to be a co-inverse (4.6).

4.1. Relative tensor products of Hilbert spaces and fiber products of von Neumann algebras [5, 17, 34, 38]. Let N be a von Neumann algebra, ψ a normal semi-finite faithful weight on N; we shall denote by H_ψ, $\mathfrak{N}_\psi, \ldots$ the canonical objects of the Tomita–Takesaki theory associated to the weight ψ.

Let α be a non-degenerate faithful representation of N on a Hilbert space $\mathcal{H}$. The set of ψ-bounded elements of the left module ${}_\alpha\mathcal{H}$ is

$$D({}_\alpha\mathcal{H}, \psi) = \{\xi \in \mathcal{H} : \exists C < \infty, \|\alpha(y)\xi\| \le C\|\Lambda_\psi(y)\|, \forall y \in \mathfrak{N}_\psi\}.$$

For any ξ in $D({}_\alpha\mathcal{H}, \psi)$, there exists a bounded operator $R^{\alpha,\psi}(\xi)$ from H_ψ to $\mathcal{H}$ such that

$$R^{\alpha,\psi}(\xi)\Lambda_\psi(y) = \alpha(y)\xi \quad \text{for all } y \in \mathfrak{N}_\psi,$$

and this operator intertwines the actions of N. If ξ and η are bounded vectors, we define the operator product

$$\langle \xi|\eta\rangle_{\alpha,\psi} = R^{\alpha,\psi}(\eta)^* R^{\alpha,\psi}(\xi),$$

which belongs to $\pi_\psi(N)'$. This last algebra will be identified with the opposite von Neumann algebra N^{o} using Tomita–Takesaki theory.

If now β is a non-degenerate faithful anti-representation of N on a Hilbert space $\mathcal{K}$, the relative tensor product $\mathcal{K}\,{}_\beta\underset{\psi}{\otimes}{}_\alpha\,\mathcal{H}$ is the completion of the algebraic tensor product $K \odot D({}_\alpha\mathcal{H}, \psi)$ by the scalar product defined by

$$(\xi_1 \odot \eta_1 | \xi_2 \odot \eta_2) = (\beta(\langle \eta_1 | \eta_2 \rangle_{\alpha,\psi})\xi_1 | \xi_2)$$

for all $\xi_1, \xi_2 \in \mathcal{K}$ and $\eta_1, \eta_2 \in D({}_\alpha\mathcal{H}, \psi)$. If $\xi \in \mathcal{K}$ and $\eta \in D({}_\alpha\mathcal{H}, \psi)$, we denote by $\xi\,{}_\beta\underset{\psi}{\otimes}{}_\alpha\,\eta$ the image of $\xi \odot \eta$ into $\mathcal{K}\,{}_\beta\underset{\psi}{\otimes}{}_\alpha\,\mathcal{H}$. Writing $\rho_\eta^{\beta,\alpha}(\xi) = \xi\,{}_\beta\underset{\psi}{\otimes}{}_\alpha\,\eta$, we get a bounded linear operator from $\mathcal{H}$ into $\mathcal{K}\,{}_\beta\underset{\nu}{\otimes}{}_\alpha\,\mathcal{H}$, which is equal to $1_\mathcal{K} \otimes_\psi R^{\alpha,\psi}(\eta)$.

Changing the weight ψ will give an isomorphic Hilbert space, but the isomorphism will not exchange elementary tensors!

We shall denote by σ_ψ the relative flip, which is a unitary sending $\mathcal{K}\,{}_\beta\underset{\psi}{\otimes}{}_\alpha\,\mathcal{H}$ onto $\mathcal{H}\,{}_\alpha\underset{\psi^o}{\otimes}{}_\beta\,\mathcal{K}$, defined by

$$\sigma_\psi(\xi\,{}_\beta\underset{\psi}{\otimes}{}_\alpha\,\eta) = \eta\,{}_\alpha\underset{\psi^o}{\otimes}{}_\beta\,\xi$$

for all $\xi \in D(\mathcal{K}_\beta, \psi^o)$ and $\eta \in D({}_\alpha\mathcal{H}, \psi)$.

If $x \in \beta(N)'$ and $y \in \alpha(N)'$, it is possible to define an operator $x\,{}_\beta\underset{\psi}{\otimes}{}_\alpha\,y$ on $\mathcal{K}\,{}_\beta\underset{\psi}{\otimes}{}_\alpha\,\mathcal{H}$, with natural values on the elementary tensors. As this operator does not depend upon the weight ψ, it will be denoted by $x\,{}_\beta\underset{N}{\otimes}{}_\alpha\,y$.

If P is a von Neumann algebra on $\mathcal{H}$ with $\alpha(N) \subset P$, and Q a von Neumann algebra on $\mathcal{K}$ with $\beta(N) \subset Q$, then we define the fiber product $Q\,{}_\beta\underset{N}{*}{}_\alpha\,P$ as $\{x\,{}_\beta\underset{N}{\otimes}{}_\alpha\,y : x \in Q', y \in P'\}'$. This von Neumann algebra can be defined independently of the Hilbert spaces on which P and Q are represented. If for $i = 1, 2$, α_i is a faithful non-degenerate homomorphism from N into P_i, and β_i is a faithful non-degenerate anti-homomorphism from N into Q_i, and Φ (resp. Ψ) a homomorphism from P_1 to P_2 (resp. from Q_1 to Q_2) such that $\Phi \circ \alpha_1 = \alpha_2$ (resp. $\Psi \circ \beta_1 = \beta_2$), then, it is possible to define a homomorphism $\Psi\,{}_{\beta_1}\underset{N}{*}{}_{\alpha_1}\,\Phi$ from $Q_1\,{}_{\beta_1}\underset{N}{*}{}_{\alpha_1}\,P_1$ into $Q_2\,{}_{\beta_2}\underset{N}{*}{}_{\alpha_2}\,P_2$.

We define a relative flip ς_N from $\mathcal{L}(\mathcal{K})\,{}_\beta\underset{N}{*}{}_\alpha\,\mathcal{L}(\mathcal{H})$ onto $\mathcal{L}(\mathcal{H})\,{}_\alpha\underset{N^o}{*}{}_\beta\,\mathcal{L}(\mathcal{K})$ by $\varsigma_N(X) = \sigma_\psi X(\sigma_\psi)^*$ for any $X \in \mathcal{L}(\mathcal{K})\,{}_\beta\underset{N}{*}{}_\alpha\,\mathcal{L}(\mathcal{H})$ and any normal semi-finite faithful weight ψ on N.

Let now U be an isometry from a Hilbert space $\mathcal{K}_1$ in a Hilbert space $\mathcal{K}_2$, which intertwines two anti-representations β_1 and β_2 of N, and let V be an isometry from a Hilbert space $\mathcal{H}_1$ in a Hilbert space $\mathcal{H}_2$, which intertwines two representations α_1 and α_2 of N. Then, it is possible to define, on linear combinations of elementary tensors, an isometry $U \underset{\psi}{{}_{\beta_1}\otimes_{\alpha_1}} V$ which can be extended to the whole Hilbert space $\mathcal{K}_1 \underset{\psi}{{}_{\beta_1}\otimes_{\alpha_1}} \mathcal{H}_1$ with values in $\mathcal{K}_2 \underset{\psi}{{}_{\beta_2}\otimes_{\alpha_2}} \mathcal{H}_2$. One can show that this isometry does not depend upon the weight ψ. It will be denoted by $U \underset{N}{{}_{\beta_1}\otimes_{\alpha_1}} V$. If U and V are unitaries, then $U \underset{N}{{}_{\beta_1}\otimes_{\alpha_1}} V$ is an unitary and $(U \underset{N}{{}_{\beta_1}\otimes_{\alpha_1}} V)^* = U^* \underset{N}{{}_{\beta_2}\otimes_{\alpha_2}} V^*$.

In [7, Chap. 11], De Commer had shown that, if N is finite-dimensional, the Hilbert space $\mathcal{K} \underset{\nu}{{}_\beta\otimes_\alpha} \mathcal{H}$ can be isometrically imbedded into the usual Hilbert tensor product $\mathcal{K} \otimes \mathcal{H}$.

4.2 Definitions. A quintuple $(N, M, \alpha, \beta, \Gamma)$ will be called a *Hopf bimodule*, following [45], [17, 6.5], if N, M are von Neumann algebras, α is a faithful non-degenerate representation of N into M, β is a faithful non-degenerate anti-representation of N into M, with commuting ranges, and Γ is an injective $*$-homomorphism from M into $M \underset{N}{{}_\beta *_\alpha} M$ such that, for all X in N,

(i) $\Gamma(\beta(X)) = 1 \underset{N}{{}_\beta\otimes_\alpha} \beta(X)$,

(ii) $\Gamma(\alpha(X)) = \alpha(X) \underset{N}{{}_\beta\otimes_\alpha} 1$,

(iii) Γ satisfies the co-associativity relation

$$(\Gamma \underset{N}{{}_\beta *_\alpha} \mathrm{id})\Gamma = (\mathrm{id} \underset{N}{{}_\beta *_\alpha} \Gamma)\Gamma$$

This last formula makes sense, thanks to the two preceeding ones and 4.1. The von Neumann algebra N will be called the *basis* of $(N, M, \alpha, \beta, \Gamma)$.

In [7, Chap. 11], De Commer had shown that, if N is finite-dimensional, the Hilbert space $L^2(M) \underset{\nu}{{}_\beta\otimes_\alpha} L^2(M)$ can be isometrically imbedded into the usual Hilbert tensor product $L^2(M) \otimes L^2(M)$ and the projection p on this closed subspace belongs to $M \otimes M$. Moreover, the fiber product $M \underset{N}{{}_\beta *_\alpha} M$ can be then identified with the reduced von Neumann algebra $p(M \otimes M)p$ and we can consider Γ as a usual coproduct $M \mapsto M \otimes M$, but with the condition $\Gamma(1) = p$.

A *co-inverse* R for a Hopf bimodule $(N, M, \alpha, \beta, \Gamma)$ is an involutive ($R^2 = \mathrm{id}$) anti-$*$-isomorphism of M satisfying $R \circ \alpha = \beta$ (and therefore $R \circ \beta = \alpha$) and $\Gamma \circ R = \varsigma_{N^o} \circ (R \underset{N}{{}_\beta *_\alpha} R) \circ \Gamma$, where ς_{N^o} is the flip from $M \underset{N^o}{{}_\alpha *_\beta} M$ onto $M \underset{N}{{}_\beta *_\alpha} M$. A Hopf bimodule is called *co-commutative* if N is abelian, $\beta = \alpha$, and $\Gamma = \varsigma \circ \Gamma$.

For an example, suppose that $\mathcal{G}$ is a measured groupoid, with $\mathcal{G}^{(0)}$ as its set of units. We denote by r and s the range and source applications from $\mathcal{G}$ to $\mathcal{G}^{(0)}$, given by $xx^{-1} = r(x)$ and $x^{-1}x = s(x)$, and by $\mathcal{G}^{(2)}$ the set of composable elements, i.e.

$$\mathcal{G}^{(2)} = \{(x, y) \in \mathcal{G}^2 : s(x) = r(y)\}.$$

Let $(\lambda^u)_{u\in\mathcal{G}^{(0)}}$ be a Haar system on $\mathcal{G}$ and ν a measure on $\mathcal{G}^{(0)}$. Let us denote by μ the measure on $\mathcal{G}$ given by integrating λ^u by ν,

$$\mu = \int_{\mathcal{G}^{(0)}} \lambda^u d\nu.$$

By definition, ν is called *quasi-invariant* if μ is equivalent to its image under the inversion $x \mapsto x^{-1}$ of $\mathcal{G}$ (see [32], [6, II.5], [30] and [1] for more details, precise definitions and examples of groupoids).

In [52–54] and [45] was associated to a measured groupoid $\mathcal{G}$, equipped with a Haar system $(\lambda^u)_{u\in\mathcal{G}^{(0)}}$ and a quasi-invariant measure ν on $\mathcal{G}^{(0)}$, a Hopf bimodule with an abelian underlying von Neumann algebra $(L^\infty(\mathcal{G}^{(0)},\nu), L^\infty(\mathcal{G},\mu), r_{\mathcal{G}}, s_{\mathcal{G}}, \Gamma_{\mathcal{G}})$, where $r_{\mathcal{G}}(g) = g\circ r$ and $s_{\mathcal{G}}(g) = g\circ s$ for all g in $L^\infty(\mathcal{G}^{(0)})$ and where $\Gamma_{\mathcal{G}}(f)$, for f in $L^\infty(\mathcal{G})$, is the function defined on $\mathcal{G}^{(2)}$ by $(s,t)\mapsto f(st)$. Thus, $\Gamma_{\mathcal{G}}$ is an involutive homomorphism from $L^\infty(\mathcal{G})$ into $L^\infty(\mathcal{G}^{(2)})$, which can be identified with $L^\infty(\mathcal{G})_s{*}_r L^\infty(\mathcal{G})$.

It is straightforward to get that the inversion of the groupoid gives a co-inverse for this Hopf bimodule structure.

4.3 Proposition ([40]). *Let $\mathbb{G}$ be a locally compact quantum group, $(N,\mathfrak{a},\widehat{\mathfrak{a}})$ a braided-commutative $\mathbb{G}$-Yetter–Drinfel'd algebra, β the injective anti-$*$-homomorphism from N into $\mathbb{G}\ltimes_{\mathfrak{a}} N$ introduced in 2.5.4, and ν a normal semi-finite faithful weight ν on N. Then the relative tensor product $(H\otimes H_\nu)\;{}_\beta\!\underset{\nu}{\otimes}_{\mathfrak{a}}\,(H\otimes H_\nu)$ can be canonically identified with $H\otimes H\otimes H_\nu$ as follows:*

(i) *For any $\eta\in H$, $p\in\mathfrak{N}_\nu$, the vector $U^{\mathfrak{a}}_\nu(\eta\otimes J_\nu\Lambda_\nu(p))$ belongs to $D({}_{\mathfrak{a}}(H\otimes H_\nu),\nu)$ and*

$$R^{\mathfrak{a},\nu}(U^{\mathfrak{a}}_\nu(\eta\otimes J_\nu\Lambda_\nu(p))) = U^{\mathfrak{a}}_\nu l_\eta J_\nu p J_\nu,$$

where l_η is the application $\zeta\to\eta\otimes\zeta$ from H_ν into $H\otimes H_\nu$. There exists a unitary V_1 from $(H\otimes H_\nu)\;{}_\beta\!\underset{\nu}{\otimes}_{\mathfrak{a}}\,(H\otimes H_\nu)$ onto $H\otimes H\otimes H_\nu$ such that

$$V_1(\Xi\;{}_\beta\!\underset{\nu}{\otimes}_{\mathfrak{a}}\,U^{\mathfrak{a}}_\nu(\eta\otimes J_\nu\Lambda_\nu(p))) = \eta\otimes\beta(p^*)\Xi \quad \textit{for all } \Xi\in H\otimes H_\nu,$$

and $V_1(X\;{}_\beta\!\underset{N}{\otimes}_{\mathfrak{a}}\,(1_H\otimes 1_{H_\nu})) = (1_H\otimes X)V_1$ for all $X\in\beta(N)'$, in particular, for $X\in\mathfrak{a}(N)$. Morover, writing $\beta^\dagger$ for the map $x^{\mathrm{o}}\mapsto\beta(x)$, we have for all $x\in N$,

$$V_1[(1_H\otimes 1_{H_\nu})\;{}_\beta\!\underset{N}{\otimes}_{\mathfrak{a}}\,(1_H\otimes x^{\mathrm{o}})] = (\mathrm{id}\otimes\beta^\dagger)(\mathfrak{a}^{\mathrm{o}}(x^{\mathrm{o}}))V_1,$$
$$V_1[(1_H\otimes 1_{H_\nu})\;{}_\beta\!\underset{N}{\otimes}_{\mathfrak{a}}\,\beta(x)] = (\mathrm{id}\otimes\beta^\dagger)(\widehat{\mathfrak{a}}^{\mathrm{o}}(x^{\mathrm{o}}))V_1.$$

(ii) *For any* $\xi \in H$, $q \in \mathfrak{N}_\nu$, *the vector* $U^{\mathfrak{a}}_\nu(\widehat{U^{\mathfrak{a}}_\nu})^*(\xi \otimes \Lambda_\nu(q))$ *belongs to* $D({}_\beta(H \otimes H_\nu), \nu^{\mathrm{o}})$ *and*

$$R^{\beta,\nu^{\mathrm{o}}}(U^{\mathfrak{a}}_\nu(\widehat{U^{\mathfrak{a}}_\nu})^*(\xi \otimes \Lambda_\nu(q))) = U^{\mathfrak{a}}_\nu(\widehat{U^{\mathfrak{a}}_\nu})^* l_\xi q.$$

There exists a unitary V_2 *from* $(H \otimes H_\nu) \;{}_\beta\!\underset{\nu}{\otimes}_{\mathfrak{a}}\; (H \otimes H_\nu)$ *onto* $H \otimes H \otimes H_\nu$ *such that*

$$V_2[U^{\mathfrak{a}}_\nu(\widehat{U^{\mathfrak{a}}_\nu})^*(\xi \otimes \Lambda_\nu(q)) \;{}_\beta\!\underset{\nu}{\otimes}_{\mathfrak{a}}\; \Xi] = \xi \otimes \mathfrak{a}(q)\Xi \quad \textit{for all } \Xi \in H \otimes H_\nu,$$

and $V_2((1_H \otimes 1_{H_\nu}) \;{}_\beta\!\underset{N}{\otimes}_{\mathfrak{a}}\; X) = (1_H \otimes X)V_2$ *for all* $X \in \mathfrak{a}(N)'$, *in particular, for* $X \in \beta(N)$.

(iii) $V_2 V_1^* = \sigma_{12}(U^{\mathfrak{a}}_\nu)_{13}(\widehat{U^{\mathfrak{a}}_\nu})_{23}(U^{\mathfrak{a}}_\nu)^*_{23} = \sigma_{12}W_{12}(\widehat{U^{\mathfrak{a}}_\nu})_{23}(U^{\mathfrak{a}}_\nu)^*_{23}W^*_{12}$.

Proof. (i) For all $n \in \mathfrak{N}_\nu$,

$$\begin{aligned} U^{\mathfrak{a}}_\nu l_\eta J_\nu p J_\nu \Lambda_\nu(n) &= U^{\mathfrak{a}}_\nu(\eta \otimes J_\nu p J_\nu \Lambda_\nu(n)) \\ &= U^{\mathfrak{a}}_\nu(\eta \otimes n J_\nu \Lambda_\nu(p)) = \mathfrak{a}(n) U^{\mathfrak{a}}_\nu(\eta \otimes J_\nu \Lambda_\nu(p)), \end{aligned}$$

which gives the proof of the first part of (i). Let now $\eta' \in H$, $p' \in \mathfrak{N}_\nu$, $\Xi' \in H \otimes H_\nu$. Then

$$\begin{aligned} \langle U^{\mathfrak{a}}_\nu(\eta' \otimes J_\nu \Lambda_\nu(p')) | U^{\mathfrak{a}}_\nu(\eta \otimes J_\nu \Lambda_\nu(p))\rangle^{\mathrm{o}}_{\mathfrak{a},\nu} &= J_\nu p^* J_\nu l^*_\eta l_{\eta'} J_\nu p' J_\nu \\ &= (\eta|\eta') J_\nu p^* p' J_\nu \end{aligned}$$

and hence

$$\begin{aligned} &(\Xi \;{}_\beta\!\underset{\nu}{\otimes}_{\mathfrak{a}}\; U^{\mathfrak{a}}(\eta \otimes J_\nu \Lambda_\nu(p)) | \Xi' \;{}_\beta\!\underset{\nu}{\otimes}_{\mathfrak{a}}\; U^{\mathfrak{a}}(\eta' \otimes J_\nu \Lambda_\nu(p'))) \\ &\qquad = (\beta(\langle U^{\mathfrak{a}}(\eta' \otimes J_\nu \Lambda_\nu(p')), U^{\mathfrak{a}}(\eta \otimes J_\nu \Lambda_\nu(p))\rangle^{\mathrm{o}}_{\mathfrak{a},\nu})\Xi|\Xi') \\ &\qquad = (\eta|\eta')(\beta(p^* p')\Xi|\Xi') \end{aligned}$$

which proves the existence of an isometry V_1 satisfying the above formula. As the image of V_1 is dense in $H \otimes H \otimes H_\nu$, we get that V_1 is unitary.

Next, let $z \in B(H)$, $x \in N$. Then

$$\begin{aligned} &(z \otimes \beta(x))V_1[\Xi \;{}_\beta\!\underset{\nu}{\otimes}_{\mathfrak{a}}\; U^{\mathfrak{a}}_\nu(\eta \otimes J_\nu \Lambda_\nu(p))] \\ &\qquad = z\eta \otimes \beta(x)\beta(p^*)\Xi \\ &\qquad = z\eta \otimes \beta((x^* p)^*)\Xi \\ &\qquad = V_1[\Xi \;{}_\beta\!\underset{\nu}{\otimes}_{\mathfrak{a}}\; U^{\mathfrak{a}}_\nu(z\eta \otimes J_\nu \Lambda_\nu(x^* p))] \\ &\qquad = V_1[\Xi \;{}_\beta\!\underset{\nu}{\otimes}_{\mathfrak{a}}\; U^{\mathfrak{a}}_\nu(z \otimes x^o)(\eta \otimes J_\nu \Lambda_\nu(p))], \end{aligned}$$

that is, $(z \otimes \beta(x))V_1 = V_1(1 \underset{\nu}{{}_\beta\otimes_\mathfrak{a}} U^\mathfrak{a}_\nu(z \otimes x^o)(U^\mathfrak{a}_\nu)^*)$. In particular,

$$(\mathrm{id} \otimes \beta^\dagger)(\mathfrak{a}^o(x^o))V_1 = V_1(1 \underset{\nu}{{}_\beta\otimes_\mathfrak{a}} U^\mathfrak{a}_\nu \mathfrak{a}^o(x^o)(U^\mathfrak{a}_\nu)^*) = V_1(1 \underset{\nu}{{}_\beta\otimes_\mathfrak{a}} (1_H \otimes x^o)),$$
$$(\mathrm{id} \otimes \beta^\dagger)(\widehat{\mathfrak{a}}^o(x^o))V_1 = V_1(1 \underset{\nu}{{}_\beta\otimes_\mathfrak{a}} U^\mathfrak{a}_\nu \widehat{\mathfrak{a}}^o(x^o)(U^\mathfrak{a}_\nu)^*) = V_1(1 \underset{\nu}{{}_\beta\otimes_\mathfrak{a}} \beta(x)).$$

(ii) We proceed as above. First, we have

$$\begin{aligned} U^\mathfrak{a}_\nu(\widehat{U^\mathfrak{a}_\nu})^* l_\xi q J_\nu \Lambda_\nu(n) &= U^\mathfrak{a}_\nu(\widehat{U^\mathfrak{a}_\nu})^*(\xi \otimes J_\nu n J_\nu \Lambda_\nu(q)) \\ &= \beta(n^*)U^\mathfrak{a}_\nu(\widehat{U^\mathfrak{a}_\nu})^*(\xi \otimes \Lambda_\nu(q)), \end{aligned}$$

which gives the proof of the first part of (ii). Let now $\xi' \in H$, $q' \in \mathfrak{N}_\nu$. Then

$$\begin{aligned} &(U^\mathfrak{a}_\nu(\widehat{U^\mathfrak{a}_\nu})^*(\xi \otimes \Lambda_\nu(q)) \underset{\nu}{{}_\beta\otimes_\mathfrak{a}} \Xi | U^\mathfrak{a}_\nu(\widehat{U^\mathfrak{a}_\nu})^*(\xi' \otimes \Lambda_\nu(q')) \underset{\nu}{{}_\beta\otimes_\mathfrak{a}} \Xi') \\ &\quad = (\mathfrak{a}(\langle U^\mathfrak{a}_\nu(\widehat{U^\mathfrak{a}_\nu})^*(\xi \otimes \Lambda_\nu(q)), U^\mathfrak{a}_\nu(\widehat{U^\mathfrak{a}_\nu})^*(\xi' \otimes \Lambda_\nu(q'))\rangle_{\beta,\nu^o})\Xi|\Xi') \\ &\quad = (\xi|\xi')(\mathfrak{a}(q'^*q)\Xi|\Xi') \end{aligned}$$

which proves the existence of an isometry V_2 satisfying the above formula. Again, as the image of V_2 is dense in $H \otimes H \otimes H_\nu$, we get (ii).

(iii) Applying (i), we get

$$\begin{aligned} &V_1[U^\mathfrak{a}_\nu(\widehat{U^\mathfrak{a}_\nu})^*(\xi \otimes \Lambda_\nu(q)) \underset{\nu}{{}_\beta\otimes_\mathfrak{a}} U^\mathfrak{a}_\nu(\eta \otimes J_\nu\Lambda_\nu(p))] \\ &\qquad = \eta \otimes \beta(p^*)U^\mathfrak{a}_\nu(\widehat{U^\mathfrak{a}_\nu})^*(\xi \otimes \Lambda_\nu(q)) \\ &\qquad = \eta \otimes U^\mathfrak{a}_\nu(\widehat{U^\mathfrak{a}_\nu})^*(\xi \otimes J_\nu p J_\nu \Lambda_\nu(q)) \\ &\qquad = \eta \otimes U^\mathfrak{a}_\nu(\widehat{U^\mathfrak{a}_\nu})^*(\xi \otimes q J_\nu \Lambda_\nu(p)), \end{aligned}$$

and, applying (ii), we get

$$\begin{aligned} &V_2[U^\mathfrak{a}_\nu(\widehat{U^\mathfrak{a}_\nu})^*(\xi \otimes \Lambda_\nu(q)) \underset{\nu}{{}_\beta\otimes_\mathfrak{a}} U^\mathfrak{a}_\nu(\eta \otimes J_\nu\Lambda_\nu(p))] \\ &\qquad = \xi \otimes \mathfrak{a}(q)U^\mathfrak{a}_\nu(\eta \otimes J_\nu\Lambda_\nu(p)) \\ &\qquad = \xi \otimes U^\mathfrak{a}_\nu(\eta \otimes q J_\nu \Lambda_\nu(p)), \end{aligned}$$

from which we easily get $(1_H \otimes \widehat{U^\mathfrak{a}_\nu}(U^\mathfrak{a}_\nu)^*)V_1 = (\sigma \otimes 1_{H\nu})(1_H \otimes (U^\mathfrak{a}_\nu)^*)V_2$. Using Corollary 3.8 (iii), we conclude

$$V_2 V_1^* = \sigma_{12}(U^\mathfrak{a}_\nu)_{13}(\widehat{U^\mathfrak{a}_\nu})_{23}(U^\mathfrak{a}_\nu)^*_{23} = \sigma_{12}W_{12}(\widehat{U^\mathfrak{a}_\nu})_{23}(U^\mathfrak{a}_\nu)^*_{23}W^*_{12}. \qquad \square$$

4.4 Theorem ([40]). *Let $\mathbb{G}$ be a locally compact quantum group and $(N, \mathfrak{a}, \widehat{\mathfrak{a}})$ a braided-commutative $\mathbb{G}$-Yetter–Drinfel'd algebra. We use the notations of 4.3.*

(i) *For $X \in \mathbb{G} \ltimes_{\mathfrak{a}} N$, let $\widetilde{\Gamma}(X) = V_1^* \tilde{\mathfrak{a}}(X) V_1$. Then this defines a normal $*$-homomorphism $\widetilde{\Gamma}$ from $\mathbb{G} \ltimes_{\mathfrak{a}} N$ into $(\mathbb{G} \ltimes_{\mathfrak{a}} N) \underset{N}{{}_\beta *_{\mathfrak{a}}} (\mathbb{G} \ltimes_{\mathfrak{a}} N)$. For all $x \in N$,*

$$\widetilde{\Gamma}(\mathfrak{a}(x)) = \mathfrak{a}(x) \underset{N}{{}_\beta \otimes_{\mathfrak{a}}} (1_H \otimes 1_{H_\nu}),$$
$$\widetilde{\Gamma}(\beta(x)) = (1_H \otimes 1_{H_\nu}) \underset{N}{{}_\beta \otimes_{\mathfrak{a}}} \beta(x),$$

and for all $y \in \widehat{M}$,

$$\widetilde{\Gamma}(y \otimes 1_{H_\nu}) = V_1^* (\widehat{\Gamma}^{\mathrm{o}}(y) \otimes 1) V_1 = V_2^* (\widehat{\Gamma}(y) \otimes 1_{H_\nu}) V_2.$$

(ii) *$(N, \mathbb{G} \ltimes_{\mathfrak{a}} N, \mathfrak{a}, \beta, \widetilde{\Gamma})$ is a Hopf bimodule.*

(iii) *We have $\tilde{\mathfrak{a}}(\beta^\dagger(x^o)) = (id \otimes \beta^\dagger)\widehat{\mathfrak{a}}^o(x^o)$, where $\beta^\dagger$ has been defined in 4.3.*

Proof. (i) and (iii) Let $x \in N$. Then 4.3 (iii) implies

$$\widetilde{\Gamma}(\mathfrak{a}(x)) = V_1^* \tilde{\mathfrak{a}}(\mathfrak{a}(x)) V_1 = V_1^* (1_H \otimes \mathfrak{a}(x)) V_1 = \mathfrak{a}(x) \underset{N}{{}_\beta \otimes_\alpha} (1_H \otimes 1_{H_\nu}),$$

in particular, $\widetilde{\Gamma}(\mathfrak{a}(x))$ lies in $(\mathbb{G} \ltimes_{\mathfrak{a}} N) \underset{N}{{}_\beta *_{\mathfrak{a}}} (\mathbb{G} \ltimes_{\mathfrak{a}} N)$.

Next, by definition,

$$\widetilde{\Gamma}(\beta(x)) = \mathrm{Ad}(V_1^* \widehat{W}_{12}^{\mathrm{o}*})[1 \otimes \beta(x)] = \mathrm{Ad}(V_1^* \widehat{W}_{12}^{\mathrm{o}*} (U_\nu^{\mathfrak{a}})_{23})[1 \otimes \widehat{\mathfrak{a}}^{\mathrm{o}}(x^{\mathrm{o}})].$$

Since $U_\nu^{\mathfrak{a}} \in M \otimes B(H_\nu)$ commutes with $\widehat{W}^{\mathrm{o}} \in \widehat{M} \otimes M'$, this is equal to

$$\begin{aligned} \mathrm{Ad}(V_1^* (U_\nu^{\mathfrak{a}})_{23} \widehat{W}_{12}^{o*})[1 \otimes \widehat{\mathfrak{a}}^{\mathrm{o}}(x^{\mathrm{o}})] &= \mathrm{Ad}(V_1^*)[(\mathrm{id} \otimes \beta^\dagger)(\widehat{\mathfrak{a}}^{\mathrm{o}}(x^{\mathrm{o}}))] \\ &= (1_H \otimes 1_{H_\nu}) \underset{N}{{}_\beta \otimes_\alpha} \beta(x), \end{aligned}$$

where we used 4.3 (i). From this calculation, one gets (iii) as well.

For $y \in \widehat{M}$, we get by definition of $\tilde{\Gamma}$

$$\begin{aligned} \tilde{\Gamma}(y \otimes 1) &= \mathrm{Ad}(V_1^*)[\tilde{\mathfrak{a}}(y \otimes 1)] \\ &= \mathrm{Ad}(V_1^*)[\widehat{\Gamma}^{\mathrm{o}}(y) \otimes 1] = \mathrm{Ad}(V_1^* W_{12})[y \otimes 1 \otimes 1]. \end{aligned}$$

By 4.3 (iii), $V_1^* W_{12} = V_2^* \sigma_{12} W_{12} (\widehat{U_\nu^{\mathfrak{a}}})_{23} (U_\nu^{\mathfrak{a}})^*_{23}$ and hence

$$\begin{aligned}\tilde{\Gamma}(y \otimes 1) &= \mathrm{Ad}(V_2^* \sigma_{12} W_{12} (\widehat{U_\nu^{\mathfrak{a}}})_{23} (U_\nu^{\mathfrak{a}})^*_{23})[y \otimes 1 \otimes 1] \\ &= \mathrm{Ad}(V_2^*)(\widehat{\Gamma}(y) \otimes 1).\end{aligned}$$

To see that $\tilde{\Gamma}(y \otimes 1)$ lies in $(\mathbb{G} \ltimes_{\mathfrak{a}} N) {}_{\beta}\underset{N}{*}_{\mathfrak{a}} (\mathbb{G} \ltimes_{\mathfrak{a}} N)$, note that for any Y in $(\mathbb{G} \ltimes_{\mathfrak{a}} N)'$,

$$\mathrm{Ad}(V_1)[Y {}_{\beta}\underset{N}{\otimes}_{\alpha} (1_H \otimes 1_{H_\nu})] = 1_H \otimes Y = \mathrm{Ad}(V_2)[(1_H \otimes 1_{H_\nu}) {}_{\beta}\underset{N}{\otimes}_{\alpha} Y]$$

by 4.3, and $1_H \otimes Y$ commutes with

$$\mathrm{Ad}(V_1)(\widetilde{\Gamma}(y \otimes 1)) = \widehat{\Gamma}^{\mathrm{o}}(y) \otimes 1 \quad \text{and} \quad \mathrm{Ad}(V_2)(\widetilde{\Gamma}(y \otimes 1)) = \widehat{\Gamma}(y) \otimes 1.$$

(ii) To get (ii), we must verify that $\widetilde{\Gamma}$ is co-associative. It is trivial to get that

$$\begin{aligned}(\widetilde{\Gamma} {}_{\beta}\underset{N}{*}_{\mathfrak{a}} \mathrm{id})\widetilde{\Gamma}(\mathfrak{a}(x)) &= \mathfrak{a}(x) {}_{\beta}\underset{N}{\otimes}_{\mathfrak{a}} (1_H \otimes 1_{H_\nu}) {}_{\beta}\underset{N}{\otimes}_{\mathfrak{a}} (1_H \otimes 1_{H_\nu}) \\ &= (\mathrm{id} {}_{\beta}\underset{N}{*}_{\mathfrak{a}} \widetilde{\Gamma})\widetilde{\Gamma}(\mathfrak{a}(x))\end{aligned}$$

for all $x \in N$.

Next, let $y \in \widehat{M}$ and consider the following diagrams,

$$\begin{tikzcd}
\widehat{M} \otimes 1_{H_\nu} \arrow[r, "\widehat{\Gamma}\otimes\mathrm{id}"] \arrow[dr, "\widetilde{\Gamma}"'] & \widehat{M} \otimes \widehat{M} \otimes 1_{H_\nu} \arrow[r, "\mathrm{id}\otimes\widetilde{\Gamma}"] \arrow[d, "\mathrm{ad}_{(V_2^*)} \otimes \mathrm{id}"] & \widehat{M} \otimes (\mathbb{G} \ltimes_{\mathfrak{a}} N) {}_{\beta}\underset{N}{*}_{\mathfrak{a}} (\mathbb{G} \ltimes_{\mathfrak{a}} N) \arrow[d, "\mathrm{ad}_{(V_2^*)} {}_{\beta}\underset{N}{*}_{\mathfrak{a}} \mathrm{id}"'] \\
& (\mathbb{G} \ltimes_{\mathfrak{a}} N) {}_{\beta}\underset{N}{*}_{\mathfrak{a}} (\mathbb{G} \ltimes_{\mathfrak{a}} N) \arrow[r, "\mathrm{id} {}_{\beta}\underset{N}{*}_{\mathfrak{a}} \widetilde{\Gamma}"'] & (\mathbb{G} \ltimes_{\mathfrak{a}} N) {}_{\beta}\underset{N}{*}_{\mathfrak{a}} (\mathbb{G} \ltimes_{\mathfrak{a}} N) {}_{\beta}\underset{N}{*}_{\mathfrak{a}} (\mathbb{G} \ltimes_{\mathfrak{a}} N)
\end{tikzcd}$$

$$\begin{tikzcd}
\widehat{M} \otimes 1_{H_\nu} \arrow[r, "\varsigma_{23}(\widehat{\Gamma}\otimes\mathrm{id})"] \arrow[dr, "\tilde{\Gamma}"'] & \widehat{M} \otimes 1_{H_\nu} \otimes \widehat{M} \arrow[r, "\widetilde{\Gamma}\otimes\mathrm{id}"] \arrow[d, "\mathrm{id}\otimes\mathrm{ad}_{(\tilde{V}_1^*)}"] & (\mathbb{G} \ltimes_{\mathfrak{a}} N) {}_{\beta}\underset{N}{*}_{\mathfrak{a}} (\mathbb{G} \ltimes_{\mathfrak{a}} N) \otimes \widehat{M} \arrow[d, "\mathrm{id} {}_{\beta}\underset{N}{*}_{\mathfrak{a}} \mathrm{ad}_{(\tilde{V}_1^*)}"'] \\
& (\mathbb{G} \ltimes_{\mathfrak{a}} N) {}_{\beta}\underset{N}{*}_{\mathfrak{a}} (\mathbb{G} \ltimes_{\mathfrak{a}} N) \arrow[r, "\widetilde{\Gamma} {}_{\beta}\underset{N}{*}_{\mathfrak{a}} \mathrm{id}"'] & (\mathbb{G} \ltimes_{\mathfrak{a}} N) {}_{\beta}\underset{N}{*}_{\mathfrak{a}} (\mathbb{G} \ltimes_{\mathfrak{a}} N) {}_{\beta}\underset{N}{*}_{\mathfrak{a}} (\mathbb{G} \ltimes_{\mathfrak{a}} N)
\end{tikzcd}$$

where $\tilde{V}_1$ denotes the composition of the unitary V_1 with the flip $\eta \otimes \xi \otimes \zeta \mapsto \xi \otimes \zeta \otimes \eta$ (for ξ, η in H and ζ in H_ν). The triangles commute by (i) and the squares commutes by definition of V_1 and V_2. Next, consider the following diagram:

$$\begin{tikzcd}[column sep=small]
 & & \widehat{M}\otimes 1_{H_\nu} \arrow[dll, "\widehat{\Gamma}\otimes\mathrm{id}"'] \arrow[drr, "\varsigma_{23}(\widehat{\Gamma}\otimes\mathrm{id})"] & & \\
\widehat{M}\otimes\widehat{M}\otimes 1_{H_\nu} \arrow[drr, "\mathrm{id}\otimes\varsigma_{23}(\widehat{\Gamma}\otimes\mathrm{id})"] \arrow[dd, "\mathrm{id}\otimes\widetilde{\Gamma}"'] & & & & \widehat{M}\otimes 1_{H_\nu}\otimes\widehat{M} \arrow[dll, "\widehat{\Gamma}\otimes\mathrm{id}\otimes\mathrm{id}"'] \arrow[dd, "\widetilde{\Gamma}\otimes\mathrm{id}"] \\
 & & \widehat{M}\otimes\widehat{M}\otimes 1_{H_\nu}\otimes\widehat{M} \arrow[dll, "\mathrm{id}\otimes\mathrm{ad}_{(\tilde{V}_1^*)}"'] \arrow[drr, "\mathrm{ad}_{(V_2^*)}\otimes\mathrm{id}"] & & \\
\widehat{M}\otimes(\mathbb{G}\ltimes_{\mathfrak{a}} N)\underset{N}{{}_\beta *_{\mathfrak{a}}}(\mathbb{G}\ltimes_{\mathfrak{a}} N) \arrow[drr, "\mathrm{ad}_{(V_2^*)}\underset{N}{{}_\beta *_{\mathfrak{a}}}\mathrm{id}"] & & & & (\mathbb{G}\ltimes_{\mathfrak{a}} N)\underset{N}{{}_\beta *_{\mathfrak{a}}}(\mathbb{G}\ltimes_{\mathfrak{a}} N)\otimes\widehat{M} \arrow[dll, "\mathrm{id}\underset{N}{{}_\beta *_{\mathfrak{a}}}\mathrm{ad}_{(\tilde{V}_1^*)}"'] \\
 & & (\mathbb{G}\ltimes_{\mathfrak{a}} N)\underset{N}{{}_\beta *_{\mathfrak{a}}}(\mathbb{G}\ltimes_{\mathfrak{a}} N)\underset{N}{{}_\beta *_{\mathfrak{a}}}(\mathbb{G}\ltimes_{\mathfrak{a}} N) & &
\end{tikzcd}$$

The upper middle cell commutes by co-associativity of $\widehat{\Gamma}$, the left and the right triangle commute by (i), and the lower middle cell commutes because the following diagram does,

$$\begin{tikzcd}
(H\otimes H_\nu)\underset{\nu}{{}_\beta\otimes_{\mathfrak{a}}}(H\otimes H_\nu)\underset{\nu}{{}_\beta\otimes_{\mathfrak{a}}}(H\otimes H_\nu) \arrow[r, "V_2\underset{N}{{}_\beta\otimes_{\mathfrak{a}}}\mathrm{id}"] \arrow[d, "\mathrm{id}\underset{N}{{}_\beta\otimes_{\mathfrak{a}}}\tilde{V}_1"'] & H\otimes((H\otimes H_\nu)\underset{\nu}{{}_\beta\otimes_{\mathfrak{a}}}(H\otimes H_\nu)) \arrow[d, "\mathrm{id}\otimes\tilde{V}_1"] \\
((H\otimes H_\nu)\underset{\nu}{{}_\beta\otimes_{\mathfrak{a}}}(H\otimes H_\nu))\otimes H \arrow[r, "V_2\otimes\mathrm{id}"'] & H\otimes(H\otimes H_\nu)\otimes H
\end{tikzcd}$$

where both compositions are given by

$$U_\nu^{\mathfrak{a}}(U_\nu^{\widehat{\mathfrak{a}}})^*(\xi\otimes\Lambda_\nu(q))\underset{\nu}{{}_\beta\otimes_{\mathfrak{a}}}\Xi\underset{\nu}{{}_\beta\otimes_{\mathfrak{a}}}U_\nu^{\mathfrak{a}}(\eta\otimes J_\nu\Lambda_\nu(p))\mapsto\xi\otimes\mathfrak{a}(q)\beta(p^*)\Xi\otimes\eta.$$

Combining everything, we can conclude that

$$(\widetilde{\Gamma}\underset{N}{{}_\beta *_{\mathfrak{a}}}\mathrm{id})\circ\widetilde{\Gamma}(y\otimes 1)=(\mathrm{id}\underset{N}{{}_\beta *_{\mathfrak{a}}}\widetilde{\Gamma})\circ\widetilde{\Gamma}(y\otimes 1). \qquad \square$$

4.5 Proposition ([40]). *Consider on the Hilbert space $H\otimes H_\nu$ the anti-linear operator:*

$$I=U_\nu^{\mathfrak{a}}(J\otimes J_\nu)U_\nu^{\widehat{\mathfrak{a}}}(U_\nu^{\mathfrak{a}})^*=U_\nu^{\mathfrak{a}}(U_\nu^{\widehat{\mathfrak{a}}})^*(J\otimes J_\nu)(U_\nu^{\mathfrak{a}})^*=U_\nu^{\mathfrak{a}}(U_\nu^{\widehat{\mathfrak{a}}})^*J_{\widetilde{\hat{\nu}}}U_\nu^{\widehat{\mathfrak{a}}}(U_\nu^{\mathfrak{a}})^*,$$

where $\widetilde{\hat{\nu}}$ denotes the dual weight of ν on the crossed product $\widehat{\mathbb{G}}\ltimes_{\widehat{\mathfrak{a}}}N$.

(i) *I is a bijective isometry and $I^2=1$.*

(ii) *$I\mathfrak{a}(x)^*I=\beta(x)$ and $I\beta(x)^*I=\mathfrak{a}(x)$ for all $x\in N$.*

(iii) $I(y^* \otimes 1)I = \widehat{R}(y) \otimes 1$ *for all* $y \in \widehat{M}$.

(iv) *If* σ_ν *denotes the flip from* $(H \otimes H_\nu) {}_\beta\!\underset{\nu}{\otimes}_{\mathfrak{a}} (H \otimes H_\nu)$ *to* $(H \otimes H_\nu) {}_{\mathfrak{a}}\!\underset{\nu^{\mathrm{o}}}{\otimes}_\beta (H \otimes H_\nu)$, *then*

$$V_2 = (J \otimes I)V_1(I \;{}_{\mathfrak{a}}\!\underset{N^{\mathrm{o}}}{\otimes}_\beta\; I)\sigma_\nu.$$

Proof. (i) The relation $U_\nu^{\widehat{\mathfrak{a}}} = J_{\widetilde{\widehat{\nu}}}(J \otimes J_\nu)$ (2.2) shows that the three expressions given for I coincide and that I is isometric, bijective, anti-linear, and equal to I^*. Moreover, the formula $I = U_\nu^{\mathfrak{a}}(U_\nu^{\widehat{\mathfrak{a}}})^* J_{\widetilde{\widehat{\nu}}} U_\nu^{\widehat{\mathfrak{a}}}(U_\nu^{\mathfrak{a}})^*$ shows that $I^2 = 1_H \otimes 1_{H_\nu}$.

(ii) We only need to prove the first equation. But by 2.5.4,

$$\begin{aligned} I\mathfrak{a}(x)^* I^* &= \mathrm{Ad}(U_\nu^{\mathfrak{a}}(U_\nu^{\widehat{\mathfrak{a}}})^*(J \otimes J_\nu)(U_\nu^{\mathfrak{a}})^*)[\mathfrak{a}(x)^*] \\ &= \mathrm{Ad}(U_\nu^{\mathfrak{a}}(U_\nu^{\widehat{\mathfrak{a}}})^*)[1 \otimes x^{\mathrm{o}}] = \beta(x). \end{aligned}$$

(iii) Using 3.8(iii) and the fact that $U_\nu^{\mathfrak{a}}$ is a representation, we find that

$$\begin{aligned} (\widehat{J} \otimes I)W_{12}(\widehat{J} \otimes I) &= \mathrm{Ad}((U_\nu^{\mathfrak{a}})_{23}(\widehat{J} \otimes J \otimes J_\nu)(U_\nu^{\widehat{\mathfrak{a}}})_{23}(U_\nu^{\mathfrak{a}})^*_{23})[W_{12}] \\ &= \mathrm{Ad}((U_\nu^{\mathfrak{a}})_{23}(\widehat{J} \otimes J \otimes J_\nu))[(U_\nu^{\mathfrak{a}})^*_{13} W_{12}] \\ &= \mathrm{Ad}((U_\nu^{\mathfrak{a}})_{23})[(U_\nu^{\mathfrak{a}})_{13} W_{12}^*] \\ &= W_{12}^*. \end{aligned}$$

For any ξ, η in H, we can conclude that

$$\begin{aligned} I(J(\omega_{\xi,\eta} \otimes \mathrm{id})(W)^* J \otimes 1)I &= I((\omega_{\widehat{J}\eta,\widehat{J}\xi} \otimes \mathrm{id})(W) \otimes 1)I \\ &= (\omega_{\xi,\eta} \otimes \mathrm{id})(W)^* \otimes 1, \end{aligned}$$

from which (iii) follows by continuity.

(iv) By (ii),

$$\begin{aligned} V_1(I \;{}_{\mathfrak{a}}\!\underset{\nu^{\mathrm{o}}}{\otimes}_\beta\; I)\sigma_\nu[U_\nu^{\mathfrak{a}}(U_\nu^{\widehat{\mathfrak{a}}})^*(\xi \otimes \Lambda_\nu(q)) \;{}_\beta\!\underset{\nu}{\otimes}_{\mathfrak{a}}\; \Xi] \\ &= V_1[I\Xi \;{}_\beta\!\underset{\nu}{\otimes}_{\mathfrak{a}}\; U_\nu^{\mathfrak{a}}(J\xi \otimes J_\nu\Lambda_\nu(q))] \\ &= J\xi \otimes \beta(q^*)I\Xi \\ &= J\xi \otimes I\mathfrak{a}(q)\Xi \\ &= (J \otimes I)V_2[U_\nu^{\mathfrak{a}}(U_\nu^{\widehat{\mathfrak{a}}})^*(\xi \otimes \Lambda_\nu(q)) \;{}_\beta\!\underset{\nu}{\otimes}_{\mathfrak{a}}\; \Xi]. \qquad \square \end{aligned}$$

4.6 Theorem ([40]). *Let* $\mathbb{G}$ *be a locally compact quantum group,* $(N, \mathfrak{a}, \widehat{\mathfrak{a}})$ *a braided-commutative* $\mathbb{G}$*-Yetter–Drinfel'd algebra and* I *the anti-linear surjective isometry constructed in* 4.5*. Then:*

(i) *For all* $z \in \mathbb{G} \ltimes_{\mathfrak{a}} N$, *let* $\widetilde{R}(z) = Iz^*I$. *Then* $\widetilde{R}$ *is an involutive anti-$*$-isomorphism of* $\mathbb{G} \ltimes_{\mathfrak{a}} N$, *and* $\widetilde{R}(\mathfrak{a}(x)) = \beta(x)$, $\widetilde{R}(\beta(x)) = \mathfrak{a}(x)$ *and* $\widetilde{R}(y \otimes 1_{H_\nu}) = \widehat{R}(y) \otimes 1_{H_\nu}$ *for all* $x \in N$ *and* $y \in \widehat{M}$.

(ii) $\widetilde{R}$ *is a co-inverse for the Hopf bimodule* $(N, \mathbb{G} \ltimes_{\mathfrak{a}} N, \mathfrak{a}, \beta, \widetilde{\Gamma})$ *constructed in 4.4.*

Proof. (i) This is just a straightforward corollary of 4.5 (ii) and (iii).

(ii) We need to prove that

$$\widetilde{\Gamma} = \varsigma_{N^{\circ}}(\widetilde{R} \underset{N}{{}_{\beta}\otimes_{\alpha}} \widetilde{R})\widetilde{\Gamma}\widetilde{R}.$$

Using (i), we find that for $x \in N$,

$$\begin{aligned}
\varsigma_{N^{\circ}}(\widetilde{R} \underset{N}{{}_{\beta}\otimes_{\alpha}} \widetilde{R})\widetilde{\Gamma}\widetilde{R}(\mathfrak{a}(x)) &= \varsigma_{N^{\circ}}(\widetilde{R} \underset{N}{{}_{\beta}\otimes_{\alpha}} \widetilde{R})\widetilde{\Gamma}(\beta(x)) \\
&= \varsigma_{N^{\circ}}(\widetilde{R} \underset{N}{{}_{\beta}\otimes_{\alpha}} \widetilde{R})((1_H \otimes 1_{H_\nu}) \underset{N}{{}_{\beta}\otimes_{\alpha}} \beta(x)) \\
&= \mathfrak{a}(x) \underset{N}{{}_{\beta}\otimes_{\mathfrak{a}}} (1_H \otimes 1_{H_\nu})
\end{aligned}$$

coincides with $\widetilde{\Gamma}(\mathfrak{a}(x))$. For $y \in \widehat{M}$, we conclude from 4.4 and 4.5 (iv) that

$$\begin{aligned}
\varsigma_{N^{\circ}}(\widetilde{R} \underset{N}{{}_{\beta}\otimes_{\alpha}} \widetilde{R})\widetilde{\Gamma}\widetilde{R}(y \otimes 1_{H_\nu}) &= \varsigma_{N^{\circ}}(\widetilde{R} \underset{N}{{}_{\beta}\otimes_{\alpha}} \widetilde{R})\widetilde{\Gamma}(\widehat{R}(y) \otimes 1_{H_\nu}) \\
&= \varsigma_{N^{\circ}}(\widetilde{R} \underset{N}{{}_{\beta}\otimes_{\alpha}} \widetilde{R})[V_2^*(\widehat{\Gamma}(\widehat{R}(y) \otimes 1_{H_\nu})V_2] \\
&= V_1^*((\widehat{R} \otimes \widehat{R})\widehat{\Gamma}(\widehat{R}(y)) \otimes 1_{H_\nu})V_1 \\
&= \widetilde{\Gamma}(y \otimes 1_{H_\nu})
\end{aligned}$$

As $\mathbb{G} \ltimes_{\mathfrak{a}} N$ is the von Neumann algebra generated by $\mathfrak{a}(N)$ and $\widehat{M} \otimes 1_{H_\nu}$, this finishes the proof of (ii). □

4.7 Lemma. *Let* $\mathbb{G}$ *be a locally compact quantum group,* $(N, \mathfrak{a}, \widehat{\mathfrak{a}})$ *a braided-commutative* $\mathbb{G}$*-Yetter–Drinfel'd algebra,* $\widetilde{\Gamma}$ *the injective $*$-homomorphism from* $\mathbb{G} \ltimes_{\mathfrak{a}} N$ *into* $(\mathbb{G} \ltimes_{\mathfrak{a}} N) \underset{N}{{}_{\beta}*_{\mathfrak{a}}} (\mathbb{G} \ltimes_{\mathfrak{a}} N)$ *defined in 4.4,* $\tilde{\mathfrak{a}}$ *the dual action of* $\widehat{\mathbb{G}}$ *on* $\mathbb{G} \ltimes_{\mathfrak{a}} N$, *and* V_1 *as in 4.3. Denote by* τ *the flip from* $(H \otimes H_\nu)_{\beta} \underset{\nu}{\otimes}_{(1\otimes\mathfrak{a})} (H \otimes H \otimes H_\nu)$ *onto* $H \otimes [(H \otimes H_\nu) \underset{\nu}{{}_{\beta}\otimes_{\mathfrak{a}}} (H \otimes H_\nu)]$ *given by*

$$\tau(\Xi_{\beta} \underset{\nu}{\otimes}_{(1\otimes\mathfrak{a})}(\xi \otimes \Xi')) = \xi \otimes \Xi \underset{\nu}{{}_{\beta}\otimes_{\alpha}} \Xi'$$

for all $\xi \in H$, $\Xi \in D({}_{\beta}(H \otimes H_\nu), \nu^{\mathrm{o}})$, $\Xi' \in D({}_{\mathfrak{a}}(H \otimes H_\nu), \nu)$. *Then:*

(i) $(\mathrm{id} \underset{N}{{}_\beta *_\mathfrak{a}} \tilde{\mathfrak{a}})\widetilde{\Gamma}(X) = \tau^*(\mathrm{id} \otimes \widetilde{\Gamma})\tilde{\mathfrak{a}}(X)\tau$ *for all* $X \in \mathbb{G} \ltimes_\mathfrak{a} N$.

(ii) $V_2\widetilde{\Gamma}(X)V_2^* = (\widehat{R} \otimes \widetilde{R})\tilde{\mathfrak{a}}(\widetilde{R}(X))$.

Proof. (i) For any $x' \in M'$, we have

$$V_1[(1_H \otimes 1_{H_\nu}) \underset{N}{{}_\beta\otimes_\mathfrak{a}} (x' \otimes 1_{H_\nu})] = (x' \otimes 1_H \otimes 1_{H_\nu})V_1.$$

As $\widehat{W}^o$ belongs to $\widehat{M} \otimes M'$, we infer

$$(1_H \otimes V_1)\tau[(1_H \otimes 1_{H_\nu})_\beta \underset{N}{\otimes}_{1\otimes\mathfrak{a}}(\widehat{W}^o \otimes 1_{H_\nu})] = (\widehat{W}^o \otimes 1_H \otimes 1_{H_\nu})(1_H \otimes V_1)\tau. \tag{3}$$

Therefore, we can conclude that for all $X \in \mathbb{G} \ltimes_\mathfrak{a} N$,

$$\begin{aligned}
(\mathrm{id} \underset{N}{{}_\beta *_\mathfrak{a}} \tilde{\mathfrak{a}})\widetilde{\Gamma}(X)
&= \mathrm{Ad}([(1_H \otimes 1_{H_\nu})_\beta \underset{N}{\otimes}_{1\otimes\mathfrak{a}}(\widehat{W}^{o*} \otimes 1_{H_\nu})]\tau^*(1_H \otimes V_1^*))[1_H \otimes \tilde{\mathfrak{a}}(X)]\\
&= \mathrm{Ad}(\tau^*(1_H \otimes V_1^*)(\widehat{W}^{o*} \otimes 1_H \otimes 1_{H_\nu}))[1_H \otimes \tilde{\mathfrak{a}}(X)]\\
&= \mathrm{Ad}(\tau^*(1_H \otimes V_1^*))[(\widehat{\Gamma}^o \otimes \mathrm{id})\tilde{\mathfrak{a}}(X)]\\
&= \mathrm{Ad}(\tau^*(1_H \otimes V_1^*))[(\mathrm{id} \otimes \tilde{\mathfrak{a}})\tilde{\mathfrak{a}}(X)]\\
&= \tau^*(\mathrm{id} \otimes \widetilde{\Gamma})\tilde{\mathfrak{a}}(X)\tau.
\end{aligned}$$

(ii) By 4.5 (iii),

$$\begin{aligned}
\mathrm{Ad}(V_2)[\widetilde{\Gamma}(X)] &= \mathrm{Ad}((J \otimes I)V_1\sigma_{\nu^o}(I \underset{N}{{}_\beta\otimes_\mathfrak{a}} I))[\widetilde{\Gamma}(X)]\\
&= \mathrm{Ad}((J \otimes I)V_1)[\widetilde{\Gamma}\widetilde{R}(X^*)]\\
&= (\widehat{R} \otimes \widetilde{R})\tilde{\mathfrak{a}}(\widetilde{R}(X)). \qquad \square
\end{aligned}$$

5. Measured quantum groupoid structure associated to a braided-commutative Yetter–Drinfel'd algebra equipped with an appropriate weight

In this chapter, after recalling the definition of a measured quantum groupoid (5.1) and describing the major data associated to a measured quantum groupoid (5.2), (5.3), we try to construct, given a braided-commutative $\mathbb{G}$-Yetter–Drinfel'd algebra $(A, \mathfrak{a}, \widehat{\mathfrak{a}})$ and a normal semi-finite faithful weight on N, a structure of a measured quantum groupoid, denoted $\mathfrak{G}(N, \mathfrak{a}, \widehat{\mathfrak{a}}, \nu)$, on the crossed product $\mathbb{G} \ltimes_\mathfrak{a} N$ or, more precisely, on the Hopf bimodule constructed in 4.6. Without any hypothesis on the normal faithful semi-finite weight ν on N, we construct a left-invariant operator-valued weight (5.4) and a right-invariant one (5.4), and we give a necessary and sufficient condition for a weight ν on N to be relatively invariant with respect to these two operator-valued weights (5.9). This condition is clearly satisfied (5.10) if ν is k-invariant with respect to $\mathfrak{a}$ (for k affiliated to $Z(M)$, or $k = \delta^{-1}$).

5.1. Definition of measured quantum groupoids [13, 24]. A *measured quantum groupoid* is an octuple $\mathfrak{G} = (N, M, \alpha, \beta, \Gamma, T, T', \nu)$ such that [13, 3.8]:

(i) $(N, M, \alpha, \beta, \Gamma)$ is a Hopf bimodule,

(ii) T is a left-invariant normal, semi-finite, faithful operator-valued weight from M to $\alpha(N)$ (to be more precise, from M^+ to the extended positive elements of $\alpha(N)$ (cf. [38, IX.4.12])), which means that, for any $x \in \mathfrak{M}_T^+$, we have

$$(\mathrm{id} \underset{\nu}{{}_\beta *_\alpha} T)\Gamma(x) = T(x) \underset{N}{{}_\beta \otimes_\alpha} 1.$$

(iii) T' is a right-invariant normal, semi-finite, faithful operator-valued weight from M to $\beta(N)$, which means that, for any $x \in \mathfrak{M}_{T'}^+$, we have

$$(T' \underset{\nu}{{}_\beta *_\alpha} \mathrm{id})\Gamma(x) = 1 \underset{N}{{}_\beta \otimes_\alpha} T'(x).$$

(iv) ν is normal semi-finite faithful weight on N, which is relatively invariant with respect to T and T', which means that the modular automorphisms groups of the weights $\Phi = \nu \circ \alpha^{-1} \circ T$ and $\Psi = \nu \circ \beta^{-1} \circ T'$ commute. The weight Φ will be called left-invariant, and Ψ right-invariant.

For example, let $\mathcal{G}$ be a measured groupoid equipped with a left Haar system $(\lambda^u)_{u \in \mathcal{G}^{(0)}}$ and a quasi-invariant measure ν on $\mathcal{G}^{(0)}$. Let us use the notations introduced in 4.2. If $f \in L^\infty(\mathcal{G}, \mu)^+$, consider the function on $\mathcal{G}^{(0)}$, $u \mapsto \int_{\mathcal{G}} f d\lambda^u$, which belongs to $L^\infty(\mathcal{G}^{(0)}, \nu)$. The image of this function by the homomorphism $r_{\mathcal{G}}$ is the function on $\mathcal{G}$, $\gamma \mapsto \int_{\mathcal{G}} f d\lambda^{r(\gamma)}$, and the application which sends f to this function can be considered as an operator-valued weight from $L^\infty(\mathcal{G}, \mu)$ to $r_{\mathcal{G}}(L^\infty(\mathcal{G}^{(0)}, \nu))$ which is normal, semi-finite and faithful. By definition of the Haar system $(\lambda^u)_{u \in \mathcal{G}^{(0)}}$, it is left-invariant in the sense of (ii). We shall denote this operator-valued weight from $L^\infty(\mathcal{G}, \mu)$ to $r_{\mathcal{G}}(L^\infty(\mathcal{G}^{(0)}, \nu))$ by $T_{\mathcal{G}}$. If we write λ_u for the image of λ^u under the inversion $x \mapsto x^{-1}$ of the groupoid $\mathcal{G}$, starting from the application which sends f to the function on $\mathcal{G}^{(0)}$ defined by $u \mapsto \int_{\mathcal{G}} f d\lambda_u$, we define a normal semifinite faithful operator-valued weight from $L^\infty(\mathcal{G}, \mu)$ to $s_{\mathcal{G}}(L^\infty(\mathcal{G}^{(0)}, \nu))$, which is right-invariant in the sense of (ii), and which we shall denote by $T_{\mathcal{G}}^{(-1)}$.

We then get that

$$(L^\infty(\mathcal{G}^{(0)}, \nu), L^\infty(\mathcal{G}, \mu), r_{\mathcal{G}}, s_{\mathcal{G}}, \Gamma_{\mathcal{G}}, T_{\mathcal{G}}, T_{\mathcal{G}}^{(-1)}, \nu)$$

is a measured quantum groupoid, which we shall denote again $\mathcal{G}$.

It can be proved [15] that any measured quantum groupoid, whose underlying von Neumann algebra is abelian, is of that type.

5.2. Pseudo-multiplicative unitary. Let $\mathfrak{G} = (N, M, \alpha, \beta, \Gamma, T, T', \nu)$ be an octuple satisfying the axioms (i), (ii) (iii) of 5.1. We shall write $H = H_\Phi$, $J = J_\Phi$ and $\gamma(n) = J\alpha(n^*)J$ for all $n \in N$.

Then [24, 3.7.3 and 3.7.4], $\mathfrak{G}$ can be equipped with a pseudo-multiplicative unitary W which is a unitary from $H \,{}_\beta\!\underset{\nu}{\otimes}_\alpha H$ onto $H \,{}_\alpha\!\underset{\nu^o}{\otimes}_\gamma H$ [13, 3.6] that intertwines α, γ, β in the following way: for all $X \in N$,

$$
\begin{aligned}
W(\alpha(X) \,{}_\beta\!\underset{N}{\otimes}_\alpha 1) &= (1 \,{}_\alpha\!\underset{N^o}{\otimes}_\gamma \alpha(X))W,\\
W(1 \,{}_\beta\!\underset{N}{\otimes}_\alpha \beta(X)) &= (1 \,{}_\alpha\!\underset{N^o}{\otimes}_\gamma \beta(X))W,\\
W(\gamma(X) \,{}_\beta\!\underset{N}{\otimes}_\alpha 1) &= (\gamma(X) \,{}_\alpha\!\underset{N^o}{\otimes}_\gamma 1)W,\\
W(1 \,{}_\beta\!\underset{N}{\otimes}_\alpha \gamma(X)) &= (\beta(X) \,{}_\alpha\!\underset{N^o}{\otimes}_\gamma 1)W.
\end{aligned}
$$

Moreover, the operator W satisfies the *pentagonal relation*

$$
(1 \,{}_\alpha\!\underset{N^o}{\otimes}_\gamma W)(W \,{}_\beta\!\underset{N}{\otimes}_\alpha 1_H) = (W \,{}_\alpha\!\underset{N^o}{\otimes}_\gamma 1)\sigma^{23}_{\alpha,\beta}(W \,{}_\gamma\!\underset{N}{\otimes}_\alpha 1)(1 \,{}_\beta\!\underset{N}{\otimes}_\alpha \sigma_{\nu^o})(1 \,{}_\beta\!\underset{N}{\otimes}_\alpha W),
$$

where $\sigma^{23}_{\alpha,\beta}$ goes from $(H \,{}_\alpha\!\underset{\nu^o}{\otimes}_\gamma H) \,{}_\beta\!\underset{\nu}{\otimes}_\alpha H$ to $(H \,{}_\beta\!\underset{\nu}{\otimes}_\alpha H) \,{}_\alpha\!\underset{\nu^o}{\otimes}_\gamma H$, and $1 \,{}_\beta\!\underset{N}{\otimes}_\alpha \sigma_{\nu^o}$ goes from $H \,{}_\beta\!\underset{\nu}{\otimes}_\alpha (H \,{}_\alpha\!\underset{\nu^o}{\otimes}_\gamma H)$ to $H \,{}_\beta\!\underset{\nu}{\otimes}_\alpha H \,{}_\gamma\!\underset{\nu}{\otimes}_\alpha H$. The operators in this formula are well defined because of the intertwining relations listed above.

Moreover, W, M and Γ are related by the following results:

(i) M is the weakly closed linear space generated by all operators of the form $(\mathrm{id} * \omega_{\xi,\eta})(W)$, where $\xi \in D({}_\alpha H, \nu)$ and $\eta \in D(H_\gamma, \nu^o)$ see [13, 3.8(vii)].

(ii) $\Gamma(x) = W^*(1 \,{}_\alpha\!\underset{N^o}{\otimes}_\gamma x)W$ for all $x \in M$ [13, 3.6].

(iii) For any x, y_1, y_2 in $\mathfrak{N}_T \cap \mathfrak{N}_\Phi$, we have [13, 3.6]

$$
(\mathrm{id} * \omega_{J_\Phi\Lambda_\Phi(y_1^* y_2), \Lambda_\Phi(x)})(W) = (\mathrm{id} \,{}_\beta\!\underset{N}{*}_\alpha \omega_{J_\Phi\Lambda_\Phi(y_2), J_\Phi\Lambda_\Phi(y_1)})\Gamma(x^*).
$$

If N is finite-dimensional, using the fact that the relative tensor products can be identified with closed subspaces of the usual Hilbert tensor product (4.1), we get that W can be considered as a partial isometry, which is multiplicative in the usual sense (i.e. such that $W_{23}W_{12} = W_{12}W_{13}W_{23}$.)

5.3. Other data associated to a measured quantum groupoid [13, 24]. Suppose that $\mathfrak{G} = (N, M, \alpha, \beta, \Gamma, T, T', \nu)$ is a measured quantum groupoid in the sense of 5.1. Let us write $\Phi = \nu \circ \alpha^{-1} \circ T$, which is a normal semi-finite faithful left-invariant weight on M. Then:

(i) There exists an anti-$*$-automorphism R on M such that

$$R^2 = \mathrm{id}, \quad R(\alpha(n)) = \beta(n) \text{ for all } n \in N, \quad \Gamma \circ R = \varsigma_{N^{\mathrm{o}}}(R \underset{N}{{}_\beta *_\alpha} R)\Gamma$$

and

$$R((\mathrm{id} * \omega_{\xi,\eta})(W)) = (\mathrm{id} * \omega_{J\eta,J\xi})(W) \quad \text{for all } \xi \in D({}_\alpha H, \nu), \eta \in D(H_\gamma, \nu^{\mathrm{o}}).$$

This map R will be called the *co-inverse*.

(ii) There exists a one-parameter group τ_t of automorphisms of M such that

$$R \circ \tau_t = \tau_t \circ R, \quad \tau_t(\alpha(n)) = \alpha(\sigma_t^\nu(n)), \quad \tau_t(\beta(n)) = \beta(\sigma_t^\nu(n)),$$
$$\Gamma \circ \sigma_t^\Phi = (\tau_t \underset{N}{{}_\beta *_\alpha} \sigma_t^\Phi)\Gamma$$

for all $t \in \mathbb{R}$ and and $n \in N$. This one-parameter group will be called the *scaling group*.

(iii) The weight ν is relatively invariant with respect to T and RTR. Moreover, R and τ_t are still the co-inverse and the scaling group of this new measured quantum groupoid, which we shall denote by

$$\underline{\mathfrak{G}} = (N, M, \alpha, \beta, \Gamma, T, RTR, \nu),$$

and for simplification we shall assume now that $T' = RTR$ and $\Psi = \Phi \circ R$.

(iv) There exists a one-parameter group γ_t of automorphisms of N such that

$$\sigma_t^T(\beta(n)) = \beta(\gamma_t(n))$$

for all $t \in \mathbb{R}$ and $n \in N$. Moreover, we get that $\nu \circ \gamma_t = \nu$.

(v) There exist a positive non-singular operator λ affiliated to $Z(M)$ and a positive non-singular operator δ affiliated with M such that

$$(D\Phi \circ R : D\Phi)_t = \lambda^{it^2/2}\delta^{it},$$

and therefore

$$(D\Phi \circ \sigma_s^{\Phi \circ R} : D\Phi)_t = \lambda^{ist}.$$

The operator λ will be called the *scaling operator*, and there exists a positive non-singular operator q affiliated to N such that $\lambda = \alpha(q) = \beta(q)$. We have $R(\lambda) = \lambda$.

The operator δ will be called the *modulus*. We have $R(\delta) = \delta^{-1}$ and $\tau_t(\delta) = \delta$ for all $t \in \mathbb{R}$, and we can define a one-parameter group of unitaries $\delta^{it} \underset{N}{{}_\beta\otimes_\alpha} \delta^{it}$ which acts naturally on elementary tensor products and satisfies for all $t \in \mathbb{R}$

$$\Gamma(\delta^{it}) = \delta^{it} \underset{N}{{}_\beta\otimes_\alpha} \delta^{it}.$$

(vi) We have $(D\Phi \circ \tau_t : D\Phi)_s = \lambda^{-ist}$, which proves that $\tau_t \circ \sigma_s^\Phi = \sigma_s^\Phi \circ \tau_t$ for all s, t in $\mathbb{R}$ and allows to define a one-parameter group of unitaries by

$$P^{it}\Lambda_\Phi(x) = \lambda^{t/2}\Lambda_\Phi(\tau_t(x)) \quad \text{for all } x \in \mathfrak{N}_\Phi.$$

Moreover, for any y in M, we get that

$$\tau_t(y) = P^{it} y P^{-it}.$$

As for the multiplicative unitary associated to a locally compact quantum group, one can prove, using this operator P, a "managing property" for W, and we shall say that the pseudo-multiplicative unitary W is *manageable*, with "managing operator" P.

As $\tau_t \circ \sigma_t^\Phi = \sigma_t^\Phi \circ \tau_t$, we get that $J_\Phi P J_\Phi = P$.

(vii) It is possible to construct a *dual* measured quantum groupoid

$$\widehat{\mathfrak{G}} = (N, \widehat{M}, \alpha, \gamma, \widehat{\Gamma}, \widehat{T}, \widehat{T}', \nu)$$

where $\widehat{M}$ is equal to the weakly closed linear space generated by all operators of the form $(\omega_{\xi,\eta} * \mathrm{id})(W)$, for $\xi \in D(H_\beta, \nu^{\mathrm{o}})$ and $\eta \in D({}_\alpha H, \nu)$, $\widehat{\Gamma}(y) = \sigma_{\nu^{\mathrm{o}}} W(y \underset{N}{{}_\beta\otimes_\alpha} 1)W^*\sigma_\nu$ for all $y \in \widehat{M}$, and the dual left operator-valued weight $\widehat{T}$ is constructed in a similar way as the dual left-invariant weight of a locally compact quantum group. Namely, it is possible to construct a normal semi-finite faithful weight $\widehat{\Phi}$ on $\widehat{M}$ such that, for all $\xi \in D(H_\beta, \nu^{\mathrm{o}})$ and $\eta \in D({}_\alpha H, \nu)$ such that $\omega_{\xi,\eta}$ belongs to I_Φ,

$$\widehat{\Phi}((\omega_{\xi,\eta} * \mathrm{id})(W)^*(\omega_{\xi,\eta} * \mathrm{id})(W)) = \|\omega_{\xi,\eta}\|_\Phi^2.$$

We can prove that $\sigma_t^{\widehat{\Phi}} \circ \alpha = \alpha \circ \sigma_t^\nu$ for all $t \in \mathbb{R}$, which gives the existence of an operator-valued weight $\widehat{T}$, which appears then to be left-invariant.

As the formula $y \mapsto Jy^*J$ $(y \in \widehat{M})$ gives a co-inverse for the coproduct $\widehat{\Gamma}$, we get also a right-invariant operator-valued weight. Moreover, the pseudo-multiplicative unitary $\widehat{W}$ associated to $\widehat{\mathfrak{G}}$ is $\widehat{W} = \sigma_\nu W^* \sigma_\nu$, its managing operator $\widehat{P}$ is equal to P, its scaling group is given by $\widehat{\tau}_t(y) = P^{it} y P^{-it}$, its scaling operator $\widehat{\lambda}$ is equal to λ^{-1}, and its one-parameter group of unitaries $\widehat{\gamma}_t$ of N is equal to γ_{-t}.

We write $\widehat{\Phi}$ for $\nu \circ \alpha^{-1} \circ \widehat{T}$, identify $H_{\widehat{\Phi}}$ with H, and write $\widehat{J} = J_{\widehat{\Phi}}$. Then $R(x) = \widehat{J} x^* \widehat{J}$ for all $x \in M$ and $W^* = (\widehat{J} \underset{N^o}{{}_\alpha\otimes_\gamma} J) W (\widehat{J} \underset{N^o}{{}_\alpha\otimes_\gamma} J)$.

Moreover, we have $\widehat{\widehat{\mathfrak{G}}} = \mathfrak{G}$.

For example, let $\mathcal{G}$ be a measured groupoid as in 5.1. The dual $\widehat{\mathcal{G}}$ of the measured quantum groupoid constructed in 5.1 (and denoted again by $\mathcal{G}$) is

$$\widehat{\mathcal{G}} = (L^\infty(\mathcal{G}^{(0)}, \nu), \mathcal{L}(\mathcal{G}), r_{\mathcal{G}}, r_{\mathcal{G}}, \widehat{\Gamma}_{\mathcal{G}}, \widehat{T}_{\mathcal{G}}, \widehat{T}_{\mathcal{G}}),$$

where $\mathcal{L}(\mathcal{G})$ is the von Neumann algebra generated by the convolution algebra associated to the groupoid $\mathcal{G}$, the coproduct $\widehat{\Gamma}_{\mathcal{G}}$ had been defined in [44, 3.3.2], and the operator-valued weight $\widehat{T}_{\mathcal{G}}$ had been defined in [44, 3.3.4]. The underlying Hopf bimodule is co-commutative.

5.4 Theorem ([40]). *Let $\mathbb{G}$ be a locally compact quantum group and $(N, \mathfrak{a}, \widehat{\mathfrak{a}})$ a braided-commutative $\mathbb{G}$-Yetter–Drinfel'd algebra. Then the normal faithful semi-finite operator-valued weight $T_{\tilde{\mathfrak{a}}}$ from $\mathbb{G} \ltimes_{\mathfrak{a}} A$ onto $\mathfrak{a}(N)$ [41, 1.3 and 2.5] is left-invariant with respect to the Hopf bimodule structure constructed in 4.6, and $\tilde{R} \circ T_{\tilde{\mathfrak{a}}} \circ \tilde{R}$ is right-invariant.*

Proof. For all positive X in $\mathbb{G} \ltimes_{\mathfrak{a}} N$, we find, using 4.7 (i) and 4.6,

$$\begin{aligned}(\mathrm{id} \underset{N}{{}_\beta *_{\mathfrak{a}}} T_{\tilde{\mathfrak{a}}})\widetilde{\Gamma}(X) &= (\mathrm{id} \underset{N}{{}_\beta *_{\mathfrak{a}}} (\widehat{\varphi} \circ \widehat{R} \otimes \mathrm{id})\tilde{\mathfrak{a}})\widetilde{\Gamma}(X)\\ &= (\widehat{\varphi} \circ \widehat{R} \otimes \mathrm{id})(\mathrm{id} \otimes \widetilde{\Gamma})\tilde{\mathfrak{a}}(X)\\ &= \widetilde{\Gamma}(T_{\tilde{\mathfrak{a}}}(X))\\ &= T_{\tilde{\mathfrak{a}}}(X) \underset{N}{{}_\beta\otimes_{\mathfrak{a}}} (1_H \otimes 1_{H_\nu})\end{aligned}$$

which proves that $T_{\tilde{\mathfrak{a}}}$ is left-invariant. Using 4.6, we get trivially that $\widetilde{R} \circ T_{\tilde{\mathfrak{a}}} \circ \widetilde{R}$ is a normal faithful semi-finite operator valued weight from $\mathbb{G} \ltimes_{\mathfrak{a}} N$ onto $\beta(N)$, which is right-invariant with respect to the coproduct $\widetilde{\Gamma}$. □

In the situation above, we shall denote by $\mathfrak{G}(N, \mathfrak{a}, \widehat{\mathfrak{a}}, \nu)$ the Hopf-bimodule $(N, \mathbb{G} \ltimes_{\mathfrak{a}} N, \mathfrak{a}, \beta, \widetilde{\Gamma})$ constructed in 4.4 (ii), equipped with its co-inverse $\widetilde{R}$ constructed in 4.6 (ii), with the left-invariant operator-valued weight $T_{\tilde{\mathfrak{a}}}$ and the right-invariant operator-vlaued weight $\widetilde{R} \circ T_{\tilde{\mathfrak{a}}} \circ \widetilde{R}$, and with the normal semi-finite faithful weight ν on N.

5.5 Proposition. *Let $\mathbb{G}$ be a locally compact quantum group, $(N, \mathfrak{a}, \widehat{\mathfrak{a}})$ a braided-commutative $\mathbb{G}$-Yetter–Drinfel'd algebra, ν a normal semi-finite faithful weight on N, D_t its Radon–Nikodym derivative with respect to $\mathfrak{a}$ (2.2) and D_t^o the Radon–Nikodym derivative of the weight ν^o on N^o with respect to the action $\mathfrak{a}^o$ (2.5.1). For all $t \in \mathbb{R}$, denote by $\widetilde{\tau}_t$ the map $\mathrm{Ad}[U_\nu^{\mathfrak{a}}(U_\nu^{\widehat{\mathfrak{a}}})^* \Delta_{\widetilde{\hat{\nu}}}^{it} U_\nu^{\widehat{\mathfrak{a}}}(U_\nu^{\mathfrak{a}})^*]$ defined on $B(H \otimes H_\nu)$, where $\widetilde{\hat{\nu}}$ is the dual weight of ν on the crossed product $\widehat{\mathbb{G}} \ltimes_{\widehat{\mathfrak{a}}} N$. Then:*

(i) $\widetilde{\tau}_t \circ \beta(x) = \beta(\sigma_t^\nu(x))$ *for all* $x \in N$ *and* $t \in \mathbb{R}$.

(ii) *for all* $t \in \mathbb{R}$, $\widetilde{\tau}_t$ *commutes with* $\operatorname{Ad} I$, *where* I *had been defined in 4.5, and, therefore* $\widetilde{\tau}_t(\mathfrak{a}(x)) = \mathfrak{a}(\sigma_t^\nu(x))$ *for all* $x \in N$ *and* $t \in \mathbb{R}$.

(iii) *Denote by* $\beta^\dagger$ *the application* $x^{\mathrm{o}} \mapsto \beta(x)$ *from* N^{o} *into* $\mathbb{G} \ltimes_{\mathfrak{a}} N$. *Then*

$$\begin{aligned}(\mathrm{id} \otimes \widetilde{\tau}_t)(W_{12}) &= \widehat{\Delta}_1^{-it}(\mathrm{id} \otimes \beta^\dagger)(D^{\mathrm{o}}_{-t})W_{12}(\mathrm{id} \otimes \mathfrak{a})(D_t)\widehat{\Delta}_1^{it} \\ &= (\tau_{-t} \otimes \beta^\dagger)(D^{\mathrm{o}}_{-t})(\mathrm{id} \otimes \widehat{\tau})(W)_{12}(\tau_{-t} \otimes \mathfrak{a})(D_t).\end{aligned}$$

(iv) $\widetilde{\tau}_t(\mathbb{G} \ltimes_{\mathfrak{a}} N) = \mathbb{G} \ltimes_{\mathfrak{a}} N$ *and* $\widetilde{\tau}_t \circ \widetilde{R} = \widetilde{R} \circ \widetilde{\tau}_t$.

Proof. (i) For any $x \in N$,

$$\begin{aligned}\widetilde{\tau}_t(\beta(x)) &= \operatorname{Ad}(U_\nu^{\mathfrak{a}}(\widehat{U_\nu^{\mathfrak{a}}})^*)[\Delta_{\widetilde{\hat{\nu}}}^{it}] \cdot \operatorname{Ad}(U_\nu^{\mathfrak{a}}(\widehat{U_\nu^{\mathfrak{a}}})^*)[1 \otimes x^{\mathrm{o}}] \cdot \operatorname{Ad}(U_\nu^{\mathfrak{a}}(\widehat{U_\nu^{\mathfrak{a}}})^*)[\Delta_{\widetilde{\hat{\nu}}}^{-it}] \\ &= \operatorname{Ad}(U_\nu^{\mathfrak{a}}(\widehat{U_\nu^{\mathfrak{a}}})^*\Delta_{\widetilde{\hat{\nu}}}^{it})[1 \otimes x^{\mathrm{o}}] \\ &= \operatorname{Ad}(U_\nu^{\mathfrak{a}}(\widehat{U_\nu^{\mathfrak{a}}})^*)[D_t(1 \otimes \sigma_t^\nu(x)^{\mathrm{o}})D_t^*] \\ &= \operatorname{Ad}(U_\nu^{\mathfrak{a}}(\widehat{U_\nu^{\mathfrak{a}}})^*)[1 \otimes \sigma_t^\nu(x)^{\mathrm{o}}] \\ &= \beta(\sigma_t^\nu(x))\end{aligned}$$

(ii) The first assertion follows from the fact that $J_{\widetilde{\hat{\nu}}}$ and $\Delta_{\widetilde{\hat{\nu}}}^{it}$ commute. To conclude that $\widetilde{\tau}_t(\mathfrak{a}(x)) = \mathfrak{a}(\sigma_t^\nu(x))$, use (i) and 4.5 (ii).

(iii) Let $t \in \mathbb{R}$. Then 2.5.2 (iii) and 2.1 imply

$$\begin{aligned}\operatorname{Ad}((\widehat{\Delta} \otimes \Delta_{\widetilde{\hat{\nu}}})^{it})[(U_\nu^{\mathfrak{a}})^*_{13}W_{12}] &= \operatorname{Ad}((\widehat{D}_t)_{23}(\widehat{\Delta} \otimes \Delta \otimes \Delta_\nu)^{it})[(U_\nu^{\mathfrak{a}})^*_{13}W_{12}] \\ &= (\widehat{D}_t)_{23}(D^{\mathrm{o}}_{-t})^*_{13}(U_\nu^{\mathfrak{a}})^*_{13}(D_t)_{13}W_{12}(\widehat{D}_t)^*_{23} \\ &= (D^{\mathrm{o}}_{-t})^*_{13}(\widehat{D}_t)_{23}(U_\nu^{\mathfrak{a}})^*_{13}(D_t)_{13}W_{12}(\widehat{D}_t)^*_{23}.\end{aligned}$$

But 2.4.4 gives that $(\mathrm{id} \otimes \widehat{\mathfrak{a}})(D_t)(\widehat{D}_t)_{23} = W_{12}^*(U_\nu^{\mathfrak{a}})_{13}(\widehat{D}_t)_{23}(U_\nu^{\mathfrak{a}})^*_{13}(D_t)_{13}W_{12}$, whence

$$(\widehat{D}_t)_{23}(U_\nu^{\mathfrak{a}})^*_{13}(D_t)_{13}W_{12}(\widehat{D}_t)^*_{23} = (U_\nu^{\mathfrak{a}})^*_{13}W_{12}(1 \otimes \widehat{\mathfrak{a}})(D_t).$$

We insert this relation above and find

$$\operatorname{Ad}((\widehat{\Delta} \otimes \Delta_{\widetilde{\hat{\nu}}})^{it})[(U_\nu^{\mathfrak{a}})^*_{13}W_{12}] = (D^{\mathrm{o}}_{-t})^*_{13} \cdot (U_\nu^{\mathfrak{a}})^*_{13}W_{12} \cdot (\mathrm{id} \otimes \widehat{\mathfrak{a}})(D_t).$$

We use this relation and $\mathrm{Ad}(1 \otimes \widehat{U_\nu^{\mathfrak{a}}}(U_\nu^{\mathfrak{a}})^*)[W_{12}] = (U_\nu^{\mathfrak{a}})^*_{13} W_{12}$ (3.8), and find

$$
\begin{aligned}
&(\mathrm{id} \otimes \widetilde{\tau}_t)(W_{12}) \\
&\qquad = \mathrm{Ad}(1_H \otimes U_\nu^{\mathfrak{a}}(\widehat{U_\nu^{\mathfrak{a}}})^* \Delta_{\widetilde{\nu}}^{it} \widehat{U_\nu^{\mathfrak{a}}}(U_\nu^{\mathfrak{a}})^*)[W_{12}] \\
&\qquad = \mathrm{Ad}(\widehat{\Delta}^{-it} \otimes U_\nu^{\mathfrak{a}}(\widehat{U_\nu^{\mathfrak{a}}})^*)[\mathrm{Ad}((\widehat{\Delta} \otimes \Delta_{\widetilde{\nu}})^{it})((U_\nu^{\mathfrak{a}})^*_{13} W_{12})] \\
&\qquad = \mathrm{Ad}(\widehat{\Delta}^{-it} \otimes U_\nu^{\mathfrak{a}}(\widehat{U_\nu^{\mathfrak{a}}})^*)[(D^{\mathrm{o}}_{-t})^*_{13} \cdot (U_\nu^{\mathfrak{a}})^*_{13} W_{12} \cdot (\mathrm{id} \otimes \widehat{\mathfrak{a}})(D_t)] \\
&\qquad = \widehat{\Delta}_1^{-it} (\mathrm{id} \otimes \beta^\dagger)(D^{\mathrm{o}}_{-t}) W_{12} (\mathrm{id} \otimes \mathfrak{a})(D_t) \widehat{\Delta}_1^{it}.
\end{aligned}
$$

(iv) For any $\omega \in M_*$, the element $\widetilde{\tau}_t[(\omega \otimes \mathrm{id})(W) \otimes 1]$ belongs to $\mathbb{G} \ltimes_{\mathfrak{a}} N$ because

$$
\widetilde{\tau}_t[(\omega \otimes \mathrm{id})(W) \otimes 1] = (\omega \circ \tau_{-t})[(\mathrm{id} \otimes \beta^\dagger)(D^{\mathrm{o}}_{-t}) W_{12} (\mathrm{id} \otimes \mathfrak{a})(D_t)].
$$

By continuity, we get that $\widetilde{\tau}_t(y \otimes 1)$ belongs to $\mathbb{G} \ltimes_{\mathfrak{a}} N$ for any $y \in \widehat{M}$. Together with (ii), we obtain that $\widetilde{\tau}_t(\mathbb{G} \ltimes_{\mathfrak{a}} N) \subseteq \mathbb{G} \ltimes_{\mathfrak{a}} N$, and, as $\widetilde{\tau}$ is a one-parameter group of automorphisms, we have $\widetilde{\tau}_t(\mathbb{G} \ltimes_{\mathfrak{a}} N) = \mathbb{G} \ltimes_{\mathfrak{a}} N$. By (ii), $\widetilde{\tau}_t$ commutes with $\widetilde{R}$. □

5.6 Lemma. *Let $\mathbb{G}$ be a locally compact quantum group, $(N, \mathfrak{a}, \widehat{\mathfrak{a}})$ a braided-commutative $\mathbb{G}$-Yetter–Drinfel'd algebra, ν a normal faithful semi-finite weight on N, D_t its Radon–Nikodym derivative with respect to $\mathfrak{a}$ (2.2) and $\tilde{\nu}$ the dual weight of ν on the crossed product $\mathbb{G} \ltimes_{\mathfrak{a}} N$. Then for all $t \in \mathbb{R}$,*

$$
(\mathrm{id} \otimes \sigma_t^{\tilde{\nu}})(W_{12}) = \delta_1^{-it} \widehat{\Delta}_1^{-it} W_{12} (\mathrm{id} \otimes \mathfrak{a})(D_t) \widehat{\Delta}_1^{it} = (\mathrm{id} \otimes \widehat{\sigma}_t)(W)_{12} (\tau_{-t} \otimes \mathfrak{a})(D_t).
$$

Proof. By [54, 3.4] and 2.2,

$$
\begin{aligned}
(\mathrm{id} \otimes \sigma_t^{\tilde{\nu}})(W_{12}) &= [D_t(\widehat{\Delta}^{it} \otimes \Delta_\nu^{it})]_{23} W_{12} [(\widehat{\Delta}^{-it} \otimes \Delta_\nu^{-it}) D_t^*]_{23} \\
&= \delta_1^{-it} \widehat{\Delta}_1^{-it} (D_t)_{23} W_{12} \widehat{\Delta}_1^{it} (D_t^*)_{23} \\
&= \delta_1^{-it} \widehat{\Delta}_1^{-it} W_{12} (\Gamma \otimes \mathrm{id})(D_t)(D_t^*)_{23} \widehat{\Delta}_1^{it} \\
&= \delta_1^{-it} \widehat{\Delta}_1^{-it} W_{12} (\mathrm{id} \otimes \mathfrak{a})(D_t) \widehat{\Delta}_1^{it}.
\end{aligned}
$$

□

5.7 Proposition. *Let $\mathbb{G}$ be a locally compact quantum group, $(N, \mathfrak{a}, \widehat{\mathfrak{a}})$ a braided-commutative $\mathbb{G}$-Yetter–Drinfel'd algebra, ν a normal faithful semi-finite weight on N, and $\tilde{\nu}$ the dual weight of ν on the crossed product $\mathbb{G} \ltimes_{\mathfrak{a}} N$. Then the one-parameter group $\widetilde{\tau}_t$ of $\mathbb{G} \ltimes_{\mathfrak{a}} N$ constructed in 5.5 satisfies, for all $t \in \mathbb{R}$,*

$$
\widetilde{\Gamma} \circ \sigma_t^{\tilde{\nu}} = (\widetilde{\tau}_t \underset{N}{{}_\beta *_{\mathfrak{a}}} \sigma_t^{\tilde{\nu}}) \circ \widetilde{\Gamma}, \quad \widetilde{\Gamma} \circ \sigma_t^{\tilde{\nu} \circ \widetilde{R}} = (\sigma_t^{\tilde{\nu} \circ \widetilde{R}} \underset{N}{{}_\beta *_{\mathfrak{a}}} \widetilde{\tau}_{-t}) \circ \widetilde{\Gamma}.
$$

Proof. Let $x \in N$ and $t \in \mathbb{R}$. Then 5.5 (ii) and 4.4 imply

$$
\begin{aligned}
\widetilde{\Gamma} \circ \sigma_t^{\tilde{\nu}}(\mathfrak{a}(x)) = \widetilde{\Gamma}(\mathfrak{a}(\sigma_t^\nu(x))) &= \mathfrak{a}(\sigma_t^\nu(x)) \underset{N}{{}_\beta \otimes_{\mathfrak{a}}} 1 \\
&= (\widetilde{\tau}_t \underset{N}{{}_\beta *_{\mathfrak{a}}} \sigma_t^{\tilde{\nu}})(\mathfrak{a}(x) \underset{N}{{}_\beta \otimes_{\mathfrak{a}}} 1) = (\widetilde{\tau}_t \underset{N}{{}_\beta *_{\mathfrak{a}}} \sigma_t^{\tilde{\nu}}) \widetilde{\Gamma}(\mathfrak{a}(x)).
\end{aligned}
$$

Next, let V_2 be the unitary from $(H \otimes H_\nu) \underset{\nu}{{}_\beta\otimes_\mathfrak{a}} (H \otimes H_\nu)$ onto $H \otimes H \otimes H_\nu$ introduced in 4.3, and denote by $\widetilde{\hat{\nu}}$ the weight on $\widehat{\mathbb{G}} \ltimes_\mathfrak{a} N$ dual to ν as before. Then

$$
\begin{aligned}
&V_2[U_\nu^\mathfrak{a}(\widehat{U_\nu^\mathfrak{a}})^*\Delta_{\widetilde{\hat{\nu}}}^{it}U^{\widehat{\mathfrak{a}}}(U_\nu^\mathfrak{a})^* \underset{N}{{}_\beta\otimes_\mathfrak{a}} \Delta_{\tilde{\nu}}^{it}]V_2^*(\xi \otimes \mathfrak{a}(q)\Xi) \\
&\qquad = V_2[U_\nu^\mathfrak{a}(\widehat{U_\nu^\mathfrak{a}})^*\Delta_{\widetilde{\hat{\nu}}}^{it}(\xi \otimes \Lambda_\nu(q)) \underset{N}{{}_\beta\otimes_\mathfrak{a}} \Delta_{\tilde{\nu}}^{it}\Xi] \\
&\qquad = V_2[U_\nu^\mathfrak{a}(\widehat{U_\nu^\mathfrak{a}})^*\widehat{D}_t(\Delta^{it}\xi \otimes \Lambda_\nu(\sigma_t^\nu(q))) \underset{N}{{}_\beta\otimes_\mathfrak{a}} \Delta_{\tilde{\nu}}^{it}\Xi] \\
&\qquad = (\mathrm{id} \otimes \mathfrak{a})(\widehat{D}_t)(\Delta^{it}\xi \otimes \mathfrak{a}(\sigma_t^\nu(q))\Delta_{\tilde{\nu}}^{it}\Xi) \\
&\qquad = (\mathrm{id} \otimes \mathfrak{a})(\widehat{D}_t)(\Delta^{it} \otimes \Delta_{\tilde{\nu}}^{it})(\xi \otimes \mathfrak{a}(q)\Xi).
\end{aligned}
$$

Let now $y \in \widehat{M}$. Then by 4.4,

$$\mathrm{Ad}(V_2)[\tilde{\Gamma}(y \otimes 1)] = \widehat{\Gamma}(y) \otimes 1 = \mathrm{Ad}(\sigma_{12}W_{12})[y \otimes 1].$$

Using these two relations and 2.4.4, we find

$$
\begin{aligned}
&\mathrm{Ad}(V_2)[(\tilde{\tau} \underset{N}{{}_\beta*_\alpha} \sigma_t^{\tilde{\nu}})(\tilde{\Gamma}(y \otimes 1))] \\
&\qquad = \mathrm{Ad}((\mathrm{id} \otimes \mathfrak{a})(\widehat{D}_t)(\Delta^{it} \otimes \Delta_{\tilde{\nu}}^{it})\sigma_{12}W_{12})[y \otimes 1 \otimes 1] \\
&\qquad = \mathrm{Ad}(\sigma_{12}W_{12}(\mathrm{id} \otimes \hat{\mathfrak{a}})(D_t)(\widehat{\Delta}^{it} \otimes \Delta_{\widetilde{\hat{\nu}}}^{it}))[y \otimes 1 \otimes 1] \\
&\qquad = \mathrm{Ad}(\sigma_{12}W_{12}(\widehat{U_\nu^\mathfrak{a}})_{23}(D_t)_{13})[\hat{\sigma}_t(y) \otimes 1 \otimes 1].
\end{aligned}
$$

By 4.3 (iii), $\sigma_{12}W_{12}(\widehat{U_\nu^\mathfrak{a}})_{23} = V_2V_1^*W_{12}(U_\nu^\mathfrak{a})_{23}$ and hence

$$
\begin{aligned}
&\mathrm{Ad}(V_1)[(\tilde{\tau} \underset{N}{{}_\beta*_\alpha} \sigma_t^{\tilde{\nu}})(\tilde{\Gamma}(y \otimes 1))] \\
&\qquad = \mathrm{Ad}(W_{12}(U_\nu^\mathfrak{a})_{23}(D_t)_{13})[\hat{\sigma}_t(y) \otimes 1 \otimes 1] \\
&\qquad = \mathrm{Ad}(W_{12}(\mathrm{id} \otimes \mathfrak{a})(D_t)(D_t)_{23})[\hat{\sigma}_t(y) \otimes 1 \otimes 1] \\
&\qquad = \mathrm{Ad}((D_t)_{23}W_{12})[\hat{\sigma}_t(y) \otimes 1 \otimes 1] \\
&\qquad = \mathrm{Ad}((D_t)_{23})[\widehat{\Gamma}^\mathrm{o}(\hat{\sigma}_t(y)) \otimes 1].
\end{aligned}
$$

On the other hand,

$$
\begin{aligned}
\mathrm{Ad}(V_1)[\tilde{\Gamma}(\sigma_t^{\tilde{\nu}}(y \otimes 1))] &= \tilde{\mathfrak{a}}(\sigma_t^{\tilde{\nu}}(y \otimes 1)) \\
&= \mathrm{Ad}((\widehat{W}^\mathrm{o}_{12})^*)[\sigma_t^{\tilde{\nu}}(y \otimes 1)] \\
&= \mathrm{Ad}((\widehat{W}^\mathrm{o}_{12})^*(D_t)_{23})[\hat{\sigma}_t(y) \otimes 1] \\
&= \mathrm{Ad}((D_t)_{23}(\widehat{W}^\mathrm{o})^*_{12})[\hat{\sigma}_t(y) \otimes 1] \\
&= \mathrm{Ad}((D_t)_{23})[(\widehat{\Gamma}^\mathrm{o}(\hat{\sigma}_t(y)) \otimes 1)],
\end{aligned}
$$

showing that $(\tilde{\tau} \underset{N}{{}_\beta*_\alpha} \sigma_t^{\tilde{\nu}})(\tilde{\Gamma}(y \otimes 1)) = \tilde{\Gamma}(\sigma_t^{\tilde{\nu}}(y \otimes 1))$.

Since $\mathbb{G} \ltimes_{\mathfrak{a}} N$ is generated by $\mathfrak{a}(N)$ and $\widehat{M} \otimes 1$, the first of the two formulas follows. Using 5.5 (iv), the second one is easy to prove from the first one. □

5.8 Corollary. *Let $\mathbb{G}$ be a locally compact quantum group, $(N, \mathfrak{a}, \widehat{\mathfrak{a}})$ a braided-commutative $\mathbb{G}$-Yetter–Drinfel'd algebra, ν a normal faithful semi-finite weight on N, and $\tilde{\nu}$ the dual weight of ν on the crossed product $\mathbb{G} \ltimes_{\mathfrak{a}} N$. Then there exists a one-parameter group γ_t of automorphisms of N such that $\sigma_t^{\tilde{\nu}}(\beta(x)) = \beta(\gamma_t(x))$.*

Proof. Using 5.7, we get that for all $x \in N$ and $t \in \mathbb{R}$,

$$\widetilde{\Gamma}(\sigma_t^{\tilde{\nu}}(\beta(x))) = (\widetilde{\tau}_t \underset{N}{{}_\beta *_{\mathfrak{a}}} \sigma_t^{\tilde{\nu}})(\widetilde{\Gamma}(\beta(x))) = (\widetilde{\tau}_t \underset{N}{{}_\beta *_{\mathfrak{a}}} \sigma_t^{\tilde{\nu}})(1 \underset{N}{{}_\beta \otimes_{\mathfrak{a}}} \beta(x)) = 1 \underset{N}{{}_\beta \otimes_{\mathfrak{a}}} \sigma_t^{\tilde{\nu}}(\beta(x))$$

from which we get the result by [24, 4.0.9]. □

5.9 Theorem. *Let $\mathbb{G}$ be a locally compact quantum group, $(N, \mathfrak{a}, \widehat{\mathfrak{a}})$ a braided-commutative $\mathbb{G}$-Yetter–Drinfel'd algebra, ν a normal faithful semi-finite weight on N, D_t the Radon–Nikodym derivative of ν with respect to the action $\mathfrak{a}$, $\tilde{\nu}$ the dual weight of ν on the crossed product $\mathbb{G} \ltimes_{\mathfrak{a}} N$, $\widetilde{\tau}_t$ the one parameter group of automorphisms of $\mathbb{G} \ltimes_{\mathfrak{a}} N$ constructed in 5.5, and γ_t the one parameter group of automorphisms of N constructed in 5.8. Let Φ_t be the automorphism of M defined by $\Phi_t(x) = \tau_t \circ \mathrm{Ad}\,\delta^{-it}$ (let us remark that Φ_t is an automorphism of $\mathbb{G}$). Then the following conditions are equivalent:*

(i) *$(\Phi_t \otimes \gamma_t)(D_s) = D_s$ for all s, t in $\mathbb{R}$.*

(ii) *$\sigma_t^{\tilde{\nu}}$ and $\widetilde{\tau}_s$ commute for all s, t in $\mathbb{R}$.*

(iii) *$\sigma_t^{\tilde{\nu}}$ and $\sigma_s^{\tilde{\nu} \circ \widetilde{R}}$ commute for all s, t in $\mathbb{R}$.*

(iv) *$\mathfrak{G}(N, \mathfrak{a}, \widehat{\mathfrak{a}}, \nu)$ is a measured quantum groupoid.*

If these conditions hold, then $\widetilde{\tau}_t$ is the scaling group of $\mathfrak{G}(N, \mathfrak{a}, \widehat{\mathfrak{a}}, \nu)$, and γ_t is the one parameter group of automorphisms of N defined in 5.3 (iv).

Proof. The restrictions of $\sigma_t^{\tilde{\nu}}$ and $\widetilde{\tau}_s$ on $\mathfrak{a}(N)$ always commute because $\sigma_t^{\tilde{\nu}} \circ \widetilde{\tau}_s(\mathfrak{a}(x)) = \mathfrak{a}(\sigma_t^{\nu} \circ \sigma_s^{\nu}(x))$ and $\widetilde{\tau}_s \circ \sigma_t^{\tilde{\nu}}(\mathfrak{a}(x)) = \mathfrak{a}(\sigma_s^{\nu} \circ \sigma_t^{\nu}(x))$ for all $x \in N$ by 5.5 (ii).

Using now 5.6, 5.5 (iii) and 2.2, we get that

$$\begin{aligned}
&(\mathrm{id} \otimes \widetilde{\tau}_s \sigma_t^{\tilde{\nu}})(W_{12}) \\
&\quad = \delta_1^{-it} \widehat{\Delta}_1^{-it} (\mathrm{id} \otimes \widetilde{\tau}_s)(W_{12})(\mathrm{id} \otimes \widetilde{\tau}_s \mathfrak{a})(D_t) \widehat{\Delta}_1^{it} \\
&\quad = \delta_1^{-it} \widehat{\Delta}_1^{-it} \widehat{\Delta}_1^{-is} (\mathrm{id} \otimes \beta^{\dagger})(D^{\mathrm{o}}_{-s}) W_{12} (\mathrm{id} \otimes \mathfrak{a})(D_s) \widehat{\Delta}_1^{is} (\mathrm{id} \otimes \mathfrak{a}\sigma_s^{\nu})(D_t) \widehat{\Delta}_1^{it} \\
&\quad = \delta_1^{-it} \widehat{\Delta}_1^{-i(s+t)} (\mathrm{id} \otimes \beta^{\dagger})(D^{\mathrm{o}}_{-s}) W_{12} (\mathrm{id} \otimes \mathfrak{a})(D_s (\tau_s \otimes \sigma_s^{\nu})(D_t)) \widehat{\Delta}_1^{i(s+t)} \\
&\quad = \delta_1^{-it} \widehat{\Delta}^{-i(s+t)} (\mathrm{id} \otimes \beta^{\dagger})(D^{\mathrm{o}}_{-s}) W_{12} (\mathrm{id} \otimes \mathfrak{a})(D_{s+t}) \widehat{\Delta}_1^{i(s+t)}
\end{aligned}$$

and, on the other hand,

$$
\begin{aligned}
&(\mathrm{id}\otimes\sigma_t^{\tilde\nu}\tilde\tau_s)(W_{12})\\
&\quad=\widehat\Delta_1^{-is}(\mathrm{id}\otimes\widetilde{\sigma_t^{\nu}\beta^\dagger})(D^{\mathrm{o}}_{-s})(\mathrm{id}\otimes\sigma_t^{\tilde\nu})(W_{12})(\mathrm{id}\otimes\widetilde{\sigma_t^{\nu}\mathfrak{a}})(D_s)\widehat\Delta_1^{is}\\
&\quad=\widehat\Delta_1^{-is}(\mathrm{id}\otimes\beta^\dagger\gamma_t^{\mathrm{o}})(D^{\mathrm{o}}_{-s})\delta_1^{-it}\widehat\Delta_1^{-it}W_{12}(\mathrm{id}\otimes\mathfrak{a})(D_t)\widehat\Delta_1^{it}(\mathrm{id}\otimes\sigma_t^{\tilde\nu}\mathfrak{a})(D_s)\widehat\Delta_1^{is}\\
&\quad=\widehat\Delta_1^{-i(s+t)}\delta_1^{-it}(\Phi_t\otimes\beta^\dagger\gamma_t^{\mathrm{o}})(D^{\mathrm{o}}_{-s})W_{12}(\mathrm{id}\otimes\mathfrak{a})(D_t(\tau_t\otimes\sigma_t^{\nu})(D_s))\widehat\Delta_1^{i(s+t)}\\
&\quad=\widehat\Delta_1^{-i(s+t)}\delta_1^{-it}(\Phi_t\otimes\beta^\dagger\gamma_t^{\mathrm{o}})(D^{\mathrm{o}}_{-s})W_{12}(\mathrm{id}\otimes\mathfrak{a})(D_{s+t})\widehat\Delta_1^{i(s+t)}.
\end{aligned}
$$

Consequently, $(\mathrm{id}\otimes\sigma_t^{\tilde\nu}\tilde\tau_s)(W_{12})=(\mathrm{id}\otimes\tilde\tau_s\sigma_t^{\tilde\nu})(W_{12})$ if and only if $(\Phi_t\otimes\gamma_t)(D_s)=D_s$, which gives the equivalence of (i) and (ii).

Let us suppose (ii). Using 5.7, we get

$$\tilde\Gamma(\sigma_t^{\tilde\nu}\sigma_s^{\tilde\nu\circ\tilde R})=(\tilde\tau_t\sigma_s^{\tilde\nu\circ\tilde R}\ {}_{\beta}{*}_{\mathfrak{a}}^{N}\ \sigma_t^{\tilde\nu}\tilde\tau_{-s})\circ\tilde\Gamma$$

and

$$\tilde\Gamma(\sigma_s^{\tilde\nu\circ\tilde R}\sigma_t^{\tilde\nu})=(\sigma_s^{\tilde\nu\circ\tilde R}\tilde\tau_t\ {}_{\beta}{*}_{\mathfrak{a}}^{N}\ \tilde\tau_{-s}\sigma_t^{\tilde\nu})\circ\tilde\Gamma,$$

and by the commutation of $\tilde\tau$ with $\sigma^{\tilde\nu}$ and with $\sigma^{\tilde\nu\circ\tilde R}$, we get (iii).

By definition of a measured quantum groupoid, we have the equivalence of (iii) and (iv). The fact that (iv) implies (ii) is given by 5.3 (vi). □

5.10 Corollary. *Let $\mathbb{G}$ be a locally compact quantum group and $(N,\mathfrak{a},\widehat{\mathfrak{a}})$ a braided-commutative $\mathbb{G}$-Yetter–Drinfel'd algebra such that one of the following conditions holds:*

(i) *N is properly infinite, or*

(ii) *$\mathfrak{a}$ is integrable, or*

(iii) *$\mathbb{G}$ is (the von Neumann version of) a compact quantum group.*

Then there exists a normal semi-finite faithful weight ν on N such that $\mathfrak{G}(N,\mathfrak{a},\widehat{\mathfrak{a}},\nu)$ is a measured quantum groupoid.

Proof. We consider the individual cases:

(i) By 3.10, there exists a normal semi-finite faithful weight ν on N, invariant under $\mathfrak{a}$; therefore its Radon–Nikodym derivative $D_t=1$, and we get the result by 5.9.

(ii) In that case, there exists a weight ν on N which is δ^{-1}-invariant with respect to $\mathfrak{a}$; so we can apply again 5.9 to get the result.

(iii) We are here in a particular case of (ii), but with $\delta=1$. □

5.11 Proposition. *Let $\mathbb{G}$ be a locally compact quantum group, $(N,\mathfrak{a},\widehat{\mathfrak{a}})$ a braided-commutative $\mathbb{G}$-Yetter–Drinfel'd algebra, ν a normal faithful semi-finite weight on N, k-invariant with respect to $\mathfrak{a}$ (with k affiliated to $Z(M)$). Then:*

(i) *the scaling group* $\widetilde{\tau}_t$ *of* $\mathfrak{G}(N, \mathfrak{a}, \widehat{\mathfrak{a}}, \nu)$ *is given by*

$$\widetilde{\tau}_t(X) = (P^{it} \otimes \Delta_\nu^{it}) X (P^{-it} \otimes \Delta_\nu^{-it})$$

for all $X \in \mathbb{G} \ltimes_{\mathfrak{a}} N$;

(ii) *the scaling operator* $\widetilde{\lambda}$ *is equal to* λ^{-1}, *where* λ *is the scaling constant of* $\mathbb{G}$, *and the managing operator* $\widetilde{P}$ *is equal to* $P \otimes \Delta_\nu$.

Proof. (i) The scaling group $\widetilde{\tau}_t$ satisfies $\widetilde{\tau}_t(\mathfrak{a}(x)) = \mathfrak{a}(\sigma_t^\nu(x))$ for all $x \in N$ (5.5 (ii)). Using now 3.3 (i), we get that $\widetilde{\tau}_t(\mathfrak{a}(x)) = (\tau_t \otimes \sigma_t^\nu)(\mathfrak{a}(x))$.

On the other hand, using 5.5 (iii) and 3.1, we get that

$$(\mathrm{id} \otimes \widetilde{\tau}_t)(W_{12}) = \widehat{\Delta}_1^{-it} R(k^{-it})_1 W_{12} k^{it} \widehat{\Delta}_1^{it} = (\tau_{-t} \otimes \mathrm{id})(W) \otimes 1 = (\mathrm{id} \otimes \widehat{\tau}_t)(W) \otimes 1.$$

So, for all $y \in \widehat{M}$, we have $\widetilde{\tau}_t(y \otimes 1) = \widehat{\tau}_t(y) \otimes 1$, from which we get (i).

(ii) The scaling operator is equal to λ^{-1} because

$$\begin{aligned}
\tilde{\nu}(\widetilde{\tau}_t(\mathfrak{a}(x^*)(y^* y \otimes 1_{H_\nu})\mathfrak{a}(x))) &= \tilde{\nu}[\mathfrak{a}(\sigma_t^\nu(x^*))(\widehat{\tau}_t(y^* y) \otimes 1_{H_\nu})\mathfrak{a}(\sigma_t^\nu(x))] \\
&= \nu(\sigma_t^\nu(x^* x))\widehat{\varphi}(\widehat{\tau}_t(y^* y)) \\
&= \lambda^{-t} \nu(x^* x)\widehat{\varphi}(y^* y) \\
&= \lambda^{-t} \tilde{\nu}(\mathfrak{a}(x^*)(y^* y \otimes 1_{H_\nu})\mathfrak{a}(x)),
\end{aligned}$$

and $\widetilde{P}$ is equal to $P \otimes \Delta_\nu$ because

$$\begin{aligned}
\Lambda_{\tilde{\nu}}(\widetilde{\tau}_t((y \otimes 1_{H_\nu})\mathfrak{a}(x))) &= \Lambda_{\tilde{\nu}}[(\widehat{\tau}_t(y) \otimes 1_{H_\nu})\mathfrak{a}(\sigma_t^\nu(x))] \\
&= \Lambda_{\widehat{\varphi}}(\widehat{\tau}_t(y)) \otimes \Lambda_\nu(\sigma_t^\nu(x)) \\
&= \lambda^{t/2}(P^{it} \otimes \Delta_\nu^{it})(\Lambda_{\widehat{\varphi}}(y) \otimes \Lambda_\nu(x)).
\end{aligned}$$
□

6. Duality

In this chapter, we prove (6.5) that, if $\mathfrak{G}(N, \mathfrak{a}, \widehat{\mathfrak{a}}, \nu)$ is a measured quantum groupoid, its dual is isomorphic to $\mathfrak{G}(N, \widehat{\mathfrak{a}}, \mathfrak{a}, \nu)$, which is therefore also a measured quantum groupoid.

6.1 Lemma. *Let* $\mathbb{G}$ *be a locally compact quantum group,* $(N, \mathfrak{a}, \widehat{\mathfrak{a}})$ *a braided-commutative* $\mathbb{G}$*-Yetter–Drinfel'd algebra and* ν *a normal faithful semi-finite weight on* N*, and let* $\mathfrak{G}(N, \mathfrak{a}, \widehat{\mathfrak{a}}, \nu)$ *be the associated Hopf-bimodule, equipped with a co-inverse, a left-invariant operator-valued weight and a right-invariant valued weight by* 4.4 *(ii),* 4.6 *and* 5.4*. Then:*

(i) *The anti-representation* γ *of* N *is given by* $\gamma(x^*) = 1_H \otimes J_\nu x J_\nu$ *for all* $x \in N$.

(ii) *For any* $\xi \in H$, $p \in \mathfrak{N}_\nu$, *the vector* $\xi \otimes \Lambda_\nu(p)$ *belongs to* $D((H \otimes H_\nu)_\gamma, \nu^{\mathrm{o}})$, *and* $R^{\gamma,\nu^{\mathrm{o}}}(\xi \otimes \Lambda_\nu(p)) = l_\xi p$, *where* l_ξ *is the linear application from* H_ν *to* $H \otimes H_\nu$ *given by* $l_\xi \zeta = \xi \otimes \zeta$ *for all* $\zeta \in H_\nu$.

(iii) *There exists a unitary* V_3 *from* $(H \otimes H_\nu) {}_{\mathfrak{a}}\!\underset{\nu^{\mathrm{o}}}{\otimes}_{\gamma} (H \otimes H_\nu)$ *onto* $H \otimes H \otimes H_\nu$ *such that*

$$V_3[\Xi \;{}_{\mathfrak{a}}\!\underset{\nu^{\mathrm{o}}}{\otimes}_{\gamma} (\xi \otimes \Lambda_\nu(p))] = \xi \otimes \mathfrak{a}(p)\Xi \quad \textit{for all } \Xi \in H \otimes H_\nu.$$

Moreover, $(1 \otimes X)V_3 = V_3(X \;{}_{\mathfrak{a}}\!\underset{N^{\mathrm{o}}}{\otimes}_{\gamma} 1)$ *for all* $X \in \mathfrak{a}(N)'$.

(iv) $V_3(I \;{}_{\beta}\!\underset{N}{\otimes}_{\mathfrak{a}} J_{\tilde{\nu}}) = (\widehat{J} \otimes I)V_1$.

Proof. (i) By definition (5.2), the left-invariant weight of $\mathfrak{G}(N, \mathfrak{a}, \widehat{\mathfrak{a}}, \nu)$ is the dual weight $\tilde{\nu}$. Therefore, by definition (5.2), and using 2.2,

$$\gamma(x^*) = J_{\tilde{\nu}}\mathfrak{a}(x)J_{\tilde{\nu}} = (\widehat{J} \otimes J_\nu)(U_\nu^{\mathfrak{a}})^*\mathfrak{a}(x)U_\nu^{\mathfrak{a}}(\widehat{J} \otimes J_\nu) = 1_H \otimes J_\nu x J_\nu.$$

(ii) This follows from the relation

$$l_\xi p J_\nu \Lambda_\nu(x) = \xi \otimes J_\nu x J_\nu \Lambda_\nu(p) = \gamma(x^*)(\xi \otimes \Lambda_\nu(p)).$$

(iii) For any $\xi' \in H$, $\Xi' \in H \otimes H_\nu$, $p' \in \mathfrak{N}_\nu$,

$$\begin{aligned}
&(\Xi \;{}_{\mathfrak{a}}\!\underset{\nu^{\mathrm{o}}}{\otimes}_{\gamma} (\xi \otimes \Lambda_\nu(p)) | \Xi' \;{}_{\mathfrak{a}}\!\underset{\nu^{\mathrm{o}}}{\otimes}_{\gamma} (\xi' \otimes \Lambda_\nu(p'))\\
&\qquad = (\mathfrak{a}(\langle \xi \otimes \Lambda_\nu(p), \xi' \otimes \Lambda_\nu(p)\rangle_{\gamma,\nu^{\mathrm{o}}})\Xi | \Xi')\\
&\qquad = (\mathfrak{a}(p'^* l_{\xi'}^* l_\xi p)\Xi | \Xi')\\
&\qquad = (\xi \otimes \mathfrak{a}(p)\Xi | \xi' \otimes \mathfrak{a}(p')\Xi'),
\end{aligned}$$

from which we get the existence of V_3 as an isometry. As it is trivially surjective, we get it is a unitary. The last formula of (iii) is trivial.

(iv) Using 4.5 (ii) and 6.1 (i), we get the existence of an anti-linear bijective isometry $I \;{}_{\beta}\!\underset{N}{\otimes}_{\mathfrak{a}} J_{\tilde{\nu}}$ from $(H \otimes H_\nu) {}_{\beta}\!\underset{\nu}{\otimes}_{\mathfrak{a}} (H \otimes H_\nu)$ onto $(H \otimes H_\nu) {}_{\mathfrak{a}}\!\underset{\nu^{\mathrm{o}}}{\otimes}_{\gamma} (H \otimes H_\nu)$ with trivial values on elementary tensors. Moreover, for any $\Xi \in H \otimes H_\nu$, $\xi \in H$, $p \in \mathfrak{N}_\nu$, analytic with respect to ν, we have, using successively 2.2, (iii), and 4.3 (i),

$$\begin{aligned}
&V_3(I \;{}_{\beta}\!\underset{N}{\otimes}_{\mathfrak{a}} J_{\tilde{\nu}})(\Xi \;{}_{\beta}\!\underset{\nu}{\otimes}_{\mathfrak{a}} U_\nu^{\mathfrak{a}}((\xi \otimes \Lambda_\nu(p)))\\
&\qquad = V_3[I\,\Xi \;{}_{\mathfrak{a}}\!\underset{\nu}{\otimes}_{\gamma} (\widehat{J}\xi \otimes J_\nu \Lambda_\nu(p))]\\
&\qquad = \widehat{J}\xi \otimes \mathfrak{a}(\sigma_{-i/2}^\nu(p^*)))I\,\Xi\\
&\qquad = \widehat{J}\xi \otimes I\beta(\sigma_{i/2}^\nu(p))\Xi\\
&\qquad = (\widehat{J} \otimes I)V_1[\Xi \;{}_{\beta}\!\underset{\nu}{\otimes}_{\mathfrak{a}} U_\nu^{\mathfrak{a}}(\xi \otimes \Lambda_\nu(p))]. \qquad \square
\end{aligned}$$

6.2 Theorem ([40]). *Let $\mathbb{G}$ be a locally compact quantum group, $(N, \mathfrak{a}, \widehat{\mathfrak{a}})$ a braided-commutative $\mathbb{G}$-Yetter–Drinfel'd algebra, ν a normal faithful semi-finite weight on N, and let $\mathfrak{G}(N, \mathfrak{a}, \widehat{\mathfrak{a}}, \nu)$ be the associated Hopf-bimodule, equipped with a co-inverse, a left-invariant operator-valued weight and a right-invariant valued weight by 4.4 (ii), 4.6 and 5.4. Let $\widetilde{W}$ be the pseudo-mutiplicative unitary associated by 5.2. Then*

$$\widetilde{W} = V_3^*(W^* \otimes 1_{H_\nu})V_1,$$

where V_1 had been defined in 4.3 and V_3 in 6.1. Moreover, for any ξ, η in H, p, q in $\mathfrak{N}_\nu$,

$$(\mathrm{id} * \omega_{U_\nu^{\mathfrak{a}}(\eta \otimes J_\nu \Lambda_\nu(p)), \xi \otimes \Lambda_\nu(q)})(\widetilde{W}) = \mathfrak{a}(q^*)[(\omega_{\eta,\xi} \otimes \mathrm{id})(W^*) \otimes 1_{H_\nu}]\beta(p^*).$$

Proof. Let x, x_1, x_2 in $\mathfrak{N}_\nu$ and y, y_1, y_2 in $\mathfrak{N}_{\widehat{\varphi}}$. Then $(y \otimes 1)\mathfrak{a}(x)$, $(y_1 \otimes 1_{H_\nu})\mathfrak{a}(x_1)$, $(y_2 \otimes 1_{H_\nu})\mathfrak{a}(x_2)$ belong to $\mathfrak{N}_{\tilde{\nu}} \cap \mathfrak{N}_{T_{\tilde{\mathfrak{a}}}}$, and by (2.2),

$$\Lambda_{\tilde{\nu}}[(y \otimes 1_{H_\nu})\mathfrak{a}(x)] = \Lambda_{\widehat{\varphi}}(y) \otimes \Lambda_\nu(x)$$

and

$$J_{\tilde{\nu}}\Lambda_{\tilde{\nu}}[(y \otimes 1_{H_\nu})\mathfrak{a}(x)] = U_\nu^{\mathfrak{a}}(\widehat{J}\Lambda_{\widehat{\varphi}}(y) \otimes J_\nu\Lambda_\nu(x))$$

$$J_{\tilde{\nu}}\Lambda_{\tilde{\nu}}[\mathfrak{a}(x_1^*)(y_1^* y_2 \otimes 1_{H_\nu})\mathfrak{a}(x_2)] = (1_H \otimes J_\nu x_1^* J_\nu)U_\nu^{\mathfrak{a}}[\widehat{J}\Lambda_{\widehat{\varphi}}(y_1^* y_2) \otimes J_\nu\Lambda_\nu(x_2)].$$

By definition of $\widetilde{W}$ (5.2), we find that for any Ξ_1, Ξ_2 in $H \otimes H_\nu$, the scalar product

$$(\widetilde{W}[\Xi_2 \,{}_\beta\!\underset{\nu}{\otimes}_{\mathfrak{a}}\, J_{\tilde{\nu}}\Lambda_{\tilde{\nu}}(\mathfrak{a}(x_1^*)(y_1^* y_2 \otimes 1_{H_\nu})\mathfrak{a}(x_2))]|\Xi_1 \,{}_\mathfrak{a}\!\underset{\nu^o}{\otimes}_\gamma\, (\Lambda_{\widehat{\varphi}}(y) \otimes \Lambda_\nu(x)))$$

is equal to

$$\begin{aligned}&(\widetilde{\Gamma}[(y \otimes 1)\mathfrak{a}(x)]^*\\ &\cdot (\Xi_2 \,{}_\beta\!\underset{\nu}{\otimes}_{\mathfrak{a}}\, U_\nu^{\mathfrak{a}}(\widehat{J}\Lambda_{\widehat{\varphi}}(y_2) \otimes J_\nu\Lambda_\nu(x_2))|\Xi_1 \,{}_\beta\!\underset{\nu}{\otimes}_{\mathfrak{a}}\, U_\nu^{\mathfrak{a}}(\widehat{J}\Lambda_{\widehat{\varphi}}(y_1) \otimes J_\nu\Lambda_\nu(x_1)))).\end{aligned}$$

Using 4.4, we get that this is equal to

$$\begin{aligned}&((\widehat{\Gamma}^{\mathrm{o}}(y^*) \otimes 1_{H_\nu})\\ &\cdot V_1[\Xi_2 \,{}_\beta\!\underset{\nu}{\otimes}_{\mathfrak{a}} U_\nu^{\mathfrak{a}}(\widehat{J}\Lambda_{\widehat{\varphi}}(y_2) \otimes J_\nu\Lambda_\nu(x_2))]|V_1[\mathfrak{a}(x)\Xi_1 \,{}_\beta\!\underset{\nu}{\otimes}_{\mathfrak{a}} (U_\nu^{\mathfrak{a}}(\widehat{J}\Lambda_{\widehat{\varphi}}(y_1) \otimes J_\nu\Lambda_\nu(x_1))),\end{aligned}$$

which, thanks to 4.3 (i), is equal to

$$((\widehat{\Gamma}^{\mathrm{o}}(y^*) \otimes 1_{H_\nu})(\widehat{J}\Lambda_{\widehat{\varphi}}(y_2) \otimes \beta(x_2^*)\Xi_2)|\widehat{J}\Lambda_{\widehat{\varphi}}(y_1) \otimes \beta(x_1^*)\mathfrak{a}(x)\Xi_1)$$

and to

$$\begin{aligned}&(\beta(x_1)((\omega_{\widehat{J}\Lambda_{\widehat{\varphi}}(y_2), \widehat{J}\Lambda_{\widehat{\varphi}}(y_1)} \otimes \mathrm{id})(\widehat{\Gamma}^{\mathrm{o}}(y^*)) \otimes 1_{H_\nu})\beta(x_2^*)\Xi_2|\mathfrak{a}(x)\Xi_1)\\ &\qquad = (\beta(x_1)((\mathrm{id} \otimes \omega_{\widehat{J}\Lambda_{\widehat{\varphi}}(y_2), \widehat{J}\Lambda_{\widehat{\varphi}}(y_1)})(\widehat{\Gamma}(y^*)) \otimes 1_{H_\nu})\beta(x_2^*)\Xi_2|\mathfrak{a}(x)\Xi_1),\end{aligned}$$

which, by 2.1, is equal to

$$(\beta(x_1)((\mathrm{id}\otimes\omega_{\widehat{J}\Lambda_{\widehat{\varphi}}(y_1^*y_2),\Lambda_{\widehat{\varphi}}(y)})(\widehat{W})\otimes 1_{H_\nu})\beta(x_2^*)\Xi_2|\mathfrak{a}(x)\Xi_1)$$
$$= (\beta(x_1)((\omega_{\widehat{J}\Lambda_{\widehat{\varphi}}(y_1^*y_2),\Lambda_{\widehat{\varphi}}(y)}\otimes\mathrm{id})(W^*)\otimes 1_{H_\nu})\beta(x_2^*)\Xi_2|\mathfrak{a}(x)\Xi_1),$$

which is, using 4.3 (i) and 6.1, equal to

$$((1_H\otimes\beta(x_1))(W^*\otimes 1_{H_\nu})$$
$$\cdot V_1(\Xi_2\ {}_\beta\!\underset{\nu}{\otimes}_{\mathfrak{a}}\ U_\nu^{\mathfrak{a}}(\widehat{J}\Lambda_{\widehat{\varphi}}(y_1^*y_2)\otimes J_\nu\Lambda_\nu(x_2)))|V_3(\Xi_1\ {}_{\mathfrak{a}}\!\underset{\nu^{\mathrm{o}}}{\otimes}_\gamma\ (\Lambda_{\widehat{\varphi}}(y)\otimes\Lambda_\nu(x)))),$$

which, using 3.8 (iv), is the scalar product of the vector

$$W_{12}^*(\mathrm{id}\otimes\beta^\dagger)(\mathfrak{a}^{\mathrm{o}}(x_1^{\mathrm{o}}))V_1(\Xi_2\ {}_\beta\!\underset{\nu}{\otimes}_{\mathfrak{a}}\ U_\nu^{\mathfrak{a}}(\widehat{J}\Lambda_{\widehat{\varphi}}(y_1^*y_2)\otimes J_\nu\Lambda_\nu(x_2)))$$

with $V_3(\Xi_1\ {}_{\mathfrak{a}}\!\underset{\nu^{\mathrm{o}}}{\otimes}_\gamma\ (\Lambda_{\widehat{\varphi}}(y)\otimes\Lambda_\nu(x)))$. But, using 4.3 (i), this vector is equal to

$$(W^*\otimes 1_{H_\nu})$$
$$\cdot V_1[(1_H\otimes 1_{H_\nu})\ {}_\beta\!\underset{N}{\otimes}_{\mathfrak{a}}\ (1_H\otimes J_\nu x_1^*J_\nu)](\Xi_2\ {}_\beta\!\underset{\nu}{\otimes}_{\mathfrak{a}}\ U_\nu^{\mathfrak{a}}(\widehat{J}\Lambda_{\widehat{\varphi}}(y_1^*y_2)\otimes J_\nu\Lambda_\nu(x_2))).$$

Finally, we get that the initial scalar product

$$(\widetilde{W}[\Xi_2\ {}_\beta\!\underset{\nu}{\otimes}_{\mathfrak{a}}\ J_{\tilde{\nu}}\Lambda_{\tilde{\nu}}(\mathfrak{a}(x_1^*)(y_1^*y_2\otimes 1)\mathfrak{a}(x_2))]|\Xi_1\ {}_{\mathfrak{a}}\!\underset{\nu^{\mathrm{o}}}{\otimes}_\gamma\ (\Lambda_{\widehat{\varphi}}(y)\otimes\Lambda_\nu(x)))$$

is equal to

$$((W^*\otimes 1_{H_\nu})$$
$$\cdot V_1[\Xi_2\ {}_\beta\!\underset{\nu}{\otimes}_{\mathfrak{a}}\ J_{\tilde{\nu}}\Lambda_{\tilde{\nu}}(\mathfrak{a}(x_1^*)(y_1^*y_2\otimes 1)\mathfrak{a}(x_2))]|V_3(\Xi_1\ {}_{\mathfrak{a}}\!\underset{\nu^{\mathrm{o}}}{\otimes}_\gamma\ (\Lambda_{\widehat{\varphi}}(y)\otimes\Lambda_\nu(x)))).$$

By density of linear combinations of elements of the form $\Lambda_{\widehat{\varphi}}(y)\otimes\Lambda_\nu(x)$ in $D((H\otimes H_\nu)_\gamma,\nu^{\mathrm{o}})$, and then of linear combinations of elements of the form $\Xi_1\ {}_{\mathfrak{a}}\!\underset{\nu^{\mathrm{o}}}{\otimes}_\gamma\ (\Lambda_{\widehat{\varphi}}(y)\otimes\Lambda_\nu(x))$ in $(H\otimes H_\nu)\ {}_{\mathfrak{a}}\!\underset{\nu^{\mathrm{o}}}{\otimes}_\gamma\ (H\otimes H_\nu)$, we get that

$$\widetilde{W}[\Xi_2\ {}_\beta\!\underset{\nu}{\otimes}_{\mathfrak{a}}\ J_{\tilde{\nu}}\Lambda_{\tilde{\nu}}(\mathfrak{a}(x_1^*)(y_1^*y_2\otimes 1)\mathfrak{a}(x_2))]$$
$$= V_3^*(W^*\otimes 1_{H_\nu})V_1[\Xi_2\ {}_\beta\!\underset{\nu}{\otimes}_{\mathfrak{a}}\ J_{\tilde{\nu}}\Lambda_{\tilde{\nu}}(\mathfrak{a}(x_1^*)(y_1^*y_2\otimes 1)\mathfrak{a}(x_2))],$$

and, with the same density arguments, we get that $\widetilde{W} = V_3^*(W^*\otimes 1_{H_\nu})V_1$. Therefore, using again 4.3 (i) and 6.1, we get that

$$(\widetilde{W}[\Xi_2\ {}_\beta\!\underset{\nu}{\otimes}_{\mathfrak{a}}\ U_\nu^{\mathfrak{a}}(\eta\otimes J_\nu\Lambda_\nu(p))]|\Xi_1\ {}_{\mathfrak{a}}\!\underset{\nu^{\mathrm{o}}}{\otimes}_\gamma\ (\xi\otimes\Lambda_\nu(q)))$$
$$= ((W^*\otimes 1_{H_\nu})V_1[\Xi_2\ {}_\beta\!\underset{\nu}{\otimes}_{\mathfrak{a}}\ U_\nu^{\mathfrak{a}}(\eta\otimes J_\nu\Lambda_\nu(p))]|V_3[\Xi_1\ {}_{\mathfrak{a}}\!\underset{\nu^{\mathrm{o}}}{\otimes}_\gamma\ (\xi\otimes\Lambda_\nu(q))])$$

is equal to

$$((W^* \otimes 1_{H_\nu})(\eta \otimes \beta(p^*)\Xi_2)|\xi \otimes \mathfrak{a}(p)\Xi_1) = (((\omega_{\eta,\xi} \otimes \mathrm{id})(W^*) \otimes 1_{H_\nu})\beta(p^*)\Xi_2|\mathfrak{a}(p)\Xi_1),$$

which finishes the proof. $\square$

6.3 Theorem. *Let $\mathbb{G}$ be a locally compact quantum group, $(N, \mathfrak{a}, \widehat{\mathfrak{a}})$ a braided-commutative $\mathbb{G}$-Yetter–Drinfel'd algebra, and ν a normal faithful semi-finite weight on N such that $\mathfrak{G}(N, \mathfrak{a}, \widehat{\mathfrak{a}}, \nu)$ is a measured quantum groupoid in the sense of 5.1. Let $\widehat{\mathfrak{G}(N, \mathfrak{a}, \widehat{\mathfrak{a}}, \nu)}$ be its dual measured quantum groupoid in the sense of 5.3, and for all $X \in \widehat{\mathbb{G}} \ltimes_{\widehat{\mathfrak{a}}} N$, let*

$$\mathcal{I}(X) = U_\nu^{\mathfrak{a}}(U_\nu^{\widehat{\mathfrak{a}}})^* X U_\nu^{\widehat{\mathfrak{a}}}(U_\nu^{\mathfrak{a}})^*.$$

Then $\mathcal{I}$ is an isomorphism of Hopf bimodule structures from $\mathfrak{G}(N, \widehat{\mathfrak{a}}, \mathfrak{a}, \nu)$ onto $\widehat{\mathfrak{G}(N, \mathfrak{a}, \widehat{\mathfrak{a}}, \nu)}$.

Proof. To prove this result, we calculate the pseudo-multplicative $\widetilde{\widetilde{W}}$ of $\mathfrak{G}(N, \widehat{\mathfrak{a}}, \mathfrak{a}, \nu)$, using 6.2 applied to $(N, \widehat{\mathfrak{a}}, \mathfrak{a}, \nu)$. We first define, as in 4.3 (i) and 6.1, a unitary $\widehat{V_1}$ from $(H \otimes H_\nu) \underset{\nu}{{}_{\widehat{\beta}}\otimes_{\widehat{\mathfrak{a}}}} (H \otimes H_\nu)$ onto $H \otimes H \otimes H_\nu$, and a unitary $\widehat{V_3}$ from $(H \otimes H_\nu) \underset{\nu^o}{{}_{\widehat{\mathfrak{a}}}\otimes_{\widehat{\gamma}}} (H \otimes H_\nu)$ onto $H \otimes H \otimes H_\nu$, where, for all $x \in N$,

$$\widehat{\beta}(x) = U_\nu^{\widehat{\mathfrak{a}}}(U_\nu^{\mathfrak{a}})^*(1_H \otimes J_\nu x^* J_\nu) U_\nu^{\mathfrak{a}}(U_\nu^{\widehat{\mathfrak{a}}})^*,$$
$$\widehat{\gamma}(x) = J_{\widetilde{\widehat{\nu}}}\widehat{\mathfrak{a}}(x^*)J_{\widetilde{\widehat{\nu}}} = 1_H \otimes J_\nu x^* J_\nu = \gamma(x),$$

$\widetilde{\widehat{\nu}}$ denoting the dual weight on $\widehat{\mathbb{G}} \ltimes_{\widehat{\mathfrak{a}}} N$ as before. More precisely, applying 4.3 (i) to $(N, \widehat{\mathfrak{a}}, \mathfrak{a}, \nu)$, we get that for any ξ, η in H and p, q in $\mathfrak{N}_\nu$,

$$\widehat{V_1}(U_\nu^{\widehat{\mathfrak{a}}}(U_\nu^{\mathfrak{a}})^* \underset{N}{{}_{\widehat{\beta}}\otimes_{\widehat{\mathfrak{a}}}} U_\nu^{\widehat{\mathfrak{a}}}(U_\nu^{\mathfrak{a}})^*)\sigma_{\nu^o}[U_\nu^{\mathfrak{a}}(\eta \otimes J_\nu\Lambda_\nu(q)) \underset{\nu^o}{{}_{\mathfrak{a}}\otimes_{\gamma}} (\xi \otimes \Lambda_\nu(p))]$$

is equal to

$$\widehat{V_1}[U_\nu^{\widehat{\mathfrak{a}}}(U_\nu^{\mathfrak{a}})^*(\xi \otimes \Lambda_\nu(p)) \underset{\nu}{{}_{\widehat{\beta}}\otimes_{\widehat{\mathfrak{a}}}} U_\nu^{\widehat{\mathfrak{a}}}(\eta \otimes J_\nu\Lambda_\nu(q))]$$
$$= \eta \otimes \widehat{\beta}(q^*) U_\nu^{\widehat{\mathfrak{a}}}(U_\nu^{\mathfrak{a}})^*(\xi \otimes \Lambda_\nu(p))$$
$$= \eta \otimes U_\nu^{\widehat{\mathfrak{a}}}(U_\nu^{\mathfrak{a}})^*(\xi \otimes J_\nu q J_\nu \Lambda_\nu(p))$$
$$= (1_H \otimes U_\nu^{\widehat{\mathfrak{a}}}(U_\nu^{\mathfrak{a}})^*)(\eta \otimes \xi \otimes p J_\nu \Lambda_\nu(q)).$$

On the other hand, using 6.1, we get that

$$\begin{aligned} V_3[U_\nu^{\mathfrak{a}}(\eta \otimes J_\nu \Lambda_\nu(q)) \underset{\nu^o}{{}_{\mathfrak{a}}\otimes_\gamma} (\xi \otimes \Lambda_\nu(p))] &= \xi \otimes \mathfrak{a}(p) U_\nu^{\mathfrak{a}}(\eta \otimes J_\nu \Lambda_\nu(q)) \\ &= \xi \otimes U_\nu^{\mathfrak{a}}(\eta \otimes p J_\nu \Lambda_\nu(q)) \\ &= (1_H \otimes U_\nu^{\mathfrak{a}})(\xi \otimes \eta \otimes p J_\nu \Lambda_\nu(q)), \end{aligned}$$

from which we get that

$$\widehat{V_1}(U_\nu^{\widehat{\mathfrak{a}}}(U_\nu^{\mathfrak{a}})^* \underset{N}{{}_{\beta}\otimes_{\widehat{\mathfrak{a}}}} U_\nu^{\widehat{\mathfrak{a}}}(U_\nu^{\mathfrak{a}})^*)\sigma_{\nu^o} = (1_H \otimes U_\nu^{\widehat{\mathfrak{a}}}(U_\nu^{\mathfrak{a}})^*)(\sigma \otimes 1_{H_\nu})(1_H \otimes (U_\nu^{\mathfrak{a}})^*)V_3.$$

Applying this result to $(N, \widehat{\mathfrak{a}}, \mathfrak{a}, \nu)$ and taking the adjoints, we find that

$$\widehat{V_3}(U_\nu^{\widehat{\mathfrak{a}}}(U_\nu^{\mathfrak{a}})^* \underset{N^o}{{}_{\mathfrak{a}}\otimes_\beta} U_\nu^{\widehat{\mathfrak{a}}}(U_\nu^{\mathfrak{a}})^*)\sigma_\nu = (1_H \otimes U_\nu^{\widehat{\mathfrak{a}}})(\sigma \otimes 1_{H\nu})(1_H \otimes U_\nu^{\widehat{\mathfrak{a}}}(U_\nu^{\mathfrak{a}})^*)V_1.$$

Applying 6.2 to $(N, \widehat{\mathfrak{a}}, \mathfrak{a}, \nu)$, we get that $\widetilde{\widetilde{W}} = \widehat{V_3}^*(\sigma \otimes 1_{H_\nu})(W \otimes 1_{H_\nu})(\sigma \otimes 1_{H_\nu})\widehat{V_1}$ and, therefore, that $\sigma_{\nu^o}[U_\nu^{\mathfrak{a}}(U_\nu^{\widehat{\mathfrak{a}}})^* \underset{N^o}{{}_{\widehat{\mathfrak{a}}}\otimes_\gamma} U_\nu^{\mathfrak{a}}(U_\nu^{\widehat{\mathfrak{a}}})^*]\widetilde{\widetilde{W}}[U_\nu^{\widehat{\mathfrak{a}}}(U_\nu^{\mathfrak{a}})^* \underset{N}{{}_\gamma\otimes_{\mathfrak{a}}} U_\nu^{\widehat{\mathfrak{a}}}(U_\nu^{\mathfrak{a}})^*]\sigma_{\nu^o}$ is equal to:

$$V_1^*(U_\nu^{\mathfrak{a}})_{23}(U_\nu^{\widehat{\mathfrak{a}}})_{23}^*(U_\nu^{\widehat{\mathfrak{a}}})_{13}^* W_{12}(U_\nu^{\widehat{\mathfrak{a}}})_{13}(U_\nu^{\mathfrak{a}})_{13}^*(U_\nu^{\mathfrak{a}})_{23}^* V_3$$

But, as $(\widehat{\Gamma} \otimes \mathrm{id})(U_\nu^{\widehat{\mathfrak{a}}}) = (U_\nu^{\widehat{\mathfrak{a}}})_{23}(U_\nu^{\widehat{\mathfrak{a}}})_{13}$, we get that $(U_\nu^{\widehat{\mathfrak{a}}})_{23}^*(U_\nu^{\widehat{\mathfrak{a}}})_{13}^* = W_{12}(U_\nu^{\widehat{\mathfrak{a}}})_{13}^* W_{12}^*$, and therefore that $(U_\nu^{\widehat{\mathfrak{a}}})_{23}^*(U_\nu^{\widehat{\mathfrak{a}}})_{13}^* W_{12}(U_\nu^{\widehat{\mathfrak{a}}})_{13} = W_{12}$. On the other hand, by the same argument, $(U_\nu^{\mathfrak{a}})_{13}^*(U_\nu^{\mathfrak{a}})_{23}^* = W_{12}^*(U_\nu^{\mathfrak{a}})_{23}^* W_{12}$. Finally, we get that

$$\sigma_{\nu^o}[U_\nu^{\mathfrak{a}}(U_\nu^{\widehat{\mathfrak{a}}})^* \underset{N^o}{{}_{\widehat{\mathfrak{a}}}\otimes_\gamma} U_\nu^{\mathfrak{a}}(U_\nu^{\widehat{\mathfrak{a}}})^*]\widetilde{\widetilde{W}}[U_\nu^{\widehat{\mathfrak{a}}}(U_\nu^{\mathfrak{a}})^* \underset{N}{{}_\gamma\otimes_{\mathfrak{a}}} U_\nu^{\widehat{\mathfrak{a}}}(U_\nu^{\mathfrak{a}})^*]\sigma_{\nu^o} = V_1^* W_{12} V_3 = \widetilde{W}^*,$$

and therefore

$$[U_\nu^{\mathfrak{a}}(U_\nu^{\widehat{\mathfrak{a}}})^* \underset{N^o}{{}_{\widehat{\mathfrak{a}}}\otimes_\gamma} U_\nu^{\mathfrak{a}}(U_\nu^{\widehat{\mathfrak{a}}})^*]\widetilde{\widetilde{W}}[U_\nu^{\widehat{\mathfrak{a}}}(U_\nu^{\mathfrak{a}})^* \underset{N}{{}_\gamma\otimes_{\mathfrak{a}}} U_\nu^{\widehat{\mathfrak{a}}}(U_\nu^{\mathfrak{a}})^*] = \sigma_\nu \widetilde{W}^* \sigma_\nu.$$

So, up to the isomorphism, the pseudo-multiplicative unitary $\widetilde{\widetilde{W}}$ of $\mathfrak{G}(N, \widehat{\mathfrak{a}}, \mathfrak{a}, \nu)$ is equal to the dual pseudo-muliplicative untary $\widehat{\widetilde{W}}$, which finishes the proof. □

6.4 Proposition. *Let $\mathbb{G}$ be a locally compact quantum group, $(N, \mathfrak{a}, \widehat{\mathfrak{a}})$ a braided-commutative $\mathbb{G}$-Yetter–Drinfel'd algebra, and ν a normal faithful semi-finite weight on N. Suppose that $\mathfrak{G}(N, \mathfrak{a}, \widehat{\mathfrak{a}}, \nu)$ is a measured quantum groupoid in the sense of 5.1, and let $\widehat{\mathfrak{G}}(N, \mathfrak{a}, \widehat{\mathfrak{a}}, \nu)$ be its dual measured quantum groupoid in the sense of 5.3.*

(i) *The co-inverse $\widetilde{R}$ constructed in 4.6 (ii) is the canonical co-inverse of the measured quantum groupoid $\mathfrak{G}(N, \mathfrak{a}, \widehat{\mathfrak{a}}, \nu)$.*

(ii) *The isomorphim of Hopf bimodules from* $\mathfrak{G}(N,\widehat{\mathfrak{a}},\mathfrak{a},\nu)$ *onto* $\widehat{\mathfrak{G}}(N,\mathfrak{a},\widehat{\mathfrak{a}},\nu)$ *constructed in 6.3 exchanges the canonical co-inverses of these Hopf-bimodules.*

Proof. (i) By 6.1 (iv), $V_3(I\ {}_\beta\underset{N}{\otimes}{}_\mathfrak{a}\ J_{\tilde\nu}) = (\widehat{J}\otimes I)V_1$. Taking adjoints, we also get $V_1(I\ {}_\mathfrak{a}\underset{N^\mathrm{o}}{\otimes}{}_\gamma\ J_{\tilde\nu}) = (\widehat{J}\otimes I)V_3$. Therefore, we get, using 6.2 and 4.5 (iii),

$$\begin{aligned}(I\ {}_\mathfrak{a}\underset{N^\mathrm{o}}{\otimes}{}_\gamma\ J_{\tilde\nu})\widetilde{W}(I\ {}_\mathfrak{a}\underset{N^\mathrm{o}}{\otimes}{}_\gamma\ J_{\tilde\nu}) &= (I\ {}_\mathfrak{a}\underset{N^\mathrm{o}}{\otimes}{}_\gamma\ J_{\tilde\nu})V_3^*(W^*\otimes 1_{H_\nu})V_1(I\ {}_\mathfrak{a}\underset{N^\mathrm{o}}{\otimes}{}_\gamma\ J_{\tilde\nu})\\ &= V_1^*(\widehat{J}\otimes I)(W^*\otimes 1_{H_\nu})(\widehat{J}\otimes I)V_3\\ &= V_1^*(W\otimes 1_{H_\nu})V_3\\ &= \widetilde{W}^*.\end{aligned}$$

For all $\Xi\in D({}_\mathfrak{a}(H\otimes H_\nu),\nu)$ and $\Xi'\in D((H\otimes H_\nu)_\gamma,\nu^\mathrm{o})$, we therefore have

$$I(\mathrm{id}*\omega_{\Xi,\Xi'})(\widetilde{W})^*I = (\mathrm{id}*\omega_{J_{\tilde\nu}\Xi',J_{\tilde\nu}\Xi})(\widetilde{W}),$$

which proves that the canonical co-inverse is given by $\widetilde{R}(X) = IX^*I$ for all $X\in\mathbb{G}\ltimes_\mathfrak{a} N$.

(ii) By 5.3, the canonical co-inverse of $\widehat{\mathfrak{G}}(N,\mathfrak{a},\widehat{\mathfrak{a}},\nu)$ is implemented by $J_{\tilde\nu}$. Using (ii) applied to $\mathfrak{G}(N,\widehat{\mathfrak{a}},\mathfrak{a},\nu)$, we therefore get that the canonical co-inverse of $\mathfrak{G}(N,\widehat{\mathfrak{a}},\mathfrak{a},\nu)$ is implemented by $\widehat{I} = U_\nu^{\widehat{\mathfrak{a}}}(U_\nu^\mathfrak{a})^*J_{\tilde\nu}U_\nu^\mathfrak{a}(U^{\widehat{\mathfrak{a}}})^*$. □

6.5 Theorem. *Let* $\mathbb{G}$ *be a locally compact quantum group,* $(N,\mathfrak{a},\widehat{\mathfrak{a}})$ *a braided-commutative* $\mathbb{G}$*-Yetter–Drinfel'd algebra, and* ν *a normal faithful semi-finite weight on* N*. Suppose that* $\mathfrak{G}(N,\mathfrak{a},\widehat{\mathfrak{a}},\nu)$ *is a measured quantum groupoid in the sense of 5.1, let* $\widehat{\mathfrak{G}}(N,\mathfrak{a},\widehat{\mathfrak{a}},\nu)$ *be its dual measured quantum groupoid in the sense of 5.3, and let* $\mathcal{I}$ *be the isomorphism of Hopf bimodule structures constructed in 6.3. Then* $\mathcal{I}$ *exchanges the left-invariant and the right-invariant operator-valued weights on* $\mathfrak{G}(N,\widehat{\mathfrak{a}},\mathfrak{a},\nu)$ *and* $\widehat{\mathfrak{G}}(N,\mathfrak{a},\widehat{\mathfrak{a}},\nu)$*. Therefore,* $\mathfrak{G}(N,\widehat{\mathfrak{a}},\mathfrak{a},\nu)$ *is also a measured quantum groupoid.*

Proof. Using 6.4 (ii), it suffices to verify that $\mathcal{I}$ exchanges the left-invariant operator valued weights, of $\mathfrak{G}(N,\widehat{\mathfrak{a}},\mathfrak{a},\nu)$ and $\widehat{\mathfrak{G}}(N,\mathfrak{a},\widehat{\mathfrak{a}},\nu)$. The left-invariant weight of $\mathfrak{G}(N,\widehat{\mathfrak{a}},\mathfrak{a},\nu)$ is the dual weight $\widetilde{\nu}$ on the crossed product $\widehat{\mathbb{G}}\ltimes_{\widehat{\mathfrak{a}}}N$. Let us denote by $\widehat{\Phi}$ the left-invariant weight of $\widehat{\mathfrak{G}}(N,\mathfrak{a},\widehat{\mathfrak{a}},\nu)$.

We apply 6.2 to $\mathfrak{G}(N,\widehat{\mathfrak{a}},\mathfrak{a},\nu)$ and get that, for any ξ in H, $z\in\mathfrak{N}_{\widehat{\varphi}}$, p,q in $\mathfrak{N}_\nu$,

$$\Big(\mathrm{id}*\omega_{U_{\tilde\nu}^{\widehat{\mathfrak{a}}}(\widehat{J}\Lambda_{\widehat{\varphi}}(z)\otimes J_\nu\Lambda_\nu(p)),\xi\otimes\Lambda_\nu(q)}\Big)(\widetilde{\widetilde{W}})\Big(\mathrm{id}*\omega_{U_{\tilde\nu}^{\widehat{\mathfrak{a}}}(\widehat{J}\Lambda_{\widehat{\varphi}}(z)\otimes J_\nu\Lambda_\nu(p)),\xi\otimes\Lambda_\nu(q)}\Big)(\widetilde{\widetilde{W}})^*$$

is equal to

$$\widehat{\mathfrak{a}}(q^*)\Big[\Big(\mathrm{id}\otimes\omega_{\widehat{J}\Lambda_{\widehat{\varphi}}(z),\xi}\Big)(W)\otimes 1\Big]\widehat{\beta}(pp^*)\Big[\Big(\mathrm{id}\otimes\omega_{\widehat{J}\Lambda_{\widehat{\varphi}}(z),\xi}\Big)(W)^*\otimes 1\Big]\widehat{\mathfrak{a}}(q),$$

where, as in 6.3, $\widetilde{\widetilde{W}}$ denotes the pseudo-multiplicative unitary associated to $\mathfrak{G}(N, \widehat{\mathfrak{a}}, \mathfrak{a}, \nu)$, and $\widehat{\beta}$ is defined, for $x \in N$, by

$$\widehat{\beta}(x) = U_\nu^{\widehat{\mathfrak{a}}}(U_\nu^{\mathfrak{a}})^*(1_H \otimes J_\nu x^* J_\nu)U_\nu^{\mathfrak{a}}(U_\nu^{\widehat{\mathfrak{a}}})^*.$$

Let us take now a family $(p_i)_{i\in I}$ in $\mathfrak{M}_\nu^+$, increasing to 1. Then, we get that

$$\widehat{\mathfrak{a}}(q^*)\Big[\Big(\mathrm{id} \otimes \omega_{\widehat{J}\Lambda_{\widehat{\varphi}}(z),\xi}\Big)(W)\Big(\mathrm{id} \otimes \omega_{\widehat{J}\Lambda_{\widehat{\varphi}}(z),\xi}\Big)(W)^* \otimes 1\Big]\widehat{\mathfrak{a}}(q)$$

is the increasing limit of

$$\Big(\mathrm{id}*\omega_{U_\nu^{\widehat{\mathfrak{a}}}(\widehat{J}\Lambda_{\widehat{\varphi}}(z)\otimes J_\nu\Lambda_\nu(p_i^{1/2})),\xi\otimes\Lambda_\nu(q)}\Big)(\widetilde{\widetilde{W}})\Big(\mathrm{id}*\omega_{U_\nu^{\widehat{\mathfrak{a}}}(\widehat{J}\Lambda_{\widehat{\varphi}}(z)\otimes J_\nu\Lambda_\nu(p_i^{1/2})),\xi\otimes\Lambda_\nu(q)}\Big)(\widetilde{\widetilde{W}})^*.$$

But, using 6.3, we get that $\Big(\mathrm{id} * \omega_{U_\nu^{\widehat{\mathfrak{a}}}(\widehat{J}\Lambda_{\widehat{\varphi}}(z)\otimes J_\nu\Lambda_\nu(p_i^{1/2})),\xi\otimes\Lambda_\nu(q)}\Big)(\widetilde{\widetilde{W}})$ is equal to

$$\begin{aligned}\mathcal{I}^{-1}&\Big[\Big(\mathrm{id} * \omega_{U_\nu^{\mathfrak{a}}(\widehat{J}\Lambda_{\widehat{\varphi}}(z)\otimes J_\nu\Lambda_\nu(p_i^{1/2})),U_\nu^{\mathfrak{a}}(U_\nu^{\widehat{\mathfrak{a}}})^*(\xi\otimes\Lambda_\nu(q))}\Big)(\sigma_{\nu^o}\widetilde{W}^*\sigma_{\nu^o})\Big]\\ &= \mathcal{I}^{-1}\Big[\Big(\omega_{U_\nu^{\mathfrak{a}}(U_\nu^{\widehat{\mathfrak{a}}})^*(\xi\otimes\Lambda_\nu(q)),U_\nu^{\mathfrak{a}}(\widehat{J}\Lambda_{\widehat{\varphi}}(z)\otimes J_\nu\Lambda_\nu(p_i^{1/2}))} * \mathrm{id}\Big)(\widetilde{W})^*\Big].\end{aligned}$$

Therefore, we get that

$$\widehat{\Phi} \circ \mathcal{I}\Big[\widehat{\mathfrak{a}}(q^*)\Big[\Big(\mathrm{id} \otimes \omega_{\widehat{J}\Lambda_{\widehat{\varphi}}(z),\xi}\Big)(W)\Big(\mathrm{id} \otimes \omega_{\widehat{J}\Lambda_{\widehat{\varphi}}(z),\xi}\Big)(W)^* \otimes 1\Big]\widehat{\mathfrak{a}}(q)\Big]$$

is the increasing limit of

$$\begin{aligned}\widehat{\Phi}&\Big[\Big(\omega_{U_\nu^{\mathfrak{a}}(U_\nu^{\widehat{\mathfrak{a}}})^*(\xi\otimes\Lambda_\nu(q)),U_\nu^{\mathfrak{a}}(\widehat{J}\Lambda_{\widehat{\varphi}}(z)\otimes J_\nu\Lambda_\nu(p_i^{1/2}))} * \mathrm{id}\Big)(\widetilde{W})^*\\ &\cdot\Big(\omega_{U_\nu^{\mathfrak{a}}(U_\nu^{\widehat{\mathfrak{a}}})^*(\xi\otimes\Lambda_\nu(q)),U_\nu^{\mathfrak{a}}(\widehat{J}\Lambda_{\widehat{\varphi}}(z)\otimes J_\nu\Lambda_\nu(p_i^{1/2}))} * \mathrm{id}\Big)(\widetilde{W})\Big],\end{aligned}$$

which, using 5.3, is equal, by definition, to the increasing limit of

$$\Big\|\omega_{U_\nu^{\mathfrak{a}}(U_\nu^{\widehat{\mathfrak{a}}})^*(\xi\otimes\Lambda_\nu(q)),U_\nu^{\mathfrak{a}}(\widehat{J}\Lambda_{\widehat{\varphi}}(z)\otimes J_\nu\Lambda_\nu(p_i^{1/2}))}\Big\|_{\tilde{\nu}}^2.$$

For $X \in \mathfrak{N}_{\tilde{\nu}}$, the scalar $\omega_{U_\nu^{\mathfrak{a}}(U_\nu^{\widehat{\mathfrak{a}}})^*(\xi\otimes\Lambda_\nu(q)),U_\nu^{\mathfrak{a}}(\widehat{J}\Lambda_{\widehat{\varphi}}(z)\otimes J_\nu\Lambda_\nu(p_i^{1/2}))}(X^*)$ is equal to

$$\begin{aligned}&(X^*U_\nu^{\mathfrak{a}}(U_\nu^{\widehat{\mathfrak{a}}})^*(\xi \otimes \Lambda_\nu(q))|U_\nu^{\mathfrak{a}}(\widehat{J}\Lambda_{\widehat{\varphi}}(z) \otimes J_\nu\Lambda_\nu(p_i^{1/2})))\\ &\quad= (U_\nu^{\mathfrak{a}}(U_\nu^{\widehat{\mathfrak{a}}})^*(\xi \otimes \Lambda_\nu(q))|XJ_{\tilde{\nu}}\Lambda_{\tilde{\nu}}[(z \otimes 1)\mathfrak{a}(p_i^{1/2})])\\ &\quad= (U_\nu^{\mathfrak{a}}(U_\nu^{\widehat{\mathfrak{a}}})^*(\xi \otimes \Lambda_\nu(q))|J_{\tilde{\nu}}(z \otimes 1)\mathfrak{a}(p_i^{1/2})J_{\tilde{\nu}}\Lambda_{\tilde{\nu}}(X))\end{aligned}$$

and, therefore,

$$\left\|\omega_{U^{\mathfrak{a}}_{\nu}(\widehat{U^{\mathfrak{a}}_{\nu}})^*(\xi\otimes\Lambda_\nu(q)),U^{\mathfrak{a}}_{\nu}(\widehat{J}\Lambda_{\widehat{\varphi}}(z)\otimes J_\nu\Lambda_\nu(p_i^{1/2}))}\right\|^2_{\widetilde{\nu}}$$
$$= \|J_{\widetilde{\nu}}\mathfrak{a}(p_i^{1/2})(z^*\otimes 1)J_{\widetilde{\nu}}U^{\mathfrak{a}}_{\nu}(\widehat{U^{\mathfrak{a}}_{\nu}})^*(\xi\otimes\Lambda_\nu(q))\|^2.$$

The limit when p_i goes to 1 is equal to

$$\begin{aligned}
\|(\widehat{J}z^*\widehat{J}\otimes 1)(\widehat{U^{\mathfrak{a}}_{\nu}})^*(\xi\otimes\Lambda_\nu(q))\|^2 &= \|(\widehat{J}z^*\widehat{J}\otimes 1)(\xi\otimes\Lambda_\nu(q))\|^2\\
&= \|\widehat{J}z^*\widehat{J}\xi\|^2\|\Lambda_\nu(q)\|^2\\
&= \left\|\omega_{\xi,\widehat{J}\Lambda_{\widehat{\varphi}}(z)}\right\|^2_{\widehat{\varphi}}\|\Lambda_\nu(q)\|^2\\
&= \left\|\Lambda_\varphi\left[\left(\mathrm{id}\otimes\omega_{\xi,\widehat{J}\Lambda_{\widehat{\varphi}}(z)}\right)(W^*)\right]\otimes\Lambda_\nu(q)\right\|^2\\
&= \left\|\Lambda_{\widetilde{\widehat{\nu}}}\left[\left(\left(\mathrm{id}\otimes\omega_{\xi,\widehat{J}\Lambda_{\widehat{\varphi}}(z)}\right)(W^*)\otimes 1_{H_\nu}\right)\widehat{\mathfrak{a}}(q)\right]\right\|^2,
\end{aligned}$$

from which we get that

$$\left\|\Lambda_{\widehat{\Phi}\circ\mathcal{I}}\left[\left(\left(\mathrm{id}\otimes\omega_{\xi,\widehat{J}\Lambda_{\widehat{\varphi}}(z)}\right)(W^*)\otimes 1_{H_\nu}\right)\widehat{\mathfrak{a}}(q)\right]\right\|^2$$
$$= \left\|\Lambda_{\widetilde{\widehat{\nu}}}\left[\left(\left(\mathrm{id}\otimes\omega_{\xi,\widehat{J}\Lambda_{\widehat{\varphi}}(z)}\right)(W^*)\otimes 1_{H_\nu}\right)\widehat{\mathfrak{a}}(q)\right]\right\|^2,$$

which proves that the left-invariant weight $\widehat{\Phi}\circ\mathcal{I}+\widetilde{\widehat{\nu}}$ is semi-finite. Using now [24, 5.2.2], we get that there exists an invertible $p \in N^+$, $p \le 1$, such that

$$(D\widetilde{\widehat{\nu}} : D(\widehat{\Phi}\circ\mathcal{I}+\widetilde{\widehat{\nu}}))_t = \beta(p)^{it}$$

for all $t \in \mathbb{R}$. So, $\beta(p)$ is invariant under the modular group $\sigma^{\widetilde{\widehat{\nu}}}$ (i.e. p is invariant under γ) and we get that

$$2\left\|\Lambda_{\widetilde{\widehat{\nu}}}\left[\left(\left(\mathrm{id}\otimes\omega_{\xi,\widehat{J}\Lambda_{\widehat{\varphi}}(z)}\right)(W^*)\otimes 1_{H_\nu}\right)\widehat{\mathfrak{a}}(q)\right]\right\|^2$$
$$\begin{aligned}
&= \left\|\Lambda_{\widehat{\Phi}\circ\mathcal{I}+\widetilde{\widehat{\nu}}}\left[\left(\left(\mathrm{id}\otimes\omega_{\xi,\widehat{J}\Lambda_{\widehat{\varphi}}(z)}\right)(W^*)\otimes 1_{H_\nu}\right)\widehat{\mathfrak{a}}(q)\right]\right\|^2\\
&= \left\|J_{\widetilde{\widehat{\nu}}}\beta(p^{-1})J_{\widetilde{\widehat{\nu}}}\Lambda_{\widetilde{\widehat{\nu}}}\left[\left(\left(\mathrm{id}\otimes\omega_{\xi,\widehat{J}\Lambda_{\widehat{\varphi}}(z)}\right)(W^*)\otimes 1_{H_\nu}\right)\widehat{\mathfrak{a}}(q)\right]\right\|^2,
\end{aligned}$$

from which we get that $p = 1/2$, and $\widetilde{\widehat{\nu}} = 1/2(\widehat{\Phi}\circ\mathcal{I}+\widetilde{\widehat{\nu}})$. Thus, $\widetilde{\widehat{\nu}} = \widehat{\Phi}\circ\mathcal{I}$. $\square$

6.6 Theorem. *Let $\mathbb{G}$ be a locally compact quantum group, $(N, \mathfrak{a}, \widehat{\mathfrak{a}})$ a braided-commutative $\mathbb{G}$-Yetter–Drinfel'd algebra, ν a normal faithful semi-finite weight on N. Let D_t be the Radon–Nikodym derivative of the weight ν with respect to the action $\mathfrak{a}$ and $\widehat{D}_t$ be the Radon–Nikodym derivative of the weight ν with respect to the action $\widehat{\mathfrak{a}}$. Then the following conditions are equivalent:*

(i) $\mathfrak{G}(N, \mathfrak{a}, \widehat{\mathfrak{a}}, \nu)$ *is a measured quantum groupoid;*

(ii) $\mathfrak{G}(N, \widehat{\mathfrak{a}}, \mathfrak{a}, \nu)$ *is a measured quantum groupoid;*

(iii) $(\tau_t Ad(\delta^{-it}) \otimes \gamma_t)(D_s) = D_s$ *for all* $s, t \in \mathbb{R}$*;*

(iv) $(\widehat{\tau}_t Ad(\widehat{\delta}^{-it}) \otimes \gamma_{-t})(\widehat{D}_s) = D_s$ *for all* $s, t \in \mathbb{R}$.

Proof. By 6.5, we know that (i) implies (ii), and is therefore equivalent to (ii). Moreover, by 5.9, we know that (i) is equivalent to (iii). Applying 5.9 to $\mathfrak{G}(N, \widehat{\mathfrak{a}}, \mathfrak{a}, \nu)$, we obtain (iv), because the one-parameter group $\widehat{\gamma}_t$ is equal to γ_{-t}. The proof that (iv) implies (ii) is the same as in 5.9, where we use again that the one-parameter group $\widehat{\gamma}_t$ of N constructed from the dual measured quantum groupoid is equal to γ_{-t} (5.3). □

6.7 Corollary. *Let $\mathbb{G}$ be a locally compact quantum group, $(N, \mathfrak{a}, \widehat{\mathfrak{a}})$ a braided-commutative $\mathbb{G}$-Yetter–Drinfel'd algebra, and ν a normal faithful semi-finite weight on N. If the weight ν is $\widehat{k}$-invariant with respect to $\widehat{\mathfrak{a}}$, for $\widehat{k}$ affiliated to the center $Z(\widehat{M})$ or $\widehat{k} = \widehat{\delta}^{-1}$, then $\mathfrak{G}(N, \mathfrak{a}, \widehat{\mathfrak{a}}, \nu)$ is a measured quantum groupoid and its dual is isomorphic to $\mathfrak{G}(N, \widehat{\mathfrak{a}}, \mathfrak{a}, \nu)$.*

Proof. We verify easily property (iv) of 6.6, and then obtain the result by 6.6 and 6.5. □

7. Examples

In this chapter, we give several examples of measured quantum groupoids constructed from a braided-commutative Yetter–Drinfel'd algebra. First, in 7.1, we show that usual transformation groupoids are indeed a particular case of this construction, which justifies the terminology. Other examples are constructed from quotient type co-ideals of compact quantum groups, in particular one is constructed from the Podleś sphere S_q^2 (7.4.5). Another example (7.5.1) is constructed from a normal closed subgroup H of a locally compact group G.

7.1. Transformation Groupoid. Let us consider a locally compact group G right acting on a locally compact space X; let us denote $\mathfrak{a}$ this action. It is well known that this leads to a locally compact groupoid $X \underset{\mathfrak{a}}{\curvearrowleft} G$, usually called a *transformation groupoid*. This groupoid is the set $X \times G$, with X as set of units, and range and source applications given by $r(x, g) = x$ and $s(x, g) = \mathfrak{a}_g(x)$, the product being

$(x,g)(\mathfrak{a}_g(x),h) = (x,gh)$, and the inverse $(x,g)^{-1} = (\mathfrak{a}_g(x),g^{-1})$ [32, 1.2.a]. This locally compact groupoid has a left Haar system [32, 2.5a], and for any measure ν on X, the lifted measure on $X \times G$ is $\nu \otimes \lambda$, where λ is the left Haar measure on G.

The measure ν is then quasi-invariant in the sense of [32] and 4.2 if and only if $\nu \otimes \lambda$ is equivalent to its image under the inversion $(x,g) \to (x,g)^{-1}$. This is equivalent [32, 3.21] to asking that, for all $g \in G$, the measure $\nu \circ \mathfrak{a}_g$ is equivalent to ν, which leads to a Radon–Nikodym $\Delta(x,g) = \frac{d\nu\circ\mathfrak{a}_{g^{-1}}}{d\nu}(x)$. Then, the Radon–Nikodym derivative between $\nu \otimes \lambda$ and its image under the inversion $(x,g) \to (x,g)^{-1}$ is $\Delta(x,g)\Delta_G(g)$, where Δ_G is the modulus of G.

Let us consider the trivial action of the dual locally compact quantum group $\widehat{G}$, defined by $\iota(f) = 1 \otimes f$ for all $f \in L^\infty(X)$. It is straightforward to verify that $(L^\infty(X),\mathfrak{a},\iota)$ is a G-Yetter–Drinfel'd algebra which is braided-commutative. The measure ν, regarded as a normal semi-finite faithful weight on $L^\infty(X)$, is evidently invariant under ι. So, by 6.7, we obtain measured quantum groupoid structures on the crossed products $G \ltimes_{\mathfrak{a}} L^\infty(X)$ and $\widehat{G} \ltimes_\iota L^\infty(X)$.

The von Neumann algebra $\widehat{G} \ltimes_\iota L^\infty(X)$ is $L^\infty(G) \otimes L^\infty(X)$, or $L^\infty(X \underset{\mathfrak{a}}{\curvearrowleft} G)$, and the structure of measured quantum groupoid is nothing but the structure given by the groupoid structure of $X \underset{\mathfrak{a}}{\curvearrowleft} G$.

The dual measured quantum groupoid $\widehat{X \underset{\mathfrak{a}}{\curvearrowleft} G}$ is the von Neumann algebra generated by the left regular representation of $X \underset{\mathfrak{a}}{\curvearrowleft} G$, which is the crossed product $G \ltimes_{\mathfrak{a}} L^\infty(X)$. Let us note that this measured quantum grouped is co-commutative, in particular, $\beta = \mathfrak{a}$ and $\gamma_t = \sigma_t^\nu = \mathrm{id}_{L^\infty(X,\nu)}$ for all $t \in \mathbb{R}$. As $\tau_t = \mathrm{Ad}(\Delta_G^{it}) = \mathrm{id}_{L^\infty(G)}$, we see that $D_t = \Delta(x,g)^{it}$ satisfies the condition of 6.6. Moreover, $\widehat{D}_t = 1$ for all $t \in \mathbb{R}$.

Therefore, we get that any transformation groupoid gives a very particular case of our "measured quantum transformation groupoids", which explains the terminology.

7.2. Basic example. Let $\mathbb{G} = (M,\Gamma,\varphi,\varphi\circ R)$ be a locally compact quantum group, $D(\mathbb{G})$ its quantum double, and let us use the notation introduced in 2.4.5. There exists an action $\mathfrak{a}_D$ of $D(\mathbb{G})$ on M such that

$$\mathfrak{a}_D(x) \otimes 1 = \Gamma_D(x \otimes 1).$$

The Yetter–Drinfel'd algebra associated to this action is given by the restrictions of the applications $\mathfrak{b}$ and $\widehat{\mathfrak{b}}$ to M, which are, respectively, the coproduct Γ (when considered as a left action of $\mathbb{G}$ on M), and the adjoint action ad of $\widehat{\mathbb{G}}$ on M given by

$$\mathrm{ad}(x) = \sigma W(x \otimes 1)W^*\sigma = \widehat{W}^*(1 \otimes x)\widehat{W}, \tag{4}$$

and we get this way the Yetter–Drinfel'd algebra (M, Γ, ad), which is the basic example given in [28]. Moreover, as

$$\begin{aligned} \varsigma\Gamma(x) &= ((R \otimes R) \circ \Gamma \circ R)(x) \\ &= (\widehat{J} \otimes \widehat{J})W^*(\widehat{J} \otimes \widehat{J})(1 \otimes x)(\widehat{J} \otimes \widehat{J})W(\widehat{J} \otimes \widehat{J}), \end{aligned} \tag{5}$$

we get that that

$$\begin{aligned} \varsigma\alpha^{\mathrm{o}}(x^{\mathrm{o}}) &= (J\widehat{J} \otimes 1)W^*(1 \otimes \widehat{J}x\widehat{J})W(\widehat{J}J \otimes 1) \\ &= (J \otimes J)W(1 \otimes J\widehat{J}x\widehat{J}J)W^*(J \otimes J) \end{aligned}$$

(where we prefer to note α the left action Γ to avoid confusion between α^{o} defined in 2.5.1 and the coproduct Γ^{o} of the locally compact quantum group $\mathbb{G}^{\mathrm{o}}$). But

$$\varsigma\, \mathrm{ad}^{\mathrm{o}}(x^{\mathrm{o}}) = (J \otimes J)W(x \otimes 1)W^*(J \otimes J)$$

from which we get that this Yetter–Drinfel'd algebra is braided-commutative.

As φ is invariant under Γ, using 5.9, we can equip the crossed products $\mathbb{G} \ltimes_{\Gamma} M$ and $\widehat{\mathbb{G}} \ltimes_{\mathrm{ad}} M$ with structures of measured quantum groupoids.

Let us describe $\widehat{\mathbb{G}} \ltimes_{\mathrm{ad}} M$ in more detail. We claim that the map $\Phi := \mathrm{Ad}((J\widehat{J} \otimes 1)\widehat{W})$ identifies $\widehat{\mathbb{G}} \ltimes_{\mathrm{ad}} M$ with $M' \otimes M$. Indeed, the first algebra is generated by elements of the form $(z \otimes 1)\,\mathrm{ad}(x)$ and $x, z \in M$, and

$$\mathrm{Ad}(\widehat{W})[(z \otimes 1)\,\mathrm{ad}(x)] = \Gamma^{\mathrm{o}}(z)(1 \otimes x) = \mathrm{Ad}(\sigma)(\Gamma(z)(x \otimes 1)).$$

But elements of the form $\Gamma(z)(x \otimes 1)$ generate $M \otimes M$, and as $\mathrm{Ad}(J\widehat{J})(M) = M'$, the assertion follows. We just saw that $\Phi(\mathrm{ad}(x)) = 1 \otimes x$, and we claim that $\Phi(\beta(x)) = x^{\mathrm{o}} \otimes 1$. Using (4) and the fact that $\widehat{W}^*$ is a cocycle for the trivial action of $\widehat{\mathbb{G}}$ on M, we get [41, 4.2]

$$U^{\mathrm{ad}}_{\phi} = \widehat{W}^*(J \otimes J)\widehat{W}(J \otimes J)$$

and therefore, using the relations $(J \otimes \widehat{J})\widehat{W}^*(J \otimes \widehat{J}) = \widehat{W}$ and $\Gamma \circ R = (R \otimes R) \circ \Gamma^{\mathrm{o}}$ (2.1),

$$\begin{aligned} \Phi(\beta(x)) &= \mathrm{Ad}((J\widehat{J} \otimes 1)\widehat{W}U^{\mathrm{ad}}_{\phi}(\widehat{J} \otimes J))[\Gamma(x)] \\ &= \mathrm{Ad}((J\widehat{J} \otimes 1)(J \otimes J)\widehat{W}(J \otimes J)(\widehat{J} \otimes J))[\Gamma(x)] \\ &= \mathrm{Ad}((\widehat{J} \otimes J)\widehat{W}(J\widehat{J} \otimes \widehat{J}\widehat{J}))[\Gamma(x)] \\ &= \mathrm{Ad}((\widehat{J}J \otimes J\widehat{J})\widehat{W}^*)[\Gamma^{\mathrm{o}}(R(x))] \\ &= \mathrm{Ad}((\widehat{J}J \otimes J\widehat{J}))[R(x) \otimes 1] \\ &= x^{\mathrm{o}} \otimes 1. \end{aligned}$$

Therefore, Φ defines an isomorphism between $\mathfrak{G}(M, \mathrm{ad}, \Gamma, \phi)$ and the pair quantum groupoid $M' \otimes M$ of Lesieur [24, 15], and induces an isomorphism between the respective duals, which are (isomorphic to) $\mathfrak{G}(M, \Gamma, \mathrm{ad}, \phi)$ and the dual pair quantum groupoid $B(H)$ constructed in [24, 15.3.7], respectively.

7.3. Quantum measured groupoid associated to an action. Let us apply 7.2 to $\widehat{\mathbf{G}}^o$. We obtain that $(\widehat{M}, \widehat{\Gamma}^o, \mathrm{ad})$ is a $\widehat{\mathbf{G}}^o$-Yetter–Drinfel'd algebra, where ad means here $\mathrm{ad}(x) = W^{c*}(1 \otimes x)W^c$. As noticed by [28, 3.1], we can extend this example to any crossed-product $\mathbf{G} \ltimes_{\mathfrak{a}} \mathbf{N}$, where $\mathfrak{a}$ is a left action of $\mathbf{G}$ on a von Neumann algebra N. Let us recall this construction. For any $X \in \mathbf{G} \ltimes_{\mathfrak{a}} \mathbf{N}$, the dual action $\widetilde{\mathfrak{a}}$ is given by

$$\widetilde{\mathfrak{a}}(X) = (\widehat{W}^{o*} \otimes 1)(1 \otimes X)(\widehat{W}^o \otimes 1).$$

Let us also write

$$\underline{\mathrm{ad}}(X) = (W^{c*} \otimes 1)(1 \otimes X)(W^c \otimes 1).$$

We first show that this formula defines an action $\underline{\mathrm{ad}}$ of $\mathbf{G}^o$ on $\mathbf{G} \ltimes_{\mathfrak{a}} \mathbf{N}$. If $X = y \otimes 1$, with $y \in \widehat{M}$, we get that $\underline{\mathrm{ad}}(1 \otimes y) = \mathrm{ad}(y) \otimes 1$, which belongs to $M' \otimes \mathbf{G} \ltimes_{\mathfrak{a}} \mathbf{N}$. If $X = \mathfrak{a}(x)$, with $x \in N$, we get that $\underline{\mathrm{ad}}(\mathfrak{a}(x)) = (W^{c*} \otimes 1)(1 \otimes \mathfrak{a}(x))(W^c \otimes 1)$, which belongs to $M' \otimes \mathbb{G} \ltimes_{\mathfrak{a}} N$; moreover, the properties of W^{c*} give then that $\underline{\mathrm{ad}}$ is an action.

To prove that $(\mathbf{G} \ltimes_{\mathfrak{a}} N, \widetilde{\mathfrak{a}}, \underline{\mathrm{ad}})$ is a $\widehat{G}^{\mathbf{o}}$-Yetter–Drinfel'd algebra, we have to check that, for any $X \in \widehat{G}^{\mathbf{o}}$,

$$\mathrm{Ad}(\sigma_{12}\widehat{W}^o_{12})(\mathrm{id} \otimes \underline{\mathrm{ad}})\widetilde{\mathfrak{a}}(X) = (\mathrm{id} \otimes \widetilde{\mathfrak{a}})\underline{\mathrm{ad}}(X).$$

To check that, it suffices to prove that $\sigma_{12}\widehat{W}^o_{12}W^{c*}_{23}\widehat{W}^{o*}_{13} = \widehat{W}^{o*}_{23}W^c_{13}$, which follows from $\widehat{W}^o = \sigma W^{c*}\sigma$ and the pentagonal relation for W^c.

7.3.1 Proposition. *Let $\mathfrak{a}$ an action of a locally compact quantum group $\mathbf{G}$ on a von Neumann algebra N and let $B = \mathbf{G} \ltimes_{\mathfrak{a}} \mathbf{N} \cap \mathfrak{a}(\mathbf{N})'$. Then the formulas*

$$\mathfrak{b}(X) = (\widehat{W}^{o*} \otimes 1)(1 \otimes X)(\widehat{W}^o \otimes 1),$$
$$\widehat{\mathfrak{b}}(X) = (W^{c*} \otimes 1)(1 \otimes X)(W^c \otimes 1)$$

define actions $\mathfrak{b}$ and $\widehat{\mathfrak{b}}$ of $\widehat{\mathbf{G}}^o$ and $\mathbf{G}^{\mathbf{c}}$, respectively, on B and $(B, \mathfrak{b}, \widehat{\mathfrak{b}})$ is a braided-commutative Yetter–Drinfel'd algebra.

Proof. As $\widetilde{\mathfrak{a}}(\mathfrak{a}(x)) = 1 \otimes \mathfrak{a}(x)$, for all $x \in N$, we get that $\mathfrak{b}$ is an action of $\widehat{\mathbf{G}}^o$ on $B = \mathbf{G} \ltimes_{\mathfrak{a}} N \cap \mathfrak{a}(N)'$.

To prove a similar result for $\widehat{\mathfrak{b}}$, we need to make a detour via the inclusion $\mathfrak{a}(N) \subset G \ltimes_{\mathfrak{a}} N$ which is depth 2 [41, 5.10]. Let ν be a normal faithful semi-finite weight on N, and $\widetilde{\nu}$ its dual weight on $\mathbf{G} \ltimes_{\mathfrak{a}} N$. Then, we have

$$\begin{aligned} J_{\widetilde{\nu}}\mathfrak{a}(N)'J_{\widetilde{\nu}} &= (\widehat{J} \otimes J_\nu)(U^{\mathfrak{a}}_\nu)^*\mathfrak{a}(N)'U^{\mathfrak{a}}_\nu(\widehat{J} \otimes J_\nu) \\ &= (\widehat{J} \otimes J_\nu)(B(H) \otimes N')(\widehat{J} \otimes J_\nu) = B(H) \otimes N \end{aligned}$$

and therefore $B(H) \otimes N \cap (\mathbf{G} \ltimes_{\mathfrak{a}} N)' = J_{\widetilde{\nu}}BJ_{\widetilde{\nu}}$.

Moreover [41, 2.6 (ii)], we have an isomorphism Φ from $B(H) \otimes N$ with $\mathbf{G}^o \ltimes_{\widetilde{\mathfrak{a}}} \mathbf{G} \ltimes_{\mathfrak{a}} N$ which sends $\mathbf{G} \ltimes_{\mathfrak{a}} N$ onto $\widetilde{\mathfrak{a}}(\mathbf{G} \ltimes_{\mathfrak{a}} N)$. Via this isomorphism, the bidual action $\widetilde{\widetilde{\mathfrak{a}}}$ of $\mathbf{G}^{oc}$ on $\mathbf{G}^o \ltimes_{\widetilde{\mathfrak{a}}} \mathbf{G} \ltimes_{\mathfrak{a}} N$ gives an action γ of $\mathbf{G}$ on $B(H_\nu) \otimes N$. As $\widetilde{\widetilde{\mathfrak{a}}}$ is invariant on $\widetilde{\mathfrak{a}}(\mathbf{G} \ltimes_{\mathfrak{a}} N)$, γ is invariant on $\mathbf{G} \ltimes_{\mathfrak{a}} N$, and its restriction to $J_{\widetilde{\nu}} B J_{\widetilde{\nu}}$ defines an action of $\mathbf{G}$ on $J_{\widetilde{\nu}} B J_{\widetilde{\nu}}$, and, thanks to this restriction, we can define an action of $\mathbf{G}^{\mathfrak{c}}$ on B. Let's have a closer look at this last action: γ is given, for any $X \in B(H) \otimes N$, by [41, 2.6 (iii)]

$$\gamma(X) = W^o_{12}(\varsigma \otimes \mathrm{id})(\mathrm{id} \otimes \mathfrak{a})(X) W^{o*}_{12} = \mathrm{Ad}[W^o_{12}(U^{\mathfrak{a}}_\nu)_{13}](X_{23}).$$

So, the opposite action of its restriction to $J_{\widetilde{\nu}} B J_{\widetilde{\nu}}$ will be implemented by

$$\begin{aligned}
&(J \otimes J_{\widetilde{\nu}}) W^o_{12}(U^{\mathfrak{a}}_\nu)_{13}(\widehat{J} \otimes J_{\widetilde{\nu}}) \\
&\qquad = (U^{\mathfrak{a}}_\nu)_{23}(J \otimes \widehat{J} \otimes J_\nu) W^o_{12}(U^{\mathfrak{a}}_\nu)_{13}(\widehat{J} \otimes \widehat{J} \otimes J_\nu)(U^{\mathfrak{a}}_\nu)^*_{23} \\
&\qquad = (U^{\mathfrak{a}}_\nu)_{23}(J \otimes \widehat{J} \otimes J_\nu) W^o_{12}(\widehat{J} \otimes \widehat{J} \otimes J_\nu)(U^{\mathfrak{a}}_\nu)^*_{13}(U^{\mathfrak{a}}_\nu)^*_{23} \\
&\qquad = (J\widehat{J})_1 (U^{\mathfrak{a}}_\nu)_{23} W_{12}(U^{\mathfrak{a}}_\nu)^*_{13}(U^{\mathfrak{a}}_\nu)^*_{23} \\
&\qquad = (J\widehat{J})_1 W_{12}
\end{aligned}$$

So, we get an action of $\mathbf{G}^c$ on B given by

$$z \mapsto \mathrm{Ad}((J\widehat{J})_1 W_{12})(1 \otimes z) = W^{c*}(1 \otimes z) W^c,$$

which is $\widehat{\mathfrak{b}}$. Thus, $\widehat{\mathfrak{b}}$ is an action of $\mathbf{G}^c$ on B, and, by restriction of $(\mathbf{G} \ltimes_{\mathfrak{a}} N, \widetilde{\mathfrak{a}}, \underline{\mathrm{ad}})$, we have obtained that $(B, \mathfrak{b}, \widehat{\mathfrak{b}})$ is a $\widehat{\mathbf{G}}^o$-Yetter–Drinfel'd algebra. Let's now prove that it is braided-commutative. Let us write $\mathcal{J}(x) = J\widehat{J}x\widehat{J}J$ for any $x \in M'$. We get that $(\mathcal{J} \otimes \mathrm{id})\widehat{\mathfrak{b}}(B)$ is included in $M \otimes B$, and, therefore, commutes with $1 \otimes \mathfrak{a}(N)$. On the other hand, we get that $(\mathcal{J} \otimes \mathrm{id})(\widehat{\mathfrak{b}}(B)) = (W \otimes 1)(1 \otimes B)(W^* \otimes 1)$ commutes with $(W^* \otimes 1)(\widehat{M} \otimes 1 \otimes 1)(W \otimes 1) = \widehat{\Gamma}^o(\widehat{M}) \otimes 1$. Therefore, we get that $(\mathcal{J} \otimes \mathrm{id})(\widehat{\mathfrak{b}}(B))$ commutes with $\widetilde{\mathfrak{a}}(\mathbf{G} \ltimes_{\mathfrak{a}} N)$, and, therefore, with $\mathfrak{b}(B)$. This finishes the proof. □

Applying now 4.4 to this braided-commutative Yetter–Drinfel'd algebra, we recover the Hopf-bimodule introduced in [13, 14.1]

7.3.2 Theorem. *Let $\mathfrak{a}$ an action of a locally compact quantum group $\mathbf{G}$ on a von Neumann algebra N, let $B = \mathbf{G} \ltimes_{\mathfrak{a}} N \cap \mathfrak{a}(N)'$, let $\mathfrak{b}$ (resp. $\widehat{\mathfrak{b}}$) be the action of $\widehat{\mathbf{G}}^o$ (resp. $\mathbf{G}^c$) on B introduced in 7.3.1, and suppose that there exists a normal semi-finite faithful weight χ on B, invariant under the modular group $\sigma^{T_{\widehat{\mathfrak{a}}}}$. Then, $\mathfrak{G}(B, \mathfrak{b}, \widehat{\mathfrak{b}}, \chi)$ is a measured quantum groupoid, which is equal to the measured quantum groupoid $\mathfrak{G}(\mathfrak{a})$ introduced in [13, 14.2].*

Proof. With the hypotheses, the measured quantum groupoid $\mathfrak{G}(\mathfrak{a})$ is constructed in [13, 14.2]; so, we get that the Hopf-bimodule constructed in 7.3.1 is a measured quantum groupoid. So, we may apply 5.9 to get that $\mathfrak{G}(B, \mathfrak{b}, \widehat{\mathfrak{b}}, \chi)$ is measured quantum groupoid equal to $\mathfrak{G}(\mathfrak{a})$. □

7.3.3 Theorem. *Let* $(N, \mathfrak{a}, \widehat{\mathfrak{a}})$ *be a* $\mathbf{G}$*-Yetter–Drinfel'd algebra with a norm faithful semi-finite weight* ν *on* N *satisfying the conditions of 5.9, which allow us to construct the measured quantum groupoid* $\mathfrak{G}(N, \mathfrak{a}, \widehat{\mathfrak{a}}, \nu)$*. Suppose that* $\beta(N) = \mathbf{G} \ltimes_{\mathfrak{a}} N \cap \mathfrak{a}(N)'$*. Then, the weight* $\nu^o \circ \beta^{-1}$ *on* $\beta(N)$ *allows us to define the measured quantum groupoid* $\mathfrak{G}(\mathfrak{a})$*, which is canonically isomorphic to* $\mathfrak{G}(N^o, \widehat{\mathfrak{a}}^o, \mathfrak{a}^o, \nu^o)$*.*

Proof. We have, for all $x \in N$ and $t \in \mathbf{R}$, $\sigma_t^{T_{\tilde{\mathfrak{a}}}}(\beta(x)) = \beta(\gamma_t(x))$. As $\nu \circ \gamma_t = \nu$, we get that the weight $\nu^o \circ \beta^{-1}$ on $\beta(N)$ allows us to define the measured quantum groupoid $\mathfrak{G}(\mathfrak{a})$. Moreover, the dual action $\widetilde{\mathfrak{a}}$ of $\widehat{\mathbf{G}}^o$ on $\mathbf{G} \ltimes_{\mathfrak{a}} N$ satisfies, for all $x \in N$, by 4.4 (iii),

$$\widetilde{\mathfrak{a}}(\beta(n)) = (\mathrm{id} \otimes \beta^{\dagger})(\widehat{\mathfrak{a}}^o(x^o)),$$

which gives that $\beta^{\dagger}$ is an isomorphism between $\widetilde{\mathfrak{a}}_{|\beta(N)} = \mathfrak{b}$ and $\widehat{\mathfrak{a}}^o$. So, the result follows. □

We are indebted to the referee who suggested us to look at the relation betwen the construction made in [13, 14.2] and the measured quantum transformation groupoids considered in this article.

7.4. Quotient type co-ideals.

7.4.1 Definitions. Let $\mathbb{G} = (M, \Gamma, \varphi, \varphi \circ R)$ and $\mathbb{G}_1 = (M_1, \Gamma_1, \varphi_1, \varphi_1 \circ R_1)$ be two locally compact quantum groups. Following [21], a *morphism* from $\mathbb{G}$ on $\mathbb{G}_1$ is a non-degenerate strict $*$-homomorphism Φ from $C_0^{\mathrm{u}}(\mathbb{G})$ on the multipliers $M(C_0^{\mathrm{u}}(\mathbf{G_1}))$ (which means that Φ extends to a unital $*$-homomorphism on $M(C_0^{\mathrm{u}}(\mathbb{G}))$ such that $\Gamma_{1,u} \circ \Phi = (\Phi \otimes \Phi)\Gamma_u$, where $\Gamma_{1,u}$ denotes the coproduct of $C_0^{\mathrm{u}}(\mathbb{G}_1)$. In [21, 10.3 and 10.8], it was shown that a morphism is equivalently given by a right action Γ_r of $\mathbb{G}_1$ on M satisfying, in addition to the action condition $(\mathrm{id} \otimes \Gamma)\Gamma_r = (\Gamma_r \otimes \mathrm{id})\Gamma_r$, also the relation $(\Gamma \otimes \mathrm{id})\Gamma_r = (\mathrm{id} \otimes \Gamma_r)\Gamma$. The morphism Φ and the action Γ_r are related by the formula

$$\Gamma_r(\pi_{\mathbb{G}}(x)) = (\pi_{\mathbb{G}} \otimes \pi_{\mathbb{G}_1} \circ \Phi)\Gamma_u(x) \quad \text{for all } x \in C_0^{\mathrm{u}}(\mathbb{G}).$$

We get as well a left action Γ_l of $\mathbb{G}_1$ on M such that $(\mathrm{id} \otimes \Gamma_l)\Gamma_l = (\Gamma_1 \otimes \mathrm{id})\Gamma_l$ and $(\mathrm{id} \otimes \Gamma)\Gamma_l = (\Gamma_l \otimes \mathrm{id})\Gamma$.

Following [11, Th. 3.6], we shall say that $\mathbb{G}_1$ is a *closed quantum subgroup* of $\mathbb{G}$ *in the sense of Woronowicz*, if, in the situation above, the $*$-homomorphism Φ is surjective. In [11, 3.3], $\mathbb{G}_1$ is called a *closed quantum subgroup* of $\mathbb{G}$ *in the sense of Vaes* if there exists an injective $*$-monomorphism γ from $\widehat{M_1}$ into $\widehat{M}$ such that $\widehat{\Gamma} \circ \gamma = (\gamma \otimes \gamma) \circ \widehat{\Gamma}_1$. Moreover, any closed quantum subgroup of $\mathbb{G}$ in the sense of Vaes is a closed quantum subgroup in the sense of Woronowicz [11, 3.5], and if $\mathbb{G}_1$ is (the von Neumann version of) a compact quantum group, then the two notions are equivalent [11, 6.1]. It is also remarked that if $\mathbb{G}$ is (the von Neumann version of) a compact quantum group, then any closed quantum subgroup of $\mathbb{G}$ is also (the von Neumann version of) a compact quantum group.

7.4.2 Proposition. *Let $\mathbb{G} = (M, \Gamma, \varphi, \varphi \circ R)$ and $\mathbb{G}_1 = (M_1, \Gamma_1, \varphi_1, \varphi_1 \circ R_1)$ be two locally compact quantum groups and Φ a surjective morphism from $\mathbb{G}$ to $\mathbb{G}_1$ in the sense of 7.4.1. Let Γ_r be the right action of $\mathbb{G}_1$ on M defined in 7.4.1, and let $N = M^{\Gamma_r} = \{x \in M : \Gamma_r(x) = x \otimes 1\}$. Then:*

(i) *$\Gamma_{|N}$ is a left action of $\mathbb{G}$ on N.*

(ii) *$\mathrm{ad}_{|N}$ is a left action of $\widehat{\mathbb{G}}$ on N.*

(iii) *$(N, \Gamma_{|N}, \mathrm{ad}_{|N})$ is a braided-commutative $\mathbb{G}$-Yetter–Drinfel'd algebra.*

(iv) *Let Γ_l be the left action of $\mathbb{G}_1$ on M defined in 7.4.1. Then its invariant algebra M^{Γ_l} is equal to $R(N)$, which is a right co-ideal of $\mathbb{G}$.*

In the situation above, we call N a *quotient type left co-ideal* of $\mathbb{G}$.

Proof. (i) Since $(\mathrm{id}\otimes\Gamma_r)\Gamma = (\Gamma\otimes\mathrm{id})\Gamma_r$ by construction, we get that for every x in $N = M^{\Gamma_r}$, the coproduct $\Gamma(x)$ belongs to $M \otimes N$.

(ii) By [21, 6.6], there exists a unique unitary $U \in M(C_0^{\mathrm{u}}(\mathbb{G}) \otimes C_0^{\mathrm{r}}(\widehat{G}))$ such that $(\Gamma_u \otimes \mathrm{id})(U) = U_{13}U_{23}$ and $(\pi_{\mathbb{G}} \otimes \mathrm{id})(U) = W$, where Γ_u denotes the comultiplication on $C_0^{\mathrm{u}}(\mathbb{G})$. Let $\widehat{U} = \varsigma(U^*) \in M(C_0^{\mathrm{r}}(\widehat{\mathbb{G}})\otimes C_0^{\mathrm{u}}(\mathbb{G}))$ and $x \in C_0^{\mathrm{u}}(\mathbb{G})$. Then $\mathrm{ad}(\pi_{\mathbb{G}}(x)) = (\mathrm{id}\otimes\pi_{\mathbb{G}})(\widehat{U}^*(1\otimes x)\widehat{U})$, and using the relation $(\mathrm{id}\otimes\Gamma_u)(\widehat{U}^*) = \widehat{U}^*_{12}\widehat{U}^*_{13}$, we find

$$\begin{aligned}(\mathrm{id}\otimes\Gamma_r)(\mathrm{ad}(\pi_{\mathbb{G}}(x))) &= (\mathrm{id}\otimes\pi_{\mathbb{G}}\otimes\pi_{\mathbb{G}_1}\Phi)((\mathrm{id}\otimes\Gamma_u)(\widehat{U}^*(1\otimes x)\widehat{U}))\\ &= (\mathrm{id}\otimes\pi_{\mathbb{G}}\otimes\pi_{\mathbb{G}_1}\Phi)(\widehat{U}^*_{12}\widehat{U}^*_{13}(1\otimes\Gamma_u(x))\widehat{U}_{13}\widehat{U}_{12})\\ &= \widehat{W}^*_{12}\widetilde{U}^*_{13}(1\otimes\Gamma_r(\pi_{\mathbb{G}}(x)))\widetilde{U}_{13}\widehat{W}_{12},\end{aligned}$$

where $\widetilde{U} = (\mathrm{id}\otimes\pi_{\mathbb{G}_1}\Phi)(V)$. By continuity, we get that for any $y \in N$,

$$(\mathrm{id}\otimes\Gamma_r)(\mathrm{ad}(y)) = \widehat{W}^*_{12}\widetilde{U}^*_{13}(1\otimes y\otimes 1)\widetilde{U}_{13}\widehat{W}_{12} = \mathrm{ad}(y)\otimes 1,$$

showing that $\mathrm{ad}(y) \in \widehat{M}\otimes N$.

(iii) This follows immediately from 2.4.

(iv) This follows easily from the fact that the unitary antipode reverses the comultiplication. □

7.4.3 Theorem. *Let $\mathbb{G} = (M, \Gamma, \varphi, \varphi \circ R)$ be a locally compact quantum group and (A_1, Γ_1) a compact quantum group which is a closed quantum subgroup in the sense of 7.4, and denote by N the quotient type co-ideal defined by this closed subgroup, as defined in 7.4.2. Then, the restriction of the weight $\varphi \circ R$ to N is semi-finite and δ^{-1}-invariant with respect to the action $\Gamma_{|N}$. Therefore, $\mathfrak{G}(N, \Gamma_{|N}, \mathrm{ad}_{|N}, \varphi \circ R_{|N})$ and $\mathfrak{G}(N, \mathrm{ad}_{|N}, \Gamma_{|N}, \varphi \circ R_{|N})$ are measured quantum groupoids, dual to each other.*

Proof. The formula $E = (\mathrm{id} \otimes \omega_1) \circ \Gamma_r$, where ω_1 is the Haar state of (A_1, Γ_1), and Γ_r is the right action of (A_1, Γ_1) on M defined in 7.4, defines a normal faithful conditional expectation from M onto $N = M^{\Gamma_r}$.

By definition of Γ_r (7.4.1), and using the right-invariance of $\varphi \circ R \circ \pi_{\mathbb{G}}$ with respect to the coproduct Γ_u of $C_0^{\mathrm{u}}(\mathbb{G})$, we get that for any $y \in C_0^{\mathrm{u}}(\mathbb{G})$, with the notations of 7.4.1,

$$\begin{aligned}\varphi \circ R \circ E(\pi_{\mathbb{G}}(y)) &= (\varphi \circ R \otimes \omega_1)\Gamma_r(\pi(y)) \\ &= (\varphi \circ R \circ \pi_{\mathbb{G}} \otimes \omega_1 \circ \pi_{\mathbb{G}_1} \circ \Phi)\Gamma_u(y) \\ &= (\varphi \circ R \circ \pi_{\mathbb{G}})(y)(\omega_1 \circ \pi_{\mathbb{G}} \circ \Phi)(1) \\ &= (\varphi \circ R \circ \pi_{\mathbb{G}})(y).\end{aligned}$$

Therefore, $\varphi \circ R \circ E(x) = \varphi \circ R(x)$ for all $x \in C_0^{\mathrm{r}}(\mathbb{G})$, and, by continuity, for all $x \in M$, which gives that this conditional expectation E is invariant under $\varphi \circ R$. Moreover, we get that $\varphi \circ R_{|N}$ is semi-finite and $\sigma_t^{\varphi \circ R} \circ E = E \circ \sigma_t^{\varphi \circ R}$.

This weight $\varphi \circ R_{|N}$ is clearly δ^{-1}-invariant with respect to $\Gamma_{|N}$. The result comes then from 5.9 and 6.5. □

7.4.4 Corollary. *Let (A, Γ) be a compact quantum group, ω its Haar state (which we can suppose to be faithful) and let $\mathbb{G} = (\pi_\omega(A)'', \Gamma, \omega, \omega)$ be the von Neumann version of (A, Γ) (2.1). Let N be a sub-von Neumann algebra N of $\pi_\omega(A)''$. Then the following conditions are equivalent:*

(i) *$\Gamma_{|N}$ is a left action of $\mathbb{G}$ on N and $\mathrm{ad}_{|N}$ is a left action of $\widehat{\mathbb{G}}$ on N.*

(ii) *There exists a quantum compact subgroup of (A, Γ) such that N is the quotient type co-ideal of $\mathbb{G}$ constructed from this quantum compact subgroup.*

If (i) and (ii) hold, then the crossed products $\mathbb{G} \ltimes_{\Gamma_{|N}} N$ and $\widehat{\mathbb{G}} \ltimes_{\mathrm{ad}_{|N}} N$ carry mutually dual structures of measured quantum groupoids $\mathfrak{G}(N, \Gamma_{|N}, \mathrm{ad}_{|N}, \omega_{|N})$ and $\mathfrak{G}(N, \mathrm{ad}_{|N}, \Gamma_{|N}, \omega_{|N})$, respectively.

Proof. The fact that (ii) implies (i) is given by 7.4.3. Suppose (i). Then N is, by 7.4.2, a quotient type co-ideal of $\mathbb{G}$, which is defined as the invariants by a right action Γ_r of a closed quantum subgroup of $\mathbb{G}$, which is (7.4.1) a compact quantum group (A_1, Γ_1). Denote its Haar state by ω_1. Then $\Gamma_r(A) \subset A \otimes A_1$, and the conditional expectation $E = (\mathrm{id} \otimes \omega_1)\Gamma_r$ which sends $\pi_\omega(A)''$ onto N, sends A onto $A \cap N$. From this it is easy to get that $A \cap N$ is weakly dense in N. But $N \cap A$ is a sub-C^*-algebra of A which is invariant under Γ and ad; therefore, using [29, Th. 3.1], we get (ii). If these conditions hold, we can apply 7.4.3. □

7.4.5. Example of a measured quantum groupoid constructed from a quotient type coideal of a compact quantum group. Let us take the compact quantum group $\mathrm{SU}_q(2)$ [48], which is the C^*-algebra generated by elements α and γ satisfying the relations

$$\alpha^*\alpha + \gamma^*\gamma = 1, \quad \alpha\alpha^* + q^2\gamma\gamma^* = 1,$$
$$\gamma\gamma^* = \gamma^*\gamma, \quad q\gamma\alpha = \alpha\gamma, \quad q\gamma^*\alpha = \alpha\gamma^*.$$

The circle group $\mathbb{T}$ appears as a closed quantum subgroup via the morphism Φ from $C_0^{\mathrm{u}}(\mathrm{SU}_q(2))$ to $C_0^{\mathrm{u}}(\mathbb{T}) = C_0(\mathbb{T})$ given by $\Phi(\alpha) = 0$ and $\Phi(\gamma) = \mathrm{id}$. Then we obtain the Podleś sphere S_q^2 as a quotient type coideal from this map [31], and mutually dual structures of measured quantum groupoids $\mathfrak{G}(S_q^2, \Gamma_{|S_q^2}, \mathrm{ad}_{|S_q^2}, \omega_{|S_q^2})$ on $\mathrm{SU}_q(2) \ltimes_{\Gamma_{|S_q^2}} S_q^2$ and $\mathfrak{G}(S_q^2, \mathrm{ad}_{|S_q^2}, \Gamma_{|S_q^2}, \omega_{|S_q^2})$ on $\widehat{\mathrm{SU}_q(2)} \ltimes_{\mathrm{ad}_{|S_q^2}} S_q^2$, respectively.

7.4.6. Further examples. Here we quickly give examples of situations in which the hypothesis of 7.4.3 are fulfilled.

Let us consider the (non-compact) quantum group $E_q(2)$ constructed by Woronowicz in [49]. In [20, 2.8.36] is proved that the circle group $\mathbb{T}$ is a closed quantum subgroup of $E_q(2)$.

In [43] is constructed the cocycle bicrossed product of two locally compact quantum groups (M_1, Γ_1) and (M_2, Γ_2), and it is proved [43, 3.5] that $(\widehat{M_1}, \widehat{\Gamma_1})$ is a closed subgroup (in the sense of Vaes) of (M, Γ). So, if (M_1, Γ_1) is a discrete quantum group, then $(\widehat{M_1}, \widehat{\Gamma_1})$ is the von Neumann version of a compact quantum group which is a closed quantum subgroup of (M, Γ).

7.5. Another example.

7.5.1 Theorem. *Let G be a locally compact group and H a closed normal subgroup of G. Then:*

(i) *The von Neumann algebra $\mathcal{L}(H)$, which can be considered as a sub-von Neumann algebra of $\mathcal{L}(G)$, is invariant under the coproduct Γ_G of $\mathcal{L}(G)$, considered as a right action of the locally compact quantum group $\widehat{G}$ on $\mathcal{L}(G)$, and under the adjoint action* ad *of G on $\mathcal{L}(G)$. Therefore, $(\mathcal{L}(H), \Gamma_{G|\mathcal{L}(H)}, \mathrm{ad}_{|\mathcal{L}(H)})$ is a braided-commutative $\widehat{G}$-Yetter–Drinfel'd algebra, which is a subalgebra of the canonical example $(\mathcal{L}(G), \Gamma_G, \mathrm{ad})$ described in 7.2.*

(ii) *The Plancherel weight φ_H on $\mathcal{L}(H)$ satisfies the conditions of 5.9, and the crossed product $\widehat{G} \ltimes_{\Gamma_{G|\mathcal{L}(H)}} \mathcal{L}(H)$ (which is isomorphic to $(\mathcal{L}(H) \cup L^\infty(G))''$) carries a structure of measured quantum groupoid*

$$\mathfrak{G}(\mathcal{L}(H), \Gamma_{G|\mathcal{L}(H)}, \mathrm{ad}_{|\mathcal{L}(H)}, \varphi_H)$$

over the basis $\mathcal{L}(H)$.

Proof. (i) Let λ_G (resp. λ_H) be the left regular representation of G (resp. H). It is well known that the application which sends $\lambda_H(s)$ to $\lambda_G(s)$, where $s \in H$, extends to an injection from $\mathcal{L}(H)$ into $\mathcal{L}(G)$, which will send the coproduct Γ_H of $\mathcal{L}(H)$ on the coproduct Γ_G of $\mathcal{L}(G)$. Let us identify $\mathcal{L}(H)$ with this sub-von Neumann algebra of $\mathcal{L}(G)$. Then for all $x \in \mathcal{L}(H)$,

$$\Gamma_G(x) = \Gamma_H(x) \in \mathcal{L}(H) \otimes \mathcal{L}(H) \subset \mathcal{L}(G) \otimes \mathcal{L}(H),$$

so that the coproduct, considered as a right action of $\widehat{G}$ on $\mathcal{L}(G)$, gives also a right action of $\widehat{G}$ on $\mathcal{L}(H)$.

Let W_G be the fundamental unitary of G, which belongs to $L^\infty(G) \otimes \mathcal{L}(G)$. The adjoint action of G on $\mathcal{L}(G)$ is given, for $x \in \mathcal{L}(G)$ by $\mathrm{ad}(x) = W_G^*(1 \otimes x)W_G$, and is therefore the function on G given by $s \mapsto \lambda_G(s)x\lambda_G(s)^*$. Hence, if $t \in H$, we get that $\mathrm{ad}(\lambda_H(s))$ is the function $s \mapsto \lambda_G(sts^{-1})$. As H is normal, sts^{-1} belongs to H, and this function takes its values in $\mathcal{L}(H)$. By density, we get that for any $x \in \mathcal{L}(H)$, $\mathrm{ad}(x)$ belongs to $L^\infty(G) \otimes \mathcal{L}(H)$, and, therefore, the restriction of the adjoint action of G to $\mathcal{L}(H)$ is an action of G on $\mathcal{L}(H)$.

(ii) The Haar weight φ_H is invariant under $\Gamma_{G|\mathcal{L}(H)}$ because $(\mathrm{id} \otimes \varphi_H)(\Gamma_G(x)) = (\mathrm{id} \otimes \varphi_H)(\Gamma_H(x)) = \varphi_H(x)1$ for all $x \in \mathcal{L}(H)^+$. We can therefore apply 5.9 to that braided-commutative Yetter–Drinfel'd algebra, equipped with this relatively invariant weight, and get (ii). Let us remark that $\widehat{G} \ltimes_{\Gamma_{G|\mathcal{L}(H)}} \mathcal{L}(H)$ is equal to $(\Gamma_G(\mathcal{L}(H)) \cup L^\infty(G) \otimes 1_{L^2(G)})''$ which we can write:

$$((J \otimes J)W_G^*(J \otimes J)(\mathcal{L}(H) \otimes 1_{L^2(G)})(J \otimes J)W_G(J \otimes J) \cup L^\infty(G) \otimes 1_{L^2(G)})''$$

which is clearly isomorphic to $(\mathcal{L}(H) \cup L^\infty(G))''$. □

7.5.2 Remark. Let us take again the hypotheses of 7.5.1, in the particular case where G is abelian. Then $\widehat{G}$ (resp. $\widehat{H}$) is a commutative locally compact group, and we have constructed a right action of $\widehat{G}$ on the set $\widehat{H}$, which leads to a transformation groupoid $\widehat{H} \curvearrowleft \widehat{G}$. Then, the measured quantum groupoid constructed in 7.5.1(ii) is just the dual of this transformation groupoid.

8. Quotient type co-ideals and Morita equivalence

In this chapter, we show that, in the case of a quotient type co-ideal N of a compact quantum group $\mathbb{G}$, the measured quantum groupoid $\widehat{\mathbb{G}} \ltimes_{\mathrm{ad}|N} N$ is Morita equivalent to the quantum subgroup $\mathbb{G}_1$ (8.3).

8.1. Definitions of actions of a measured quantum groupoid and Morita equivalence.

8.1.1 Definition ([16, 2.4]). Let $\mathfrak{G} = (N, M, \alpha, \beta, \Gamma, T, T', \nu)$ be a measured quantum groupoid, and let A be a von Neumann algebra.

A *right action* of $\mathfrak{G}$ on A is a couple $(b, \underline{\mathfrak{a}})$, where:

(i) b is an injective anti-$*$-homomorphism from N into A;

(ii) $\underline{\mathfrak{a}}$ is an injective $*$-homomorphism from A into $A \underset{N}{{}_b*_\alpha} M$;

(iii) b and $\underline{\mathfrak{a}}$ satisfy

$$\underline{\mathfrak{a}}(b(n)) = 1 \underset{N}{{}_b\otimes_\alpha} \beta(n) \quad \text{for all } n \in N,$$

which allow us to define $\underline{\mathfrak{a}} \underset{N}{{}_b*_\alpha} \mathrm{id}$ from $A \underset{N}{{}_b*_\alpha} M$ into $A \underset{N}{{}_b*_\alpha} M \underset{N}{{}_\beta*_\alpha} M$, and

$$(\underline{\mathfrak{a}} \underset{N}{{}_b*_\alpha} \mathrm{id})\underline{\mathfrak{a}} = (\mathrm{id} \underset{N}{{}_b*_\alpha} \Gamma)\underline{\mathfrak{a}}.$$

If there is no ambiguity, we shall say that $\underline{\mathfrak{a}}$ is the right action.

So, a measured quantum groupoid $\mathfrak{G}$ can act only on a von Neumann algebra A which is a right module over the basis N.

Moreover, if M is abelian, then $\underline{\mathfrak{a}}(b(n)) = 1 \underset{N}{{}_b\otimes_\alpha} \beta(n)$ commutes with $\underline{\mathfrak{a}}(x)$ for all $n \in N$ and $x \in A$, so that $b(N)$ is in the center of A. As in that case (5.1) the measured quantum groupoid comes from a measured groupoid $\mathcal{G}$, we have $N = L^\infty(\mathcal{G}^{(0)}, \nu)$, and A can be decomposed as $A = \int_{\mathcal{G}^{(0)}} A^x d\nu(x)$.

The invariant subalgebra $A^{\underline{\mathfrak{a}}}$ is defined by

$$A^{\underline{\mathfrak{a}}} = \{x \in A \cap b(N)' : \underline{\mathfrak{a}}(x) = x \underset{N}{{}_b\otimes_\alpha} 1\}.$$

As $A^{\underline{\mathfrak{a}}} \subset b(N)'$, A (and $L^2(A)$) is a $A^{\mathfrak{a}}$-N^{o}-bimodule. If $A^{\underline{\mathfrak{a}}} = \mathbb{C}$, the action $(b, \underline{\mathfrak{a}})$ (or, simply $\underline{\mathfrak{a}}$) is called *ergodic*.

Let us write, for any $x \in A^+$, $T_{\underline{\mathfrak{a}}}(x) = (\mathrm{id} \underset{\nu}{{}_b*_\alpha} \Phi)\underline{\mathfrak{a}}(x)$. This formula defines a normal faithful operator-valued weight from A onto $A^{\underline{\mathfrak{a}}}$, and the action $\underline{\mathfrak{a}}$ will be called *integrable* if $T_{\underline{\mathfrak{a}}}$ is semi-finite [15, 6.11, 12, 13 and 14].

The *crossed product* of A by $\mathfrak{G}$ via the action $\underline{\mathfrak{a}}$ is the von Neumann algebra generated by $\underline{\mathfrak{a}}(A)$ and $1 \underset{N}{{}_b\otimes_\alpha} \widehat{M}'$ [13, 9.1] and is denoted by $A \rtimes_{\underline{\mathfrak{a}}} \mathfrak{G}$. There exists [13, 9.3] an integrable dual action $(1 \underset{N}{{}_b\otimes_\alpha} \widehat{\alpha}, \underline{\tilde{\mathfrak{a}}})$ of $(\widehat{\mathfrak{G}})^{\mathrm{c}}$ on $A \rtimes_{\underline{\mathfrak{a}}} \mathfrak{G}$.

We have $(A \rtimes_{\underline{\mathfrak{a}}} \mathfrak{G})^{\underline{\tilde{\mathfrak{a}}}} = \mathfrak{a}(A)$ [13, 11.5], and, therefore, the normal faithful semi-finite operator-valued weight $T_{\underline{\tilde{\mathfrak{a}}}}$ sends $A \rtimes_{\mathfrak{a}} \mathfrak{G}$ onto $\mathfrak{a}(A)$. Starting with a normal semi-finite weight ψ on A, we can thus construct a *dual weight* $\tilde{\psi}$ on $A \rtimes_{\underline{\mathfrak{a}}} \mathfrak{G}$ by the formula $\tilde{\psi} = \psi \circ \underline{\mathfrak{a}}^{-1} \circ T_{\underline{\tilde{\mathfrak{a}}}}$ [15, 13.2].

Moreover [13, 13.3], the linear set generated by all the elements $(1 \underset{N}{_b\otimes_\alpha} a)\underline{\mathfrak{a}}(x)$, where $x \in \mathfrak{N}_\psi$ and $a \in \mathfrak{N}_{\widehat{\Phi}^c} \cap \mathfrak{N}_{\widehat{T}^c}$, is a core for $\Lambda_{\tilde{\psi}}$, and one can identify the GNS representation of $A \rtimes_{\underline{\mathfrak{a}}} \mathfrak{G}$ associated to the weight $\tilde{\psi}$ with the natural representation on $H_\psi \underset{\nu}{_b\otimes_\alpha} H$ by writing

$$\Lambda_{\tilde{\psi}}[(1 \underset{N}{_b\otimes_\alpha} a)\underline{\mathfrak{a}}(x)] = \Lambda_\psi(x) \underset{\nu}{_b\otimes_\alpha} \Lambda_{\widehat{\Phi}^c}(a),$$

which leads to the identification of $H_{\tilde{\psi}}$ with $H_\psi \underset{\nu}{_b\otimes_\alpha} H$.

Let us suppose now that the action $\underline{\mathfrak{a}}$ is integrable. Let ψ_0 be a normal semi-finite weight on $A^{\underline{\mathfrak{a}}}$, and let us write $\psi_1 = \psi_0 \circ T_{\underline{\mathfrak{a}}}$. If we write $V = J_{\tilde{\psi}_1}(J_{\psi_1} \underset{N^o}{_a\otimes_\beta} J_{\widehat{\Phi}})$, we get a representation of $\mathfrak{G}$ which implements $\mathfrak{a}$ and which we shall call the *standard implementation* of $\mathfrak{a}$ ([16, 3.2] and [15, 8.6]).

Moreover, there exists then a canonical isometry G from $H_{\psi_1} \underset{\psi_0}{_s\otimes_r} H_{\psi_1}$ into $H_{\psi_1} \underset{\nu}{_b\otimes_\alpha} H$ such that, for any $x \in \mathfrak{N}_{T_\mathfrak{a}} \cap \mathfrak{N}_{\psi_1}$, $\zeta \in D((H_{\psi_1})_b, \nu^o)$ and e in $\mathfrak{N}_\Phi$,

$$(1 \underset{N}{_b\otimes_\alpha} J_\Phi e J_\Phi) G(\Lambda_{\psi_1}(x) \underset{\psi_0}{_s\otimes_r} \zeta) = \mathfrak{a}(x)(\zeta \underset{\nu}{_b\otimes_\alpha} J_\Phi \Lambda_\Phi(e)),$$

where r is the canonical injection of $A^{\underline{\mathfrak{a}}}$ into A, and $s(x) = J_{\psi_1} x^* J_{\psi_1}$ for all $x \in A^{\underline{\mathfrak{a}}}$. There exists a surjective $*$-homomorphism $\pi_{\underline{\mathfrak{a}}}$ from the crossed product $(A \rtimes_{\underline{\mathfrak{a}}} \mathfrak{G})$ onto $s(A^{\underline{\mathfrak{a}}})'$, defined, for all X in $A \rtimes_\mathfrak{a} \mathfrak{G}$ by $\pi_{\underline{\mathfrak{a}}}(X) \underset{A^\mathfrak{a}}{_s\otimes_r} 1 = G^* X G$. It should be noted that this algebra $s(A^{\underline{\mathfrak{a}}})'$ is the basic construction for the inclusion $A^{\underline{\mathfrak{a}}} \subseteq A$ [16, 3.6]. If the operator G is unitary (or, equivalently, the $*$-homomorphism $\pi_{\underline{\mathfrak{a}}}$ is an isomorphism), then the action $\underline{\mathfrak{a}}$ is called a *Galois action* [16, 3.11] and the unitary $\widetilde{G} = \sigma_\nu G$ its *Galois unitary*.

8.1.2 Definition ([15, 6.1]). A *left action* of $\mathfrak{G}$ on a von Neumann algebra A is a couple $(a, \underline{\mathfrak{b}})$, where

(i) a is an injective *-homomorphism from N into A;

(ii) $\underline{\mathfrak{b}}$ is an injective $*$-homomorphism from A into $M \underset{N}{_\beta *_a} A$;

(iii) $\underline{\mathfrak{b}}(a(n)) = \alpha(n) \underset{N}{_\beta\otimes_a} 1$ for all $n \in N$, and $(\mathrm{id} \underset{N}{_\beta *_a} \underline{\mathfrak{b}})\underline{\mathfrak{b}} = (\Gamma \underset{N}{_\beta *_a} \mathrm{id})\underline{\mathfrak{b}}$.

Then, it is clear that $(a, \varsigma_N \underline{\mathfrak{b}})$ is a right action of $\mathfrak{G}^o$ on A. Conversely, if $(b, \underline{\mathfrak{a}})$ is a left action of $\mathfrak{G}$ on A, then, $(b, \varsigma_N \underline{\mathfrak{a}})$ is a left action of $\mathfrak{G}^o$ on A.

The invariant subalgebra $A^{\underline{\mathfrak{b}}}$ is defined by

$$A^{\underline{\mathfrak{b}}} = \{x \in A \cap a(N)' : \underline{\mathfrak{b}}(x) = 1 \underset{N}{_\beta\otimes_a} x\},$$

and $T_{\underline{\mathfrak{b}}} = (\Phi \circ R \underset{\nu}{_\beta *_a} \mathrm{id})\underline{\mathfrak{b}}$ is a normal faithful operator-valued weight from A onto $A^{\underline{\mathfrak{b}}}$. The action $\underline{\mathfrak{b}}$ will be called *integrable* if $T_{\underline{\mathfrak{b}}}$ is semi-finite. It is clear that $\underline{\mathfrak{b}}$ is integrable if and only if $\varsigma_N \underline{\mathfrak{b}}$ is integrable, and *Galois* if and only if $\varsigma_N \underline{\mathfrak{b}}$ is Galois.

8.1.3 Definition ([16, 2.4]). Let $(b, \underline{\mathfrak{a}})$ be a right action of

$$\mathfrak{G}_1 = (N_1, M_1, \alpha_1, \beta_1, \Gamma_1, T_1, T_1', \nu_1)$$

on a von Neumann algebra A and $(a, \underline{\mathfrak{b}})$ a left action of

$$\mathfrak{G}_2 = (N_2, M_2, \alpha_2, \beta_2, \Gamma_2, T_2, T_2', \nu_2)$$

on A such that $a(N_2) \subset b(N_1)'$ We shall say that the actions $\underline{\mathfrak{a}}$ and $\underline{\mathfrak{b}}$ *commute* if

$$b(N_1) \subseteq A^{\underline{\mathfrak{b}}}, \quad a(N_2) \subseteq A^{\underline{\mathfrak{a}}}, \quad (\underline{\mathfrak{b}}\ {}_b{*}_{\alpha_1}_{N_1}\ \mathrm{id})\underline{\mathfrak{a}} = (\mathrm{id}\ {}_{\beta_2}{*}_a{}_{N_2}\ \underline{\mathfrak{a}})\underline{\mathfrak{b}}.$$

Let us remark that the first two properties allow us to write the fiber products $\underline{\mathfrak{b}}\ {}_b\underset{N_1}{*}_{\alpha_1}\ \mathrm{id}$ and $\mathrm{id}\ {}_{\beta_2}\underset{N_2}{*}_a\ \underline{\mathfrak{a}}$.

8.1.4 Definition ([16, 6.5]). For $i = 1, 2$, let $\mathfrak{G}_i = (N_i, M_i, \alpha_i, \beta_i, T_i, T_i', \nu_i)$ be a measured quantum groupoid. We shall say that $\mathfrak{G}_1$ is *Morita equivalent* to $\mathfrak{G}_2$ if there exists a von Neumann algebra A, a Galois right action $(b, \mathfrak{a})$ of $\mathfrak{G}_1$ on A, and a Galois left action $(a, \mathfrak{b})$ of $\mathfrak{G}_2$ on A such that

(i) $A^{\mathfrak{a}} = a(N_2)$, $A^{\mathfrak{b}} = b(N_1)$, and the actions $(b, \mathfrak{a})$ and $(a, \mathfrak{b})$ commute;

(ii) the modular automorphism groups of the normal semi-finite faithful weights $\nu_1 \circ b^{-1} \circ T_{\mathfrak{b}}$ and $\nu_2 \circ a^{-1} \circ T_{\mathfrak{a}}$ commute.

Then A (or, more precisely, $(A, b, \mathfrak{a}, a, \mathfrak{b})$) will be called an *imprimitivity bi-comodule* for $\mathfrak{G}_1$ and $\mathfrak{G}_2$.

8.2 Proposition. *Let N be a quotient type co-ideal of (the von Neumann version of a) compact quantum group $\mathbb{G} = (M, \Gamma, \omega, \omega)$, and let us consider the measured quantum groupoid $\mathfrak{G}(N, \mathrm{ad}_{|N}, \Gamma_{|N}, \omega_{|N})$ constructed in 7.4.4.*

(i) *There exists a unitary V_4 from $H\ {}_{R_{|N}}\underset{\omega_{|N}}{\otimes}_{\mathrm{ad}_{|N}}\ (H \otimes H_{\omega_{|N}})$ onto $H \otimes H$ such that*

$$V_4\Big(\xi\ {}_{R_{|N}}\underset{\omega_{|N}}{\otimes}_{\mathrm{ad}_{|N}}\ U^{\mathrm{ad}_{|N}}_{\omega_{|N}}\big(\eta \otimes J_{\omega_{|N}}\Lambda_{\omega_{|N}}(x^*)\big)\Big) = R(x)\xi \otimes \eta$$

for all $x \in N$ and ξ, η in H. Moreover, for all $z \in R(N)'$, $x \in N$ and $y \in B(H)$,

$$V_4\Big(z\ {}_{R_{|N}}\underset{N}{\otimes}_{\mathrm{ad}_{|N}}\ 1\Big) = (z \otimes 1_H)V_4,$$

$$V_4\Big(1_H\ {}_{R_{|N}}\underset{N}{\otimes}_{\mathrm{ad}_{|N}}\ U^{\mathrm{ad}_{|N}}_{\omega_{|N}}(y \otimes x^{\mathrm{o}})(U^{\mathrm{ad}_{|N}}_{\omega_{|N}})^*\Big) = (R(x) \otimes y)V_4.$$

(ii) *Let $y \in M$ and $\underline{\mathfrak{a}}(y) = V_4^*\Gamma(y)V_4$. Then $\underline{\mathfrak{a}}(y)$ belongs to*

$$M \ {}_{R_{|N}}\!\underset{N}{*}_{\mathrm{ad}_{|N}} \ (\widehat{\mathbb{G}} \ltimes_{\mathrm{ad}_{|N}} N).$$

(iii) *Let $x \in N$. Then $\underline{\mathfrak{a}}(R(x)) = 1 \ {}_{R_{|N}}\!\underset{N}{\otimes}_{\mathrm{ad}_{|N}} \widehat{\beta}(x)$, where $\widehat{\beta}$ is the canonical anti-representation of the basis N into $\widehat{\mathbb{G}} \ltimes_{\mathrm{ad}_{|N}} N$.*

(iv) *$(R_{|N}, \underline{\mathfrak{a}})$ is a right action of $\mathfrak{G}(N, \mathrm{ad}_{|N}, \Gamma_{|N}, \omega_{|N})$ on M.*

(v) *The action $\underline{\mathfrak{a}}$ is ergodic, and integrable. More precisely, the canonical operator-valued weight $T_{\underline{\mathfrak{a}}}$ is equal to the Haar state ω.*

(vi) *The action $\underline{\mathfrak{a}}$ is Galois and its Galois unitary is $V_4^* W^* \sigma$.*

Proof. (i) By 4.3 (i) applied to the braided-commutative $\widehat{\mathbb{G}}$-Yetter–Drinfel'd algebra $(N, \mathrm{ad}_{|N}, \Gamma_{|N})$, we get that $U^{\mathrm{ad}_{|N}}_{\omega_{|N}}(\eta \otimes J_{\omega_{|N}} \Lambda_{\omega_{|N}}(x^*))$ belongs to

$$D((H \otimes H_{\omega_{|N}})_{\mathrm{ad}_{|N}}, \omega_{|N})$$

and that

$$R^{\mathrm{ad}_{|N}, \omega_{|N}}(U^{\mathrm{ad}_{|N}}_{\omega_{|N}}(\eta \otimes J_{\omega_{|N}} \Lambda_{\omega_{|N}}(x^*))) = U^{\mathrm{ad}_{|N}}_{\omega_{|N}} l_\eta J_{\omega_{|N}} x^* J_{\omega_{|N}}.$$

Therefore, using standard arguments, we get an isometry V_4 given by the formula above. As its image is trivially dense in $H \otimes H$, we get that V_4 is unitary. The commutation relations are straightforward.

(ii) Thanks to the commutation property in (i), $\underline{\mathfrak{a}}(y)$ belongs to

$$M \ {}_{R_{|N}}\!\underset{N}{*}_{\mathrm{ad}_{|N}} \ B(H \otimes H_{\omega_{|N}}).$$

By 2.5 (i),

$$\begin{aligned}(\widehat{\mathbb{G}} \ltimes_{\mathrm{ad}_{|N}} N)' &= U^{\mathrm{ad}_{|N}}_{\omega_{|N}}(\widehat{\mathbb{G}}^{\mathrm{o}} \ltimes_{\mathrm{ad}^{\mathrm{o}}_{|N}} N^{\mathrm{o}})(U^{\mathrm{ad}_{|N}}_{\omega_{|N}})^* \\ &= U^{\mathrm{ad}_{|N}}_{\omega_{|N}}(M' \otimes 1 \cup \mathrm{ad}^{\mathrm{o}}_{|N}(N^{\mathrm{o}}))''(U^{\mathrm{ad}_{|N}}_{\omega_{|N}})^*.\end{aligned}$$

On one hand, the commutation relations in (i) imply

$$1 \ {}_{R_{|N}}\!\underset{\omega_{|N}}{\otimes}_{\mathrm{ad}_{|N}} U^{\mathrm{ad}_{|N}}_{\omega_{|N}}(M' \otimes 1)(U^{\mathrm{ad}_{|N}}_{\omega_{|N}})^* = V_4^*(1_H \otimes M')V_4,$$

which evidently commutes with $\underline{\mathfrak{a}}(M) = V_4^*\Gamma(M)V_4$. On the other hand, if $z \in \widehat{M}$ and $x \in N$, then

$$\begin{aligned}V_4\Big(1 \ {}_{R_{|N}}\!\underset{\omega_{|N}}{\otimes}_{\mathrm{ad}_{|N}} U^{\mathrm{ad}_{|N}}_{\omega_{|N}}(z \otimes x^{\mathrm{o}})(U^{\mathrm{ad}_{|N}}_{\omega_{|N}})^*\Big)V_4^* &= \widehat{J} x^* \widehat{J} \otimes z \\ &= (\widehat{J} J \otimes 1)\sigma(z \otimes x^{\mathrm{o}})\sigma(J\widehat{J} \otimes 1)\end{aligned}$$

and hence

$$
\begin{aligned}
V_4\Big(1\ {}_{R_{|N}}\underset{\omega_{|N}}{\otimes}{}_{\mathrm{ad}_{|N}}\ U^{\mathrm{ad}_{|N}}_{\omega_{|N}}\ \mathrm{ad}^{\mathrm{o}}_{|N}(N^{\mathrm{o}})(U^{\mathrm{ad}_{|N}}_{\omega_{|N}})^*\Big)V_4^* &= (\widehat{J}J\otimes 1)\sigma\,\mathrm{ad}^{\mathrm{o}}_{|N}(N^{\mathrm{o}})\sigma(J\widehat{J}\otimes 1)\\
&= (\widehat{J}\otimes J)\sigma\,\mathrm{ad}_{|N}(N)\sigma(\widehat{J}\otimes J)\\
&= (\widehat{J}\otimes J)W(N\otimes 1_H)W^*(\widehat{J}\otimes J)\\
&= W^*(R(N)\otimes 1_H)W
\end{aligned}
$$

which commutes with $\Gamma(M) = W^*(1_H\otimes M)W$.

Therefore, $\underline{\mathfrak{a}}(y)$ commutes with $1\ {}_{R_{|N}}\underset{N}{\otimes}{}_{\mathrm{ad}_{|N}}\ (\widehat{\mathbb{G}}\ltimes_{\mathrm{ad}_{|N}} N)'$.

(iii) Using 2.5.4 applied to $(\widehat{\mathbb{G}}, \mathrm{ad}_{|N}, \Gamma_{|N})$, we get that $\widehat{\beta}(x) = U^{\mathrm{ad}_{|N}}_{\omega_{|N}}\alpha^{\mathrm{o}}(x^{\mathrm{o}})(U^{\mathrm{ad}_{|N}}_{\omega_{|N}})^*$, where we write $\alpha = \Gamma_{|N}$ and $\alpha^{\mathrm{o}}(x^{\mathrm{o}}) = (R\otimes .^{\mathrm{o}})\Gamma(x)\in M\otimes N^{\mathrm{o}}$ to avoid confusion with Γ^{o}. Then the commutation relations in (i) imply that

$$
V_4\Big(1_H\ {}_{R_{|N}}\underset{N}{\otimes}{}_{\mathrm{ad}_{|N}}\ \widehat{\beta}(x)\Big)V_4^* = V_4\Big(1_H\ {}_{R_{|N}}\underset{N}{\otimes}{}_{\mathrm{ad}_{|N}}\ U^{\mathrm{ad}_{|N}}_{\omega_{|N}}\alpha^{\mathrm{o}}(x^{\mathrm{o}})(U^{\mathrm{ad}_{|N}}_{\omega_{|N}})^*\Big)V_4^*
$$

is equal to $\varsigma(R\otimes R)(\Gamma(x)) = \Gamma(R(x)) = V_4\underline{\mathfrak{a}}(R(x))V_4^*$.

(iv) Let us first fix notation. We denote by

$$
\varsigma\circ\widetilde{\mathrm{ad}_{|N}}\colon \widehat{\mathbb{G}}\ltimes_{\mathrm{ad}_{|N}} N\to(\widehat{\mathbb{G}}\ltimes_{\mathrm{ad}_{|N}} N)\otimes M
$$

the dual action followed by the flip. Standard arguments show that there exists a unitary

$$
V_5\colon (H\otimes H_{\omega_{|N}})_{\widehat{\beta}}\underset{\omega_{|N}}{\otimes}{}_{\mathrm{ad}_{|N}}(H\otimes H_{\omega_{|N}})\to H\otimes H_{\omega_{|N}}\otimes H
$$

such that

$$
V_5\Big(\Xi_{\widehat{\beta}}\underset{\omega_{|N}}{\otimes}{}_{\mathrm{ad}_{|N}} U^{\mathrm{ad}_{|N}}_{\omega_{|N}}(\eta\otimes\Lambda_{\omega_{|N}}(x^*))\Big) = \widehat{\beta}(x)\Xi\otimes\eta
$$

for all $\Xi\in H\otimes H_{\omega_{|N}}$, $\eta\in H$, $x\in N$.

We need to prove commutativity of the following diagram,

$$
\begin{tikzcd}
M \arrow[r, "\Gamma"] \arrow[d, "\Gamma"'] \arrow[dr, phantom, "(1)"] & M\otimes M \arrow[r, "\mathrm{ad}_{V_4^*}"] \arrow[d, "\Gamma\otimes\mathrm{id}"] \arrow[dr, phantom, "(2)"] & M\ {}_R\underset{N}{*}{}_{\mathrm{ad}}(\widehat{\mathbb{G}}\ltimes N) \arrow[d, "\Gamma*\mathrm{id}"] \\
M\otimes M \arrow[r, "\mathrm{id}\otimes\Gamma"'] \arrow[d, "\mathrm{ad}_{V_4^*}"'] \arrow[dr, phantom, "(3)"] & M\otimes M\otimes M \arrow[r, "\mathrm{id}\otimes\mathrm{ad}_{V_4^*}"'] \arrow[d, "\mathrm{ad}_{V_4^*}\otimes\mathrm{id}"] \arrow[dr, phantom, "(4)"] & (M\otimes M)_{(\Gamma\circ R)}\underset{N}{*}{}_{\mathrm{ad}}(\widehat{\mathbb{G}}\ltimes N) \arrow[d, "\mathrm{ad}_{V_4^*}*\mathrm{id}"] \\
M\ {}_R\underset{N}{*}{}_{\mathrm{ad}}(\widehat{\mathbb{G}}\ltimes_{\mathrm{ad}} N) \arrow[r, "\mathrm{id}*\varsigma\widetilde{\mathrm{ad}}"'] & M\ {}_R\underset{N}{*}{}_{(\mathrm{ad}\otimes 1)}((\widehat{\mathbb{G}}\ltimes_{\mathrm{ad}} N)\otimes M) \arrow[r, "\mathrm{id}*\mathrm{ad}_{V_5^*}"'] & M\ {}_R\underset{N}{*}{}_{\mathrm{ad}}(\widehat{\mathbb{G}}\ltimes_{\mathrm{ad}} N)_{\widehat{\beta}}\underset{N}{*}{}_{\mathrm{ad}}(\widehat{\mathbb{G}}\ltimes N),
\end{tikzcd}
\tag{$*$}
$$

where we dropped the subscripts from R and ad.

Commutativity of cells (1) and (2) is evident or easy.

Let us show that cell (3) commutes. By definition,

$$(\varsigma \circ \widetilde{\mathrm{ad}_{|N}})(X) = \widehat{W}^{\mathrm{c}}_{13}(X \otimes 1)(\widehat{W}^{\mathrm{c}}_{13})^*$$

for all $X \in \widehat{\mathbb{G}} \ltimes_{\mathrm{ad}_{|N}} N$, where

$$\widehat{W}^{\mathrm{c}} = (\widehat{J} \otimes \widehat{J})\widehat{W}(\widehat{J} \otimes \widehat{J}) \in \widehat{M}' \otimes M,$$

and $\Gamma(x) = \widehat{W}^{\mathrm{c}}(x \otimes 1)\widehat{W}^{\mathrm{c}}$ for all $x \in M$. Therefore,

$$(\mathrm{ad}_{V_4^*} \otimes \mathrm{id})((\mathrm{id} \otimes \Gamma)(Y)) = \mathrm{ad}_{(V_4^* \otimes 1_H)(1_H \otimes \widehat{W}^{\mathrm{c}})}(Y \otimes 1), \tag{6}$$

$$(\mathrm{id} * \varsigma \circ \widetilde{\mathrm{ad}_{|N}})(\mathrm{ad}_{V_4^*}(Y)) = \mathrm{ad}_{(1_{R_{|N}} \underset{\omega_{|N}}{\otimes} {}_{(\mathrm{ad}_{|N} \otimes 1)} \widehat{W}^{\mathrm{c}}_{13})(V_4^* \otimes 1_H)}(Y \otimes 1) \tag{7}$$

for all $Y \in M \otimes M$. To prove that the two expressions coincide, it suffices to show that the following diagram $(**)$ commutes:

$$\begin{tikzcd}
H_{R_{|N}} \underset{\omega_{|N}}{\otimes} {}_{\mathrm{ad}_{|N}} (H \otimes H_{\omega_{|N}}) \otimes H \arrow[r, "1_{R_{|N}} \underset{\omega_{|N}}{\otimes} {}_{(\mathrm{ad}_{|N} \otimes 1)} \widehat{W}^{\mathrm{c}}_{13}"] \arrow[d, "V_4 \otimes 1_H"'] & H_{R_{|N}} \underset{\omega_{|N}}{\otimes} {}_{\mathrm{ad}_{|N}} (H \otimes H_{\omega_{|N}}) \otimes H \arrow[d, "V_4 \otimes 1_H"] \\
H \otimes H \otimes H \arrow[r, "\widehat{W}^{\mathrm{c}}_{23}"'] & H \otimes H \otimes H
\end{tikzcd} \tag{$**$}$$

But since the first legs of $U^{\mathrm{ad}_{|N}}_{\omega_{|N}} \in \widehat{M} \otimes B(H_{\omega_{|N}})$ and $\widehat{W}^{\mathrm{c}} \in (\widehat{M})' \otimes M$ commute,

$$\begin{aligned}
(V_4 \otimes 1_H)&(1_{R_{|N}} \underset{\omega_{|N}}{\otimes} {}_{(\mathrm{ad}_{|N} \otimes 1)} \widehat{W}^{\mathrm{c}}_{13})(\xi_{R_{|N}} \underset{\omega_{|N}}{\otimes} {}_{\mathrm{ad}_{|N}} U^{\mathrm{ad}_{|N}}_{\omega_{|N}}(\eta \otimes x^{\mathrm{o}}\Lambda_{\omega_{|N}}(1)) \otimes \vartheta) \\
&= (V_4 \otimes 1_H)(\xi_{R_{|N}} \underset{\omega_{|N}}{\otimes} {}_{(\mathrm{ad}_{|N} \otimes 1)} (U^{\mathrm{ad}_{|N}}_{\omega_{|N}})_{12} \widehat{W}^{\mathrm{c}}_{13}(\eta \otimes x^{\mathrm{o}}\Lambda_{\omega_{|N}}(1) \otimes \vartheta)) \\
&= R(x)\xi \otimes \widehat{W}^{\mathrm{c}}(\eta \otimes \vartheta) \\
&= \widehat{W}^{\mathrm{c}}_{23}(V_4 \otimes 1_H)(\xi_{R_{|N}} \underset{\omega_{|N}}{\otimes} {}_{\mathrm{ad}_{|N}} U^{\mathrm{ad}_{|N}}_{\omega_{|N}}(\eta \otimes x^{\mathrm{o}}\Lambda_{\omega_{|N}}(1)) \otimes \vartheta).
\end{aligned}$$

for all $\vartheta \in H$. Therefore, diagram $(**)$ commutes, the expressions (6) and (7) coincide, and cell (3) commutes.

To see that cell (4) commutes as well, consider the following diagram:

$$\begin{tikzcd}
H_{R_{|N}} \underset{\omega_{|N}}{\otimes} {}_{\mathrm{ad}_{|N}} (H \otimes H_{\omega_{|N}})_{\widehat{\beta}} \underset{\omega_{|N}}{\otimes} {}_{\mathrm{ad}_{|N}} (H \otimes H_{\omega_{|N}}) \arrow[r, "1 \otimes V_5"] \arrow[d, "V_4 \otimes 1"'] & H_{R_{|N}} \underset{\omega_{|N}}{\otimes} {}_{\mathrm{ad}_{|N}} (H \otimes H_{\omega_{|N}}) \otimes H \arrow[d, "V_4 \otimes 1"] \\
(H \otimes H)_{(\Gamma \circ R_{|N})} \underset{\omega_{|N}}{\otimes} {}_{\mathrm{ad}_{|N}} (H \otimes H_{\omega_{|N}}) \arrow[r, "1 \otimes V_4"'] & H \otimes H \otimes H
\end{tikzcd}$$

We show that this diagram commutes, and then cell (4) commutes as well. We first compute

$$(V_4 \otimes 1)(1 \otimes V_5)\Big(\xi\, {}_{R_{|N}} \underset{\omega_{|N}}{\otimes}{}_{\mathrm{ad}_{|N}} U^{\mathrm{ad}_{|N}}_{\omega_{|N}}(\eta \otimes x^{\mathrm{o}} \Lambda_{\omega_{|N}}(1))\, {}_{\widehat{\beta}} \underset{\omega_{|N}}{\otimes}{}_{\mathrm{ad}_{|N}} U^{\mathrm{ad}_{|N}}_{\omega_{|N}}(\vartheta \otimes y^{\mathrm{o}} \Lambda_{\omega_{|N}}(1))\Big).$$

We use (iii) and find that this vector is equal to

$$(V_4 \otimes 1)\Big(\xi\, {}_{R_{|N}} \underset{\omega_{|N}}{\otimes}{}_{\mathrm{ad}_{|N}}\, \widehat{\beta}(y) U^{\mathrm{ad}_{|N}}_{\omega_{|N}}(\eta \otimes x^{\mathrm{o}} \Lambda_{\omega_{|N}}(1)) \otimes \vartheta\Big)$$

and therefore

$$\begin{aligned}(\Gamma(R(y)) \otimes 1)(V_5 \otimes 1)\Big(\xi\, {}_{R_{|N}} \underset{\omega_{|N}}{\otimes}{}_{\mathrm{ad}_{|N}} U^{\mathrm{ad}_{|N}}_{\omega_{|N}}(\eta \otimes x^{\mathrm{o}} \Lambda_{\omega_{|N}}(1)) \otimes \vartheta\Big)& \\ = (\Gamma(R(y)) \otimes 1)(R(x)\xi \otimes \eta \otimes \vartheta).&\end{aligned}$$

On the other hand,

$$\begin{aligned}&(1 \otimes V_4)(V_4 \otimes 1)\\ &\quad\cdot \Big(\xi\, {}_{R_{|N}} \underset{\omega_{|N}}{\otimes}{}_{\mathrm{ad}_{|N}} U^{\mathrm{ad}_{|N}}_{\omega_{|N}}(\eta \otimes x^{\mathrm{o}} \Lambda_{\omega_{|N}}(1))\, {}_{\widehat{\beta}} \underset{\omega_{|N}}{\otimes}{}_{\mathrm{ad}_{|N}} U^{\mathrm{ad}_{|N}}_{\omega_{|N}}(\vartheta \otimes y^{\mathrm{o}} \Lambda_{\omega_{|N}}(1))\Big)\end{aligned}$$

is equal to

$$\begin{aligned}(1 \otimes V_4)\Big((R(x)\xi \otimes \eta)\, {}_{(\Gamma \circ R_{|N})} \underset{\omega_{|N}}{\otimes}{}_{\mathrm{ad}_{|N}} U^{\mathrm{ad}_{|N}}_{\omega_{|N}}(\vartheta \otimes y^{\mathrm{o}} \Lambda_{\omega}(1))\Big)& \\ = \Gamma(R(y))(R(x)\xi \otimes \eta) \otimes \vartheta&\end{aligned}$$

as well, which finishes the proof of (iv).

(v) Let $y \in M \cap R(N)'$ and assume $\underline{\mathfrak{a}}(y) = y\, {}_{R_{|N}} \underset{N}{\otimes}{}_{\mathrm{ad}_{|N}} 1$. Then by (i),

$$\Gamma(y) V_4 = V_4\Big(y\, {}_{R_{|N}} \underset{N}{\otimes}{}_{\mathrm{ad}_{|N}} 1\Big) = (y \otimes 1_H) V_4$$

and hence $\Gamma(y) = y \otimes 1_H$, whence y is a scalar and $\underline{\mathfrak{a}}$ is ergodic.

The canonical operator-valued weight $T_{\underline{\mathfrak{a}}}$ is equal to $(\mathrm{id}_{R_{|N}} \underset{N}{*}{}_{\mathrm{ad}_{|N}} \widehat{\Phi}) \circ \underline{\mathfrak{a}}$, where $\widehat{\Phi} = \omega \circ \mathrm{ad}^{-1} \circ T_{\widetilde{\mathrm{ad}_{|N}}}$, and $T_{\widetilde{\mathrm{ad}_{|N}}}$ is the left-invariant weight from $\widehat{\mathbb{G}} \ltimes_{\mathrm{ad}_{|N}} N$ to $\mathrm{ad}(N)$, i.e. the operator-valued weight arising from the dual action on $\widehat{\mathbb{G}} \ltimes_{\mathrm{ad}_{|N}} N$, that is, $(\omega \otimes \mathrm{id}) \circ \widetilde{\mathrm{ad}_{|N}}$. In fact, these operator-valued weights are conditional expectations.

We write $T_{\widetilde{\mathrm{ad}_{|N}}} = (\mathrm{id} \otimes \omega) \circ \varsigma \widetilde{\mathrm{ad}_{|N}}$ and use commutativity of the cells (1) and (3) in diagram $(*)$, and find that for any $x \in M^+$,

$$
\begin{aligned}
&(\mathrm{id}_{R_{|N}} \underset{\omega_{|N}}{*} {}_{\mathrm{ad}_{|N}} T_{\widetilde{\mathrm{ad}_{|N}}}) \circ \underline{\mathfrak{a}}(x) \\
&\qquad = (\mathrm{id}_{R_{|N}} \underset{\omega_{|N}}{*} {}_{\mathrm{ad}_{|N}} T_{\widetilde{\mathrm{ad}_{|N}}}) \circ \mathrm{ad}_{V_4^*} \circ \Gamma(x) \\
&\qquad = ((\mathrm{id}_{R_{|N}} \underset{\omega_{|N}}{*} {}_{\mathrm{ad}_{|N}} \mathrm{id}) \otimes \omega) \circ (\mathrm{id}_{R_{|N}} \underset{\omega_{|N}}{*} {}_{\mathrm{ad}_{|N}} \varsigma \widetilde{\mathrm{ad}_{|N}}) \circ \mathrm{ad}_{V_4^*} \circ \Gamma(x) \\
&\qquad = ((\mathrm{id}_{R_{|N}} \underset{\omega_{|N}}{*} {}_{\mathrm{ad}_{|N}} \mathrm{id}) \otimes \omega) \circ (\mathrm{ad}_{V_4^*} \otimes \mathrm{id}) \circ \Gamma^{(2)}(x) \\
&\qquad = \mathrm{ad}_{V_4^*} \circ (\mathrm{id} \otimes \mathrm{id} \otimes \omega) \circ \Gamma^{(2)}(x) \\
&\qquad = \mathrm{ad}_{V_4^*} \circ (1_{M \otimes M} \cdot \omega)(x) \\
&\qquad = 1_{(M \, {}_{R_{|N}} \underset{\omega_{|N}}{*} {}_{\mathrm{ad}_{|N}} \widehat{\mathbb{G}} \ltimes_{\mathrm{ad}_{|N}} N)} \cdot \omega(x),
\end{aligned}
$$

where $\Gamma^{(2)} = (\Gamma \otimes \mathrm{id}) \circ \Gamma$ and, for any von Neumann algebra P, $1_P \cdot \omega$ denotes the positive application $x \mapsto \omega(x) 1_P$. Therefore, we get (v).

As $\mathfrak{a}$ is integrable and ergodic, by [16, 3.8] or 8.1.2 , there exists an isometry G from $H \otimes H$ to $H_{R_{|N}} \underset{\omega_{|N}}{\otimes} {}_{\mathrm{ad}_{|N}} H_{\omega_{|N}}$ such that, for all $\zeta \in D(H_{R|N}, (\omega_{|N})^{\mathrm{o}})$, $x \in M$ and $e \in \widehat{\mathbb{G}} \ltimes_{\mathrm{ad}_{|N}} N$,

$$
\Big(1_{R_{|N}} \underset{N}{\otimes} {}_{\mathrm{ad}_{|N}} J_{\widehat{\Phi}} e J_{\widehat{\Phi}}\Big) G(x \Lambda_\omega(1) \otimes \zeta) = \underline{\mathfrak{a}}(x) \Big(\zeta_{R_{|N}} \underset{\omega_{|N}}{\otimes} {}_{\mathrm{ad}_{|N}} J_{\widehat{\Phi}} \Lambda_{\widehat{\Phi}}(e)\Big).
$$

Let $y^* \in M$ and let us take $e = y^* \otimes 1 \in \widehat{\mathbb{G}} \ltimes_{\mathrm{ad}_{|N}} N$. The relation $J_{\widehat{\Phi}} = U_{\omega_{|N}}^{\mathrm{ad}_{|N}} (J \otimes J_{\omega_{|N}})$ implies $J_{\widehat{\Phi}} e J_{\widehat{\Phi}} = U_{\omega_{|N}}^{\mathrm{ad}_{|N}} (y^{\mathrm{o}} \otimes 1) (U_{\omega_{|N}}^{\mathrm{ad}_{|N}})^*$ and

$$
\begin{aligned}
U_{\omega_{|N}}^{\mathrm{ad}_{|N}} (y^{\mathrm{o}} \Lambda_\omega(1) \otimes \Lambda_{\omega_{|N}}(1)) &= U_{\omega_{|N}}^{\mathrm{ad}_{|N}} (J y^* \Lambda_\omega(1) \otimes \Lambda_{\omega_{|N}}(1)) \\
&= U_{\omega_{|N}}^{\mathrm{ad}_{|N}} (J \otimes J_{\omega_{|N}}) \Lambda_{\widehat{\Phi}}(e) \\
&= J_{\widehat{\Phi}} \Lambda_{\widehat{\Phi}}(e).
\end{aligned}
$$

We then get that for all $\xi \in H$, $z \in M$, the vector

$$
(1 \, {}_{R_{|N}} \underset{N}{\otimes}{}_{\mathrm{ad}_{|N}} \, J_{\widehat{\Phi}} e J_{\widehat{\Phi}}) V_4^* (\xi \otimes z \Lambda_\omega(1))
$$

is equal to

$$
\begin{aligned}
&\Big(1_{R_{|N}} \underset{N}{\otimes} {}_{\mathrm{ad}_{|N}} U_{\omega_{|N}}^{\mathrm{ad}_{|N}} (y^{\mathrm{o}} \otimes 1) (U_{\omega_{|N}}^{\mathrm{ad}_{|N}})^*\Big) \Big(\xi_{R_{|N}} \underset{\omega_{|N}}{\otimes} {}_{\mathrm{ad}_{|N}} U_{\omega_{|N}}^{\mathrm{ad}_{|N}} (z \Lambda_\omega(1) \otimes \Lambda_{\omega_{|N}}(1))\Big) \\
&\qquad\qquad = \xi_{R_{|N}} \underset{\omega_{|N}}{\otimes} {}_{\mathrm{ad}_{|N}} U_{\omega_{|N}}^{\mathrm{ad}_{|N}} (y^{\mathrm{o}} z \Lambda_\omega(1) \otimes \Lambda_{\omega_{|N}}(1))) \\
&\qquad\qquad = V_4^* (\xi \otimes y^{\mathrm{o}} z \Lambda_\omega(1)).
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
\Big(1_{R_{|N}} \underset{N}{\otimes} {}_{\mathrm{ad}_{|N}} J_{\widehat{\Phi}} e J_{\widehat{\Phi}}\Big) V_4^* W^* \sigma(x\Lambda_\omega(1) \otimes \zeta) \\
&= V_4^*(1 \otimes y^{\mathrm{o}}) W^*(\zeta \otimes x\Lambda_\omega(1)) \\
&= V_4^*(1 \otimes y^{\mathrm{o}}) \Gamma(x)(\zeta \otimes \Lambda_\omega(1)) \\
&= V_4^* \Gamma(x)(\zeta \otimes y^{\mathrm{o}}\Lambda_\omega(1)) \\
&= \underline{\mathfrak{a}}(x) V_4^*(\zeta \otimes y^{\mathrm{o}}\Lambda_\omega(1)) \\
&= \underline{\mathfrak{a}}(x)\Big(\zeta_{R_{|N}} \underset{\omega_{|N}}{\otimes} {}_{\mathrm{ad}_{|N}} U^{\mathrm{ad}_{|N}}_{\omega_{|N}} (y^{\mathrm{o}}\Lambda_\omega(1) \otimes \Lambda_{\omega_{|N}}(1))\Big) \\
&= \underline{\mathfrak{a}}(x)\Big(\zeta_{R_{|N}} \underset{\omega_{|N}}{\otimes} {}_{\mathrm{ad}_{|N}} J_{\widehat{\Phi}} \Lambda_{\widehat{\Phi}}(e)\Big).
\end{aligned}
$$

Thus, we get that $(1_{R_{|N}} \underset{N}{\otimes} {}_{\mathrm{ad}_{|N}} J_{\widehat{\Phi}} e J_{\widehat{\Phi}}) V_4^* W^* \sigma = (1_{R_{|N}} \underset{N}{\otimes} {}_{\mathrm{ad}_{|N}} J_{\widehat{\Phi}} e J_{\widehat{\Phi}}) G$ for all $e = y^* \otimes 1$, and so $G = V_4^* W^* \sigma$. $\square$

8.3 Theorem. *Let $\mathbb{G} = (M, \Gamma, \omega, \omega)$ be a (von Neumann version of a) compact quantum group, $\mathbb{G}_1$ a compact quantum subgroup, and N the quotient type co-ideal. Then the von Neumann algebra M, equipped with the right Galois action $(R_{|N}, \underline{\mathfrak{a}})$ of $\widehat{\mathbb{G}} \ltimes_{\mathrm{ad}_{|N}} N$ constructed in 8.2 and the left Galois action Γ_l of $\mathbb{G}_1$ defined in 7.4, is an imprimitivity bimodule which is a Morita equivalence between the compact quantum group $\mathbb{G}_1$ and the measured quantum groupoid $\mathfrak{G}(N, \mathrm{ad}_{|N}, \Gamma_{|N}, \omega_{|N})$.*

Proof. Let $x \in M$. Commutativity of the cells (1) and (2) in diagram $(*)$ implies that

$$\Big(\Gamma \underset{N}{{}_{R_{|N}} *_{\mathrm{ad}_{|N}}} \mathrm{id}\Big)\underline{\mathfrak{a}}(x) = (\mathrm{id} \otimes \underline{\mathfrak{a}})\Gamma(x)$$

and applying $(\pi \otimes \mathrm{id}) \underset{N}{{}_{R_{|N}} *_{\mathrm{ad}_{|N}}} \mathrm{id}$ to this relation, we get:

$$\Big(\Gamma_l \underset{N}{{}_{R_{|N}} *_{\mathrm{ad}_{|N}}} \mathrm{id}\Big)\underline{\mathfrak{a}}(x) = (\mathrm{id} \otimes \underline{\mathfrak{a}})\Gamma_l(x),$$

which is the commutativity of the right Galois action $(R_{|N}, \underline{\mathfrak{a}})$ of $\widehat{\mathbb{G}} \ltimes_{\mathrm{ad}_{|N}} N$ and the left Galois action Γ_l of $\mathbb{G}_1$.

Moreover, we had got in 8.2 that the canonical operator-valued weight $T_{\underline{\mathfrak{a}}}$ was the Haar state ω. Let ω_1 be the Haar state of $\mathbb{G}_1$. Then the canonical operator-valued weight T_{Γ_l} is equal to $(\omega_1 \circ \pi \otimes \mathrm{id})\Gamma$, which is, in fact, a conditional expectation from M into $M^{\Gamma_l} = R(N)$. Composed with the state $\omega_{|N} \circ R = \omega_{|R(N)}$, we get $(\omega_1 \circ \pi \otimes \omega)\Gamma = \omega_1(\pi(1))\omega = \omega$. Therefore, using 8.1.4, we get the result. $\square$

8.4 Corollary. *The measured quantum groupoid $\widehat{\mathrm{SU}_q(2)} \ltimes_{\mathrm{ad}_{|S_q^2}} S_q^2$ constructed in 7.4.5 is Morita equivalent to $\mathbb{T}$.*

Proof. Apply 8.3 to 7.4.5. □

8.5 Corollary ([33]). *Let G be a compact group and G_1 a compact subgroup of G. The the right action of G on G/G_1 defines a transformation groupoid $(G/G_1) \curvearrowleft G$ and this groupoid is Morita equivalent to G_1.*

Proof. The canonical surjective $*$-homomorphism from $L^\infty(G)$ onto $L^\infty(G_1)$ gives to $L^\infty(G/G_1)$ a structure of a quotient type co-ideal. The restriction of the coproduct $\Gamma_{L^\infty(G)}$ to $L^\infty(G/G_1)$ is just the right action of G on G/G_1, and the measured quantum groupoid $G \ltimes_\Gamma L^\infty(G/G_1)$ is the dual of the groupoid $(G/G_1) \curvearrowleft G$. Therefore, by 7.1, its dual is just the abelian von Neumann algebra $L^\infty((G/G_1) \curvearrowleft G)$, and, by 8.3, we get the result. □

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