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Derivatives in noncommutative calculus and deformation property of quantum algebras

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Abstract. The aim of the paper is twofold. First, we introduce analogs of (partial) derivatives on certain noncommutative algebras, including some enveloping algebras and their "braided counterparts" — the so-called modified Reflection Equation algebras. With the use of the mentioned derivatives we construct an analog of the de Rham complex on these algebras. Second, we discuss deformation property of some quantum algebras and show that contrary to a commonly held view, in the so-called q-Witt algebra there is no analog of the PBW property. In this connection, we discuss different forms of the Jacobi condition related to quadratic-linear algebras.

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1. Introduction

In our recent publications [7, 10] we introduced the notion of partial derivatives on some noncommutative (NC) algebras, in particular, on the enveloping algebras of the Lie algebras $gl(m)$ and on their super and *braided* (see below) analogs. These partial derivatives differ from their classical counterparts by the form of the Leibniz rule.

In this connection a natural question arises: given a NC algebra A, which operators acting on this algebra can be considered as appropriate analogs of partial derivatives? This question is pertinent if A is a deformation (quantization) of the symmetric algebra $\mathrm{Sym}(V)$ of a vector space V or its super or braided analog. In this paper we give an answer to this question for the enveloping algebras of some Lie algebras.

Note that the answer depends on a given Lie algebra. Nevertheless, once such partial derivatives are introduced, we are able to define an analog of the de Rham complex on the corresponding enveloping algebra. Compared with all known

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approaches to the problem of defining such a complex, our method leads to objects possessing a good deformation property[1]. However, the terms of our complex are endowed with a *one-sided* A-module structure, whereas using the classical Leibniz rule for the de Rham operator requires a two-sided A-module structure. Also, we define the notion of the Weyl algebra $\mathcal{W}(U(\mathfrak{g}))$ generated by the enveloping algebra of a given Lie algebra and the corresponding partial derivatives and give some example of these Weyl algebras.

Besides, we generalize all considered objects (partial derivatives, Weyl algebra, de Rham complex) to the Reflection Equation algebra and its modified version. This algebra and all related objects are called *braided* since they arise from braidings (see Section 4). For a more precise meaning of this term the reader is referred to [10]. The explicit construction of the braided objects mentioned above is one of the purposes of the present paper. It is worth noting, that certain deformations of the usual derivative are known since long ago, for instance, the q-derivative (also called the Jackson derivative) defined by

$$\partial_q(f(t)) = \frac{f(qt) - f(t)}{t(q-1)} \tag{1.1}$$

and the difference operator

$$\partial_\hbar(f(t)) = \frac{f(t+\hbar) - f(t)}{\hbar} \tag{1.2}$$

(called below the $\hbar$-derivative), or their slight modifications. It is tempting to use them in order to introduce analogs of algebras whose construction is based on the usual derivative. The most known examples are the q-Witt and q-Virasoro algebras.

The other purpose of the paper is to study the deformation property of the enveloping algebra of the q-Witt algebra. In Section 5 we show that contrary to the claim of [11], the PBW property fails for this enveloping algebra. The same is true for the enveloping algebra of the $\hbar$-Witt algebra, constructed with the use of the $\hbar$-derivative instead of the usual one. Our reasoning is based on the paper [16] where a version of the Jacobi condition useful for dealing with quadratic algebras and their quadratic-linear deformations is exhibited[2]. This condition is necessary for the PBW property and since it is not satisfied for the aforementioned algebras, we arrive to our conclusion.

In this connection we discuss other forms of the Jacobi condition which are useful for generalizing some other objects and operators associated with Lie algebras,

[1]For finitely generated quadratic-linear(-constant) algebras (in particular, enveloping ones) we deal with, this property means that an analog of the PBW theorem is valid for them and homogeneous components of the corresponding quadratic algebra have stable dimensions (at least for a generic value of the deformation parameter). Note that nowadays the term "PBW property" is often used as a synonym of our "good deformation property". We prefer to reserve this term for a deformation of quadratic algebras by linear(-constant) terms.

[2]A version of this construction covering quadratic-linear-constant deformations of quadratic algebras was considered in [1].

namely, the Chevalley–Eilenberg complex and the adjoint representation. It is worth noting, that in general these forms of the Jacobi condition are not equivalent and each of them plays its own role in the theory of quadratic(-linear) algebras.

The paper is organized as follows. In the next section we compare different ways of associating a differential algebra with the enveloping algebra of $gl(m)$. In Section 3 we discuss a generalization of this construction onto the enveloping algebras of some other Lie algebras. In Section 4 we extend our construction to the Reflection Equation algebra. Here, the central problem consists in a convenient definition of the algebra generated by the differentials of the generators of the initial algebra such that the corresponding de Rham operator d meets the usual property $d^2 = 0$. In Section 5 we consider the aforementioned versions of the Witt algebra and show that the PBW property fails in their enveloping algebras. We complete the paper (Section 6) with a discussion on different forms of the Jacobi condition related to different generalizations of the Lie algebra notion.

2. Partial derivatives on $U(gl(m))$: different approaches

In what follows we deal with different deformations of the symmetric algebra $\mathrm{Sym}(\mathfrak{g})$, where $\mathfrak{g}$ is a Lie algebra. Our main example is $\mathfrak{g} = gl(m)_\hbar$, where the subscribe $\hbar$ means that the parameter $\hbar$ is introduced as a multiplier in the $gl(m)$ Lie bracket. As usual, we fix a basis $\{n_i^j\}$, $1 \le i, j \le m$, in $gl(m)$. The Lie brackets in this basis read

$$[n_i^j, n_k^l] = \hbar(n_i^l \delta_k^j - n_k^j \delta_i^j), \qquad 1 \le i, j, k, l, \le m.$$

In each homogenous component of the algebra $\mathrm{Sym}(gl(m))$ we fix a basis consisting of symmetric elements, i.e. those invariant with respect to the action of the symmetric group. Denote $\{e_\beta\}$ the corresponding basis of the whole algebra $\mathrm{Sym}(gl(m))$. Any element e_β is a polynomial in the generators of the algebra $\mathrm{Sym}(gl(m))$. A similar basis in the filtered quadratic-linear algebra $U(gl(m)_\hbar)$ will be denoted $\{\hat{e}_\beta\}$. The element $\hat{e}_\beta$ can be obtained by replacing the generators of $\mathrm{Sym}(gl(m))$ in the polynomial e_β by the corresponding generators of the algebra $U(gl(m)_\hbar)$.

Now, consider a linear map

$$\alpha : \mathrm{Sym}(gl(m)) \to U(gl(m)_\hbar)$$

defined on the above bases as follows

$$\alpha(e_\beta) = \hat{e}_\beta.$$

This map is the central ingredient of the Weyl quantization method.

Using this map we can push forward any operator

$$\mathcal{Q} : \mathrm{Sym}(gl(m)) \to \mathrm{Sym}(gl(m))$$

to that

$$\mathcal{Q}_\alpha : U(gl(m)_\hbar) \to U(gl(m)_\hbar)$$

as follows

$$\mathcal{Q}_\alpha = \alpha \circ \mathcal{Q} \circ \alpha^{-1}.$$

In particular, we can push forward partial derivatives from the algebra $\mathrm{Sym}(gl(m))$ to that $U(gl(m)_\hbar)$ and consider them as an appropriate noncommutative analog of the usual partial derivatives. And visa versa, any operator defined in the algebra $U(gl(m)_\hbar)$ can be pulled back to $\mathrm{Sym}(gl(m))$. For instance, the product in the algebra $U(gl(m)_\hbar)$ being pulled back to the algebra $\mathrm{Sym}(gl(m))$ is called $\star$-product (induced from $U(gl(m)_\hbar)$). This product is often used in a quantization of dynamical models (see [12]). In such models (for example, the Schrödinger one) the kinetic part composed of momenta is classical but the usual product of coordinate functions is replaced by the $\star$-product.

Equivalently, these models can be treated in terms of the algebra $U(gl(m)_\hbar)$ but then the partial derivatives (momenta) of the kinetic part should be replaced by their images with respect to the map α. Emphasize that these images are not subject to any form of the Leibniz rule. However, if such a "derivative" acts on a totally symmetrized element, the usual Leibniz rule can be applied.

Note that other methods of defining the map α (for instance, the Wick one) also can be used. However, the way exhibited above (in fact, the Weyl quantization method) gives rise to a $GL(m)$-covariant map.

By contrast to this method of defining analogs of partial derivatives on $U(gl(m)_\hbar)$ ours consists in modifying the Leibnitz rule. The modified Leibniz rule can be realized via the coproduct defined on the partial derivatives as follows[3]

$$\Delta(\partial_i^j) = \partial_i^j \otimes 1 + 1 \otimes \partial_i^j + \hbar \sum_k \partial_k^j \otimes \partial_i^k. \tag{2.1}$$

Hereafter, we use the notation $\partial_i^j = \partial_{n^i_j}$ for the partial derivative in the element n^i_j. Thus, we set by definition that

$$\partial_i^j(n_k^l) = \delta_i^l \delta_k^j, \tag{2.2}$$

i.e. this action is nothing but the pairing of the dual bases $\{n_i^j\}$ and $\{\partial_i^j\}$. Besides, we naturally assume the derivatives to be linear operators killing elements of the ground field $\mathbb{K}$ which is assumed to be $\mathbb{C}$ or $\mathbb{R}$ depending on the context. Then, using the coproduct (2.1), one can extend the action of the derivatives on polynomials in the generators.

An equivalent form of the Leibniz rule, exhibited in [10], consists in the following. Consider an associative product $n_i^j \circ n_k^l = \delta_k^j n_i^l$ in the Lie algebra $gl(m)$. Note that

[3]This form of the Leibniz rule was found by S.Meljanac and Z.Škoda.

$[n_i^j, n_k^l] = n_i^j \circ n_k^l - n_k^l \circ n_i^j$. Then, in addition to (2.2), the action of a derivative on a quadratic monomial in generators is defined as follows:

$$\partial_i^j (n_a^b\, n_c^d) = \partial_i^j (n_a^b)\, n_c^d + n_a^b\, \partial_i^j (n_c^d) + \hbar\, \partial_i^j (n_a^b \circ n_c^d).$$

In general, the action of a derivative ∂_i^j on a p-th order monomial $n_{i_1}^{j_1} \dots n_{i_p}^{j_p}$ gives rise to a sum of monomials whose order varies from zero (a constant term) to $p-1$. In this sum the $(p-k)$-th order component $(1 \le k \le p)$ is composed from all monomials, each of them being obtained by the pairing of ∂_i^j and the $\circ$-product of a subset of k elements from the initial monomial. Besides, the sum of all such $(p-k)$-th order monomials has a multiplier $\hbar^{k-1}$. We illustrate this rule by an example of a third order monomial:

$$\begin{aligned}\partial_i^j (n_a^b\, n_c^d\, n_k^l) = {} & \partial_i^j (n_a^b)\, n_c^d\, n_k^l + n_a^b\, \partial_i^j (n_c^d)\, n_k^l + n_a^b\, n_c^d\, \partial_i^j (n_k^l) \\ & + \hbar\big(\partial_i^j (n_a^b \circ n_c^d)\, n_k^l + \partial_i^j (n_a^b \circ n_k^l)\, n_c^d + n_a^b\, \partial_i^j (n_c^d \circ n_k^l)\big) \\ & + \hbar^2 \partial_i^j (n_a^b \circ n_c^d \circ n_k^l).\end{aligned}$$

Note that the partial derivatives commute with each other. Denote $\mathcal{D}$ the unital algebra generated by the partial derivatives. It becomes a bi-algebra being equipped with the coproduct defined on the generators by formula (2.1) and the counit $\varepsilon : \mathcal{D} \to \mathbb{K}$ defined in the usual way: it kills all generators ∂_i^j and maps $1_{\mathcal{D}}$ (the unit of $\mathcal{D}$) into the unit of the field.

The above coproduct allows one to introduce the so-called *permutation relations* between the partial derivatives and elements of the algebra $\mathcal{U} = U(gl(m)_\hbar)$ by the following rule

$$\partial_i^j \otimes n_k^l = (\partial_i^j)_1 \triangleright n_k^l \otimes (\partial_i^j)_2, \quad \text{where} \quad \Delta(\partial_i^j) = (\partial_i^j)_1 \otimes (\partial_i^j)_2$$

in Sweedler's notation. Also, the notation $\triangleright$ stands for the action of an operator on an element. Explicitly these permutation relations read:

$$\partial_i^j \otimes n_k^l - n_k^l \otimes \partial_i^j = \delta_i^l \delta_k^j\, 1_{\mathcal{U}} \otimes 1_{\mathcal{D}} + \hbar\, 1_{\mathcal{U}} \otimes (\partial_i^l \delta_k^j - \partial_k^j \delta_i^l).$$

These permutation relations can be presented in a matrix form as follows

$$D_1\, P\, N_1\, P - P\, N_1\, P\, D_1 = P + \hbar(D_1\, P - P\, D_1). \tag{2.3}$$

Here $D = \|\partial_i^j\|$ and $N = \|n_i^j\|$ are the matrices composed of the elements ∂_i^j and n_i^j respectively (the low index labels the lines) and $A_1 = A \otimes I$ for any matrix A, I being the unit matrix. Also, P stands for the matrix of the usual flip. Besides, we omit the factors $1_{\mathcal{U}}$, $1_{\mathcal{D}}$ and the sign $\otimes$.

The algebra generated by two subalgebras $U(gl(m)_\hbar)$ and $\mathcal{D}$, equipped with the permutation relations (2.3), is called the *Weyl algebra* and is denoted $\mathcal{W}(U(gl(m)_\hbar)$.

Note that for $\hbar = 0$ we get the usual Weyl algebra generated by $\mathrm{Sym}(gl(m))$ and the usual partial derivatives in the generators[4].

The above permutation relations have been obtained via a passage to a limit $q \to 1$ in the permutation relations for the modified Reflection Equation algebra under assumption that the Hecke symmetry is a deformation of the usual flip. In general, the permutation relations themselves can be used for introducing partial derivatives. In order to define the action of a derivative ∂_i^j on an element $a \in U(gl(m)_\hbar)$ one proceeds as follows. One permutes the factors in the product $\partial_i^j \otimes a$ by means of the permutation relations and applies the counit to the right factor of the final element belonging to the tensor product $U(gl(m)_\hbar) \otimes \mathcal{D}$. This counit to be usual: it kills all monomials in the generators apart from constant ones and maps $1_{U(gl(m)_\hbar)}$ to $1_{\mathbb{K}}$.

Concluding this section, we resume that there are three ways of defining our partial derivatives on the algebra $U(gl(m)_\hbar)$. One of them is based on using the coproduct (2.1), another one uses the product $\circ$ in the algebra $U(gl(m)_\hbar)$. The third way is based on the related permutation relations. Similar ways also exist on the enveloping algebras of the Lie super-algebras $gl(m|n)_\hbar$ and their "braided" analogs related to involutive braidings (see [10]). In Section 4 we consider similar algebras related to non-involutive (namely, Hecke type) braidings. For them the only way based on permutation relations is known.

3. Partial derivatives on other enveloping algebras

The methods of defining partial derivatives on the algebra $U(gl(m)_\hbar)$ should be modified for other enveloping algebras. Consider some examples.

Let $\mathfrak{g}_h$ be a Lie subalgebra of the Lie algebra $gl(m)_\hbar$. In general, the above method of defining the partial derivatives via the product $\circ$ fails since this product is not well-defined in the subalgebra $\mathfrak{g}_h$. As for the coproduct (2.1) or the permutation relations (2.3), in general they cannot be restricted to the algebra $U(\mathfrak{g}_h)$ (see example below).

Nevertheless, the partial derivatives in elements of $\mathfrak{g}_h$ are well defined as operators. To show this, we fix a complementary subspace W to $\mathfrak{g}_h$ that is $gl(m)_\hbar = \mathfrak{g}_h \oplus W$ as vector spaces. Then we chose a basis in the space $gl(m)_\hbar$ subordinated to the direct sum $\mathfrak{g}_h \oplus W$ (i.e. composed from some bases of $\mathfrak{g}_h$ and W). In the space generated by the partial derivatives we pass to the dual basis.

Let x_i be an element of the chosen basis of $\mathfrak{g}_h$. Then, on applying the derivative ∂^{x_i} to a monomial composed of elements from $\mathfrak{g}_h$, we get a polynomial also possessing this property though in the coproduct (2.1) presented in the new basis of $gl(m)$ certain external derivatives (i.e. derivatives in elements from W) enter.

Now, consider some subalgebras of the Lie algebra $gl(2)_\hbar$ or more precisely, of

[4]Physicists prefer to call this algebra the Heisenberg one.

its compact form $u(2)_\hbar$. On fixing in the latter algebra the standard basis $\{t, x, y, z\}$ such that

$$[x, y] = \hbar z, \quad [y, z] = \hbar x, \quad [z, x] = \hbar y, \quad [t, x] = [t, y] = [t, z] = 0,$$

and taking in the dual space the basis $\partial^t, \partial^x, \partial^y, \partial^z$, we get the following permutation relations

$$\partial^t\, t - t\, \partial^t = 1 + \frac{\hbar}{2}\, \partial^t \qquad \partial^t\, x - x\, \partial^t = -\frac{\hbar}{2}\, \partial^x \qquad \partial^t\, y - y\, \partial^t = -\frac{\hbar}{2}\, \partial^y$$
$$\partial^t\, z - z\, \partial^t = -\frac{\hbar}{2}\, \partial^z$$

$$\partial^x\, t - t\, \partial^x = \frac{\hbar}{2}\, \partial^x \qquad \partial^x\, x - x\, \partial^x = 1 + \frac{\hbar}{2}\, \partial^t \quad \partial^x\, y - y\, \partial^x = \frac{\hbar}{2}\, \partial^z$$
$$\partial^x\, z - z\, \partial^x = -\frac{\hbar}{2}\, \partial^y$$

$$\partial^y\, t - t\, \partial^y = \frac{\hbar}{2}\, \partial^y \qquad \partial^y\, x - x\, \partial^y = -\frac{\hbar}{2}\, \partial^z \qquad \partial^y\, y - y\, \partial^y = 1 + \frac{\hbar}{2}\, \partial^t$$
$$\partial^y\, z - z\, \partial^y = \frac{\hbar}{2}\, \partial^x$$

$$\partial^z\, t - t\, \partial^z = \frac{\hbar}{2}\, \partial^z \qquad \partial^z\, x - x\, \partial^z = \frac{\hbar}{2}\, \partial^y \qquad \partial^z\, y - y\, \partial^z = -\frac{\hbar}{2}\, \partial^x$$
$$\partial^z\, z - z\, \partial^z = 1 + \frac{\hbar}{2}\, \partial^t.$$

First, consider the Lie subalgebra $su(2)_\hbar \subset u(2)_\hbar$. As follows from [7] for any polynomial of the form $f(x, y, z) = f_1(x)\, f_2(y)\, f_3(z)$ the action of the partial derivative ∂^x is defined by

$$\partial^x(f) = 2\hbar^{-1}\, (B(f_1)\, A(f_2)\, A(f_3) + A(f_1)\, B(f_2)\, B(f_3)),$$

where

$$A(f(v)) = \frac{1}{2}\Big(f(v - i\,\hbar/2) + f(v + i\,\hbar/2)\Big),$$
$$B(f(v)) = \frac{i}{2}\Big(f(v - i\,\hbar/2) - f(v + i\,\hbar/2)\Big).$$

Note that though the quantity $i = \sqrt{-1}$ enters these formulae, the result is real provided f has real coefficients and $\hbar \in \mathbb{R}$. Similar formulae are valid for the derivatives ∂^y, ∂^z. Consequently, we have

$$\partial^x(f),\ \partial^y(f),\ \partial^z(f) \in U(su(2)_\hbar).$$

Definition 3.1. Let $\mathfrak{g}$ be a Lie algebra and $U(\mathfrak{g})$ its enveloping algebra. Choose a basis $\{x_i\}$, $1 \le i \le m$ in $\mathfrak{g}$, and denote by $c^k_{i,j}$ the structure constants of $\mathfrak{g}$ in this basis. Introduce an algebra $\mathcal{W}(U(\mathfrak{g}))$ generated by $U(\mathfrak{g})$ and a commutative algebra $\mathcal{D}$ with generators ∂^l, $1 \le l \le m$, subject to the permutation relations

$$[\partial^i, x_j] = -[x_j, \partial^i] = b^i_{j,k}\,\partial^k + \delta^i_j. \tag{3.1}$$

We call the algebra $\mathcal{W}(U(\mathfrak{g}))$ the Weyl algebra if the Jacobi identity is valid for the bracket

$$[\,,\,] : \wedge^2(W) \to W \oplus \mathbb{K}, \quad W = \mathfrak{g} \oplus \operatorname{span}(\partial^i),$$

defined by the initial Lie bracket on $\mathfrak{g}$, by the trivial bracket on $\mathcal{D}$ and by the bracket (3.1) on $\operatorname{span}(\partial^i) \otimes \mathfrak{g}$ and on $\mathfrak{g} \otimes \operatorname{span}(\partial^i)$.

Note that the Jacobi identity must be adapted to the case when the image of the bracket belongs to $W \oplus \mathbb{K}$. In fact, we have only to satisfy the relations

$$[\partial^p, [x_i, x_j]] = [[\partial^p, x_i], x_j] - [[\partial^p, x_j], x_i],$$

or in terms of the structure constants

$$c^k_{i,j} b^p_{k,l} = b^p_{i,k} b^k_{j,l} - b^p_{j,k} b^k_{i,l}, \quad c^p_{i,j} = b^p_{i,j} - b^p_{j,i}.$$

Note that in virtue of the PBW theorem the graded algebra $\operatorname{Gr}\mathcal{W}(U(\mathfrak{g}))$ is canonically isomorphic to the commutative algebra generated by the elements x_i and ∂^j.

It is straightforward checking that the algebra $\mathcal{W}(U(u(2)_\hbar))$ is a Weyl algebra in the sense of the above definition. However, an attempt to define a similar Weyl algebra for the enveloping algebra $U(su(2)_\hbar)$ as a quotient of $\mathcal{W}(U(u(2)_\hbar))$ fails. Indeed, it suffices to check that the relation

$$[\partial^x, [x, y]] = [[\partial^x, x], y] - [[\partial^x, y], x]$$

fails, if we assume that $\partial^t = 0$. This example shows that in order to define the partial derivatives on an enveloping algebra we have, in general, to consider the Weyl algebra related to a larger Lie algebra.

It is not the case for the subalgebra $\mathfrak{g} \subset u(2)_\hbar$ generated by the elements t and x. This Lie algebra is commutative. So, its enveloping algebra coincides with $\operatorname{Sym}(\mathfrak{g})$ and the corresponding partial derivatives and the Weyl algebra can be defined in the classical way. However, considering the subalgebra of $\mathcal{W}(U(u(2)_\hbar))$ generated by the elements $t, x, \partial^t, \partial^x$ we get another Weyl algebra corresponding to the same Lie algebra $\mathfrak{g}$. Thus, we get two different Weyl algebras related to the algebra $\mathfrak{g}$.

In general, for the two-dimensional commutative Lie algebra the permutation relations must be of the form

$$\begin{aligned} \partial^t t - t\partial^t &= 1 + (a_1\partial^t + b_1\partial^x), & \partial^t x - x\partial^t &= (a_2\partial^t + b_2\partial^x), \\ \partial^x t - t\partial^x &= (c_1\partial^t + d_1\partial^x), & \partial^x x - x\partial^x &= 1 + (c_2\partial^t + d_2\partial^x). \end{aligned}$$

It would be interesting to classify all possible families of constants $a_1, \ldots, d_2$ giving rise to the Weyl algebras on the two-dimensional commutative Lie algebra (as well as on two-dimensional noncommutative one). Two examples above correspond to the following families of these constants. In the classical case all constants are trivial. In the other one nontrivial constants are: $a_1 = d_1 = -b_2 = c_2 = \frac{\hbar}{2}$.

We get a little bit more general Weyl algebra by putting

$$b_1 = c_1 = a_2 = 0, \quad a_1 = d_1 = c_2$$

(here we do not impose any restriction on b_2).

In conclusion, we want to mention the following fact. Though the way of introducing the partial derivatives on an enveloping algebra via permutation relations is not universal, it is apparently more general than that based on the coproduct defined on the algebra $\mathcal{D}$. This observation is also valid for braided algebras considered in the next section. Thus, we have not succeeded in finding a coproduct corresponding to the permutation relations (4.7).

4. Braided Weyl algebras and related de Rham complex

In this section we consider a *braided analog* of the enveloping algebra $U(gl(m)_\hbar)$ and differential calculus on it. By braided analog we mean the so-called Reflection Equation (RE) algebra in its modified form. Let us recall the definition of this algebra.

Let V be a vector space over the ground field $\mathbb{K}$ and $R : V^{\otimes 2} \to V^{\otimes 2}$ be a linear invertible operator satisfying the braid relation

$$R_{12}\, R_{23}\, R_{12} = R_{23}\, R_{12}\, R_{23}$$

where $R_{12} = R \otimes I$, $R_{23} = I \otimes R$ and I is the identity operator (in an equivalent form it is also called the *quantum Yang–Baxter equation*). Such an operator R is called a *braiding*. If a braiding R is subject to an additional condition

$$(R - q\, I)(R + q^{-1}\, I) = 0, \quad q \in \mathbb{K},$$

it is called a *Hecke symmetry* provided $q \neq 1$ and an *involutive symmetry* provided $q = 1$.

By modified Reflection Equation algebra we mean a unital algebra generated by elements n_i^j, $1 \leq i, j \leq m$, subject to the system of relations

$$R\, N_1\, R\, N_1 - N_1\, R\, N_1\, R = \hbar\, (R\, N_1 - N_1\, R), \quad \hbar \in \mathbb{K} \tag{4.1}$$

where $N = \|n_i^j\|$ and $N_1 = N \otimes I$. We omit the term "modified" if $\hbar = 0$. The algebra (4.1) will be denoted $\mathcal{N}(q, \hbar)$ provided $\hbar \neq 0$ or $\mathcal{N}(q)$ provided $\hbar = 0$.

Below we assume R to be a *skew-invertible* Hecke symmetry. This means that there exists an operator $\Psi : V^{\otimes 2} \to V^{\otimes 2}$ such that

$$\mathrm{Tr}_2 R_{12}\Psi_{23} = \mathrm{Tr}_2 \Psi_{12} R_{23} = P_{13}, \tag{4.2}$$

where Tr stands for the usual trace, and indices label the spaces where the operators act. In what follows we shall need the operators

$$B = \mathrm{Tr}_1 \Psi_{12}, \qquad C = \mathrm{Tr}_2 \Psi_{12}. \tag{4.3}$$

As a direct consequence of the definition of the operator Ψ we have

$$\mathrm{Tr}_1 B_1 R_{12} = I, \qquad \mathrm{Tr}_2 C_2 R_{12} = I. \tag{4.4}$$

The operator B is supposed to be invertible. Then, as was shown in [6],

$$B \cdot C = q^{-2m} I, \quad \mathrm{Tr} B = \mathrm{Tr} C = q^{-m} \frac{q^m - q^{-m}}{q - q^{-1}}. \tag{4.5}$$

Note, that the algebra (4.1) is filtered. We call it the quadratic-linear one since it is defined by the quadratic-linear relations. We treat this algebra as a *braided analog* of the enveloping algebra $U(gl(m)_\hbar)$. Indeed, it is possible to define a *braided Lie bracket* such that the modified RE algebra has the sense of the enveloping algebra of the corresponding braided Lie algebra (see [8]). Furthermore, the algebra $\mathcal{N}(q, \hbar)$ has the following properties (see [6]):

1. If R comes from the quantum group (QG) $U_q(sl(m))$, the algebra $\mathcal{N}(q, \hbar)$ has a good deformation property. This means that the homogeneous components $\mathcal{N}^k(q)$, $k = 0, 1, 2, \ldots$, of the algebra $\mathcal{N}(q)$ have the classical dimensions, i.e.

$$\dim \mathcal{N}^k(q) = \dim U^k(gl(m)_\hbar)$$

for any k and a generic q. Also, for the algebra $\mathcal{N}(q, \hbar)$ there is a sort of the PBW theorem ensuring that the associated graded algebra $\mathrm{Gr}\,(\mathcal{N}(q, \hbar))$ is isomorphic to $\mathcal{N}(q)$ (for a discussion on the PBW property see the next section).

2. This algebra can be equipped with a braided bi-algebra structure (see [7] for a definition). This structure is determined by the usual counit and the coproduct such that for $\hbar = 1$ it has the form

$$\Delta(n_i^j) = n_i^j \otimes 1 + 1 \otimes n_i^j - (q - q^{-1}) \sum_k n_i^k \otimes n_k^j.$$

However, similarly to super-algebras, the product of two such elements $\Delta(n_i^j)\Delta(n_k^l)$ involves an operator transposing two middle factors in the product. This operator results from the initial Hecke symmetry.

3. The category of finite dimensional representations of this algebra is similar to that of $U(gl(m))$ or $U(gl(m|n))$ (depending on R), an analog of the adjoint representation included.

4. The structure of the center of $\mathcal{N}(q,\hbar)$ is similar to that of $U(gl(m))$ (or $U(gl(m|n))$).

5. If R comes from $U_q(sl(m))$, the algebra $\mathcal{N}(q,\hbar)$ is covariant with respect to the action of this QG. In general, a similar property can be formulated via a coaction of the RTT algebra on the algebra $\mathcal{N}(q,\hbar)$.

Note that if R is an involutive symmetry, the algebra $\mathcal{N}(q,\hbar)$ turns into the enveloping algebra of a generalized Lie algebra, introduced by one of the authors in the 80's.

The point is that a braided analog of the partial derivatives can be introduced on the algebra $\mathcal{N}(q,\hbar)$ (first it was done in [7]). This enable us to define a braided analog of the Weyl algebra corresponding to the algebra $\mathcal{N}(q,\hbar)$.

Definition 4.1. *The braided Weyl algebra* $\mathcal{W}(\mathcal{N}(q,\hbar))$ is an associative unital algebra generated by two subalgebras $\mathcal{N}(q,\hbar)$ and $\mathcal{D}$ provided that the following conditions are satisfied:

(1) As a vector space the algebra $\mathcal{W}(\mathcal{N}(q,\hbar))$ is isomorphic to $\mathcal{N}(q,\hbar)\otimes\mathcal{D}$.

(2) The subalgebra $\mathcal{D}$ is generated by elements ∂_i^j, $1\le i,j\le m$, subject to the following relations

$$R^{-1}D_1R^{-1}D_1 - D_1R^{-1}D_1R^{-1} = 0, \tag{4.6}$$

where $D = \|\partial_i^j\|$ and $D_1 = D\otimes I$.

(3) The permutation relations between the generators n_i^j, $1\le i,j\le m$, of the subalgebra $\mathcal{N}(q,\hbar)$ and the generators ∂_i^j of the subalgebra $\mathcal{D}$ are as follows

$$D_1RN_1R - RN_1R^{-1}D_1 = R + \hbar D_1R. \tag{4.7}$$

In the limit $q=1$ (provided that R is a deformation of the usual flip P) the relations (4.6) turn into the commutativity conditions for the generators ∂_i^j and the equalities (4.7) turn into the permutation relations (2.3). Note that the relations (4.6) and (4.7) have been introduced in [7].

Now, we want to give an operator meaning to the elements of the subalgebra $\mathcal{D}$ since we intend to interpret the generators ∂_i^j as analogs of partial derivatives. The permutation relations (4.7) in the above Definition 4.1 allows one to define an action of the subalgebra $\mathcal{D}$ on the subalgebra $\mathcal{N}(q,\hbar)$ by the same method as above. More precisely, we have to apply the counit with the same properties. On the level of generators this action is as follows:

$$\partial_i^j(n_k^p) = \delta_i^p B_k^j, \tag{4.8}$$

where $\|B_i^j\|$ is the matrix of the operator B introduced in (4.3).

Let us point out that this action gives a representation of the subalgebra $\mathcal{D}$ in the *algebra* $\mathcal{N}(q,\hbar)$, that is the action (4.8), extended on the whole algebra $\mathcal{N}(q,\hbar)$ via (4.7), respects the algebraic structure of $\mathcal{N}(q,\hbar)$. In the classical limit $q=1$ the operator ∂_i^j turns into the usual partial derivative in n_j^i: $\partial_i^j = \partial/\partial n_j^i$.

Our next aim is to define the space of differential forms on the algebra $\mathcal{N}(q,\hbar)$ and to introduce an analog of the de Rham operator on it. First of all, we need a braided analog of the external algebra generated by the differentials dn_i^j. In the classical case it is identified with the skew-symmetric algebra $\bigwedge(gl(m))$. In our current setting we define the corresponding analog $\bigwedge_q$ by quotienting the free tensor algebra, generated by the linear space span(dn_i^j), over the ideal generated by the matrix elements of the left hand side of the equality

$$R_{12}\Omega_1\hat{\Psi}_{12}\Omega_1 + \Omega_1\hat{\Psi}_{12}\Omega_1 R_{12}^{-1} = 0. \tag{4.9}$$

Here as usual $\Omega_1 = \Omega \otimes I$ and the matrix elements Ω_i^j of the $m\times m$ matrix $\Omega = \|\Omega_i^j\|$ are the linear combinations of the differentials:

$$\Omega_i^j = (B^{-1})_i^k\, dn_k^j, \tag{4.10}$$

where the summation over the repeated index is understood. The symbol $\hat{\Psi}$ stands for the following operator

$$\hat{\Psi}_{12} = \Psi_{21} + (q-q^{-1})q^{2m}B_1C_2.$$

Using the definitions and properties (4.2)–(4.5) one can easily verify that

$$\mathrm{Tr}_1\hat{\Psi}_{12}R_{13}^{-1} = P_{23} = \mathrm{Tr}_1\hat{\Psi}_{21}R_{31}^{-1}. \tag{4.11}$$

Below, we give a motivation for the definition (4.9) of the algebra $\bigwedge_q$.

Consider the product $\mathfrak{D}_q = \bigwedge_q \otimes \mathcal{N}(q,\hbar)$ which is a right $\mathcal{N}(q,\hbar)$-module. In order to convert this module into an associative algebra, we have to introduce some permutation relations between the algebras $\bigwedge_q$ and $\mathcal{N}(q,\hbar)$. However, we do not use this structure and shall consider the space $\mathfrak{D}_q$ as a one-sided $\mathcal{N}(q,\hbar)$-module only. Elements of this module are called *braided differentials*. Elements of the right $\mathcal{N}(q,\hbar)$-module $\mathfrak{D}_q^k = \bigwedge_q^k \otimes \mathcal{N}(q,\hbar)$ are called *braided k-differentials*. Here, as usual, $\bigwedge_q^k$ stands for the k-th degree homogenous component of the quadratic algebra $\bigwedge_q$.

Now, define the braided analog of the de Rham operator $d : \mathfrak{D}_q^k \to \mathfrak{D}_q^{k+1}$. Let

$$\omega = \omega_0 \otimes f, \quad f \in \mathcal{N}(q,\hbar), \ \ \omega_0 \in \bigwedge_q^k$$

be a k-differential: $\omega \in \mathfrak{D}_q^k$. Then we set by definition:

$$d\,\omega = \omega_0 \otimes \sum_{i,j} \Omega_i^j \otimes \partial_j^i(f) \in \mathfrak{D}_q^{k+1}. \tag{4.12}$$

In fact, the map d consists in inserting the element $\sum_{i,j} \Omega_i^j \otimes \partial_j^i$ inside of the k-form ω with subsequent application of the partial derivatives to the element $f \in \mathcal{N}(q,\hbar)$. The following claim is the main motivation of our definition of the algebra $\bigwedge_q$ via formula (4.9).

Proposition 4.2. *The usual property $d^2 = 0$ holds.*

Proof. First, let us recall some facts from the theory of monoidal categories. Let $\mathfrak{A}$ be a monoidal rigid category of finite dimensional vector spaces and U be its object. Let U^* be its right dual (see [2] for detail). This means that there exists an *evaluation map* $U \otimes U^* \to \mathbb{K}$ and a *coevaluation map* $\mathbb{K} \to U^* \otimes U$ which are in a sense coordinated. Let $\{u_i\}$ be a basis of U and $\{u^j\}$ be its right dual, i.e. $\langle u_i, u^j \rangle = \delta_i^j$. Then as follows from the definition, the coevaluation map is generated by $1 \mapsto \sum_k u^k \otimes u_k$.

Now, consider a subspace $I \subset U^{\otimes 2}$, playing the role of skew-symmetric subspace, and the quadratic algebra $\mathrm{Sym}(U) = T(U)/\langle I \rangle$, playing the role of the symmetric algebra of the space U. Also, consider the subspace $I^\perp \subset (U^*)^{\otimes 2}$ orthogonal to I with respect to the pairing

$$\langle x \otimes y, z \otimes v \rangle = \langle x, v \rangle \langle y, z \rangle, \quad x, y \in U, \ z, v \in U^*. \tag{4.13}$$

The algebra $\bigwedge(U^*) = T(U^*)/\langle I^\perp \rangle$ plays the role of the skew-symmetric algebra of the space U^*.

Let us form a complex

$$\delta : \bigwedge^k (U^*) \otimes \mathrm{Sym}^l(U) \to \bigwedge^{k+1} (U^*) \otimes \mathrm{Sym}^{l+1}(U),$$

where the map δ is defined by

$$u^{j_1} \otimes \cdots \otimes u^{j_k} \otimes u_{i_1} \otimes \cdots \otimes u_{i_l} \overset{\delta}{\mapsto} u^{j_1} \otimes \cdots \otimes u^{j_k} \otimes \sum_m (u^m \otimes u_m) \otimes u_{i_1} \otimes \cdots \otimes u_{i_l}.$$

Let us emphasize that the map δ consists in introducing the unit 1 inside of the product $\bigwedge^k (U^*) \otimes \mathrm{Sym}^l(U)$ with subsequent applying the coevaluation operator to the unit.

Lemma 4.3. *The following property holds: $\delta^2 = 0$.*

Proof. In order to prove the lemma, we have to show that the element

$$\sum_{m,n} u^m \otimes u^n \otimes u_n \otimes u_m, \tag{4.14}$$

which corresponds to the operator δ applied twice, vanishes in the product

$$\bigwedge^2 (U^*) \otimes \mathrm{Sym}^2(U).$$

Let $\{z_i\}$ be a basis of the subspace $I \subset U^{\otimes 2}$ and $\{\overline{z^j}\}$ be a basis of the subspace $I^\perp \subset (U^*)^{\otimes 2}$. Let us complete the former basis up to a basis $\{z_i, \overline{z_j}\}$ of the whole space $U^{\otimes 2}$ and the latter one up to a basis $\{z^i, \overline{z^j}\}$ of the space $(U^*)^{\otimes 2}$ so that the basis $\{z^i, \overline{z^j}\}$ be the right dual of that $\{z_i, \overline{z_j}\}$. Then the element (4.14) can be presented as follows

$$\sum_i z^i \otimes z_i + \sum_j \overline{z^j} \otimes \overline{z_j}.$$

It is clear that this element vanishes in the product $\bigwedge^2(U^*) \otimes \mathrm{Sym}^2(U)$.

Now, go back to the proposition. We treat the space $\mathrm{span}(\partial_i^j)$ as an object U from the above lemma and the space $\mathrm{span}(dn_i^j)$ as its right dual U^*. Besides, the basis $\{\Omega_i^j\}$ (4.10) of the space U^* is the right dual to $\{\partial_k^l\}$ with respect to the pairing

$$\langle D_1, \Omega_2 \rangle = P_{12} \quad \text{or} \quad \langle \partial_i^j, \Omega_k^p \rangle = \delta_i^p\, \delta_k^j. \tag{4.15}$$

In fact, if we identify $\mathrm{span}(n_k^l)$ and $\mathrm{span}(dn_k^l)$ as linear spaces, this pairing is nothing but the action of the partial derivatives on the generators of the RE algebra in the spirit of the classical differential calculus. It is true in virtue of the definition of the elements Ω_k^p and formula (4.8).

The role of the subspace $I \subset U^{\otimes 2}$ is played by the left hand side of (4.6), giving rise to the RE algebra but with the braiding R^{-1} instead of R. The only claim has to be shown is that the left hand side of (4.9) is just $I^\perp$. In order to prove this, we fix the basis $X_{12} = R^{-1} D_1 R^{-1} D_1$ in the space U and that $X_{12}^* = \Omega_1 \hat{\Psi} \Omega_1 R$ in the space U^*.

Lemma 4.4. *The basis X_{12}^* is right dual to X_{12}, that is $\langle X_{12}, X_{34}^* \rangle = P_{13} P_{24}$.*

Proof. The claim of the lemma is verified by a direct calculation on the base of (4.11), (4.13) and (4.15):

$$\begin{aligned}
\langle X_{12}, X_{34}^* \rangle &= \langle R_{12}^{-1} D_1 R_{12}^{-1} \langle D_1, \Omega_3 \rangle \hat{\Psi}_{34} \Omega_3 R_{34} \rangle \\
&= \langle R_{12}^{-1} D_1 R_{12}^{-1}, \hat{\Psi}_{14} \Omega_1 R_{14} \rangle P_{13} \\
&= R_{12}^{-1} \mathrm{Tr}_0\big(P_{01} R_{12}^{-1} \hat{\Psi}_{14} \langle D_0, \Omega_1 \rangle\big) R_{14} P_{13} \\
&= R_{12}^{-1} \mathrm{Tr}_0\big(P_{01} R_{12}^{-1} \hat{\Psi}_{14} P_{01}\big) R_{14} P_{13} \\
&= R_{12}^{-1} \mathrm{Tr}_0\big(R_{02}^{-1} \hat{\Psi}_{04}\big) R_{14} P_{13} \\
&= R_{12}^{-1} P_{24} R_{14} P_{13} = P_{24} P_{13}.
\end{aligned}$$

Introduce now two operators $Q : U^{\otimes 2} \to U^{\otimes 2}$ and $Q' : U^{\otimes 2} \to U^{\otimes 2}$ defined as follows

$$Q(R^{-1} D_1 R^{-1} D_1) = D_1 R^{-1} D_1 R^{-1}, \quad Q'(R^{-1} D_1 R^{-1} D_1) = D_1 R^{-1} D_1 R.$$

Similar operators were considered in [9]. They were also used in the construction of a differential calculus on a q-Minkowski space algebra in [14, 15]. It is easy to see that Q and Q' commute with each other and satisfy the following relation

$$(I - Q)(I + Q') = 0.$$

Also, the subspace $I \subset U^{\otimes 2}$ defined by (4.6) can be written as follows

$$(I - Q)(R^{-1} D_1 R^{-1} D_1) = 0.$$

Whereas, the equation

$$(I + Q')(R^{-1} D_1 R^{-1} D_1) = 0$$

defines the subspace of $U^{\otimes 2}$ which is complementary to I. In fact, the subspace $I^{\perp}$ is just $(I + Q'^{*})(U^{*})^{\otimes 2}$ where Q'^{*} stands for the conjugate operator to Q'.

This shows that the spaces defined respectively by the left hand side of (4.6) and (4.9) are orthogonal to each other[5]. The fact that they are maximal (i.e. the latter space include all elements orthogonal to the former space) can be shown from considering the dimensions of these spaces (first, for a generic q with subsequent passage to all q).

This completes constructing an analog of the de Rham complex corresponding to the modified RE algebra. □

Remark 4.5. Note that we precise no way of completing either the basis of the subspace I or that of the space $I^{\perp}$. Nevertheless, there exists the "most natural" choice to do so or, equivalently, to fix a complimentary subspace to I in the space $U^{\otimes 2}$. If the space $I \subset U^{\otimes 2}$ is defined by the left hand side of (4.1) then we define its complementary subspace as

$$R\, N_1 R\, N_1 + N_1 R\, N_1 R^{-1}. \tag{4.16}$$

The quotient of the tensor algebra of U over the ideal generated by this subspace is often treated to be a braided analog of the skew-symmetric algebra of U (see [9]).

5. q-Witt algebra: deformation property

In this section we deal with the so-called q-Witt algebra (q is assumed to be generic). This algebra is usually defined in the same way as the classical Witt algebra is but with the q-derivative (1.1) instead of the usual one. Let us precise that $\partial_q(x^k) = k_q x^{k-1}$, $k \in \mathbb{Z}$. Hereafter, we use the notation $m_q = \frac{q^m - 1}{q - 1}$.

[5]This can be also verified by a direct calculation, similar to that in the proof of Lemma 4.4.

Note that the Leibniz rule for the q-derivative reads

$$\partial_q(f(x)g(x)) = (\partial_q f(x))g(x) + f(qx)\partial_q g(x), \tag{5.1}$$

whereas its permutation relation with the generator x is:

$$\partial_q\, x - q\, x\, \partial_q = 1. \tag{5.2}$$

Below, we do not use the Leibniz rule (5.1) (see remark at the end of the section).

Now, similarly to the usual Witt algebra, consider the operators

$$e_k = x^{k+1}\,\partial_q, \quad k \in \mathbb{Z}$$

acting on the algebra $\mathbb{K}[x, x^{-1}]$. These operators act on the elements x^l as follows

$$e_k(x^l) = l_q\, x^{k+l}, \quad l \in \mathbb{Z}$$

and are subject to the relations

$$q^{m+1}e_m e_n - q^{n+1}e_n e_m - ((n+1)_q - (m+1)_q)e_{m+n} = 0. \tag{5.3}$$

These relations are usually considered (see [11] and the references therein) as a motivation for introducing the following "q-Lie bracket"

$$U \otimes U \to U : \; e_m \otimes e_n \mapsto [e_m, e_n] = ((n+1)_q - (m+1)_q)e_{m+n}, \tag{5.4}$$

where $U = \mathrm{span}(e_k)$ is the space of all finite linear combinations of the elements e_k. Then by q-Witt algebra one means the space U endowed with the q-Lie bracket (5.4), which is assumed, of course to be a bilinear operator. We denote this q-Witt algebra $\mathcal{W}_q$. Its enveloping algebra $U(\mathcal{W}_q)$ is defined to be the quotient of the free tensor algebra of the space U over the ideal generated by the left hand side of (5.3).

Emphasize that the bracket (5.4) is well-defined on the whole space $U^{\otimes 2}$. This bracket has the following properties:

(1) The "q-skew-symmetry":

$$[e_m, e_n] = -[e_n, e_m];$$

(2) The "q-Jacobi relation":

$$(1+q^k)[e_k, [e_l, e_m]] + (1+q^l)[e_l, [e_m, e_k]] + (1+q^m)[e_m, [e_k, e_l]] = 0.$$

The first relation entails the element $e_m \otimes e_n + e_n \otimes e_m$ to be sent to zero by the bracket. Consequently, we have two subspaces in the space $U^{\otimes 2}$

$$\begin{aligned} I_+ &= \mathrm{span}(e_m \otimes e_n + e_n \otimes e_m), \\ I_- &= I = \mathrm{span}(q^{m+1}e_m \otimes e_n - q^{n+1}e_n \otimes e_m), \end{aligned} \tag{5.5}$$

which are analogs of symmetric and skew-symmetric subspaces (in fact, the symmetric one is classical).

Below, we deal with the PBW theorem in the form suggested in [16]. Namely, let U be a finite dimensional vector space over the field $\mathbb{K}$ and $I \subset U^{\otimes 2}$ be a subspace. Consider an operator $[\ ,\ ] : I \to U$ satisfying two conditions

(1) $[\ ,\ ]_{12} - [\ ,\ ]_{23} : \quad I \otimes U \bigcap U \otimes I \to I$;

(2) $[\ ,\ ] \circ ([\ ,\ ]_{12} - [\ ,\ ]_{23}) : \quad I \otimes U \bigcap U \otimes I \to 0$,

where $[\ ,\ ]_{12} = [\ ,\ ] \otimes I$, $[\ ,\ ]_{23} = I \otimes [\ ,\ ]$, I is the identity operator and $\circ$ means the composition of the maps. (Below, we omit this symbol.)

If, in addition, the quadratic algebra $\mathcal{A} = T(U)/\langle I \rangle$ is Koszul then the associated graded algebra $\mathrm{Gr}\,\mathcal{A}_{[\,,]}$ where $\mathcal{A}_{[\,,]} = T(U)/\langle I - [\,,\,]I \rangle$ is canonically isomorphic to $\mathcal{A}$. Here $\langle I \rangle$ stands for the ideal generated by a set I and by $I - [\,,\,]I$ we mean the family of elements $u - [\,,\,]u,\ u \in I$.

This is just the PBW theorem under the form of [16]. Below, the call the both conditions listed above the *Jacobi-PP condition*.

Emphasize that the subspace $I \otimes U \bigcap U \otimes I \subset U^{\otimes 3}$ is an analog of the space of third degree skew-symmetric elements. Note, that the bracket is defined only on the subspace I. Thus, the first of the above conditions (which means that the bracket maps $I \otimes U \bigcap U \otimes I$ into I) ensures a possibility to apply the bracket once more.

Let us also show that the first condition above (without assuming the algebra $T(U)/\langle I \rangle$ to be Koszul) is necessary for the canonical isomorphism. Consider the element

$$([\,,\,]_{12} - [\,,\,]_{23})Z, \tag{5.6}$$

where Z is an arbitrary element belonging to $I \otimes U \bigcap U \otimes I$. Since the element $Z - Z$ equals to 0 in the algebra $\mathcal{A}_{[\,,]}$, its image under replacing factors from $I \otimes U$ (resp., $U \otimes I$) by the terms $[\,,\,]_{12}Z$ (resp., $[\,,\,]_{23}Z$) is also trivial in the algebra $\mathcal{A}_{[\,,]}$. If nevertheless, the term (5.6) does not belong to I, we have that there is an element which is trivial in $\mathrm{Gr}\mathcal{A}_{[\,,]}$ and is nontrivial in $\mathcal{A}$. Consequently, the canonical isomorphism of the algebras $\mathrm{Gr}\mathcal{A}_{[\,,]}$ and $\mathcal{A}$ does not exist.

Remark 5.1. Note that to describe the space $I^{(3)} = I \otimes U \bigcap U \otimes I \subset U^{\otimes 3}$ explicitly is not an easy deal in general. However, if the subspace $I \subset U^{\otimes 2}$ is generated by elements of the form

$$e_i e_j - c(i,j) e_j e_i, \quad c(i,j) \neq 0 \ \ \forall\, i, j \tag{5.7}$$

the space $I^{(3)}$ is easy to describe. First, consider the case $\dim U = 3$. Let $\{x, y, z\}$ be a basis of the space U. We set

$$I = \mathrm{span}(xy - ayx,\ yz - bzy,\ zx - cxz), \quad a\,b\,c \neq 0.$$

Then the space $I^{(3)}$ is one-dimensional and is generated by the following element

$$\begin{aligned} Z(x,y,z) &= c(xy - ayx)z + a(yz - bzy)x + b(zx - cxz)y \\ &= bz(xy - ayx) + cx(yz - bzy) + ay(zx - cxz). \end{aligned} \tag{5.8}$$

If $\dim U > 3$, the space $I^{(3)}$ is generated by all elements $Z(e_k, e_l, e_m)$, each of them being associated with a triple e_k, e_l, e_m.

Now, go back to the q-Witt algebra. This algebra is infinite dimensional. However, if by U we mean all finite linear combinations of the generators $\{e_i\}$, and by $U^{\otimes k}$ we also mean the finite linear combinations of $e_{i_1} \otimes e_{i_2} \otimes \cdots \otimes e_{i_k}$, then we can extend our reasoning to this case.

Namely, denote the vector space of finite linear combinations of elements $q^{k+1}e_k\, e_l - q^{l+1}e_l\, e_k$ by I and consider an element $Z \in I \otimes U \bigcap U \otimes I$ of the following form (we omit the sign $\otimes$ between the elements e_k):

$$\begin{aligned} Z &= q^{l+m+2}(q^{l+1}e_l e_m - q^{m+1}e_m e_l)e_k + q^{m+k+2}(q_{m+1}e_m e_k - q^{k+1}e_k e_m)e_l \\ &\qquad + q^{k+l+2}(q_{k+1}e_k e_l - q^{l+1}e_l e_k)e_m \\ &= q^{2(m+1)}e_m(q^{k+1}e_k e_l - q^{l+1}e_l e_k) + q^{2(k+1)}e_k(q^{l+1}e_l e_m - q^{m+1}e_m e_l) \\ &\qquad + q^{2(l+1)}e_l(q^{m+1}e_m e_k - q^{k+1}e_k e_m). \end{aligned}$$

Compute the images of this element under the maps $[\,,\,]_{12}$ and $[\,,\,]_{23}$ correspondingly. We have

$$\begin{aligned} [\,,\,]_{12}Z &= q^{l+m+2}((m+1)_q - (l+1)_q)e_{l+m}e_k \\ &\quad + q^{m+k+2}((k+1)_q - (m+1)_q)e_{m+k}e_l \\ &\quad + q^{k+l+2}((l+1)_q - (k+1)_q)e_{k+l}e_m, \\ [\,,\,]_{23}Z &= q^{2(m+1)}((l+1)_q - (k+1)_q)e_m e_{k+l} \\ &\quad + q^{2(k+1)}((m+1)_q - (l+1)_q)e_k e_{l+m} \\ &\quad + q^{2(l+1)}(((k+1)_q - (m+1)_q)e_l e_{m+k}. \end{aligned}$$

Let us assume that the numbers $k, l, m, k+l, k+m$ and $l+m$ are pairwise distinct. Then the difference $[\,,\,]_{12}Z - [\,,\,]_{23}Z$ belongs to I iff it is so for the element

$$q^{l+1}q^{m+1}((m+1)_q - (l+1)_q)e_{l+m}e_k - q^{2(k+1)}((m+1)_q - (l+1)_q)e_k e_{l+m} \tag{5.9}$$

and for two similar elements obtained by cyclic permutations $k \to l \to m$. However, it is evident that for a generic q the element (5.9) does not belong to I since the two-dimensional vector

$$(q^{l+1}q^{m+1}((m+1)_q - (l+1)_q),\ -q^{2(k+1)}((m+1)_q - (l+1)_q))$$

composed of the coefficients of the element (5.9) is not collinear to the vector $(q^{l+m+1},\ -q^{k+1})$ unless $k = 0$.

Thus, the first of the above conditions is not satisfied. Consequently, the algebra $\mathrm{Gr}(U(\mathcal{W}_q))$ is not isomorphic to $T(U)/\langle q^{k+1}e_k e_l - q^{l+1}e_l e_k\rangle$.

Remark 5.2. If we introduce a parameter $\hbar$ as a multiplier in the right hand side of the bracket (5.4) of the q-Witt algebra we get a two parametric analog of the usual Witt algebra. On putting $\hbar = 0$ we get a quadratic algebra $T(U)/\langle q^{k+1}e_k e_l - q^{l+1}e_l e_k\rangle$

which possesses a good deformation property. Since this quadratic algebra is infinite dimensional, we should precise the meaning of this property. The ordered monomials

$$e_1^{k_1} e_2^{k_2} \cdots e_l^{k_l}, \quad k_1 + k_2 + \cdots + k_l = k$$

form a basis of its k-th degree homogeneous component. It can be considered as a quantization of the corresponding Poisson structure defined by

$$\{e_k, e_l\} = (l - k) e_k\, e_l.$$

Nevertheless, the algebra $U(\mathcal{W}_q)$ is not a two-parameter quantization of a Poisson pencil. This is due to the fact that the passage from the mentioned quadratic algebra to its filtered (quadratic-linear) analog is not a deformation.

In a similar way we can introduce another analog of the Witt algebra, called $\hbar$-Witt one. In its construction the usual derivative is replaced by its difference analog (1.2). The permutation relation with x reads

$$\partial_\hbar x - x \partial_\hbar = 1 + \hbar \partial_\hbar.$$

Note that the algebra generated by x and $\partial_\hbar$ is a Weyl algebra but it is not so for the algebra generated by x and ∂_q. (However, it is a Weyl algebra in the sense of a more general definition exhibited in the next section.)

Now, consider operators $e_k = e^{ikx} \partial_\hbar$, $k \in \mathbb{Z}$ acting onto the space of real continuous functions. Using the permutation relation

$$\partial_\hbar\, e^{ikx} - e^{ik(x+\hbar)}\, \partial_\hbar = \frac{e^{ik\hbar} - 1}{\hbar}\, e^{ikx},$$

we get the following:

$$e^{ik\hbar}\, e_k\, e_l - e^{il\hbar}\, e_l\, e_k = \frac{e^{il\hbar} - e^{ik\hbar}}{\hbar}\, e_{k+l}.$$

So, denoting $q = e^{i\hbar}$, we can see that the difference of this structure from that described above is unessential.

Also, it is tempting to introduce an $\hbar$-analog of the Lie bracket by putting

$$[e_k, e_l] = \frac{e^{il\hbar} - e^{ik\hbar}}{\hbar}\, e_{k+l}.$$

Finally, for the reason presented above, the PBW theorem in the enveloping algebra of this "generalized Lie algebra" fails. The detail is left to the reader.

Two above analogs of the usual derivatives being put together give rise to a $(q, \hbar)$-counterpart of the derivatives defined by

$$\partial_{q,\hbar}(f(x)) = \frac{f(qx + \hbar) - f(x)}{(q - 1)x + \hbar}.$$

This $(q,\hbar)$-derivative have the following permutation relation with x:

$$\partial_{q,\hbar}\, x - qx\, \partial_{q,\hbar} = 1 + \hbar\partial_{q,\hbar}.$$

Note that this $(q,\hbar)$-derivative can be deduced from the q-derivative via the change of the generator $x \to x + \frac{\hbar}{q-1}$.

Also, note that besides the above analogs of the usual derivative, there are their slight modifications

$$\tilde{\partial}_q(f(x)) = \frac{f(qx) - f(q^{-1}x)}{(q-q^{-1})x}, \qquad \tilde{\partial}_\hbar(f(x)) = \frac{f(x+\hbar) - f(x-\hbar)}{2\hbar}.$$

These operators do not give rise to any Weyl algebra on the function space in one variable. Nevertheless, the operator $\tilde{\partial}_\hbar$ appears in the frameworks of the Weyl algebra $\mathcal{W}(U(u(2)_\hbar))$ and its commutative subalgebra considered in Section 3.

We complete this section by the following observation. The q-Witt algebra contains a subalgebra looking like the enveloping algebra of the Lie algebra $sl(2)$. Namely, consider the subalgebra generated by three elements $e_{-1} = \partial_q$, $e_0 = x\partial_q$ and $e_1 = x^2\partial_q$. They are subject to the following relations

$$\begin{aligned} e_{-1}e_0 - q\, e_0\, e_{-1} &= e_{-1}, \\ e_{-1}e_1 - q^2 e_1 e_{-1} &= (1+q)\, e_0, \\ e_0\, e_1 - q\, e_1 e_0 &= e_1. \end{aligned} \tag{5.10}$$

This quadratic-linear algebra was considered in [13] in the frameworks of the so-called Hom–Lie algebras. This notion is based on the modified Leibniz rule (5.1). In a similar manner the notions of Hom-associative algebras, Hom–Poisson algebras etc., were introduced. Emphasize that the enveloping algebra of a Hom–Lie algebra is not an associative one but Hom-associative. By contrast, our approach is based only on the permutation relations between derivative(s) and generator(s) of a given algebra. Thus, the algebra defined by the relations (5.10) is a usual associative algebra. We claim that this algebra has the good deformation property. This property will be proven in the next section for a larger family of quadratic-linear algebras.

6. Other roles and forms of Jacobi condition for quadratic-linear algebras

In the previous section we presented a form of the Jacobi condition, which is useful for proving or denying the PBW property of a given quadratic-linear algebra. Nevertheless, if such an object is the enveloping algebra of a Lie algebra, the Jacobi condition enables one to construct the adjoint representation of this object. Besides, the construction of the Chevalley–Eilenberg complex associated with this Lie algebra is mainly based on the Jacobi identity.

Below, we discuss other forms the Jacobi condition which are useful for generalizing the notions of the adjoint representation and the Chevalley–Eilenberg complex for quadratic-linear algebras. However, first we give some examples of such algebras subject to Jacobi-PP condition.

Let again U be a three dimensional space and $\{x, y, z\}$ be its basis. Consider the quadratic-linear algebra generated by these generators subject to the relations

$$xy - ayx - l_1 = 0, \quad yz - bzy - l_2 = 0, \quad zx - cxz - l_3 = 0,$$

where $a, b, c \in \mathbb{K}$ are nontrivial constant and l_1, l_2, l_3 are some elements of U. As usual, we also consider the corresponding quadratic algebra which is obtained by setting $l_1 = l_2 = l_3 = 0$.

It would be interesting to classify all families (a, b, c, l_1, l_2, l_3) such that the Jacobi-PP condition for the corresponding quadratic-linear algebras would be valid. We restrict ourselves to two particular cases.

The first case is $sl(2)$ like. We assume that

$$l_1 = kx, \quad l_2 = lz, \quad l_3 = my, \quad k, l, m \in \mathbb{K}, \ klm \neq 0.$$

It is easy to see that the Jacobi-PP condition is valid iff $b = a$ and $l = k$. Thus, the relations on the generators of the corresponding algebra become

$$xy - ayx = kx, \ yz - azy = kz, \ zx - cxz = my. \tag{6.1}$$

This algebra can be treated as a multiparameter deformation of the commutative algebra $\mathrm{Sym}(U)$. It can be easily seen that the algebra defined by (5.10) is a particular case of the family (6.1). Indeed, by identifying $e_{-1} = x, e_0 = y, e_1 = z$ we get (5.10) if in (6.1) we put

$$a = q, \quad c = q^{-2}, \quad m = -(q^{-1} + q^{-2}).$$

Since the quadratic algebra corresponding to the quadratic-linear algebra defined by (6.1) is Koszul, the latter algebra meets the PBW property.

The second case is $su(2)$ like. We assume that

$$l_1 = kz, \quad l_2 = lx, \quad l_3 = my, \quad k, l, m \in \mathbb{K}, \ klm \neq 0.$$

The Jacobi-PP condition is fulfilled iff $a = b = c$. Also, by a change of a basis (over the field $\mathbb{K} = \mathbb{C}$) we can get that $k = l = m = 1$. Thus, we assume that the generators are bound by the following relations

$$xy - ayx = z, \quad yz - azy = x, \quad zx - axz = y. \tag{6.2}$$

Since the corresponding quadratic algebra is Koszul, we conclude that the algebra defined by (6.2) has the good deformation property. Thus, by introducing a

parameter $\hbar$ in front of the right hand side, we get a two parameter deformation of the same algebra $\mathrm{Sym}(U)$.

Now, we pass to considering other forms of the Jacobi condition. Consider a quadratic algebra $\mathcal{A} = T(U)/\langle I\rangle$, $I \subset U^{\otimes 2}$ and a quadratic-linear one $\mathcal{A}_{[\,,\,]}$ (we use the notations of the previous section). Assume that the Jacobi-PP condition for the bracket $[\,,\,] : I \to U$ is valid. However, this condition cannot be written in the form

$$[\,,\,][\,,\,]_{12} = [\,,\,][\,,\,]_{23} \quad \text{on} \quad I^{(3)} = I \otimes U \bigcap U \otimes I, \tag{6.3}$$

because the images of the operators $[\,,\,]_{12}$ and $[\,,\,]_{23}$ acting on $I^{(3)}$ do not belong in general to I (but their difference does by assumption). Thus, the sides of (6.3) are not well defined separately.

In order to make this object more similar to a usual Lie algebra we assume that in the space $U^{\otimes 2}$ there is a complementary subspace I_+ (playing the role of the symmetric subspace) where the bracket acts trivially. Consequently, the bracket becomes well defined on the whole space $U^{\otimes 2}$. Thus, we have the following data $(U, I, I_+, [\,,\,] : U^{\otimes 2} \to U)$ with complementary subspaces $I \oplus I_+ = U^{\otimes 2}$ and such that the image of the subspace I_+ under the map $[\,,\,]$ is trivial. Thus, the both sides of (6.3) are well defined.

Definition 6.1. We say that the data $(U, I, I_+, [\,,\,] : U^{\otimes 2} \to U)$ meets the strong Jacobi condition if the Jacobi-PP condition is valid for the corresponding quadratic-linear algebra, and the both sides of (6.3) are trivial.

Note that the strong Jacobi identity enables us to define an analog of the Chevalley–Eilenberg complex composed of the terms

$$I^{(k)} = I \otimes U^{\otimes(k-2)} \bigcap U \otimes I \otimes U^{\otimes(k-3)} \bigcap \cdots \bigcap U^{\otimes(k-3)} \otimes I \otimes U \bigcap U^{\otimes(k-2)} \otimes I$$

with the differential $d = [\,,\,]_{12}$. The relation $d^2 = 0$ follows immediately from the fact that $[\,,\,][\,,\,]_{12} = 0$. Observe that the subspaces $I^{(k)} \subset U^{\otimes k}$ are analogs of the spaces of skew-symmetric elements. Also, observe that the result of applying the usual Chevalley–Eilenberg operator and that $d = [\,,\,]_{12}$ to a skew-symmetric element differ by a nontrivial factor. Thus, our complex can be considered as a generalization of the Chevalley–Eilenberg one.

Now, we go back to the above examples and examine the problem: whether the corresponding algebras can be completed with convenient subspaces $I_+ \subset U^{\otimes 2}$ such that the new data $(U, I, I_+, [\,,\,] : U^{\otimes 2} \to U)$ meets the strong Jacobi condition.

First, consider the $sl(2)$ like algebra. By a straightforward computation it is possible to check that the strong Jacobi condition is met if I_+ contains the elements y^2 and $cxz + azx$.

As for the $su(2)$ like algebra, we have that the strong Jacobi identity is valid for the data $(U, I, I_+, [\,,\,] : U^{\otimes 2} \to U)$ iff the subspace I_+ contains the term $x^2 + y^2 + z^2$.

In particular, we can put

$$I_+ = \mathrm{span}(x^2,\ y^2,\ z^2,\ xy+\alpha yx,\ yz+\alpha zy,\ zx+\alpha xz), \quad \alpha \in \mathbb{K},\ \alpha \neq 0. \tag{6.4}$$

Let us discuss now a form of the Jacobi condition for quadratic-linear algebras enabling us to construct an analog of the adjoint representation.

Definition 6.2. We say that the data $(U, I, I_+, [\,,\,] : U^{\otimes 2} \to U)$ with complementary I and I_+, such that $[\,,\,]I_+ = 0$, is an almost Lie algebra, if there exists a nontrivial number $p \in \mathbb{K}$ such that the map $p\,[\,,\,]$ defines left and right representations of the quadratic-linear algebra

$$T(U)/\langle I - [\,,\,]I\rangle. \tag{6.5}$$

Explicitly, this condition means that

$$[\,,\,][\,,\,]_{12} = p[\,,\,][\,,\,]_{23} \quad \text{on } I \otimes U \tag{6.6}$$

and

$$[\,,\,][\,,\,]_{23} = p[\,,\,][\,,\,]_{12} \quad \text{on } U \otimes I. \tag{6.7}$$

Also, emphasize that for the enveloping algebra of a Lie algebra the normalizing factor p equals 2 since, if we define the bracket $[\,,\,]$ from the system

$$[\,,\,](x_i \otimes x_j - x_j \otimes x_i) = c^k_{i,j} x_k, \quad [\,,\,](x_i \otimes x_j + x_j \otimes x_i) = 0,$$

we get a half of the usual Lie bracket. In general, the factor p has to be found.

Let us turn to the above examples and look for the values of the parameters entering the defining relations of these algebras such that these algebras acquire almost Lie algebra structures.

First, consider the $su(2)$ like algebra. Let us assume that the subspace I_+ is given by (6.4). Then we get a bracket defined on the whole space $U^{\otimes 2}$ and having the following multiplication table

$$[x, x] = 0, \quad [x, y] = \frac{\alpha z}{\gamma}, \quad [y, x] = \frac{-z}{\gamma}, \quad \text{c.p.}$$

where $\gamma = a + \alpha$.

Now, we are able to define the left action (denoted $\triangleright$) of the space U onto itself which is multiple (with the factor p) of the above bracket action. We have

$$x \triangleright x = 0, \quad x \triangleright y = \frac{p\alpha z}{\gamma}, \quad x \triangleright z = \frac{-py}{\gamma}$$

and so on.

Thus, we can represent x, y and z as operators acting in the space $U = \mathrm{span}(x, y, z)$. We want to classify families of parameters such that the relation (6.6) is satisfied. Straightforward computing implies the following relations on the parameters a, α and p:

$$p = \alpha\,\gamma, \quad a\alpha^2 p = \gamma, \quad \gamma = a + \alpha.$$

Treating α as an independent parameter we can express other parameters in terms of α:

$$a = \alpha^{-3}, \quad \gamma = \alpha + \alpha^{-3}, \quad p = \alpha^2 + \alpha^{-2}.$$

It is easy to see that the same parameters are convenient for the relation (6.7). Thus, we get a family of almost Lie algebra structures, parameterized by α.

Note that the family of the data satisfying the strong Jacobi condition is larger than this family since the parameters a entering (6.2) and α entering (6.4) are not related.

Now, pass to the $sl(2)$ type algebra. We assume that the subspace I_+ has the following form

$$I_+ = \mathrm{span}(x^2,\ y^2,\ z^2,\ xy + \alpha yx,\ yz + \beta zy,\ zx + \beta xy).$$

By a change of the basis we can get $k = 1, l = 2$ (see (6.1)). By tedious but straightforward computations it can be shown that the data $(U, I, I_+, [\,,\,] : U^{\otimes 2} \to U)$ determines an almost Lie structure iff $a = b = \alpha = \beta = 1$, $p = 2$, i.e. when it corresponds to the usual $sl(2)$ Lie structure.

Other examples interesting from the viewpoint of the different forms of the Jacobi condition arise from braidings. In the early 80's one of the authors (D.G.) introduced the notion of generalized Lie algebras associated with involutive symmetries (see [3,4]). A $gl(m)$ type example can be constructed as follows.

Let $R : V^{\otimes 2} \to V^{\otimes 2}$ be a skew-invertible (see Section 4) involutive symmetry. Then it can be extended up to an involutive symmetry

$$R_{\mathrm{End}} : \mathrm{End}(V)^{\otimes 2} \to \mathrm{End}(V)^{\otimes 2}.$$

Besides, in the space $\mathrm{End}(V)$ there is a usual product (composition) of endomorphisms

$$\mathrm{End}(V) \ni X, Y \mapsto X \circ Y \in \mathrm{End}(V).$$

Introduce the following bracket

$$[\,,\,] : \mathrm{End}(V)^{\otimes 2} \to \mathrm{End}(V), \quad [X, Y] = X \circ Y - \circ R_{\mathrm{End}}(X, Y).$$

Consider the quadratic-linear algebra defined by the relations

$$X_i\, X_j - R_{\mathrm{End}}(X_i, X_j) = [X_i, X_j] \tag{6.8}$$

where $\{X_i\}$ is a basis of the space $\mathrm{End}(V)$. We claim that for this filtered algebra the Jacobi-PP condition is valid. Moreover, the corresponding quadratic algebra

$$T(\mathrm{End}(V))/\langle I \rangle,$$

where $I = \mathrm{span}(X_i\, X_j - R_{\mathrm{End}}(X_i, X_j))$ is Koszul. This ensures the PBW property for the algebra defined by (6.8).

In contrast with the above examples where the bracket was only defined on a subspace $I \subset U^{\otimes 2}$, now we have defined this bracket on the whole space $U^{\otimes 2}$, $U = \mathrm{End}(V)$. Consequently, we have no ambiguity in choosing of the subspace I_+. Namely, we have

$$I_+ = \mathrm{span}(X_i X_j + R_{\mathrm{End}}(X_i, X_j)).$$

It is not difficult to see that the bracket above acts on this space trivially. We claim that for the data $(U, I, I_+, [\,,\,] : U^{\otimes 2} \to U)$ the strong Jacobi identity is also valid. Moreover, this data defines an almost Lie algebra structure with $p = 1$.

Remark 6.3. Here $p = 1$ since $[\,,\,](X_i \otimes X_j - R_{\mathrm{End}}(X_i, X_j)) = 2[X_i, X_j]$. Also note that for the Jacobi-PP condition, as well as for the strong Jacobi condition, the normalizing factor p does not matter.

If R is a skew-invertible Hecke symmetry, a similar construction mutatis mutandis can be also defined. Without going into detail, we only give an explicit form of the subspaces $I,\ I_+ \subset U^{\otimes 2}$ where $U = \mathrm{End}(V)$. Namely, I is spanned by the entries of the left hand side matrix from (4.1) and I_+ is spanned by those of the matrix (4.16). Also, we assume that the corresponding bracket maps each entry of the left hand side matrix from (4.1) to the corresponding entry of the right hand side matrix and is trivial on the subspace I_+. We claim that the corresponding quadratic-linear algebra (which is nothing but the modified RE algebra) meets the Jacobi-PP condition. Also, for a generic q the PBW property is valid in it (see [5,6]). Besides, the corresponding data $(U, I, I_+, [\,,\,] : U^{\otimes 2} \to U)$ defines an almost Lie algebra structure. However, we do not know whether the strong Jacobi condition is valid for this data.

Now, we want to discuss a possible generalization of the notion of the Weyl algebra from Section 3 to quadratic-linear algebras.

Let U be a finite-dimensional vector space with a basis $\{x_i\}$. As usual, consider a quadratic algebra $\mathcal{A} = T(U)/\langle I\rangle$, where I is a subspace of $U^{\otimes 2}$, and introduce its quadratic-linear counterpart $\mathcal{A}_{[\,,\,]} = T(U)/\langle I - [\,,\,]I\rangle$, where $[\,,\,] :\ I \to U$ is a linear map (called bracket). Also, consider the quadratic algebra $\mathcal{B} = T(U^*)/\langle J\rangle$ where $J \subset (U^*)^{\otimes 2}$. Let ∂^i be the basis of the space U^*, dual to the basis $\{x_i\}$: $\langle \partial^i, x_j\rangle = \delta_i^j$. Let some permutation relations of the form

$$\partial^i x_j - \alpha^{i,k}_{j,l} x_k \partial^l = b^i_{j,k}\partial^k + \delta^i_j \tag{6.9}$$

be given. The space spanned by the left hand side of these elements will be denoted K. Assume that these permutation relations are compatible with the algebras $\mathcal{A}_{[\,,\,]}$ and $\mathcal{B}$, i.e. modulo these permutation relations any element of the product $\mathcal{B} \otimes \mathcal{A}_{[\,,\,]}$ can be converted into an element of $\mathcal{A}_{[\,,\,]} \otimes \mathcal{B}$. Then the product $\mathcal{B} \otimes \mathcal{A}_{[\,,\,]}$ can be equipped with an associative algebra structure.

Definition 6.4. This associative algebra is called the Weyl algebra on the algebra $\mathcal{A}_{[\,,\,]}$.

Practically, a verification of the fact that the permutation relations (6.9) are compatible with the structures of the algebras $\mathcal{A}_{[\,,\,]}$ and $\mathcal{B}$ (i.e. that they preserve the

ideals generated by defining relations) should be done via the Jacobi condition in the form of [1], which is a generalization of the Jacobi-PP one. Namely, this condition must be applied to the subspace

$$I \oplus J \oplus K \subset (U \oplus U^*)^{\otimes 2}$$

endowed with the bracket, which equals $[\,,\,]$ on I, trivial on J and defined by (6.9) on K.

Note, that the Weyl algebras introduced in Section 3 and the braided Weyl algebra from Section 4 are covered by this definition. In the latter case we have to change the dual basis in the space spanned by the partial derivatives. Our last definition also covers the algebra defined by (5.2). In this case the subspaces I and J are trivial and K is generated by the left hand side of (5.2).

Completing the paper, we want to observe the following. Let $\mathcal{A}$ and $\mathcal{B}$ be two unital associative algebras and there is a representation $\mathcal{B} \otimes \mathcal{A} \to \mathcal{A}$ of the algebra $\mathcal{B}$. If the algebra $\mathcal{B}$ has a bi-algebra structure (may be braided) then it is possible to define permutation relations

$$\mathcal{B} \otimes \mathcal{A} \to \mathcal{A} \otimes \mathcal{B}$$

via the method exhibited above. Inversely, if the algebra $\mathcal{B}$ is endowed with a counit only and the algebras are equipped with some permutation relations, it is possible to define an action of $\mathcal{B}$ onto $\mathcal{A}$ (i.e. a representation of of $\mathcal{B}$ in $\mathcal{A}$). However in general, we cannot find an appropriate coproduct in the algebra $\mathcal{B}$. It is just the case of the braided Weyl algebra from Section 4.

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