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Elliptic and transversally elliptic index theory from the viewpoint of KK-theory

Gennadi Kasparov*

Abstract. The main purpose of the paper is to develop a KK-theory approach to index theory of operators transversally elliptic with respect to a proper group action. The principal novelty is the use of fields of Clifford algebras along the orbits. Index formulas that we prove allow to calculate index from symbol and vice-versa using KK-product. This has a deeper meaning: there exists Poincaré duality between the symbol and the index when they are defined as elements of the appropriate KK-groups.

The main results on transversally elliptic operators are contained in the second half of the paper. In the first part we develop the necessary technique and prove index theorems for elliptic operators. This is done in the equivariant setting with respect to a proper group action, the manifolds and groups are *not* assumed to be compact. At the end of the paper we give applications of our methods to transversally elliptic operators on singular foliations.

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1. Introduction

The development of index theory in the 1960s was closely related with the development of K-theory. The first proof of the Atiyah–Singer index theorem based entirely on K-theory was given in [8], and soon after that an idea to define the analytical index of an elliptic operator as an element of a K-homology group was suggested by Atiyah in [6]. One of the forms of an index theorem for the K-homological index was given in [25], and later made more precise in [27–29]. The theorem states that the K-homological index is the intersection product of the K-theoretic symbol and the so called Dolbeault (or Dirac) element which depends only on the manifold. No embedding of the manifold into a Euclidean space is needed. The formula is valid for non-compact manifolds in the equivariant form: with a proper action of a locally compact group.

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In the present paper, on our way to new results for transversally elliptic operators which constitute the second half of the paper, we first develop the necessary technique in the elliptic operator case and prove the above mentioned index theorems. Logically this is reasonable because a great deal of the technical details in the elliptic and transversally elliptic cases are the same. We use a new method of proof, starting from what we call the "inverse index theorem", i.e. a theorem which gives a formula for the K-theoretic symbol via the K-theoretic index. This method employs the full strength of KK-theory, and emphasizes the role of Poincaré duality between the K-theoretic symbol $[\sigma_A]$ of an elliptic operator A and its index $[A]$: when the symbol and the index are defined as elements of the appropriate KK-groups, they carry exactly the same information. In particular, the "direct" index theorem, which expresses $[A]$ through $[\sigma_A]$, is an easy corollary of the "inverse" index theorem.

The proof of the inverse index theorem and several other results makes use of a certain rotation trick, similar to Atiyah's proof of the Bott periodicity [5]. This kind of argument was used for the proof of a certain form of Poincaré duality in [30] and will be used here again. We prove an appropriate form of Poincaré duality in Section 4, and a more general result in Section 7 (Theorem 7.8).

There are many other proofs of the K-homological index formula, for instance, [14, 16, 17, 19, 21]. These proofs use other methods, usually the tangent groupoid approach and the associated pseudo-differential operator extension (explicitly or implicitly), and sometimes also an embedding into a Euclidean space.

An important role in our approach is played by what we call the *Clifford symbol* and *Clifford index*. Although in the case of elliptic operators the most natural K-theoretic symbol is associated with the algebra $C_0(TX)$ (where TX is the tangent manifold for the manifold X), one can also use the KK-equivalent algebra $\mathrm{Cl}_\tau(X)$ of continuous sections of the field of Clifford algebras associated with the tangent bundle $T(X)$ (cf. [30, Definition 4.1], where this algebra is denoted $C_\tau(X)$). The Clifford symbol associated with $\mathrm{Cl}_\tau(X)$ is useful in general technical constructions, such as the proof of the inverse index theorem, but may be especially useful in dealing with Dirac operators (cf. Proposition 3.10).

The transversally elliptic operators are discussed in the second half of the paper. Historically, transversally elliptic operators were first introduced by M. Atiyah in [7] for operators on compact manifolds. The main emphasis there was on the so called distributional index. Subsequent publications of N. Berline and M. Vergne [11, 12] were devoted to the index theorem related with this index (for operators on compact manifolds). In the present paper we develop more comprehensive K-theoretic invariants for operators on complete Riemannian manifolds. It is very likely that the index formula [12] for the distributional index can be obtained from our K-theoretic invariants. This will require some further work on translating K-theoretic formulas into the language of cohomology.

In the case of transversally elliptic operators, the natural K-theoretic symbol is associated with a non-separable commutative C^*-algebra $\mathfrak{S}_\Gamma(X)$ (defined in

Section 6). However, this symbol is equivalent to a Clifford symbol associated with much smaller (and easier to deal with) algebras: either $\mathrm{Cl}_\Gamma(TX) = C_0(TX) \otimes_{C_0(X)} \mathrm{Cl}_\Gamma(X)$, or $\mathrm{Cl}_{\tau\oplus\Gamma}(X) = \mathrm{Cl}_\tau(X) \hat{\otimes}_{C_0(X)} \mathrm{Cl}_\Gamma(X)$, where $\mathrm{Cl}_\Gamma(X)$ is the algebra of Clifford sections of $T(X)$ along the orbits of the group action (Definition 7.1). All transversally elliptic operators are supposed to be invariant under a proper action of a Lie group G. The Clifford index is defined as an element of the K-homology group $K^*_G(\mathrm{Cl}_\Gamma(X))$. The Clifford symbol and index are related by the Clifford index theorems 8.12 and 8.14. They are also Poincaré dual via the Poincaré duality Theorem 7.8.

Another version of the index theorem is Theorem 8.18. It calculates the index of a transversally elliptic operator as an element of the K-homology group of the crossed product algebra $C^*(G, C_0(X))$. This index is most closely related with the distributional index in the compact manifold case (cf. [24]). The two index theorems, 8.14 and 8.18, are actually equivalent in view of Theorem 8.15.

At the end of the paper (Section 9) we give applications of our methods to operators related with a certain class of singular foliations. The index theorem 9.5 for transversally elliptic operators is a non-equivariant version of Theorem 8.14, although we do require a certain equivariance for our operators (see Definition 9.1). An equivariant version of the index theorem must probably include a groupoid action.

The paper is organized as follows. In Section 2 we review the construction of basic KK-elements for elliptic operators: Dirac and local Bott (or dual Dirac) elements, including the Clifford versions of those. Section 3 contains the necessary facts about pseudo-differential operators, the definition of the K-theoretic symbol and index for elliptic operators and the definition of the Clifford symbol. The main results for elliptic operators: index theorems and Poincaré duality, are in Section 4.

In Section 5 we give a proof of the index theorem for equivariant elliptic operators with index in $K_*(C^*(G))$. This result first appeared in [27] without full proof. It is used in the literature as a basis for the statement of the Baum–Connes conjecture. We also give a definition of an index with value in $K_*(C^*(G))$ for first order G-invariant elliptic differential operators. A new approach of V. Lafforgue [34] to the description of the discrete series representations for semisimple Lie groups uses both the above index theorem and the $K_*(C^*(G))$-index for Dirac operators.

Section 6 introduces the symbol and the K-homology index for transversally elliptic operators. Basic KK-elements for transversally elliptic operators are constructed in Section 7. The general form of Poincaré duality related with continuous fields of Clifford algebras is proved at the end of Section 7.

Section 8 contains the main index theorems in the transversally elliptic case: 8.12, 8.14, 8.18. There are several preliminary results which are contained in this section and used in the proof of the index theorems, most essential of them is the construction of the orbital Dirac element.

The singular foliation case is discussed in Section 9.

Notation. The algebra of continuous complex-valued functions vanishing at infinity on a locally compact space X is denoted by $C_0(X)$, the subspace of compactly supported smooth functions by $C_c^\infty(X)$. The notation $C_0(\cdot)$ is also used for continuous, vanishing at infinity, sections of vector bundles and continuous fields of vector spaces. All Clifford algebras that we use are associated with *positive* quadratic forms. The class of pseudo-differential operators that will be used in all main results of the paper is the Hörmander $\rho = 1, \delta = 0$ class.

Whenever integration over a locally compact group G is involved, we always use the left Haar measure. Note that the modular function (associated with the right translation) is defined differently in different books. We denote the modular function by μ and adopt the definition of Bourbaki: if dg is the left Haar measure and $s \in G$, then $d(gs) = \mu(s)dg$.

In everything related with Hilbert modules, KK-theory, and KK-product, we use the notation of [30]. Because Clifford algebras play significant role in the paper, we always use KK-theory of $\mathbf{Z}_2$-graded C^*-algebras. *A small simplification in notation will consist in using notation $\mathcal{R}K(X; B)$ for the group $\mathcal{R}KK(X; C_0(X), B)$* (cf. [30, Definition 2.19]), *when B is a $C_0(X)$-algebra* (cf. [30, Definition 1.5]). (We recall that by [30, Definition 2.19], the group $\mathcal{R}KK(X; A, B)$ differs from $KK(A, B)$ only in the following additional requirement: if $(\mathcal{E}, T)$ is a KK-cycle, then for any $f \in C_0(X), a \in A, b \in B, e \in \mathcal{E}$, one has: $(fa)eb = ae(fb)$.)

Remark. In general, the same C^*-algebra B equipped with two different $C_0(X)$-module structures represents two different objects in the category of $C_0(X)$-algebras. However, the two corresponding groups $\mathcal{R}K(X; B)$ may appear to be isomorphic by definition of $\mathcal{R}K(X; B)$. The simplest case is when X is compact. Then $\mathcal{R}K(X; B)$ is just isomorphic to $K_0(B)$. Another case, often used in this paper, is when B is a $C_0(X \times X)$-algebra and U is an open neighborhood of the diagonal of $X \times X$ such that both projections of the closure $\bar{U}$ to X are proper maps. In this case, the algebra $C_0(U) \cdot B$ has two different $C_0(X)$-module structures corresponding to the action of $C_0(X)$ over the first or the second tensor copy of $C_0(X) \otimes C_0(X)$, but it is easy to see that by [30, Definition 2.19] cited above, the corresponding groups $\mathcal{R}K(X; C_0(U) \cdot B)$ are the same.

2. Basic *KK*-elements

In this section we will recall the definition of some basic KK-elements that will be used in the statement and the proof of the index theorems for elliptic operators.

Everywhere in this paper, X is a complete Riemannian manifold equipped with a proper isometric action of a locally compact group G. We denote by $T(X)$ the tangent bundle of X, by $T^*(X)$ the cotangent bundle and by $p : T^*(X) \to X$ the projection. We will not actually distinguish much between $T(X)$ and $T^*(X)$ and will often identify them using the Riemannian metric of X. We will usually denote covectors on X by ξ. The tangent manifold will be denoted TX, and the projection $TX \to X$ also by p.

First, we recall from [30, Section 4] the definition of the elements $[\Theta_X]$ and $[d_X]$. We start with the algebra of Clifford sections.

Definition 2.1. Let us denote by $\mathcal{V}$ a real vector bundle over X equipped with a G-invariant Riemannian metric and by $\mathrm{Cliff}(\mathcal{V}, Q)$ the Clifford algebra bundle associated with the quadratic form $Q(v) = \|v\|^2$ on $\mathcal{V}$. We define the norm on fibers of this bundle by means of the embedding of $\mathrm{Cliff}(\mathcal{V}, Q)$ into the bundle of endomorphisms $\mathcal{L}(\Lambda^*(\mathcal{V}))$ of the exterior algebra of $\mathcal{V}$. The embedding is given on the fibers of $\mathcal{V} \subset \mathrm{Cliff}(\mathcal{V}, Q)$ by $v \mapsto \mathrm{ext}\,(v) + \mathrm{int}\,(v)$. Here ext is the exterior product operator and int its adjoint. Let $\mathrm{Cl}_{\mathcal{V}}(X)$ be the complexification of the algebra of continuous sections of $\mathrm{Cliff}(\mathcal{V}, Q)$ over X, vanishing at infinity of X. With the sup-norm on sections, this is a C^*-algebra. It will be denoted $\mathrm{Cl}_{\mathcal{V}}(X)$. Until Section 7 we will use only a particular case of this algebra when $\mathcal{V} = \tau = T^*(X)$. This algebra will be denoted $\mathrm{Cl}_\tau(X)$.

Definition 2.2. The Dirac element $[d_X] \in K^0_G(\mathrm{Cl}_\tau(X))$ is defined as follows. Let $H = L^2(\Lambda^*(X))$ be the Hilbert space of complex-valued L^2-forms on X graded by the even-odd form decomposition. The homomorphism $\mathrm{Cl}_\tau(X) \to \mathcal{L}(H)$ is given on (real) covector fields by the Clifford multiplication operators $v \mapsto \mathrm{ext}\,(v) + \mathrm{int}\,(v)$. The (unbounded) operator d_X is the operator of exterior derivation on H. The operator $D_X = d_X + d_X^*$ is essentially self-adjoint. The pair $(H, D_X(1 + D_X^2)^{-1/2})$ defines the Dirac element $[d_X]$ (cf. [30, 4.2]).

Another element that we need is the local dual Dirac element (cf. [30, 4.4]). In order to define it, we use a continuous family of convex open balls $\{U_x, x \in X\}$ in X such that in each set U_x, any two points are joined by a unique minimal geodesic. We also require that the radii r_x of these balls vary smoothly with $x \in X$ and the function $x \mapsto r_x$ is G-invariant. (See [32, 4.3.6] or [30, 4.3]. A G-invariant function r_x is constructed in [30], but can also be obtained by averaging any non-G-invariant r_x over G using the properness of the G-action on X.) This family of balls actually defines an open neighborhood U of the diagonal in

$$X \times X: \; U = \{(x, y) \in X \times X,\; \rho(x, y) < r_x\},$$

where ρ is the distance function. *We will always assume that both coordinate projections of the closure of U into X are proper maps.* By definition, U is G-invariant with respect to the diagonal action of G on $X \times X$.

Now we define a radial covector field Θ_x on U_x which at the point $y \in U_x$ is given by $\Theta_x(y) = \rho(x, y) d_y(\rho)(x, y)/r_x$, where d_y means the exterior derivative in y. (Note that $d_y(\rho)(x, y)$ is the covector dual to the unit tangent vector at the end of the geodesic joining x and y.) Clearly, the norm of Θ_x goes to 1 at the boundary of U_x, so the operator of Clifford multiplication by Θ_x, considered as an element of $\mathrm{Cl}_\tau(U_x)$, and denoted also by Θ_x, will have the property $\Theta_x^2 - 1 \in C_0(U_x)$.

As mentioned in the introduction, when B is a $C_0(X)$-algebra (see [30, Definition 1.5]), we will use notation $\mathcal{R}K(X;B)$ for the group $\mathcal{R}KK(X;C_0(X),B)$ defined in [30, 2.19].

Definition 2.3. The above family of Clifford multiplication operators, parametrized by $x \in X$, defines an element $[\Theta_{X,1}]$ of the group $\mathcal{R}K^G(X;C_0(X)\otimes \mathrm{Cl}_\tau(X))$, where $C_0(X)$ acts on $C_0(X)\otimes \mathrm{Cl}_\tau(X)$ by multiplication over the first tensor multiple (cf. [30, Definition 4.4]).

Actually $[\Theta_{X,1}] \in \mathcal{R}K^G(X;C_0(U)\cdot C_0(X)\otimes \mathrm{Cl}_\tau(X))$. It follows from the remark at the end of the Introduction that we may also consider this element as an element of the groups $\mathcal{R}K^G(X;C_0(U)\cdot C_0(X)\otimes \mathrm{Cl}_\tau(X))$ and $\mathcal{R}K^G(X;C_0(X)\otimes \mathrm{Cl}_\tau(X))$, where $C_0(X)$ acts on $C_0(X)\otimes \mathrm{Cl}_\tau(X)$ by multiplication over the second tensor multiple. In this case, we will denote this element $[\Theta_{X,2}]$.

An important property of the Dirac and the local dual Dirac elements is the following result [30, Theorem 4.8]:

Theorem 2.4. $[\Theta_{X,1}]\otimes_{\mathrm{Cl}_\tau(X)}[d_X] = 1_X \in \mathcal{R}K^G(X;C_0(X)) = RK^0_G(X)$.

Another important fact that we need is that the C^*-algebras $C_0(T^*(X))$ and $\mathrm{Cl}_\tau(X)$ are KK-equivalent. This KK-equivalence (which may be called the Thom isomorphism (cf. [26, Theorem 5.8]) is implemented by the fiberwise Dirac and Bott elements. For any $x \in X$, we will denote the Clifford algebra fiber of $\mathrm{Cl}_\tau(X)$ at the point x by Cl_{τ_x}.

Definition 2.5. For any $x \in X$, consider the Hilbert module $H_x = L^2(T^*_x(X))\otimes \mathrm{Cl}_{\tau_x}$ over Cl_{τ_x}. The Dirac operator on this Hilbert module is defined by

$$D_x = \sum_{k=1}^{\dim X}(-i)c(e_k)\partial/\partial\xi_k$$

in any orthonormal basis $\{e_k\}$ of $T^*_x(X)$, where $c(e_k)$ are the left Clifford multiplication operators and ξ_k are the coordinates of $T^*_x(X)$ in the basis $\{e_k\}$. We denote by F_x the operator $D_x/(1+D_x^2)^{-1/2}$ on H_x. The family of Hilbert modules H_x, parametrized by $x \in X$, defines a Hilbert module of continuous sections over the algebra $\mathrm{Cl}_\tau(X)$, vanishing at infinity of X, which will be denoted by $\mathcal{E}$. The grading of $\mathcal{E}$ is defined by the grading of the algebra $\mathrm{Cl}_\tau(X)$. The family of operators F_x gives an operator Φ of degree 1 on $\mathcal{E}$. The algebra $C_0(TX)$ acts on $\mathcal{E}$ on the left by multiplication. The pair $(\mathcal{E},\Phi)$ is an element of the group $\mathcal{R}KK^G(X;C_0(TX),\mathrm{Cl}_\tau(X))$. We will denote it $[d_\xi]$ and call the fiberwise Dirac element.

Definition 2.6. Let $\Lambda^*_{\mathbf{C}}(TX)$ be the bundle $p^*(\Lambda^*(X))\otimes \mathbf{C}$ over the manifold TX and $C_0(\Lambda^*_{\mathbf{C}}(TX))$ the Hilbert module of continuous sections of this bundle vanishing at infinity of TX. The left action of $\mathrm{Cl}_\tau(X)$ on $C_0(\Lambda^*_{\mathbf{C}}(TX))$ is given on (real) covector fields by the Clifford multiplication operators $v \mapsto \operatorname{ext}(v) + \operatorname{int}(v)$, and the operator β is the Bott operator given by

$$\beta(\xi) = i(\operatorname{ext}(\xi) - \operatorname{int}(\xi))/(1 + \|\xi\|^2)^{1/2},$$

for any $(x, \xi) \in TX$. The pair $(C_0(\Lambda^*_{\mathbf{C}}(TX)), \beta)$ defines an element

$$[\mathcal{B}_\xi] \in \mathcal{R}KK^G(X; \mathrm{Cl}_\tau(X), C_0(TX)).$$

It will be called the fiberwise Bott element.

Theorem 2.7. *We have the following relations:*

(1) $[d_\xi] \otimes_{\mathrm{Cl}_\tau(X)} [\mathcal{B}_\xi] = 1_{C_0(TX)}$.

(2) $[\mathcal{B}_\xi] \otimes_{C_0(TX)} [d_\xi] = 1_{\mathrm{Cl}_\tau(X)}$.

Proof. We will consider the products fiberwise over X. Let $\Lambda^*_{\mathbf{C}}(T_x(X))$ be the complexified exterior algebra of $T_x(X)$. The Hilbert module over Cl_{τ_x} for the product $[\mathcal{B}_\xi] \otimes_{C_0(T^*(X))} [d_\xi]$ is

$$\mathfrak{H}_x = L^2(T^*_x(X)) \otimes \Lambda^*_{\mathbf{C}}(T^*_x(X)) \hat{\otimes} \mathrm{Cl}_{\tau_x},$$

equipped with the left action of Cl_{τ_x} by Clifford multiplication over the tensor multiple $\Lambda^*_{\mathbf{C}}(T^*_x(X))$. The operator can be written as

$$1 \hat{\otimes} (1 + D_x^2)^{-1/2} \cdot \frac{i(\operatorname{ext}(\xi) - \operatorname{int}(\xi)) \hat{\otimes} 1}{(1 + \|\xi\|^2)^{1/2}} + (1 \hat{\otimes} F_x).$$

We use the $\hat{\otimes}$ sign in the formulas here only to distinguish between different Clifford multiplication operators acting over the two tensor multiples of $\Lambda^*_{\mathbf{C}}(T^*_x(X)) \hat{\otimes} \mathrm{Cl}_{\tau_x}$.

There exists a natural left action of the Clifford algebra $\mathrm{Cl}_{\tau_x} \hat{\otimes} \mathrm{Cl}_{\tau_x}$ on $\Lambda^*_{\mathbf{C}}(T^*_x(X)) \hat{\otimes} \mathrm{Cl}_{\tau_x}$. Using the rotation of $T^*_x(X) \oplus T^*_x(X)$ which flips the two copies of $T^*_x(X)$, we define the rotation homotopy of this action. The formula for the rotation is

$$(x, y) \mapsto (\cos t \cdot x - \sin t \cdot y,\ \sin t \cdot x + \cos t \cdot y), \quad 0 \le t \le \pi/2.$$

In this way we obtain the rotation homotopy of the identity automorphism of the algebra $\mathrm{Cl}_{\tau_x} \hat{\otimes} \mathrm{Cl}_{\tau_x}$ into the automorphism $\gamma_{\pi/2}$ which on the subspace $T^*_x(X) \oplus T^*_x(X)$ sends $(x, y) \mapsto (-y, x)$.

We apply this homotopy to our KK-product element. The automorphism $\gamma_{\pi/2}$ transforms the left action of the algebra Cl_{τ_x} on $\Lambda^*_{\mathbf{C}}(T_x(X))$ into the identity action of Cl_{τ_x} on itself. The Dirac operator $D_x = \sum_{k=1}^{\dim X} (-i) c(e_k) \partial/\partial\xi_k$ transforms into $\sum_{k=1}^{\dim X} i(\operatorname{ext}(e_k) + \operatorname{int}(e_k))\partial/\partial\xi_k$. The $i(\operatorname{ext}(\xi) - \operatorname{int}(\xi)) \hat{\otimes} 1$ part stays unchanged (we do not apply any homotopy to it).

As a result of this homotopy, our operator transforms into the operator of the form $S_1 \hat{\otimes} 1$ on the Hilbert module $L^2(\Lambda^*(T_x^*(X))) \hat{\otimes} \mathrm{Cl}_{\tau_x}$. The operator S_1 is an index 1 operator on $L^2(\Lambda^*(T_x^*(X)))$, so our KK-product is equal to $1_{\mathrm{Cl}_\tau(X)}$ in the group $\mathcal{R}KK^G(X; \mathrm{Cl}_\tau(X), \mathrm{Cl}_\tau(X))$.

The product of the fiberwise Bott and Dirac elements in the opposite order is also the identity by a similar rotation trick. □

Next we need to define the Dolbeault and the local Bott elements for the manifold TX. The manifold TX is an almost complex manifold: the cotangent space $T^*_{(x,\xi)}(TX)$ at any point (x, ξ) is the direct sum of the two orthogonal subspaces isomorphic to $T_x^*(X)$: the horizontal space $p^*(T_x^*(X))$ and the space dual to the vertical tangent space $T_\xi(T_x(X))$. This allows to define the complex structure on the tangent space of the manifold TX for any $(x, \xi) \in TX$. We will identify the complex cotangent bundle $T^*(TX)$ with the complexification of the lifted cotangent bundle of X : $p^*(T^*(X)) \otimes \mathbf{C}$. The exterior algebra of this bundle was already denoted by $\Lambda^*_{\mathbf{C}}(TX)$ in 2.6.

Since the manifold TX is an almost complex manifold, it is in particular a Spin^c-manifold. The canonical Dirac operator $\mathcal{D}$ on TX is called the Dolbeault operator. The coordinates in the cotangent space $T^*_{(x,\xi)}(TX)$ will be denoted by (η, ζ).

Definition 2.8. We define the symbol of the Dolbeault operator $\mathcal{D}$ as

$$\sigma_{\mathcal{D}}(x, \xi, \eta, \zeta) = (\mathrm{ext}\,\zeta + \mathrm{int}\,\zeta) + i(\mathrm{ext}\,\eta - \mathrm{int}\,\eta)$$

on the exterior algebra bundle $\Lambda^*_{\mathbf{C}}(TX)$. The grading of $\Lambda^*_{\mathbf{C}}(TX)$ is given by the even-odd form decomposition. $\mathcal{D}$ is a G-invariant first order differential operator which acts on sections of the bundle $\Lambda^*_{\mathbf{C}}(TX)$. It has degree 1 with respect to the grading. When considered as an operator on L^2-sections, $\mathcal{D}$ is essentially self-adjoint (see [40]). The operator $F = \mathcal{D}/(1 + \mathcal{D}^2)^{-1/2}$ is bounded. The algebra $C_0(TX)$ acts on $L^2(\Lambda^*_{\mathbf{C}}(TX))$ by multiplication. The pair $(L^2(\Lambda^*_{\mathbf{C}}(TX)), F)$ defines the K-homology class $[\mathcal{D}_X] \in K^0_G(C_0(TX))$.

The property that $f \cdot (1 + \mathcal{D}^2)^{-1/2}$ is a compact operator for any $f \in C_0(TX)$ follows from the Rellich lemma. For the commutator property between f and F, we use the standard reasoning of the proof of Lemma 4.2 [30].

Note that the algebras $C_0(TX)$ and $\mathrm{Cl}_\tau(TX)$ are Morita equivalent, or better to say, related by Clifford periodicity. So we could use the element $[d_{TX}]$ instead of the Dolbeault element, although this is less convenient.

Definition 2.9. Set $[\mathcal{B}_{X,k}] = [\Theta_{X,k}] \otimes_{\mathrm{Cl}_\tau(X)} [\mathcal{B}_\xi] \in \mathcal{R}K^G(X; C_0(X) \otimes C_0(TX))$ for $k = 1, 2$. These are the local Bott elements corresponding to the above defined elements $[\Theta_{X,1}]$ and $[\Theta_{X,2}]$. The action of $C_0(X)$ on $C_0(X) \otimes C_0(TX)$ goes over the k-th tensor multiple.

A concrete realization of the Bott elements goes as follows. Consider the family of tangent manifolds $\{TU_x\} \subset TX$ parametrized by $x \in X$. The coordinates in each TU_x are (y,ξ), where $y \in U_x$ and ξ a tangent vector at y. Each U_x is homeomorphic to a ball in $\mathbf{R}^n$, so TU_x is homeomorphic to $\mathbf{C}^n$. The natural Bott element in TU_x is defined by the Bott operator

$$\mathcal{B}_{TU_x} = (\mathrm{ext}\,(\Theta_x(y)) + \mathrm{int}\,(\Theta_x(y))) \\ + i(1-\|\Theta_x(y)\|^2)^{1/2}(1+\|\xi\|^2)^{-1/2}(\mathrm{ext}\,\xi - \mathrm{int}\,\xi)$$

acting on sections of the complex exterior algebra bundle $\Lambda^*_{\mathbf{C}}(TU_x)$ over TU_x. The operator $\mathcal{B}_{TU_x}$ has degree 1 with respect to the natural grading of $\Lambda^*_{\mathbf{C}}(TU_x)$, and it is clear that $1-(\mathcal{B}_{TU_x})^2$ is an operator of multiplication by a function which vanishes at infinity of TU_x. The family of these pointwise Bott elements defines the Bott element of the group $\mathcal{R}K^G(X; C_0(X) \otimes C_0(TX))$.

Theorem 2.10. *We have the following relations:*

(1) $[\mathcal{D}_X] = [d_\xi] \otimes_{\mathrm{Cl}_\tau(X)} [d_X]$.

(2) $[\mathcal{B}_{X,1}] \otimes_{C_0(TX)} [\mathcal{D}_X] = 1_X \in \mathcal{R}K^G(X; C_0(X)) = RK^0_G(X)$.

Proof. For the proof of the first relation, see the proof of formula (1) in the proof of Theorem 5.4 [30, pp. 186–187]. The second relation follows from the first one together with Theorems 2.4, 2.7 and Definition 2.9. □

3. Elliptic operators

This section contains some known facts and notation related with elliptic operators. At the end of the section, we give a definition of the Clifford symbol. Recall that X is a complete Riemannian manifold, $T(X)$ its tangent bundle, $T^*(X)$ the cotangent bundle, and $p : T^*(X) \to X$ the projection. Covectors on X are usually denoted by ξ.

We will consider pseudo-differential operators on X, with proper support, of the Hörmander class ρ, δ (see [22]), under the usual assumptions: $1-\rho \le \delta < \rho \le 1$. The assumption $1-\rho \le \delta$ is actually important only for the good behavior of this class of operators under diffeomorphisms and for the construction of pseudo-differential operators on manifolds. It is usually not needed for operators on a Euclidean space.

Our main results will be stated and proved for the class of $\rho = 1, \delta = 0$ *operators.* However, for technical reasons, in Section 8 we will use the case of $\rho < 1, \delta = 0$ operators on a Euclidean space.

The usual definition of pseudo-differential operators on the manifold X goes as follows (cf. [22, 36]) . First one defines operators of order m on $\mathbf{R}^n$ by the formula:

$$Au(x) = (2\pi)^{-n} \int_{\mathbf{R}^n \times \mathbf{R}^n} e^{i(x-y,\xi)} a(x,y,\xi) u(y)\, dy d\xi.$$

The symbol function a of an operator of order m belongs to the Hörmander class ρ, δ if for any compact subset $K \subset \mathbf{R}^n$ and any multi-indices α, β, γ, the function a satisfies the following condition:

$$\|\partial^\beta/\partial x^\beta \quad \partial^\gamma/\partial y^\gamma \quad \partial^\alpha/\partial\xi^\alpha \quad a(x,y,\xi)\| \le C_{\alpha,\beta,\gamma,K}(1+\|\xi\|)^{m-\rho|\alpha|+\delta|\beta|+\delta|\gamma|}$$

for $x, y \in K$, with some constants $C_{\alpha,\beta,\gamma,K}$ which depend on a and K. An operator belongs to the class $\Psi^m_{\rho,\delta}(\mathbf{R}^n)$ of all pseudo-differential operators of order m on $\mathbf{R}^n$ if it is a sum of an operator given by the above formula and an integral operator with a smooth kernel.

An operator $A : C_c^\infty(X) \to C^\infty(X)$ belongs to $\Psi^m_{\rho,\delta}(X)$ if its restriction to any open subset X_1 of X diffeomorphic to $\mathbf{R}^n$ belongs to $\Psi^m_{\rho,\delta}(X_1)$, and the distributional kernel of A is smooth outside the diagonal of $X \times X$. The operator A has *proper support* if the projections of the support of the distributional kernel of A to both factors of $X \times X$ are proper maps.

In this paper we will use a coordinate-free presentation of pseudo-differential operators $A : C_c^\infty(X) \to C_c^\infty(X)$ in the form (cf. [22, 2.3]):

$$Au(x) = (2\pi)^{-\dim X} \int_{T_x^*(X)} \int_X e^{i\Phi(x,y,\xi_x)} a(x,y,\xi_x) u(y)\, \mathrm{d}y \mathrm{d}\xi_x, \tag{1}$$

where Φ is the phase function, a the symbol function, ξ_x a covector at $x \in X$. The symbol function $a \in C^\infty(T^*(X) \times X)$ is assumed to be 0 for all $(x, y) \in X \times X$ outside of the neighborhood U of the diagonal of $X \times X$ (we will use the same neighborhood U as in Section 2). So we need to define the phase function Φ for $(x, y) \in X \times X$ only in this neighborhood U. Note that U maps diffeomorphically onto a neighborhood of the zero section of $T(X)$ via the inverse of the exponential map. Under this identification the point $(x, y) \in X \times X$ corresponds to the vector $\overrightarrow{v(y,x)} \in T_x(X)$ (like $x - y$ in $\mathbf{R}^n$), and we define the phase function by $\Phi(x, y, \xi_x) = \langle \overrightarrow{v(y,x)}, \xi_x \rangle$. The integral in the formula (1) is taken first over the y variable (over $y \in U_x$) and then over $\xi_x \in T_x^*(X)$.

An important remark is that any pseudo-differential operator on X can be presented as a sum of an operator given by the formula (1) and an integral operator with smooth kernel. An operator (1) with the symbol function which is 0 in a neighborhood of the diagonal $x = y$ is an integral operator with smooth kernel. In particular, the operator (1) depends on the choice of the neighborhood U only up to integral operators with smooth kernel. Also, if the symbol function a of an operator of order m is 0 on the diagonal $x = y$, then this operator actually has order $\le m + \delta - \rho$ (i.e. there exists another symbol function of lower order); see [36, Proposition 2.4].

The symbol of a properly supported pseudo-differential operator A given by formula (1) is defined in a sufficiently small Euclidean neighborhood by

$$\sigma_A(x,\xi) = \exp(-i(x,\xi)) A \exp(i(x,\xi)).$$

The symbol σ_A belongs to $S^m_{\rho,\delta}(X)$, the linear space of symbols of order m operators of the Hörmander class ρ, δ, if it satisfies in any Euclidean neighborhood a similar condition as the symbol function a above, namely:

$$\|\partial^\beta/\partial x^\beta \;\; \partial^\alpha/\partial \xi^\alpha \;\; \sigma_A(x,\xi)\| \le C_{\alpha,\beta,K}(1+\|\xi\|)^{m-\rho|\alpha|+\delta|\beta|},$$

for any compact subset $K \subset X$ and all $x \in K$, with some constants $C_{\alpha,\beta,K}$ which depend on σ_A and K. The symbol is well defined modulo the subspace of lower order symbols $S^{m-\rho+\delta}_{\rho,\delta}(X)$, and modulo the same subspace, one has: $\sigma_A(x,\xi) = a(x,x,\xi)$.

To construct an operator A from its symbol, let $p_1 : TX \times X \to TX$ be the projection onto the first factor. Let us choose a function χ on $X \times X$ such that $0 \le \chi \le 1$; $\chi(x,y) = 1$ for all (x,y) in a neighborhood U_1 of the diagonal, $U_1 \subset U$; and $\chi = 0$ outside of the neighborhood U. We define $a(x,y,\xi) = \chi(x,y)p_1^*(\sigma)(x,y,\xi)$ and use this function a as the symbol function in the definition of the operator A.

Usually it is necessary to consider pseudo-differential operators on vector bundles over X. We will assume that all vector bundles are equipped with Hermitian metric. Let E^0 and E^1 be vector bundles over X. The symbol function a is a continuous section of the external tensor product bundle $p^*(E^1) \otimes (E^0)^*$ over $TX \times X$, where $(E^0)^*$ means the Hermitian dual of E^0. The symbol σ_A of the operator A is a section of the bundle $p^*(E^1 \otimes (E^0)^*)$, where $E^1 \otimes (E^0)^*$ is now the internal tensor product (of vector bundles over X). The symbol will be considered as a homomorphism of vector bundles: $p^*(E^0) \to p^*(E^1)$. The local inequalities defining the ρ, δ class of operators remain the same as above.

We will consider a natural generalization of the above to the equivariant setting with respect to an action of a locally compact group G. *We will always assume that G acts properly on X and isometrically on X and on all vector bundles. The pseudo-differential operator A will usually be assumed G-invariant (except Section 9 where no G-action is required).*

Note that if the symbol σ_A is G-invariant, there is a G-invariant pseudo-differential operator A with the symbol σ_A. To construct such operator, one has to use the G-invariant phase and symbol functions (as well as the function χ above) in the integral formula (1) which defines A. This is always possible since G acts on X properly and isometrically.

We turn now to the questions of boundedness and compactness of our operators on L^2-spaces. Since we do not assume compactness of X or X/G, a pseudo-differential operator A of order zero, as it is defined above, can be unbounded as an operator on L^2-sections. (Boundedness of order zero G-invariant operators in the case when X/G is compact will be proved in Section 5.) However, we will show below (Proposition 3.5) that there always exists a bounded operator with the symbol which differs only by lower order symbols. In the statements of our main results, we will usually be making an assumption of boundedness. This will be enough to ensure that the analytical index is well defined (Lemma 3.7).

We will state now some known results on L^2-boundedness and compactness of pseudo-differential operators. We start with the following well known result (cf. [23, 18.1.12]):

Lemma 3.1. *An integral operator with a continuous kernel k, which has the property that both $\int_X \|k(x,y)\|\mathrm{d}x$ and $\int_X \|k(x,y)\|\mathrm{d}y$ are bounded by a constant $C_1 > 0$, is a bounded operator of norm $\leq C_1$ on $L^2(X)$.*

The next theorem [22, Theorem 2.2.1] is most useful in all applications related with L^2-boundedness:

Theorem 3.2. *Let P be a properly supported pseudo-differential operator of order 0, and suppose that its symbol σ_P is bounded at infinity in the cotangent direction by a constant $C > 0$, i.e. for any compact subset $K \subset X$ and any $x \in K$,*

$$\limsup_{\xi\to\infty} \|\sigma_P(x,\xi)\| < C.$$

Then there are another pseudo-differential operator B of order 0 with proper support and a self-adjoint integral operator R with continuous kernel and proper support such that

$$P^*P + B^*B - C^2 = R. \tag{$*$}$$

Moreover, if the support of the distributional kernel of P is compact in $X \times X$, then the operators B and R will also have distributional kernels with compact supports. In this case all operators P, B, R are bounded.

The assertion follows from the following proposition ([22, Proposition 2.2.2]; [36, Proposition 6.1]).

Proposition 3.3. *Let Q be a properly supported pseudo-differential operator of order 0, such that $Q^* = Q$, and for any compact subset $K \subset X$,*

$$\lim_{\|\xi\|\to\infty} \inf_{x\in K} \operatorname{Re}\sigma_Q(x,\xi) > 0.$$

Then there exists a properly supported pseudo-differential operator B of order 0 such that $R = B^*B - Q$ is an integral operator with continuous kernel.

Since we will need some technical details of this result, we include a sketch of proof.

Sketch of proof. The operator B is constructed as a sum: $\sum_{j=0}^m B_j$, where the operators B_j of order $j(\delta-\rho)$ are defined inductively. At the beginning of the induction, one finds an order 0 pseudo-differential operator B_0 such that $Q - B_0^*B_0$ has order $\delta - \rho$. The symbol $\sigma_{B_0}(x,\xi)$ is real-valued. If $B_0,\dots,B_{j-1}$ are already constructed so that $D_j = \sum_{k=0}^{j-1} B_k$ satisfies the condition that $R_j = Q - D_j^*D_j$

has order $j(\delta-\rho)$, one can choose the operator B_j with the symbol which satisfies the equation $2\sigma_{B_j}(x,\xi)\sigma_{B_0}(x,\xi)=\sigma_{R_j}(x,\xi)$ for large ξ.

We do not need an infinite induction as in [22] and an infinite asymptotic sum of B_j's. We can stop as soon as the remainder R_j becomes an integral operator with continuous kernel. This happens when the order of R_j becomes less than $-\dim X$. This operator R_j will be taken as R. □

Since we consider G-invariant operators, we also need the following improvement of Theorem 3.2:

Lemma 3.4. *In the assumptions of Theorem 3.2, let the operator P be G-invariant. Then the operators B and R can also be chosen G-invariant.*

Proof. Since the G-action is proper, we can use the following averaging procedure for the symbol: $\tilde{\sigma}_A=\int_G g(\mathfrak{c})g(\sigma_A)\mathrm{d}g$. Here $\mathfrak{c}$ is a cut-off function on X, i.e. a smooth non-negative function such that the support of $\mathfrak{c}$ has compact intersection with any G-compact subset of X, and for any $x\in X$, $\int_G \mathfrak{c}(g^{-1}x)\mathrm{d}g=1$. (The notation $g(\mathfrak{c})(x)$ means $\mathfrak{c}(g^{-1}x)$, and $g(\sigma_A)=g\sigma_A g^{-1}$.)

Now we will use the same inductive procedure as in the proof of Proposition 3.3, sketched above, with the following modification: each time the successive symbol σ_{B_i} has been defined, we first average it as explained above (which will not change σ_{B_i} modulo lower order symbols), and then use the integral formula (1) to construct the operator B_i. This will ensure that all operators B_i are G-invariant, and so B and R will also be G-invariant. □

Proposition 3.5. *Let A be a properly supported G-invariant pseudo-differential operator of order 0 on a vector bundle E over X defined by formula (1). Assume that A satisfies the assumption of Theorem 3.2 on its symbol. Then there exists an L^2-bounded G-invariant operator A' on E such that the symbol of $A-A'$ belongs to $S^{\delta-\rho}_{\rho,\delta}(E)$.*

Proof. Using Theorem 3.2 and Lemma 3.4, we obtain properly supported G-invariant operators B and R such that $A^*A+B^*B-C^2=R$. Here R is an integral operator with a continuous kernel $k(x,y)$.

To simplify the proof, we will first consider two special cases. If we forget about the group action, we can take a covering $\{U_i\}$ of X of finite covering dimension and the corresponding partition of unity $\sum_i \alpha_i^2=1$. Let us replace the operator A with $A'=\sum_i \alpha_i A\alpha_i$. Using the operator inequality $b^*dc+c^*db\le b^*db+c^*dc$ for operators b,c,d with $d\ge 0$, it is easy to show that $\|A'^*A'\|\le \mathrm{const}\|\sum_i \alpha_i A^*A\alpha_i\|$, where the constant depends only on the covering dimension. By Theorem 3.2, the norm $\|\sum_i \alpha_i A^*A\alpha_i\|$ does not exceed $C^2+\|\sum_i \alpha_i R\alpha_i\|$. The latter norm can be evaluated using Lemma 3.1, and can be made arbitrarily small if the covering is sufficiently small. The symbol of the operator A' modulo symbols of order $\delta-\rho$ is the same as the symbol of A because $\sum_i \alpha_i^2=1$.

Next, let us consider the case when X is a homogeneous space G/M, where M is a compact subgroup. Let $U = V \cdot (M)$ be a neighborhood of the point (M), where V is a small neighborhood of the identity in G. Take a smooth function ϕ_U on X with the support in U, such that $0 \le \phi_U \le 1$. We require that the pullback of ϕ_U to G must be a function which takes value 1 on a subset of VM of at least half of the measure of the set VM. Put $\mathfrak{c}_U = \phi_U^{1/2}(\int_G g(\phi_U)\,\mathrm{d}g)^{-1/2}$. Then $\int_G g(\mathfrak{c}_U)^2\,\mathrm{d}g = 1$. Note that

$$\int_G \mathfrak{c}_U g(\mathfrak{c}_U)\,\mathrm{d}g = \int_G (\phi_U)^{1/2} g(\phi_U)^{1/2}\,\mathrm{d}g \cdot \left(\int_G g(\phi_U)\,\mathrm{d}g\right)^{-1}$$
$$\le \int_G g(\phi_U)^{1/2}\,\mathrm{d}g \cdot \left(\int_G g(\phi_U)\,\mathrm{d}g\right)^{-1},$$

because $\phi_U \le 1$. Both functions $\int_G g(\phi_U)\,\mathrm{d}g$ and $\int_G g(\phi_U)^{1/2}\,\mathrm{d}g$ are constant functions on X. Calculating them at the point $x = (M)$, we see that in both cases the integral goes only over the set $\{g \in G \mid g^{-1} \in VM\}$, so by the above assumption on the value 1 of the function ϕ_U, the integral $\int_G \mathfrak{c}_U g(\mathfrak{c}_U)\,\mathrm{d}g$ is bounded by a constant which does not depend of the choice of the neighborhood V, if V is small enough.

Now we define $A' = \int_G g(\mathfrak{c}_U) A g(\mathfrak{c}_U)\,\mathrm{d}g$. The operator inequality already used in the first special case, and the above bound on $\int_G \mathfrak{c}_U g(\mathfrak{c}_U)\,\mathrm{d}g$ imply the estimate: $\|A'^* A'\| \le \mathrm{const}\|\int_G g(\mathfrak{c}_U) A^* A g(\mathfrak{c}_U)\,\mathrm{d}g\|$. By Theorem 3.2, the norm $\|\int_G g(\mathfrak{c}_U) A^* A g(\mathfrak{c}_U)\,\mathrm{d}g\|$ does not exceed $C^2 + \|\int_G g(\mathfrak{c}_U) R g(\mathfrak{c}_U)\,\mathrm{d}g\|$. The latter norm can be evaluated using Lemma 3.1. Let $k(x, y)$ be the kernel of the operator R. Then the kernel of $\int_G g(\mathfrak{c}_U) R g(\mathfrak{c}_U)\,\mathrm{d}g$ at the point (x, y) is $\int_G \mathfrak{c}_U(g^{-1}x) k(x, y) \mathfrak{c}_U(g^{-1}y)\,\mathrm{d}g$. The integral $h_U(x, y) = \int_G \mathfrak{c}_U(g^{-1}x)\mathfrak{c}_U(g^{-1}y)\,\mathrm{d}g$ is a G-invariant function on $X \times X$ (under the diagonal action), which vanishes outside of $G \cdot (U \times U)$. We have:

$$h_U(x, y) \le \left(\int_G (\mathfrak{c}_U^2(g^{-1}x)\,\mathrm{d}g\right)^{1/2} \cdot \left(\int_G (\mathfrak{c}_U^2(g^{-1}y)\,\mathrm{d}g\right)^{1/2} = 1$$

(by the Cauchy–Schwarz inequality). The function $k(x, y)$ is also a G-invariant continuous function on $X \times X$. Now it is clear that the product $h_U(x, y)k(x, y)$ satisfies the assumptions of Lemma 3.1 with the bound C_1 arbitrarily small if V is small enough. The symbol of the operator A' modulo symbols of order $\delta - \rho$ is the same as the symbol of A because $\int_G g(\mathfrak{c}_U)^2\,\mathrm{d}g = 1$.

In the general case, the proof is a combination of the previous two special cases. We consider a covering $\{W_i\}$ of X/G of finite covering dimension and the corresponding partition of unity $\sum_i \alpha_i^2 = 1$ on X/G. We may assume that each W_i corresponds to a small tubular neighborhood of an orbit in X (and will denote this tubular neighborhood by W_i). This tubular neighborhood is diffeomorphic to $G \times_{M_i} Z_i$, where M_i is a stability subgroup of G at some point x_i, and Z_i is a small open ball in X orthogonal to the orbit at this point. Such tubular neighborhood has a natural projection onto G/M_i. Now we choose a small neighborhood U_i as in the second special case above and construct a function $\mathfrak{c}_{U_i}$ as it was done in that special case. We lift this function $\mathfrak{c}_{U_i}$ to the tubular neighborhood W_i using the natural projection. The operator A' is defined by the formula:

$$A' = \sum_i \alpha_i A'_i \alpha_i, \quad \text{where } A'_i = \int_G g(\mathfrak{c}_{U_i}) A g(\mathfrak{c}_{U_i}) \, \mathrm{d}g.$$

If all W_i and all U_i are small enough, the reasoning in the second special case will give an upper bound for the norm of each A'_i independent of i. Arguing as in the first special case, the obtain boundedness of A'. The symbol of A' coincides with the symbol of A modulo symbols of order $\delta - \rho$. □

A slightly different treatment of pseudo-differential calculus is given in [23, Section 18.1] in the case when the manifold X is the Euclidean space $\mathbf{R}^n$. The class of $\rho = 1, \delta = 0$ symbols of order m is defined globally by the condition:

$$\|\partial^\beta/\partial x^\beta \quad \partial^\alpha/\partial \xi^\alpha \quad \sigma_A(x,\xi)\| \leq C_{\alpha,\beta}(1 + \|\xi\|)^{m-|\alpha|},$$

for all $x, \xi \in \mathbf{R}^n$, with some constants $C_{\alpha,\beta}$ which depend on σ_A.

This stronger assumption allows to prove L^2-boundedness of order 0 pseudo-differential operators defined by the formula:

$$Au(x) = (2\pi)^{-n} \int\int e^{i(x-y,\xi)} \sigma_A(x,\xi) u(y) \, \mathrm{d}y \mathrm{d}\xi$$

(see [23, Theorem 18.1.11]). The proof of boundedness is very similar to the proof of Theorem 3.2, but it uses the additional fact that integral kernels of operators of order $-(n+1)$ satisfy the assumptions of Lemma 3.1. This is a consequence of the stronger assumption on the class of operators.

An important feature of this approach is that *no properness of support is required for this class of operators*. We will use this class of operators in the proof of index theorems in Sections 4 and 8. Also we will need the following theorem [23, Theorem 18.1.11′] which ensures L^2-boundedness for pseudo-differential operators on a Euclidean space with an explicit estimate of the norm (and without the assumption of properness of support, or even the ρ, δ class condition).

Theorem 3.6. *Let $\sigma(x,\xi)$ be a measurable function on $\mathbf{R}^n \times \mathbf{R}^n$ which is smooth as a function of x for any fixed ξ. If for some $M < \infty$,*

$$\sum_{|\alpha| \leq n+1} \int |\partial^\alpha/\partial x^\alpha \, \sigma(x,\xi)| \, \mathrm{d}x \leq M$$

for any $\xi \in \mathbf{R}^n$, then the pseudo-differential operator

$$Au(x) = (2\pi)^{-n} \int\int e^{i(x-y,\xi)} \sigma(x,\xi) u(y) \, \mathrm{d}y \mathrm{d}\xi$$

is bounded on $L^2(\mathbf{R}^n)$ with the norm $\leq \mathrm{const} \cdot M$.

We come now to ellipticity. *Suppose the operator* A *has order* 0. *We will call* A elliptic *if* $\|\sigma_A(x,\xi)\sigma_A^*(x,\xi)-1\| \to 0$ *and* $\|\sigma_A^*(x,\xi)\sigma_A(x,\xi)-1\| \to 0$ *uniformly in* $x \in X$ *on compact subsets of* X *as* $\xi \to \infty$ *in* TX. *Here* σ_A^* *means the Hermitian adjoint symbol.*

Remark. Our definition of ellipticity corresponds to the requirement that the symbol must be unitary at infinity in ξ. The usual requirement is just invertibilily of the symbol at infinity in ξ. However, if σ_A is invertible at infinity in ξ, it can be normalized by replacing it with $(\sigma_A\sigma_A^*)^{-1/2}\sigma_A$. The new symbol is unitary at infinity in ξ and homotopic to the initial one.

In dealing with elliptic operators, we will consider $E = E^0 \oplus E^1$ as a $\mathbf{Z}_2$-graded bundle. We replace our initial operator A with a self-adjoint (grading) degree 1 operator $\left(\begin{smallmatrix} 0 & A^* \\ A & 0 \end{smallmatrix}\right)$ on $E = E^0 \oplus E^1$. This new operator will now be called A. The condition of ellipticity becomes: $\|\sigma_A^2(x,\xi)-1\| \to 0$ uniformly in x on compact subsets of X when $\xi \to \infty$. The space of continuous sections of the vector bundle $p^*(E)$ over TX vanishing at infinity will be denoted by $C_0(p^*(E))$. The algebra $C_0(X)$ acts on $C_0(p^*(E))$ by multiplication.

The pair $(C_0(p^*(E)),\sigma_A)$ *defines an element* $[\sigma_A] \in \mathcal{R}K_0(X;C_0(TX))$ *which we will call the* K*-theoretic symbol of* A. In the G-equivariant situation we get an element $[\sigma_A] \in \mathcal{R}K_0^G(X;C_0(TX))$.

On the other hand, if an operator A acting on a non-graded vector bundle E is self-adjoint, then the same construction gives an element $[\sigma_A] = (C_0(p^*(E)),\sigma_A) \in \mathcal{R}K_1^G(X;C_0(TX))$. These two cases lead to two different index theorems. We will usually state and prove our index theorems for the graded case. The ungraded case is similar.

All our operators act on *complex* vector bundles. By considering operators on real vector bundles or on complex vector bundles with a Real involution, one can obtain 8 cases of an index theorem.

We come now to the definition of the K-homological analytical index of an elliptic operator. Whether $L^2(E)$ is a graded or a non-graded Hilbert space, the algebra $C_0(X)$ acts on $L^2(E)$ by pointwise multiplication.

Lemma 3.7. *Assume that an elliptic operator* A *of order 0 is self-adjoint, properly supported, and* L^2*-bounded. Then the multiplication operators* $a \in C_0(X)$ *commute with* A *modulo compact operators, and for any* $a \in C_0(X)$*, the operator* $a(1-A^2)$ *is compact. So there is an index element*

$$[A] = (L^2(E),A) \in K_G^*(C_0(X)).$$

(Here K^* *means* K^0 *or* K^1 *depending on whether we consider the graded or the non-graded case.)*

Proof. The proof is based on Theorem 3.2. Since the operator A is assumed to be bounded, it is enough to prove compactness of operators $Aa - aA$ and $a(1-A^2)$

for functions $a \in C_0(X)$ with compact support. The commutator $Aa - aA$ is a pseudo-differential operator of negative order. Taking any of the two operators $Aa - aA$ or $a(1 - A^2)$ as P, we see that the operator P satisfies the assumptions of Theorem 3.2, including the compactness of support of its distributional kernel, and with the constant C arbitrary small. The kernel of the operator R in this case has compact support in $X \times X$, therefore the operator R is compact. This implies compactness of the operator P, and therefore compactness of $Aa - aA$ and $a(1 - A^2)$. □

Remark. The index element $[A]$ coincides with the index element $[A']$ for the operator A' constructed in Proposition 3.5 because the difference $A - A'$ is of negative order.

We end this section with the definition of a "Clifford symbol" of an elliptic operator.

Definition 3.8. The Clifford symbol of an order 0 elliptic G-invariant operator A on the manifold X is defined as

$$[\sigma_A^{\mathrm{cl}}] = [\sigma_A] \otimes_{C_0(TX)} [d_\xi] \in \mathcal{R}K_*^G(X; \mathrm{Cl}_\tau(X)).$$

The Clifford symbol is very useful for Dirac type operators. We will show that in this case, the Clifford symbol is determined by the corresponding Clifford module bundle.

Let E be a $\mathbf{Z}_2$-graded complex G-vector bundle over X equipped with a Hermitian metric and with a $\mathbf{Z}_2$-graded left action of the Clifford algebra bundle $\mathrm{Cliff}(\tau, Q)$ (of Definition 2.1). We assume that E^0 and E^1 are orthogonal and the Clifford multiplication operators $c(\xi)$ are self-adjoint, degree 1 operators on E for any $\xi \in T_x^*(X)$. This action induces a G-equivariant left action of the algebra $\mathrm{Cl}_\tau(X)$ on the $C_0(X)$-module $C_0(E)$ of continuous sections of E vanishing at infinity. The Dirac operator D is a first order G-invariant differential operator on smooth sections of E with the symbol $\sigma_D(x, \xi) = c(\xi)$ (cf. [10, Section 3.3]).

Definition 3.9. Consider the left action of $\mathrm{Cl}_\tau(X) \hat{\otimes}_{C_0(X)} \mathrm{Cl}_\tau(X)$ on the Hilbert module $C_0(E) \hat{\otimes}_{C_0(X)} \mathrm{Cl}_\tau(X)$ over $\mathrm{Cl}_\tau(X)$. For any point $x \in X$, let $\tau_x = T_x^*(X)$, and let Cl_{τ_x} be the Clifford algebra fiber of $\mathrm{Cliff}(\tau, Q)$ over x. Since $\mathrm{Cl}_{\tau_x} \hat{\otimes} \mathrm{Cl}_{\tau_x} \simeq \mathcal{L}(\Lambda^*(\tau_x))$, there is a projection P in the algebra of multipliers $\mathcal{M}(\mathrm{Cl}_\tau(X) \hat{\otimes}_{C_0(X)} \mathrm{Cl}_\tau(X))$, which is the projection onto $\Lambda^0(\tau_x)$ for any point $x \in X$. We denote by M the space $P \cdot (C_0(E) \hat{\otimes}_{C_0(X)} \mathrm{Cl}_\tau(X))$ which is a right $\mathbf{Z}_2$-graded $\mathrm{Cl}_\tau(X)$-Hilbert module.

Proposition 3.10. *Let $\sigma_A = \sigma_D(1 + \sigma_D^2)^{-1/2}$ be the normalized to order 0 symbol of the Dirac operator D on E. Then the Clifford symbol $[\sigma_A^{\mathrm{cl}}]$ corresponding to the symbol σ_A is equal to the element $(M, 0) \in \mathcal{R}K_0^G(X; \mathrm{Cl}_\tau(X))$.*

Proof. We will use the unbounded operator version of the KK-product of Definition 3.8, i.e. we will take the product of σ_D and d_ξ without normalizing them to order 0. Moreover, we will calculate this product point-wise, for every point $x \in X$. We denote the fiber of E at $x \in X$ by E_x.

Let $e_1, \dots, e_n$ be an orthonormal basis in $\tau_x = T_x^*(X)$. The family of unbounded operators for the KK-product which defines the Clifford symbol is

$$\Phi_x = \sum_{k=1}^{n} ((c(e_k)\hat{\otimes}1)\xi_k - i(1\hat{\otimes}c(e_k))\partial/\partial\xi_k), \quad x \in X,$$

acting on the family of Hilbert spaces $H_x = L^2(\tau_x) \otimes E_x \hat{\otimes} \mathrm{Cl}_{\tau_x}$, where $\{\xi_k\}$ are the coordinates in τ_x corresponding to the orthonormal basis $\{e_k\}$, and $c(e_k)$ is the left Clifford multiplication by e_k. The operator Φ_x^2 is equal to

$$\sum_k (\xi_k^2 - \partial^2/\partial\xi_k^2) + \sum_k i(c(e_k)\hat{\otimes}c(e_k)).$$

This is the harmonic oscillator operator. Our KK-product element splits into the direct sum of a degenerate KK-element, which lives on the direct summand $H_x' \subset H_x$ spanned by the positive eigenvalues of Φ_x^2, and the KK-element which lives on the kernel of Φ_x^2. This kernel is spanned by the vectors $e^{-\|\xi\|^2/2} \otimes v$, where $v \in E_x \hat{\otimes} \mathrm{Cl}_{\tau_x}$ belongs to the eigenspace corresponding to the lowest eigenvalue $-n$ of the operator $\sum_k i(c(e_k)\hat{\otimes}c(e_k))$ acting on $E_x \hat{\otimes} \mathrm{Cl}_{\tau_x}$. The projection onto this eigenspace is given by the projection P of Definition 3.9 at the point $x \in X$. $\square$

4. Index theorems and Poincaré duality for elliptic operators

Our first result is the following.

Theorem 4.1 (Inverse Clifford index theorem). *Let X be a complete Riemannian manifold and G a second countable locally compact group which acts on X properly and isometrically. Let A be a properly supported, G-invariant, L^2-bounded elliptic operator on X of order* 0*, of the Hörmander class $\rho = 1, \delta = 0$. Then*

$$[\sigma_A^{\mathrm{cl}}] = [\Theta_{X,2}] \otimes_{C_0(X)} [A] \in \mathcal{R}K_*^G(X; \mathrm{Cl}_\tau(X)).$$

Proof. We will treat the graded case, the non-graded case is similar.

The elliptic operator A acts on sections of a vector bundle E over X. The right-hand side KK-product can be written as the pair $(\mathcal{F}, S)$ where the Hilbert module $\mathcal{F}$ is $C_0(U) \cdot L^2(E)\hat{\otimes}\mathrm{Cl}_\tau(X)$ and the operator S is defined as the family $\{S_y\}$ of pseudo-differential operators (parametrized by $y \in X$):

$$S_y = 1\hat{\otimes}\Theta_x(y) + (1 - \Theta_x^2(y))^{1/4} A (1 - \Theta_x^2(y))^{1/4}\hat{\otimes}1.$$

Here we consider $L^2(E)\hat{\otimes}\,\mathrm{Cl}_\tau(X)$ as the family of Hilbert spaces $L^2(E)\hat{\otimes}\,\mathrm{Cl}_{\tau_y}$, where Cl_{τ_y} is the fiber of $\mathrm{Cl}_\tau(X)$ over y.

The corresponding family of symbols is given by

$$1\hat{\otimes}\Theta_x(y) + (1-\Theta_x^2(y))^{1/2}\sigma_A(x,\xi)\hat{\otimes}1.$$

This family is elliptic, of the Hörmander class $\rho = 1$, $\delta = 0$. The family of symbols of $1 - S^2$ is equal (modulo lower order terms) to $(1-\Theta_x^2(y))((1-\sigma_A^2(x,\xi))\hat{\otimes}1)$.

We will perform a homotopy of the pair $\{(\mathcal{F}, S)\}$ to the one which represents the element $[\sigma_A^{\mathrm{cl}}]$ in several stages. By definition, our covector field $\Theta_x(y)$ has length 1 for (x, y) on the boundary of the neighborhood U, where $\rho(x, y) = r_x$. First we find a G-invariant positive smooth function r'_y of $y \in X$ such that the neighborhood of the diagonal $U' = \{(x, y), \rho(x, y) < r'_y\}$ is contained in U (see [30, Remark at the end of Subsection 4.3]). We deform $\Theta_x(y)$ so that it becomes of length 1 on the boundary of U' and outside of U'. The ball of radius r'_y in X centered at y will be denoted by U'_y. To simplify notation we will denote the function $(1-\Theta_x^2(y))^{1/4}$ by $\chi_y(x)$. It is clear that this function is 0 outside U'_y.

Our first task is to produce a homotopy of the portion $B_y = \chi_y(x)A\chi_y(x)$ of the operator S_y. Up to operators of negative order, the operator B_y can be given by the formula (1) of Section 3:

$$B_y u(x) = (2\pi)^{-\dim X}\int_{T_x^*(X)}\int_X e^{i\Phi(x,z,\xi_x)}\chi_y(x)\sigma_A(x,\xi_x)\chi_y(z)u(z)\,\mathrm{d}z\mathrm{d}\xi_x.$$

For any point y of the parameter space X, the operator B_y actually acts on L^2-sections of the vector bundle E over U'_y because $\chi_y(x) = 0$ outside of U'_y. The multiplication part $1\hat{\otimes}\Theta_x(y)$ of the operator S_y is also degenerate (has square 1) outside U'_y, so we can restrict the operator S_y to $L^2(E|_{U'_y})\hat{\otimes}\,\mathrm{Cl}_{\tau_y}$.

Let us now use the diffeomorphism between the neighborhood U' and the neighborhood $\tilde{U}$ of the zero section of the tangent manifold TX provided by the exponential map. For any $y \in X$, we identify the ball $U'_y \subset X$ of radius r'_y around y with the Euclidean ball $\tilde{U}_y$ of the same radius in $T_y(X)$. We claim that modulo operators of negative order, this diffeomorphism transforms the operator B_y into the operator $\tilde{B}_y$ over $\tilde{U}_y$ given by the formula:

$$(\tilde{B}_y u)(x) = (2\pi)^{-\dim X}\int_{T_y^*(X)}\int_{\tilde{U}_y} e^{i(x-z,\eta)}\chi_y(x)\sigma_A(x,\eta)\chi_y(z)u(z)\,\mathrm{d}z\mathrm{d}\eta,$$

where now $x, z \in T_y(X)$, $\eta \in T_y^*(X)$, and the function $\chi_y(x)$ is transplanted to TX in an obvious way.

Indeed, let us first look at the phase function Φ. It follows from the definition given in Section 3 that in the coordinate presentation corresponding to the dual bases of $T_x(X)$ and $T_x^*(X)$, the phase function $\Phi(x, z, \xi)$ has the form $\sum_{i,j}\phi_{i,j}(x,z)v_i\xi_j$, where $v \in T_x(X)$ is the tangent vector to the minimal geodesic joining z and x (and v_i are its coordinates); $\xi \in T_x^*(X)$ (and ξ_j are its coordinates); and $\phi_{i,j}(x,x) = \delta_{i,j}$. It is easy to see that under our diffeomorphism, these properties will be preserved. Therefore, in the Euclidean coordinates, Φ will have the form $(x-z, \phi(x,z)(\xi_x))$, where $\phi(x,z)$ is the map given by the matrix $\phi_{i,j}(x,z)$, such that $\phi(x,x)$ is the identity map for any x. Making the change of variables in the integral, $\eta = \phi(x,z)(\xi_x)$, we transform $\Phi(x,z,\xi_x)$ into $(x-z,\eta)$ without changing the symbol $\sigma_A(x,\eta)$ up to symbols of negative order (since $\phi(x,x)$ is the identity map for any x).

The symbol σ_A of the operator A transforms from the manifold U' to $\tilde{U}$ without any change at all if we again neglect operators of negative order. Since all our operators have distributional kernels with compact support, the operators we neglect are compact operators.

Finally, we want to replace the symbol $\sigma_A(x,\eta)$ in the integral expression for $\tilde{B}_y$ by $\sigma_A(y,\eta)$, i.e. by the symbol constant in $x \in T_y(X)$. Note that the point y is the zero point of $T_y(X)$, so the obvious homotopy for this is $t \mapsto \sigma_A((1-t)x,\eta)$, $0 \le t \le 1$, $x \in \tilde{U}_y$. The norm-continuity of the operator $\tilde{B}_y$ along this homotopy follows from Theorem 3.6. (Recall that uniformness of norm continuity for homotopy of KK-cycles is required only on compact subsets in the parameter y.)

If we denote now by A_y the operator on $T_y(X)$ with the symbol $\sigma_A(y,\eta)$, the operator S_y will become

$$S_y = 1\hat{\otimes}\Theta_x(y) + (1-\Theta_x^2(y))^{1/4} A_y (1-\Theta_x^2(y))^{1/4}\hat{\otimes}1.$$

In the following we will often use notation $c(\cdot)$ to denote Clifford multiplication operators.

The final step is to replace (by homotopy) the neighborhood $\tilde{U}$ with the whole manifold TX. For any $y \in X$, the variable $\theta \in T_y(X)$ will replace $x-y$. In particular, the Clifford multiplication operator $\Theta_x(y)$ will be replaced by $c(-\theta)(1+\|\theta\|^2)^{-1/2}$. With this change, our KK-product will become a family of elliptic operators on the family of Hilbert spaces $L^2(T_y^*(X)) \otimes E_y \hat{\otimes} \mathrm{Cl}_{\tau_y}$ parametrized by $y \in X$, where E_y is the fiber of E over $y \in X$. The corresponding family of symbols is

$$1\hat{\otimes}c(-\theta)(1+\|\theta\|^2)^{-1/2} + (1+\|\theta\|^2)^{-1/2}\sigma_A(y,\eta)\hat{\otimes}1.$$

Here η is the derivation variable, θ the multiplication variable, and y the parameter.

At this point, we remark that the family of operators A_y is strongly continuous in the parameter y. This is easily seen from the continuity of the function $\sigma_A(y,\eta)$, which is dual to A_y via the Fourier transform. Moreover, for any family of operators T_y of the same kind with symbols $\sigma_T(y,\eta)$ vanishing at infinity in η (e.g. for $T_y = 1 - A_y^2$), we have norm-continuity in y, because their symbols vary norm-continuously as functions $y \mapsto \sigma_T(y,\eta) \in C_0(\mathbf{R}^{\dim X})$.

Now let us look at the left-hand side of the formula in the statement of the theorem: $[\sigma_A^{\mathrm{cl}}] = [\sigma_A] \otimes_{C_0(TX)} [d_\xi]$. It can be represented by the family of elliptic operators, on the same family of Hilbert spaces, with the symbols:

$$1 \hat{\otimes} c(\eta)(1 + \|\eta\|^2)^{-1/2} + (1 + \|\eta\|^2)^{-1/2} \sigma_A(y, \theta) \hat{\otimes} 1.$$

(We have replaced here the operator F_y of Definition 2.5 by the pseudo-differential operator with the symbol $c(\eta)(1 + \|\eta\|^2)^{-1/2}$. Note that the operators with the symbols $\sigma_A(y, \eta)$ and $c(\eta)(1 + \|\eta\|^2)^{-1/2}$ are not properly supported.)

The two sides of the formula in the statement of the theorem differ now only by the substitution $\theta \mapsto -\eta,\ \eta \mapsto \theta$. We will produce a rotation homotopy which joins them. But first we will redefine the ingredients of the two formulas.

In the rest of the proof we will use the standard representation of the Lie algebra L_{2n+1} of the n-dimensional Heisenberg group Γ_{2n+1} on $L^2(T_y^*(X))$, where $n = \dim X$. It induces the representation of $L_{2n+1} \otimes \mathcal{L}(E_y) \hat{\otimes} \mathrm{Cl}_{\tau_y}$ on $L^2(T_y^*(X)) \otimes E_y \hat{\otimes} \mathrm{Cl}_{\tau_y}$. This representation is defined by the coordinate multiplication operators and the corresponding coordinate derivation operators on $L^2(T_y^*(X))$, and we consider $\{\theta_j\}$ as the multiplication variables and $\{\eta_k\}$ as the derivation variables, with the commutation relations $[\eta_k, \theta_j] = \delta_{k,j}$. The corresponding unitary representation of Γ_{2n+1} maps $C^*(\Gamma_{2n+1}) \otimes \mathcal{L}(E_y) \hat{\otimes} \mathrm{Cl}_{\tau_y}$ into compact operators on $L^2(T_y^*(X)) \otimes E_y \hat{\otimes} \mathrm{Cl}_{\tau_y}$. All ingredients in the above two displayed formulas, namely, $c(\eta)(1 + \|\eta\|^2)^{-1/2}$ and $c(\theta)(1 + \|\theta\|^2)^{-1/2}$, as well as $\sigma_A(y, \eta)$ and $\sigma_A(y, \theta)$, are bounded continuous functions of the unbounded multipliers of $C^*(\Gamma_{2n+1}) \otimes \mathcal{L}(E_y) \hat{\otimes} \mathrm{Cl}_{\tau_y}$ given by the above coordinate multiplication and derivation operators. They belong to the multiplier algebra $\mathcal{M}(C^*(\Gamma_{2n+1})) \otimes \mathcal{L}(E_y) \hat{\otimes} \mathrm{Cl}_{\tau_y},\ y \in X$ (cf. [35, Proposition 4.1]).

Now we define a homotopy of automorphisms of the Heisenberg group using the rotation in the η, θ variables given by

$$\eta_k \mapsto \alpha \eta_k + \beta \theta_k, \quad \theta_k \mapsto \alpha \theta_k - \beta \eta_k,$$

where $\alpha = \cos t,\ \beta = \sin t,\ 0 \leq t \leq \pi/2$. Here we consider η_k, θ_k, and ρ as the generators of the Lie algebra of the Heisenberg group, with the commutation relations: $[\eta_k, \theta_j] = \delta_{k,j}\rho$. A simple calculation shows that $[\alpha\eta_k + \beta\theta_k, \alpha\theta_j - \beta\eta_j] = \delta_{k,j}\rho$, so the automorphisms are well defined. Composing this homotopy of automorphisms of the Heisenberg group with the representation of $C^*(\Gamma_{2n+1}) \otimes \mathcal{L}(E_y) \hat{\otimes} \mathrm{Cl}_{\tau_y}$ on $L^2(T_y^*(X)) \otimes E_y \hat{\otimes} \mathrm{Cl}_{\tau_y}$ (mapping ρ to the operator 1), we get the homotopy of our family of operators (parametrized by (y, t)), which on the symbol level can be written as

$$1 \hat{\otimes} c(-\alpha\theta + \beta\eta)(1 + \|\alpha\theta - \beta\eta\|^2)^{-1/2} + (1 + \|\alpha\theta - \beta\eta\|^2)^{-1/2} \sigma_A(y, \alpha\eta + \beta\theta) \hat{\otimes} 1.$$

There are two equivalent ways to interpret the ingredients of this formula, for example, the operator with the symbol $\sigma_A(y, \alpha\eta + \beta\theta) \hat{\otimes} 1$. One can either use the above representation of the C^*-algebra of the Heisenberg group, or, alternatively, one can use the operator functional calculus and apply the matrix function $\sigma_A(y,\cdot)\hat{\otimes}1$ to the unbounded self-adjoint multiplier $M_{\alpha,\beta}$ with the symbol $\alpha\eta+\beta\theta$. (The multiplier $M_{\alpha,\beta}$ belongs to the image of the Lie algebra of the Heisenberg group.) In both approaches, it is clear that the above homotopy of operators is strongly continuous.

Note that if $f(\theta)$ and $g(\eta)$ are two continuous functions which both vanish at infinity, then the operator with the symbol $f(\theta)g(\eta)$ is compact because it belongs to the image of $C^*(\Gamma_{2n+1})\otimes\mathcal{L}(E_y)\hat{\otimes}\,\mathrm{Cl}_{\tau_y}$ (i.e. by the Rellich lemma). Applying the above automorphism of the Heisenberg group, we see that $f(\alpha\theta-\beta\eta)g(\alpha\eta+\beta\theta)$ is also a compact operator on $L^2(T_y^*(X))\otimes E_y\hat{\otimes}\,\mathrm{Cl}_{\tau_y}$.

We can also easily calculate the graded commutator of the operators represented by the symbols $1\hat{\otimes}c(\eta)$ and $\sigma_A(y,\theta)\hat{\otimes}1$. It is given by the operator with the symbol $\sum_k(\partial\sigma_A(y,\theta)/\partial\theta_k)\hat{\otimes}c(e_k)$, where e_k is the basis vector corresponding to the k-th coordinate. Since this symbol has negative order, it vanishes at infinity in θ. It follows by the argument of [30, Lemma 4.2] that the operators with the symbols $1\hat{\otimes}c(\eta)(1+\|\eta\|^2)^{-1/2}$ and $(1+\|\eta\|^2)^{-1/2}\sigma_A(y,\theta)$ commute modulo compact operators (all commutators are graded).

Again using the above automorphism of the Heisenberg group, we see that the operators with the symbols $1\hat{\otimes}c(-\alpha\theta+\beta\eta)(1+\|\alpha\theta-\beta\eta\|^2)^{-1/2}$ and $(1+\|\alpha\theta-\beta\eta\|^2)^{-1/2}\sigma_A(y,\alpha\eta+\beta\theta)\hat{\otimes}1$ also commute modulo compact operators on $L^2(T_y^*(X))\otimes E_y\hat{\otimes}\,\mathrm{Cl}_{\tau_y}$.

Combining these observations with the fact that $\sigma_A(y,\eta)$ is an elliptic symbol (i.e. $1-\sigma_A^2(y,\eta)$ vanishes at infinity in the cotangent direction), we see that the above homotopy is well defined. This finishes the proof. □

We come now to the Atiyah–Singer index theorem:

Theorem 4.2. *Let X be a complete Riemannian manifold and G a second countable locally compact group which acts on X properly and isometrically. Let A be a properly supported, G-invariant, L^2-bounded elliptic operator on X of order* 0*, of the Hörmander class $\rho=1,\delta=0$. Then*

$$[A]=[\sigma_A]\otimes_{C_0(TX)}[\mathcal{D}_X]\in K_G^*(C_0(X))$$

and

$$[A]=[\sigma_A^{\mathrm{cl}}]\otimes_{\mathrm{Cl}_\tau(X)}[d_X]\in K_G^*(C_0(X)).$$

Proof. The second formula follows from Theorem 4.1 by applying $\otimes_{\mathrm{Cl}_\tau(X)}[d_X]$ to both sides of the formula in the statement of Theorem 4.1 and using Theorem 2.4. (There is no difference between $[\Theta_{X,1}]$ and $[\Theta_{X,2}]$ here because the $C_0(X)$-structure is forgotten.) The first formula follows from the second one and Theorem 2.10. □

Theorem 4.3 (Inverse index theorem). *In the assumptions of Theorems 4.1 and 4.2, one has*

$$[\sigma_A] = [\mathcal{B}_{X,2}] \otimes_{C_0(X)} [A] \in \mathcal{R}K^G_*(X; C_0(TX)),$$

where $[\mathcal{B}_{X,2}] \in \mathcal{R}KK^G(X; C_0(X) \otimes C_0(TX))$ *is the local Bott element (Definition 2.9).*

Proof. Apply $\otimes_{\mathrm{Cl}_\tau(X)}[\mathcal{B}_\xi]$ to both sides of the formula in the statement of Theorem 4.1 and use Theorem 2.7. □

Remark 4.4. Theorem 4.2 can be considered as a reduction of the general index theorem for the manifold X to the special case of Dirac type operators. We can interpret the right-hand side of the first formula of Theorem 4.2 as a K-homology class represented by a twisted Dolbeault operator. More precisely, let us replace the manifold X by the double, $\Sigma(X)$, of the manifold BX of unit balls in TX. The two copies of BX are glued together along their common boundary of unit spheres fiberwise. This gives a smooth almost complex manifold $\Sigma(X)$ and a projection $\pi : \Sigma(X) \to X$. The Dolbeault operator of this manifold defines an element of $K^0_G(\Sigma(X))$. In the graded case, the symbol $[\sigma_A]$ of the initial operator A on X can be considered as an element of the group $\mathcal{R}K^G_0(X; \Sigma(X))$ as follows. On the sphere bundle SX, which is the boundary of the disk bundle $B(X)$, the symbol serves as a gluing map between the bundles $p^*(E^0)$ and $p^*(E^1)$. Gluing the two bundles along the sphere bundle $S(X)$ we obtain a vector bundle F over $\Sigma(X)$. The desired element $[\sigma_A] \in \mathcal{R}K^G_0(X; \Sigma(X))$ is a formal difference $[F]-[\pi^*(E^1)]$. Now the right-hand side of the first formula of Theorem 4.2 becomes the K-homology class represented by the Dolbeault operator on $\Sigma(X)$ twisted by the difference of vector bundles F and $\pi^*(E^1)$. So Theorem 4.2 says that the "analytical index" of this twisted Dolbeault operator maps into $[A]$ (which is the "analytical index" of the operator A) under the projection $\pi_* : K^0_G(C_0(\Sigma(X))) \to K^0_G(C_0(X))$. It is easy to show that the "topological indices" of these two operators are also related by the projection π_*. (The "topological index" of the operator A is given by the right side of the index formula of Theorem 4.2. The "topological index" of the twisted Dolbeault operator is defined similarly.)

Remark 4.5 (Cohomological form of the index formula). In the situation without group action, the index theorem 4.2 for closed manifolds X transforms into the well-known cohomological formula for the index [9, Theorem 2.12], by applying the Chern character. One needs only to know that the Chern character commutes with the intersection product and that *the Chern character of the Dolbeault element is Poincaré dual in TX to the Todd class of the complexification of the tangent bundle of X*. (Poincaré duality between cohomology and homology is just the cap-product with the fundamental class.)

Here is some explanation. Let X be a closed manifold. Embed it into $\mathbf{R}^m$. Let us denote the normal bundle for this embedding by N. Then TX will embed into

$T\mathbf{R}^m = \mathbf{C}^m$ with the normal bundle $N \otimes \mathbf{C}$. Let W be the tubular neighborhood of TX in $\mathbf{C}^m$. Naturally, W is fibered over TX and this fibration is diffeomorphic to the complex vector bundle $N \otimes \mathbf{C}$.

To obtain the Dolbeault element of TX one needs to take the Dolbeault element of $\mathbf{C}^m$, restrict it to W, and take the KK-product with the fiberwise Bott element of W in the direction of the fibers of $N \otimes \mathbf{C}$. When we do that, the Dolbeault part in the fiber direction of $N \otimes \mathbf{C}$ turns into $1_{N\otimes\mathbf{C}}$ (= trivial line bundle). So the result will be the Dolbeault element of TX.

The Chern character of the Dolbeault element of the manifold W (which comes from the Dolbeault element of $\mathbf{C}^m$ by restriction) is just the fundamental class of W. This is related with the fundamental class of TX by the Thom isomorphism in homology. On the other hand, if we denote the Thom isomorphism in cohomology for the bundle $N \otimes \mathbf{C}$ over TX by $\psi_{N\otimes\mathbf{C}}$, and the fiberwise Bott element of $N \otimes \mathbf{C}$ by $\beta_{N\otimes\mathbf{C}}$, then by definition (see [9, pp. 552–556]), we have: $Td(T(X) \otimes \mathbf{C}) = Td(N \otimes \mathbf{C})^{-1} = \psi^{-1}_{N\otimes\mathbf{C}}(\mathrm{ch}(\beta_{N\otimes\mathbf{C}}))$.

So the (co)homological index formula of [9] for $\mathrm{ind}(A) = \mathrm{ch}([A])$ in homology $H_*(TX) \otimes \mathrm{Q} = H_*(X) \otimes \mathrm{Q}$ comes as follows:

$$\begin{aligned}\mathrm{ind}(A) = \mathrm{ch}([\sigma_A]) \cap \mathrm{ch}(\mathcal{D}_X) &= \mathrm{ch}([\sigma_A] \cdot \beta_{N\otimes\mathbf{C}}) \cap [W] \\ &= \psi^{-1}_{N\otimes\mathbf{C}}(\mathrm{ch}([\sigma_A] \cdot \beta_{N\otimes\mathbf{C}})) \cap [TX] \\ &= \mathrm{ch}([\sigma_A]) \cdot \psi^{-1}_{N\otimes\mathbf{C}}(\mathrm{ch}(\beta_{N\otimes\mathbf{C}})) \cap [TX] \\ &= \mathrm{ch}([\sigma_A]) \cdot Td(T(X) \otimes \mathbf{C}) \cap [TX].\end{aligned}$$

Finally in this section we are going to show that from the point of view of KK-theory, the K-theoretic index $[A] \in K^*_G(C_0(X))$ of an elliptic operator carries the same amount of information as its symbol $[\sigma_A]$. This will follow from the Poincaré duality isomorphism of the next theorem.

A certain form of Poincaré duality, called "first Poincaré duality" has been proved in [30, Theorem 4.9]. The theorem which we prove below may be called "second Poincaré duality". A more general result which implies both these theorems will be proved in Section 7 (Theorem 7.8).

Theorem 4.6 (Second Poincaré duality). *For any separable $G - C^*$-algebras A and B,*

$$\begin{aligned}KK^G_*(A \otimes C_0(X), B) &\simeq \mathcal{R}KK^G_*(X; A \otimes C_0(X), B \otimes C_0(TX)) \\ &\simeq \mathcal{R}KK^G_*(X; A \otimes C_0(X), B \hat{\otimes} \mathrm{Cl}_\tau(X)).\end{aligned}$$

Proof. To simplify notation we will deal only with the case $A = B = \mathbf{C}$. Also since $C_0(TX)$ is KK-equivalent to $\mathrm{Cl}_\tau(X)$, we have to prove only that

$$K^*_G(C_0(X)) \simeq \mathcal{R}K^G_*(X; \mathrm{Cl}_\tau(X)).$$

We will prove that the homomorphisms

$$\mu = [\Theta_{X,2}] \otimes_{C_0(X)} : K_G^*(C_0(X)) \to \mathcal{R}K_*^G(X; \mathrm{Cl}_\tau(X))$$

and

$$\nu = \otimes_{\mathrm{Cl}_\tau(X)}[d_X] : \mathcal{R}K_*^G(X; \mathrm{Cl}_\tau(X)) \to KK_*^G(C_0(X), \mathrm{Cl}_\tau(X)) \to K_G^*(C_0(X))$$

are inverses of each other. Associativity of the product and Theorem 2.4 easily imply that $\nu \cdot \mu = \mathrm{id}$. (Again, note that there is no difference between $[\Theta_{X,1}]$ and $[\Theta_{X,2}]$ here.)

In the opposite direction, let $\alpha \in \mathcal{R}K_*^G(X; \mathrm{Cl}_\tau(X))$. We need to deal with the triple product $[\Theta_{X,2}] \otimes \alpha \otimes [d_X]$, so by definition, we have to look at the KK-product of the following three groups:

$$\begin{gathered}\mathcal{R}KK^G(X; C_0(X), C_0(X) \otimes \mathrm{Cl}_\tau(X)) \\ \otimes_{C_0(X) \otimes \mathrm{Cl}_\tau(X)} \mathcal{R}KK_*^G(X; C_0(X) \otimes \mathrm{Cl}_\tau(X), \mathrm{Cl}_\tau(X) \hat{\otimes} \mathrm{Cl}_\tau(X)) \\ \otimes_{\mathrm{Cl}_\tau(X) \hat{\otimes} \mathrm{Cl}_\tau(X)} \mathcal{R}KK^G(X; \mathrm{Cl}_\tau(X) \hat{\otimes} \mathrm{Cl}_\tau(X), \mathrm{Cl}_\tau(X)).\end{gathered}$$

We have replaced here the elements α and $[d_X]$ with the elements $\alpha \otimes 1$ and $[d_X] \otimes 1$, which belong to the second and third groups of this triple KK-product respectively. Concerning the $C_0(X)$-structure of the tensor product arguments of the $\mathcal{R}KK$-groups involved: we assume here that $C_0(X)$ acts by multiplication over the *second* tensor multiples of all arguments which are tensor products. Note that for the element $\alpha \otimes 1$, this does not correspond to the initial $C_0(X)$-action.

Recall now that the element $[\Theta_{X,2}]$ actually comes from the group

$$\mathcal{R}KK^G(X; C_0(X), C_0(U) \cdot (C_0(X) \otimes \mathrm{Cl}_\tau(X))).$$

This allows us to restrict all four tensor product arguments of our KK-product to the neighborhood $U \subset X \times X$ of the diagonal by multiplying them similarly by $C_0(U)$. Moreover, we can (and later in this proof we will) change U to a smaller neighborhood of the diagonal. This will require only a homotopy of the element $[\Theta_{X,2}]$.

Now we will use the "rotation" homotopy: if $(x, y) \in U$, and $p_t(x, y)$, $t \in [1, 2]$, is the unique geodesic segment which joins x and y in X, then the family of maps $(x, y) \mapsto p_t(x, y)$, $1 \le t \le 2$, joins the two projections p_1 and $p_2 : U \to X$. For the first $\mathcal{R}KK$-group of our product, this homotopy shifts the $C_0(X)$-action from the second to the first tensor multiple (and we change the notation $[\Theta_{X,2}]$ to $[\Theta_{X,1}]$). For the $\mathcal{R}KK$-group in the middle, the $C_0(X)$-action in the second tensor multiples of both arguments will be replaced by the $C_0(X)$-action in the first tensor multiples. As a result, we recover the initial $C_0(X)$-action for the $\alpha \otimes 1$ element. For the last $\mathcal{R}KK$-group, we need *not only* to shift the $C_0(X)$-action from the second tensor multiple to the first one but also to replace the $[d_X] \otimes 1$ element which lives in the first tensor multiple of $\mathrm{Cl}_\tau(X) \hat{\otimes} \mathrm{Cl}_\tau(X)$ with $1 \otimes [d_X]$ which lives in the second tensor multiple. To do this, we will adapt the proof of Lemma 4.6 [30].

The map $p_t : U \to X$ is a smooth regular submersion. Let us denote the fiber of this map over the point $x \in X$ by $V_{x,t}$. This is a smooth Riemannian submanifold of U. For $t = 1$, it coincides with $x \times U_x \subset U$, and for $t = 2$, with the intersection of $U \cap (X \times x)$.

Because the map $p_t : T_{(x,y)}(U) \to T_{p_t(x,y)}(X)$ is surjective, the normalized map $q_t^* : T^*_{p_t(x,y)}(X) \to T^*_{(x,y)}(U)$, where $q_t^* = p_t^*(p_t p_t^*)^{-1/2}$, is an isometric embedding. The image of q_t^* at any point $(x, y) \in U$ is orthogonal to the cotangent space $T^*_{(x,y)}(V_{p_t(x,y),t})$, and the direct sum of these two spaces, $T^*_{(x,y)}(V_{p_t(x,y),t}) \oplus \operatorname{Im} q_t^*$, is the whole cotangent space $T^*_{(x,y)}(U)$.

Let $\tau_z = T_z^*(X)$, and Cl_{τ_z} be the fiber of $\mathrm{Cl}_\tau(X)$ at the point $z \in X$. We identify $\tau_{p_t(x,y)}$ with $\operatorname{Im} q_t^*$ at the point $(x, y) \in U$. We define now the Hilbert module $\mathcal{T}_t$ over $\mathrm{Cl}_\tau(X)$ as the Hilbert module of continuous sections (vanishing at infinity) of the field of Hilbert spaces $L^2(\Lambda^*(V_{z,t}))\hat{\otimes}\,\mathrm{Cl}_{\tau_z}$ parametrized by $z \in X$. The algebra $\mathrm{Cl}_\tau(X)\hat{\otimes}\,\mathrm{Cl}_\tau(X) \simeq \mathrm{Cl}_\tau(X \times X)$ acts on this Hilbert module on the left as follows. The fiber of this algebra over the point (x, y) is the Clifford algebra generated by $T^*_{(x,y)}(U)$, so it is enough to specify the Clifford action of $T^*_{(x,y)}(U)$. This linear space decomposes as $T^*_{(x,y)}(V_{p_t(x,y),t}) \oplus \operatorname{Im} q_t^*$. The Clifford action of the first summand on $\Lambda^*(T^*_{(x,y)}(V_{p_t(x,y),t}))$ is given by $v \mapsto \operatorname{ext}(v) + \operatorname{int}(v)$, the Clifford action of the second summand on $\mathrm{Cl}_{\tau_{p_t(x,y)}}$ is just Clifford multiplication.

Now we want to construct the $d + d^*$ operator on all manifolds $V_{x,t}$. However, since these manifolds are not complete, this requires a small technical trick. Let us consider a neighborhood U' of the diagonal of $X \times X$ defined as $\{(x, y),\ \rho(x, y) < r_x/2\}$. Clearly, $U' \subset U$ and $V'_{x,t} = U' \cap V_{x,t} \subset V_{x,t}$. Now we change the Riemannian metric on the complement $V_{x,t} - V'_{x,t}$ so that each $V_{x,t}$ becomes a complete Riemannian manifold. (We can use averaging over G with the cut-off function in order to ensure that this procedure is equivariant with respect to the G-action.) The operators $d + d^*$ on these complete manifolds $V_{x,t}$ are used to define elements $[d_{V'_{x,t}}] \in K^0(\mathrm{Cl}_\tau(V'_{x,t}))$ for all manifolds $V'_{x,t}$.

To make sure that the above construction gives the required homotopy of the element $[d_X] \otimes 1$, we apply the reasoning of the proof of Lemma 4.7 and Theorem 4.8 of [30] which shows that the elements $[d_{V'_{x,t}}]$ vary continuously in t and x when x is in a compact subset of X. More precisely, the operators $F_{x,t} = (d_{V'_{x,t}} + d^*_{V'_{x,t}})/(1 + (d_{V'_{x,t}} + d^*_{V'_{x,t}})^{-1/2})$ vary strongly continuously and $a(F^2_{x,t} - 1)$ and $[a, F_{x,t}]$ vary continuously in norm for any $a \in C_0(U'_t) \cdot (\mathrm{Cl}_\tau(X)\hat{\otimes}\,\mathrm{Cl}_\tau(X))$. Also the reasoning in the proof of Theorem 4.8 of [30] shows that for any $a \in C_0(U'_t) \cdot (\mathrm{Cl}_\tau(X)\hat{\otimes}\,\mathrm{Cl}_\tau(X))$, the operators $aF_{x,t}a$ depend on the Riemannian metric of $V_{x,t} - V'_{x,t}$ only up to compact operators. The commutation up to compact operators with the left action of $C_0(U'_t) \cdot (\mathrm{Cl}_\tau(X)\hat{\otimes}\,\mathrm{Cl}_\tau(X))$ follows by the reasoning of [30, Lemma 4.2].

After this homotopy is performed, the element $[d_X] \otimes 1$ of the group $\mathcal{R}KK^G(X; C_0(U)\cdot\mathrm{Cl}_\tau(X)\hat{\otimes}\,\mathrm{Cl}_\tau(X), \mathrm{Cl}_\tau(X))$ becomes $1\otimes[d_X]$. The calculation of our triple KK-product now becomes easy: the product $\alpha \otimes [d_X]$ is just the cup-product of these two elements. Using Theorem 2.4, we get $[\Theta_{X,1}] \otimes \alpha \otimes [d_X] = \alpha$. $\square$

Remark 4.7. Comparing the statement of the Index theorem 4.2 and the proof of the Poincaré duality Theorem 4.6 we see that the homomorphism ν carries the Clifford symbol element $[\sigma_A^{\mathrm{cl}}]$ of an elliptic operator A into its index element $[A]$. The same is true for the symbol element $[\sigma_A]$ as well. This means that the symbol and the index of an elliptic operator are related by Poincaré duality.

5. G-invariant elliptic operators with index in $K_*(C^*(G))$

In this section we deduce from Theorem 4.2 an index theorem with index taking value in $K_*(C^*(G))$ (Theorem 5.6). This result was published without full proof in [27, Theorem 3], and was widely used since then. We will give a full proof here. We will also define an index with value in $K_*(C^*(G))$ for first order G-invariant elliptic differential operators. This has applications to the realization of discrete series representations of Lie groups. Theorem 5.8 below essentially coincides with [27, Theorem 2] the proof of which was also only sketched in [27].

Everywhere in this section we assume that X/G is compact.

Let $C_c(X)$ (respectively, $C_c(E)$ – for a vector bundle E over X) be the space of continuous functions (resp., sections) with compact support on X. We will use notation $C_c(G)$ for the space of continuous functions with compact support on G. $C_c(G)$ is equipped with the convolution product and the involution given by $f^*(g) = \bar{f}(g^{-1})\mu(g^{-1})$, where μ is the modular function of G. We define on $C_c(E)$ the structure of a pre-Hilbert module over $C_c(G)$ by the formulas:

$$(e \cdot b)(x) = \int_G g(e)(x) \cdot b(g^{-1}) \cdot \mu(g)^{-1/2}\, \mathrm{d}g \in C_c(E),$$
$$(e_1, e_2)(g) = \mu(g)^{-1/2} \int_X (e_1(x), g(e_2)(x))\, \mathrm{d}x \in C_c(G),$$

for $e, e_1, e_2 \in C_c(E)$, $b \in C_c(G)$. The scalar product under the integral is the Hermitian scalar product on E.

In order to show positivity of the above inner product, we will embed $C_c(E)$ into another Hilbert module, $C^*(G, L^2(E))$ (see [30, 3.8]), which is defined as follows. The algebra $C_c(G)$ acts on $C_c(G, L^2(E))$ on the right by the usual convolution product, and the inner product of elements $f_1, f_2 \in C_c(G, L^2(E))$ is given by $(f_1, f_2)(t) = \int_G (f_1(s), f_2(st))_{L^2(E)}\, \mathrm{d}s \in C_c(G)$.

We define the map $i : C_c(E) \to C_c(G, L^2(E))$ by the formula:

$$i(e)(g) = \mu(g)^{-1/2} \cdot \mathfrak{c}^{1/2} \cdot g(e),$$

where $\mathfrak{c}$ is a cut-off function on X, i.e. a smooth non-negative function with compact support and the property that for any $x \in X$, $\int_G \mathfrak{c}(g^{-1}x)\,dg = 1$. The map i preserves the inner product. This implies the positivity of the inner product on $C_c(E)$.

There is also a map in the opposite direction $q : C_c(G, C_c(E)) \to C_c(E)$ given by

$$q(f) = \int_G g(\mathfrak{c})^{1/2} \cdot g(f(g^{-1})) \cdot \mu(g)^{-1/2}\,dg,$$

where for $f \in C_c(G, C_c(E))$, we set $g(f(g^{-1}))(x) = f(g^{-1})(g^{-1}x)$, $x \in X$. The map q is the adjoint of i : $(i(e), f) = (e, q(f))$.

The completion of $C_c(E)$ in the norm $\|(e,e)\|^{1/2}_{C^*(G)}$ gives a Hilbert $C^*(G)$-module which will be denoted by $\mathcal{E}$. Clearly $\mathcal{E}$ is a Hilbert submodule and a direct summand of $C^*(G, L^2(E))$. It is easy to check that the operator $i \cdot q$, which is the orthogonal projection onto this submodule, is given by the left convolution with the idempotent element $[\mathfrak{c}] = (\mathfrak{c} \cdot g(\mathfrak{c}) \cdot \mu(g)^{-1})^{1/2} \in C_c(G, C_c(X)) \subset C^*(G, C_0(X))$. (The left convolution with an element $b \in C_c(G, C_0(X))$ on $C_c(G, L^2(E))$ is defined by the formula: $(bf)(t) = \int_G b(s)s(f(s^{-1}t))ds$, where $f \in C_c(G, L^2(E))$.) Note that the K-theory class of the projection $[\mathfrak{c}] \in K_0(C^*(G, C_0(X)))$ does not depend on the choice of a cut-off function $\mathfrak{c}$ because the set of cut-off functions is convex.

Any properly supported pseudo-differential operator A on a vector bundle E defines a linear map $C_c^\infty(E) \to C_c^\infty(E)$. We want to prove that a G-invariant operator A of order 0 gives a bounded operator on the Hilbert module $\mathcal{E}$ defined above.

We will need the averaging procedure for the operator A suggested in [15]:

$$\tilde{A} = \int_G g(\mathfrak{c})g(A)\,dg,$$

where $\mathfrak{c}$ is a cut-off function and, as usual, $g(\mathfrak{c})(x) = \mathfrak{c}(g^{-1}x)$ and $g(A) = gAg^{-1}$.

This averaging is well defined because of the following proposition which is a restatement of Proposition 1.4 of [15] in our more general situation.

Proposition 5.1. *Let P be a bounded operator on $L^2(E)$, where E is a vector bundle over X, such that the support of the distributional kernel of P is contained in $K \times K$ for some compact subset $K \subset X$. Then the average $\int_G g(P)\,dg$ is well defined as a bounded operator on $L^2(E)$ and its norm does not exceed $c \cdot \|P\|$, where the constant c depends only on K.*

Proof. We define $F(g) = g(P)u$, with $u \in C_c^\infty(E)$, as in the proof of Proposition 1.4 of [15]. Because the action of G on X is proper, the conditions of [15, Lemma 1.5] are satisfied, and the rest of the proof goes as in [15]. □

Let us first consider L^2-boundedness of G-invariant operators.

Lemma 5.2. *Assume that X/G is compact, and let A be a properly supported G-invariant pseudo-differential operator of order 0 on a vector bundle E over X. Assume that its symbol is bounded at infinity in the cotangent direction by a constant $C > 0$, as in Theorem 3.2. Then A defines a bounded operator on $L^2(E)$.*

Proof. For any cut-off function $\mathfrak{c}$, the operator $\mathfrak{c}A$ satisfies the assumptions of Theorem 3.2, including the compactness of support of its distributional kernel. Therefore $\mathfrak{c}A$ is bounded. Proposition 5.1 shows that the average of $\mathfrak{c}A$ over G is L^2-bounded. But this average is just A. □

Lemma 5.3. *Let B be a bounded positive operator on $L^2(E)$ with a compactly supported distributional kernel. Then the inner product $(e, (\int_G s(B)ds)(e))_{\mathcal{E}} \in C^*(G)$ is well defined and positive for any $e \in C_c(E)$.*

Proof. For any $e \in C_c(E)$ and $s \in G$, we have:

$$(s(e), B(s(e)))_{L^2(E)} = (B^{1/2}(s(e)), B^{1/2}(s(e)))_{L^2(E)}.$$

Because the left-hand side has compact support in $s \in G$, and this support depends only on the support of B and e, the function $G \to L^2(E) : s \mapsto B^{1/2}(s(e))$ has compact support in G.

It follows that for any unitary representation of the group G in a Hilbert space H and any $h \in H$, the vector $v = \int_G \mu(s)^{-1/2} B^{1/2}(s(e)) \otimes s(h)\,\mathrm{d}s$ is well defined in $L^2(E) \otimes H$. The following expression is equal to $\|v\|^2$ and therefore is positive:

$$\begin{aligned}
&\int_G \int_G \mu(s)^{-1/2}\mu(t)^{-1/2}(s(e), B(t(e)))_{L^2(E)} \cdot (s(h), t(h))_H \,\mathrm{d}s\mathrm{d}t \\
&\quad = \int_G \int_G \mu(s)^{-1/2}\mu(t)^{-1/2}(e, s^{-1}(B)(s^{-1}t(e)))_{L^2(E)} \cdot (h, s^{-1}t(h))_H \,\mathrm{d}s\mathrm{d}t \\
&\quad = \int_G \int_G \mu(s)^{-1}\mu(t)^{-1/2}(e, s^{-1}(B)(t(e)))_{L^2(E)} \cdot (h, t(h))_H \,\mathrm{d}s\mathrm{d}t \\
&\quad = \int_G \int_G \mu(t)^{-1/2}(e, s(B)(t(e)))_{L^2(E)} \cdot (h, t(h))_H \,\mathrm{d}s\mathrm{d}t \\
&\quad = \int_G \left(e, \left(\int_G s(B)\,\mathrm{d}s\right)(e)\right)_{\mathcal{E}}(t) \cdot (h, t(h))_H \,\mathrm{d}t.
\end{aligned}$$

This means that $(e, (\int_G s(B)ds)(e))_{\mathcal{E}}$ is a positive operator in all unitary representations of G. (The action of an element $f \in C_c(G)$ on H is defined by $f(h) = \int_G f(g)g(h)\,\mathrm{d}g$ for any $h \in H$.) □

We will use the following notation: $\mathcal{L}(\mathcal{E})$ is the C^*-algebra of all bounded operators $\mathcal{E} \to \mathcal{E}$ which have adjoint.

Proposition 5.4. *Let A be an operator on $C_c(E)$ which is $L^2(E)$-bounded, G-invariant, and has a properly supported distributional kernel. Then A defines an element of $\mathcal{L}(\mathcal{E})$ with the norm $\leq$ const $\cdot \|A\|$, where $\|A\|$ is the L^2-norm of A, and the constant depends only on the supports of the operators $\mathfrak{c}A^*A + A^*A\mathfrak{c}$ and $\mathfrak{c}AA^* + AA^*\mathfrak{c}$, where $\mathfrak{c}$ is any cut-off function of our choice.*

Proof. The operator $A_1 = (\mathfrak{c}A^*A + A^*A\mathfrak{c})/2$ is bounded on $L^2(E)$ by $\|A\|^2\|\mathfrak{c}\|$, self-adjoint, and has distributional kernel with compact support. Let $\mathfrak{c}_1$ be a non-negative, compactly supported function on X such that $\mathfrak{c}_1 = 1$ on the support of A_1. The operator $B = \mathfrak{c}_1^2\|A\|^2\|\mathfrak{c}\| - A_1$ is positive, bounded, and has distributional kernel with compact support. The function $\int_G g(\mathfrak{c}_1^2)\,dg$ is bounded on X by some constant M. It is easy to see from the definition of the inner product on $\mathcal{E}$ that the operator of multiplication by any bounded G-invariant function f is bounded on $\mathcal{E}$ with the norm $\leq \|f\|$. Because $\int_G g(A_1)\,dg = A^*A$, applying Lemma 5.3 to the operator B we get for any $e \in C_c(E)$, $(A(e), A(e))_{\mathcal{E}} = (e, A^*A(e))_{\mathcal{E}} \leq M\|A\|^2\|\mathfrak{c}\|(e,e)_{\mathcal{E}}$.

The lower bound M of $\int_G g(\mathfrak{c}_1^2)\,dg$ over all functions $\mathfrak{c}_1$ satisfying the above assumptions depends only on the support of $\mathfrak{c}A^*A + A^*A\mathfrak{c}$. Obviously, A^* is also bounded (by the same reasoning), and since A^* is the adjoint of A with respect to the inner product of $\mathcal{E}$, we get $A \in \mathcal{L}(\mathcal{E})$. □

Now we want to find out what the "rank one" operators θ_{e_1,e_2} on $C_c(E)$ are. Recall that the "rank one" operators on a Hilbert module are defined by $\theta_{e_1,e_2}(e) = e_1(e_2, e)$, where e, e_1, e_2 are elements of the Hilbert module. In our case, let $e, e_1, e_2 \in C_c(E)$. Then it is easy to calculate that

$$\theta_{e_1,e_2}(e)(x) = \int_X \left(\int_G \theta_{g(e_1)(x),g(e_2)(y)}\,dg \right) e(y)\,dy,$$

where on the right-hand side the same notation θ is used for the rank one operators on the Hilbert space $L^2(E)$. We see that "rank one" operators on $C_c(E)$ are integral operators with G-invariant continuous kernels and proper support.

In the above integral expression, the operator θ_{e_1,e_2} on $\mathcal{E}$ is presented as the average over G of the operator θ_{e_1,e_2} on $L^2(E)$. Let S be a finite sum of rank one operators θ_{e_i,e_j} on $L^2(E)$ with all $e_i, e_j \in C_c(E)$. Propositions 5.1 and 5.4 show that the norm $\|\int_G g(S)\,dg\|_{\mathcal{E}}$ is bounded by const $\cdot \|S\|_{L^2(E)}$.

By definition, $\mathcal{K}(\mathcal{E})$ is the closure of linear combinations of "rank one" operators on $\mathcal{E}$ in the operator norm of $\mathcal{L}(\mathcal{E})$.

Any integral operator R with a G-invariant continuous kernel and proper support defines an element of $\mathcal{K}(\mathcal{E})$, because it can be approximated in the norm of $\mathcal{L}(\mathcal{E})$ by finite linear combinations of operators $\theta_{e_i,e_j} \in \mathcal{K}(\mathcal{E})$. Indeed, $\mathfrak{c}R$ can be approximated by a finite linear combination of rank one operators θ_{e_i,e_j} on $L^2(E)$ with all $e_i, e_j \in C_c(E)$. It remains to take the average over G and apply Propositions 5.1 and 5.4.

Proposition 5.5. *Let X be a complete Riemannian manifold and G a locally compact group which acts on X properly and isometrically with compact quotient X/G. If the symbol of a G-invariant properly supported operator A of order 0 is bounded at infinity in the cotangent direction by a constant $C > 0$, then $A \in \mathcal{L}(\mathcal{E})$, and the norm of A as an element of $\mathcal{L}(\mathcal{E})/\mathcal{K}(\mathcal{E})$ does not exceed C. If the symbol of A is zero at infinity, then $A \in \mathcal{K}(\mathcal{E})$.*

Proof. Using Theorem 3.2 and Lemma 3.4, we obtain G-invariant operators B and R which satisfy the relation: $A^*A + B^*B - C^2 = R$ on $C_c(E)$. All operators A, B, R here are elements of $\mathcal{L}(\mathcal{E})$ by Proposition 5.4, and $R \in \mathcal{K}(\mathcal{E})$ by the above discussion. This implies the assertion. □

Assume now that A is elliptic. As in Section 3, we will consider two cases: in the graded case we replace the operator A with the self-adjoint operator $\left(\begin{smallmatrix} 0 & A^* \\ A & 0 \end{smallmatrix}\right)$ on $E = E^0 \oplus E^1$. In the non-graded case we assume that A is self-adjoint from the beginning, acting on sections of the vector bundle E.

The ellipticity condition of Section 3 and Proposition 5.5 imply that $A^2 - 1$ is compact on $\mathcal{E}$. *This means that A defines an element of $K_*(C^*(G))$ which we will call the index of A and denote* $\mathrm{ind}_{C^*(G)}(A)$. The index theorem below computes this element from the symbol. Note first that Theorem 4.2 already computes the K-homological index $[A] \in K^*_G(C_0(X))$ from the symbol $[\sigma_A]$.

Now recall that for any two $G - C^*$-algebras $\mathcal{A}$ and $\mathcal{B}$ there is a natural homomorphism

$$j^G : KK^G(\mathcal{A}, \mathcal{B}) \to KK(C^*(G, \mathcal{A}), C^*(G, \mathcal{B}))$$

(cf. [30, Theorem 3.11]). Let us apply the homomorphism

$$j^G : KK^G_*(C_0(X), \mathbf{C}) \to KK_*(C^*(G, C_0(X)), C^*(G))$$

to the element $[A]$ and take the intersection product $[\mathfrak{c}]\otimes_{C^*(G,C_0(X))}$ with the projection $[\mathfrak{c}] \in K_0(C^*(G, C_0(X)))$ defined above. We get an element of the group $K_*(C^*(G))$. The next theorem will show that this element is equal to $\mathrm{ind}_{C^*(G)}(A)$.

In order to state this index theorem, we need also the appropriate version of the symbol. Applying the above construction to the symbol $[\sigma_A] \in KK^G_*(C_0(X), C_0(TX))$, we get an element

$$[\tilde{\sigma}_A] = [\mathfrak{c}] \otimes_{C^*(G,C_0(X))} j^G([\sigma_A]) \in K_*(C^*(G, C_0(TX))).$$

In fact, the latter group is isomorphic to $\mathcal{R}K^G_*(X; C_0(TX))$ (cf. [31, Theorem 5.4]), and upon this identification, we recover the initial symbol $[\sigma_A]$.

Note also that the homomorphism j^G applied to the Dolbeault element $[\mathcal{D}_X]$, defined in Section 2, gives an element

$$j^G([\mathcal{D}_X]) \in KK(C^*(G, C_0(TX)), C^*(G)).$$

Theorem 5.6. *Let X be a complete Riemannian manifold and G a second countable locally compact group which acts on X properly and isometrically with compact quotient X/G. Let A be a properly supported G-invariant elliptic operator on X of order* 0*, of the Hörmander class $\rho = 1, \delta = 0$. Then*

$$\begin{aligned}\mathrm{ind}_{C^*(G)}(A) &= [\mathfrak{c}] \otimes_{C^*(G,C_0(X))} j^G([A]) \\ &= [\tilde{\sigma}_A] \otimes_{C^*(G,C_0(TX))} j^G([\mathcal{D}_X]) \in K_*(C^*(G)).\end{aligned}$$

Proof. The map j^G applied to the operator A acting on $L^2(E)$ gives an operator $\tilde{A}$ on the Hilbert module $C^*(G, L^2(E))$ over $C^*(G)$. The compression P of the operator $\tilde{A}$ to the Hilbert submodule $\mathcal{E}$ is equal to $q\tilde{A}i = \int_G g(\mathfrak{c}^{1/2}A\mathfrak{c}^{1/2})\,\mathrm{d}g$. Let us compare this new operator with our initial operator A acting on $\mathcal{E}$. Their difference is

$$\int_G g(\mathfrak{c}^{1/2}A\mathfrak{c}^{1/2})\,\mathrm{d}g - A = \int_G g(\mathfrak{c}^{1/2}(A\mathfrak{c}^{1/2} - \mathfrak{c}^{1/2}A))\,\mathrm{d}g.$$

According to Proposition 5.1, both operators

$$P = \int_G g(\mathfrak{c}^{1/2}A\mathfrak{c}^{1/2})\,\mathrm{d}g \quad \text{and} \quad Q = \int_G g(\mathfrak{c}^{1/2}(A\mathfrak{c}^{1/2} - \mathfrak{c}^{1/2}A))\,\mathrm{d}g$$

are well defined and bounded on $L^2(E)$. (Recall that A has proper support.)

Note that the operator $\mathfrak{c}^{1/2}(A\mathfrak{c}^{1/2} - \mathfrak{c}^{1/2}A)$ is compact on $L^2(E)$ and has compact support (because $\mathfrak{c}$ is a continuous function with compact support). Therefore it can be approximated by a finite linear combination D of rank one operators θ_{e_i,e_j} on $L^2(E)$ with all $e_i, e_j \in C_c(E)$. The operator $\int_G g(D)\,\mathrm{d}g$ is well defined and close in norm to the operator Q, both as an operator on $L^2(E)$ and on $\mathcal{E}$, by Propositions 5.1 and 5.4. Since by definition $\int_G g(D)\,\mathrm{d}g \in \mathcal{K}(\mathcal{E})$, we deduce that $Q \in \mathcal{K}(\mathcal{E})$ as well. Hence, $P - A \in \mathcal{K}(\mathcal{E})$.

On the other hand, the compression of $\tilde{A}$ to $\mathcal{E}$ corresponds to the intersection product with the projection $[\mathfrak{c}]$. This proves the first equality. The second equality follows from the functoriality of the homomorphism j^G and Theorem 4.2. □

Remark 5.7. Suppose that the group G is almost connected, and let K be a maximal compact subgroup. By Abels' theorem [1] and [30, Theorem 5.8] (or [29, 7.4]), the element $\mathrm{ind}_{C^*(G)}(A)$ belongs to the so called γ-part of the group $K_*(C^*(G))$ (see [30, 5.10]). If the action of K on the tangent space of the manifold G/K at the point (K) is spin, then by [30, 5.10], this γ-part is isomorphic to $K_{*+n}(C^*(K))$, where $n = \dim G/K$, so $\mathrm{ind}_{C^*(G)}(A)$ determines a certain element of the group $K_{*+n}(C^*(K))$. Apparently, there should exist a formula which calculates explicitly this element. It is a non-trivial problem to find such formula. (cf. also Remark 5.10 below.)

We will discuss now G-invariant first order elliptic differential operators and define their index as an element of $K_*(C^*(G))$.

Theorem 5.8. *In the assumptions of Theorem 5.6 concerning G and X, let $D : C_c^\infty(E) \to C_c^\infty(E)$ be a formally self-adjoint G-invariant first-order elliptic differential operator on a vector bundle E over the manifold X. Then both operators $D \pm i$ have dense range as operators on the Hilbert module $\mathcal{E}$ defined at the beginning of Section 5, and the operators $(D \pm i)^{-1}$ are bounded and belong to $\mathcal{K}(\mathcal{E})$. The operator $D(1 + D^2)^{-1/2} \in \mathcal{L}(\mathcal{E})$ is Fredholm and defines an element* $\operatorname{ind}_{C^*(G)}(D) \in K_*(C^*(G))$.

Proof. Only the following assertion of the theorem is non-trivial: *for large enough $\lambda \geq 1$, both spaces $(D \pm i\lambda)(C_c(E))$ are dense in $\mathcal{E}$.*

Assuming that this is proved, let us prove the remaining assertions. For any $e \in C_c^\infty(E)$, one has: $((D \pm i\lambda)e, (D \pm i\lambda)e) = (De, De) + \lambda^2\|e\|^2$. Since the operator $D \pm i\lambda$ has dense range on $\mathcal{E}$, it has an inverse $k_{\pm\lambda} = (D \pm i\lambda)^{-1}$ which is bounded on $\mathcal{E}$, with the norm $\leq \lambda^{-1}$.

Now let $\lambda \geq 1$, and put $h_\pm = k_{\pm\lambda}(1 \mp i(\lambda - 1)k_{\pm\lambda})^{-1}$. (The operator $1 \mp i(\lambda - 1)k_{\pm\lambda}$ is invertible because $\|k_{\pm\lambda}\| \leq \lambda^{-1}$.) An obvious calculation shows that $h_\pm$ is a two-sided inverse for $D \pm i$.

Note that both operators $k_{\pm\lambda} \in \mathcal{K}(\mathcal{E})$. In fact, since operators $D \pm i\lambda$ are properly supported, order 1 elliptic operators of the Hörmander class $\rho = 1, \delta = 0$, they have parametrices $P_{\pm\lambda}$ which are order -1, properly supported, G-invariant operators of the Hörmander class $\rho = 1, \delta = 0$, i.e. $P_{\pm\lambda}(D \pm i\lambda) - 1$ are also order -1, properly supported, G-invariant operators of the Hörmander class $\rho = 1, \delta = 0$. Multiplying $P_{\pm\lambda}(D \pm i\lambda) - 1$ on the right by $(D \pm i\lambda)^{-1}$, we see that: $P_{\pm\lambda} - (D \pm i\lambda)^{-1}$ has negative order. In view of Proposition 5.5, this means that both $P_{\pm\lambda}$ and $P_{\pm\lambda} - (D \pm i\lambda)^{-1}$ belong to $\mathcal{K}(\mathcal{E})$, which proves our assertion.

The last assertion of the theorem is clear.

For the proof of the density of the range assertion we will use the method adapted from [40] where the L^2 case is treated. We will start with some notation and a lemma.

On a complete Riemannian manifold X, let $\rho(x)$ be the distance function from a fixed point $x_0 \in X$ to $x \in X$. This function is Lipschitz, so the exterior derivative $d\rho$ exists almost everywhere and $\|d\rho\| \leq 1$ where it exists. We will also use a Gromov type family $\{a_\varepsilon | \, \varepsilon > 0\}$ of smooth functions with compact support on X with the following properties: $0 \leq a_\varepsilon \leq 1$, $\sup_{x\in X} \|da_\varepsilon(x)\| \leq \varepsilon$, and $X = \cup_{\varepsilon>0}\{a_\varepsilon^{-1}(1)\}$.

Lemma 5.9. *Let X be a complete Riemannian manifold, E a vector bundle over X, and $D : C_c^\infty(E) \to C_c^\infty(E)$ a formally self-adjoint first-order elliptic differential operator. Let σ_D be the symbol of D, and let us denote $\sup_{x\in X, \|\xi\|\leq 1} \|\sigma_D(x, \xi)\|$ by $\|\sigma_D\|$. For any $\kappa, \lambda \in \mathbf{R}$, $\lambda \neq 0$, put $T_{\kappa,\lambda} = D + i\kappa\sigma_D(x, d\rho(x)) + i\lambda$. Assume that $2|\kappa| \cdot \|\sigma_D\| < |\lambda|$. Then $T_{\kappa,\lambda}(C_c^\infty(E))$ is dense in $L^2(E)$, and there exists an inverse $T_{\kappa,\lambda}^{-1}$ which is a bounded operator on $L^2(E)$.*

Proof. If a vector $u \in L^2(E)$ is orthogonal to the range of $T_{-\kappa,-\lambda}$, then it is in the kernel of $T_{\kappa,\lambda}$ in the sense of distributions. By the regularity theorem for elliptic equations (existence of a parametrix), u is smooth. We have:

$$T_{\kappa,\lambda}(a_\varepsilon u) = a_\varepsilon T_{\kappa,\lambda}(u) - i\sigma_D(x, da_\varepsilon)u = -i\sigma_D(x, da_\varepsilon)u.$$

When $\varepsilon \to 0$, the right-hand side goes to 0. But

$$\begin{aligned}\|T_{\kappa,\lambda}(a_\varepsilon u)\|^2 &= ((D + i\kappa\sigma_D(x, d\rho(x)))(a_\varepsilon u), (D + i\kappa\sigma_D(x, d\rho(x)))(a_\varepsilon u)) \\ &\qquad + 2\lambda\kappa \mathrm{Re}\,(\sigma_D(x, d\rho(x))(a_\varepsilon u), a_\varepsilon u) + \lambda^2\|a_\varepsilon u\|^2 \\ &\geq (\lambda^2 - 2|\lambda\kappa| \cdot \|\sigma_D\|) \cdot \|(a_\varepsilon u)\|^2.\end{aligned}$$

Since $a_\varepsilon u \to u$ when $\varepsilon \to 0$, we get $u = 0$.

This proves that the range of $T_{-\kappa,-\lambda}$ is dense. But since for any $v \in C_c^\infty(E)$, one has $\|T_{-\kappa,-\lambda}(v)\|^2 \geq (\lambda^2 - 2|\lambda\kappa| \cdot \|\sigma_D\|)\|v\|^2$ by the same estimate as above, the operator $T_{-\kappa,-\lambda}$ is invertible on $L^2(E)$. □

End of proof of Theorem 5.8. Let us choose κ positive and large enough so that the following function of $g \in G$: $\mu(g)^{-1/2} \int_X e^{-\kappa\rho(x)-\kappa\rho(g^{-1}x)}\,\mathrm{d}x$, belongs to $L^1(G)$. This is always possible: because the G-action on X is proper and isometric, and X/G is compact, all sectional curvatures of X are uniformly bounded, so the volume of balls in X, as a function of the radius, grows at most exponentially. Since D is G-invariant and X/G is compact, we can also choose λ so that $2\kappa\|\sigma_D\| < \lambda$.

Let $v \in C_c^\infty(E)$. Because the operator $D \pm i\lambda = T_{0,\pm\lambda}$ is invertible on $L^2(E)$, one can solve the equation $(D \pm i\lambda)(u) = v$ in $L^2(E)$. But the operator $T_{\kappa,\pm\lambda}$ is also invertible on $L^2(E)$, and one can solve the equation $T_{\kappa,\pm\lambda}(u_1) = e^{\kappa\rho(x)}v$ as well. Calculating $(D \pm i\lambda)(u - e^{-\kappa\rho(x)}u_1)$ we get 0. This means that $u = e^{-\kappa\rho(x)}u_1$, where $u_1 \in L^2(E)$.

Let us calculate now $(u, u)_{\mathcal{E}}$. We have for any $g \in G$:

$$\begin{aligned}(u, u)_{\mathcal{E}}(g) &= \mu(g)^{-1/2} \int_X (e^{-\kappa\rho(x)}u_1(x), e^{-\kappa\rho(g^{-1}x)}g(u_1)(x))\,\mathrm{d}x \\ &= \mu(g)^{-1/2} \int_X e^{-\kappa\rho(x)-\kappa\rho(g^{-1}x)}(u_1(x), g(u_1)(x))\,\mathrm{d}x \\ &\leq \mu(g)^{-1/2} \int_X e^{-\kappa\rho(x)-\kappa\rho(g^{-1}x)}\,\mathrm{d}x \cdot \|u_1\|^2_{L^2(E)}.\end{aligned}$$

The latter function of g belongs to $L^1(G)$ by the previous choice of κ.

The same calculation shows that

$$((1-a_\varepsilon)u, (1-a_\varepsilon)u)_{\mathcal{E}}(g) \leq \mu(g)^{-1/2} \int_X e^{-\kappa\rho(x)-\kappa\rho(g^{-1}x)}\,\mathrm{d}x \cdot \|(1-a_\varepsilon)u_1\|^2_{L^2(E)},$$

so when $\varepsilon \to 0$, the elements $a_\varepsilon u \in C_c(E)$ converge in $\mathcal{E}$, and the limit is u. But since $(D \pm i\lambda)(a_\varepsilon u) = a_\varepsilon(D \pm i\lambda)(u) - i\sigma_D(x, da_\varepsilon)u$, we also get $(D \pm i\lambda)(a_\varepsilon u) \to (D \pm i\lambda)(u) = v$ (because $\|\sigma_D(x, da_\varepsilon)u\|_{\mathcal{E}} \to 0$ by the same calculation as above, and $a_\varepsilon v = v$ for small ε by the definition of a_ε). □

Remark 5.10. Theorem 5.8 has an important application. It ties up the realization of discrete series representations of the group G with the K-theory of $C^*_{\mathrm{red}}(G)$ (cf. [27, 34]). Let G be a semisimple Lie group and K its maximal compact subgroup. It is known that discrete series representations of G are realized on the kernels of Dirac operators: Let D be a Dirac operator with the kernel carrying a discrete series representation. Let us use the natural map $r : C^*(G) \to C^*_{\mathrm{red}}(G)$ and consider the operator $D(1 + D^2)^{-1/2}$ as an operator on the corresponding Hilbert module over $C^*_{\mathrm{red}}(G)$. Then the kernel of the operator $D(1 + D^2)^{-1/2}$ is contained in the grading degree 0 and represents a finitely generated projective module M over $C^*_{\mathrm{red}}(G)$. On the one hand, this projective module corresponds to an element $[M] = r_*(\mathrm{ind}_{C^*(G)}(D))$ of the group $K_0(C^*_{\mathrm{red}}(G))$. On the other hand, the corresponding representation of G on the space $M \otimes_{C^*_{\mathrm{red}}(G)} L^2(G)$ is the discrete series representation defined by the orthogonal projection in $L^2(G)$ onto the kernel of D.

Unfortunately, there seems to be much less clarity about the relation of Dirac operators with the realization of discrete series representations for general unimodular Lie groups, although the basic K-theory result on $K_*(C^*_{\mathrm{red}}(G))$ [13] exists already for some time.

6. Transversally elliptic operators

We come now to transversally elliptic operators. Let X be a complete Riemannian manifold equipped with a smooth, proper, isometric action of a Lie group G. *We continue to assume that all pseudo-differential operators belong to the Hörmander class $\rho = 1, \delta = 0$, unless stated otherwise.*

Let $x \in X$ and $f_x : G \to X$ be the map defined by $g \mapsto g(x)$. We will denote by $\mathfrak{g}$ the Lie algebra of the group G. Let $f'_x : \mathfrak{g} \to T_x(X)$ be the tangent (first derivative) map of f_x at the identity of G, and $f'^*_x : T^*_x(X) \to \mathfrak{g}^*$ the dual map. It is easy to see that for any $x \in X,\ g \in G,\ v \in \mathfrak{g}$, one has: $g(f'_x(v)) = f'_{g(x)}(Ad(g)(v))$.

Let us consider the trivial vector bundle $\mathfrak{g}_X = X \times \mathfrak{g}$ over X with the G-action given by $(x, v) \mapsto (g(x), Ad(g)(v))$. Because the G-action on X is proper, there exists a G-invariant Riemannian metric on $\mathfrak{g}_X$. Equivalently, one can say that there exists a smooth map from X to the space of Euclidean norms on $\mathfrak{g}$: $x \mapsto \|\cdot\|_x$, such that for any $x \in X,\ g \in G,\ v \in \mathfrak{g}$, one has: $\|Ad(g)(v)\|_{g(x)} = \|v\|_x$. From the above formulas, it is clear that the map $f' : \mathfrak{g}_X \to T(X)$, defined by $(x, v) \mapsto f'_x(v)$ at any $x \in X$, is G-equivariant. Note that by multiplying our Riemannian metric on $\mathfrak{g}_X$ by a certain strictly positive G-invariant function, we can also arrange that the following condition is satisfied: for any $v \in \mathfrak{g}$, $\|f'_x(v)\| \leq \|v\|_x$. We will assume that this is the case, and so we have $\|f'_x\| \leq 1$ for any $x \in X$.

We identify $\mathfrak{g}_X$ with its dual bundle via the Riemannian metric and define a G-invariant map $\{\varphi_x = f'_x f'^*_x,\ x \in X : T^*(X) \to T^*(X)\}$. This also allows to define a G-invariant quadratic form q on covectors $\xi \in T_x^*(X)$ by $q_x(\xi) = (f'_x f'^*_x(\xi), \xi) = \|f'^*_x(\xi)\|_x^2$. It follows from the above that $q_x(\xi) \leq \|\xi\|^2$ for any covector ξ. *Note that a covector $\xi \in T_x^*(X)$ is orthogonal to the orbit passing through x if and only if $q_x(\xi) = 0$, or iff $\varphi_x(\xi) = 0$.*

We define a self-adjoint second order G-invariant differential operator Δ_G (the Laplacian in the orbit direction on X) as an operator with the symbol $\sigma_{\Delta_G}(x, \xi) = q_x(\xi)$.

In the definition of a transversally elliptic symbol, we will use the conventions of Section 3 for elliptic symbols. Namely, we will consider two cases: in the graded case, we replace the operator A with $\left(\begin{smallmatrix} 0 & A^* \\ A & 0 \end{smallmatrix}\right)$ acting on the vector bundle $E = E^0 \oplus E^1$. In the non-graded case, A is just self-adjoint from the beginning, acting on sections of the vector bundle E.

The definition of a transversally elliptic symbol is a non-trivial point of the theory. We will call the "naïve definition" the following definition based on [7]. Consider the closed subset $T_G X = \{(x, \xi) \in TX : q_x(\xi) = 0\}$ of TX. A G-invariant properly supported pseudo-differential operator A of order 0 is called transversally elliptic if $\|\sigma_A^2(x, \xi) - 1\| \to 0$ uniformly in $x \in X$ on compact subsets of X as $(x, \xi) \to \infty$ in $T_G X$.

The definition of the K-theory symbol class with the naïve definition is completely similar to the elliptic case. Let us denote the space of continuous sections of the vector bundle $p^*(E)$ over $T_G X$ vanishing at infinity by $C_0(p^*(E)_G)$. The algebra $C_0(X)$ acts on $C_0(p^*(E)_G)$ by multiplication. Thus we get an element

$$[\sigma_A] = (C_0(p^*(E)_G), \sigma_A) \in KK_*^G(C_0(X), C_0(T_G X))$$

which we call the naïve K-theoretic symbol of A.

However, the algebra $C_0(T_G X)$ is not good enough in order to state an index theorem. This was noticed in the previous work on the subject. For example in [11], the notion of a "good" symbol was introduced [11, Definitions 7 and 9, p. 21]; see also [33].

We will adopt the following definition which we call the "technical definition" of transverse ellipticity.

Definition 6.1. A properly supported G-invariant pseudo-differential operator A of order 0 on a vector bundle E over X is called transversally elliptic if its symbol σ_A satisfies the following condition: for any compact subset $K \subset X$ and any $\varepsilon > 0$, there exists $c > 0$ such that for any $x \in K$, $\xi \in T_x^*(X)$,

$$\|\sigma_A^2 - 1\|_{(x,\xi)} \leq c \cdot (1 + q_x(\xi))(1 + \|\xi\|^2)^{-1} + \varepsilon.$$

The notation $\|\cdot\|_{(x,\xi)}$ here and later means the norm of an endomorphism of the vector bundle $p^*(E)$ over TX at the point (x, ξ).

In order to define the K-theory symbol class with the technical definition, we first need to introduce an analog of the algebra of scalar symbols of negative order which will replace $C_0(T_G X)$. We define this algebra of symbols as a subalgebra of the algebra of bounded continuous functions $C_b(TX)$.

Definition-Lemma 6.2. The symbol algebra $\mathfrak{S}_G(X)$ is the norm-closure in $C_b(TX)$ of the set of all smooth, bounded, compactly supported in the x-variable functions $b(x,\xi)$ on TX, which in the local coordinates satisfy the following condition (1), as well as one of the conditions (2)–(3):

(1) The exterior derivative $d_x b(x,\xi)$ is norm-bounded uniformly in ξ, and for the exterior derivative d_ξ there is an estimate: $\|d_\xi b(x,\xi)\| \leq C\cdot(1+\|\xi\|)^{-1}$ with the constant C which depends only on b and not on (x,ξ);

(2) The restriction of $b(x,\xi)$ to $T_G X$ belongs to $C_0(T_G X)$;

(3) For any $\varepsilon > 0$, there exists $c > 0$ such that for any $x\in X,\ \xi\in T_x^*(X)$,

$$|b(x,\xi)| \leq c\cdot(1+q_x(\xi))(1+\|\xi\|^2)^{-1}+\varepsilon.$$

Under the assumption that the condition (1) is satisfied, the conditions (2) and (3) are equivalent.

Remarks. 1. The condition (1) is just a weak version of the Hörmander property for the $\rho = 1, \delta = 0$ class of symbols of order 0.

2. In both Definitions 6.1 and 6.2, $q_x(\xi)$ can be replaced with $\|\varphi_x(\xi)\|^2$.

Proof. (3) $\to$ (2) For any covector $\xi\in T_x^*(X)$ which is orthogonal to the orbit passing through x, we have $q_x(\xi) = 0$. So $b(x,\xi)$ vanishes at infinity in ξ.

(2) $\to$ (3) Assume that the restriction of the function $b(x,\xi)\in C_b(TX)$ to $T_G X$ belongs to $C_0(T_G X)$. Suppose that condition (3) is not satisfied. Since by definition $b(x,\xi)$ vanishes at infinity in x, there exists a compact subset $K\subset X$, a number $\varepsilon > 0$, a sequence of positive numbers $c_i\to\infty$, and a sequence of points $(x_i,\xi_i)\in TX$, $x_i\in K$, $i = 1,2,\ldots$, such that $|b(x_i,\xi_i)| \geq c_i(1+q_{x_i}(\xi_i))(1+\|\xi_i\|^2)^{-1}+\varepsilon$. Because K is compact, we can pick a convergent subsequence out of $\{x_i\}$. Let us assume that the initial sequence $\{x_i\}$ is already convergent and converges to a point $x_0\in X$.

Since the function $b(x,\xi)$ is bounded, the sequence $\{\xi_i\}$ cannot be bounded. Passing to a subsequence, let us assume that $\|\xi_i\|\to\infty$. Put $\eta_i = \xi_i/\|\xi_i\|$. The sequence $\{\eta_i\}$ is bounded, so passing again to a subsequence, let us assume that $\{(x_i,\eta_i)\}$ is convergent and (x_0,η_0) is the limit of this sequence. Obviously, $\|\eta_0\| = 1$. We have:

$$\begin{aligned}\lim_{i\to\infty}(1+q_{x_i}(\xi_i))(1+\|\xi_i\|^2)^{-1} &= \lim_{i\to\infty} q_{x_i}(\xi_i)\|\xi_i\|^{-2}\\ &= \lim_{i\to\infty} q_{x_i}(\eta_i) = q_{x_0}(\eta_0).\end{aligned}$$

Because $c_i\to\infty$ and the sequence $|b(x_i,\xi_i)|$ is bounded, we get $q_{x_0}(\eta_0) = 0$. This means that $(x_0,\eta_0)\in T_G^*(X)$. Therefore also $(x_0,\|\xi_i\|\eta_0)\in T_G^*(X)$ for any i.

By condition (2), $b(x_0, \|\xi_i\|\eta_0)$ tends to 0 when $i \to \infty$. But $|b(x_i, \xi_i) - b(x_0, \|\xi_i\|\eta_0)|$ admits a simple estimate (based on the Taylor formula together with condition (1) on b), which shows that $b(x_i, \xi_i) \to 0$ as well. This leads to a contradiction with our assumption that $|b(x_i, \xi_i)| \geq \varepsilon$. $\square$

Let E be a vector bundle on X, and consider the norm-closure of the set of all smooth bounded sections of the vector bundle $p^*(E)$ over TX which satisfy the conditions of Definition 6.2 (with $|b(x, \xi)|$ replaced with $\|b\|_{(x,\xi)}$). Call it $\mathfrak{S}_G(E)$. Then $\mathfrak{S}_G(E)$ is a Hilbert module over $\mathfrak{S}_G(X)$, with the obvious action of $\mathfrak{S}_G(X)$ by multiplication and the pointwise inner product given by the Hermitian metric of $p^*(E)$. The algebra $C_0(X)$ acts on $\mathfrak{S}_G(E)$ by multiplication.

Definition 6.3. Let A be a self-adjoint, properly supported, G-invariant pseudo-differential operator A of order 0 on X of the Hörmander class $\rho = 1, \delta = 0$. If its symbol σ_A satisfies the condition of Definition 6.1, then this symbol is a bounded operator on the Hilbert module $\mathfrak{S}_G(E)$ and $f(\sigma_A^2 - 1) \in \mathcal{K}(\mathfrak{S}_G(E))$ for any $f \in C_0(X)$. Therefore σ_A defines an element $[\sigma_A] = (\mathfrak{S}_G(E), \sigma_A) \in \mathcal{R}K_*^G(X; \mathfrak{S}_G(X))$.

A K-homological index $\operatorname{ind}(A) \in K^*(C^*(G, C_0(X)))$ for a transversally elliptic operator A of order 0 was introduced by P. Julg [24]. (He considered only the case of compact X and G.) The action of the group G and the pointwise multiplication by functions of $C_0(X)$ on $L^2(E)$ combine together in a covariant representation which produces a representation of the algebra $C^*(G, C_0(X))$ on $L^2(E)$. *Note that as in Section 3 we assume L^2-boundedness of A (if X/G is compact, this is true by Lemma 5.2).*

Proposition 6.4. *The pair $(L^2(E), A)$ defines an element of the group*

$$K^*(C^*(G, C_0(X))),$$

denoted by $\operatorname{ind}(A)$.

Proof. One has to prove that for any $h \in C^*(G, C_0(X))$, the operators $h(1 - A^2)$ and $hA - Ah$ are compact. In order to prove compactness of these operators, we approximate any element $h \in C^*(G, C_0(X))$ by finite sums $\sum_i e_i(g)a_i(x)$ where $e_i \in C_c^\infty(G)$ and $a_i \in C_c^\infty(X)$. Then we need to prove that for any $e \in C_c^\infty(G)$ and $a \in C_c^\infty(X)$, the operators $a(1 - A^2)e$ and $e(aA - Aa)$ are compact. (Note that since A is G-invariant, A and e commute.) For the second operator, the compactness follows from Lemma 3.7. For the first operator we need a couple of lemmas.

Lemma 6.5. *Let T be a self-adjoint, properly supported pseudo-differential operator of order* 0 *with the symbol $\sigma_T(x, \xi)$. Denote by F a self-adjoint, properly supported pseudo-differential operator with the symbol $\sigma_F(x, \xi) = (1 + q_x(\xi))(1 + \|\xi\|^2)^{-1}$. Assume that for any compact subset $K \subset X$ and any $\varepsilon > 0$, there exists $c > 0$ such that for any $x \in K$, $\xi \in T_x^*(X)$, $\|\sigma_T\|_{(x,\xi)} \leq c \cdot \sigma_F(x, \xi) + \varepsilon$. Then for any $\varepsilon' > 0$,* *there exist $c_1, c_2 > 0$ and two self-adjoint properly supported integral operators with continuous kernel, R_1 and R_2, such that*

$$-(c_1 F + \varepsilon' + R_1) \leq T \leq c_2 F + \varepsilon' + R_2.$$

In particular, for $T = 1 - A^2$, we get:

$$-(c_1 F + \varepsilon' + R_1) \leq (1 - A^2) \leq c_2 F + \varepsilon' + R_2.$$

Proof. Taking any $\varepsilon' > \varepsilon$, we get

$$\lim_{\xi \to \infty} \inf_{x \in K} \operatorname{Re}\,[c\sigma_F(x,\xi) + \varepsilon' - \sigma_T(x,\xi)] > 0.$$

According to Proposition 3.3, there exist a properly supported pseudo-differential operator B of order 0 and a self-adjoint properly supported integral operator R such that $B^*B + T = cF + \varepsilon' + R$. This gives the upper inequality. The same reasoning works for $-T$ instead of T, which gives the lower inequality.

In the case of $T = 1 - A^2$, we can replace $\sigma_A^2(x,\xi)$ with $\sigma_{A^2}(x,\xi)$ because the difference between the two is of negative order. The result for $1 - A^2$ follows. □

Lemma 6.6. *Let $\{v_k\}$ be a basis for the Lie algebra $\mathfrak{g}$ and $\{v_k^*\}$ the dual basis for $\mathfrak{g}^*$. Define the operator $d_G : C_c^\infty(E) \to C_c^\infty(E \otimes \mathfrak{g}_X^*)$ by $b \mapsto \sum_k \partial b/\partial v_k \otimes v_k^*$ where $\partial b/\partial v_k$ means the derivative along the one-parameter subgroup of G corresponding to the vector v_k. Then for any element $e \in C_c^\infty(G)$ and any $b \in C_c^\infty(E)$ we have: $d_G(eb) = d(e)b$, where eb is the convolution product of e and b and d is the exterior derivative on the group G.*

Proof. The convolution product is defined by $(eb)(x) = \int_G e(g)b(g^{-1}(x))\,dg$. The assertion follows by a simple change of variables in the integral, using the fact that all derivatives $\partial b/\partial v_i$ are derivatives along one-parameter subgroups at the identity of G. □

End of proof of Proposition 6.4. Now we can prove that for any $e \in C_c^\infty(G)$ and $a \in C_c^\infty(X)$, the operator $a(1 - A^2)e$ is compact. In view of Lemma 6.5, it is enough to prove that $a \cdot F \cdot e$ is compact. (Note that aR_1 and aR_2 are compact.)

Both operators Δ_G and Δ (with symbols $q_x(\xi)$ and $\|\xi\|^2$ respectively) belong to the Hörmander class of $\rho = 1, \delta = 0$ operators. We have $\Delta_G = d_G^* d_G$ modulo operators of order 1. Indeed, the symbol of d_G at the point (x,ξ) is equal to

$$i \sum_k (\xi, f_x'(v_k))\operatorname{ext}(v_k^*) = i \sum_k (f_x'^*(\xi), v_k)\operatorname{ext}(v_k^*) = i \cdot \operatorname{ext}(f_x'^*(\xi)).$$

So the symbol of the adjoint operator d_G^* is equal to $-i \cdot \operatorname{int}(f_x'^*(\xi))$, and the symbol of $d_G^* d_G$ at the point (x,ξ) is $(f_x'^*(\xi), f_x'^*(\xi)) = q_x(\xi)$ modulo symbols of order 1.

Now we can present F as $d_G^*(1+\Delta)^{-1} d_G$ modulo operators of negative order. In view of Lemma 6.6, the composition $d_G \cdot e$ is the convolution with $d(e)$, and therefore a bounded operator. Since $a \cdot d_G^*(1 + \Delta)^{-1}$ is compact, our assertion follows. □

Remark 6.7. As pointed out by P. Julg [24], in the case when G and X are compact, one can use the map $K^*(C^*(G, C(X))) \to K^*(C^*(G))$ induced by the map $X \to$ point to obtain from $\operatorname{ind}(A)$ the distributional index of A defined by Atiyah [7]. The distribution is determined by its Fourier coefficients $(\ldots, a_i, \ldots) \in \prod \mathbf{Z} = K^0(C^*(G))$. The product goes over all irreducible representations of G, and all coefficients may be non-zero.

7. Fields of Clifford algebras and Poincaré duality

We will treat index theorems in the transversally elliptic case much in the same way as in the case of elliptic operators. So we need essentially the same basic elements as those constructed in Section 2. This time it will be a little bit more technical to construct them. Some constructions of this section will apply also in a more general case of singular foliations which will be discussed in Section 9. One of the principal results of the present section is the general Poincaré duality Theorem 7.8.

We start with a minor technical problem: the algebra $\mathfrak{S}_G(X)$ is not separable. In order to deal with it using KK-theory, we need to use a separable version of KK-theory introduced in [38, pp. 571–572]. This means that we have to replace all KK-groups that involve non-separable C^*-algebras B as their first argument by the inverse limit KK-groups:

$$KK_{\mathrm{sep}}(B, D) = \varprojlim_{B_1 \subset B} KK(B_1, D),$$

where the inverse limit is taken over all separable C^*-subalgebras $B_1 \subset B$. (In the case with a G-action, we consider separable G-subalgebras.) *We will usually keep the notation $KK(B, D)$ for these new groups.* The new KK-groups have the same properties as the former ones, and on the category of separable C^*-algebras coincide with the former KK-groups.

Note that the symbol $[\sigma_A]$ of any transversally elliptic operator A can be considered as an element of a KK-group where both arguments are separable, i.e. we can always consider $[\sigma_A]$ as an element of $\mathcal{R}K^G_*(X, B_1)$ with $B_1 \subset \mathfrak{S}_G(X)$ separable.

Turning to fields of Clifford algebras, we are going to define an "orbit-wise" version of the algebra $\mathrm{Cl}_\tau(X)$. Let $\mathcal{V}$ be a real G-vector bundle over X equipped with a G-invariant Riemannian metric and $\Gamma = \{\Gamma_x,\ x \in X\}$ a continuous field of vector subspaces of the fibers of $\mathcal{V}$. Recall that this means that there is given a set of "continuous sections" $\{x \mapsto v_x \in \Gamma_x\}$, which generate subspaces Γ_x at all points $x \in X$ and satisfy the condition that the norms $\|v_x\|$ are continuous in x.

Technically we need some more restrictions on Γ. Let $\mathcal{W}$ be another real G-vector bundle over X equipped with a G-invariant Riemannian metric and $f : \mathcal{V} \to \mathcal{W}$ a continuous G-map of vector bundles. We assume that $\|f\| \leq 1$, i.e. for any

$x \in X,\ \|f_x\| \leq 1$. Let $f^* : \mathcal{W} \to \mathcal{V}$ be the Riemannian dual map. The two fields of subspaces

$$\Gamma_{\mathcal{V}} = \{\mathrm{Im}\, f_x^* \subset \mathcal{V},\ x \in X\} \quad \text{and} \quad \Gamma_{\mathcal{W}} = \{\mathrm{Im}\, f_x \subset \mathcal{W},\ x \in X\}$$

are isometrically isomorphic. The isomorphism is provided by the isometry $(ff^*)^{-1/2} f : \mathrm{Im}\, f^* \to \mathrm{Im}\, f$. (*Note that we do not assume that f is invertible.*) The continuous sections of these two fields are

$$\{f(v),\ v \in C_0(\mathcal{V})\} \quad \text{or} \quad \{f^*(v),\ v \in C_0(\mathcal{W})\}.$$

We will denote by Γ any of these two fields of vector spaces.

Main example. At the beginning of Section 6 we have constructed a G-vector bundle $\mathcal{V} = \mathfrak{g}_X$ over X equipped with a G-invariant Riemannian metric and a smooth G-map $f' : \mathfrak{g}_X \to \tau = T(X)$. This defines the field Γ in the case when Γ_x are tangent spaces to the orbits of the G-action on X.

Recall now Definition 2.1 of the algebra $\mathrm{Cl}_{\mathcal{V}}(X)$.

Definition 7.1. The C^*-algebra generated by $C_0(X) \subset \mathrm{Cl}_{\mathcal{V}}(X)$ together with the subspace $\{f^*(w),\ w \in C_0(\mathcal{W})\}$ in $\mathrm{Cl}_{\mathcal{V}}(X)$ will be denoted by $\mathrm{Cl}_{\Gamma}(X)$. A similar C^*-algebra generated by $C_0(X) \subset \mathrm{Cl}_{\mathcal{W}}(X)$ together with the subspace $\{f(v),\ v \in C_0(\mathcal{V})\}$ in $\mathrm{Cl}_{\mathcal{W}}(X)$ is isomorphic to $\mathrm{Cl}_{\Gamma}(X)$ because the spaces of continuous sections $\{f(v),\ v \in C_0(\mathcal{V})\}$ and $\{f^*(w),\ w \in C_0(\mathcal{W})\}$ are isometrically isomorphic. (We will not distinguish between these two C^*-algebras).

Since $\mathrm{Cl}_{\Gamma}(X)$ is a $C_0(X)$-algebra, we can define $\mathrm{Cl}_{\Gamma}(TX) = \mathrm{Cl}_{\Gamma}(X) \otimes_{C_0(X)} C_0(TX)$. The algebra $\mathrm{Cl}_{\tau}(X) \hat{\otimes}_{C_0(X)} \mathrm{Cl}_{\Gamma}(X)$ will be denoted by $\mathrm{Cl}_{\tau \oplus \Gamma}(X)$.

Let us also consider the $C_0(X)$-submodule of $C_0(\Lambda^*_{\mathbb{C}}(\mathcal{V}))$ generated by the subspace $C_0(X) = C_0(\Lambda^0_{\mathbb{C}}(\mathcal{V}))$ and all wedge products of elements $\{f^*(w),\ w \in C_0(\mathcal{W})\}$. We will denote the closure of this submodule in $C_0(\Lambda^*_{\mathbb{C}}(\mathcal{V}))$ by $C_0(\Lambda^*_{\Gamma}(X))$. (This is not in general a direct summand of $C_0(\Lambda^*_{\mathbb{C}}(\mathcal{V}))$.) A similar $C_0(X)$-submodule of $C_0(\Lambda^*_{\mathbb{C}}(\mathcal{W}))$ generated by $C_0(X) = C_0(\Lambda^0_{\mathbb{C}}(\mathcal{W}))$ and all wedge products of elements $\{f(v),\ v \in C_0(\mathcal{V})\}$ is isomorphic to $C_0(\Lambda^*_{\Gamma}(X))$, again because the spaces of continuous sections $\{f(v)\}$ and $\{f^*(w)\}$ are isometrically isomorphic.

The Hilbert module $C_0(\Lambda^*_{\Gamma}(X))$ provides Morita equivalence between the C^*-algebras $\mathcal{K}(C_0(\Lambda^*_{\Gamma}(X)))$ and $C_0(X)$. There are two natural homomorphisms $\mathrm{Cl}_{\Gamma}(X) \to \mathcal{K}(C_0(\Lambda^*_{\Gamma}(X)))$ given on continuous sections of Γ by the maps $v \mapsto \mathrm{ext}\,(v) + \mathrm{int}\,(v)$ and $v \mapsto i(\mathrm{ext}\,(v) - \mathrm{int}\,(v))$. They generate the natural isomorphism: $\mathrm{Cl}_{\Gamma}(X) \hat{\otimes}_{C_0(X)} \mathrm{Cl}_{\Gamma}(X) \simeq \mathcal{K}(C_0(\Lambda^*_{\Gamma}(X)))$.

Notation. We will generalize now Definition 6.2 of the algebra $\mathfrak{S}_G(X)$ to the case of an arbitrary field Γ defined by the map $f : \mathcal{V} \to \tau$, where $\mathcal{W} = \tau = T(X)$. We put for any $x \in X$: $\varphi_x = f_x f_x^*$. This returns us to the notation of the beginning of Section 6. In particular, for any such field Γ, we define the quadratic form q_x on covectors $\xi \in T_x^*(X)$ by $q_x(\xi) = \|f_x^*(\xi)\|^2$. The algebra $\mathfrak{S}_{\Gamma}(X)$ is defined according to Definition 6.2 (without item (2)), and the Hilbert module $\mathfrak{S}_\Gamma(E)$ is defined as $\mathfrak{S}_G(E)$ before Definition 6.3. (An analog of Atiyah's symbol space $T_G(X)$ can also be defined as in Section 6, and denoted $T_\Gamma(X)$, although we will not use it. The statement of Lemma 6.2 about the equivalence of conditions (2) and (3) remains true.)

We want to show now that in the case of the field Γ defined by the map $f : \mathcal{V} \to \tau$, the three algebras $\mathfrak{S}_\Gamma(X)$, $\mathrm{Cl}_{\tau\oplus\Gamma}(X)$ and $\mathrm{Cl}_\Gamma(TX)$ are KK-equivalent (more precisely, KK^G_{sep}-equivalent) as $G - C_0(X)$-algebras. We need fiberwise Bott and Dirac elements which implement the KK-equivalence.

Definition 7.2. The fiberwise Bott element

$$[\mathcal{B}_{\xi,\Gamma}] \in \mathcal{R}KK^G(X; \mathrm{Cl}_{\tau\oplus\Gamma}(X), \mathfrak{S}_\Gamma(X))$$

is given by the pair $(\mathfrak{S}_\Gamma(E), \beta)$, where $E = \Lambda^*_{\mathbb{C}}(TX)$ (cf. Definition 2.6), the left action of $\mathrm{Cl}_{\tau\oplus\Gamma}(X)$ on $\mathfrak{S}_\Gamma(E)$ is defined on (real) covectors by

$$\xi_1 \oplus \xi_2 \mapsto \mathrm{ext}\,(\xi_1) + \mathrm{int}\,(\xi_1) + i(\mathrm{ext}\,(\xi_2) - \mathrm{int}\,(\xi_2)).$$

Here ξ_2 is a section of $\Gamma \subset \tau = T(X)$. The operator β is the Bott operator defined by

$$\beta(\xi) = i(\mathrm{ext}\,(\xi) - \mathrm{int}\,(\xi))/(1 + \|\xi\|^2)^{1/2}.$$

It is easy to check that the commutators between $\mathrm{Cl}_{\tau\oplus\Gamma}(X)$ and β belong to $\mathcal{K}(\mathfrak{S}_\Gamma(E))$ as required.

The definition of the fiberwise Dirac element goes as follows. Let $\mathrm{Cl}_{\tau_x\oplus\Gamma_x}$ be the fiber of the algebra $\mathrm{Cl}_{\tau\oplus\Gamma}$ at the point $x \in X$. For any $x \in X$, consider the Hilbert module

$$H_x = L^2(T^*_x(X)) \otimes \mathrm{Cl}_{\tau_x\oplus\Gamma_x} \simeq L^2(T^*_x(X)) \otimes \mathrm{Cl}_{\tau_x} \hat{\otimes} \mathrm{Cl}_{\Gamma_x}$$

over $\mathrm{Cl}_{\tau_x\oplus\Gamma_x}$. There is a left action of the algebra $\mathfrak{S}_\Gamma(X)$ on $L^2(T^*_x(X)$ given by the point-wise multiplication. The Hilbert module of continuous sections, vanishing at infinity of X, of the family of Hilbert modules H_x is a Hilbert module over the algebra $\mathrm{Cl}_{\tau\oplus\Gamma}(X)$. It will be denoted by $\mathcal{H}$. The grading of $\mathcal{H}$ is defined by the grading of the algebra $\mathrm{Cl}_{\tau\oplus\Gamma}(X)$.

As in Section 2, we have the Dirac operator on $L^2(T^*_x(X)) \otimes \mathrm{Cl}_{\tau_x} \hat{\otimes} \mathrm{Cl}_{\Gamma_x}$, namely $D_x = \sum_{k=1}^{\dim X} (-i)c(e_k)\partial/\partial\xi_k \hat{\otimes} 1$ in any orthonormal basis $\{e_k\}$ of $\tau_x = T^*_x(X)$. We normalize D_x by setting $F_x = D_x(1 + D_x^2)^{-1/2}$. Now we will modify this operator by an additional summand.

Let us denote by $\mathfrak{f}_\Gamma$ the element of $\mathcal{M}(\mathrm{Cl}_\Gamma(TX))$ given by the continuous covector field $\{\mathfrak{f}_{\Gamma_x}(\xi) = \varphi_x(\xi)(1 + \|\varphi_x(\xi)\|^2)^{-1/2},\ x \in X\}$ over TX. Note that

$$1 - \mathfrak{f}^2_{\Gamma_x} = (1 + \|\varphi_x(\xi)\|^2)^{-1},$$

so $(1-\mathfrak{f}_\Gamma^2)b \in C_0(TX)$ for any $b \in \mathfrak{S}_\Gamma(X)$ by the condition (3) of Definition 6.2. The Clifford multiplication operator by $\mathfrak{f}_{\Gamma_x}(\xi)$ on $L^2(T_x^*(X)) \otimes \mathrm{Cl}_{\tau_x} \hat{\otimes} \mathrm{Cl}_{\Gamma_x}$ will still be denoted by $\mathfrak{f}_{\Gamma_x}(\xi)$. Because $\varphi_x(\xi)$ is a linear function of ξ, the graded commutator $[D_x, \mathfrak{f}_{\Gamma_x}]$ is an operator of multiplication (containing no derivatives) by a (matrix) function vanishing at infinity in ξ. By the reasoning of [30, Lemma 4.2], it follows that the graded commutator $[F_x, \mathfrak{f}_{\Gamma_x}]$ is compact. Similarly, one proves that the commutators of F_x with multiplication by elements of $\mathfrak{S}_\Gamma(X)$ are compact (using condition (1) of Definition 6.2). Since $(1-\mathfrak{f}_{\Gamma_x}^2)b \in C_0(TX)$ for any $b \in \mathfrak{S}_\Gamma(X)$, the operator $(1-F_x^2)(1-\mathfrak{f}_{\Gamma_x}^2)b$ is compact. This fiberwise construction gives global G-invariant operators F and $\mathfrak{f}_\Gamma$ on $\mathcal{H}$.

Definition 7.3. The fiberwise Dirac operator on $\mathcal{H}$ is defined as

$$\Phi = (1-F^2)^{1/2} \cdot \mathfrak{f}_\Gamma + F.$$

The pair $(\mathcal{H}, \Phi)$ is an element of the group $\mathcal{R}KK^G(X; \mathfrak{S}_\Gamma(X), \mathrm{Cl}_{\tau\oplus\Gamma}(X))$, and therefore an element of $\mathcal{R}KK^G_{sep}(X; \mathfrak{S}_\Gamma(X), \mathrm{Cl}_{\tau\oplus\Gamma}(X))$ as well. We will denote it $[d_{\xi,\Gamma}]$.

The element $[\mathfrak{f}_\Gamma] \in \mathcal{R}KK^G(X; \mathfrak{S}_\Gamma(X), \mathrm{Cl}_\Gamma(TX))$ is defined by the natural homomorphism $\mathfrak{S}_\Gamma(X) \to \mathcal{M}(\mathrm{Cl}_\Gamma(TX))$, together with the operator $\mathfrak{f}_\Gamma \in \mathcal{M}(\mathrm{Cl}_\Gamma(TX))$.

Theorem 7.4. *The algebras $\mathfrak{S}_\Gamma(X)$, $\mathrm{Cl}_\Gamma(TX)$, and $\mathrm{Cl}_{\tau\oplus\Gamma}(X)$ are KK^G_{sep}-equivalent. The equivalence of the algebras $\mathfrak{S}_\Gamma(X)$ and $\mathrm{Cl}_{\tau\oplus\Gamma}(X)$ is implemented by the elements $[d_{\xi,\Gamma}]$ and $[\mathcal{B}_{\xi,\Gamma}]$ which are inverses of each other. The element $[d_{\xi,\Gamma}]$ is equal to $[\mathfrak{f}_\Gamma] \otimes_{\mathrm{Cl}_\Gamma(TX)} ([d_\xi] \otimes_{C_0(X)} 1_{\mathrm{Cl}_\Gamma(X)})$, where*

$$[d_\xi] \otimes_{C_0(X)} 1_{\mathrm{Cl}_\Gamma(X)} \in \mathcal{R}KK^G(X; \mathrm{Cl}_\Gamma(TX), \mathrm{Cl}_{\tau\oplus\Gamma}(X)).$$

The equivalence between $\mathfrak{S}_\Gamma(X)$ and $\mathrm{Cl}_\Gamma(TX)$ is implemented by the element $[\mathfrak{f}_\Gamma]$.

Proof. The KK-equivalence between the algebras $\mathrm{Cl}_{\tau\oplus\Gamma}(X)$ and $\mathrm{Cl}_\Gamma(TX)$ follows from the KK-equivalence between $\mathrm{Cl}_\tau(X)$ and $C_0(TX)$ (Theorem 2.7) by tensoring it over $C_0(X)$ (i.e. $\otimes_{C_0(X)}$) with $\mathrm{Cl}_\Gamma(X)$. We have to prove the KK-equivalence of $\mathfrak{S}_\Gamma(X)$ and $\mathrm{Cl}_{\tau\oplus\Gamma}(X)$. The proof will be similar to the proof of Theorem 2.7.

We will consider the products fiberwise over X. The Hilbert module over $\mathrm{Cl}_{\tau\oplus\Gamma}(X)$ for the product $[\mathcal{B}_{\xi,\Gamma}] \otimes_{\mathfrak{S}_\Gamma(X)} [d_{\xi,\Gamma}]$ is

$$\mathfrak{H}_x = L^2(T_x^*(X)) \otimes \Lambda^*_{\mathbb{C}}(T_x^*(X)) \hat{\otimes} \mathrm{Cl}_{\tau_x\oplus\Gamma_x},$$

equipped with the left action of $\mathrm{Cl}_{\tau_x\oplus\Gamma_x}$ by Clifford multiplication over the $\Lambda^*_{\mathbb{C}}(T_x^*(X))$ tensor multiple. The operator can be written as

$$1 \hat{\otimes} (1+D_x^2)^{-1/2} \cdot \frac{i(\mathrm{ext}(\xi) - \mathrm{int}(\xi)) \hat{\otimes} 1 + 1 \hat{\otimes} c(\varphi_x(\xi))}{(1+\|\xi\|^2 + \|\varphi_x(\xi)\|^2)^{1/2}} + (1 \hat{\otimes} F_x).$$

We use the $\hat{\otimes}$ sign in the formulas here only to distinguish between operators acting over the two tensor multiples of $\Lambda^*_{\mathbf{C}}(T^*_x(X))\hat{\otimes}\,\mathrm{Cl}_{\tau_x\oplus\Gamma_x}$. The Clifford multiplication in the formulas for the operators $\mathfrak{f}_{\Gamma_x}$ and F_x corresponds to the left Clifford multiplication over $\mathrm{Cl}_{\tau_x\oplus\Gamma_x}$.

There exists a natural left action of the Clifford algebra $\mathrm{Cl}_{\tau_x\oplus\Gamma_x}\hat{\otimes}\,\mathrm{Cl}_{\tau_x\oplus\Gamma_x}$ on $\Lambda^*_{\mathbf{C}}(T^*_x(X))\hat{\otimes}\,\mathrm{Cl}_{\tau_x\oplus\Gamma_x}$. Using the rotation of $T^*_x(X)\oplus T^*_x(X)$ which flips the two copies of $T^*_x(X)$, we define the rotation homotopy of this action. The formula for the rotation is

$$(x,y)\mapsto(\cos t\cdot x-\sin t\cdot y,\sin t\cdot x+\cos t\cdot y),\quad 0\le t\le\pi/2.$$

It leads to the rotation homotopy of the identity automorphism of the algebra $\mathrm{Cl}_{\tau_x\oplus\Gamma_x}\hat{\otimes}\,\mathrm{Cl}_{\tau_x\oplus\Gamma_x}$ into the automorphism $\gamma_{\pi/2}$ which on the subspace $T^*_x(X)\oplus T^*_x(X)$ sends $(x,y)\mapsto(-y,x)$.

We apply this homotopy to our KK-product element. The automorphism $\gamma_{\pi/2}$ transforms the left action of the algebra $\mathrm{Cl}_{\tau_x\oplus\Gamma_x}$ on $\Lambda^*_{\mathbf{C}}(T^*_x(X))$ into the identity action of $\mathrm{Cl}_{\tau_x\oplus\Gamma_x}$ on itself. The Dirac operator part $D_x=\sum_{k=1}^{\dim X}(-i)c(e_k)\partial/\partial\xi_k$ transforms into $\sum_{k=1}^{\dim X}i(\mathrm{ext}\,(e_k)+\,\mathrm{int}\,(e_k))\partial/\partial\xi_k$. The homotopy of the

$$i(\mathrm{ext}\,(\xi)-\,\mathrm{int}\,(\xi))\hat{\otimes}1+1\hat{\otimes}c(\varphi_x(\xi))$$

part is given by

$$i[\mathrm{ext}\,(\xi+\sin t\cdot\varphi_x(\xi))-\,\mathrm{int}\,(\xi+\sin t\cdot\varphi_x(\xi))]\hat{\otimes}1+1\hat{\otimes}\cos t\cdot c(\varphi_x(\xi)).$$

Both this part and the Dirac part have to be normalized by dividing them by $(1+(\cdot))^{1/2}$ in order to obtain bounded operators.

As a result of this homotopy, our operator transforms into the operator which has the form $S_1\hat{\otimes}1$ with respect to the tensor product decomposition $L^2(\Lambda^*(T^*_x(X)))\hat{\otimes}\,\mathrm{Cl}_{\tau_x\oplus\Gamma_x}$ of $\mathfrak{H}_x$. The operator $S_1\in\mathcal{L}(L^2(\Lambda^*(T^*_x(X))))$ is homotopic to the usual KK-product of the Bott and Dirac operators because the map $\xi\mapsto\xi+\varphi_x(\xi)$ is homotopic to the identity map $\xi\mapsto\xi$ by the linear homotopy. So our KK-product is equal to $1\in\mathcal{R}KK^G(X;\mathrm{Cl}_{\tau\oplus\Gamma}(X),\mathrm{Cl}_{\tau\oplus\Gamma}(X))$.

The product of fiberwise Bott and Dirac elements in the opposite direction is also the identity by a similar rotation trick. We leave it to the reader to deduce that the KK-equivalence between $\mathfrak{S}_\Gamma(X)$ and $\mathrm{Cl}_\Gamma(TX)$ is implemented by the element $[\mathfrak{f}_\Gamma]$. □

Now we want to define analogs of the elements $[\Theta_{X,1}]$ and $[\Theta_{X,2}]$ of Section 2, namely elements $[\Theta'_{X,\Gamma}]$, $[\Theta_{X,\Gamma,1}]$, and $[\Theta_{X,\Gamma,2}]$. This will be done in full generality of Definition 7.1, for an arbitrary field Γ defined by a map $f:\mathcal{V}\to\mathcal{W}$. We start with a remark.

Remark 7.5. If A and B are any $G-C_0(X)$-algebras, then the KK-groups

$$\mathcal{R}KK^G(X;A\hat{\otimes}_{C_0(X)}\mathrm{Cl}_\Gamma(X),B)\quad\text{and}\quad\mathcal{R}KK^G(X;A,B\hat{\otimes}_{C_0(X)}\mathrm{Cl}_\Gamma(X))$$

are isomorphic. Indeed, applying the map

$$\begin{aligned}\hat{\otimes}_{C_0(X)} \mathrm{Cl}_\Gamma(X) : \mathcal{R}KK^G(X; A\hat{\otimes}_{C_0(X)} \mathrm{Cl}_\Gamma(X), B) \\ \to \mathcal{R}KK^G(X; A\hat{\otimes}_{C_0(X)} \mathrm{Cl}_\Gamma(X)\hat{\otimes}_{C_0(X)} \mathrm{Cl}_\Gamma(X), B\hat{\otimes}_{C_0(X)} \mathrm{Cl}_\Gamma(X))\end{aligned}$$

and using the isomorphism and Morita equivalence of Definition 7.1, we map the first of these groups into the second one. The inverse map is similar.

This has an immediate application. Let U be the neighborhood of the diagonal of $X \times X$ defined in Section 2. Using the possibility to shift $\hat{\otimes}_{C_0(X)} \mathrm{Cl}_\Gamma(X)$ from one argument of the $\mathcal{R}KK$-group to the other, together with the remark at the end of the introduction, we see that the group $\mathcal{R}KK^G(X; C_0(X), C_0(U) \cdot C_0(X)\hat{\otimes} \mathrm{Cl}_\tau(X))$ is isomorphic to $\mathcal{R}KK^G(X; \mathrm{Cl}_\Gamma(X), C_0(U) \cdot \mathrm{Cl}_\Gamma(X)\hat{\otimes} \mathrm{Cl}_\tau(X))$ with the action of $C_0(X)$ over the first tensor multiple. Similarly, the group

$$\mathcal{R}KK^G(X; C_0(X), C_0(U) \cdot \mathrm{Cl}_\Gamma(X)\hat{\otimes} \mathrm{Cl}_{\tau\oplus\Gamma}(X))$$

is isomorphic to the group

$$\mathcal{R}KK^G(X; \mathrm{Cl}_\Gamma(X), C_0(U) \cdot \mathrm{Cl}_\Gamma(X)\hat{\otimes} \mathrm{Cl}_\tau(X)),$$

with the action of $C_0(X)$ over the second tensor multiple.

To define the element $[\Theta'_{X,\Gamma}] \in \mathcal{R}KK^G(X; C_0(X), \mathrm{Cl}_\Gamma(X)\hat{\otimes} \mathrm{Cl}_{\tau\oplus\Gamma}(X))$, we remark first that the algebra $\mathrm{Cl}_\Gamma(X)\hat{\otimes} \mathrm{Cl}_\Gamma(X)$ is isomorphic the algebra $\mathrm{Cl}_{\Gamma\times\Gamma}(X \times X)$. The latter is defined by the map $f \times f : \mathcal{V} \times \mathcal{V} \to \mathcal{W} \times \mathcal{W}$ over $X \times X$. By restricting $\mathrm{Cl}_{\Gamma\times\Gamma}(X \times X)$ to the diagonal of $X \times X$, we get a surjective homomorphism: $\mathrm{Cl}_\Gamma(X)\hat{\otimes} \mathrm{Cl}_\Gamma(X) \to \mathrm{Cl}_\Gamma(X)\hat{\otimes}_{C_0(X)} \mathrm{Cl}_\Gamma(X)$. Since the latter algebra is isomorphic to $\mathcal{K}(C_0(\Lambda^*_\Gamma(X)))$, there exists a canonical G-invariant projection in the multiplier algebra $\mathcal{M}(\mathrm{Cl}_\Gamma(X)\hat{\otimes}_{C_0(X)} \mathrm{Cl}_\Gamma(X))$ defined as the projection onto $C_0(X)$, the zero degree exterior forms in $C_0(\Lambda^*_\Gamma(X))$. We lift this projection to a self-adjoint element of $\mathcal{M}(\mathrm{Cl}_\Gamma(X)\hat{\otimes} \mathrm{Cl}_\Gamma(X))$ and then make it G-invariant by averaging over G with a cut-off function. (The averaged element is still a lifting of the initial projection.) In a small G-invariant neighborhood U' of the diagonal of $X \times X$ (which, we assume, satisfies the properties of the neighborhood U of Section 2), the lifted element will be close to a projection, and we will correct it in order to make it a self-adjoint G-invariant projection in $\mathcal{M}(C_0(U') \cdot \mathrm{Cl}_\Gamma(X)\hat{\otimes} \mathrm{Cl}_\Gamma(X))$ by using the standard formula to obtain a projection out of an almost projection. This projection will be denoted P_0.

Definition 7.6. The element $[\Theta'_{X,\Gamma}] \in \mathcal{R}KK^G(X; C_0(X), \mathrm{Cl}_\Gamma(X)\hat{\otimes} \mathrm{Cl}_{\tau\oplus\Gamma}(X))$ is defined as follows. We will use the neighborhood U' and the projection P_0 constructed above. We adjust the covector field $\Theta = \{\Theta_x\}$ defining the element $[\Theta_X]$ so that it grows to norm 1 at the boundary of the neighborhood U' and define the operator $\Theta' \in C_0(U') \cdot \mathrm{Cl}_\Gamma(X)\hat{\otimes} \mathrm{Cl}_{\tau\oplus\Gamma}(X)$ as the compression of $1\hat{\otimes}\Theta$ by the projection P_0. This gives the element $[\Theta'_{X,\Gamma}] \in \mathcal{R}KK^G(X; C_0(X), C_0(U') \cdot \mathrm{Cl}_\Gamma(X) \hat{\otimes} \mathrm{Cl}_{\tau \oplus \Gamma}(X))$. (The action of $C_0(X)$ may go over the first or the second tensor multiple.)

We define $[\Theta_{X,\Gamma,1}] \in \mathcal{R}KK^G(X; \mathrm{Cl}_\Gamma(X), C_0(U') \cdot \mathrm{Cl}_\Gamma(X) \hat{\otimes} \mathrm{Cl}_\tau(X))$ as the element corresponding to the element $[\Theta_{X,1}]$ (of Definition 2.3) via the map $\hat{\otimes}_{C_0(X)} \mathrm{Cl}_\Gamma(X)$, with the action of $C_0(X)$ going over the first tensor multiple.

The element $[\Theta_{X,\Gamma,2}] \in \mathcal{R}KK^G(X; \mathrm{Cl}_\Gamma(X), C_0(U') \cdot \mathrm{Cl}_\Gamma(X) \hat{\otimes} \mathrm{Cl}_\tau(X))$ is defined as the element corresponding to $[\Theta'_{X,\Gamma}]$ via the isomorphism of Remark 7.5, with the action of $C_0(X)$ going over the second tensor multiple.

Lemma 7.7. *The elements* $[\Theta_{X,\Gamma,1}]$ *and* $[\Theta_{X,\Gamma,2}]$ *are equal after restriction to the group* $KK^G(\mathrm{Cl}_\Gamma(X), C_0(U') \cdot \mathrm{Cl}_\Gamma(X) \hat{\otimes} \mathrm{Cl}_\tau(X))$.

Proof. We will use the rotation homotopy (see the proof of Theorem 4.6 above and [30, Lemma 4.6]). Recall that the element $[\Theta_{X,\Gamma,2}]$ is obtained from $[\Theta'_{X,\Gamma}]$ by taking the product $\hat{\otimes}_{C_0(X)} \mathrm{Cl}_\Gamma(X)$ and then using Morita equivalence. To avoid additional notational complications, we will forget about Morita equivalence and use notation $\mathrm{Cl}_\Gamma(X) \hat{\otimes}_{C_0(X)} \mathrm{Cl}_\Gamma(X)$ instead of $\mathcal{K}(C_0(\Lambda^*_\Gamma(X)))$. So we will work with the group

$$\mathcal{R}KK^G(X; \mathrm{Cl}_\Gamma(X), C_0(U') \cdot \mathrm{Cl}_\Gamma(X) \hat{\otimes} (\mathrm{Cl}_{\tau \oplus \Gamma}(X) \hat{\otimes}_{C_0(X)} \mathrm{Cl}_\Gamma(X))).$$

The action of the first argument of this $\mathcal{R}KK^G$-group, $\mathrm{Cl}_\Gamma(X)$, goes over the last tensor multiple of $\mathrm{Cl}_\Gamma(X) \hat{\otimes} (\mathrm{Cl}_{\tau \oplus \Gamma}(X) \hat{\otimes}_{C_0(X)} \mathrm{Cl}_\Gamma(X))$. We need to shift it by rotation to the first tensor multiple.

Let $(x, y) \in U'$ and $p_t(x, y)$, $t \in [1, 2]$, be the unique geodesic segment joining points x and y in X. The family of maps $U' \to X : (x, y) \mapsto p_t(x, y)$, $1 \le t \le 2$, joins the two projections p_1 and $p_2 : U' \to X$. Therefore the two vector bundles over U' : $\mathcal{T}_1 = \mathcal{V} \times \mathcal{V} = p_1^*(\mathcal{V}) \oplus p_2^*(\mathcal{V})$ and $\mathcal{T}_2 = p_t^*(\mathcal{V} \oplus \mathcal{V})$ are G-equivariantly isomorphic, and this family of isomorphisms is norm-continuous in the parameter t. In particular, there is a homotopy of G-embeddings $\phi_t : \mathcal{V} \to \mathcal{V} \times \mathcal{V}$ such that ϕ_1 is the embedding onto the first multiple of $\mathcal{V} \times \mathcal{V}$, and ϕ_2 – onto the second multiple. We will denote the orthogonal projection $\mathcal{V} \times \mathcal{V} \to \phi_t(\mathcal{V})$ by Q_t.

To adjust this construction to our situation, we have to work actually with the vector bundle $\mathcal{V} \times (\mathcal{V} \oplus \mathcal{V})$ over $U' \subset X \times X$. First, we lift our maps ϕ_t and Q_t into the vector bundle $\mathcal{V} \times (\mathcal{V} \oplus \mathcal{V})$. We define the new ϕ_t as an embedding of $\mathcal{V}$ into the first and third copies of $\mathcal{V}$ corresponding to ϕ_t defined above, and the new Q_t will correspond to Q_t defined above on the first and third copies of $\mathcal{V}$ and will be 0 on the second copy of $\mathcal{V}$. The kernel $\mathcal{R}_t$ of this new Q_t will be isomorphic to $\mathcal{V} \oplus \mathcal{V}$ over the diagonal of $X \times X$ for all t. The image of Q_t corresponds to the first copy of $\mathcal{V}$ when $t = 1$ and to the third copy when $t = 2$.

At this point, we remark that we can also perform the same constructions for the vector bundle $\mathcal{W}$, as well as for the bundle map $f : \mathcal{V} \to \mathcal{W}$. This allows us to apply the above homotopy to the field of vector spaces $\Gamma \times (\Gamma \oplus \Gamma)$.

Now we apply the construction of the projection in the algebra $\mathcal{M}(C_0(U')\cdot \mathrm{Cl}_\Gamma(X)\hat{\otimes}\,\mathrm{Cl}_\Gamma(X))$ given before Definition 7.6 to the algebra $\mathrm{Cl}_{\Gamma\times\Gamma}(X\times X)$ associated with the vector bundle defined by the kernel $\mathcal{R}_t$ of Q_t. The resulting projection for the algebra $C_0(U')\cdot \mathrm{Cl}_\Gamma(X)\hat{\otimes}(\mathrm{Cl}_{\tau\oplus\Gamma}(X)\hat{\otimes}_{C_0(X)}\mathrm{Cl}_\Gamma(X))$ will be denoted by $P_{0,t}$.

The homotopy of homomorphisms

$$\mathrm{Cl}_\Gamma(X)\to C_0(U')\cdot \mathrm{Cl}_\Gamma(X)\hat{\otimes}(\mathrm{Cl}_{\tau\oplus\Gamma}(X)\hat{\otimes}_{C_0(X)}\mathrm{Cl}_\Gamma(X))$$

on covectors is defined by mapping covectors (isometrically) into the corresponding covectors in the image of Q_t using the embedding ϕ_t defined above, normalized as usual into $(\phi_t\phi_t^*)^{-1/2}\phi_t$. The projection $P_{0,t}$ is used to compress the operator Θ (of Definition 7.6). This gives the homotopy of operators. (The operator Θ itself is not changing along the homotopy.)

When $t=1$, the projection $P_{0,t}$ acts over $\mathrm{Cl}_{\tau\oplus\Gamma}(X)\hat{\otimes}_{C_0(X)}\mathrm{Cl}_\Gamma(X)$ and cuts it to $\mathrm{Cl}_\tau(X)\hat{\otimes}_{C_0(X)}\mathcal{K}(\Lambda^0_\Gamma(X))=\mathrm{Cl}_\tau(X)$, so for $t=1$ we get the element $[\Theta_{X,\Gamma,1}]$. □

Our next result is the Poincaré duality which generalizes both the Poincaré duality of Theorem 4.6 and the Poincaré duality of [30, Theorem 4.9]. It will serve the same purpose as Theorem 4.6: to show that the symbol and the index of a transversally elliptic operator carry the same information.

Theorem 7.8 (Poincaré duality). *For any separable $G-C^*$-algebras A and B,*

$$\begin{aligned}KK^G_*(A\hat{\otimes}\,\mathrm{Cl}_\Gamma(X),B)&\simeq \mathcal{R}KK^G_*(X;A\hat{\otimes}C_0(X),B\hat{\otimes}\,\mathrm{Cl}_{\tau\oplus\Gamma}(X))\\&\simeq \mathcal{R}KK^G_*(X;A\hat{\otimes}C_0(X),B\hat{\otimes}\,\mathrm{Cl}_\Gamma(TX)).\end{aligned}$$

Proof. As in the proof of Theorem 4.6, to simplify notation, we will deal only with the case of $A=B=\mathbf{C}$. Also because of the KK-equivalences of Theorem 7.4, we need to prove only that $K^*_G(\mathrm{Cl}_\Gamma(X)))\simeq \mathcal{R}K^G_*(X;\mathrm{Cl}_{\tau\oplus\Gamma}(X))$, or rather that $K^*_G(\mathrm{Cl}_\Gamma(X))\simeq \mathcal{R}KK^G_*(X;\mathrm{Cl}_\Gamma(X),\mathrm{Cl}_\tau(X))$ (in view of Remark 7.5).

The proof is essentially identical to the proof of Theorem 4.6. We define the homomorphisms

$$\mu=[\Theta_{X,\Gamma,2}]\otimes_{\mathrm{Cl}_\Gamma(X)}:K^*_G(\mathrm{Cl}_\Gamma(X))\to \mathcal{R}KK^G_*(X;\mathrm{Cl}_\Gamma(X),\mathrm{Cl}_\tau(X))$$

and

$$\nu=\otimes_{\mathrm{Cl}_\tau(X)}[d_X]:\mathcal{R}KK^G_*(X;\mathrm{Cl}_\Gamma(X),\mathrm{Cl}_\tau(X))\to K^*_G(\mathrm{Cl}_\Gamma(X))).$$

We apply Lemma 7.7 and Theorem 2.4 to show that $\nu\cdot\mu=\mathrm{id}$.

The other composition $\mu\cdot\nu$ is treated as follows. We fix some $\alpha\in \mathcal{R}KK^G_*(X;\mathrm{Cl}_\Gamma(X),\mathrm{Cl}_\tau(X))$. As in the proof of Theorem 4.6, we will have to consider the following triple KK-product:

$$\begin{aligned}&\mathcal{R}KK^G(X; C_0(X), \mathrm{Cl}_\Gamma(X) \hat{\otimes} \mathrm{Cl}_{\tau\oplus\Gamma}(X))\\ &\quad \hat{\otimes}_{\mathrm{Cl}_\Gamma(X)\hat{\otimes}\mathrm{Cl}_{\tau\oplus\Gamma}(X)} \mathcal{R}KK^G_*(X; \mathrm{Cl}_\Gamma(X) \hat{\otimes} \mathrm{Cl}_{\tau\oplus\Gamma}(X), \mathrm{Cl}_\tau(X) \hat{\otimes} \mathrm{Cl}_{\tau\oplus\Gamma}(X))\\ &\quad \otimes_{\mathrm{Cl}_\tau(X)\hat{\otimes}\mathrm{Cl}_{\tau\oplus\Gamma}(X)} \mathcal{R}KK^G(X; \mathrm{Cl}_\tau(X) \hat{\otimes} \mathrm{Cl}_{\tau\oplus\Gamma}(X), \mathrm{Cl}_{\tau\oplus\Gamma}(X)).\end{aligned}$$

The first KK-group of this product contains the element $[\Theta'_{X,\Gamma}]$; the second one, the element $\alpha \otimes 1$; the third one, the element $[d_X] \otimes 1$. Note that we have replaced here the element $[\Theta_{X,\Gamma,2}]$ in the first position with $[\Theta'_{X,\Gamma}]$. So we have to expect the product to be equal to the same α, but in $\mathcal{R}K^G_*(X; \mathrm{Cl}_{\tau\oplus\Gamma}(X))$.

Now the "rotation" homotopy can be carried out exactly in the same way as in the proof of Theorem 4.6. □

Corollary 7.9. *When $\Gamma = \tau \oplus \Gamma'$, we have:*

$$K^*_G(\mathrm{Cl}_{\tau\oplus\Gamma'}(X)) \simeq \mathcal{R}K^*_G(X; \mathrm{Cl}_{\Gamma'}(X)),$$

and when $\Gamma = \tau$, we have:

$$K^*_G(\mathrm{Cl}_\tau(X)) \simeq RK^*_G(X)$$

(which is the "first Poincaré duality" [30, Theorem 4.9]).

8. Index theorems for transversally elliptic operators

This section contains the main results for transversally elliptic operators with respect to a group action. In particular, the field Γ will be defined as the field of tangent spaces to the orbits of the action of the group G (cf. Sections 6 and 7).

We start with some remarks about K-homology groups $K^*(C^*(G, C_0(X)))$, $K^*(C^*(G, \mathrm{Cl}_\Gamma(X)))$, and other similar groups that we will use below.

First of all, for any $G-C^*$-algebra B, there exists a natural unitary representation $u : G \to \mathcal{M}(C^*(G, B))$ given on elements $b \in C_c(G, B)$ by $(u_g \cdot b)(h) = g(b(g^{-1}h))$, for any $g, h \in G$. In particular, for the algebra $B = C_0(X)$, if an element $b \in C_c(G, C_0(X))$ is presented as a function $b(h, x)$, we have: $(u_g \cdot b)(h, x) = b(g^{-1}h, g^{-1}x)$. The group G acts on $C^*(G, B)$ by conjugation with this unitary representation.

Let us compare now the groups $K^*(C^*(G, B))$ and $K^*_G(C^*(G, B))$.

Lemma 8.1. *The group $K^*(C^*(G, B))$ is a direct summand in $K^*_G(C^*(G, B))$ characterized by the following condition: suppose a triple $\alpha = (H, \psi, T)$ represents an element of $K^*_G(C^*(G, B))$, where H is a G-Hilbert space and $\psi : C^*(G, B) \to \mathcal{L}(H)$ a homomorphism, and assume that ψ is non-degenerate* *and corresponds to a covariant representation* (v, ψ_B) *of* (G, B) *in* H*. Then* α *belongs to the subgroup* $K^*(C^*(G, B))$ *if the* G*-action on* H *is defined by the representation* v*.*

Proof. The projection map $K^*_G(C^*(G, B)) \to K^*(C^*(G, B))$ is the natural "forgetting" map. The embedding $K^*(C^*(G, B)) \to K^*_G(C^*(G, B))$ is constructed as follows. Let a triple $\alpha = (H, \psi : C^*(G, B) \to \mathcal{L}(H), T)$ represent an element of $K^*(C^*(G, B))$. We can always assume that ψ is non-degenerate (cf. [30, Lemma 2.8]), and then ψ defines a covariant representation (v, ψ_B) of (G, B) on the Hilbert space H. It is easy to check that $\psi(u_g \cdot b)l = v_g \cdot \psi(b)l$, for any $g \in G$, $l \in H$, $b \in C_c(G, B)$, which means that the homomorphism ψ is equivariant with respect to the conjugation actions by u and v on $C^*(G, B)$ and $\mathcal{L}(H)$ respectively. For simplicity, we will denote the operator $v_g T v_g^{-1}$ by $g(T)$.

The only condition that we have to check in order to show that $\alpha \in K^*_G(C^*(G, B))$ is that for any $g \in G$ and $b \in C_c(G, B)$, $(g(T) - T)\psi(b) \in \mathcal{K}(H)$. This is equivalent to $(Tv_g - v_g T)\psi(b) \in \mathcal{K}(H)$, and easily follows from the relation $v_g \cdot \psi(b) = \psi(u_g \cdot b)$ mentioned above. The norm-continuity of $g \mapsto (g(T) - T)\psi(a)$ for $a \in C^*(G, B)$ follows from [39, Theorem 1.4.1]. □

Remark. From now on, everywhere in this section, we will consider the group $K^*(C^*(G, B))$ as a subgroup and direct summand of $K^*_G(C^*(G, B))$.

Next, we want to define the orbital Dirac operator D_Γ. We need it in order to construct certain KK-theory elements, e.g. an element of the group $KK^G(C_0(X), C^*(G, \mathrm{Cl}_\Gamma(X)))$. However, these will be elements of some *different* groups, and we will first introduce notation related with this.

Definition 8.2. Let E be a (graded) finite-dimensional G-vector bundle over X equipped with a G-invariant Hermitian metric. Then there is a natural representation $\eta_E : C^*(G, C_0(X)) \to \mathcal{L}(L^2(E))$. We will denote the image of $C^*(G, C_0(X))$ under this representation by $C^*_E(G, C_0(X))$. If $E_1 \subset E_2$ is an equivariant and isometric embedding of vector bundles, then $\ker \eta_{E_2} \subset \ker \eta_{E_1}$, so there is a natural map $C^*_{E_2}(G, C_0(X)) \to C^*_{E_1}(G, C_0(X))$. We obtain an inverse system of C^*-algebras parametrized by the inductive system of vector bundles on X. We will often use the notation $C^*_E(G, C_0(X))$ to indicate some generic vector bundle E, not a particular one.

Similarly, by considering representations of the algebra $C^*(G, \mathrm{Cl}_\Gamma(X))$ on the spaces $L^2(\Lambda^*(X) \hat{\otimes} E)$ we get an inverse system of algebras $C^*_E(G, \mathrm{Cl}_\Gamma(X))$.

Remark. The purpose of taking quotients of $C^*(G, C_0(X))$ can be explained as follows. Suppose, X is a one-orbit space G/M. Then $C^*(G, C_0(X)) \simeq \mathcal{K}(L^2(G/M)) \otimes C^*(M)$ [18]. We want to replace the infinite-dimensional algebra $C^*(M)$ by its image in some finite-dimensional representation W, so we consider the image of $C^*(G, C_0(X))$ in $L^2(E)$, where E is the vector bundle $G \times_M W$ over $X = G/M$.

We turn now to the definition of the operator D_Γ. The unitary representation $u : G \to \mathcal{M}(C^*(G, \mathrm{Cl}_\Gamma(X)))$ defined above will be used in an essential way here. We will also use the smooth subalgebra $Cl^\infty_{\Gamma,c}(X) \subset \mathrm{Cl}_\Gamma(X)$ generated in $\mathrm{Cl}_\Gamma(X)$ by $C_c^\infty(X)$ and the image of the map $f' : C_c^\infty(\mathfrak{g}_X) \to C_c^\infty(T(X))$, where $\mathfrak{g}_X$ is the vector bundle used in the definition of Γ (cf. Section 6 and "Main example" before Definition 7.1).

Now let us fix a finite-dimensional vector bundle E as in Definition 8.2. We will include the subscript "E" in all crossed product notation $C^*(G, \cdot)$ and $C_c^\infty(G, \cdot)$ that follows.

Definition 8.3. Let $v_1, \dots, v_{\dim G}$ be an orthonormal basis in $\mathfrak{g}$ for the norm $\|\cdot\|_x$ at the point x, and $g_k(t)$ the one-parameter subgroup of G corresponding to the vector v_k. Denote by $\partial/\partial v_k$ the corresponding derivation operators on $C^\infty_{E,c}(G, \mathrm{Cl}^\infty_{\Gamma,c}(X))$: $\partial/\partial v_k(b) = \mathrm{d}/\mathrm{d}t(u_{g_k(t)}(b))|_{t=0}$ for any $b \in C^\infty_{E,c}(G, \mathrm{Cl}^\infty_{\Gamma,c}(X))$ at the point $x \in X$. We define the orbital Dirac operator D_Γ on the space $C^\infty_{E,c}(G, \mathrm{Cl}^\infty_{\Gamma,c}(X))$ as $\sum_{k=1}^{\dim G}(-i)c(f'_x(v_k))\partial/\partial v_k$, where $c(\cdot)$ means Clifford multiplication.

Lemma 8.4. *D_Γ is a G-invariant operator on the space $C^\infty_{E,c}(G, \mathrm{Cl}^\infty_{\Gamma,c}(X))$. It defines an unbounded multiplier of $C^*_E(G, \mathrm{Cl}_\Gamma(X))$, i.e. this operator has the property of formal self-adjointness: $(D_\Gamma b_1)^* b_2 = b_1^* D_\Gamma b_2$, for any $b_1, b_2 \in C^\infty_{E,c}(G, Cl^\infty_\Gamma(X))$, and the operators $D_\Gamma \pm i$ have dense range in $C^*_E(G, \mathrm{Cl}_\Gamma(X))$. In particular, $(D_\Gamma \pm i)^{-1}$, $(1 + D_\Gamma^2)^{-1}$ and $D_\Gamma(1 + D_\Gamma^2)^{-1/2}$ are bounded multipliers of $C^*(G, \mathrm{Cl}_\Gamma(X))$. Moreover, for any $a \in C_0(X)$, the operators $a(D_\Gamma \pm i)^{-1}$ and $a(1 + D_\Gamma^2)^{-1}$ belong to $C^*_E(G, \mathrm{Cl}_\Gamma(X))$.*

Proof. The formal self-adjointness and G-invariance of D_Γ are easy calculations. Let us turn to the density of the range of $D_\Gamma \pm i$. This is local with respect to X/G, i.e. it is enough to prove this for a small neighborhood of one orbit. Let $x \in X$ and let M be the stability subgroup at the point x. Then the orbit passing through x is isomorphic to G/M. A small closed G-invariant tubular neighborhood V of this orbit in X is diffeomorphic to $G \times_M Z$, where Z is a closed ball in X orthogonal at the point x to the orbit passing through x.

Let us proceed by induction on the dimension of the space X. Let ∂V be the boundary of the neighborhood V. This is a G-space, so the Dirac operator D_Γ restricts to $C^\infty_{E,c}(G, \mathrm{Cl}^\infty_{\Gamma,c}(\partial V))$. Note that $\partial V = G \times_M \partial Z$, where ∂Z is the boundary sphere of Z. For any concentric sphere $S \subset Z$, the G-space $G \times_M S$ carries an isomorphic action of G. Let us denote by $D_{\Gamma,r}$ the operator D_Γ restricted to $G \times_M S$ for the sphere S of radius r.

Assume that the operator $D_\Gamma \pm i$ has dense range in $C^*_E(G, \mathrm{Cl}_\Gamma(\partial V))$. This will imply the existence of the inverse operator of C^*-norm 1, and the same will be true on $C^*(G, \mathrm{Cl}_\Gamma(G \times_M S))$ for all concentric spheres S of the ball Z. If we identify the two spheres, S_1 of the radius r_1 and S_2 of the radius r_2, with each other, then D_{Γ,r_1} and D_{Γ,r_2} will differ only by a certain scaling of the Clifford multiplication operators $c(f_x'(v_k))$, and this scaling will depend linearly on the radius. (More precisely, there will be no scaling in the direction of $v_k \in \mathfrak{m}^\perp$, where $\mathfrak{m}$ is the Lie algebra of the group M, and the linear scaling in the direction of $v_k \in \mathfrak{m}$.) Since

$$(D_{\Gamma,r_1} \pm i)^{-1} - (D_{\Gamma,r_2} \pm i)^{-1} = (D_{\Gamma,r_1} \pm i)^{-1}(D_{\Gamma,r_2} - D_{\Gamma,r_1})(D_{\Gamma,r_2} \pm i)^{-1},$$

the operator $(D_{\Gamma,r} \pm i)^{-1}$ depends norm-continuously on r and has norm-limit when $r \to 0$. (The norm-limit exists because by considering $C_E^*(G,\cdot)$ instead of $C^*(G,\cdot)$, we have reduced $C^*(M)$ to its finite-dimensional representation, as explained in the remark above.) This implies that the operator $D_\Gamma \pm i$ on $C_{E,c}^\infty(G, \mathrm{Cl}_{\Gamma,c}^\infty(G \times_M Z))$ also has dense range.

Since the space ∂V has dimension $\dim X - 1$, proceeding by induction we come to the case of $\dim Z = 0$. This is the case of a homogeneous space $X = G/M$, where M is the stability subgroup at the point x. In this case, the algebra $C_E^*(G, \mathrm{Cl}_\Gamma(X))$ is a subalgebra of the algebra of compact operators on $L^2(\tilde{E})$ for some finite-dimensional vector bundle $\tilde{E}$ over X, and the Dirac operator D_Γ corresponds to the usual Dirac operator on this vector bundle. So $D_\Gamma \pm i$ has dense range (cf. [40], or Lemma 5.9 above).

To prove the last assertion of the lemma, we will use the same induction on dimension. We have seen already that on the space $V = G \times_M Z$, the operator $(D_{G,r} \pm i)^{-1}$ restricted to $C_{E,c}^\infty(G, \mathrm{Cl}_{\Gamma,c}^\infty(G \times_M S))$ for the sphere S of radius r depends norm-continuously on r and has a norm-limit when $r \to 0$. If we assume that for some $a \in C_0(X)$, the operators $a(D_\Gamma \pm i)^{-1}$, restricted to $G \times_M \partial V$, belong to $C_E^*(G, \mathrm{Cl}_\Gamma(G \times_M \partial V))$, then the same will be true for $C_E^*(G, \mathrm{Cl}_\Gamma(G \times_M S))$ for all concentric spheres S, and it will follow that $a(D_\Gamma \pm i)^{-1} \in C_E^*(G, \mathrm{Cl}_\Gamma(G \times_M Z))$.

By induction, we come again to the case of $\dim Z = 0$, i.e. $X = G/M$, which corresponds to the case of the usual Dirac operator on a vector bundle. So the assertion follows from the Rellich lemma. $\square$

There is another version of the operator D_Γ which we will also use. It is obtained by replacing in the Definition 8.3 the algebra $\mathrm{Cl}_\Gamma(X)$ with the algebra of compact operators on the Hilbert module $C_0(\Lambda_\Gamma^*(X))$, i.e. with $\mathrm{Cl}_\Gamma(X)\hat{\otimes}_{C_0(X)} \mathrm{Cl}_\Gamma(X)$ (cf. Definition 7.1). This operator D_Γ acts by the formula of Definition 8.3 using the Clifford variables of the second tensor multiple $\mathrm{Cl}_\Gamma(X)$. In the exterior algebra notation, it corresponds to the formula: $\sum_{k=1}^{\dim G}(\mathrm{ext}\,(f_x'(v_k)) - \mathrm{int}\,(f_x'(v_k))\partial/\partial v_k$. In addition to this, there is an action of $\mathrm{Cl}_\Gamma(X)$ on $C_E^*(G, C_0(\Lambda_\Gamma^*(X))) = C_0(\Lambda_\Gamma^*(X))\hat{\otimes}_{C_0(X)} C_E^*(G, C_0(X))$ over the first tensor multiple of $\mathrm{Cl}_\Gamma(X)\hat{\otimes}_{C_0(X)} \mathrm{Cl}_\Gamma(X)$. In the exterior algebra notation this corresponds to the action given on continuous sections of Γ by $v \mapsto \mathrm{ext}\,(v) + \mathrm{int}\,(v)$.

Still another variation that we also need is D_Γ acting as unbounded multiplier on the algebra $C_E^*(G, \mathrm{Cl}_{\tau\oplus\Gamma}(X))$ by the formula of Definition 8.3 using the Clifford variables of $\mathrm{Cl}_\Gamma(X)$. We will use all three versions of it.

Set $F_\Gamma = D_\Gamma(1 + D_\Gamma^2)^{-1/2}$ and consider this operator as an element of $\mathcal{M}(C_E^*(G, \mathrm{Cl}_\Gamma(X)))$, or $\mathcal{M}(C_E^*(G, \mathrm{Cl}_{\tau\oplus\Gamma}(X)))$, or $\mathcal{M}(C_E^*(G, C_0(\Lambda_\Gamma^*(X))))$ according to Lemma 8.4. The (graded) commutators of this operator with the left multiplication by elements $a \in C_0(X)$, or $a \in \mathrm{Cl}_\tau(X)$, or $a \in \mathrm{Cl}_\Gamma(X)$ belong to $C_E^*(G, \mathrm{Cl}_\Gamma(X))$, or $C_E^*(G, \mathrm{Cl}_{\tau\oplus\Gamma}(X))$, or $C_E^*(G, C_0(\Lambda_\Gamma^*(X)))$ respectively. Indeed, for example in the second case, for any $a \in \mathrm{Cl}_\tau^\infty(X)$ with compact support, $[D_\Gamma, a] \in \mathrm{Cl}_{\tau\oplus\Gamma}^\infty(X)$ and has compact support. So the claim follows by the same method as in [30, Lemma 4.2], using Lemma 8.4 instead of the Rellich lemma.

Definition 8.5. Set $F_\Gamma = D_\Gamma(1 + D_\Gamma^2)^{-1/2} \in \mathcal{M}(C_E^*(G, \mathrm{Cl}_\Gamma(X)))$ and define the element $[D_\Gamma] \in KK^G(C_0(X), C_E^*(G, \mathrm{Cl}_\Gamma(X)))$ as the pair $(C_E^*(G, \mathrm{Cl}_\Gamma(X)), F_\Gamma)$, where $C_0(X)$ acts on $C_E^*(G, \mathrm{Cl}_\Gamma(X))$ on the left by multiplication.

Similarly, we define the element $[D_\Gamma] \in KK^G(\mathrm{Cl}_\Gamma(X), C_E^*(G, C_0(X)))$ as the pair $(C_0(\Lambda_\Gamma^*(X))\hat{\otimes}_{C_0(X)} C_E^*(G, C_0(X)), F_\Gamma)$. Here the action of the algebra $\mathrm{Cl}_\Gamma(X)$ on $C_0(\Lambda_\Gamma^*(X))$ is given by $v \mapsto \mathrm{ext}\,(v) + \mathrm{int}\,(v)$, where v is a section of Γ.

Also we define the element $[D_{\Gamma,\tau}] \in KK^G(\mathrm{Cl}_\tau(X), C_E^*(G, \mathrm{Cl}_{\tau\oplus\Gamma}(X)))$ as the pair $(C_E^*(G, \mathrm{Cl}_{\tau\oplus\Gamma}(X)), F_\Gamma)$, where $\mathrm{Cl}_\tau(X)$ acts on $C_E^*(G, \mathrm{Cl}_{\tau\oplus\Gamma}(X))$ by left multiplication.

Definition 8.6. We define the group $\varprojlim KK^G(A, C_E^*(G, \mathrm{Cl}_{\Gamma'}(X)))$, for any graded $G-C^*$-algebra A, as the inverse limit of $KK^G(A, C_E^*(G, \mathrm{Cl}_{\Gamma'}(X)))$ over the inverse system of Definition 8.2. The KK-elements constructed in Definition 8.5 represent elements of $\varprojlim KK^G(A, C_E^*(G, \mathrm{Cl}_{\Gamma'}(X)))$, when $\Gamma' = \Gamma$, or $\Gamma' = \Gamma \oplus \Gamma$, or $\Gamma' = \tau \oplus \Gamma$ respectively, and $A = C_0(X)$, or $A = \mathrm{Cl}_\Gamma(X)$, or $A = \mathrm{Cl}_\tau(X)$ respectively.

Definition 8.7. Similarly, we define the group $\varinjlim K_G^*(C_E^*(G, \mathrm{Cl}_{\Gamma'}(X)))$. The inductive limit here is taken over the inverse system of Definition 8.2. All index elements $\mathrm{ind}(A)$ of Definition 6.4 (for Γ' trivial) and the element $[d_{X,\Gamma}]$ of the next Definition 8.8 (for $\Gamma' = \Gamma$) live in these groups because the representation in a Hilbert space used in the definition of these elements always factors through some $C_E^*(G, \mathrm{Cl}_{\Gamma'}(X))$. Note that there exists a natural KK-product pairing between the corresponding groups $\varprojlim KK^G(A, C_E^*(G, \mathrm{Cl}_{\Gamma'}(X)))$ and these groups $\varinjlim K_G^*(C_E^*(G, \mathrm{Cl}_{\Gamma'}(X)))$.

Remark. Lemma 8.1 also applies to the algebras $C_E^*(G, \mathrm{Cl}_{\Gamma'}(X))$. This means that $\varinjlim K^*(C_E^*(G, \mathrm{Cl}_{\Gamma'}(X)))$ is a direct summand of $\varinjlim K_G^*(C_E^*(G, \mathrm{Cl}_{\Gamma'}(X)))$.

Before we proceed further, we will define one more important element, the Dirac element $[d_{X,\Gamma}] \in K^0(C^*(G, \mathrm{Cl}_{\tau\oplus\Gamma}(X))) \subset K_G^0(C^*(G, \mathrm{Cl}_{\tau\oplus\Gamma}(X)))$. As mentioned in the previous definition, it can also be considered as an element of the inductive limit group $\varinjlim K^0(C_E^*(G, \mathrm{Cl}_{\tau\oplus\Gamma}(X)))$.

Definition-Lemma 8.8. The Dirac element $[d_{X,\Gamma}] \in K^0(C^*(G, \mathrm{Cl}_{\tau\oplus\Gamma}(X)))$ is given by the pair (H, F_X), where $H = L^2(\Lambda^*(X))$, the action of $\mathrm{Cl}_{\tau\oplus\Gamma}(X)$ on H

is defined on (real) covectors by

$$\xi_1 \oplus \xi_2 \mapsto \operatorname{ext}(\xi_1) + \operatorname{int}(\xi_1) + i(\operatorname{ext}(\xi_2) - \operatorname{int}(\xi_2)).$$

Here ξ_1 is a section of $\tau = T(X)$ and ξ_2 is a section of Γ. The representation of $C^*(G, \mathrm{Cl}_{\tau\oplus\Gamma}(X))$ on H is induced by the covariant representation defined by the action of G and the above action of $\mathrm{Cl}_{\tau\oplus\Gamma}(X)$, the operator d_X is the exterior derivative operator, and $F_X = (d_X + d_X^*)(1 + (d_X + d_X^*)^2)^{-1/2}$.

Proof. We need to show that the pair (H, F_X) satisfies the conditions required for an element of the group $K^0(C^*(G, \mathrm{Cl}_{\tau\oplus\Gamma}(X)))$. Although the commutation properties of the operator F_X with $\mathrm{Cl}_\tau(X)$ are known to be good (cf. [30, Lemma 4.2]), the commutation properties with $\mathrm{Cl}_\Gamma(X)$ are not so good. However, because we are working here with the algebra $C^*(G, \mathrm{Cl}_\Gamma(X))$, we must replace $a = \operatorname{ext}(\xi_2) - \operatorname{int}(\xi_2) \in \mathrm{Cl}_\Gamma(X)$ with ae, where $e \in C_c^\infty(G)$. Moreover, we can assume that a is smooth, with compact support. The commutator $[d_X + d_X^*, ae]$ is equal $[d_X + d_X^*, a]e$ because $d_X + d_X^*$ is G-invariant.

The commutator $[d_X + d_X^*, a]$ is an order 1 differential operator in the *orbit direction*, with compact support. So modulo bounded operators, it can be presented as a sum $\sum_k a_k \partial/\partial v_k$, where $\partial/\partial v_k$ are derivatives along one-parameter subgroups of G (as in Lemma 6.6) and a_k Clifford multiplication operators. This means that the operator $[d_X + d_X^*, a]e$ is bounded (see Lemma 6.6). The distributional kernel of this operator has compact support. The compactness of commutators of the operator F_X with the elements of the algebra $C^*(G, \mathrm{Cl}_\Gamma(X))$ now follows by the reasoning of [30, Lemma 4.2]. $\square$

Now we will clarify the relation between the elements $[D_{\Gamma,\tau}]$, $[d_{X,\Gamma}]$ and the element $[d_X] \in K_G^0(\mathrm{Cl}_\tau(X))$ (of Definition 2.2).

Theorem 8.9. $[D_{\Gamma,\tau}] \otimes_{C_E^*(G,\mathrm{Cl}_{\tau\oplus\Gamma}(X))} [d_{X,\Gamma}] = [d_X]$, *where* $E = T^*(X)$.

Proof. Let us introduce some notation. Set $H = L^2(\Lambda^*(X))$. We will denote the operator $d_X + d_X^*$ on H by D_X and $D_X(1 + D_X^2)^{-1/2}$ by F_X, the image of the operator D_Γ under the covariant representation of $C^*(G, \mathrm{Cl}_{\tau\oplus\Gamma}(X))$ in H (Definition 8.8) will be denoted by D_Γ, and the operator $D_\Gamma(1 + D_\Gamma^2)^{-1/2}$ by F_Γ.

Let $J = \mathcal{K}(H)$ and A_1 be the subalgebra of $\mathcal{L}(H)$ generated by J and the image of the algebra $C^*(G, \mathrm{Cl}_{\tau\oplus\Gamma}(X))$ in $\mathcal{L}(H)$. Set A_2 to be the subalgebra of $\mathcal{L}(H)$ generated by $(1 + D_X^2)^{-1}$ and $[F_\Gamma, F_X]$. We define $\Delta \subset \mathcal{L}(H)$ as the subalgebra of $\mathcal{L}(H)$ generated by the operators F_X, F_Γ and the algebra $\mathrm{Cl}_{\tau\oplus\Gamma}(X)$. Applying [30, Theorem 1.4] we obtain operators M_1 and M_2 such that $M_1^2 + M_2^2 = 1$, $M_i a_i \in J$ for $a_i \in A_i$, and $[M_i, a] \in J$ for $a \in \Delta$. We define the product operator as $M_1 F_\Gamma + M_2 F_X$.

We want to prove that this operator is homotopic to F_X, using [37, Lemma 11]. This requires to show that the graded commutator of these two operators, multiplied by any $a \in \mathrm{Cl}_\tau(X)$ on the left and by a^* on the right, is a positive operator modulo J. So we have to prove that for any $a \in \mathrm{Cl}_\tau(X)$,

$$M_1^{1/2} a [F_\Gamma, F_X] a^* M_1^{1/2} \geq 0 \ (\mathrm{mod}\ J).$$

Let us first compute the symbol of the operator D_Γ. As in the proof of Proposition 6.4, we have:

$$\begin{aligned} \sigma_{D_\Gamma}(x,\xi) &= \sum_k (\xi, f_x'(v_k)) i (\mathrm{ext}\,(f_x'(v_k)) - \mathrm{int}\,(f_x'(v_k))) \\ &= \sum_k (f_x'^*(\xi), v_k) i (\mathrm{ext}\,(f_x'(v_k)) - \mathrm{int}\,(f_x'(v_k))) \\ &= i (\mathrm{ext}\,(\varphi_x(\xi)) - \mathrm{int}\,(\varphi_x(\xi))). \end{aligned}$$

(The i comes from the action of $\mathrm{Cl}_{\tau\oplus\Gamma}(X)$ on H defined in 8.8.) Since the symbol of the operator D_X is $i(\mathrm{ext}\,(\xi) - \mathrm{int}\,(\xi))$, the symbol of the operator $[D_\Gamma, D_X]$ (modulo symbols of order 1) is equal to $2(\varphi_x(\xi), \xi)$. Let $\mathfrak{d}_\Gamma$ be a first order differential operator with the symbol $\mathrm{ext}\,(\varphi_x^{1/2}(\xi)) - \mathrm{int}\,(\varphi_x^{1/2}(\xi))$. Then the symbol of $\mathfrak{d}_\Gamma^* \mathfrak{d}_\Gamma$ is $\|\varphi_x^{1/2}(\xi)\|^2 = (\varphi_x(\xi), \xi)$ (modulo symbols of order 1). It means that $\mathfrak{D} = [D_\Gamma, D_X] - 2\mathfrak{d}_\Gamma^* \mathfrak{d}_\Gamma$ is an operator of order 1. To simplify further notation we will denote $[D_\Gamma, D_X]$ by $D_{\Gamma,X}$.

Now we will use the integral representations

$$\begin{aligned} F_X &= (2/\pi) \int_0^\infty D_X (1 + \lambda^2 + D_X^2)^{-1}\, \mathrm{d}\lambda; \\ F_\Gamma &= (2/\pi) \int_0^\infty D_\Gamma (1 + \mu^2 + D_\Gamma^2)^{-1}\, \mathrm{d}\mu, \end{aligned}$$

(cf. [30, Lemma 4.2]) in order to present the operator $[F_\Gamma, F_X]$ in the form similar to the one used in the proof of Theorem 5.4 of [30, p. 187]:

$$\begin{aligned} (4/\pi^2) \int_0^\infty \int_0^\infty (& k_{G,\mu} k_{X,\lambda} D_{\Gamma,X} k_{X,\lambda} k_{G,\mu} + h_{G,\mu} k_{X,\lambda} D_{\Gamma,X} k_{X,\lambda} h_{G,\mu} \\ & + k_{G,\mu} h_{X,\lambda} D_{\Gamma,X} h_{X,\lambda} k_{G,\mu} + h_{G,\mu} h_{X,\lambda} D_{\Gamma,X} h_{X,\lambda} h_{G,\mu})\, \mathrm{d}\lambda\, \mathrm{d}\mu, \end{aligned}$$

where

$$\begin{aligned} k_{X,\lambda} &= (1 + \lambda^2)^{1/2} (1 + \lambda^2 + D_X^2)^{-1}, \\ k_{G,\mu} &= (1 + \mu^2)^{1/2} (1 + \mu^2 + D_\Gamma^2)^{-1}, \\ h_{X,\lambda} &= D_X (1 + \lambda^2 + D_X^2)^{-1}, \\ h_{G,\mu} &= D_\Gamma (1 + \mu^2 + D_\Gamma^2)^{-1}. \end{aligned}$$

If we replace $D_{\Gamma,X}$ in the above expression with $\mathfrak{D}$, all integrals over λ will converge in the strong topology and give bounded operators. Indeed, $-c(1+D_X)^{1/2} \leq \mathfrak{D} \leq c(1+D_X)^{1/2}$ for some $c > 0$ because D_X is elliptic and $\mathfrak{D}$ is a differential operator of order 1, and

$$\int_0^\infty (1+D_X)^{1/2}(1+\lambda^2+D_X^2)^{-1}\,\mathrm{d}\lambda = \pi/2.$$

The second integrals, in μ, will converge in norm, and because $M_1 a(1+\mu^2+D_\Gamma^2)^{-1}$ is a compact operator, the result will be a compact operator.

It remains to show that by replacing $\mathcal{D}_{\Gamma,X}$ with $2\mathfrak{d}_\Gamma^*\mathfrak{d}_\Gamma$ in the same integral expression we get a positive operator. This is obvious since $2\mathfrak{d}_\Gamma^*\mathfrak{d}_\Gamma$ is positive. The integral converges in the strong topology because there is convergence both for $\mathcal{D}_{\Gamma,X}$ and $\mathfrak{D}$. □

Definition 8.10. We define the Clifford index $[A]^{\mathrm{cl}} \in K_G^*(\mathrm{Cl}_\Gamma(X))$ of a transversally elliptic operator A acting on sections of a vector bundle E over X as $[D_\Gamma] \otimes_{C_E^*(G,C_0(X))} \mathrm{ind}(A)$. The element $[D_\Gamma]$ is considered here as an element of $\varprojlim KK^G(\mathrm{Cl}_\Gamma(X), C_E^*(G, C_0(X)))$ (Definition 8.6).

Recall that the symbol of A is an element $[\sigma_A] \in \mathcal{R}K_*^G(X; \mathfrak{S}_\Gamma(X))$ (Definition 6.3).

Definition 8.11. We define the Clifford symbol of the operator A as

$$[\sigma_A^{\mathrm{cl}}] = [\sigma_A] \otimes_{\mathfrak{S}_\Gamma(X)} [d_{\xi,\Gamma}] \in \mathcal{R}K_*^G(X; \mathrm{Cl}_{\tau\oplus\Gamma}(X))$$

(cf. Theorem 7.4).

Theorem 8.12 (Inverse Clifford index theorem). *Let X be a complete Riemannian manifold and G a Lie group which acts on X properly and isometrically. Let A be a properly supported, G-invariant, L^2-bounded transversally elliptic operator on X of order* 0*, of the Hörmander class $\rho = 1, \delta = 0$. Then*

$$[\sigma_A^{\mathrm{cl}}] = [\Theta'_{X,\Gamma}] \otimes_{\mathrm{Cl}_\Gamma(X)} [A]^{\mathrm{cl}} \in \mathcal{R}K_*^G(X; \mathrm{Cl}_{\tau\oplus\Gamma}(X)).$$

Proof. The product $[A]^{\mathrm{cl}} = [D_\Gamma] \otimes_{C_E^*(G,C_0(X))} \mathrm{ind}(A)$ is given by the pair $(\mathcal{M}', S')$, where $\mathcal{M}' = C_0(\Lambda_\Gamma^*(X)) \hat{\otimes}_{C_0(X)} L^2(E)$, $S' = M_1 \cdot \tilde{A} + M_2 \cdot F_\Gamma \hat{\otimes} 1$, the operator F_Γ is defined in 8.5, $\tilde{A}$ is a K-theoretic A-connection (see [37, Definition 8] or [30, 2.6]), and M_1, M_2 are the operators entering the KK-product construction (see [30, Theorem 2.11]), $M_1^2 + M_2^2 = 1$. (The sign $\hat{\otimes}$ is used here only to keep track of the grading of operators acting over $C_0(\Lambda_\Gamma^*(X))$ and $L^2(E)$. All derivatives act over $L^2(E)$.)

The right-hand side KK-product in the statement of the theorem is represented by the triple product:

$$[\Theta'_{X,\Gamma}] \otimes_{\mathrm{Cl}_\Gamma(X)} [D_\Gamma] \otimes_{C_E^*(G,C_0(X))} \mathrm{ind}(A).$$

The Hilbert module for the triple product is

$$C_0(U') \cdot (C_0(\Lambda^*_\Gamma(X))\hat{\otimes}_{C_0(X)} L^2(E))\hat{\otimes}\,\mathrm{Cl}_{\tau\oplus\Gamma}(X).$$

Here the algebra $C_0(X)$ acts over $\mathrm{Cl}_{\tau\oplus\Gamma}(X)$, and the Clifford multiplication part of the operator D_Γ acts over $C_0(\Lambda^*_\Gamma(X))$. The projection P_0 of Definition 7.6 acts over $C_0(\Lambda^*_\Gamma(X))\hat{\otimes}\,\mathrm{Cl}_{\tau\oplus\Gamma}(X)$. Let us use notation $\mathrm{Cl}_\Gamma(X)\hat{\otimes}_{C_0(X)}\mathrm{Cl}_\Gamma(X)$ instead of $C_0(\Lambda^*_\Gamma(X))$ (as we did in the proof of Lemma 7.7). This will simplify the explanation. The Clifford multiplication part of the operator D_Γ acts over the second copy of $\mathrm{Cl}_\Gamma(X)$, and the projection P_0 acts over the first copy of $\mathrm{Cl}_\Gamma(X)$ and over $\mathrm{Cl}_{\tau\oplus\Gamma}(X)$.

We can now use the rotation homotopy similar to the one in the proof of Lemma 7.7, but applied to the Clifford variables only. This homotopy shifts the Clifford multiplication of the operator D_Γ to $\mathrm{Cl}_{\tau\oplus\Gamma}(X)$, and the action of the projection P_0 to $C_0(\Lambda^*_\Gamma(X))$. Since $P_0 \cdot C_0(\Lambda^*_\Gamma(X)) = C_0(\Lambda^0_\Gamma(X)) = C_0(X)$, our Hilbert module for the triple product becomes $C_0(U') \cdot L^2(E)\hat{\otimes}\,\mathrm{Cl}_{\tau\oplus\Gamma}(X)$.

Considering $L^2(E)$ as the space of L^2-sections over X, we reserve notation x for the variable point on this copy of X. Points on the second copy of X (corresponding to $\mathrm{Cl}_{\tau\oplus\Gamma}(X)$) will be denoted by y. The operator S for the triple product can now be written as the family of operators

$$(1\hat{\otimes}\Theta_x(y) + (1-\Theta_x^2(y))^{1/4}(M_1\cdot A\hat{\otimes}1 + M_2\cdot 1\hat{\otimes}F_\Gamma)(1-\Theta_x^2(y))^{1/4}),$$

parametrized by $y \in X$, where $\Theta_x(y)$ is the Clifford multiplication by the covector $\Theta_x(y)$. The Clifford multiplication part of the operator F_Γ acts over $\mathrm{Cl}_{\tau\oplus\Gamma}(X)$.

Now we will follow the proof of Theorem 4.1. We will consider the operator A as the family of operators A'_y (parametrized by $y \in X$) in the first stage of the homotopy in the proof of Theorem 4.1. We have to pass through the same stages of homotopy of the operator A as in the proof of Theorem 4.1. We will arrive at the family of operators A_y on the neighborhood $\tilde{U}$ of the zero section of the tangent vector bundle $T(X)$ over X with the symbols $\sigma_A(y,\eta)$ (independent of x). The symbol of the differential operator D_Γ on $L^2(T^*_y(X))\hat{\otimes}\,\mathrm{Cl}_{\Gamma_y}$ is $c(\varphi_y(\eta))$, where $\varphi_y = f'_y f'^*_y$ (see the proof of Theorem 8.9; the notation $c(\cdot)$ means Clifford multiplication). So the symbol of F_Γ on $\tilde{U}$ is $c(\varphi_y(\eta))(1+\|\varphi_y(\eta)\|^2)^{-1/2}$.

The resulting KK-product operator will be given by the family of operators

$$S_{1,y} = 1\hat{\otimes}\Theta_x(y) + (1-\Theta_x^2(y))^{1/4}(M_1\cdot A_y\hat{\otimes}1 + M_2\cdot 1\hat{\otimes}F_\Gamma)(1-\Theta_x^2(y))^{1/4}.$$

Again following the proof of Theorem 4.1, we can replace the neighborhood $\tilde{U}$ with TX and the Clifford multiplication operator $\Theta_x(y)$ with $c(-\theta)(1+\|\theta\|^2)^{-1/2}$, where $\theta \in T_y(X)$ replaces the previous variable $x-y$, as in the proof of Theorem 4.1.

We arrive at the family of operators

$$S_{2,y} = 1\hat{\otimes}c(-\theta)(1+\|\theta\|^2)^{-1/2} + (1+\|\theta\|^2)^{-1/4}(M_1\cdot A_y\hat{\otimes}1 + M_2\cdot 1\hat{\otimes}F_\Gamma)(1+\|\theta\|^2)^{-1/4}$$

on the field of Hilbert spaces $L^2(T_y^*(X))\otimes E_y\hat{\otimes}\mathrm{Cl}_{\tau_y}\hat{\otimes}\mathrm{Cl}_{\Gamma_y}$ parametrized by $y\in X$.

We will present now M_1, M_2 as a family of pseudo-differential operators. Although the dimension of the family $\{\mathrm{Cl}_{\Gamma_y}\}$ may vary with $y\in Y$, we always have an embedding $\mathrm{Cl}_{\Gamma_y}\subset\mathrm{Cl}_{\tau_y}$, and this allows us to apply all pseudo-differential techniques (of Section 3). Note that the notion of ellipticity defined in Section 3 applies to the current situation without any change.

We want to write S_2 as a family of elliptic pseudo-differential operators with the symbols:

$$1\hat{\otimes}c(-\theta)(1+\|\theta\|^2)^{-1/2} + (1+\|\theta\|^2)^{-1/2}(\sigma_{M_1}(y,\eta)\cdot\sigma_A(y,\eta)\hat{\otimes}1 + \sigma_{M_2}(y,\eta)\cdot 1\hat{\otimes}c(\varphi_y(\eta))(1+\|\varphi_y(\eta)\|^2)^{-1/2}).$$

In this notation, η stands for the derivation variable and θ for the multiplication variable.

The functions σ_{M_1} and σ_{M_2} used in this KK-product are defined as follows. Let $\nu:\mathbf{R}\to\mathbf{R}$ be a smooth function, $0\le\nu(t)\le 1$, such that $\nu(t)=0$ for $t\le 0$, and $\nu(t)=1$ for $t\ge 1$. In particular, all derivatives of ν vanish both at 0 and at 1, so for any integers $m,n\ge 0$ there exist positive real numbers $c_{m,n}$ such that $|d^n\nu(t)/dt^n|\le c_{m,n}|t|^m$ and $|d^n(1-\nu(t))/dt^n|\le c_{m,n}|1-t|^m$. Put

$$\sigma_{M_1}(y,\eta)=\nu(\|\eta\|^2\cdot(\|\eta\|^2+(\|\varphi_y(\eta)\|^2+1)^{3/2})^{-1}),\quad \sigma_{M_2}=(1-\sigma_{M_1}^2)^{1/2}.$$

It is easy to deduce from the above that for any $n\ge 0$,

$$\sigma_{M_1}(y,\eta)\le\mathrm{const}(n)\cdot[\|\eta\|^2\cdot(\|\eta\|^2+(1+\|\varphi_y(\eta)\|^2)^{3/2})^{-1}]^n,$$
$$\sigma_{M_2}(y,\eta)\le\mathrm{const}(n)\cdot[(1+\|\varphi_y(\eta)\|^2)^{3/2}(\|\eta\|^2+(1+\|\varphi_y(\eta)\|^2)^{3/2})^{-1}]^n.$$

Estimating the consecutive derivatives of σ_{M_1} and σ_{M_2}, we get a decrease by the factor $(1+\|\eta\|^2)^{-1/3}$ with each derivative, i.e.

$$|\partial^\alpha/\partial\eta^\alpha\sigma_{M_i}(y,\eta)|\le C_\alpha(1+\|\eta\|)^{-2|\alpha|/3}\quad\text{for } i=1,2.$$

All first derivatives of $\|\varphi_y(\eta)\|^2$ are bounded by $\mathrm{const}\cdot(1+\|\varphi_y(\eta)\|^2)^{1/2}$ for some const >0. Therefore when derivatives in η are taken of $\sigma_{M_1}(y,\eta)\cdot\sigma_A(y,\eta)\hat{\otimes}1$ or $\sigma_{M_2}(y,\eta)\cdot 1\hat{\otimes}c(\varphi_y(\eta))(1+\|\varphi_y(\eta)\|^2)^{-1/2}$, we get a decrease in the estimate by the factor $(1+\|\eta\|^2)^{-1/3}$ with each derivative as well. So both $\sigma_{M_1}(y,\eta)\cdot\sigma_A(y,\eta)\hat{\otimes}1$ and $\sigma_{M_2}(y,\eta)\cdot 1\hat{\otimes}c(\varphi_y(\eta))(1+\|\varphi_y(\eta)\|^2)^{-1/2}$ belong to the Hörmander class

$\rho = 2/3, \ \delta = 0$. This means that the symbols of all operators of the family S_2 belong to the Hörmander class $\rho = 2/3, \ \delta = 0$.

We reiterate here the remark made in the proof of Theorem 4.1: the family of operators with the symbols

$$\sigma_{M_1}(y,\eta)\cdot\sigma_A(y,\eta)\hat{\otimes}1 + \sigma_{M_2}(y,\eta)\cdot 1\hat{\otimes}c(\varphi_y(\eta))(1+\|\varphi_y(\eta)\|^2)^{-1/2}$$

is strongly continuous in the parameter y because their symbols are continuous (Fourier transform).

The symbol of $1 - S_{2,y}^2$ is equal (modulo lower order terms) to

$$(1+\|\theta\|^2)^{-1}(\sigma_{M_1}^2(y,\eta)((1-\sigma_A(y,\eta)^2)\hat{\otimes}1) + \sigma_{M_2}^2(y,\eta)(1+\|\varphi_y(\eta)\|^2)^{-1}).$$

Because $\sigma_{M_1}^2(y,\eta)((1-\sigma_A(y,\eta)^2)\hat{\otimes}1)$ and $\sigma_{M_2}^2(y,\eta)(1+\|\varphi_y(\eta)\|^2)^{-1})$, according to Definition 6.1 and the second remark after Definition 6.2, vanish at infinity in η, the operator $S_{2,y}$ is elliptic, Fredholm for any $y \in X$. Moreover, the family $y \mapsto 1-S_{2,y}^2$ is norm-continuous because its symbols are norm-continuous as functions of y (cf. the same remark in the proof of Theorem 4.1).

To deal with the left-hand side of the index formula in the statement of the theorem, we will use the expression: $[d_{\xi,\Gamma}] = [\mathfrak{f}_\Gamma] \otimes_{\mathrm{Cl}_\Gamma(TX)} ([d_\xi] \otimes_{C_0(X)} 1_{\mathrm{Cl}_\Gamma(X)})$ (Theorem 7.4). Then the left-hand-side of the index formula becomes the triple product: $[\sigma_A] \otimes_{\mathfrak{S}_\Gamma(X)} [\mathfrak{f}_\Gamma] \otimes_{\mathrm{Cl}_\Gamma(TX)} ([d_\xi] \otimes_{C_0(X)} 1_{\mathrm{Cl}_\Gamma(X)})$. This product is represented by the family of elliptic operators with the symbols

$$\begin{aligned}1\hat{\otimes}c(\eta)(1+\|\eta\|^2)^{-1/2} + (1+\|\eta\|^2)^{-1/2}(\sigma_{M_1}(y,\theta)\cdot\sigma_A(y,\theta)\hat{\otimes}1 \\ + \sigma_{M_2}(y,\theta)\cdot 1\hat{\otimes}c(\varphi_y(\theta))(1+\|\varphi_y(\theta)\|^2)^{-1/2})\end{aligned}$$

on the same family of Hilbert spaces as the right-hand side of the index formula. Variables η and θ are dual for the pseudo-differential calculus: η serves as the derivation variable and θ as the multiplication one.

The proof is completed by the rotation in the (η,θ) variables (for each point $y \in X$) exactly as it was done in the proof of Theorem 4.1. □

Definition 8.13. Let A be a transversally elliptic operator (acting on sections of the vector bundle E) with the symbol element $[\sigma_A] \in \mathcal{R}K_*^G(X;\mathfrak{S}_\Gamma(X))$. We define the tangent-Clifford symbol as

$$[\sigma_A^{\mathrm{tcl}}] = [\sigma_A] \otimes_{\mathfrak{S}_\Gamma(X)} [\mathfrak{f}_\Gamma] \in \mathcal{R}K_*^G(X;\mathrm{Cl}_\Gamma(TX)),$$

where the element $\mathfrak{f}_\Gamma \in \mathcal{R}KK^G(X;\mathfrak{S}_\Gamma(X),\mathrm{Cl}_\Gamma(TX))$ is defined in 7.3.

The tangent-Clifford symbol can be presented as the pair $(\mathcal{E}_\Gamma, S_\Gamma)$ consisting of the Hilbert module $\mathcal{E}_\Gamma = C_0(p^*(E))\hat{\otimes}_{C_0(TX)}\mathrm{Cl}_\Gamma(TX)$ and the operator S_Γ given by the following family parametrized by $y \in X$:

$$\sigma_{M_1}(y,\xi)\cdot\sigma_A(y,\xi)\hat{\otimes}1 + \sigma_{M_2}(y,\xi)\cdot 1\hat{\otimes}c(\varphi_y(\xi))(1+\|\varphi_y(\xi)\|^2)^{-1/2},$$

where

$$\sigma_{M_1}(y,\xi) = \|\xi\|(\|\xi\|^2 + (1+\|\varphi_y(\xi)\|^2)^{3/2})^{-1/2},$$
$$\sigma_{M_2}(y,\xi) = (1+\|\varphi_y(\xi)\|^2)^{3/4}(\|\xi\|^2 + (1+\|\varphi_y(\xi)\|^2)^{3/2})^{-1/2}.$$

(cf. the proof of Theorem 8.12).

Remark. In the proof of Theorem 8.12, we used a more complicated formula for the operators $\sigma_{M_1}, \sigma_{M_2}$ because we needed the Hörmander ρ, δ condition. To define the operator S_Γ as an element of a KK-theory group, we do not need this condition.

In the next theorem, according to Remark 7.5, we will consider $[\sigma_A^{\mathrm{cl}}]$ as an element of the group $\mathcal{R}KK_*^G(X; \mathrm{Cl}_\Gamma(X), \mathrm{Cl}_\tau(X))$ and $[\sigma_A^{\mathrm{tcl}}]$ as an element of the group $\mathcal{R}KK_*^G(X; \mathrm{Cl}_\Gamma(X), C_0(TX))$. The Clifford index theorem for transversally elliptic operators can be stated as follows:

Theorem 8.14 (Clifford index theorem). *In the assumptions of Theorem 8.12,*

$$[A]^{\mathrm{cl}} = [\sigma_A^{\mathrm{cl}}] \otimes_{\mathrm{Cl}_\tau(X)} [d_X] \in K_G^*(\mathrm{Cl}_\Gamma(X)),$$

and

$$[A]^{\mathrm{cl}} = [\sigma_A^{\mathrm{tcl}}] \otimes_{C_0(TX)} [\mathcal{D}_X] \in K_G^*(\mathrm{Cl}_\Gamma(X)),$$

where $[\mathcal{D}_X]$ is the Dolbeault element of Definition 2.8.

Proof. Using Remark 7.5, we can rewrite the inverse index formula of Theorem 8.12 as follows:

$$[\sigma_A^{\mathrm{cl}}] = [\Theta_{X,\Gamma,2}] \otimes_{\mathrm{Cl}_\Gamma(X)} [A]^{\mathrm{cl}} \in \mathcal{R}KK_*^G(X; \mathrm{Cl}_\Gamma(X), \mathrm{Cl}_\tau(X)).$$

Now we drop in this formula all $\mathcal{R}$'s in the $\mathcal{R}KK$-groups, replace $[\Theta_{X,\Gamma,2}]$ with $[\Theta_{X,\Gamma,1}]$ using Lemma 7.7, and apply $\otimes_{\mathrm{Cl}_\tau(X)}[d_X]$ to both sides of the formula. This gives the first of our index formulas in view of Theorem 2.4. The second index formula reduces to the first one in view of Definitions 8.11, 8.13 and Theorems 2.10 and 7.4. □

Remark. The statement of Remark 4.7 can be repeated now for transversally elliptic operators: the Clifford symbol and index of a transversally elliptic operator are related by Poincaré duality 7.8.

Theorem 8.15. *The map* $\mathrm{ind}(A) \mapsto [A]^{\mathrm{cl}}$ *of Definition 8.10 is an isomorphism of the subgroup* $\varinjlim K^*(C_E^*(G, C_0(X)))$ *of* $\varinjlim K_G^*(C_E^*(G, C_0(X)))$ *onto the group* $K_G^*(\mathrm{Cl}_\Gamma(X))$ *(cf. Definition 8.7 and Lemma 8.1).*

Proof. The homomorphism $\varinjlim K^*_G(C^*_E(G, C_0(X))) \to K^*_G(\mathrm{Cl}_\Gamma(X))$ is given by the product with the element $[D_\Gamma] \in \varprojlim KK^G(\mathrm{Cl}_\Gamma(X), C^*_E(G, C_0(X)))$ (Definitions 8.5, 8.6). The map in the opposite direction is defined as the following composition:

$$\begin{aligned} K^*_G(\mathrm{Cl}_\Gamma(X)) &\to \mathcal{R}KK^G_*(X; C_0(X), \mathrm{Cl}_{\tau\oplus\Gamma}(X)) \\ &\to KK^G_*(C^*_E(G, C_0(X)), C^*_E(G, \mathrm{Cl}_{\tau\oplus\Gamma}(X))) \to K^*_G(C^*_E(G, C_0(X))), \end{aligned}$$

where the first map is the product $[\Theta'_{X,\Gamma}]\otimes_{\mathrm{Cl}_\Gamma(X)}$, the second map is the homomorphism j^G (of [30, Theorem 3.11]), and the third map is the product with the element $[d_{X,\Gamma}] \in K^0_G(C^*(G, \mathrm{Cl}_{\tau\oplus\Gamma}(X)))$ (Definition 8.8) considered here as an element of the group $K^0_G(C^*_E(G, \mathrm{Cl}_{\tau\oplus\Gamma}(X)))$. (The homomorphism j^G is applied here to an $\mathcal{R}KK$-group, which allows us to take $KK^G_*(C^*_E(G, \cdot), C^*_E(G, \cdot))$ as its target group.) This composition maps $K^*_G(\mathrm{Cl}_\Gamma(X))$ into the subgroup $\varinjlim K^*(C^*_E(G, C_0(X)))$ of $\varinjlim K^*_G(C^*_E(G, C_0(X)))$ because the image satisfies the condition given in Lemma 8.1.

To show that the composition

$$K^*_G(\mathrm{Cl}_\Gamma(X)) \to \varinjlim K^*_G(C^*_E(G, C_0(X))) \to K^*_G(\mathrm{Cl}_\Gamma(X))$$

is the identity, we need the following lemma.

Lemma 8.16. *Let* $\alpha \in \mathcal{R}K^G_*(X; \mathrm{Cl}_{\tau\oplus\Gamma}(X))$. *Then the triple product*

$$[D_\Gamma] \otimes_{C^*_E(G, C_0(X))} j^G(\alpha) \otimes_{C^*_E(G, \mathrm{Cl}_{\tau\oplus\Gamma}(X))} [d_{X,\Gamma}] \in K^*_G(\mathrm{Cl}_\Gamma(X))$$

is equal to the product $\tilde{\alpha} \otimes_{\mathrm{Cl}_\tau(X)} [d_X]$, *where* $\tilde{\alpha} \in \mathcal{R}KK^G_*(X; \mathrm{Cl}_\Gamma(X), \mathrm{Cl}_\tau(X))$ *corresponds to* α *via the isomorphism of Remark 7.5.*

Proof of Lemma 8.16. Let $\alpha = (\mathcal{E}, \Phi)$. The triple product in the statement of the lemma can be presented as a pair $(\mathcal{T}, \mathcal{S})$, where the Hilbert module

$$\mathcal{T} = C_0(\Lambda^*_\Gamma(X)) \hat{\otimes}_{C_0(X)} \mathcal{E} \hat{\otimes}_{\mathrm{Cl}_{\tau\oplus\Gamma}(X)} L^2(\Lambda^*(X)).$$

We have omitted here the crossed product notation $C^*(G, \cdot)$ for the first two Hilbert modules: the resulting Hilbert module $\mathcal{T}$ does not depend on this.

Using equivariant stabilization [31, Proposition 5.5], we will assume that $\mathcal{E} = l^2(\mathbf{Z}) \otimes \mathrm{Cl}_\tau(X) \hat{\otimes} \mathrm{Cl}_\Gamma(X)$. Then the Hilbert module $\mathcal{T}$ becomes

$$\mathcal{T} = C_0(\Lambda^*_\Gamma(X)) \hat{\otimes}_{C_0(X)} [l^2(\mathbf{Z}) \otimes \mathrm{Cl}_\tau(X) \hat{\otimes}_{C_0(X)} \mathrm{Cl}_\Gamma(X)] \hat{\otimes}_{\mathrm{Cl}_{\tau\oplus\Gamma}(X)} L^2(\Lambda^*(X)).$$

The operator $\mathcal{S}$ for our triple product can be presented in the form:

$$\mathcal{S} = M_1(F_\Gamma \hat{\otimes} 1 \hat{\otimes} 1) + M_2(1 \hat{\otimes} \Phi \hat{\otimes} 1) + M_3 \tilde{F}_X.$$

Notation F_Γ, F_X comes from Definitions 8.5 and 8.8, $\tilde F_X$ is a K-theoretic connection for $1\hat\otimes 1\hat\otimes F_X$, and M_1, M_2, M_3 are the operators entering in the construction of the KK-product (with the sum of squares equal to 1. See [16, 37] or [30, 2.6], for the definition of a K-theoretic connection.)

The second product in the statement of the lemma, can also be written as a triple product (using the formula for $[d_X]$ from Theorem 8.9):

$$\tilde\alpha \otimes_{\mathrm{Cl}_\tau(X)} [D_{\Gamma,\tau}] \otimes_{C^*_E(G,\mathrm{Cl}_{\tau\oplus\Gamma}(X))} [d_{X,\Gamma}].$$

It can be presented as a pair $(\mathcal{T}, \mathcal{S}')$, where $\mathcal{T}$ is the same Hilbert module, except that the tensor multiples are grouped differently. The element $\tilde\alpha$ lives on the part corresponding to the Hilbert module $C_0(\Lambda^*_\Gamma(X))\hat\otimes_{C_0(X)} \mathrm{Cl}_\tau(X) \otimes l^2(\mathbf{Z})$, the element $[D_\Gamma]$ lives on the next tensor multiple: $\mathrm{Cl}_\Gamma(X)$, and the element $[d_X]$ lives on $L^2(\Lambda^*(X))$. So the operator $\mathcal{S}'$ can be written as

$$\mathcal{S}' = M_1'(\Phi\hat\otimes 1\hat\otimes 1) + M_2'(1\hat\otimes F_\Gamma\hat\otimes 1) + M_3 \tilde F_X.$$

(Note that in both triple products the operator F_Γ is defined using Lie group derivatives. The tensor signs indicate only the action of Clifford multiplication.)

It is clear now that the differences between the two triple products can be eliminated by a rotation homotopy. If we use the isomorphism

$$\mathcal{K}(C_0(\Lambda^*_\Gamma(X))) \simeq \mathrm{Cl}_\Gamma(X)\hat\otimes_{C_0(X)}\hat\otimes\,\mathrm{Cl}_\Gamma(X),$$

then the rotation (of Clifford variables only) on the diagonal of $X\times X\times X$ must turn the second copy of $\mathrm{Cl}_\Gamma(X)$ in $\mathcal{K}(C_0(\Lambda^*_\Gamma(X)))$ into the copy of $\mathrm{Cl}_\Gamma(X)$ which sits inside the square bracket in the previous expression for $\mathcal{T}$. $\square$

Returning to the proof of Theorem 8.15, we see now that the composition

$$K^*_G(\mathrm{Cl}_\Gamma(X)) \to \varinjlim K^*_G(C^*_E(G, C_0(X))) \to K^*_G(\mathrm{Cl}_\Gamma(X))$$

is the identity because it coincides with the composition $\nu\cdot\mu$ in the proof of Theorem 7.8.

It remains to show that the composition of homomorphisms indicated at the beginning of the proof is an isomorphism of $K^*_G(\mathrm{Cl}_\Gamma(X))$ *onto* the subgroup $\varinjlim K^*(C^*_E(G, C_0(X)))$. First, let us consider the case of $X = G/M$, where M is a compact subgroup. In this case, $\Gamma = \tau$, and the first map in the composition is $K^*_G(\mathrm{Cl}_\tau(X)) \simeq RK^*_G(X)$ (an isomorphism by Corollary 7.9). The group $RK^*_G(G/M)$ is isomorphic to $R^*(M)$ (see e.g. [30, Theorem 3.6, Part 2]; here $R^0(M)$ is the usual representation ring of M, and $R^1(M) = 0$). On the other hand, in the projective system of C^*-algebras $C^*_E(G, C_0(G/M))$, the vector bundles E are induced from the projective system of all finite-dimensional representations of M (cf. Remark after Definition 8.2). Therefore, the group $\varinjlim K^*(C^*_E(G, C_0(X)))$, as an inductive limit of all finitely generated subgroups of $R^*(M)$, coincides with $R^*(M)$.

To see that the composition of homomorphisms that we consider is an isomorphism onto $\varinjlim K^*(C^*_E(G, C_0(X)))$, we must first mention that the element $[d_{X,\Gamma}]$, in the special case of $X = G/M$, is given by the pair $(L^2(\Lambda^*(G/M)), 0)$, because the image of $C^*(G, Cl_{\tau\oplus\tau}(G/M))$ in $\mathcal{L}(L^2(\Lambda^*(G/M)))$ is contained in $\mathcal{K}(L^2(\Lambda^*(G/M)))$. So by the Clifford periodicity, $[d_{X,\Gamma}]$ is defined by the map of the subalgebra $C^*(G, C_0(G/M))$ into $\mathcal{K}(L^2(G/M))$. This map corresponds to the trivial one-dimensional representation $C^*(M) \to \mathbf{C}$ if we present the algebra $C^*(G, C_0(G/M))$ as $C^*(M) \otimes \mathcal{K}(L^2(G/M))$ [18]. In particular, if the vector bundle E is trivial one-dimensional, then $C^*_E(G, C_0(X)) \simeq \mathcal{K}(L^2(G/M))$.

Now we remark that the ring $RK^*_G(G/M) \simeq R^*(M)$ acts on the group $\varinjlim K^*(C^*_E(G, C_0(X))) \simeq R^*(M)$ by multiplication (via the composition with elements of the group $KK^G_*(C^*_E(G, C_0(X)), C^*_E(G, C_0(X))))$. Multiplying the element $1 \in R^*(M)$ by the whole ring $R^*(M)$, we get the whole ring $R^*(M)$.

Turning now to the general case, we will use the Mayer–Vietoris exact sequence (see [20, Section 3]) and induction on dimension of X to show that the product with the element $[D_\Gamma]$ gives an isomorphism of $\varinjlim K^*(C^*_E(G, C_0(X)))$ onto $K^*_G(\mathrm{Cl}_\Gamma(X))$. If V_1 and V_2 are two open G-invariant subsets of X, we have two Mayer–Vietoris exact sequences: the one relating the groups $\varinjlim K^*(C^*_E(G, C_0(\cdot)))$, and the other — the corresponding groups $K^*_G(\mathrm{Cl}_\Gamma(\cdot))$. The 5-lemma proves our assertion for $V_1 \cup V_2$ if we already have it for V_1, V_2 and $V_1 \cap V_2$. If X/G is compact, we can cover X by a finite number of open tubular neighborhoods of different orbits. This reduces the assertion to one tubular neighborhood. In general the assertion also reduces to this case by an additional use of a $\lim^1$ exact sequence of K^*-groups related with an exhaustive increasing sequence of G-invariant open subsets in X.

Next, let us reduce our assertion to lower dimension in the case when $X = V$ is a tubular neighborhood of one orbit. Let us denote this orbit by $O \simeq G/M$, where M is the stability subgroup at a point $x \in O$. We have: $V - O \simeq \partial V \times (0, 1)$, where ∂V is the boundary of V. Hence, we get an exact sequence of C^*-algebras:

$$0 \to C_0(\partial V \times (0, 1)) \to C_0(V) \to C_0(O) \to 0,$$

which induces similar exact sequences for the corresponding C^*-algebras $C^*_E(G, C_0(\cdot))$, as well as for the corresponding C^*-algebras $\mathrm{Cl}_\Gamma(\cdot)$. The product with the element $[D_\Gamma]$ maps the associated six term exact sequences of K-homology groups one into the other. Because both O and ∂V have lower dimension than V, the inductive assertion is proved. □

Definition 8.17. The Dolbeault element $[\mathcal{D}_{X,\Gamma}] \in K^0_G(C^*(G, \mathfrak{S}_\Gamma(X)))$ is defined as the product of $j^G([d_{\xi,\Gamma}]) \in KK_G(C^*(G, \mathfrak{S}_\Gamma(X)), C^*(G, \mathrm{Cl}_{\tau\oplus\Gamma}(X)))$ and $[d_{X,\Gamma}] \in K^0_G(C^*(G, \mathrm{Cl}_{\tau\oplus\Gamma}(X)))$.

The Clifford Dolbeault element $[\mathcal{D}^{\mathrm{cl}}_{X,\Gamma}] \in K^0_G(C^*(G, \mathrm{Cl}_\Gamma(TX)))$ is the product of $j^G([d_\xi] \otimes_{C_0(X)} 1_{\mathrm{Cl}_\Gamma(X)}) \in KK_G(C^*(G, \mathrm{Cl}_\Gamma(TX)), C^*(G, \mathrm{Cl}_{\tau\oplus\Gamma}(X)))$ and $[d_{X,\Gamma}] \in K^0_G(C^*(G, \mathrm{Cl}_{\tau\oplus\Gamma}(X)))$.

An explicit description of the element $[\mathcal{D}^{\mathrm{cl}}_{X,\Gamma}]$ goes as follows: Let $\mathcal{D}$ be the Dolbeault operator of Definition 2.8. We keep the Hilbert space and the operator $\mathcal{D}$ as in Definition 2.8, but extend the action of $C_0(TX)$ on $L^2(\Lambda^*_{\mathbb{C}}(TX))$ to the action of $\mathrm{Cl}_\Gamma(TX)$. On (real) covectors of the Clifford part, this action is given by: $\eta \mapsto i(\mathrm{ext}\,(\eta) - \mathrm{int}\,(\eta))$ (where η is a section of $\Gamma \subset \tau = T(X)$). The resulting pair gives the element $[\mathcal{D}^{\mathrm{cl}}_{X,\Gamma}]$. (See the proof in 8.8 for the commutator properties.)

Theorem 8.18 (Index theorem). *Let X be a complete Riemannian manifold and G a Lie group which acts on X properly and isometrically. Let A be a properly supported, G-invariant, L^2-bounded transversally elliptic operator on X of order* 0*, of the Hörmander class $\rho = 1, \delta = 0$. Then*

$$\mathrm{ind}(A) = j^G([\sigma^{\mathrm{cl}}_A]) \otimes_{C^*(G,\mathrm{Cl}_{\tau\oplus\Gamma}(X))} [d_{X,\Gamma}] \in K^*(C^*(G, C_0(X))),$$

and

$$\mathrm{ind}(A) = j^G([\sigma_A]) \otimes_{C^*(G,\mathfrak{S}_\Gamma(X))} [\mathcal{D}_{X,\Gamma}] \in K^*(C^*(G, C_0(X))),$$

and

$$\mathrm{ind}(A) = j^G([\sigma^{\mathrm{tcl}}_A]) \otimes_{C^*(G,\mathrm{Cl}_\Gamma(TX))} [\mathcal{D}^{\mathrm{cl}}_{X,\Gamma}] \in K^*(C^*(G, C_0(X))).$$

We will give two proofs. The first of them uses Theorem 8.15, the second one does not use it. In both proofs we need to treat only the first index formula because all three formulas are equivalent.

First proof. Let us apply to both sides of the first index formula the isomorphism of Theorem 8.15: $\varinjlim K^*(C^*_E(G, C_0(X))) \simeq K^*_G(\mathrm{Cl}_\Gamma(X))$ given by the product with the element $[D_\Gamma] \in \varprojlim KK^G(\mathrm{Cl}_\Gamma(X), C^*_E(G, C_0(X)))$ (Definitions 8.5, 8.6). Lemma 8.16 shows that we get precisely the first index formula of Theorem 8.14. Therefore, in view of Theorem 8.15, Theorem 8.18 is equivalent to Theorem 8.14. $\square$

Second proof. We will be using the well known fact that $C^*(G, B_1\hat{\otimes}B_2) \simeq C^*(G, B_1)\hat{\otimes}B_2$ if the G-action on the algebra B_2 is inner. "Inner" means that there is a unitary representation v of the group G in the multiplier algebra $\mathcal{M}(B_2)$ such that the G-action on B_2 is given by the conjugation with v (example: $B_2 = C^*(G, B_2')$). [The proof of this fact is easy: if (u, ψ) is a covariant representation of $(G, B_1\hat{\otimes}B_2)$, then $(\{g \mapsto (1 \otimes \psi(v(g^{-1})))u(g)\}, \psi)$ is a covariant representation of $(G, B_1\hat{\otimes}B_2^{\mathrm{triv}})$, where B_2^{triv} is B_2 with the trivial action of G.]

Let us rewrite the right side of the first index formula above using the expression for $[\sigma^{\mathrm{cl}}_A]$ from Theorem 8.12 and the expression for $[A]^{\mathrm{cl}}$ from Definition 8.10. We get:

$$j^G([\Theta'_{X,\Gamma}] \otimes_{\mathrm{Cl}_\Gamma(X)\hat{\otimes}\mathrm{Cl}_{\tau\oplus\Gamma}(X)} (([D_\Gamma] \otimes_{C^*_E(G,C_0(X))} \mathrm{ind}(A)) \otimes 1_{\mathrm{Cl}_{\tau\oplus\Gamma}(X)})) \\ \otimes_{C^*(G,\mathrm{Cl}_{\tau\oplus\Gamma}(X))} [d_{X,\Gamma}].$$

This can again be rewritten as the following triple product:

$$j^G([\Theta'_{X,\Gamma}]) \otimes_{C^*(G,\mathrm{Cl}_\Gamma(X)\hat{\otimes}\mathrm{Cl}_{\tau\oplus\Gamma}(X))} j^G([D_\Gamma]\otimes 1_{\mathrm{Cl}_{\tau\oplus\Gamma}(X)}) \\ \otimes_{C^*_E(G,C_0(X))\otimes C^*(G,\mathrm{Cl}_{\tau\oplus\Gamma}(X))} (\mathrm{ind}(A)\otimes[d_{X,\Gamma}]),$$

where

$$j^G([D_\Gamma]\otimes 1_{\mathrm{Cl}_{\tau\oplus\Gamma}(X)}) \in KK(C^*(G,\mathrm{Cl}_\Gamma(X)\hat{\otimes}\mathrm{Cl}_{\tau\oplus\Gamma}(X)), \\ C^*_E(G,C_0(X))\otimes C^*(G,\mathrm{Cl}_{\tau\oplus\Gamma}(X))),$$

and

$$\mathrm{ind}(A)\otimes[d_{X,\Gamma}] \in K^*(C^*_E(G,C_0(X))\otimes C^*(G,\mathrm{Cl}_{\tau\oplus\Gamma}(X))).$$

We want to compare this with another triple product:

$$j^G([\Theta_X]) \otimes_{C^*(G,C_0(X)\otimes\mathrm{Cl}_\tau(X))} j^G(1_{C_0(X)}\otimes[D_{\Gamma,\tau}]) \\ \otimes_{C^*(G,C_0(X))\otimes C^*_E(G,\mathrm{Cl}_{\tau\oplus\Gamma}(X))} (\mathrm{ind}(A)\otimes[d_{X,\Gamma}]),$$

where

$$j^G(1_{C_0(X)}\otimes[D_{\Gamma,\tau}]) \in KK(C^*(G,C_0(X)\otimes\mathrm{Cl}_\tau(X)), \\ C^*(G,C_0(X))\otimes C^*_E(G,\mathrm{Cl}_{\tau\oplus\Gamma}(X))).$$

Theorems 8.9 and 2.4 show that this latter triple product is equal to $\mathrm{ind}(A)$. Therefore to prove the theorem, it is enough to show that the products

$$j^G([\Theta'_{X,\Gamma}]) \otimes_{C^*(G,\mathrm{Cl}_\Gamma(X)\hat{\otimes}\mathrm{Cl}_{\tau\oplus\Gamma}(X))} j^G([D_\Gamma]\otimes 1_{\mathrm{Cl}_{\tau\oplus\Gamma}(X)})$$

and

$$j^G([\Theta_X]) \otimes_{C^*(G,C_0(X)\otimes\mathrm{Cl}_\tau(X))} j^G(1_{C_0(X)}\otimes[D_{\Gamma,\tau}])$$

are equal in $KK(C^*(G,C_0(X)), C^*_E(G,C_0(X))\otimes C^*_E(G,\mathrm{Cl}_{\tau\oplus\Gamma}(X)))$.

For the following proof we will replace in the previous formulas some algebras $C_0(X)$ with the Morita equivalent algebras $\mathcal{K}(C_0(\Lambda^*_\Gamma(X)))$. We will consider $[\Theta'_{X,\Gamma}]$ as an element of $KK^G(C_0(X), C_0(U')\cdot\mathrm{Cl}_\Gamma(X)\hat{\otimes}\mathrm{Cl}_{\tau\oplus\Gamma}(X))$, $[\Theta_X]$ as an element of $KK^G(C_0(X), C_0(U')\cdot C_0(X)\hat{\otimes}\mathrm{Cl}_\tau(X))$, and

$$[D_\Gamma]\otimes 1 \in KK^G(\mathrm{Cl}_\Gamma(X)\hat{\otimes}\mathrm{Cl}_{\tau\oplus\Gamma}(X), C^*_E(G,\mathcal{K}(C_0(\Lambda^*_\Gamma(X))))\hat{\otimes}\mathrm{Cl}_{\tau\oplus\Gamma}(X)),$$
$$1\otimes[D_{\Gamma,\tau}] \in KK^G(C_0(X))\hat{\otimes}\mathrm{Cl}_\tau(X), \mathcal{K}(C_0(\Lambda^*_\Gamma(X)))\hat{\otimes}C^*_E(G,\mathrm{Cl}_{\tau\oplus\Gamma}(X))).$$

In the last KK-group, the map $C_0(X)\to\mathcal{K}(C_0(\Lambda^*_\Gamma(X)))$ is given by the embedding onto $\mathcal{K}(C_0(\Lambda^0_\Gamma(X)))$.

Both $j^G([D_\Gamma]\otimes 1)$ and $j^G(1\otimes[D_{\Gamma,\tau}])$ are elements of the KK-groups whose second argument is the algebra

$$C^*_E(G,\mathcal{K}(C_0(\Lambda^*_\Gamma(X))))\hat{\otimes}C^*_E(G,\mathrm{Cl}_{\tau\oplus\Gamma}(X)).$$

There are two orbital Dirac operators on this algebra: corresponding to the first and second copies of X. These two Dirac operators D_1 and D_2 anti-commute, so we can define the following rotation homotopy: $\cos t \cdot D_1 + \sin t \cdot D_2$ for $0 \leq t \leq \pi/2$.

Note that the Hilbert module for both KK-products is

$$\mathcal{E} = C_0(U') \otimes_{C_0(X)\otimes C_0(X)} C_E^*(G, C_0(\Lambda_\Gamma^*(X)))\hat{\otimes} C_E^*(G, \mathrm{Cl}_{\tau\oplus\Gamma}(X))$$

In the rest of the proof, we replace $\mathcal{K}(C_0(\Lambda_\Gamma^*(X)))$ with $\mathrm{Cl}_\Gamma(X)\hat{\otimes}_{C_0(X)} \mathrm{Cl}_\Gamma(X)$.

We will use the rotation homotopy of the proof of Lemma 7.7. In our case here, the vector bundle $\mathcal{V}$ is the trivial bundle $\mathfrak{g}_X$ (see Section 6). The Clifford variables of the Hilbert module $\mathcal{E}$ come from the vector space $\mathfrak{g} \oplus \mathfrak{g} \oplus \mathfrak{g}$ via the map $\mathfrak{g}_X \oplus \mathfrak{g}_X \oplus \mathfrak{g}_X \to \Gamma \times \Gamma \times \Gamma$, where $\mathfrak{g}$ is the Lie algebra of G. In the first of our products, the Clifford multiplication part of the operator D_Γ acts over the second copy of Γ, and in the second product – over the third copy of Γ. On the other hand, the projection P_0 of Definition 7.6 (for the first product) acts over the first and third copies of Γ.

We already have the rotation homotopy for the operator D_Γ. It is defined by the rotation of the second copy of $\mathfrak{g}$ into the third one. The orthogonal complement of the rotating subspace $\mathfrak{g}$ is isomorphic to $\mathfrak{g} \oplus \mathfrak{g}$. We define the homotopy of the projection P_0 using this subspace. At the end of this homotopy, the resulting projection P_0 will act over the first and second copies of Γ. So this projection will cut $C_0(\Lambda_\Gamma^*(X))$ to $C_0(\Lambda_\Gamma^0(X))$, thus reducing it to $C_0(X)$. We will arrive at the second product.

The construction of the required homotopy is completed using the KK-product technique of Theorem 2.11 [30]. □

Remark 8.19. In the same way as in Remark 5.7, we can apply Abels' theorem [1]: there exists a continuous G-map $\psi : X \to G/K$. Let us assume that X/G is *compact*. Then the map ψ is obviously proper. Applying the homomorphism

$$\psi_* : K^*(C^*(G, C_0(X))) \to K^*(C^*(G, C_0(G/K))) \simeq K^*(C^*(K))$$

to the K-homological index $\mathrm{ind}(A)$ of the operator A, we obtain an element of the group $K^*(C^*(K))$.

This element may be called the *distributional index* of the operator A. In the case when G and X are compact (i.e. when $G = K$), this was already explained in Remark 6.7.

We will give now a couple of examples illustrating Theorem 8.18.

Example 8.20. First, let us consider the case of a homogeneous space $X = G/M$, where M is a compact subgroup of G. If M_x is the stability subgroup at the point $x \in X$ and $\mathfrak{m}_x$ the Lie algebra of M_x, then the map $f'_x : \mathfrak{g}/\mathfrak{m}_x \to T_x(X)$ is an isometric isomorphism, and $q_x(\xi) = \|\xi\|_x$ for any $x \in X$. The condition (3) of Definition 6.2 means only that the symbol is bounded in the ξ-direction. Therefore the algebra $\mathfrak{S}_\Gamma(X)$ is homotopy equivalent (using a linear contraction of TX onto X) to the algebra $C_0(X) = C_0(T_G(X))$. The algebra $\mathrm{Cl}_{\tau\oplus\Gamma}(X)$ is Morita equivalent to $C_0(X)$ as well.

Any transversally elliptic operator on X is homotopic to the zero operator A (which is transversally elliptic in this case). Its index $\mathrm{ind}(A) \in K^0(C^*(G, C_0(X))) \simeq K^0(C^*(M))$ is defined by the pair $(\pi, 0)$, where the homomorphism

$$\pi : C^*(G, C_0(G/M)) \simeq C^*(M) \otimes \mathcal{K}(L^2(G/M)) \to \mathcal{K}(L^2(G/M))$$

corresponds to the trivial one-dimensional representation of M.

The symbol element $[\sigma_A]$, as well as the Clifford symbol element $[\sigma_A^{\mathrm{cl}}]$, both are equal to $1_X \in \mathcal{R}K_0^G(X; C_0(X))$. The Clifford index $[A]^{\mathrm{cl}} \in K_G^0(\mathrm{Cl}_\Gamma(X)) = K_G^0(\mathrm{Cl}_\tau(X))$ corresponds to $1 \in R(M)$ via the isomorphism $K_G^0(\mathrm{Cl}_\tau(X)) \simeq RK_G^0(X) \simeq R(M)$ (which is the Poincaré duality of Corollary 7.9 combined with the isomorphism of Theorem 3.6 of [30]).

Example 8.21. Now a more sophisticated example. We will discuss the index of Atiyah's operator (cf. [12, p. 58]). The space X will be $\mathbf{C}^1$; the group $G = S^1 \subset \mathbf{C}$ acts on X by rotations; the graded vector bundle over X is $E = X \times (\mathbf{C} \oplus \mathbf{C})$ (where the first direct summand is E^0, and the second is E^1). The group action on E is given by $g(z, v_0, v_1) = (g(z), v_0, g(v_1))$, where $g \in G$, and the application of g to v_1 means again rotation. The operator A is a self-adjoint operator of degree 1 on E. Its part acting from E^0 to E^1 is given by $z + \partial/\partial\bar{z}$.

We will consider $\xi \in T_z^*(X) \simeq \mathbf{C}$ as a complex-valued covector. A simple calculation shows that the map φ_z (of Section 6) is given by $\varphi_z(\xi) = iz\mathrm{Im}\,(\bar{z}\xi)$, and $q_z(\xi) = |\mathrm{Im}\,\bar{z}\xi|^2$. The symbol of the part of the operator A acting from E^0 to E^1 is $\sigma_A(z, \xi) = z + i\xi$. It follows that σ_A^2 is a 2×2 diagonal matrix with entries equal $|z|^2 + |\xi|^2 - 2\mathrm{Im}\,(\bar{z}\xi)$.

We will choose an unusual normalization of the symbol of A to order 0. Namely we will divide it not by $(1 + |\xi|^2)^{1/2}$, but by $(1 + |z|^2 + |\xi|^2)^{1/2}$. We will denote the resulting operator of order 0 by A_0. From the above calculations, it easily follows that

$$\|\sigma_{A_0}^2 - 1\|_{(z,\xi)} \leq 2(1 + q_z(\xi))/(1 + |z|^2 + |\xi|^2).$$

This is much stronger than the estimate of Definition 6.1. In fact, this implies that $C^*(G)(A_0^2 - 1) \subset \mathcal{K}(L^2(E)$ (as if $C_0(X)$ is replaced by $\mathbf{C}$). To prove this inclusion, we can repeat the proof of Proposition 6.4, omitting the element $a \in C_c^\infty(X)$ used there, and replacing the operator Δ by the operator with the symbol $1 + |z|^2 + |\xi|^2$. Applying Shubin's calculus of pseudo-differential operators with polynomial symbols on Euclidean space [36, Ch. 4], we get the required assertion.

It is clear now that the element $\mathrm{ind}(A_0)$ comes to $K^*(C^*(G, C_0(X)))$ by restriction from $K^*(C^*(G, C(S^2)))$, where S^2 is the one-point compactification of X, the Riemann sphere. Mapping this sphere to a point gives an element of $K^*(C^*(G))$, i.e. the distributional index, which was the primary object of study in [7, 12]. We will also concentrate in this example on the distributional index.

The analytical distributional index of A_0, as an element of $K^*(C^*(G))$, is easy to calculate. The kernel of A acting from E^0 to E^1 calculates as follows: if $(z + \partial/\partial\bar{z})f = 0$, then $\exp(z\bar{z})f$ is in the kernel of $\partial/\partial\bar{z}$, so $\exp(z\bar{z})f$ is holomorphic. Therefore f is a product of $\exp(-z\bar{z})$ and a holomorphic function. A similar argument shows that the cokernel of A is 0. So the distributional index is $\sum_{m=0}^{\infty} e^{im\theta}$. (We use here Remark 6.7.)

Let us discuss now the topological distributional index. We can write the tangent-Clifford symbol of A in the form:

$$\sigma_A^{\mathrm{tcl}}(z,\xi) = \begin{pmatrix} 0 & \bar{z} - i\bar{\xi} \\ z + i\xi & 0 \end{pmatrix} \hat{\otimes} 1 + 1 \hat{\otimes} \begin{pmatrix} 0 & -i\bar{z} \\ iz & 0 \end{pmatrix} \operatorname{Im}(\bar{z}\xi).$$

This is a non-normalized version of the formula in Definition 8.13. The second summand corresponds to the $\varphi_z(\xi)$-part calculated earlier in this example.

It is easy to see that for the so defined $\sigma_A^{\mathrm{tcl}}(z,\xi)$, we have: $\|\sigma_A^{\mathrm{tcl}}(z,\xi)\|^2 \to \infty$ when $(z,\xi) \to \infty$ in TX. This means that after the appropriate normalization to order 0, as above, $\sigma_A^{\mathrm{tcl}}(z,\xi)$ defines an element $[\sigma_{A_0}^{\mathrm{tcl}}] \in K_0^G(\mathrm{Cl}_\Gamma(TX))$. The usual tangent-Clifford symbol (of Definition 8.13) comes from this element by the natural map $K_0^G(\mathrm{Cl}_\Gamma(TX)) \to \mathcal{R}K_0^G(X, \mathrm{Cl}_\Gamma(TX))$.

Applying the last formula of Theorem 8.18 to the element $[\sigma_{A_0}^{\mathrm{tcl}}]$, we will get an element of $K^*(C^*(G))$. This is the topological distributional index of A_0. Let us calculate it. In the KK-product formula of Theorem 8.18, all operators act on the Hilbert space $H = C_0(p^*(E))\hat{\otimes}_{C_0(TX)} L^2(\Lambda_{\mathbf{C}}^*(TX))$. We will work with the unbounded version of the KK-product, which means that the operator for the product of $[\sigma_{A_0}^{\mathrm{tcl}}]$ with the Dolbeault element will be just the sum of $\sigma_A^{\mathrm{tcl}}(z,\xi)$ and the (graded) Dolbeault operator for TX. Note that the first summand of $\sigma_A^{\mathrm{tcl}}(z,\xi)$ acts on H over $C_0(p^*(E))$, the second summand acts according to Definition 8.17 which gives the formula for the action of the Clifford part of $\mathrm{Cl}_\Gamma(TX)$, and the Dolbeault operator acts over $L^2(\Lambda_{\mathbf{C}}^*(TX))$.

After the operator for the product is defined, we can forget about the algebra $\mathrm{Cl}_\Gamma(TX)$ and must remember only that our product is an element of $K^*(C^*(G))$. We can transform now (by homotopy) the symbol $\sigma_A^{\mathrm{tcl}}(z,\xi)$ into

$$\begin{pmatrix} 0 & \bar{z} - i\bar{\xi} \\ z + i\xi & 0 \end{pmatrix} \hat{\otimes} 1 + 1 \hat{\otimes} \begin{pmatrix} 0 & \bar{\xi} - i\bar{z} \\ \xi + iz & 0 \end{pmatrix}.$$

In order to explain how the second summand acts on $L^2(\Lambda_{\mathbf{C}}^*(TX))$, let us choose the new variables in $\mathbf{C}^2$: $u = z + i\xi$, $v = \xi + iz$. This will split the (graded) Dolbeault operator on $\mathbf{C}^2$ into the product of two one-dimensional Dolbeault operators: $\mathcal{D}_u \hat{\otimes} 1 + 1 \hat{\otimes} \mathcal{D}_v$. The second summand of the deformed $\sigma_A^{\mathrm{tcl}}(z,\xi)$ will use the same Clifford variable as the operator $\mathcal{D}_v$. This is consistent with the initial action according to Definition 8.17.

We remark now that the first matrix summand in the last expression for $\sigma_A^{\mathrm{tcl}}(z,\xi)$ is the one-dimensional Bott element for the first copy of $\mathbf{C} \oplus \mathbf{C} = \mathbf{C}^2$ (corresponding to the coordinate u). This Bott element has its own Clifford variable, anti-commuting with those of $\mathcal{D}_u$ and $\mathcal{D}_v$. The product of the Bott and Dolbeault elements in the variable u is equal to 1, and we can omit u.

On the other hand, for the remaining variable v, the product is the same operator as the original one, $z + \partial/\partial\bar{z}$, but in the variable v, namely, $v + \partial/\partial\bar{v}$. Therefore, the topological distributional index of the operator A_0 equals the analytical distributional index of the operator $v + \partial/\partial\bar{v}$, that is $\sum_{m=0}^{\infty} e^{im\theta}$.

9. Operators on singular foliations

Some of the techniques developed in Sections 6–8 allow to treat, with little additional effort, a more general case of operators transversally elliptic with respect to a singular foliation on a complete Riemannian manifold X. In this section we will present some results which can be obtained in this direction. Namely, we will discuss analogs of Clifford index theorems 8.12 and 8.14. At the end of the section, we also suggest a conjecture related with leaf-wise operators.

We will consider only the case of a foliation defined by a Lie algebroid.

A foliation is defined by a Lie algebroid as follows (see e.g. [2, Example 1.3]). An algebroid $\mathcal{V}$ is a vector bundle over the manifold X together with the (*anchor*) map of vector bundles $f : \mathcal{V} \to \tau = T(X)$. By definition, the space of smooth sections of $\mathcal{V}$ is equipped with the Lie bracket $[\cdot,\cdot]$ which satisfies the compatibility conditions: the map $f : C_c^\infty(\mathcal{V}) \to C_c^\infty(\tau)$ preserves the Lie bracket (where the Lie bracket on vector fields is the classical one), and $[v, aw] = a[v,w] + f(v)(a) \cdot w$, for $v, w \in C_c^\infty(\mathcal{V}), a \in C_c^\infty(X)$.

In general, a singular foliation $\mathcal{F}$ on X is a locally finitely generated submodule $\mathcal{F} \subset C_c^\infty(\tau)$ stable under the Lie bracket [2, Definition 1.1]. In our case, $\mathcal{F}$ is defined as the image of $C_c^\infty(\mathcal{V})$ under the anchor map f. Any compactly supported vector field on X defines a one-parameter group of diffeomorphisms of X. So vector fields which belong to $\mathcal{F}$ generate a certain group $\exp \mathcal{F}$ of diffeomorphisms of X. The leaves of $\mathcal{F}$ are defined as orbits of this group (cf. [2, Definition 1.7]).

We will use all basic KK-elements which were constructed in Sections 2 and 7. *In particular, we will assume that $\mathcal{V}$ (and, obviously, τ) carry Riemannian metrics, and the anchor map f is bounded in the sup-norm over X.* The image of the anchor map f will be considered as a continuous field Γ (see the discussion preceding Definition 7.1). We will use all notation introduced in Section 7 related with this.

Assumptions on pseudo-differential operators in the transversally elliptic case will be the same as stated in Section 3 (i.e. all operators belong to the $\rho = 1, \delta = 0$

class) and in Section 6, Definition 6.1, except for the assumption of G-invariance used in Section 6. (We will define a new property of invariance in a moment.) Definitions 6.1–6.3 pass to the foliation case almost without change. We use the Lie algebroid $\mathcal{V}$ instead of $\mathfrak{g}_X$ in the Definition 6.2 (see Section 7, Definition 7.1 and the Notation after it.)

Now we introduce the property of $\mathcal{F}$-invariance for pseudo-differential operators. Let A be a properly supported pseudo-differential operator of order 0 which acts on sections of a vector bundle E. In the previous parts of the paper where we considered the case of operators invariant with respect to an action of a Lie group G, we could define G-invariance as $[L_v, A] = 0$, where L_v is a first order differential operator defined by a vector field on X corresponding to an element v of the Lie algebra $\mathfrak{g}$. However, if we take an arbitrary vector field v in the orbit direction (i.e. $v \in \mathcal{F}$), then $v = \sum_k a_k v_k$, where $a_k \in C^\infty(X)$ and $v_k \in \mathfrak{g}$. The corresponding first order differential operator L_v will be $\sum_k a_k L_{v_k}$, and the commutator $[L_v, A]$ will not be 0 but a linear combination $\sum_k B_k L_{v_k}$, where B_k are pseudo-differential operators of order -1 or less.

Recall now Definitions 6.2–6.3 and Notation following Definition 7.1. It is easy to see that the symbols of operators $\sum_k B_k L_{v_k}$ described above belong to $\mathcal{K}(\mathfrak{S}_\Gamma(E))$. This justifies the definition that follows.

Let us denote by $\nabla : C^\infty(E) \to C^\infty(E \otimes \tau^*)$ a covariant derivative, and by $\nabla_v : C^\infty(E) \to C^\infty(E)$ its composition with $\operatorname{int}(v)$, where $v \in C_c^\infty(\tau)$.

Definition 9.1. An operator A of order 0 will be called $\mathcal{F}$-invariant if for any $v \in \mathcal{F}$, the commutator $[\nabla_v, A]$ is a pseudo-differential operator with symbol in $\mathcal{K}(\mathfrak{S}_\Gamma(E))$.

We remark that equivariance required by Definition 9.1 reduces to G-equivariance in the case of a proper action of a group G (Section 6) by averaging over G with a cut-off function.

Now we need to discuss a construction of a lifting of A to an operator on the vector bundle $\Lambda^*(\tau)\hat{\otimes}E$, as well as to an operator on the Hilbert space $\mathcal{H} = C_0(\Lambda^*_\Gamma(X))\hat{\otimes}_{C_0(X)} L^2(E))$. We will also see whether $\mathcal{F}$-invariance of A is affected by such lifting.

We assume here that A is L^2-bounded. The lifting of A to $\Lambda^*(\tau)\hat{\otimes}E$ can be done by taking a trivialization of E, i.e. presenting X as $\cup_i U_i$, where U_i are open sets in X with compact closure, such that $E_{|U_i}$ is trivial. Then one can use a partition of unity and transition functions for E in order to glue together operators $1\hat{\otimes}A$ defined in each neighborhood of this trivialization. More precisely, if ρ_{ij} are transition functions for E and $\sum_i \alpha_i = 1$ is a partition of unity, then on any open set U_j, the lifted operator will be $\sum_i \alpha_i \rho_{ij}(1\hat{\otimes}A)\rho_{ij}^{-1}$, which coincides with $(1\hat{\otimes}A)|_{U_j}$ modulo operators of order -1. The resulting operator $\tilde{A}_\tau$ on $L^2(\Lambda^*(\tau)\hat{\otimes}E)$ is defined modulo operators of order -1, and it is clear that $\tilde{A}_\tau$ will be $\mathcal{F}$-invariant (because on each U_j, the operator $(1\hat{\otimes}A)|_{U_j}$ is $\mathcal{F}$-invariant).

We can perform the same construction for the Hilbert module $C_0(\Lambda^*_\Gamma(X))$ in place of $C_0(\Lambda^*(\tau))$. On each subspace $(C_0(\Lambda^*_\Gamma(X)) \cdot C_0(U_i))\hat{\otimes}_{C_0(U_i)} L^2(E)$ the operator $\tilde{A}$ is defined as $1\hat{\otimes}A$. Then we glue these pieces together as above. This gives the operator $\tilde{A}$ on the Hilbert space $\mathcal{H}$.

The space $\mathcal{H}$ is a closed subspace of the space $\mathcal{H}' = L^2(\Lambda^*(\tau)\hat{\otimes}E)$. We can consider a restriction of the operator $\tilde{A}_\tau$ to an operator $\mathcal{H} \to \mathcal{H}'$. Obviously, this operator differs from the operator $\tilde{A} : \mathcal{H} \to \mathcal{H} \subset \mathcal{H}'$ at most by an operator which becomes compact when multiplied by any continuous function $a \in C_c(X)$.

Now we define the Clifford K-theoretic symbol $[\sigma_A^{\mathrm{cl}}]$ of a transversally elliptic operator A as an element of the (non-equivariant) group $\mathcal{R}K_*(X; \mathrm{Cl}_{\tau\oplus\Gamma}(X))$ according to Definition 8.11. We will define a (non-equivariant) Clifford index element $[A]^{\mathrm{cl}}$ directly as a product prescribed by Definition 8.10. This will allow us to avoid technicalities related with using foliation C^*-algebras, both in the definition of $\mathrm{ind}(A)$ and in the definition of the leaf-wise Dirac element.

We will use below a shorter notation $\Lambda^*(\mathcal{F})$ for the $C_c^\infty(X)$-submodule of $\Omega_c^*(X)$ generated by $C_c^\infty(X)$ and all wedge products of elements of $\mathcal{F}$. (This replaces notation $C_c^\infty(\Lambda^*_\Gamma(X))$ used in Sections 7 and 8.) The notation $C_0(\Lambda^*_\Gamma(X))$ remains as in Section 7.

Definition 9.2. Let $v_1, \dots, v_{\dim\mathcal{V}}$ be smooth sections of $\mathcal{V}$ which form a full orthonormal frame in fibers of $\mathcal{V}$ over an open subset W of X. Define the Dirac operator $D_\mathcal{F} : C_c^\infty(\Lambda^*(\tau)) \to C_c^\infty(\Lambda^*(\tau))$ over W as $\sum_{k=1}^{\dim\mathcal{V}}(\mathrm{ext}\,(f(v_k)) - \mathrm{int}\,(f(v_k)))L_{f(v_k)}$, where $L_{f(v_k)}$ is the Lie derivative on $C_c^\infty(\Lambda^*(\tau))$ in the direction of $f(v_k)$. This operator maps the subspace $\Lambda^*(\mathcal{F}|_W)$ into itself. Another choice of the orthonormal frame may affect the Dirac operator at most by adding a summand which is a (differential) operator of order 0. Therefore, $D_\mathcal{F}$ is defined globally over X up to a summand which is a multiplication operator.

The operator $D_\mathcal{F}$ is a leaf-wise elliptic differential operator of order 1 (cf. [3]). In the natural representations on the Hilbert spaces $L^2(\Lambda^*(\tau))$ and $C_0(\Lambda^*_\Gamma(X))\hat{\otimes}_{C_0(X)}L^2(X)$, it gives a regular unbounded operator (cf. [3, Section 6]) The symbol of the operator $D_\mathcal{F}$ on $C_c^\infty(\Lambda^*(\tau))$ (modulo operators of order 0) is calculated as in the proof of Theorem 8.9: $\sigma_{D_\mathcal{F}}(x,\xi) = i(\mathrm{ext}\,(\varphi_x(\xi)) - \mathrm{int}\,(\varphi_x(\xi)))$ for any $x \in X$, where $\varphi_x = f_x f_x^*$ (see Notation after Definition 7.1). Replacing $D_\mathcal{F}$ with $(D_\mathcal{F} + D_\mathcal{F}^*)/2$, we get a self-adjoint operator with the same symbol (again modulo order 0).

If E is a ($\mathbf{Z}_2$-graded) vector bundle over X, one can similarly define a Dirac operator on $C_c^\infty(\Lambda^*(\tau)\hat{\otimes}E)$ using the same trivialization and gluing procedure for E as above. We obtain an unbounded, essentially self-adjoint, first order differential operator $D_\mathcal{F}$ on $L^2(\Lambda^*(\tau)\hat{\otimes}E)$ and on $C_0(\Lambda^*_\Gamma(X))\hat{\otimes}_{C_0(X)}L^2(E)$. Let us denote by $T_\mathcal{F}$ the bounded operator $D_\mathcal{F}(1 + D_\mathcal{F}^2)^{-1/2}$ on the latter space.

Definition-Lemma 9.3. Let A be an L^2-bounded, properly supported pseudo-differential operator of order 0 which acts on sections of the vector bundle E.

We assume that A is $\mathcal{F}$-invariant and transversally elliptic with respect to the field Γ corresponding to the foliation $\mathcal{F}$. Notation $\tilde{A}_\tau$ will mean a lifting of A to $\Lambda^*(\tau)\hat{\otimes}E$, and $\tilde{A}$ will mean a lifting of A to $C_0(\Lambda^*_\Gamma(X))\hat{\otimes}_{C_0(X)}L^2(E)$. There exists an element $[A]^{\mathrm{cl}} \in K^*(\mathrm{Cl}_\Gamma(X))$ given by the pair $(\mathcal{H}, S)$, where $\mathcal{H} = C_0(\Lambda^*_\Gamma(X))\hat{\otimes}_{C_0(X)}L^2(E)$, and the operator $S = M_1 \cdot T_\mathcal{F} + M_2 \cdot \tilde{A}$. The algebra $\mathrm{Cl}_\Gamma(X)$ acts on $\mathcal{H}$ via its action on $\mathcal{K}(C_0(\Lambda^*_\Gamma(X)))$ defined on covector fields $\xi \in \Gamma$ by $\xi \to \mathrm{ext}\,(\xi) + \mathrm{int}\,(\xi)$. The operators M_1, M_2 are the operators entering the KK-product construction [30, Theorem 2.11].

Proof. We will use the notation introduced above: $\mathcal{H}' = L^2(\Lambda^*(\tau)\hat{\otimes}E)$. As we already explained, since A is $\mathcal{F}$-invariant, the same is true for the lifting of A to $\Lambda^*(\tau)\hat{\otimes}E$. An easy calculation shows that the symbol of the operator $[D_\mathcal{F}, \tilde{A}_\tau]$ on $\Lambda^*(\tau)\hat{\otimes}E$ locally belongs to $\mathcal{K}(\mathfrak{S}_\Gamma(\Lambda^*(\tau)\hat{\otimes}E))$.

For the construction of operators M_1, M_2 we will proceed as in the proof of Theorem 2.11 of [30]. We need to choose three C^*-subalgebras $J, \mathcal{A}_1, \mathcal{A}_2 \subset \mathcal{L}(\mathcal{H})$. We take $J = \mathcal{K}(\mathcal{H})$. The subalgebra $\mathcal{A}_1$ is generated by J and elements $a(1 + D^2_\mathcal{F})^{-1}, aD_\mathcal{F}(1 + D^2_\mathcal{F})^{-1}$, for all $a \in \mathcal{K}(C_0(\Lambda^*_\Gamma(X)))$. The subalgebra $\mathcal{A}_2$ is generated by J and elements $1 - \tilde{A}^2$, $[T_\mathcal{F}, \tilde{A}]$. The reasoning given in the proof of Lemma 4.2 of [30] shows that for any $a \in \mathrm{Cl}_\Gamma(X)$, the graded commutator $[a, T_\mathcal{F}]$ is in $\mathcal{A}_1$.

Now we claim that $\mathcal{A}_1\cdot(1-\tilde{A}^2) \subset J$ and $\mathcal{A}_1\cdot[D_\mathcal{F}, \tilde{A}] \subset J$. In view of the previous discussion on the construction of the operator $\tilde{A}$, it is enough to show that for operators on $\mathcal{H}'$ we have: $\mathcal{A}_1\cdot(1-\tilde{A}^2_\tau) \subset \mathcal{K}(\mathcal{H}')$ and $\mathcal{A}_1\cdot[D_\mathcal{F}, \tilde{A}_\tau] \subset \mathcal{K}(\mathcal{H}')$. This immediately follows from Lemma 6.5, where we take $F = D_\mathcal{F}(1 + \Delta)^{-1}D_\mathcal{F} + (1 + \Delta)^{-1}$ and $T = 1 - \tilde{A}^2$ or $T = [D_\mathcal{F}, \tilde{A}]$. (The symbol of $D^2_\mathcal{F}$ on $\Lambda^*(\tau)\hat{\otimes}E$ is $\|\varphi_x(\xi)\|^2$ modulo symbols of order 1. Remark 2 after Definition 6.2 allows to replace $q_x(\xi)$ with $\|\varphi_x(\xi)\|^2$.)

The operator $a[(1 + D^2_\mathcal{F})^{-1}, \tilde{A}]$, for $a \in \mathcal{K}(C_0(\Lambda^*(X)))$, can be presented as

$$a(1 + D^2_\mathcal{F})^{-1}[\tilde{A}, D_\mathcal{F}]D_\mathcal{F}(1 + D^2_\mathcal{F})^{-1} - aD_\mathcal{F}(1 + D^2_\mathcal{F})^{-1}[\tilde{A}, D_\mathcal{F}](1 + D^2_\mathcal{F})^{-1}.$$

In view of the above, this operator is compact. The same is true for $a[D_\mathcal{F}(1 + D^2_\mathcal{F})^{-1}, \tilde{A}]$ (using a similar expression). From the compactness of $a[(1 + D^2_\mathcal{F})^{-1}, \tilde{A}]$, we also deduce compactness of $a[(1 + D^2_\mathcal{F})^{-1/2}, \tilde{A}]$. It easily follows that the operator $a(1 + D^2_\mathcal{F})^{-1/2}[T_\mathcal{F}, \tilde{A}]$ is compact for any $a \in \mathcal{K}(C_0(\Lambda^*(X)))$. Therefore, $\mathcal{A}_1 \cdot \mathcal{A}_2 \subset J$.

Note that $[a, \tilde{A}] \in J$, for all $a \in \mathrm{Cl}_\Gamma(X)$, so all elements $[a, \tilde{A}]$ belong to $\mathcal{A}_2$. Following further the proof of Theorem 2.11 of [30], we have to check that all graded commutators $[\tilde{A}, \mathcal{A}_1]$, $[a, \mathcal{A}_1]$, $[T_\mathcal{F}, \mathcal{A}_1]$ belong to $\mathcal{A}_1$ for any $a \in \mathrm{Cl}_\Gamma(X)$. This is already verified for $[\tilde{A}, \mathcal{A}_1]$. For $[a, \mathcal{A}_1]$ and $[T_\mathcal{F}, \mathcal{A}_1]$ this fact is obvious. To finish the proof, it remains to apply Theorem 1.4 of [30]. □

Now we can state analogs of Theorems 8.12 and 8.14. Proofs are exactly the same.

Theorem 9.4 (Inverse Clifford index theorem). *Let X be a complete Riemannian manifold and $\mathcal{F}$ a singular foliation on X defined by a Lie algebroid. Let A be a properly supported, L^2-bounded, $\mathcal{F}$-invariant transversally elliptic operator on X of order 0, of the Hörmander class $\rho = 1, \delta = 0$. Then the elements $[\sigma_A^{\mathrm{cl}}]$ and $[\Theta'_{X,\Gamma}] \otimes_{\mathrm{Cl}_\Gamma(X)} [A]^{\mathrm{cl}}$ coincide in the group $\mathcal{R}K_*(X; \mathrm{Cl}_{\tau\oplus\Gamma}(X))$.*

We will consider $[\sigma_A^{\mathrm{cl}}]$ as an element of $\mathcal{R}KK_*(X; \mathrm{Cl}_\Gamma(X), \mathrm{Cl}_\tau(X))$ for the next theorem, according to Remark 7.5.

Theorem 9.5 (Clifford index theorem). *In the assumptions of Theorem 9.4,*

$$[A]^{\mathrm{cl}} = [\sigma_A^{\mathrm{cl}}] \otimes_{\mathrm{Cl}_\tau(X)} [d_X] \in K^*(\mathrm{Cl}_\Gamma(X))$$

and

$$[A]^{\mathrm{cl}} = [\sigma_A^{\mathrm{tcl}}] \otimes_{C_0(TX)} [\mathcal{D}_X] \in K^*(\mathrm{Cl}_\Gamma(X)).$$

Here $[\sigma_A^{\mathrm{tcl}}] \in \mathcal{R}KK_*(X; \mathrm{Cl}_\Gamma(X), C_0(TX))$ is defined as in 8.13 (and using Remark 7.5).

The elements $[\sigma_A^{\mathrm{cl}}]$ and $[A]^{\mathrm{cl}}$ are Poincaré dual according to Theorem 7.8.

Theorems 9.4 and 9.5 are *non-equivariant* analogs of Theorems 8.12 and 8.14 in the singular foliation case that we consider. An equivariant version must probably include a groupoid action. However, we point out to a certain difference between the group action case where we had to assume properness of group action and the foliation case where we have no groupoid action and no properness assumption.

Finally, I want to state a conjecture concerning the index of leaf-wise elliptic operators. (Again we will consider only the case of a foliation defined by a Lie algebroid.) The pseudo-differential calculus for leaf-wise operators is developed in [3]. In [4], the symbol $[\sigma_A]$ and the index, which we will denote $\mathrm{ind}(A)$, are defined as elements of suitable K-theory groups. The element $[\sigma_A]$ essentially is an element of the group $\mathcal{R}KK_*(X; C_0(X), C_0(\mathcal{F}^*))$, where the space $\mathcal{F}^*$ is called in [3] the "cotangent bundle". (The argument $C_0(X)$ of the KK-group enters here because we do not assume our operators to be trivial at infinity.) In our (Lie algebroid) case the corresponding Clifford symbol $[\sigma_A^{\mathrm{cl}}]$ will be an element of the group $\mathcal{R}KK_*(X; C_0(X), \mathrm{Cl}_\Gamma(X)) = \mathcal{R}K_*(X; \mathrm{Cl}_\Gamma(X))$.

The index $\mathrm{ind}(A)$, according to [3, 4], can be defined as an element of the group $KK(C_0(X), C^*(X, \mathcal{F}))$, where $C^*(X, \mathcal{F})$ is the foliation algebra (again $C_0(X)$ enters here because we do not assume our operators to be trivial at infinity).

To state a conjectural index formula, we need a leaf-wise Dirac element $[D_\Gamma] \in KK^*(\mathrm{Cl}_\Gamma(X), C^*(X, \mathcal{F}))$. It can be defined as in 8.3, 8.5 and 9.2, using the pseudo-differential calculus of [3].

Conjecture 9.6.

$$\mathrm{ind}(A) = [\sigma_A^{\mathrm{cl}}] \otimes_{\mathrm{Cl}_\Gamma(X)} [D_\Gamma],$$

where $[D_\Gamma]$ is the leafwise Dirac element.

We remark that the Clifford symbol groups of leaf-wise elliptic and transversally elliptic operators, $\mathcal{R}K_*(X;\mathrm{Cl}_\Gamma(X))$ and $\mathcal{R}K_*(X;\mathrm{Cl}_{\tau\oplus\Gamma}(X))$, are in a certain sense dual. There is a natural pairing defined by the external product of Proposition 2.21 of [30] combined with the Poincaré duality isomorphism:

$$\mathcal{R}K_*(X;\mathrm{Cl}_\Gamma(X)) \otimes \mathcal{R}K_*(X;\mathrm{Cl}_{\tau\oplus\Gamma}(X)) \to \mathcal{R}K_*(X;\mathrm{Cl}_\tau(X)) \simeq K^*(C_0(X)).$$

Using the Poincaré duality 7.8, we get the pairing for the corresponding analytical Clifford index groups:

$$K^*(\mathrm{Cl}_{\tau\oplus\Gamma}(X)) \otimes K^*(\mathrm{Cl}_\Gamma(X)) \to K^*(C_0(X)).$$

When X is compact, this gives the pairing with values in $K^*(\mathbf{C})$.

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G. Kasparov, Department of Mathematics, Vanderbilt University, 1326 Stevenson Center, Nashville, TN 37240, USA

E-mail: gennadi.kasparov@vanderbilt.edu