J. Noncommut. Geom. 10 (2016), 1243–1268
DOI 10.4171/JNCG/259

Journal of Noncommutative Geometry

About the obstacle to proving the Baum–Connes conjecture without coefficient for a non-cocompact lattice in Sp_4 in a local field

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Abstract. We introduce property $(T_{\mathrm{Schur}}, G, K)$ and prove it for some non-cocompact lattice in Sp_4 in a local field of finite characteristic. We show that property $(T_{\mathrm{Schur}}, G, K)$ for a non-cocompact lattice Γ in a higher rank almost simple algebraic group in a local field is an obstacle to proving the Baum–Connes conjecture without coefficient for Γ with known methods, and this is stronger than the well-known fact that Γ does not have the property of rapid decay (property (RD)). It is the first example (as announced in [7]) for which all known (as of March, 2015) methods for proving the Baum–Connes conjecture without coefficient fail.

Mathematics Subject Classification (2010). 46L80; 19K56.

Keywords. Baum–Connes conjecture, Kazhdan's property (T), the property of rapid decay.

1. Introduction

N. Higson, V. Lafforgue, and G. Skandalis constructed counterexamples to the Baum–Connes conjecture for discrete group actions on commutative C^* algebras using Gromov's groups which do not uniformly embed into Hilbert space [3]. V. Lafforgue introduced strong Banach property (T) [5,6], proved it for $SL_3(\mathbb{Q}_p)$, and constructed the first example of expander graphs which do not embed uniformly into any Banach space of non-trivial type. Other examples of expander graphs non-embeddable in Banach spaces of non-trivial type or Banach spaces of weaker properties have been found [11,13,15]. In [8], V. Lafforgue introduced property (T_{Schur}) (which is stronger than strong property (T) [5]) and proved that it is an obstacle to proving Baum–Connes conjecture for $SL_3(\mathbb{Q}_p)$ with commutative coefficient containing $C_0(SL_3(\mathbb{Q}_p))$ with known methods. In this article, we introduce property $(T_{\mathrm{Schur}}, G, K)$ in Definition 1.1 as an analogue of property (T_{Schur}), prove it for the non-cocompact lattice $\Gamma = Sp_4(\mathbb{F}_q[\pi^{-1}])$ of $Sp_4(\mathbb{F}_q((\pi)))$ in Theorem 1.2 (which is the main result of this article), and show that it is an obstacle to proving Baum–Connes conjecture

*This work is part of the author's Ph.D. thesis in Université Paris Diderot - Paris 7.

without coefficients for the lattice Γ with known methods, which is stronger than the well-known fact that Γ does not have the property of rapid decay (property (RD)). It is the first example for which all known methods for proving Baum–Connes conjecture *without coefficient* fail.

We begin with some notations, and then state the main result of this article.

Let G be non-compact locally compact topological group, $K \subsetneq G$ a compact subgroup. Let $H \subseteq G$ be a non-compact closed subgroup. Let $\ell : G \to \mathbb{R}_{\geq 0}$ be a continuous length function on G. Denote by B_n the ball of radius n in G.

For any continuous function $c \in C(G)$ on G, we introduce the following notation for the norm of the Schur product by c on the subspace

$$C(H \cap B_n) = \{f \in C_c(H), \operatorname{supp}(f) \subset B_n\}$$

of functions on H with supports in B_n,

$$\begin{aligned} &\| \operatorname{Schur}_c|_{C(H\cap B_n)}\| \\ &\qquad = \| \operatorname{Schur}_c |_{C(H\cap B_n)}\|_{\mathcal{L}(C_r^*(H))} \\ &\qquad = \sup\{\| \operatorname{Schur}_c f\|_{C_r^*(H)}, f \in C_c(H), \operatorname{supp}(f) \subset B_n, \|f\|_{C_r^*(H)} \leq 1\}, \end{aligned}$$

where $\operatorname{Schur}_c f \in C_c(H)$ denotes the Schur product

$$\operatorname{Schur}_c f(h) = c(h) f(h), \forall h \in H.$$

Definition 1.1. We say that H has property (T_{Schur}, G, K) if for any continuous length function $\ell : G \to \mathbb{R}_{\geq 0}$, there exists $s_0 > 0$ such that $\forall s \in [0, s_0)$ there exists a continous function $\phi \in C_0(G)$ vanishing at infinity, such that $\forall C > 0$ and for any family of K-biinvariant functions $c \in C(G)$ with the following uniform Schur condition

$$\| \operatorname{Schur}_c |_{C(H\cap B_n)}\|_{\mathcal{L}(C_r^*(H))} \leq C e^{sn}, \forall n \in \mathbb{N}, \tag{1.1}$$

there exists a limit $c_\infty \in \mathbb{C}$ to which c tends uniformly rapidly

$$|c(g) - c_\infty| \leq C\phi(g), \forall g \in G.$$

Let $\mathbb{F}_q$ be a finite field of cardinality q which is not divided by 2 (this assumption is needed in the proofs due to technical reasons - we do not intend to discuss the case of characteristic 2, since one example is sufficient to elaborate the obstacle to proving the conjecture). Let G be $Sp_4(\mathbb{F}_q((\pi)))$ over the local field $\mathbb{F}_q((\pi))$, $K = Sp_4(\mathbb{F}_q[[\pi]])$ a maximal compact subgroup of G. Let Γ be the non-cocompact lattice $Sp_4(\mathbb{F}_q[\pi^{-1}])$ in G. Let $H \subsetneq \Gamma$ be the unipotent subgroup consisting of elements of the form

$$\begin{pmatrix} 1 & * & * & * \\ & 1 & * & * \\ & & 1 & * \\ & & & 1 \end{pmatrix} \in \Gamma.$$

The following is the main result of this article.

Theorem 1.2. *The unipotent group H has property (T_{Schur}, G, K) as in Definition 1.1. As a consequence, the non-cocompact lattice $\Gamma = Sp_4(\mathbb{F}_q[\pi^{-1}]) \subsetneq G = Sp_4(\mathbb{F}_q((\pi)))$ also has property (T_{Schur}, G, K) .*

Remark. That Γ has property (T_{Schur}, G, K) follows from that H has property (T_{Schur}, G, K) . Indeed, it is clear from Definition 1.1 that for any two discrete subgroups $H \subsetneq H' \subseteq G$, if H has property (T_{Schur}, G, K) , then H' also has property (T_{Schur}, G, K) .

H is an amenable group and thus satisfies the Baum–Connes conjecture. Theorem 1.2 says that the constant function 1 on H cannot be deformed among K-biinvariant functions on G satisfying the Schur type condition (1.1), whereas it does not prevent the possibilities of deformations among other functions.

For the non-cocompact lattice $\Gamma = Sp_4(\mathbb{F}_q[\pi^{-1}])$, property (T_{Schur}, G, K) is an obstacle of known methods for proving Baum–Connes conjecture (without coefficient). It plays the role of property (T_{Schur}) for a locally compact group G (e.g. $SL_3(\mathbb{Q}_p)$) relative to some open compact subgroup (e.g. $SL_3(\mathbb{Z}_p)$) as in [8] being an obstacle of known methods for proving Baum–Connes conjecture for G with commutative coefficient containing $C_0(G)$. Indeed, if a lattice in a higher rank almost simple algebraic group has property (T_{Schur}, G, K) , then it does not have property (RD) (Proposition 2.3). A. Valette conjectured that any cocompact lattice in a simple algebraic group over a local field has property (RD), thus cocompact lattice wouldn't seem to have property (T_{Schur}, G, K) . What is shown in this article is stronger: property (T_{Schur}, G, K) for a higher rank lattice Γ prevents the existence of any reasonable dense subalgebra of $C_r^*(\Gamma)$ for known methods for proving the Baum–Connes conjecture (Proposition 2.1), in particular the Jolissaint algebra for property (RD). We recall known methods for proving Baum–Connes conjecture for a lattice in a reductive group over a local field in Section 2, and prove in Proposition 2.1 that the conditions of these methods are not satisfied for any lattice with property (T_{Schur}, G, K) (by adapting the arguments in [8] to our situation).

We will give two proofs of Theorem 1.2 in Sections 3 and 4, respectively.

The first proof is more in line with the arguments in [9], and is therefore more transparent and more checkable. What is different from [9] is the use of two families of parameters (i.e. $n = 2$ in Lemma 3.3) which yields an improved estimate q^{-j} in the second inequality of Proposition 3.2.

The second proof makes use of two families of functions on H with exponentially small C_r^* norms, and yields a slightly better constant s_0 in Definition 1.1. The first family of functions corresponding to the abelian subgroups are already constructed and the corresponding estimate is obtained in [7], which we include in this article using the same arguments. We construct in this article the other family of functions corresponding to the discrete Heisenberg subgroup. The improved estimate in q^{-j}

is obtained using harmonic analysis on the Heisenberg group over the ring of polynomials $\mathbb{F}_q[\pi^{-1}]$.

Acknowledgements. I would like to thank Vincent Lafforgue for his useful suggestions and precise explanations of various aspects on this problem. I thank Georges Skandalis for the discussion on algebra A_θ. I thank Mikael de la Salle for several corrections to this article, especially for pointing out a mistake in the parameterization of the group of Heisenberg type.

2. An obstacle to proving the Baum–Connes conjecture without coefficient

In this section, G is an arbitary reductive group over a local field, $K \subsetneq G$ a maximal compact subgroup, $\ell : G \to \mathbb{R}_{\geq 0}$ a continuous length function, and $\Gamma \subsetneq G$ a lattice.

Adapting conditions $(\tilde{D}) = (D1) + (D2) + (D3) + (D4)$ in [8] to our situation, we list the following conditions $(\tilde{D}') = (D1) + (D2) + (D3) + (D4')$ to include all known methods to prove Baum–Connes conjecture for Γ. More precisely, conditions $(D1), (D2), (D3)$ are just specialization to $A = \mathbb{C}$ of conditions $(D1)$, $(D2)$, $(D3)$ [8], and condition $(D4')$ is a variant of $(D4)$ [8] in which we require that the representations of Γ for the homotopy come from representations of G.

(D1) For any $s > 0$, there exists $C_s > 0$, and a Banach subalgebra $\mathcal{B}_s \subset C_r^*(\Gamma)$ containing $\mathbb{C}(\Gamma)$ as a dense subalgebra, such that $\forall n \in \mathbb{N}, \forall f \in \mathbb{C}(\Gamma)$ with $\mathrm{supp}(f) \subset B_n$,

$$\|f\|_{\mathcal{B}_s} \leq C_s e^{sn} \|f\|_{C_r^*(\Gamma)},$$

where $B_n \subset \Gamma$ denotes the ball of length n.

(D2) There exist a homotopy $(E, \pi, T) \in E^{\mathrm{ban}}_{\Gamma,?}(\mathbb{C}, \mathbb{C}[0,1])$ (? indicates that there is no restriction on the norms of the representations of Γ) from 1 to the γ element, and $(\tilde{E}, \tilde{\pi}, \tilde{T}) \in E^{\mathrm{ban}}(\mathcal{B}_s, C_r^*(\Gamma)[0,1])$ with $(\tilde{E}^<, \tilde{E}^>)$ containing $(\mathbb{C}(\Gamma, E^<), \mathbb{C}(\Gamma, E^>))$ as a dense subspace, such that the embeddings

$$i_s : \mathbb{C}(\Gamma) \hookrightarrow \mathcal{B}_s, i_r : \mathbb{C}(\Gamma, \mathbb{C}[0,1]) \hookrightarrow C_r^*(\Gamma, \mathbb{C}[0,1]),$$
$$i^< : \mathbb{C}(\Gamma, E^<) \hookrightarrow \tilde{E}^<, i^> : \mathbb{C}(\Gamma, E^>) \hookrightarrow \tilde{E}^>$$

and

$$\langle \cdot, \cdot \rangle : \tilde{E}^< \times \tilde{E}^> \to C_r^*(\Gamma)[0,1]$$

satisfy: $\forall f \in \mathbb{C}(\Gamma), \varphi \in \mathbb{C}(\Gamma, \mathbb{C}[0,1]), F_1 \in \mathbb{C}(\Gamma, E^<), F_2 \in \mathbb{C}(\Gamma, E^>)$,

$$i^<(\varphi F_1) = i_r(\varphi) i^<(F_1), i^<(F_1 f) = i^<(F_1) i_s(f),$$
$$i^>(f F_2) = i_s(f) i^>(F_2), i^>(F_2 \varphi) = i^>(F_2) i_r(\varphi),$$

and

$$i_r(\langle F_1, F_2 \rangle) = \langle i^<(F_1), i^>(F_2) \rangle.$$

(D3) The $\mathbb{C}[0,1]$ pair $(E^<, E^>)$ is in isometric duality as defined in [8, Definition 2.13], i.e. the maps

$$E^< \to \mathcal{L}_{\mathbb{C}[0,1]}(E^>, \mathbb{C}[0,1]), E^> \to \mathcal{L}_{\mathbb{C}[0,1]}(E^<, \mathbb{C}[0,1])$$

are isometries.

(D4′) The representation E of Γ is restriction of some representation of G on which K acts by isometries, and $\forall x \in E^>, \xi \in E^<$,

$$\|e_0 \otimes x\|_{\tilde{E}^>} \leq \|x\|_{E^>}, \|e_0 \otimes \xi\|_{\tilde{E}^<} \leq \|\xi\|_{E^<},$$

where $e_0 \in \mathbb{C}(\Gamma)$ is the Dirac function at the neutral element (and thus $e_0 \otimes x \in \mathbb{C}(\Gamma, E^>), e_0 \otimes \xi \in \mathbb{C}(\Gamma, E^<)$).

Proposition 2.1. *Suppose Γ has property (T_{Schur}, G, K) as in Definition 1.1. Then for any continuous length function ℓ on G, the 4-tuple G, K, ℓ, Γ do not satisfy $(\tilde{D}') = (D1) + (D2) + (D3) + (D4')$.*

The proof of Proposition 2.1 is an adaptation of the proof of Proposition 4.2 in [8] to our situation.

Lemma 2.2. *Suppose Γ satisfies $(D1) + (D2) + (D4')$. Then $\forall s > 0$, there exists $C_s > 1$, such that $\forall x \in E^>, \xi \in E^<$ both of norms ≤ 1, putting $c_t(\gamma) = \langle \xi, \pi_t(\gamma)x \rangle$, $\forall \gamma \in \Gamma$, we have $\forall n \in \mathbb{N}, \forall t \in [0,1]$,*

$$\| \operatorname{Schur}_{c_t} |_{\mathbb{C}(\Gamma \cap B_n)} \| \leq C_s e^{sn}.$$

Proof. For any $f \in \mathbb{C}(\Gamma)$ we have the following fundamental calculation

$$\begin{aligned} \langle e_0 \otimes \xi, f(e_0 \otimes x) \rangle &= \sum_{\gamma \in \Gamma} f(\gamma) \langle \xi, \pi_t(\gamma)x \rangle e_\gamma \\ &= \operatorname{Schur}_{c_t} f \in \mathbb{C}(\Gamma). \end{aligned}$$

By condition $(D2)$ and $(D4')$,

$$\begin{aligned} \| \operatorname{Schur}_{c_t} f \|_{C_r^*(\Gamma)} &\leq \|e_0 \otimes \xi\|_{\tilde{E}^<} \|f\|_{\mathcal{B}_s} \|e_0 \otimes x\|_{\tilde{E}^>} \\ &\leq \|\xi\|_{E^<} \|f\|_{\mathcal{B}_s} \|x\|_{E^>} \leq \|f\|_{\mathcal{B}_s}. \end{aligned}$$

When $\operatorname{supp} f \subset B_n$, by condition $(D1)$,

$$\| \operatorname{Schur}_{c_t} f \|_{C_r^*(\Gamma)} \leq C_s e^{sn} \|f\|_{C_r^*(\Gamma)}. \qquad \square$$

Proof of Proposition 2.1. We prove it by contradiction. Suppose Γ, G, K, ℓ satisfy $(\tilde{D}')$ and Γ has property (T_{Schur}, G, K) . By Lemma 2.2 we see that when ξ, x are both K-invariant, $c_t(g)$ is a Cauchy sequence

$$|c_t(g) - c_t(g')| \leq C_s(\phi(g) + \phi(g')), \forall g, g' \in G.$$

By condition (D3) we have

$$\sup_{\substack{\|\xi\|_{E^<}\le 1,\\ K\text{-invariant},\\ t\in[0,1]}} |\langle \xi, (\pi_t(g) - \pi_t(g'))x\rangle| = \|\langle \cdot, (\pi(e_K e_g) - \pi(e_K e_{g'}))x\rangle\|_{\mathcal{L}(E^<,\mathbb{C}[0,1])}$$

$$= \|(\pi(e_K e_g) - \pi(e_K e_{g'}))x\|_{E^>},$$

where e_K is the characteristic function of $K \subset G$. As a consequence, $\pi(e_K e_g e_K)$ is also a Cauchy sequence

$$\|\pi(e_K e_g e_K) - \pi(e_K e_{g'} e_K)\|_{\mathcal{L}_{\mathbb{C}[0,1]}(E^<)} \le C_s(\phi(g) + \phi(g')).$$

For the same reason

$$\|\pi(e_K e_g e_K) - \pi(e_K e_{g'} e_K)\|_{\mathcal{L}_{\mathbb{C}[0,1]}(E^>)} \le C_s(\phi(g) + \phi(g')).$$

Denote by P the limit of $\pi(e_K e_g e_K)$. We see that $g'kg$ tends to infinity when g' tends to infinity since $\ell(gkg') \ge \ell(g') - \ell(g^{-1})$. Therefore, we have

$$e_K e_g P = \lim_{g'} \pi\Big(e_K \int_K e_{gkg'} dk e_K\Big) = P,$$

and

$$P^2 = \lim_{g} \lim_{g'} \pi\Big(e_K \int_K e_{gkg'} dk e_K\Big) = P.$$

Moreover, when E_t is a Hilbert space and (π_t, E_t) is a unitary representation of G, $P_t \in \mathcal{L}(E_t)$ is the projection onto G-invariant vectors $P_t E_t = E_t^G$. Indeed, $\forall x \in P_t E_t, \forall g \in G$,

$$\|\pi(e_g)x - \pi(e_K e_g)x\|^2 = \|\pi(e_g)x\|^2 - \|\pi(e_K e_g)x\|^2 = \|x\|^2 - \|x\|^2 = 0,$$

we have

$$x = \pi(e_K e_g)x = \pi(e_g)x.$$

$P \in \mathcal{L}_{\mathbb{C}[0,1]}(E)$ and consequently $PTP \in \mathcal{L}_{\mathbb{C}[0,1]}(E)$. We denote by $\operatorname{Im} P$ the $\mathbb{C}[0,1]$-pair whose underlying Banach spaces are the images of $E^<, E^>$ under the maps $P^<, P^>$. We have that $(\operatorname{Im} P, PTP) \in E^{\mathrm{ban}}(\mathbb{C}, \mathbb{C}[0,1])$. Indeed, $[e_K e_g e_K, T] \in \mathcal{K}_{\mathbb{C}[0,1]}(E)$, as a consequence $[P, T] \in \mathcal{K}_{\mathbb{C}[0,1]}(E)$. Moreover,

$$P - (PTP)^2 P = P(1 - T^2) + P[P,T]T + PTP[P,T] \in \mathcal{K}_{\mathbb{C}[0,1]}(E),$$

which means $\mathrm{Id}_{\operatorname{Im} P} - (PTP)^2 \in \mathcal{K}_{\mathbb{C}[0,1]}(\operatorname{Im} P)$.

Now (π_0, E_0) is the trivial representation of G ($E_0 = \mathbb{C}$), and (π_1, E_1) is a unitary representation of G without G-invariant vectors. $P_0 T_0 P_0 : \mathbb{C} \to 0$ has index 1 whereas $P_1 T_1 P_1 : 0 \to 0$ has index 0, this is a contradiction and the proposition is proved. □

Now let G be a semisimple group over a local field, $K \subsetneq G$ a maximal compact subgroup, $\Gamma \subsetneq G$ a lattice.

Let $\ell : G \to \mathbb{R}_{\geq 0}$ be the K biinvariant length function induced from the G-invariant Riemannian metric from the symmetric space or the Bruhat–Tits building associated to G. Recall that when the split rank of G is ≥ 2, Γ has Kazhdan's property (T) and thus is finitely generated. The word metric and Riemannian metric on Γ are bi-Lipschitz [12].

When Γ has property (RD), it is shown in [8] that G, K, ℓ, Γ satisfy conditions $(\tilde{D}')$ above, and thus do not fulfil the condition in Definition 1.1. We give a direct proof of this fact.

Proposition 2.3. *If Γ has property $(T_{\mathrm{Schur}}, G, K)$ as in Definition 1.1, then Γ does not have property (RD) for any continuous length function restricted from G. In particular when the split rank of G is ≥ 2, Γ does not have property (RD) for the word length.*

Proof. Suppose that Γ has property (RD) with respect to the polynomial $P(n) = Rn^D$ for some $R, D \geq 0$.

Denote by χ_{B_m} the characteristic function of B_m (ball of radius m) for $m \in \mathbb{N}$. For $f \in \mathbb{C}(\Gamma)$ with $\operatorname{supp} f \subset B_n$, we have

$$\begin{aligned}\| \mathrm{Schur}_{\chi_{B_m}} f \|_{C_r^*(\Gamma)} &\leq R \min(m,n)^D \| \mathrm{Schur}_{\chi_{B_m}} f \|_{\ell^2(\Gamma)} \leq Rn^D \| f \|_{\ell^2(\Gamma)} \\ &\leq Rn^D \| f \|_{C_r^*(\Gamma)}, \forall m \in \mathbb{N}.\end{aligned}$$

Namely, for any $s > 0$, there exists $C_s > 0$ such that

$$\| \mathrm{Schur}_{\chi_{B_m}} |_{\mathbb{C}(\Gamma \cap B_n)} \| \leq C_s e^{sn}, \forall n \in \mathbb{N}.$$

Now let $s \in (0, s_0)$, by property $(T_{\mathrm{Schur}}, G, K)$,

$$|\chi_{B_m}(\gamma)| \leq C_s \phi(\gamma), \forall m \in \mathbb{N}, \forall \gamma \in \Gamma,$$

which is a contradiction to the assumption that $\phi \in \mathbb{C}_0(\Gamma)$ is a function vanishing at infinity. □

3. First proof of Theorem 1.2

First let us be more precise on the notations.

Let by $\mathbb{F}_q$ a finite field of characteristic different from 2 with cardinality q. Denote by $F = \mathbb{F}_q((\pi))$ the local field of Laurent series in π with coefficients in $\mathbb{F}_q$, $\mathcal{O} = \mathbb{F}_q[[\pi]]$ the ring of formal series in π, i.e. the ring of integers of F.

Let $G = Sp_4(F)$, i.e. the symplectic group of 4 by 4 matrices $A \in M_4(F)$ satisfying $AJ^tA = J$, where ${}^tA$ denotes the transpose of A and

$$J = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}.$$

Let $K = Sp_4(\mathcal{O})$, a maximal compact subgroup of G.

Now let $\Gamma = Sp_4(\mathbb{F}_q[\pi^{-1}])$. By the well-known reduction theory of Harish-Chandra–Borel–Behr–Harder, Γ is lattice in G, and by Godement's compactness criterion (see [14, IV 1.4] in the case of characteristic p), it is non-cocompact.

Denote by H the unipotent subgroup in Γ consisting of elements of the form

$$\begin{pmatrix} 1 & a_{12} & a_{13} & a_{14} \\ & 1 & a_{23} & a_{24} \\ & & 1 & a_{34} \\ & & & 1 \end{pmatrix} \in \Gamma.$$

Note that for $a_{ij} \in F, 1 \leq i < j \leq 4$, a matrix of this form is in $Sp_4(F)$ if and only if $a_{12} + a_{34} = 0$ and $a_{13} - a_{12}a_{23} - a_{24} = 0$.

We define an explicit length function on G. Denote by $D(i, j)$ the diagonal element

$$D(i, j) = \begin{pmatrix} \pi^{-i} & & & \\ & \pi^{-j} & & \\ & & \pi^{j} & \\ & & & \pi^{i} \end{pmatrix} \in G$$

for any (i, j) in the Weyl chamber $\Lambda = \{(i, j) \in \mathbb{N}, i \geq j\}$. By Cartan decomposition, Λ is in bijection with the double coset $K\backslash G/K$ via the map $(i, j) \mapsto KD(i, j)K, (i, j) \in \Lambda$. Let $\ell : G \to \mathbb{N}$ be the length function on G defined by $\ell(kD(i, j)k') = i + j, (i, j) \in \Lambda, k, k' \in K$. It is clear that ℓ is bi-Lipschitz equivalent to the length function induced from the distance on the Bruhat–Tits building associated to G. For any continuous length function $\ell' : G \to \mathbb{R}_{\geq 0}$, there exists $\kappa > 0$ such that $\ell' \leq \kappa(\ell + 1)$.

It is clear that H surjects onto the double cosets $K\backslash G/K$, since

$$\begin{pmatrix} 1 & 0 & 0 & \pi^{-j} \\ & 1 & \pi^{-i} & 0 \\ & & 1 & 0 \\ & & & 1 \end{pmatrix} \in KD(i, j)K.$$

The following theorem is a more precise statement of Theorem 1.2 in the introduction.

Theorem 3.1. *Let* $s_0 = \frac{\log q}{6}$. *There exists a constant* $C_q > 0$ *depending only on* q*, such that the following holds. For any* $s \in [0, s_0)$ *and* $C > 0$*, for any* K*-biinvariant function* $c \in C(G)$ *with the following Schur condition*

$$\| \operatorname{Schur}_c |_{\mathbb{C}(H \cap B_n)} \| \leq C e^{sn}, \quad \forall n \in \mathbb{N}, \tag{3.1}$$

there exists a limit $c_\infty \in \mathbb{C}$ *to which* c *tends exponentially fast*

$$|c(g) - c_\infty| \leq C C_q e^{-\ell(g)(s_0 - s)/4}, \quad \forall g \in G.$$

Proof of Theorem 1.2 from Theorem 3.1. Let ℓ' be any length function on G. There exists $\kappa > 0$ such that $\forall g \in G, \ell'(g) \leq \kappa(\ell(g) + 1)$. With $s_0' = s_0/\kappa, s' \in [0, s_0')$ and $\phi'(g) = C_q e^{s\kappa} e^{-\ell(g)(s_0 - s\kappa)/4}$, $\forall g \in G$, Theorem. 1.2 is proved. □

Proposition 3.2. *For any* K *biinvariant function* $c \in C(G)$*, we have*

$$|c(D(i,j)) - c(D(i,j+1))| \leq 2q^{-(i-j)/2} \| \operatorname{Schur}_c |_{\mathbb{C}(H \cap B_{2i})} \|, \tag{3.2}$$

for any $(i,j) \in \Lambda$ *with* $i \geq 1$ *and*

$$|c(D(i,j)) - c(D(i+1,j-1))| \leq 2q^3 \cdot q^{-j} \| \operatorname{Schur}_c |_{\mathbb{C}(H \cap B_{i+j})} \|, \tag{3.3}$$

for any $(i,j) \in \Lambda$ *with* $j \geq 3$.

Proof of Theorem 3.1 from Proposition 3.2. The argument is very similar to that in the proof of Proposition 3.1 by Lemma 3.3 and 3.4 in [11]. By hypothesis

$$\| \operatorname{Schur}_c |_{\mathbb{C}(H \cap B_n)} \|_{C_r^*(H)} \leq C e^{sn}, \forall n \in \mathbb{N},$$

the two inequalities in Proposition 3.2 imply respectively

$$|c(D(i,j)) - c(D(i,j+1))| \leq 2q^{-(i-j)/2} C e^{2si}, \tag{3.4}$$

$$|c(D(i,j)) - c(D(i+1,j-1))| \leq 2q^3 \cdot q^{-j} C e^{s(i+j)}. \tag{3.5}$$

Combing the two inequalities above we have

$$|c(D(3j,j)) - c(D(3j+3,j+1))| \leq C C_q e^{-(\log q - 6s)j}.$$

By moving along the line $i = 3j$ in the Weyl chamber as illustrated below, we have

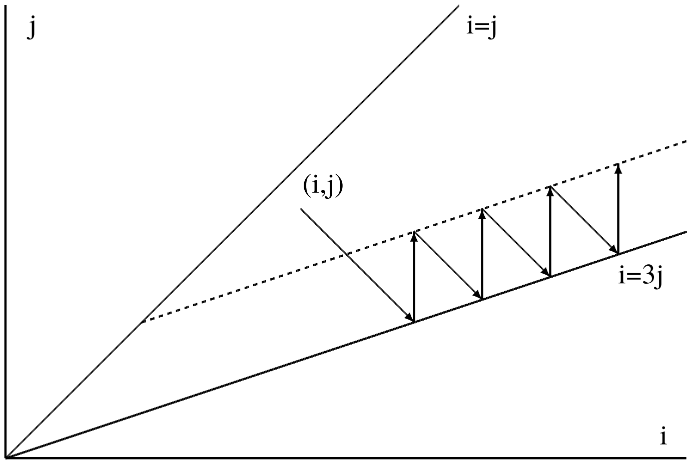
$$|c(D(3j,j)) - c_\infty| \leq C C_q e^{-(\log q - 6s)j}.$$

When $i \geq 3j$,

$$|c(D(i,j)) - c_\infty)| \leq C C_q e^{-(\log q - 6s)i/3} \leq C C_q e^{-(\log q - 6s)(i+j)/4}.$$

When $i \leq 3j$,

$$|c(D(i,j)) - c_\infty)| \leq C C_q e^{-(\log q - 6s)(i+j)/4}.$$



□

To prove Proposition 3.2, we quote the following lemma in [10], which will be applied in the proof to some finite subgroups in H.

Lemma 3.3 ([10, Lemma 4.9]). *Let $m, n \in \mathbb{N}^*, k \in \{1, 2, \ldots, m\}$. Let H be a locally compact amenable group, $\alpha, \beta : (\mathcal{O}/\pi^m\mathcal{O})^{n+1} \to H$ two maps. Let $f \in C_c(H)$ satisfying*

$$f(\alpha(a_1, \ldots, a_n, b)\beta(x_1, \ldots, x_n, y)) = \lambda, \text{ if } y = \sum_{i=1}^{n} a_i x_i + b + \pi^k \in \mathcal{O}/\pi^m\mathcal{O},$$

and

$$f(\alpha(a_1, \ldots, a_n, b)\beta(x_1, \ldots, x_n, y)) = \mu, \text{ if } y = \sum_{i=1}^{n} a_i x_i + b + \pi^{k-1} \in \mathcal{O}/\pi^m\mathcal{O}.$$

Then

$$|\lambda - \mu| \leq 2q^{-nk/2}\|f\|_{MA(H)}$$

where

$$\|f\|_{MA(H)} = \sup\{\|\operatorname{Schur}_f(\varphi)\|_{C_r^*(H)}, \|\varphi\|_{C_r^*(H)} \leq 1\}.$$

Let us remark that when H is an arbitrary locally compact group, the lemma above still holds if in the conclusion $\|f\|_{MA(H)}$ is replaced by

$$\|f\|_{M_0A(H)} = \sup_{B\ H-C^*\text{-algebra}} \{\|\operatorname{Schur}_f(\varphi)\|_{C_r^*(H,B)}, \|\varphi\|_{C_r^*(H,B)} \leq 1\}.$$

But we do not need this generality in the proof of Proposition 3.2.

Proof of the first inequality in Proposition 3.2. We adapt the arguments in the proof of Lemma 2.1 in [9] to our situation by discretizing the matrices.

Denote by $[\cdot] : \mathbb{F}_q((\pi)) \to \mathbb{F}_q[\pi^{-1}]$ the integral part of an element, i.e.

$$[a_i\pi^{-i} + a_{i-1}\pi^{-i+1} + \cdots + a_1\pi^{-1} + a_0 + a_{-1}\pi + \cdots] = a_i\pi^{-i} + a_{i-1}\pi^{-i+1} + \cdots + a_1\pi + a_0, \forall a_* \in \mathbb{F}_q.$$

Let $\sigma : \mathcal{O}/\pi^{i+1}\mathcal{O} \to \mathcal{O} = \mathbb{F}_q[[\pi]]$ be any section. Define $\alpha, \beta : (\mathcal{O}/\pi^{i+1}\mathcal{O})^2 \to H$ by

$$\alpha(a,b) = \begin{pmatrix} 1 & 0 & [\pi^{-i}\sigma(a)] & [\pi^{-i}\sigma(a^2-b)] \\ 0 & 1 & \pi^{-i} & [\pi^{-i}\sigma(a)] \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

$$\beta(x,y) = \begin{pmatrix} 1 & 0 & [\pi^{-i}\sigma(x/2)] & [\pi^{-i}\sigma(x^2/4+y)] \\ 0 & 1 & 0 & [\pi^{-i}\sigma(x/2)] \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

for any $a, b, x, y \in \mathcal{O}/\pi^{i+1}\mathcal{O}$.

Compute $\alpha(a,b)\beta(x,y)$

$$= \begin{pmatrix} 1 & 0 & [\pi^{-i}(\sigma(x/2)+\sigma(a))] & [\pi^{-i}(\sigma(x^2/4+y)+\sigma(a^2-b))] \\ 0 & 1 & \pi^{-i} & [\pi^{-i}(\sigma(x/2)+\sigma(a))] \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

We see that $\forall a, b, x, y \in \mathcal{O}/\pi^{i+1}\mathcal{O}$, $\|\alpha(a,b)\beta(x,y)\| = q^i$ (the $(2,3)$ matrix element achieves the maximal norm). Moreover, we see that

$$\det\begin{pmatrix} [\pi^{-i}(\sigma(x/2)+\sigma(a))] & [\pi^{-i}(\sigma(x^2/4+y)+\sigma(a^2-b))] \\ \pi^{-i} & [\pi^{-i}(\sigma(x/2)+\sigma(a))] \end{pmatrix} = -\pi^{-2i}(y-ax-b) \bmod \pi^{-i+1}\mathcal{O}.$$

So for any $l \in \{0, 1, \ldots, i\}$, when

$$y - ax - b \in \pi^l\mathcal{O}^\times/\pi^{i+1}\mathcal{O} \tag{3.6}$$

(where $\mathcal{O}^\times$ denotes the group of units of $\mathcal{O}$), we have

$$\| \wedge^2 (\alpha(a,b)\beta(x,y))\| = q^{2i-l}.$$

That is to say for any $l \in \{0, 1, \ldots, i\}$, when $a, b, x, y \in \mathcal{O}/\pi^{i+1}\mathcal{O}$ satisfy 3.6, we have

$$\alpha(a,b)\beta(x,y) \in KD(i, i-l)K.$$

Now denote for any $n \in \mathbb{N}^*$,

$$H_1^n = \{\begin{pmatrix} 1 & 0 & x & z \\ & 1 & y & x \\ & & 1 & 0 \\ & & & 1 \end{pmatrix}, x, y, z \in \mathbb{F}_q[\pi^{-1}], |x|, |y|, |z| \le q^n\},$$

which is a finite subgroup of H. Note that the images of α and β both lie in H_1^i. Apply Lemma 3.3 to $n = 1, m = i + 1, k = i - j$, the finite group H_1^i, α, β as above, and $f = c|_{H_1^i}, \lambda = c(D(i, j)), \mu = c(D(i, j + 1))$, we have

$$\begin{aligned}|c(D(i, j)) - c(D(i, j + 1))| &\le 2q^{-\frac{i-j}{2}} \| \mathrm{Schur}_{c|_{H_1^i}} \| \\ &\le 2q^{-\frac{i-j}{2}} \| \mathrm{Schur}_c |_{\mathbb{C}(H \cap B_{2i})} \|.\end{aligned}$$

The last inequality is due to the facts that $H_1^i \subset H \cap B_{2i}$ and $\|f\|_{C_r^*(H_1^i)} = \|f\|_{C_r^*(H)}, \forall f \in \mathbb{C}(H_1^i) \subset \mathbb{C}(H)$. □

Proof of the second inequality in Proposition 3.2. We will use discretization as in the proof of the first inequality, and improve the matrices in the proof of Lemma 2.2 in [9]. This improvement allows us to use the case of $n = 2$ of Lemma 3.3 (whereas in the proof of the first inequality only $n = 1$ can be used), resulting in the better factor q^{-j}.

We first write the matrices that are useful in both cases when $i + j$ is even and when $i + j$ is odd. Let $m \in \mathbb{N}$. Let $\sigma : \mathcal{O}/\pi^{m+1}\mathcal{O} \to \mathcal{O}$ be any section. Define $\alpha, \beta : (\mathcal{O}/\pi^{m+1}\mathcal{O})^3 \to H$ by

$$\alpha(a_1, a_2, b) = \begin{pmatrix} 1 & -[\pi^{-m-1}(1 + \pi\sigma(a_1))] & [\pi^{-m-1}(1 + \pi\sigma(a_2))] & -[\pi^{-2m}\sigma(b)] \\ 0 & 1 & 0 & [\pi^{-m-1}(1 + \pi\sigma(a_2))] \\ 0 & 0 & 1 & [\pi^{-m-1}(1 + \pi\sigma(a_1))] \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

$$\beta(x_1, x_2, y) = \begin{pmatrix} 1 & [\pi^{-m}\sigma(x_2)] & [\pi^{-m}\sigma(x_1)] & \pi^{-m-1}[\pi^{-m}(\sigma(x_1) + \sigma(x_2))] + [\pi^{-2m}\sigma(y)] \\ 0 & 1 & 0 & [\pi^{-m}\sigma(x_1)] \\ 0 & 0 & 1 & -[\pi^{-m}\sigma(x_2)] \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

for $a_1, a_2, b, x_1, x_2, y \in \mathcal{O}/\pi^{m+1}\mathcal{O}$.

Compute

$$\alpha(a_1, a_2, b)\beta(x_1, x_2, y) = \begin{pmatrix} 1 & -\xi_1 & \xi_2 & \eta \\ 0 & 1 & 0 & \xi_2 \\ 0 & 0 & 1 & \xi_1 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

where

$$\xi_1 = [\pi^{-m-1}(1 + \pi\sigma(a_1) - \pi\sigma(x_2))],$$
$$\xi_2 = [\pi^{-m-1}(1 + \pi\sigma(a_2) + \pi\sigma(x_1))],$$
$$\eta = [\pi^{-2m}(\sigma(y) - \sigma(b))] - [\pi^{-m}\sigma(a_1)][\pi^{-m}\sigma(x_1)] - [\pi^{-m}\sigma(a_2)][\pi^{-m}\sigma(x_2)]$$
$$= \pi^{-2m}(y - b - a_1x_1 - a_2x_2) \bmod \pi^{-m+1}\mathcal{O}.$$

Let us now prove the estimate when $i + j \in 2\mathbb{N}$. Let

$$m = (i + j)/2 - 1.$$

We see that for any $a_1, a_2, b, x_1, x_2, y \in \mathcal{O}/\pi^{m+1}\mathcal{O}$,

$$\| \wedge^2 \big(\alpha(a_1, a_2, b)\beta(x_1, x_2, y)\big)\| = q^{2m+2} = q^{i+j}.$$

Moreover, when

$$y - (a_1x_1 + a_2x_2 + b) \in \pi^l\mathcal{O}^\times/\pi^{m+1}\mathcal{O}, l \in \{0, 1, \dots, m-1\}, \tag{3.7}$$

we have

$$\|\alpha(a_1, a_2, b)\beta(x_1, x_2, y)\| = q^{2m-l}.$$

Summarizing, for $a_1, a_2, b, x_1, x_2, y \in \mathcal{O}/\pi^{m+1}\mathcal{O}$ satisfying (3.7) above, we have

$$\alpha(a_1, a_2, b)\beta(x_1, x_2, y) \in KD(i + j - 2 - l, l + 2)K.$$

Define for $n \in \mathbb{N}$,

$$H_2^n = \{\begin{pmatrix} 1 & x & y & z \\ & 1 & 0 & y \\ & & 1 & -x \\ & & & 1 \end{pmatrix}, x, y, z \in \mathbb{F}_q[\pi^{-1}], |x|, |y| \le q^n, |z| \le q^{2n}\}.$$

It is also a finite subgroup of H. Note that the images of α and β are both in $H_2^{(i+j)/2}$. Apply Lemma 3.3 to $n = 2, m = (i + j)/2, k = j - 2, H_2^{(i+j)/2}$, and α, β as above, and $f = c|_{H_2^{(i+j)/2}}, \lambda = c(D(i, j)), \mu = c(D(i + 1, j - 1))$, and since $H_2^{(i+j)/2} \subset H \cap B_{i+j}$, we have

$$|c(D(i, j)) - c(D(i + 1, j - 1))| \le 2q^{-(j-2)}\| \mathrm{Schur}_{c|_{H_2^{(i+j)/2}}} \|$$
$$\le 2q^2 \cdot q^{-j}\| \mathrm{Schur}_c |_{\mathbb{C}(H\cap B_{i+j})}\|.$$

Now prove the estimate when $i + j \in 2\mathbb{N} + 1$. In this case let

$$m = (i + j - 1)/2 - 1.$$

Define the embedding $\iota : \mathbb{F}_q \to H$ by

$$\iota(\varepsilon) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ & 1 & \pi^{-1}\varepsilon & 0 \\ & & 1 & 0 \\ & & & 1 \end{pmatrix}, \quad \forall \varepsilon \in \mathbb{F}_q.$$

Define $H_2'^n, n \in \mathbb{N}$ as the following subgroup of H_2,

$$H_2'^n = \{\begin{pmatrix} 1 & x & y & z \\ & 1 & 0 & y \\ & & 1 & -x \\ & & & 1 \end{pmatrix} |x| \le q^n, |y| \le, q^{n+1}, |z| \le q^{2n+1}\}.$$

$H_2'^n$ is stable under the conjugate action of $\iota(\mathbb{F}_q)$. Define $\tilde{H}_2^n$ to be the finite subgroup $\tilde{H}_2^n = \iota(\mathbb{F}_q) \cdot H_2'^n$.

Now let $\tilde{\alpha} : (\mathcal{O}/\pi^{m+1}\mathcal{O})^3 \to H$ be the map defined by

$$\tilde{\alpha}(a_1, a_2, b) = \iota(1)\alpha(a_1, a_2, b), \quad \forall a_1, a_2, b \in \mathcal{O}/\pi^{m+1}\mathcal{O}.$$

By easy computation we see that $\forall a_1, a_2, b, x_1, x_2, y \in \mathcal{O}/\pi^{m+1}\mathcal{O}$,

$$\tilde{\alpha}(a_1, a_2, b)\beta(x_1, x_2, y) = \begin{pmatrix} 1 & -\xi_1 & \xi_2 & \eta \\ 0 & 1 & \pi^{-1} & \xi_2 + \pi^{-1}\xi_1 \\ 0 & 0 & 1 & \xi_1 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

We see

$$\| \wedge^2 (\tilde{\alpha}(a_1, a_2, b)\beta(x_1, x_2, y))\| = q^{2m+3} = q^{i+j}.$$

And when

$$y - (a_1x_1 + a_2x_2 + b) \in \pi^l\mathcal{O}^\times/\pi^{m+1}\mathcal{O}, l \in \{0, 1, \dots, m-1\},$$

we have

$$\|\tilde{\alpha}(a_1, a_2, b)\beta(x_1, x_2, y)\| = q^{2m-l}.$$

Namely in this case, we obtain the decomposition

$$\tilde{\alpha}(a_1, a_2, b)\beta(x_1, x_2, y) \in KD(i + j - l - 3, l + 3)K.$$

The images of $\tilde{\alpha}$ and β are both in $\tilde{H}_2^{(i+j-1)/2}$. Now apply Lemma 3.3 to $n = 2, m = (i + j - 1)/2, k = j - 3, \tilde{H}_2^{(i+j-1)/2}, \tilde{\alpha}, \beta$ and $f = c|_{\tilde{H}_2^{(i+j-1)/2}}$, $\lambda = c(D(i, j)), \mu = c(D(i + 1, j - 1))$, and since $\tilde{H}_2^{(i+j-1)/2} \subset H \cap B_{i+j}$, we have

$$\begin{aligned} |c(D(i, j)) - c(D(i + 1, j - 1))| &\le 2q^{-(j-3)}\| \mathrm{Schur}_{c|_{\tilde{H}_2^{(i+j-1)/2}}} \| \\ &\le 2q^3 \cdot q^{-j}\| \mathrm{Schur}_c |_{\mathbb{C}(H\cap B_{i+j})}\|. \end{aligned}$$

□

4. Second proof of Theorem 1.2

In this section, another proof of Theorem 1.2, more precisely of Theorem 3.1, is given by showing the following proposition (a slightly improved version of Proposition 3.2 in the first inequality).

Proposition 4.1. *For any K biinvariant function $c \in C(G)$, we have*

$$|c(D(i,j)) - c(D(i,j+1))| \leq 2q^{-(i-j)/2} \| \operatorname{Schur}_c |_{\mathbb{C}(H \cap B_{n_1})} \|, \tag{4.1}$$

for any $(i,j) \in \Lambda$ with $i \geq 1$ where

$$n_1 = \max(\ell(D(i,j)), \ell(D(i,j+1))) = i + j + 1$$

and

$$|c(D(i,j)) - c(D(i+1,j-1))| \leq 2q^2 \cdot q^{-j} \| \operatorname{Schur}_c |_{\mathbb{C}(H \cap B_{n_2})} \|, \tag{4.2}$$

for any $(i,j) \in \Lambda$ with $j \geq 3$ where

$$n_2 = \max(\ell(D(i,j)), \ell(D(i+1,j-1))) = i + j.$$

Let H_1 be the abelian subgroup in H

$$H_1 = \{h_1(x,y,z) = \begin{pmatrix} 1 & 0 & x & z \\ & 1 & y & x \\ & & 1 & 0 \\ & & & 1 \end{pmatrix}, x, y, z \in \mathbb{F}_q[\pi^{-1}]\},$$

and let H_2 be the subgroup of Heisenberg type in H

$$H_2 = \{h_2(x,y,z) = \begin{pmatrix} 1 & x & y/2 & z \\ & 1 & 0 & y/2 \\ & & 1 & -x \\ & & & 1 \end{pmatrix}, x, y, z \in \mathbb{F}_q[\pi^{-1}]\}.$$

The group law is as follows:

$$h_2(x,y,z)h_2(x',y',z') = h_2(x+x', y+y', z+z'+\frac{1}{2}(xy'-yx')).$$

The proof of the two inequalities in Proposition 4.1 relies on the construction as follows of two family of explicit functions on H_1 and H_2 respectively.

Denote by $[\cdot] : \mathbb{F}_q((\pi)) \to \mathbb{F}_q[\pi^{-1}]$ the integral part of an element as defined in the previous section. Now fix $(i,j) \in \Lambda$. Define

$$h_{1,i,j} = \mathop{\mathbb{E}}_{a \in \mathcal{O}/\pi^i \mathcal{O}} e_{h_1([\pi^{-i}a], \pi^{-i}, [\pi^{-i}a^2] + \pi^{-j})},$$

$$h_{2,i,j} = \mathop{\mathbb{E}}_{a,b,c \in \mathcal{O}/\pi^i \mathcal{O}} e_{h_2([\pi^{-m}(1+\pi a)], [\pi^{-m}b], [\pi^{-i}(1+\pi c)])},$$

where $m = m_{i,j}$ is the integral part of $(i+j)/2$, i.e. when $i+j \in 2\mathbb{N}$, $m = (i+j)/2$, when $i + j \in 2\mathbb{N} + 1$, $m = (i + j - 1)/2$. More explicitly,

$$h_1([\pi^{-i}a], \pi^{-i}, [\pi^{-i}a^2] + \pi^{-j}) = \begin{pmatrix} 1 & 0 & [\pi^{-i}a] & [\pi^{-i}a^2] + \pi^{-j} \\ & 1 & \pi^{-i} & [\pi^{-i}a] \\ & & 1 & 0 \\ & & & 1 \end{pmatrix}, \quad a \in \mathcal{O}/\pi^i\mathcal{O}.$$

$$h_2([\pi^{-m}(1+\pi a)], [\pi^{-m}b], [\pi^{-i}(1+\pi c)]) = \begin{pmatrix} 1 & [\pi^{-m}(1+\pi a)] & [\pi^{-m}b]/2 & [\pi^{-i}(1+\pi c)] \\ & 1 & 0 & [\pi^{-m}b]/2 \\ & & 1 & -[\pi^{-m}(1+\pi a)] \\ & & & 1 \end{pmatrix}, \quad a, b, c \in \mathcal{O}/\pi^i\mathcal{O}.$$

The explicit functions are defined as

$$\Delta_{1,i,j} = h_{1,i,j} - h_{1,i,j+1} \in \mathbb{C}H_1,$$
$$\Delta_{2,i,j} = h_{2,i,j} - h_{2,i+1,j-1} \in \mathbb{C}H_2.$$

Proposition 4.2.

$$\|\Delta_{1,i,j}\|_{C_r^*(H_1)} \leq 2q^{-(i-j)/2} \tag{4.3}$$
$$\|\Delta_{2,i,j}\|_{C_r^*(H_2)} \leq 2q^2 \cdot q^{-j} \tag{4.4}$$

Proof of Proposition 4.1 from Proposition 4.2. Recall that for any

$$g = (g_{\alpha,\beta})_{1\leq\alpha,\beta\leq 4} \in G, \quad g \in KD(i,j)K$$

for $(i, j) \in \Lambda$ if and only if

$$\|g\| = \max_{1\leq\alpha,\beta\leq 4} |g_{\alpha,\beta}| = q^i$$

and

$$\| \wedge^2 g\| = \max_{1\leq\alpha_1,\beta_1,\alpha_2,\beta_2\leq 4} |g_{\alpha_1,\beta_1}g_{\alpha_2,\beta_2} - g_{\alpha_1,\beta_2}g_{\alpha_2,\beta_1}| = q^{i+j}.$$

By definition we have

$$h_1([\pi^{-i}a], \pi^{-i}, [\pi^{-i}(a^2 + \pi^{i-j})]) \in H_1 \cap KD(i,j)K,$$

i.e.

$$\operatorname{supp} h_{1,i,j} \subset H_1 \cap KD(i,j)K.$$

Since H is amenable, we have

$$\begin{aligned}|c(D(i,j)) - c(D(i,j+1))| &= \Big|\sum_{h\in H} c(h)\Delta_{1,i,j}(h)\Big| \\ &\le \|\operatorname{Schur}_c(\Delta_{1,i,j})\|_{C_r^*(H)} \\ &\le \|\operatorname{Schur}_c|_{\mathbb{C}(H\cap B_{n_1})}\|\|\Delta_{1,i,j}\|_{C_r^*(H)}.\end{aligned}$$

Now that H_1 is a subgroup of H, we have that $\|\Delta_{1,i,j}\|_{C_r^*(H)} = \|\Delta_{1,i,j}\|_{C_r^*(H_1)}$, so the first inequality is proved. The second inequality requires a bit more computation.

First, when $i+j\in 2\mathbb{N}$, we have by definition

$$\operatorname{supp} h_{2,i,j} \subset H_2 \cap KD(i,j)K,$$

$$\begin{aligned}|c(D(i,j)) - c(D(i+1,j-1))| &= \Big|\sum_{h\in H} c(h)\Delta_{2,i,j}(h)\Big| \\ &\le \|\operatorname{Schur}_c(\Delta_{2,i,j})\|_{C_r^*(H)} \\ &\le \|\operatorname{Schur}_c|_{\mathbb{C}(H\cap B_{n_2})}\|\|\Delta_{2,i,j}\|_{C_r^*(H)},\end{aligned}$$

and

$$\|\Delta_{2,i,j}\|_{C_r^*(H)} = \|\Delta_{2,i,j}\|_{C_r^*(H_2)}.$$

When $i+j\in 2\mathbb{N}+1$,

$$\begin{aligned}&\iota(1)h_2([\pi^{-m}(1+\pi a)],[\pi^{-m}b],[\pi^{-i}(1+\pi c)]) \\ &\quad= \begin{pmatrix}1 & [\pi^{-m}(1+\pi a)] & [\pi^{-m}b]/2 & [\pi^{-i}(1+\pi c)] \\ & 1 & \pi^{-1} & [\pi^{-m}b]/2-\pi^{-1}[\pi^{-m}(1+\pi a)] \\ & & 1 & -[\pi^{-m}(1+\pi a)] \\ & & & 1\end{pmatrix} \\ &\qquad\qquad \in KD(i,j)K,\end{aligned}$$

$\forall a,b,c\in\mathcal{O}/\pi^i\mathcal{O}$, where $m=(i+j-1)/2$ as before, and

$$\iota(\varepsilon) = \begin{pmatrix}1 & 0 & 0 & 0 \\ & 1 & \varepsilon\pi^{-1} & 0 \\ & & 1 & 0 \\ & & & 1\end{pmatrix}, \quad \varepsilon\in\mathbb{F}_q.$$

Finally, we have

$$\begin{aligned}|c(D(i,j)) - c(D(i+1,j-1))| &= \Big|\sum_{h\in H} c(h)\Delta_{2,i,j}(\iota(1)h)\Big| \\ &\le \|\operatorname{Schur}_c(L_{\iota(-1)}\Delta_{2,i,j})\|_{C_r^*(H)} \\ &\le \|\operatorname{Schur}_c|_{\mathbb{C}(H\cap B_{n_2})}\|\|L_{\iota(-1)}\Delta_{2,i,j}\|_{C_r^*(H)},\end{aligned}$$

and

$$\|L_{\iota(-1)}\Delta_{2,i,j}\|_{C_r^*(H)} = \|\Delta_{2,i,j}\|_{C_r^*(H_2)},$$

so the second inequality is proved. □

Now it suffices to show Proposition 4.2, whose proof unlike Proposition 3.2 does not rely on Lemma 4.9 in [10].

Proof of inequality (4.3) *in Proposition 4.2.* Here we follow [7].

Lemma 4.3 (Norm of quadratic Gauss sum). *If the character $\eta \in \widehat{\mathcal{O}/\pi^\ell\mathcal{O}}$ is non-degenerate (i.e. $\eta|_{\pi^{\ell-1}\mathcal{O}/\pi^\ell\mathcal{O}} \neq 1$), then*

$$\Big|\mathop{\mathbb{E}}_{a\in\mathcal{O}/\pi^\ell\mathcal{O}} \eta(a^2)\Big| = q^{-\ell/2}.$$

Proof.

$$\Big|\mathop{\mathbb{E}}_{a\in\mathcal{O}/\pi^\ell\mathcal{O}} \eta(a^2)\Big|^2 = \Big|\mathop{\mathbb{E}}_{a,b\in\mathcal{O}/\pi^\ell\mathcal{O}} \eta(a^2-b^2)\Big| = \Big|\mathop{\mathbb{E}}_{a,b\in\mathcal{O}/\pi^\ell\mathcal{O}} \eta\big((a-b)(a+b)\big)\Big|.$$

Since q is odd, we can introduce new variables $x = a+b$, $y = a-b$ which is an invertible linear transform on $(\mathcal{O}/\pi^\ell\mathcal{O})^2$, thus

$$\Big|\mathop{\mathbb{E}}_{a\in\mathcal{O}/\pi^\ell\mathcal{O}} \eta(a^2)\Big|^2 = \Big|\mathop{\mathbb{E}}_{x,y\in\mathcal{O}/\pi^\ell\mathcal{O}} \eta(xy)\Big| = q^{-\ell}. \qquad \square$$

Since H_1 is an abelian group, $\forall \varphi \in \mathbb{C}H_1$,

$$\|\varphi\|_{C_r^*(H_1)} = \sup_{\chi\in\widehat{H_1}} \big|\chi(\varphi)\big|.$$

Fix $\chi \in \widehat{H_1}$, and suppose $\chi_1, \chi_2, \chi_3 \in \widehat{\mathbb{F}_q[\pi^{-1}]}$ such that $\forall x, y, z \in \mathbb{F}_q[\pi^{-1}]$, $\chi(h_1(x, y, z)) = \chi_1(x)\chi_2(y)\chi_3(z)$.

$$\chi(\Delta_{1,i,j}) = \chi_2(\pi^{-i})\big(\chi_3(\pi^{-j}) - \chi_3(\pi^{-j-1})\big) \mathop{\mathbb{E}}_{a\in\mathcal{O}/\pi^i\mathcal{O}} \chi_1\big([\pi^{-i}a]\big)\chi_3\big([\pi^{-i}a^2]\big).$$

We see that unless $\chi(\Delta_{1,i,j}) = 0$, we have $\mathrm{Ker}\big(\chi_3\big([\pi^{-i}\cdot]\big)\big) \subset \pi^{i-j}\mathcal{O}$, and $\mathrm{Ker}\big(\chi_1\big([\pi^{-i}\cdot]\big)\big) \supset \mathrm{Ker}\big(\chi_3\big([\pi^{-i}\cdot]\big)\big)$ (see footnote[1]). Consequently, there exists $\theta \in \mathcal{O}$, such that $\chi_1\big([\pi^{-i}\cdot]\big) = \chi_3\big([\pi^{-i}\theta\cdot]\big)$.

[1] If we replace $h_{1,i,j}$ by the function

$$h'_{1,i,j} = \mathop{\mathbb{E}}_{a,b,c\in\mathcal{O}/\pi^i\mathcal{O}} e_{h_1([\pi^{-i}a],[\pi^{-i}(1+\pi b)],[\pi^{-i}a^2+\pi^{-j}(1+\pi c)])},$$

we can then locate the support of $\Delta'_{1,i,j} = h'_{1,i,j} - h'_{1,i,j+1}$ more precisely, i.e. we have

$$\mathrm{Ker}\big(\chi_3([\pi^{-i}\cdot])\big) = \pi^{i-j}\mathcal{O} \text{ or } \pi^{i-j+1}\mathcal{O},$$

$$\mathrm{Ker}\big(\chi_1([\pi^{-i}\cdot])\big) \supset \pi^{i-j+1}\mathcal{O} \quad \text{and} \quad \mathrm{Ker}\big(\chi_2([\pi^{-i}\cdot])\big) \supset \pi\mathcal{O}.$$

But this is not very useful in the current situation.

Now we have

$$|\chi(\Delta_{1,i,j})| \le 2\Big| \mathop{\mathbb{E}}_{a\in\mathcal{O}/\pi^i\mathcal{O}} \chi_3\big([\pi^{-i}(\theta a + a^2)]\big)\Big|.$$

Since q is odd, $(\theta/2)^2 + \theta a + a^2 = (\theta/2 + a)^2, |\chi_3\big([\pi^{-i}(\theta/2)^2]\big)| = 1$, we have by Lemma 4.3,

$$|\chi(\Delta_{1,i,j})| \le 2\Big| \mathop{\mathbb{E}}_{a\in\mathcal{O}/\pi^i\mathcal{O}} \chi_3\big([\pi^{-i}(\theta/2 + a)^2]\big)\Big| \le 2q^{-(i-j)/2}$$

(see footnote[2]). □

Proof of inequality (4.4) *in Proposition 4.2*. Let $\chi, \chi' \in \widehat{\mathbb{F}_q[\pi^{-1}]}$, $\chi \ne 0$. We define a unitary representation of $\rho_{\chi,\chi'} : H_2 \to \mathcal{U}(\ell^2(\mathbb{F}_q[\pi^{-1}]))$ by

$$\rho_{\chi,\chi'}(h_2(a,b,c))f(x) = f(x+a)\chi(xb)\chi\Big(c + \frac{1}{2}ab\Big)\chi'(b)$$

(it is well defined since q is odd). □

V. Lafforgue suggested the following formula for calculating the C_r^* norms on H_2.

Lemma 4.4. $\forall \varphi \in \mathbb{C}H_2$,

$$\|\varphi\|_{C_r^*(H_2)} = \sup_{\chi,\chi'\in\widehat{\mathbb{F}_q[\pi^{-1}]},\chi\ne 0} \|\rho_{\chi,\chi'}(\varphi)\|_{\mathcal{L}(\ell^2(\mathbb{F}_q[\pi^{-1}]))}.$$

Remarks. (1) Being a counterpart of H_2 in a number field, the following discrete Heisenberg group

$$H_2(\mathbb{Z}) = \{h_2(x,y,z) = \begin{pmatrix} 1 & x & y/2 & z \\ & 1 & 0 & y/2 \\ & & 1 & -x \\ & & & 1 \end{pmatrix}, x, z \in \mathbb{Z}, y \in 2\mathbb{Z}\}$$

also admits a similar formula for the C_r^* norms. More precisely, for $\theta, \theta' \in [0,1)$, define the unitary representation $\rho_{\theta,\theta'} : H_2(\mathbb{Z}) \to \mathcal{U}(\ell^2(\mathbb{Z}))$ of central character θ by a similar formula

$$\rho_{\theta,\theta'}(h_2(a,b,c))f(x) = f(x+a)e^{2\pi i\theta xb}e^{2\pi i\theta(c+\frac{1}{2}ab)}e^{2\pi i\theta' b},$$

then we have $\forall \varphi \in \mathbb{C}(H_2(\mathbb{Z}))$,

$$\|\varphi\|_{C_r^*(H_2(\mathbb{Z}))} = \sup_{\theta,\theta'\in[0,1)} \|\rho_{\theta,\theta'}(\varphi)\|_{\mathcal{L}(\ell^2(\mathbb{Z}))}.$$

[2] In fact, it suffices to prove this final inequality for rational $\chi \in \widehat{H_1}$, i.e. there exist $\psi \in \widehat{F}$ vanishing on $\pi\mathcal{O}$ and being non zero on $\mathcal{O}$, and $t_1, t_2, t_3 \in \mathbb{F}_q[\pi^{-1}]$ such that $\chi_i(\cdot) = \psi(t_i\cdot)$, $i = 1, 2, 3$. Note that $\psi(t\cdot)$ vanishes on $\pi^{\ell+1}\mathcal{O}$ and is non zero on $\pi^\ell\mathcal{O}$ if and only if $|t| = q^\ell$. With this notation, $|t_3| \ge q^{j-1}$, $|t_1| \le |t_3|$ unless $\chi(\Delta_{1,i,j}) = 0$.

(2) The formula for the C_r^* norm in the first remark can be reduced to those irrational $\theta \in [0,1)\backslash\mathbb{Q}$ and $\theta' = 0$, i.e. $\forall \varphi \in \mathbb{C}(H_2(\mathbb{Z}))$,

$$\|\varphi\|_{C_r^*(H_2(\mathbb{Z}))} = \sup_{\theta\in[0,1)\backslash\mathbb{Q}} \|\rho_{\theta,0}(\varphi)\|_{\mathcal{L}(\ell^2(\mathbb{Z}))} .$$

(The analogues formula also holds for H_2, but we don't use it the proof). Indeed, when θ is irrational, $\rho_{\theta,0}(H_2(\mathbb{Z}))$ generates algebra A_θ which is a simple C^* algebra, i.e. any representation of A_θ is faithful. Moreover, for any C^* algebra A and any representation $\sigma_1, \sigma_2 : A \to \mathcal{L}(H)$, we have

$$\mathrm{Ker}\sigma_1 \subset \mathrm{Ker}\sigma_2 \Leftrightarrow \|\sigma_1(a)\|_{\mathcal{L}(H)} \geq \|\sigma_2(a)\|_{\mathcal{L}(H)}, \quad \forall a \in A.$$

By applying the previous fact to the representation of A_θ generated by $\rho_{\theta,\theta'}(H_2(\mathbb{Z}))$, we have

$$\|\rho_{\theta,\theta'}(\varphi)\| = \|\rho_{\theta,0}(\varphi)\|, \forall \theta' \in [0,1), \quad \forall \varphi \in \mathbb{C}(H_2(\mathbb{Z})).$$

Proof of Lemma 4.4. Let $N_2 \supset H_2$ be the following Heisenberg group

$$N_2 = \{h_2(x,y,z) = \begin{pmatrix} 1 & x & y/2 & z \\ & 1 & 0 & y/2 \\ & & 1 & -x \\ & & & 1 \end{pmatrix}, x, y, z \in F = \mathbb{F}_q((\pi))\},$$

and for a character $\eta \in \hat{F}\backslash\{0\}$, denote by $\rho_\eta : N_2 \to \mathcal{U}(L^2(F))$ the representation defined by

$$\rho_\eta(h_2(a,b,c))f(x) = f(x+a)\eta(xb)\eta\Big(c + \frac{1}{2}ab\Big), a, b, c, x \in F.$$

Let D be the fundamental domain $D = \pi\mathcal{O}$ for the translation of $\mathbb{F}_q[\pi^{-1}]$ on F. We have an isomorphism of representations of H

$$\rho_\eta|_{H_2} \simeq \int_D^\oplus \rho_{\eta|_{\mathbb{F}_q[\pi^{-1}]}, \eta|_{\mathbb{F}_q[\pi^{-1}]}(\delta\cdot)} d\delta$$

defined by

$$L^2(F) \xrightarrow{\sim} \int_D^\oplus \ell^2(\mathbb{F}_q[\pi^{-1}]) d\delta,$$
$$\phi \mapsto \Big(\big[r \mapsto \phi(r+\delta)\big] \in \ell^2\Big(\mathbb{F}_q\big[\pi^{-1}\big]\Big)\Big)_{\delta\in D},$$

where $\eta|_{\mathbb{F}_q[\pi^{-1}]}(\delta\cdot)$ denotes the character $[\gamma \mapsto \eta(\delta\gamma)] \in \widehat{\mathbb{F}_q[\pi^{-1}]}$. □

We write the action of $\Delta_{2,i,j}$ in the following form

$$\rho_{\chi,\chi'}(\Delta_{2,i,j})f(x)$$
$$= C \mathop{\mathbb{E}}_{a,b\in\mathcal{O}}\Big(f\Big(x+\big[\pi^{-m}(1+\pi a)\big]\Big)\chi\Big(\Big(x+\frac{1}{2}\big[\pi^{-m}(1+\pi a)\big]\Big)\big[\pi^{-m}b\big]\Big)\chi'\Big(\big[\pi^{-m}b\big]\Big)\Big),$$

where
$$C = \mathop{\mathbb{E}}_{c\in\mathcal{O}}\Big(\chi\big([\pi^{-i}(1+\pi c)]\big) - \chi\big([\pi^{-i-1}(1+\pi c)]\big)\Big).$$

We use constantly the following basic fact in the proof: for any finite abelian group A and any unitary character $\eta \in \widehat{A}$, we have

$$\begin{aligned} &\mathop{\mathbb{E}}_{a\in A} \eta(a) = 1 \text{ when } \eta \in \widehat{A} \text{ is trivial;} \\ &\mathop{\mathbb{E}}_{a\in A} \eta(a) = 0 \text{ when } \eta \in \widehat{A} \text{ is non-trivial.} \end{aligned} \tag{4.5}$$

Lemma 4.5. *If $C \neq 0$, then*

$$\chi|_{[\pi^{-i+1}\mathcal{O}]} = 1 \in \widehat{[\pi^{-i+1}\mathcal{O}]} \quad \textit{and} \quad \chi|_{[\pi^{-i-1}\mathcal{O}]} \neq 1 \in \widehat{[\pi^{-i-1}\mathcal{O}]}.$$

Proof of Lemma 4.5. If $\chi|_{[\pi^{-i-1}\mathcal{O}]}$ is trivial, then $C = 0$ since $\chi(z_1) - \chi(z_2) = 0$, $\forall z_1 \in [\pi^{-i-1}\mathcal{O}]$, $z_2 \in [\pi^{-i}\mathcal{O}]$. On the other hand, if $\chi|_{[\pi^{-i+1}\mathcal{O}]}$ is non-trivial, then by (4.5)

$$\mathop{\mathbb{E}}_{c\in\mathcal{O}} \chi\Big(\big[\pi^{-i}(1+\pi c)\big]\Big) = \chi(\pi^{-i}) \mathop{\mathbb{E}}_{z\in[\pi^{-i+1}\mathcal{O}]} \chi(z) = 0,$$

$$\mathop{\mathbb{E}}_{c\in\mathcal{O}} \chi\Big(\big[\pi^{-i-1}(1+\pi c)\big]\Big)\chi(\pi^{-i-1}) \mathop{\mathbb{E}}_{z\in[\pi^{-i}\mathcal{O}]} \chi(z) = 0,$$

and therefore $C = 0$. □

The matrix of $\rho_{\chi,\chi'}(\Delta_{2,i,j})$ is block diagonal and each block corresponds to a coset $x_0 + [\pi^{-m}\mathcal{O}]$, $x_0 \in \mathbb{F}_q[\pi^{-1}]$. Indeed, the action of $\rho_{\chi,\chi'}(\Delta_{2,i,j})$ on $\ell^2(\mathbb{F}_q[\pi^{-1}])$ only concerns translations of elements in $[\pi^{-m}\mathcal{O}]$ and scalars.

It remains to show that each block of $\rho_{\chi,\chi'}(\Delta_{2,i,j})$ associated to the coset $x_0 + [\pi^{-m}\mathcal{O}]$ has norm $\leq 2q^{2-j}$,

$$\|\rho_{\chi,\chi'}(\Delta_{2,i,j})|_{\ell^2(x_0+[\pi^{-m}\mathcal{O}])}\|_{\mathcal{L}(\ell^2(x_0+[\pi^{-m}\mathcal{O}]))} \leq 2q^{2-j}. \tag{$*$}$$

Now fix a coset $x_0 + [\pi^{-m}\mathcal{O}]$ for some $x_0 \in \mathbb{F}_q[\pi^{-1}]$. We provide two proofs of ($*$). The two proofs are related, but the author thinks that both have merits and it might be useful to write them down.

First proof of ($*$). Denote by E_ε the subset $x_0+\pi^{-m}\varepsilon+[\pi^{-m+1}\mathcal{O}] \subset x_0+[\pi^{-m}\mathcal{O}]$ for $\varepsilon \in \mathbb{F}_q$. We have a disjoint union decomposition

$$x_0 + [\pi^{-m}\mathcal{O}] = \sqcup_{\varepsilon\in\mathbb{F}_q} E_\varepsilon.$$

For each $\varepsilon \in \mathbb{F}_q$, $\rho_{\chi,\chi'}(\Delta_{2,i,j})$ sends $\ell^2(E_\varepsilon)$ to $\ell^2(E_{\varepsilon-1})$, and thus the action of $\rho_{\chi,\chi'}(\Delta_{2,i,j})$ on $\ell^2(x_0 + [\pi^{-m}\mathcal{O}])$ has the following form of block matrix

$$\begin{pmatrix} 0 & * & 0 & \cdots & 0 \\ 0 & 0 & * & \cdots & 0 \\ & & \cdots & & \\ 0 & 0 & 0 & \cdots & * \\ * & 0 & 0 & \cdots & 0 \end{pmatrix},$$

where each block $*$ has size q^{m-1} and corresponds to the action $\ell^2(E_\varepsilon) \to \ell^2(E_{\varepsilon-1})$.

The following lemma claims that after appropriate identification of E_ε and $E_{\varepsilon-1}$ the block $*$ corresponding to ε is $Cq^{-2m+1+i}(\simeq q^{-j})$ times the projection from $\ell^2(E_\varepsilon)$ onto $[\pi^{m-i}\mathcal{O}]$ invariant vectors in $\ell^2(E_\varepsilon)$, and thus our inequality follows. More precisely, by identifying $x_0 + \pi^{-m}(\varepsilon - 1) - y + y_\varepsilon \in E_{\varepsilon-1}$ and $x_0 + \pi^{-m}\varepsilon + y \in E_\varepsilon$, $\rho_{\chi,\chi'}(\Delta_{2,i,j})$ sends $\delta_{x_0+\pi^{-m}\varepsilon+y}$ to

$$Cq^{-2m+1+i} \mathop{\mathbb{E}}_{\alpha\in\mathcal{O}} \delta_{x_0+\pi^{-m}\varepsilon+y+[\pi^{m-i}\alpha]},$$

and thus has norm less than $2q^{-2m+1+i} \le 2q^{2-j}$. □

Remark. The identification of $E_{\varepsilon-1}$ and E_ε via

$$x_0 + \pi^{-m}(\varepsilon - 1) + y_\varepsilon - y \to x_0 + \pi^{-m}\varepsilon + y$$

corresponds to the fact that $A_{x,y}$ is an anti-diagonal in the second proof below (the center of the anti-diagonal is $x_0 + \pi^{-m}(\varepsilon - \frac{1}{2}) + \frac{1}{2}y_\varepsilon$).

Lemma 4.6. *If $\rho_{\chi,\chi'}(\Delta_{2,i,j})|_{\ell^2(E_\varepsilon)} \neq 0 \in \mathcal{L}(\ell^2(E_\varepsilon), \ell^2(E_{\varepsilon-1}))$, then there exists $y_\varepsilon \in [\pi^{-m+1}\mathcal{O}]$, such that $\forall y \in [\pi^{-m+1}\mathcal{O}]$*

$$\rho_{\chi,\chi'}(\Delta_{2,i,j}) f(x_0 + \pi^{-m}(\varepsilon - 1) + y) = Cq^{-2m+1+i} \mathop{\mathbb{E}}_{\alpha\in\mathcal{O}} f(x_0 + \pi^{-m}\varepsilon - y + y_\varepsilon + [\pi^{m-i}\alpha]),$$

for any $i \ge j \ge 2$.

Proof of Lemma 4.6. By hypothesis there exist $f_0 \in \ell^2(E_\varepsilon)$ and $y_0 \in [\pi^{-m+1}\mathcal{O}]$ such that

$$\rho_{\chi,\chi'}(\Delta_{2,i,j}) f_0(x_0 + \pi^{-m}(\varepsilon - 1) + y_0) = C \mathop{\mathbb{E}}_{a,b\in\mathcal{O}} \Big(f_0\big(x_0 + \pi^{-m}\varepsilon + y_0 + [\pi^{-m+1}a)]\big) \cdot \chi\big((x_0 + \pi^{-m}(\varepsilon - 1) + y_0 + \frac{1}{2}[\pi^{-m}(1 + \pi a)]\big)[\pi^{-m}b]\big)\chi'\big([\pi^{-m}b]\big)\Big) \neq 0.$$

By fixing a and averaging over b we see that there exists $a_0 \in \mathcal{O}$ such that

$$\chi\big((x_0 + \pi^{-m}(\varepsilon - 1) + y_0 + \frac{1}{2}[\pi^{-m}(1 + \pi a_0)])[\pi^{-m}b]\big)\chi'([\pi^{-m}b]) = 1, \quad \forall b \in \mathcal{O}. \tag{4.6}$$

Set $y_\varepsilon = [\pi^{-m+1}a_0] + 2y_0 \in [\pi^{-m+1}\mathcal{O}]$.

By definition $\forall f \in \ell^2(E_\varepsilon)$, $y \in [\pi^{-m+1}\mathcal{O}]$ we have

$$\begin{aligned}\rho_{\chi,\chi'}(\Delta_{2,i,j})f(x_0 + \pi^{-m}(\varepsilon - 1) + y) &= C \mathop{\mathbb{E}}_{a,b\in\mathcal{O}}\Big(f(x_0 + \pi^{-m}\varepsilon + y + [\pi^{-m+1}a])\\ &\cdot \chi\big((x_0 + \pi^{-m}(\varepsilon - 1) + y + \frac{1}{2}[\pi^{-m}(1 + \pi a)])[\pi^{-m}b]\big)\chi'([\pi^{-m}b])\Big),\end{aligned}$$

by equality (4.6) it equals

$$= C \mathop{\mathbb{E}}_{a,b\in\mathcal{O}}\Big(f(x_0+\pi^{-m}\varepsilon+y+[\pi^{-m+1}a])\chi\big((y-y_0+\frac{1}{2}[\pi^{-m+1}(a-a_0)])[\pi^{-m}b]\big)\Big),$$

by the change of variables $a' = a-a_0+2\pi^{m-1}(y-y_0)$ (where $2\pi^{m-1}(y-y_0) \in \mathbb{F}_q$) it equals

$$= C \mathop{\mathbb{E}}_{a',b\in\mathcal{O}}\Big(f(x_0 + \pi^{-m}\varepsilon - y + y_\varepsilon + [\pi^{-m+1}a'])\chi\big(\frac{1}{2}[\pi^{-m+1}a'][\pi^{-m}b]\big)\Big).$$

By hypotheses $C \neq 0$, and by Lemma 4.5 we have $\chi|_{[\pi^{-i+1}\mathcal{O}]} = 1$, $\chi|_{[\pi^{-i-1}\mathcal{O}]} \neq 1$. There are two cases: $\chi|_{[\pi^{-i}\mathcal{O}]} = 1$ and $\chi|_{[\pi^{-i}\mathcal{O}]} \neq 1$.

When $\chi|_{[\pi^{-i}\mathcal{O}]} = 1$, $\chi|_{[\pi^{-i-1}\mathcal{O}]} \neq 1$, we have $\chi([\pi^{-m}c]/2) = 1, \forall c \in \mathcal{O}$ and then

$$\chi\Big(\frac{1}{2}[\pi^{-m+1}a'][\pi^{-m}b]\Big) = \chi\Big(\frac{1}{2}[\pi^{-2m+1}a'b]\Big), \forall a', b \in \mathcal{O}.$$

Thus

$$\begin{aligned}\rho_{\chi,\chi'}(\Delta_{2,i,j})f(x_0 + \pi^{-m}(\varepsilon - 1) + y) &= C \mathop{\mathbb{E}}_{a',b\in\mathcal{O}}\Big(f(x_0 + \pi^{-m}\varepsilon - y + y_\varepsilon + [\pi^{-m+1}a'])\chi\big(\frac{1}{2}[\pi^{-2m+1}a'b]\big)\Big).\end{aligned}$$

Being a Fourier transform for the non-degenerate character $[\alpha \mapsto \chi(\frac{1}{2}[\pi^{-2m+1}\alpha])] \in \widehat{\mathcal{O}/\pi^{2m-1-i}\mathcal{O}}$, it equals

$$Cq^{-2m+1+i} \mathop{\mathbb{E}}_{\alpha\in\mathcal{O}} f(x_0 + \pi^{-m}\varepsilon - y + y_\varepsilon + [\pi^{m-i}\alpha]).$$

The case when $\chi|_{[\pi^{-i+1}\mathcal{O}]} = 1$, $\chi|_{[\pi^{-i}\mathcal{O}]} \neq 1$ can be handled similarly. □

This ends the first proof of (∗).

Second proof of $(*)$ *(due to V. Lafforgue).*

Lemma 4.7. *If* $\chi|_{[\pi^{-i+1}\mathcal{O}]} = 1 \in \widehat{[\pi^{-i+1}\mathcal{O}]}$ *and* $\chi|_{[\pi^{-i-1}\mathcal{O}]} \neq 1 \in \widehat{[\pi^{-i-1}\mathcal{O}]}$*, then unless* $w \in [\pi^{-(i-m)}\mathcal{O}]$*, we have that* $\forall w \in [\pi^{-m}\mathcal{O}]$*,*

$$\mathop{\mathbb{E}}_{z\in[\pi^{-m}\mathcal{O}]} \chi\Big(\frac{1}{2}wz\Big) = 0,$$

or equivalently by (4.5)

$$\Big[z \mapsto \chi\Big(\frac{1}{2}wz\Big)\Big] \in \widehat{[\pi^{-m}\mathcal{O}]} \text{ is non-trivial.}$$

Proof of Lemma 4.7. We prove it by contradiction. Suppose $w = \pi^{m-i-\alpha}w_0 \in [\pi^{m-i-\alpha}\mathcal{O}^\times]$, $w_0 \in \mathbb{F}_q + \cdots + \pi^{-m+i+\alpha}\mathbb{F}_q, \alpha \in \{1, 2, \ldots, 2m-i\}$, such that

$$\chi|_{w[\pi^{-m}\mathcal{O}]} = 1 \in \widehat{[\pi^{-m}\mathcal{O}]}.$$

We have

$$\chi|_{\pi^{-i-1}w_0(\mathbb{F}_q+\mathbb{F}_q\pi)} = 1.$$

Indeed, since $1 \le \alpha \le m$, we have

$$\begin{aligned}\pi^{-i-1}w_0(\mathbb{F}_q + \mathbb{F}_q\pi) &= \pi^{-i-\alpha}w_0(\mathbb{F}_q\pi^{\alpha-1} + \mathbb{F}_q\pi^{\alpha}) \\ &\subset \pi^{-i-\alpha}w_0(\mathbb{F}_q + \mathbb{F}_q\pi + \cdots + \mathbb{F}_q\pi^m) = w[\pi^{-m}\mathcal{O}].\end{aligned}$$

Now $\forall \varepsilon_1, \varepsilon_2 \in \mathbb{F}_q$, there exist $\varepsilon_1', \varepsilon_2' \in \mathbb{F}_q$ such that $\varepsilon_1 + \varepsilon_2\pi \in w_0(\varepsilon_1' + \varepsilon_2'\pi) + \pi^2\mathcal{O}$. Since $\chi|_{[\pi^{-i+1}\mathcal{O}]} = 1$, we have

$$\chi(\pi^{-i-1}(\varepsilon_1 + \varepsilon_2\pi)) = \chi(\pi^{-i-1}w_0(\varepsilon_1' + \varepsilon_2'\pi)) = 1.$$

As a consequence $\chi|_{[\pi^{-i-1}\mathcal{O}]} = 1$, which is a contradiction to the hypothesis in the lemma. □

Let $A = (A_{x,y})_{x,y\in x_0+[\pi^{-m}\mathcal{O}]}$ be the matrix of the block of $\rho_{\chi,\chi'}(\Delta_{2,i,j})$ associated to $\ell^2(x_0 + [\pi^{-m}\mathcal{O}])$.

We will show that $\|A\|_{\mathcal{L}(\ell^2(x_0+[\pi^{-m}\mathcal{O}]))} \le 2q^{1-j}$.

First we have $A_{x,y} = 0$ unless $y \in x + \pi^{-m} + [\pi^{-m+1}\mathcal{O}]$, and in this case, (since $|[\pi^{-m}\mathcal{O}]| = q^{m+1}$)

$$A_{x,y} = Cq^{-m-1} \mathop{\mathbb{E}}_{z\in[\pi^{-m}\mathcal{O}]} \chi\Big(\frac{x+y}{2}z\Big)\chi'(z),$$

i.e. by (4.5)

$$A_{x,y} = Cq^{-m-1} \text{ when } \Big[z \mapsto \chi\Big(\frac{x+y}{2}z\Big)\chi'(z)\Big] \in \widehat{[\pi^{-m}\mathcal{O}]} \text{ is trivial, and}$$

$$A_{x,y} = 0 \text{ when } \Big[z \mapsto \chi\Big(\frac{x+y}{2}z\Big)\chi'(z)\Big] \in \widehat{[\pi^{-m}\mathcal{O}]} \text{ is non-trivial.}$$

Now suppose x, y, y' are elements in $x_0 + [\pi^{-m}\mathcal{O}]$ such that both $A_{x,y}$ and $A_{x,y'}$ are non-zero. By taking ratio we see that $[z \mapsto \chi(\frac{1}{2}(y-y')z)] \in [\pi^{-m}\mathcal{O}]$ is a trivial character. By Lemma 4.7 we see that $y - y' \in [\pi^{-(i-m)}\mathcal{O}]$.

By the same argument for $x, x', y \in x_0 + [\pi^{-m}\mathcal{O}]$ such that both $A_{x,y}$ and $A_{x',y}$ are non-zero, we have $x - x' \in [\pi^{-(i-m)}\mathcal{O}]$.

Therefore, each line and column in A has at most $|[\pi^{-(i-m)}\mathcal{O}]| = q^{i-m+1}$ non-zero coefficients. Each coefficient in A has absolute value at most $2q^{-m-1}$. The ℓ^2 norm of A is at most $2q^{-m-1} \cdot q^{i-m+1} = 2q^{i-2m} \leq 2q^{-j+1}$, and so is the operator norm of A. □

Remark 1. By the same argument, for $x, x', y, y' \in x_0 + [\pi^{-m}\mathcal{O}]$ such that both $A_{x,y}$ and $A_{x',y'}$ are non-zero, we have $(x+y) - (x'+y') \in [\pi^{-(i-m)}\mathcal{O}]$. It means that A is a block "anti-diagonal".

Remark 2. Following the previous remark, we can write the action of A in the following form (supposing $A_{x,y} \neq 0$)

$$Af(x') = \sum_{y' \in x_0 + [\pi^{-m}\mathcal{O}]} A_{x',y'} f(y') = \sum_{\alpha \in \mathcal{O}} A_{x',x+y-x'+[\pi^{m-i}\alpha]} f(x+y-x'+[\pi^{m-i}\alpha]),$$

where $A_{x',x+y-x'+[\pi^{m-i}\alpha]} = 0$ or Cq^{-m-1}, which means that A is roughly (the precise formula requires a more detailed analysis on Lemma 4.7) Cq^{i-2m} times the projection onto $[\pi^{m-i}\mathcal{O}]$-invariant functions in $\ell^2(x_0 + [\pi^{m-i}\mathcal{O}])$, after identifying x' to $x+y-x'$, corresponding to the arguments in the first proof above.

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Received 03 March, 2015; revised 22 March, 2016

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