J. Noncommut. Geom. 10 (2016), 1559–1587
DOI 10.4171/JNCG/266
Journal of Noncommutative Geometry

K-theory for the crossed products by certain actions of $\mathbb{Z}^2$

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Abstract. We investigate the K-theory of crossed product C^*-algebras by actions of $\mathbb{Z}^2$. Given a $\mathbb{Z}^2$-action, we associate to it a homomorphism between certain subquotients of the K-theory of the underlying C^*-algebra, which we call the obstruction homomorphism. This homomorphism together with the K-theory of the underlying algebra and the induced action in K-theory determine the K-theory of the associated crossed product C^*-algebra up to group extension problems. A concrete description of this obstruction homomorphism is provided as well. We give examples of $\mathbb{Z}^2$-actions, where the associated obstruction homomorphisms are non-trivial. One class of examples comprises certain outer $\mathbb{Z}^2$-actions on Kirchberg algebras, which act trivially on KK-theory. This relies on a classification result by Izumi and Matui. A second class of examples consists of certain pointwise inner $\mathbb{Z}^2$-actions. One instance is given as a natural action on the group C^*-algebra of the discrete Heisenberg group. We also compute the K-theory of the corresponding crossed product. A general and concrete construction yields various examples of pointwise inner $\mathbb{Z}^2$-actions on amalgamated free product C^*-algebras with non-trivial obstruction homomorphisms. Among these, there are actions that are universal, in a suitable sense, for pointwise inner $\mathbb{Z}^2$-actions with non-trivial obstruction homomorphisms. We also compute the K-theory of the crossed products associated with these universal C^*-dynamical systems.

Mathematics Subject Classification (2010). 46L55, 46L80; 46L40, 46L09.

Keywords. Crossed product C^*-algebras, $\mathbb{Z}^n$-actions, K-theory, free products of C^*-algebras.

1. Introduction

The study of group actions on C^*-algebras and their associated crossed product C^*-algebras plays an important role within the field of operator algebra theory. Beside the fact that many interesting and prominent C^*-algebras arise naturally as crossed products, their importance is also due to the various connections to other fields such as representation theory, index theory and topological dynamical systems.

*Supported by: SFB 878 *Groups, Geometry and Actions*, ERC AdG 267079, and the Villum Fonden project grant "Local and global structures of groups and their algebras" (2014–2018).

One of the most important invariants for crossed product C*-algebras is topological K-theory. However, given an action of a locally compact group on a C*-algebra, it is often very difficult to compute the K-theory of the corresponding crossed product, even if the K-theory of the underlying C*-algebra is well understood. One approach to the computation of the K-theory of the reduced crossed product, which is accessible via topological methods, is proposed by the famous Baum–Connes Conjecture [2,3]. The conjecture is known to hold for a strikingly large class of groups. In this context, we emphasize the deep work of Higson and Kasparov [12] on groups with the Haagerup property, and of Lafforgue [16] on hyperbolic groups.

For actions of the integer group, the celebrated Pimsner–Voiculescu exact sequence [20] is a very powerful tool to compute the K-theory of the corresponding crossed products. Given a C*-dynamical system $(A, \alpha, \mathbb{Z})$, it gives in particular rise to a short exact sequence

$$0 \longrightarrow \operatorname{coker}(K_*(\alpha) - \mathrm{id}) \longrightarrow K_*(A \rtimes_\alpha \mathbb{Z}) \longrightarrow \ker(K_{*+1}(\alpha) - \mathrm{id}) \longrightarrow 0.$$

This shows that $K_*(A \rtimes_\alpha \mathbb{Z})$ is determined by the action $K_*(\alpha) : \mathbb{Z} \curvearrowright K_*(A)$ up to a group extension problem.

It is natural to ask to what extends this property of $\mathbb{Z}$-actions generalises to actions of $\mathbb{Z}^n$. More concretely, we can ask the following:

Question. Given a C*-dynamical system $(A, \alpha, \mathbb{Z}^n)$, does the K-theory of $A \rtimes_\alpha \mathbb{Z}^n$ only depend on $K_*(\alpha) : \mathbb{Z}^n \curvearrowright K_*(A)$ up to group extension problems?

In this paper, we provide examples of $\mathbb{Z}^2$-actions giving a negative answer to this question. To this end, we associate to a given C*-dynamical system $(A, \alpha, \mathbb{Z}^2)$ a group homomorphism $d_*(\alpha)$ between certain subquotients of $K_*(A)$, which we call the *obstruction homomorphism*. It is constructed using the Pimsner–Voiculescu sequence and its naturality property. This homomorphism together with the action $K_*(\alpha) : \mathbb{Z}^2 \curvearrowright K_*(A)$ determines $K_*(A \rtimes_\alpha \mathbb{Z}^2)$ up to group extension problems. The name for $d_*(\alpha)$ justifies since it obstructs $K_*(A \rtimes_\alpha \mathbb{Z}^2)$ to be solely determined by $K_*(\alpha) : \mathbb{Z}^2 \curvearrowright K_*(A)$ up to group extension problems.

The instances of $\mathbb{Z}^2$-actions with negative answer to the above question that we discuss in this work all act trivially on K-theory, but give rise to non-trivial obstruction homomorphisms. The first class of examples consists of certain $\mathbb{Z}^2$-actions on Kirchberg algebras. The fact that many such actions induce non-trivial obstruction homomorphisms turns out to be a consequence of Izumi's and Matui's classification of outer locally KK-trivial $\mathbb{Z}^2$-actions on Kirchberg algebras, see [13].

As a second class, we consider pointwise inner $\mathbb{Z}^2$-actions. Contrary to the naïve expectation, we find examples with non-trivial obstruction homomorphisms even within this class of actions. This is even more remarkable, as $\mathbb{Z}^2$-actions arising from group representations into the unitary group of the underlying C*-algebra all give rise to isomorphic crossed products. An instructive example, which is also of interest in its own right, is given as a natural pointwise inner $\mathbb{Z}^2$-action on the group

C*-algebra of the discrete Heisenberg group. This C*-algebra has already obtained a great deal of attention, whereat we point out the thorough investigation of Anderson and Paschke [1]. We conclude this paper by giving a general construction of pointwise inner $\mathbb{Z}^2$-actions on certain amalgamated free product C*-algebras. Among these, we find actions that are universal, in a suitable sense, for non-trivial obstruction homomorphisms coming from pointwise inner $\mathbb{Z}^2$-actions. We also compute the K-theory of the crossed products associated with these universal actions.

The paper is organised as follows. In Section 2, we provide concrete lifts for images under the boundary map of the Pimsner–Voiculescu sequence $\rho_* : K_*(A \rtimes_\alpha \mathbb{Z}) \to K_{*+1}(A)$. The lifts for ρ_1 are well-known and easily found by using the partial isometry picture of the index map. They are all given as, what we call, generalised Bott elements associated with a commuting pair of a projection and a unitary. Finding suitable lifts for ρ_0 is less obvious. These are given as generalised Bott elements in the sense of Exel [11]. To define these lifts, we use a result by Dadarlat [9], which provides a better suitable description of the K_1-group for a unital C*-algebra.

In Section 3, we define the obstruction homomorphism associated with a $\mathbb{Z}^2$-action. We use the results of Section 2 to give a concrete description of this homomorphism in terms of generalised Bott elements.

In Section 4, we make use of Izumi's and Matui's classification result [13] to show the existence of $\mathbb{Z}^2$-actions on Kirchberg algebras, whose associated obstruction homomorphisms do not vanish. Given a Kirchberg algebra A, we show that their classification invariant of a locally KK-trivial $\mathbb{Z}^2$-action on A, which is an element in $KK(A, SA)$, descends to the associated obstruction homomorphism, which basically amounts to a homomorphism $K_*(A) \to K_{*+1}(A)$. They prove that every element in $KK(A, SA)$ is realised as the invariant of such a $\mathbb{Z}^2$-action, provided that A is stable. Consequently, if A moreover satisfies the universal coefficient theorem (UCT) by Rosenberg and Schochet [21], then every homomorphism $K_*(A) \to K_{*+1}(A)$ occurs as the obstruction homomorphism of some $\mathbb{Z}^2$-action on A.

In Section 5, we provide examples of pointwise inner $\mathbb{Z}^2$-actions, which induce non-trivial obstruction homomorphisms. We first consider the group C*-algebra of the discrete Heisenberg group, $C^*(H_3)$, equipped with a natural pointwise inner action. This action is universal in the sense that every pointwise inner $\mathbb{Z}^2$-action on a unital C*-algebra B gives rise to an equivariant and unital $*$-homomorphism $C^*(H_3) \to B$. We show that the associated obstruction homomorphism is non-trivial and compute the K-theory of the corresponding crossed product. It turns out that this crossed product is not isomorphic in K-theory to the crossed product of $C^*(H_3)$ by the trivial $\mathbb{Z}^2$-action. Finally, we present a general method of constructing pointwise inner actions, which induce non-trivial obstruction homomorphisms. All occurring C*-algebras are given as amalgamated free products of the form $A *_{C(\mathbb{T})} B$. We require that A is equipped with a poinwise inner action, which has the property that the commutator of the two implementing unitaries has full spectrum. Moreover, B is supposed to contain a central unitary with full spectrum, which gets identified with the commutator. The action of A extends to a pointwise inner action on the amalgamated free product, and an additional, relatively mild K-theoretical assumption on B ensures that the associated obstruction homomorphism is non-trivial. Among the constructed examples, we find C*-dynamical systems which are universal, in a suitable sense, for non-trivial obstruction homomorphisms associated with pointwise inner $\mathbb{Z}^2$-actions. We compute the K-theory of the crossed products associated with these universal C*-dynamical systems by employing a six-term exact sequence for amalgamated free products by Thomsen [22].

Acknowledgements. This paper is based on results of my doctoral thesis completed at Westfälische Wilhelms-Universität Münster in 2014. I would like to thank my advisor Joachim Cuntz for his guidance and several helpful discussions. Moreover, I am grateful to Siegfried Echterhoff, Dominic Enders, Sven Raum, Nicolai Stammeier, Gábor Szabó, Christian Voigt, Christoph Winges, and Wilhelm Winter for inspiring dicussions held in the course of my doctoral studies. I am thankful to Dominic Enders, Sven Raum, Nicolai Stammeier, Gábor Szabó, and Christoph Winges for their valuable feedback on preliminary versions. Finally, I should also like to thank Chris Phillips for pointing out to me the existence of certain K-theoretically trivial actions $\alpha : \mathbb{Z}^2 \curvearrowright A$ with the property that $A \rtimes_\alpha \mathbb{Z}^2$ and $A \rtimes_{\mathrm{id}} \mathbb{Z}^2$ have different K-theory.

2. The boundary map of the Pimsner–Voiculescu sequence

Let us first recall Exel's [11] definition and some basic properties of Bott elements associated with almost commuting unitaries. Let $\mathcal{T}$ denote the Toeplitz algebra and $\mathcal{K}$ the algebra of compact operators on a separable infinite dimensional Hilbert space. Consider the following extension of C*-algebras

$$0 \longrightarrow \mathcal{K} \otimes \mathcal{C}(\mathbb{T}) \longrightarrow \mathcal{T} \otimes \mathcal{C}(\mathbb{T}) \longrightarrow \mathcal{C}(\mathbb{T}) \otimes \mathcal{C}(\mathbb{T}) \longrightarrow 0$$

induced by the canonical surjection $\mathcal{T} \to \mathcal{C}(\mathbb{T})$. Up to a sign, the Bott element $\mathfrak{b} \in K_0(\mathcal{C}(\mathbb{T}^2))$ is characterised by the property that its image under the corresponding index map $\rho_0 : K_0(\mathcal{C}(\mathbb{T}) \otimes \mathcal{C}(\mathbb{T})) \to K_1(\mathcal{C}(\mathbb{T}))$ is a generator for $K_1(\mathcal{C}(\mathbb{T}))$. We fix the convention that $\rho_0(\mathfrak{b}) = [z]$, with $z := \mathrm{id}_{\mathcal{C}(\mathbb{T})}$.

For $\varepsilon \geq 0$, Exel [11] defines the *soft torus* A_ε as the universal C*-algebra

$$A_\varepsilon := \mathrm{C}^*(u_\varepsilon, v_\varepsilon \text{ unitaries } : \|[u_\varepsilon, v_\varepsilon]\| \leq \varepsilon),$$

where $[u_\varepsilon, v_\varepsilon] = u_\varepsilon v_\varepsilon - v_\varepsilon u_\varepsilon$ denotes the commutator of u_ε and v_ε. It is obvious from the definition that $A_0 = \mathcal{C}(\mathbb{T}^2)$, and that for $\varepsilon \geq 2$ the soft torus A_ε coincides with the full group C*-algebra of the free group in two generators. There is a canonical surjective $*$-homomorphism

$$\varphi_\varepsilon : A_\varepsilon \to \mathcal{C}(\mathbb{T}^2) \quad \text{with } \varphi_\varepsilon(u_\varepsilon) := z_1, \varphi_\varepsilon(v_\varepsilon) := z_2,$$

where z_1 and z_2 denote the projections onto the first and second coordinate, respectively. By [11, Theorem 2.4], $K_*(\varphi_\varepsilon)$ is an isomorphism whenever $\varepsilon < 2$, and in this case we define

$$\mathfrak{b}_\varepsilon := K_0(\varphi_\varepsilon)^{-1}(\mathfrak{b}) \in K_0(A_\varepsilon).$$

Definition 2.1 (cf. [11]). Let $0 < \varepsilon < 2$, B a unital C*-algebra, and $u, v \in B$ unitaries satisfying $\| [u, v] \| \leq \varepsilon$. The universal property of the soft torus A_ε yields a unique $*$-homomorphism $\varphi : A_\varepsilon \to B$ with $\varphi(u_\varepsilon) = u$ and $\varphi(v_\varepsilon) = v$. Then the *Bott element* associated with u and v is defined as

$$\kappa(u, v) := K_0(\varphi)(\mathfrak{b}_\varepsilon) \in K_0(B).$$

Note that $\kappa(u, v)$ is independent of ε as long as $\| [u, v] \| \leq \varepsilon$. By definition, $\kappa(z_1, z_2) = \mathfrak{b} \in K_0(\mathcal{C}(\mathbb{T}^2))$. If $\varphi : A \to B$ is a unital $*$-homomorphism and $u, v \in A$ are unitaries with $\| [u, v] \| < 2$, then

$$K_0(\varphi)(\kappa(u, v)) = \kappa(\varphi(u), \varphi(v)).$$

For small $\varepsilon > 0$, the Bott element $\kappa(u, v)$ is given (up to a sign) by the following description due to Loring [17]. Consider the real-valued functions $f, g, h \in \mathcal{C}(\mathbb{T})$ defined as

$$f(e^{2\pi i t}) = \begin{cases} 1 - 2t, & \text{if } 0 \leq t \leq 1/2, \\ -1 + 2t, & \text{if } 1/2 \leq t \leq 1, \end{cases}$$
$$g(e^{2\pi i t}) = \begin{cases} (f(\exp(2\pi i t)) - f(\exp(2\pi i t))^2)^{1/2}, & \text{if } 0 \leq t \leq 1/2, \\ 0, & \text{if } 1/2 \leq t \leq 1, \end{cases}$$
$$h(e^{2\pi i t}) = \begin{cases} 0, & \text{if } 0 \leq t \leq 1/2, \\ (f(\exp(2\pi i t)) - f(\exp(2\pi i t))^2)^{1/2}, & \text{if } 1/2 \leq t \leq 1, \end{cases}$$

and set

$$e(u, v) := \begin{pmatrix} f(v) & g(v) + h(v)u \\ g(v) + u^* h(v) & 1 - f(v) \end{pmatrix} \in M_2(B).$$

One checks that $e(u, v)$ is always self-adjoint and a projection whenever u and v commute. Loring observed in [17, Proposition 3.5] that there is a universal constant $\delta > 0$ such that whenever $\| [u, v] \| < \delta$, then the spectrum of $e(u, v)$ does not contain $1/2$. In this case, $\chi_{[1/2,\infty)}(e(u, v)) \in M_2(B)$ is a projection, and Loring's Bott element is given as

$$[\chi_{[1/2,\infty)}(e(u, v))] - [1] \in K_0(B).$$

The Bott elements also have the following well-known properties. We leave the proof to the reader.

Proposition 2.2. *Let B be a unital* C*-*algebra and $u, v, u_1, v_1, \ldots, u_n, v_n \in B$ unitaries. Then the following statements hold true:*

(i) *If $u_t \in B$ is a homotopy of unitaries with $\| [u_t, v] \| < 2$ for all $t \in [0,1]$, then $\kappa(u_0, v) = \kappa(u_1, v)$.*

(ii) *If $\| [u_i, v_i] \| < 2$ for $i = 1, \ldots, n$, then*

$$\kappa(\mathrm{diag}(u_1, \ldots, u_n), \mathrm{diag}(v_1, \ldots, v_n)) = \sum_{i=1}^{n} \kappa(u_i, v_i).$$

(iii) *If $\sum_{i=1}^{n} \| [u, v_i] \| < 2$, then $\kappa(u, v_1 v_2 \ldots v_n) = \sum_{i=1}^{n} \kappa(u, v_i)$.*

(iv) *If $\| [u, v] \| < 2$, then $\kappa(u, v) = -\kappa(u, v^*) = -\kappa(v, u)$.*

For a unital, purely infinite and simple C*-algebra A, Elliott and Rørdam showed in [10, Theorem 2.2.1] that every element $x \in K_0(A)$ is a Bott element $x = \kappa(u, v)$ for some pair of commuting unitaries $u, v \in A$ (with full spectrum). On the other hand, if A is a unital C*-algebra admitting a tracial state τ, then every Bott element in $K_0(A)$ associated with exactly commuting unitaries vanishes under $K_0(\tau)$. In fact, if τ_2 denotes the induced (unnormalized) trace on $M_2(A)$, then $\tau_2(e(u, v)) = 1$.

It will turn out to be convenient to consider the following analogous notion of Bott elements in the K_1-group of a unital C*-algebra.

Notation 2.3. Let A be a unital C*-algebra. Let $p \in A$ be a projection and $u \in A$ a unitary commuting with p. Then $pup + 1 - p \in A$ is a unitary, and we define the *Bott element* associated with p and u as

$$\kappa(p, u) := [pup + 1 - p] \in K_1(A).$$

Observe that the Bott isomorphism $K_0(A) \xrightarrow{\cong} K_1(SA)$ indeed sends $[p]$ to $\kappa(p, z)$.

Now let A be a C*-algebra and $\alpha \in \mathrm{Aut}(A)$ an automorphism on A. Recall the *Pimsner–Voiculescu exact sequence* [20]

$$\begin{array}{ccccc}
K_0(A) & \xrightarrow{K_0(\alpha)-\mathrm{id}} & K_0(A) & \xrightarrow{K_0(j)} & K_0(A \rtimes_\alpha \mathbb{Z}) \\
\uparrow \rho_1 & & & & \downarrow \rho_0 \\
K_1(A \rtimes_\alpha \mathbb{Z}) & \xleftarrow[K_1(j)]{} & K_1(A) & \xleftarrow[K_1(\alpha)-\mathrm{id}]{} & K_1(A)
\end{array}$$

with $j : A \hookrightarrow A \rtimes_\alpha \mathbb{Z}$ denoting the canonical embedding. It is natural in the sense that for any equivariant $*$-homomorphism $\varphi : (A, \alpha, \mathbb{Z}) \to (B, \beta, \mathbb{Z})$, the following diagram commutes

$$\begin{array}{ccccccc} K_*(A) & \xrightarrow{K_*(\alpha)-\mathrm{id}} & K_*(A) & \longrightarrow & K_*(A \rtimes_\alpha \mathbb{Z}) & \xrightarrow{\rho_*} & K_{*+1}(A) \\ \downarrow{\scriptstyle K_*(\varphi)} & & \downarrow{\scriptstyle K_*(\varphi)} & & \downarrow{\scriptstyle K_*(\check{\varphi})} & & \downarrow{\scriptstyle K_{*+1}(\varphi)} \\ K_*(B) & \xrightarrow{K_*(\beta)-\mathrm{id}} & K_*(B) & \longrightarrow & K_*(B \rtimes_\beta \mathbb{Z}) & \xrightarrow{\rho_*} & K_{*+1}(B) \end{array}$$

where $\check{\varphi} : A \rtimes_\alpha \mathbb{Z} \to B \rtimes_\beta \mathbb{Z}$ denotes the natural extension of φ to a $*$-homomorphism between the crossed products. Furthermore, we recall that the boundary maps of the Pimsner–Voiculescu sequence coincide (at least up to an application of the stabilisation isomorphism) with the ones of the six-term exact sequence associated with the *Toeplitz extension*

$$0 \longrightarrow \mathcal{K} \otimes A \longrightarrow \mathcal{T}(A, \alpha) \longrightarrow A \rtimes_\alpha \mathbb{Z} \longrightarrow 0,$$

where

$$\mathcal{T}(A, \alpha) := \mathrm{C}^*(v \otimes au \ : \ a \in A) \subset \mathcal{T} \otimes (A \rtimes_\alpha \mathbb{Z})$$

is the *crossed Toeplitz-algebra* associated with α, see also [8]. Here, $v \in \mathcal{T}$ denotes the canonical isometry and $u \in \mathcal{M}(A \rtimes_\alpha \mathbb{Z})$ denotes the canonical unitary in the multiplier algebra of $A \rtimes_\alpha \mathbb{Z}$ implementing α.

Notation 2.4. For a C^*-algebra A and $n \in \mathbb{N}$, let $\mathcal{P}_n(A)$ denote the set of projections in $M_n(A)$. If A is unital, then $\mathcal{U}_n(A)$ denotes the set of unitaries in $M_n(A)$. Furthermore, if $\alpha \in \mathrm{Aut}(A)$ is an automorphism, then we write $\alpha^{(n)} := \alpha \otimes \mathrm{id} \in \mathrm{Aut}(A \otimes M_n(\mathbb{C}))$. Similarly, we define $a^{(n)} := a \otimes 1_n \in A \otimes M_n(\mathbb{C})$ for $a \in A$.

Assume that A is unital. We now describe preimages of the boundary map $\rho_1 : K_1(A \rtimes_\alpha \mathbb{Z}) \to K_0(A)$ of the Pimsner–Voiculescu sequence. Observe that every element $g \in K_0(A)$ can be expressed as $g = [p] - [1_n]$ for some $p \in \mathcal{P}_m(A)$ and $n \geq 0$. It is obvious that $g \in \ker(K_0(\alpha) - \mathrm{id})$ if and only if $[p] \in \ker(K_0(\alpha) - \mathrm{id})$. Hence, it suffices to describe lifts for elements of the form $[p] \in \mathrm{im}(\rho_1) = \ker(K_0(\alpha) - \mathrm{id})$.

Proposition 2.5. *Let A be a unital C^*-algebra, $\alpha \in \mathrm{Aut}(A)$, and $p \in \mathcal{P}_k(A)$ a projection satisfying $[p] \in \ker(K_0(\alpha) - \mathrm{id})$. By the standard picture of $K_0(A)$, we find $l, m \geq 0$ and a unitary $w \in \mathcal{U}_n(A)$ such that*

$$\alpha^{(n)}(q) = wqw^*,$$

where $n := k + l + m$ and $q := \mathrm{diag}(p, 1_l, 0_m) \in \mathcal{P}_n(A)$. Then

$$\rho_1(\kappa(q, w^* u^{(n)}) - [u^{(l)}]) = [p].$$

Proof. Assume first that $k = 1$ and $l = m = 0$. It is easy to verify that

$$y := v \otimes (pw^*up) + 1 \otimes (1 - p) \in \mathcal{T}(A, \alpha)$$

is an isometry that lifts $pwu^*p+1-p \in A \rtimes_\alpha \mathbb{Z}$. Using the partial isometry picture of the index map, one computes

$$\begin{aligned} \rho_1(\kappa(p, w^*u)) &= [1-yy^*] - [1-y^*y] = [1-yy^*] \\ &= [(1-vv^*)\otimes p] \in K_0(\mathcal{K}\otimes A). \end{aligned}$$

By the stabilisation isomorphism $K_0(A) \cong K_0(\mathcal{K}\otimes A)$, we deduce that

$$\rho_1(\kappa(p, w^*u)) = [p] \in K_0(A).$$

Now, let $q \in \mathcal{P}_n(A)$ and $w \in \mathcal{U}_n(A)$ be as in the statement. The canonical isomorphism $\eta : M_n(A) \rtimes_{\alpha^{(n)}} \mathbb{Z} \xrightarrow{\cong} A \rtimes_\alpha \mathbb{Z} \otimes M_n(\mathbb{C})$ gives rise to a commutative diagram

$$\begin{array}{ccc} K_*(M_n(A) \rtimes_{\alpha^{(n)}} \mathbb{Z}) & \xrightarrow{\rho_*^{(n)}} & K_{*+1}(A) \\ {\scriptstyle K_*(\eta)}\downarrow & \nearrow_{\rho_*} & \\ K_*(A \rtimes_\alpha \mathbb{Z}) & & \end{array}$$

relating the boundary maps of the respective Pimsner–Voiculescu sequences. Hence,

$$\rho_1(\kappa(q, w^*u^{(n)})) = (\rho_1 \circ K_1(\eta))(\kappa(q, w^*u)) = \rho_1^{(n)}(\kappa(q, w^*u)) = [q].$$

It follows that

$$\rho_1(\kappa(q, w^*u^{(n)}) - [u^l]) = [q] - [1_l] = [\mathrm{diag}(p, 1_l)] - [1_l] = [p]. \qquad \square$$

For the boundary map $\rho_0 : K_0(A \rtimes_\alpha \mathbb{Z}) \to K_1(A)$, we use an alternative picture for $K_1(A)$, derived from the natural identification $K_*(A) \cong KK(\mathcal{C}(\mathbb{T}), A)$ together with Dadarlat's result [9, Theorem A] applied to $\mathbb{T}$.

Theorem 2.6 (cf. [9]). *Let A be a unital* C*-*algebra and $u, v \in \mathcal{U}(A)$ two unitaries. Then $[u] = [v] \in K_1(A)$ if and only if for every $\varepsilon > 0$, there exist $k \geq 1$, $\lambda_1, \ldots, \lambda_k \in \mathbb{T}$, and a unitary $w \in M_{k+1}(A)$ such that*

$$\|w(\mathrm{diag}(u, \lambda_1, \ldots, \lambda_k))w^* - \mathrm{diag}(v, \lambda_1, \ldots, \lambda_k)\| \leq \varepsilon.$$

For $0 < \varepsilon < 2$, define the universal C*-algebra

$$T_\varepsilon := \mathrm{C}^*\left(s \text{ isometry}, u \text{ unitary} \;:\; \|\,[s,u]\,\| \leq \varepsilon,\; uss^* = ss^*u\right).$$

Consider the extension of C*-algebras

$$0 \longrightarrow I_\varepsilon \longrightarrow T_\varepsilon \longrightarrow A_\varepsilon \longrightarrow 0$$

induced by the canonical surjection $T_\varepsilon \to A_\varepsilon$ mapping s to u_ε and u to v_ε.

We denote the exponential map of the associated six-term exact sequence by $\rho_\varepsilon : K_0(A_\varepsilon) \to K_1(I_\varepsilon)$. Set $e := 1 - ss^* \in I_\varepsilon$. As u and ss^* commute, $eu + 1 - e$ defines a unitary in the minimal unitization of I_ε.

Lemma 2.7. *The exponential map $\rho_\varepsilon : K_0(A_\varepsilon) \to K_1(I_\varepsilon)$ satisfies*

$$\rho_\varepsilon(\mathfrak{b}_\varepsilon) = [eu + 1 - e].$$

Proof. Using that u and e commute, one computes

$$\begin{pmatrix} s^* & 0 \\ e & s \end{pmatrix} \begin{pmatrix} eu + 1 - e & 0 \\ 0 & u \end{pmatrix} \begin{pmatrix} s & e \\ 0 & s^* \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & eu + sus^* \end{pmatrix}.$$

As $\|[s, u]\| \le \varepsilon$,

$$\left\| \begin{pmatrix} 1 & 0 \\ 0 & eu + sus^* \end{pmatrix} - \begin{pmatrix} 1 & 0 \\ 0 & u \end{pmatrix} \right\| \le \varepsilon < 2.$$

Thus, the above computation shows that in $K_1(T_\varepsilon)$,

$$\left[\begin{pmatrix} eu + 1 - e & 0 \\ 0 & u \end{pmatrix} \right] = \left[\begin{pmatrix} 1 & 0 \\ 0 & eu + sus^* \end{pmatrix} \right] = \left[\begin{pmatrix} 1 & 0 \\ 0 & u \end{pmatrix} \right],$$

and hence $[eu + 1 - e] = 0 \in K_1(T_\varepsilon)$. Consequently,

$$[eu + 1 - e] \in \operatorname{im}(\rho_\varepsilon) \subset K_1(I_\varepsilon),$$

so that we find $n \in \mathbb{Z}$ such that $\rho_\varepsilon(n \cdot \mathfrak{b}_\varepsilon) = [eu + 1 - e]$.

Consider now the commutative diagram with exact rows

$$\begin{array}{ccccccccc} 0 & \longrightarrow & I_\varepsilon & \longrightarrow & T_\varepsilon & \longrightarrow & A_\varepsilon & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \psi'_\varepsilon} & & \downarrow{\scriptstyle \psi_\varepsilon} & & \downarrow{\scriptstyle \varphi_\varepsilon} & & \\ 0 & \longrightarrow & \mathcal{K} \otimes \mathcal{C}(\mathbb{T}) & \longrightarrow & \mathcal{T} \otimes \mathcal{C}(\mathbb{T}) & \longrightarrow & \mathcal{C}(\mathbb{T}) \otimes \mathcal{C}(\mathbb{T}) & \longrightarrow & 0 \end{array}$$

induced by the surjective $*$-homomorphism $\psi_\varepsilon : T_\varepsilon \to \mathcal{T} \otimes \mathcal{C}(\mathbb{T})$ given by $\psi_\varepsilon(s) = v \otimes 1$ and $\psi_\varepsilon(u) = 1 \otimes z$. Naturality of K-theory yields a commutative diagram

$$\begin{array}{ccc} K_0(A_\varepsilon) & \xrightarrow{\rho_\varepsilon} & K_1(I_\varepsilon) \\ {\scriptstyle K_0(\varphi_\varepsilon)}\downarrow & & \downarrow{\scriptstyle K_1(\psi'_\varepsilon)} \\ K_0(\mathcal{C}(\mathbb{T}) \otimes \mathcal{C}(\mathbb{T})) & \xrightarrow{\rho_0} & K_1(\mathcal{C}(\mathbb{T})) \end{array}$$

Thus,

$$\begin{aligned} n \cdot K_1(\psi'_\varepsilon) \circ \rho_\varepsilon(\mathfrak{b}_\varepsilon) &= K_1(\psi'_\varepsilon)([eu + 1 - e]) = [z] = \rho_0(\mathfrak{b}) \\ &= \rho_0 \circ K_0(\varphi_\varepsilon)(\mathfrak{b}_\varepsilon) = K_1(\psi'_\varepsilon) \circ \rho_\varepsilon(\mathfrak{b}_\varepsilon), \end{aligned}$$

which implies that $n = 1$. This completes the proof. □

Proposition 2.8. *Let A be a unital* C**-algebra, $\alpha \in \mathrm{Aut}(A)$, and $x \in \mathcal{U}_k(A)$ a unitary satisfying $[x] \in \ker(K_1(\alpha) - \mathrm{id})$. An application of Theorem 2.6 yields $l \geq 0$, $\lambda_1, \ldots, \lambda_l \in \mathbb{T}$, and $w \in \mathcal{U}_m(A)$ satisfying*

$$\left\| \alpha^{(m)}(y) - wyw^* \right\| < 2,$$

where $m := k + l$ and $y := \mathrm{diag}(x, \lambda_1, \ldots, \lambda_l) \in \mathcal{U}_m(A)$. Then

$$\rho_0(\kappa(w^* u^{(m)}, y)) = [x].$$

Proof. First assume that $k = 1$ and $l = 0$. For suitably chosen $0 < \varepsilon < 2$, there exists a $*$-homomorphism

$$\psi : T_\varepsilon \to \mathcal{T}(A, \alpha), \quad \psi(s) = v \otimes w^* u, \; \psi(u) = 1 \otimes x.$$

This homomorphism fits into the commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & I_\varepsilon & \longrightarrow & T_\varepsilon & \longrightarrow & A_\varepsilon & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \psi'} & & \downarrow{\scriptstyle \psi} & & \downarrow{\scriptstyle \varphi} & & \\ 0 & \longrightarrow & \mathcal{K} \otimes A & \longrightarrow & \mathcal{T}(A, \alpha) & \longrightarrow & A \rtimes_\alpha \mathbb{Z} & \longrightarrow & 0 \end{array}$$

where φ is given by $\varphi(u_\varepsilon) = w^* u$ and $\varphi(v_\varepsilon) = x$. By Lemma 2.7 and stability of K-theory,

$$\begin{aligned} \rho_0(\kappa(w^* u, x)) &= \rho_0 \circ K_0(\varphi)(\mathfrak{b}_\varepsilon) = K_1(\psi') \circ \rho_\varepsilon(\mathfrak{b}_\varepsilon) \\ &= K_1(\psi')([eu + 1 - e]) = [x] \in K_1(A). \end{aligned}$$

If $w, y \in \mathcal{U}_m(A)$ are as in the statement, then by a similar reasoning as in the proof of Proposition 2.5,

$$\begin{aligned} \rho_0(\kappa(w^* u^{(m)}, y)) &= (\rho_0 \circ K_0(\eta))(\kappa(w^* u, y)) \\ &= \rho_0^{(m)}(\kappa(w^* u, y)) = [y] = [x]. \end{aligned}$$

□

3. The obstruction homomorphism

Notation 3.1. For a C*-dynamical system $(A, \alpha, \mathbb{Z}^2)$, we write α_1 and α_2 for the automorphisms corresponding to $(1,0)$ and $(0,1)$, respectively. Moreover, we denote by $\breve{\alpha}_2 : A \rtimes_{\alpha_1} \mathbb{Z} \to A \rtimes_{\alpha_1} \mathbb{Z}$ the natural extension of α_2 to an automorphism of $A \rtimes_{\alpha_1} \mathbb{Z}$.

Given a C*-dynamical system $(A, \alpha, \mathbb{Z}^2)$, we denote by

$$\mathrm{k}_*(\alpha_2) : \ker(K_*(\alpha_1) - \mathrm{id}) \to \ker(K_*(\alpha_1) - \mathrm{id}),$$
$$\mathrm{co}_*(\alpha_2) : \mathrm{coker}(K_*(\alpha_1) - \mathrm{id}) \to \mathrm{coker}(K_*(\alpha_1) - \mathrm{id})$$

the endomorphisms induced by $K_*(\alpha_2) - \mathrm{id}$. We remark that these are well-defined as α_1 and α_2 commute. By naturality of the Pimsner–Voiculescu sequence, the equivariant $*$-automorphism $\alpha_2 : (A, \alpha_1, \mathbb{Z}) \to (A, \alpha_1, \mathbb{Z})$ gives rise to the following commutative diagram with exact rows

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathrm{coker}(K_*(\alpha_1) - \mathrm{id}) & \longrightarrow & K_*(A \rtimes_{\alpha_1} \mathbb{Z}) & \xrightarrow{\rho_*} & \ker(K_{*+1}(\alpha_1) - \mathrm{id}) & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \mathrm{co}_*(\alpha_2)} & & \downarrow{\scriptstyle K_*(\breve{\alpha}_2) - \mathrm{id}} & & \downarrow{\scriptstyle \mathrm{k}_{*+1}(\alpha_2)} & & \\ 0 & \longrightarrow & \mathrm{coker}(K_*(\alpha_1) - \mathrm{id}) & \longrightarrow & K_*(A \rtimes_{\alpha_1} \mathbb{Z}) & \xrightarrow{\rho_*} & \ker(K_{*+1}(\alpha_1) - \mathrm{id}) & \longrightarrow & 0 \end{array} \tag{3.1}$$

Applying the snake lemma (see e.g. [23, 1.3.2]) to this diagram, we obtain a group homomorphism $d_*(\alpha) : S_*(\alpha) \to T_{*+1}(\alpha)$, where

$$S_*(\alpha) := \ker(K_*(\alpha_1) - \mathrm{id}) \cap \ker(K_*(\alpha_2) - \mathrm{id}),$$
$$T_*(\alpha) := K_*(A) \big/ \langle \mathrm{im}(K_*(\alpha_1) - \mathrm{id}), \mathrm{im}(K_*(\alpha_2) - \mathrm{id}) \rangle .$$

We call $d_*(\alpha)$ the *obstruction homomorphism* associated with α. It satisfies the following naturality property.

Proposition 3.2. *If $\varphi : (A, \alpha, \mathbb{Z}^2) \to (B, \beta, \mathbb{Z}^2)$ is an equivariant $*$-homomorphism between* C*-*dynamical systems, then the following diagram commutes*

$$\begin{array}{ccc} S_*(\alpha) & \xrightarrow{d_*(\alpha)} & T_{*+1}(\alpha) \\ {\scriptstyle K_*(\varphi)}\downarrow & & \downarrow{\scriptstyle K_{*+1}(\varphi)} \\ S_*(\beta) & \xrightarrow{d_*(\beta)} & T_{*+1}(\beta) \end{array}$$

Proof. This follows directly from the naturality of the Pimsner–Voiculescu sequence and the naturality of the snake lemma exact sequence. □

The K-theory of $A \rtimes_\alpha \mathbb{Z}^2$ is determined by $(K_*(A), K_*(\alpha_1), K_*(\alpha_2), d_*(\alpha))$ up to group extension problems. In this sense, $d_*(\alpha)$ is indeed an obstruction for that $K_*(A \rtimes_\alpha \mathbb{Z}^2)$ only depends on $K_*(A)$, $K_*(\alpha_1)$, and $K_*(\alpha_2)$ up to group extension problems. To see this, observe first that the Pimsner–Voiculescu sequence for $(A \rtimes_{\alpha_1} \mathbb{Z}, \check{\alpha}_2, \mathbb{Z})$ induces a short exact sequence

$$0 \longrightarrow \operatorname{coker}(K_*(\check{\alpha}_2) - \mathrm{id}) \longrightarrow K_*(A \rtimes_\alpha \mathbb{Z}^2) \xrightarrow{\rho_*} \ker(K_{*+1}(\check{\alpha}_2) - \mathrm{id}) \longrightarrow 0.$$

Moreover, by definition of the obstruction homomorphism, the snake lemma exact sequence associated with (3.1) splits into two extensions, namely

$$0 \longrightarrow \ker(\mathrm{co}_*(\alpha_2)) \longrightarrow \ker(K_*(\check{\alpha}_2) - \mathrm{id}) \longrightarrow \ker(d_{*+1}(\alpha)) \longrightarrow 0$$

and

$$0 \longrightarrow \operatorname{coker}(d_{*+1}(\alpha)) \longrightarrow \operatorname{coker}(K_*(\check{\alpha}_2) - \mathrm{id}) \longrightarrow \operatorname{coker}(\mathrm{k}_{*+1}(\alpha_2)) \longrightarrow 0.$$

We proceed with a concrete description of the obstruction homomorphism in terms of generalised Bott elements.

Theorem 3.3. *Let A be a unital* C*-*algebra and $\alpha : \mathbb{Z}^2 \curvearrowright A$ an action. Let $p \in \mathcal{P}_k(A)$ be a projection satisfying $[p] \in S_0(\alpha)$. Find $l, m \geq 0$ and unitaries $v, w \in \mathcal{U}_n(A)$ with*

$$\alpha_1^{(n)}(q) = vqv^* \quad \textit{and} \quad \alpha_2^{(n)}(q) = wqw^*,$$

where $n := k + l + m$ and $q := \mathrm{diag}(p, 1_l, 0_m) \in \mathcal{P}_n(A)$. Then

$$d_0(\alpha)([p]) = \left[\kappa(q, w^*\alpha_2^{(n)}(v)^*\alpha_1^{(n)}(w)v)\right] \in T_1(\alpha).$$

Proof. If $\rho_1 : K_1(A \rtimes_{\alpha_1} \mathbb{Z}) \to K_0(A)$ denotes the index map of the Pimsner–Voiculescu sequence for α_1, then Proposition 2.5 yields

$$\rho_1\left(\kappa(q, v^*u^{(n)}) - [u^{(l)}]\right) = [p].$$

One computes that

$$\begin{aligned}(K_0(\check{\alpha}_2) - \mathrm{id})(\kappa(q, v^*u^{(n)}) - [u^{(l)}]) &= (K_0(\check{\alpha}_2) - \mathrm{id})(\kappa(q, v^*u^{(n)})) \\ &= \kappa(\alpha_2^{(n)}(q), \alpha_2^{(n)}(v)^*u^{(n)}) - \kappa(q, v^*u^{(n)}) \\ &= \kappa(wqw^*, \alpha_2^{(n)}(v)^*u^{(n)}) + \kappa(q, u^{(n)*}v) \\ &= \kappa(q, w^*\alpha_2^{(n)}(v)^*u^{(n)}w) + \kappa(q, u^{(n)*}v) \\ &= \kappa(q, w^*\alpha_2^{(n)}(v)^*u^{(n)}wu^{(n)*}v) \\ &= \kappa(q, w^*\alpha_2^{(n)}(v)^*\alpha_1^{(n)}(w)v) \in K_1(A).\end{aligned}$$

By definition of the snake lemma homomorphism, we get that

$$d_0(\alpha)([p]) = \left[\kappa(q, w^*\alpha_2^{(n)}(v)^*\alpha_1^{(n)}(w)v)\right] \in T_1(\alpha). \qquad \square$$

Observe that $d_0(\alpha)$ is completely determined by Theorem 3.3. In fact, for any $g \in S_0(\alpha)$, there is a projection $p \in \mathcal{P}_m(A)$ and some $n \geq 0$ such that $g = [p] - [1_n] \in K_0(A)$. In this situation, $[p] \in S_0(\alpha)$ and $d_0(\alpha)(g) = d_0(\alpha)([p])$.

For the description of $d_1(\alpha)$, we need the following perturbation result.

Lemma 3.4. *Let $0 < \varepsilon < \frac{2}{3}$. Let A be a unital* C*-*algebra and $u, \tilde{u}, v \in \mathcal{U}(A)$ unitaries satisfying*

$$\|u - \tilde{u}\|, \ \| [u, v] \| \leq \varepsilon.$$

Then there is a homotopy $u_t \in \mathcal{U}(A)$ with $u_0 = u$, $u_1 = \tilde{u}$, and $\| [u_t, v] \| \leq 3\varepsilon$ for all $t \in [0, 1]$.

Proof. Since $\|u - \tilde{u}\| < \frac{2}{3}$, the spectrum of $u^*\tilde{u}$ does not contain -1. Therefore, we can define $h := -i \log(u^*\tilde{u}) \in A$, where log denotes the principal branch of the logarithm. This yields a continuous path of unitaries $u_t := u \exp(ith) \in A$, $t \in [0, 1]$, with $u_0 = u$ and $u_1 = u \exp(\log(u^*\tilde{u})) = \tilde{u}$. For $s, t \in [0, 1]$,

$$\|u_s - u_t\| = \|1 - \exp(i(s-t)h)\| \leq \|1 - \exp(ih)\| = \|u - \tilde{u}\| \leq \varepsilon.$$

One now computes

$$\| [u_t, v] \| \leq \|u_t v - uv\| + \|v u_t - vu\| + \| [u, v] \| \leq 3\varepsilon.$$ □

Theorem 3.5. *Let A be a unital* C*-*algebra and $\alpha : \mathbb{Z}^2 \curvearrowright A$ an action. Let $v \in \mathcal{U}_k(A)$ be a unitary satisfying $[v] \in S_1(\alpha)$. By Theorem 2.6, there are $l \geq 0$, $\lambda_1, \dots, \lambda_l \in \mathbb{T}$, and unitaries $x, y \in \mathcal{U}_m(A)$ such that*

$$\left\|\alpha_1^{(m)}(w) - xwx^*\right\|, \ \left\|\alpha_2^{(m)}(w) - ywy^*\right\| < \frac{1}{2},$$

where $m := k + l$ and $w := \operatorname{diag}(v, \lambda_1, \dots, \lambda_l) \in \mathcal{U}_m(A)$. Then

$$d_1(\alpha)([v]) = \left[\kappa(y^*\alpha_2^{(m)}(x)^*\alpha_1^{(m)}(y)x, w)\right] \in T_0(\alpha).$$

Proof. Using the isomorphism $M_m(A \rtimes_{\alpha_1} \mathbb{Z}) \cong M_m(A) \rtimes_{\alpha_1^{(m)}} \mathbb{Z}$, we compute that

$$\left\|[x^*u^{(m)}, w]\right\| = \left\|u^{(m)}wu^{(m)^*} - xwx^*\right\| = \left\|\alpha_1^{(m)}(w) - xwx^*\right\| < \frac{1}{2}.$$

By Proposition 2.8, the boundary map $\rho_0 : K_0(A \rtimes_{\alpha_1} \mathbb{Z}) \to K_1(A)$ of the Pimsner–Voiculescu sequence for α_1 satisfies

$$\rho_0(\kappa(x^*u^{(m)}, w)) = [v].$$

The naturality of the Bott elements and Proposition 2.2(iv) yield

$$\begin{aligned}(K_0(\check{\alpha}_2) - \mathrm{id})(\kappa(x^*u^{(m)}, w)) &= \kappa(\alpha_2^{(m)}(x)^*u^{(m)}, \alpha_2^{(m)}(w)) - \kappa(x^*u^{(m)}, w) \\ &= \kappa(\alpha_2^{(m)}(x)^*u^{(m)}, \alpha_2^{(m)}(w)) + \kappa(u^{(m)^*}x, w).\end{aligned}$$

Since

$$\|\alpha_2^{(m)}(w) - ywy^*\|, \ \|[\alpha_2^{(m)}(x)^* u^{(m)}, \alpha_2^{(m)}(w)]\| < \frac{1}{2},$$

we can apply Lemma 3.4 and find a homotopy $w_t \in \mathcal{U}_m(A)$ between $\alpha_2^{(m)}(w)$ and ywy^* such that

$$\|[w_t, \alpha_2^{(m)}(x)^* u^{(m)}]\| < \frac{3}{2} \quad \text{for all } t \in [0,1].$$

By Proposition 2.2(i) and the naturality of the Bott elements, we obtain that

$$\begin{aligned}\kappa(\alpha_2^{(m)}(x)^* u^{(m)}, \alpha_2^{(m)}(w)) &= \kappa(\alpha_2^{(m)}(x)^* u^{(m)}, ywy^*) \\ &= \kappa(y^*\alpha_2^{(m)}(x)^* u^{(m)} y, w).\end{aligned}$$

Moreover,

$$\|[u^{(m)*} x, w]\| + \|[y^*\alpha_2^{(m)}(x)^* u^{(m)} y, w]\| < \frac{1}{2} + \frac{3}{2} = 2,$$

and therefore Proposition 2.2(iii) yields

$$\begin{aligned}(K_0(\check{\alpha}_2) - \mathrm{id})(\kappa(x^* u^{(m)}, w)) &= \kappa(y^*\alpha_2^{(m)}(x)^* u^{(m)} y, w) + \kappa(u^{(m)*} x, w) \\ &= \kappa(y^*\alpha_2^{(m)}(x)^* u^{(m)} y u^{(m)*} x, w) \\ &= \kappa(y^*\alpha_2^{(m)}(x)^* \alpha_1^{(m)}(y) x, w) \in K_0(A).\end{aligned}$$

By the definition of the snake lemma homomorphism, it follows that

$$d_1(\alpha)([v]) = [\kappa(y^*\alpha_2^{(m)}(x)^* \alpha_1^{(m)}(y) x, w)] \in T_0(\alpha). \qquad \square$$

4. Locally KK-trivial $\mathbb{Z}^2$-actions on Kirchberg algebras

Let A be a C*-algebra and $\alpha : \mathbb{Z}^2 \curvearrowright A$ an action. We present another description of the obstruction homomorphism $d_*(\alpha)$, making use of the *mapping torus* for α_1,

$$\mathcal{M}_{\alpha_1}(A) := \{ f \in \mathcal{C}([0,1], A) \ : \ f(1) = \alpha_1(f(0)) \}.$$

Consider the short exact sequence

$$0 \longrightarrow SA \longrightarrow \mathcal{M}_{\alpha_1}(A) \xrightarrow{\mathrm{ev}_0} A \longrightarrow 0$$

induced by evaluation at 0. It is well-known that the Pimsner–Voiculescu exact sequence can be derived from the six-term exact sequence corresponding to this extension. More concretely, there is an isomorphism $K_*(A \rtimes_{\alpha_1} \mathbb{Z}) \cong K_{*+1}(\mathcal{M}_{\alpha_1}(A))$ making the following diagram commute

$$\begin{array}{ccc}
K_*(A) & \xrightarrow{K_*(j)} & K_*(A \rtimes_{\alpha_1} \mathbb{Z}) \\
\big\downarrow & \overset{\cong}{\nearrow} & \big\downarrow{\scriptstyle \rho_*} \\
K_{*+1}(\mathcal{M}_{\alpha_1}(A)) & \xrightarrow[K_{*+1}(\mathrm{ev}_0)]{} & K_{*+1}(A)
\end{array}$$

where the left vertical map is given by composition of the natural map $K_{*+1}(SA) \to K_{*+1}(\mathcal{M}_{\alpha_1}(A))$ with the Bott isomorphism. We refer to [6] and [18] for details.

Denote by $\tilde{\alpha}_2 \in \mathrm{Aut}(\mathcal{M}_{\alpha_1}(A))$ the automorphism given by

$$\tilde{\alpha}_2(f)(t) := \alpha_2(f(t)), \quad f \in \mathcal{M}_{\alpha_1}(A),\ t \in [0,1].$$

This indeed is a self map of $\mathcal{M}_{\alpha_1}(A)$, as α_1 and α_2 commute. The isomorphism $K_*(A \rtimes_{\alpha_1} \mathbb{Z}) \cong K_{*+1}(\mathcal{M}_{\alpha_1}(A))$ is natural, so that in particular, the diagram

$$\begin{array}{ccc}
K_*(A \rtimes_{\alpha_1} \mathbb{Z}) & \xrightarrow{\cong} & K_{*+1}(\mathcal{M}_{\alpha_1}(A)) \\
{\scriptstyle K_*(\breve{\alpha}_2)}\big\downarrow & & \big\downarrow{\scriptstyle K_{*+1}(\tilde{\alpha}_2)} \\
K_*(A \rtimes_{\alpha_1} \mathbb{Z}) & \xrightarrow{\cong} & K_{*+1}(\mathcal{M}_{\alpha_1}(A))
\end{array}$$

is commutative. It now follows from the construction of the obstruction homomorphism that the snake lemma homomorphism associated with the diagram

$$\begin{array}{ccccc}
\mathrm{coker}(K_{*+1}(\alpha_1) - \mathrm{id}) & \hookrightarrow & K_*(\mathcal{M}_{\alpha_1}(A)) & \xrightarrow{K_*(\mathrm{ev}_0)} & \ker(K_*(\alpha_1) - \mathrm{id}) \\
\big\downarrow{\scriptstyle \mathrm{co}_{*+1}(\alpha_2)} & & \big\downarrow{\scriptstyle K_*(\tilde{\alpha}_2) - \mathrm{id}} & & \big\downarrow{\scriptstyle \mathrm{k}_*(\alpha_2)} \\
\mathrm{coker}(K_{*+1}(\alpha_1) - \mathrm{id}) & \hookrightarrow & K_*(\mathcal{M}_{\alpha_1}(A)) & \xrightarrow{K_*(\mathrm{ev}_0)} & \ker(K_*(\alpha_1) - \mathrm{id})
\end{array}$$

with exact rows coincides with $d_*(\alpha)$.

Now assume that α_1 is homotopic to id_A in $\mathrm{Aut}(A)$. Fix a homotopy $\beta_t \in \mathrm{Aut}(A)$ between $\beta_0 = \alpha_1$ and $\beta_1 = \mathrm{id}_A$ and consider the induced $*$-automorphism $\varphi \in \mathrm{Aut}(A \otimes \mathcal{C}(\mathbb{T}))$ given by

$$\varphi(f)(\exp(2\pi i t)) = \left(\beta_t \circ \alpha_2 \circ \beta_t^{-1}\right)(f(\exp(2\pi i t))),$$

for $f \in A \otimes \mathcal{C}(\mathbb{T})$ and $t \in [0,1]$. Note that φ is well-defined since α_1 and α_2 commute. Obviously, φ restricts to an automorphism $\varphi' : SA \xrightarrow{\cong} SA$ and fits into the following commutative diagram with split-exact rows

$$\begin{array}{ccccccccc} & & & & & \overset{j}{\longleftarrow} & & & \\ 0 & \longrightarrow & SA & \longrightarrow & A \otimes \mathcal{C}(\mathbb{T}) & \xrightarrow{\mathrm{ev}_1} & A & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \varphi'} & & \downarrow{\scriptstyle \varphi} & & \downarrow{\scriptstyle \alpha_2} & & \\ 0 & \longrightarrow & SA & \longrightarrow & A \otimes \mathcal{C}(\mathbb{T}) & \xrightarrow{\mathrm{ev}_1} & A & \longrightarrow & 0 \\ & & & & & \underset{j}{\longleftarrow} & & & \end{array} \tag{4.1}$$

Here, $j : A \hookrightarrow A \otimes \mathcal{C}(\mathbb{T})$ denotes the canonical embedding. Futhermore, there is a $*$-isomorphism $\psi : \mathcal{M}_{\alpha_1}(A) \xrightarrow{\cong} A \otimes \mathcal{C}(\mathbb{T})$ satisfying

$$\psi(f)(\exp(2\pi i t)) = (\beta_t \circ \alpha_1^{-1})(f(t)), \quad f \in \mathcal{M}_{\alpha_1}(a),\ t \in [0,1].$$

Its restriction $\psi' \in \mathrm{Aut}(SA)$ is homotopic to id_{SA} via $\psi'_s \in \mathrm{Aut}(SA)$ given by

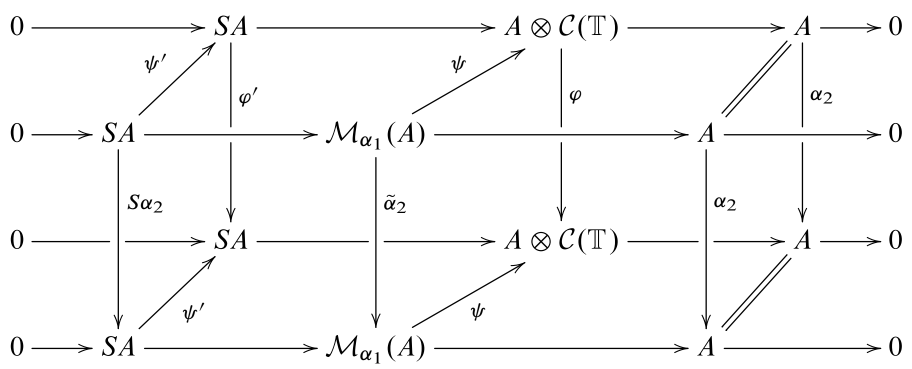
$$\psi'_s(f)(t) = (\beta_{st} \circ \alpha_1^{-1})(f(t)), \quad f \in SA,\ s,t \in [0,1].$$

Proposition 4.1. *Let A be a* C*-*algebra and* $\alpha : \mathbb{Z}^2 \curvearrowright A$ *an action such that* α_1 *is homotopic to* id_A *in* $\mathrm{Aut}(A)$. *Then the obstruction homomorphism* $d_*(\alpha) : K_*(A) \to K_{*+1}(A)$ *satisfies*

$$d_*(\alpha) = (K_*(\varphi) - \mathrm{id}) \circ K_*(j),$$

where we use the Bott isomorphism to identify $K_{*+1}(A) \cong K_*(SA) \subset K_*(A \otimes \mathcal{C}(\mathbb{T}))$.

Proof. The isomorphism $\psi : \mathcal{M}_{\alpha_1}(A) \xrightarrow{\cong} A \otimes \mathcal{C}(\mathbb{T})$ and (4.1) induce the following commutative diagram



Observe that all occurring rows are split-exact. Apply K-theory to the whole diagram, use Bott periodicity for the left hand square involving the suspensions of A and subtract the respective identity morphism at each vertical arrow. The snake lemma homomorphism of the resulting front diagram is $d_*(\alpha)$, and the one of the back side diagram is $(K_*(\varphi)-\mathrm{id})\circ K_*(j)$. The naturality of the snake lemma homomorphism and the fact that ψ' acts trivially on K-theory yield that these two homomorphisms coincide. □

Assume now that $\alpha : \mathbb{Z}^2 \curvearrowright A$ is an action on a Kirchberg algebra. We follow [13] and say that α is *locally KK-trivial* if $KK(\alpha_1) = KK(\alpha_2) = 1_A$. Moreover, we call two actions $\alpha, \beta : \mathbb{Z}^2 \curvearrowright A$ *KK-trivially cocycle conjugate* if there exists an α-cocyle u, that is, a map $u : \mathbb{Z}^2 \to \mathcal{U}(\mathcal{M}(A))$ into the unitary group of the multiplier algebra of A satisfying $u_g\alpha_g(u_h) = u_{gh}$ for all $g, h \in \mathbb{Z}^2$, and an automorphism $\mu \in \mathrm{Aut}(A)$ with $KK(\mu) = 1_A$ such that $\mathrm{Ad}(u_g)\circ\alpha_g = \mu\circ\beta_g\circ\mu^{-1}$ for all $g \in \mathbb{Z}^2$.

Given a locally KK-trivial action α on a Kirchberg algebra A, Izumi and Matui [13] associate an element $\Phi(\alpha) \in KK(A, SA)$ as follows. Since $KK(\alpha_1) = 1_A \in KK(A, A)$, [19, Theorem 4.1.1] yields a homotopy $\beta_t \in \mathrm{Aut}(A \otimes \mathcal{K})$ between $\beta_0 = \alpha_1 \otimes \mathrm{id}_{\mathcal{K}}$ and $\beta_1 = \mathrm{id}_{A\otimes\mathcal{K}}$. As above, we define the automorphism $\varphi \in \mathrm{Aut}(A \otimes \mathcal{K} \otimes \mathcal{C}(\mathbb{T}))$ by

$$\varphi(f)(\exp(2\pi i t)) = \left(\beta_t \circ (\alpha_2 \otimes \mathrm{id}_{\mathcal{K}}) \circ \beta_t^{-1}\right)(f(\exp(2\pi i t)))$$

for $f \in A \otimes \mathcal{K} \otimes \mathcal{C}(\mathbb{T})$ and $t \in [0, 1]$. Denote by $j : A \otimes \mathcal{K} \hookrightarrow A \otimes \mathcal{K} \otimes \mathcal{C}(\mathbb{T})$ the canonical embedding. Using stability of the KK-bifunctor and the fact that $KK(\alpha_2) = 1_A$, we obtain that

$$\Phi(\alpha) := KK(\varphi \circ j) - KK(j) \in KK(A, SA) \subset KK(A, A \otimes \mathcal{C}(\mathbb{T})).$$

If

$$\gamma_* : KK(A, SA) \to \mathrm{Hom}(K_*(A), K_{*+1}(A))$$

denotes the natural homomorphism, then Izumi's and Matui's result is given as follows.

Theorem 4.2 (cf. [13]). *Let A be a unital Kirchberg algebra. The assignment $\alpha \mapsto \Phi(\alpha)$ induces a well-defined bijection between the following two sets:*

(i) *KK-trivial cocycle conjugacy classes of locally KK-trivial outer $\mathbb{Z}^2$-actions on A;*

(ii) $\{x \in KK(A, SA) \ : \ \gamma_0(x)([1]) = 0 \in K_1(A)\}$.

If A is a stable Kirchberg algebra, then the statement remains true when we take $KK(A, SA)$ as a classifying invariant.

By the definition of γ_*,

$$\gamma_*(\Phi(\alpha)) = K_*(\varphi\circ j)-K_*(j) = (K_*(\varphi)-\mathrm{id})\circ K_*(j) \in \mathrm{Hom}(K_*(A), K_{*+1}(A))$$

for any locally KK-trivial $\mathbb{Z}^2$-action α. Hence, $\gamma_*(\Phi(\alpha))$ coincides with the obstruction homomorphism associated with $(A\otimes\mathcal{K}, \alpha\otimes\mathrm{id}_{\mathcal{K}}, \mathbb{Z}^2)$ by Proposition 4.1. Using stability of K-theory and Proposition 3.2, we conclude that $\gamma_*(\Phi(\alpha)) = d_*(\alpha)$. Combining this observation with Theorem 4.2, we draw the following consequence.

Corollary 4.3. *Let A be a unital Kirchberg algebra satisfying the UCT. Let $\eta_* : K_*(A) \to K_{*+1}(A)$ be a homomorphisms with $\eta_0([1]) = 0$. Then there is a locally KK-trivial $\mathbb{Z}^2$-action α on A such that $d_*(\alpha) = \eta_*$. Moreover, $K_*(A \rtimes_\alpha \mathbb{Z}^2)$ fits into a six-term exact sequence*

$$\begin{array}{ccccc} K_1(A) \oplus K_0(A) & \xrightarrow{\eta_1 \oplus 0} & K_0(A) \oplus K_1(A) & \longrightarrow & K_0(A \rtimes_\alpha \mathbb{Z}^2) \\ \uparrow & & & & \downarrow \\ K_1(A \rtimes_\alpha \mathbb{Z}^2) & \longleftarrow & K_1(A) \oplus K_0(A) & \xleftarrow[\eta_0 \oplus 0]{} & K_0(A) \oplus K_1(A) \end{array}$$

If A is a stable Kirchberg algebra satisfying the UCT, then the statement remains true if the condition on the class of the unit is removed.

Proof. Since A satisfies the UCT, we find some element $x \in KK(A, SA)$ satisfying $\gamma_*(x) = \eta_*$. Observe that if A is unital, then the condition $\gamma_0(x)([1]) = 0$ is satisfied by assumption. Theorem 4.2 yields a locally KK-trivial action α with $\Phi(\alpha) = x$, and hence

$$\eta_* = \gamma_*(x) = \gamma_*(\Phi(\alpha)) = d_*(\alpha).$$

As $\alpha_1 \otimes \mathrm{id}_{\mathcal{K}}$ is homotopic to $\mathrm{id}_{A \otimes \mathcal{K}}$, we have that $K_i(A \rtimes_{\alpha_1} \mathbb{Z}) \cong K_0(A) \oplus K_1(A)$ for $i = 0, 1$. The claim now follows from the Pimsner–Voiculescu sequence for $(A \rtimes_{\alpha_1} \mathbb{Z}, \check{\alpha}_2, \mathbb{Z})$. □

5. Pointwise inner $\mathbb{Z}^2$-actions with non-trivial obstruction homomorphisms

Definition 5.1. Let A be a unital C*-algebra and $n \in \mathbb{N}$. We say that an action $\alpha : \mathbb{Z}^n \curvearrowright A$ is *pointwise inner* if α_g is an inner automorphisms for all $g \in \mathbb{Z}^n$. If $n = 2$, $\alpha_1 = \mathrm{Ad}(v)$, and $\alpha_2 = \mathrm{Ad}(w)$ for unitaries $v, w \in \mathcal{U}(A)$, then we call $u(\alpha) := v^* w^* v w \in \mathcal{U}(A)$ the *commutator* associated with α.

It is easy to check that $u(\alpha) \in \mathcal{Z}(A)$ and that $u(\alpha)$ does not depend on the choice of the implementing unitaries. Here and in the following, $\mathcal{Z}(A)$ denotes the center of A.

Next, we give a description of the obstruction homomorphism associated with a pointwise inner $\mathbb{Z}^2$-action.

Corollary 5.2. *Let A be a unital* C**-algebra and $\alpha : \mathbb{Z}^2 \curvearrowright A$ a pointwise inner action. Let $n \geq 1$, $x \in \mathcal{U}_n(A)$, and $p \in \mathcal{P}_n(A)$. Then*

$$d_0(\alpha)([p]) = \kappa(p, u(\alpha)^{(n)}) \quad \text{and} \quad d_1(\alpha)([x]) = \kappa(u(\alpha)^{(n)}, x).$$

Proof. One computes that

$$w^* \alpha_2(v)^* \alpha_1(w) v = w^* w v^* w^* v w v^* v = v^* w^* v w = u(\alpha).$$

As the automorphisms $\alpha_1 = \mathrm{Ad}(v)$ and $\alpha_2 = \mathrm{Ad}(w)$ satisfy

$$\alpha_1^{(n)}(p) = v^{(n)} p v^{(n)*} \quad \text{and} \quad \alpha_2^{(n)}(p) = w^{(n)} p w^{(n)*},$$

Theorem 3.3 implies that

$$d_0(\alpha)([p]) = \kappa(p, u(\alpha)^{(n)}).$$

The proof for $d_1(\alpha)([x])$ follows similarly from Theorem 3.5. □

Consequently, $d_*(\alpha)$ can only be non-trivial if the unitary $u(\alpha)$ has full spectrum. Otherwise, $u(\alpha)$ is connected to 1 by unitaries in $\mathrm{C}^*(u(\alpha)) \subseteq \mathcal{Z}(A)$. In fact, we have the following result.

Proposition 5.3. *Let A be a unital C^*-algebra and $\alpha_1, \dots, \alpha_n$ pairwise commuting inner automorphisms on A. Let $\alpha : \mathbb{Z}^n \curvearrowright A$ denote the induced pointwise inner action. Assume that for $i, j \in \{1, \dots, n\}$, the commutator associated with the $\mathbb{Z}^2$-action generated by α_i and α_j is homotopic to 1 in $\mathcal{U}(\mathcal{Z}(A))$. Then $K_*(A \rtimes_\alpha \mathbb{Z}^n) \cong K_*(A \otimes \mathcal{C}(\mathbb{T}^n))$.*

Proof. The proof goes by induction over $n \in \mathbb{N}$. For a single automorphism, this is trivial since $A \rtimes_{\mathrm{Ad}(v)} \mathbb{Z} \cong A \otimes \mathcal{C}(\mathbb{T})$. Assume now that the statement is true for $n-1$. Denote by $\check{\alpha}$ the $\mathbb{Z}^{n-1}$-action on $A \rtimes_{\alpha_1} \mathbb{Z}$ induced by $\check{\alpha}_2, \dots, \check{\alpha}_n$. For $i, j \in \{1, \dots, n\}$, let $v(i, j)$ denote the commutator associated with the $\mathbb{Z}^2$-action generated by α_i and α_j. As inner automorphisms fix the center pointwise, it holds that $v(i, j) \in \mathcal{Z}(A) \subseteq \mathcal{Z}(A \rtimes_{\alpha_1} \mathbb{Z})$. Hence, $\alpha_2, \dots, \alpha_n$ give rise to a pointwise inner action $\alpha' : \mathbb{Z}^{n-1} \curvearrowright A \rtimes_{\alpha_1} \mathbb{Z}$. Observe that α' is as in the statement, so that we can apply the induction hypothesis to it. For $i = 1, \dots, n$, we find homotopies $w_{t,i} \in \mathcal{U}(\mathcal{Z}(A))$, $t \in [0, 1]$, with $w_{0,i} = v(i, 1)$ and $w_{1,i} = 1$. Since these homotopies lie in the centre of $A \rtimes_{\alpha_1} \mathbb{Z}$, we can define automorphisms

$$\varphi_{t,i} : A \rtimes_{\alpha_1} \mathbb{Z} \xrightarrow{\cong} A \rtimes_{\alpha_1} \mathbb{Z}, \quad \varphi_{t,i}(a) = \alpha_i(a),\ \varphi_{t,i}(u) = w_{t,i} u.$$

For all $s, t \in [0, 1]$ and $i, j = 2, \dots, n$, one computes that

$$\varphi_{s,i} \circ \varphi_{t,j}(u) = w_{t,j} w_{s,i} u = w_{s,i} w_{t,j} u = \varphi_{t,j} \circ \varphi_{s,i}(u).$$

Thus, the $\varphi_{t,i}$ define a homotopy between the $\mathbb{Z}^n$-actions $\check{\alpha}$ and α'. In particular, the corresponding crossed products have isomorphic K-theory. By the induction hypothesis, we therefore obtain that

$$\begin{aligned} K_*(A \rtimes_\alpha \mathbb{Z}^n) &\cong K_*(A \rtimes_{\alpha_1} \mathbb{Z} \rtimes_{\check{\alpha}} \mathbb{Z}^{n-1}) \cong K_*(A \rtimes_{\alpha_1} \mathbb{Z} \rtimes_{\alpha'} \mathbb{Z}^{n-1}) \\ &\cong K_*((A \rtimes_{\alpha_1} \mathbb{Z}) \otimes \mathcal{C}(\mathbb{T}^{n-1})) \cong K_*(A \otimes \mathcal{C}(\mathbb{T}^n)). \end{aligned}$$ □

The next result shows that there are certain restrictions on obstruction homomorphisms associated with pointwise inner $\mathbb{Z}^2$-actions.

Proposition 5.4. *Let A be a unital* C*-*algebra and* $\alpha : \mathbb{Z}^2 \curvearrowright A$ *a pointwise inner action. Then* $d_{*+1}(\alpha) \circ d_*(\alpha) = 0$.

Proof. Using the canonical isomorphism

$$K_*(\mathcal{C}(\mathbb{T}) \oplus \mathcal{C}(\mathbb{T})) \cong K_*(\mathcal{C}(\mathbb{T})) \oplus K_*(\mathcal{C}(\mathbb{T})),$$

we compute

$$\kappa(z \oplus z, z \oplus 1) = \kappa(z, z) \oplus \kappa(z, 1) = 0 \ \in K_0(\mathcal{C}(\mathbb{T})) \oplus K_0(\mathcal{C}(\mathbb{T})).$$

Given a projection $p \in \mathcal{P}_n(A)$, there exists a unital $*$-homomorphism $\varphi : \mathcal{C}(\mathbb{T}) \oplus \mathcal{C}(\mathbb{T}) \to M_n(A)$ satisfying $\varphi(z \oplus z) = u(\alpha)^{(n)}$ and $\varphi(1 \oplus 0) = p$. By Corollary 5.2, we get that

$$\begin{aligned} d_1(\alpha)(\kappa(p, u(\alpha)^{(n)})) &= \kappa(u(\alpha)^{(n)}, pu(\alpha)^{(n)}p + 1_n - p) \\ &= K_0(\varphi)(\kappa(z \oplus z, z \oplus 1)) \\ &= 0. \end{aligned}$$

This shows that $d_1(\alpha) \circ d_0(\alpha) = 0$. Applying this to $(A \otimes \mathcal{C}(\mathbb{T}), \alpha \otimes \mathrm{id}, \mathbb{Z}^2)$, we conclude that the obstruction homomorphism for $(SA, S\alpha, \mathbb{Z}^2)$ also has this property. Bott periodicity thus shows that $d_0(\alpha) \circ d_1(\alpha) = 0$. □

5.1. A natural action on the group C*-algebra of the discrete Heisenberg group.

Recall the discrete Heisenberg group

$$H_3 := \langle r, s \ : \ rsr^{-1}s^{-1} \text{ is central} \rangle$$

and its associated (full) group C*-algebra

$$\mathrm{C}^*(H_3) := \mathrm{C}^*(u, v \text{ unitaries } : \ uvu^*v^* \text{ is central}).$$

Consider the pointwise inner $\mathbb{Z}^2$-action α on $\mathrm{C}^*(H_3)$ given by $\alpha_1 = \mathrm{Ad}(u)$ and $\alpha_2 = \mathrm{Ad}(v)$. The associated C*-dynamical system $(\mathrm{C}^*(H_3), \alpha, \mathbb{Z}^2)$ is universal in the following sense. Let B be a unital C*-algebra and $\beta : \mathbb{Z}^2 \curvearrowright B$ a pointwise inner action. If $\beta_1 = \mathrm{Ad}(x)$ and $\beta_2 = \mathrm{Ad}(y)$, then there is a unital and equivariant $*$-homomorphism $\varphi : (\mathrm{C}^*(H_3), \alpha, \mathbb{Z}^2) \to (B, \beta, \mathbb{Z}^2)$ satisfying $\varphi(u) = x$ and $\varphi(v) = y$.

The Heisenberg group also admits the following description as a semidirect product

$$H_3 = \mathbb{Z}^2 \rtimes_{\tilde{\sigma}} \mathbb{Z}, \quad \text{with } \tilde{\sigma}(e_1) = e_1, \tilde{\sigma}(e_2) = e_1 + e_2.$$

Hence, the $*$-automorphism $\sigma \in \mathrm{Aut}(\mathcal{C}(\mathbb{T}^2))$ satisfying $\sigma(z_1) = z_1$ and $\sigma(z_2) = z_1 z_2$ gives rise to an isomorphism

$$\mathrm{C}^*(H_3) \xrightarrow{\cong} \mathcal{C}(\mathbb{T}^2) \rtimes_\sigma \mathbb{Z}, \quad u \mapsto u, \ v \mapsto z_2,$$

where $u \in \mathcal{C}(\mathbb{T}^2) \rtimes_\sigma \mathbb{Z}$ denotes the canonical unitary implementing σ. Using this identification, the action α is given by $\alpha_1 = \mathrm{Ad}(u)$ and $\alpha_2 = \mathrm{Ad}(z_2)$, and the commutator associated with α satisfies

$$u(\alpha) = u^* z_2^* u z_2 = z_1 z_2^* z_2 = z_1 \in \mathcal{C}(\mathbb{T}^2) \rtimes_\sigma \mathbb{Z}.$$

The Bott projection $e := e(z_1, z_2) \in M_2(\mathcal{C}(\mathbb{T}^2))$ is unitarily equivalent to $\sigma^{(2)}(e)$, see [1, Section 1]. So, we find a unitary $x \in \mathcal{U}_2(\mathcal{C}(\mathbb{T}^2))$ with $\sigma^{(2)}(e) = xex^*$, which in turn gives rise to a Bott element $\kappa(e, x^* u^{(2)}) \in K_1(\mathrm{C}^*(H_3))$.

Let us recall the K-theory of $\mathrm{C}^*(H_3)$, which was determined in [1, Proposition 1.4(a)]. For the reader's convenience, we also provide a short proof.

Proposition 5.5. *The K-theory of $\mathrm{C}^*(H_3)$ is given by*

$$K_0(\mathrm{C}^*(H_3)) = \mathbb{Z}^3\big[\kappa(z_1, z_2), \kappa(z_1, u), [1]\big],$$
$$K_1(\mathrm{C}^*(H_3)) = \mathbb{Z}^3\big[[z_2], [u], \kappa(e, x^* u^{(2)})\big].$$

Proof. As $K_0(\sigma) = \mathrm{id}$, the Pimsner–Voiculescu sequence for σ is of the form

$$\begin{array}{ccccc} K_0(\mathcal{C}(\mathbb{T}^2)) & \xrightarrow{\;0\;} & K_0(\mathcal{C}(\mathbb{T}^2)) & \longrightarrow & K_0(\mathcal{C}(\mathbb{T}^2) \rtimes_\sigma \mathbb{Z}) \\ {\scriptstyle \rho_1}\uparrow & & & & \downarrow{\scriptstyle \rho_0} \\ K_1(\mathcal{C}(\mathbb{T}^2) \rtimes_\sigma \mathbb{Z}) & \longleftarrow & K_1(\mathcal{C}(\mathbb{T}^2)) & \xleftarrow[K_1(\sigma)-\mathrm{id}]{} & K_1(\mathcal{C}(\mathbb{T}^2)) \end{array}$$

Moreover, $\ker(K_1(\sigma) - \mathrm{id}) = \mathbb{Z}[z_1]$. This yields

$$K_0(\mathrm{C}^*(H_3)) \cong K_1(\mathrm{C}^*(H_3)) \cong \mathbb{Z}^3.$$

We also conclude that the natural inclusions $K_0(\mathcal{C}(\mathbb{T}^2)) \hookrightarrow K_0(\mathcal{C}(\mathbb{T}^2) \rtimes_\sigma \mathbb{Z})$ and $\mathbb{Z}[z_2] \hookrightarrow K_1(\mathcal{C}(\mathbb{T}^2) \rtimes_\sigma \mathbb{Z})$ are split-injective. Proposition 2.5 implies that $\rho_1([u]) = [1]$ and $\rho_1(\kappa(e, x^* u^{(2)})) = [e]$. This shows the assertion for $K_1(\mathrm{C}^*(H_3))$. Analogously, the claim for $K_0(\mathrm{C}^*(H_3))$ follows since $\rho_0(\kappa(z_1, u)) = [z_1]$ by Proposition 2.8. □

Theorem 5.6. *The obstruction homomorphism associated with the natural action $\alpha : \mathbb{Z}^2 \curvearrowright \mathrm{C}^*(H_3)$ is non-trivial. The K-theory of the crossed product $\mathrm{C}^*(H_3) \rtimes_\alpha \mathbb{Z}^2$ satisfies*

$$K_0(\mathrm{C}^*(H_3) \rtimes_\alpha \mathbb{Z}^2) \cong K_1(\mathrm{C}^*(H_3) \rtimes_\alpha \mathbb{Z}^2) \cong \mathbb{Z}^{10}.$$

In particular, $\mathrm{C}^*(H_3) \rtimes_\alpha \mathbb{Z}^2$ and $\mathrm{C}^*(H_3) \otimes \mathcal{C}(\mathbb{T}^2)$ are not isomorphic in K-theory.

Proof. By applying Corollary 5.2 to $d_1(\alpha) : K_1(\mathrm{C}^*(H_3)) \to K_0(\mathrm{C}^*(H_3))$, we obtain

$$d_1(\alpha)([z_2]) = \kappa(u(\alpha), z_2) = \kappa(z_1, z_2) \quad \text{and} \quad d_1(\alpha)([u])) = \kappa(z_1, u).$$

The discussion in Section 2 shows that the canonical trace on $\mathrm{C}^*(H_3)$ prevents $[1] \in K_0(\mathrm{C}^*(H_3))$ from being a Bott element associated with two exactly commuting unitaries. Since α is pointwise inner, every element in the image of $d_1(\alpha)$ is representable in such a way. Hence,

$$\mathrm{im}(d_1(\alpha)) = \mathbb{Z}^2 [\kappa(z_1, z_2),\ \kappa(z_1, u)] \subset K_0(\mathrm{C}^*(H_3)),$$

which sits inside $K_0(\mathrm{C}^*(H_3))$ as a direct summand.

It holds that $d_0(\alpha)([1]) = 0$, and by Proposition 5.4, we also have that $d_0(\alpha) \circ d_1(\alpha) = 0$. It now follows from Proposition 5.5 that $d_0(\alpha) = 0$.

If $G_0 := K_0(\mathrm{C}^*(H_3))$ and $G_1 := K_1(\mathrm{C}^*(H_3))$, then the Pimsner-Voiculescu sequence for $(\mathrm{C}^*(H_3) \rtimes_{\alpha_1} \mathbb{Z}, \check{\alpha}_2, \mathbb{Z})$ is of the form

$$\begin{array}{ccccc} G_1 \oplus G_0 & \xrightarrow{d_1(\alpha)\oplus 0} & G_0 \oplus G_1 & \longrightarrow & K_0(\mathrm{C}^*(H_3) \rtimes_\alpha \mathbb{Z}^2) \\ \uparrow & & & & \downarrow \\ K_1(\mathrm{C}^*(H_3) \rtimes_\alpha \mathbb{Z}^2) & \longleftarrow & G_1 \oplus G_0 & \xleftarrow{0} & G_0 \oplus G_1 \end{array} \tag{5.1}$$

The result now follows by splitting up this six-term exact sequence into two extension, and then comparing the ranks of the occurring abelian groups. □

We also find pointwise inner $\mathbb{Z}^2$-actions on $\mathrm{C}^*(H_3)$ whose corresponding crossed products have torsion in K-theory.

Corollary 5.7. *Let $m, n \in \mathbb{N}$ and denote by $\bar{\alpha}$ the pointwise inner $\mathbb{Z}^2$-action on $\mathrm{C}^*(H_3)$ generated by α_1^m and α_2^n. Then*

$$K_0(\mathrm{C}^*(H_3) \rtimes_{\bar{\alpha}} \mathbb{Z}^2) \cong \mathbb{Z}^{10} \oplus \mathbb{Z}/mn\mathbb{Z} \oplus \mathbb{Z}/mn\mathbb{Z},$$
$$K_1(\mathrm{C}^*(H_3) \rtimes_{\bar{\alpha}} \mathbb{Z}^2) \cong \mathbb{Z}^{10}.$$

Proof. A similar proof as in Theorem 5.6 shows that the K-theory of $\mathrm{C}^*(H_3) \rtimes_{\bar{\alpha}} \mathbb{Z}^2$ fits into the exact sequence (5.1) with $d_1(\alpha)$ replaced by $d_1(\bar{\alpha}) = mn \cdot d_1(\alpha)$. □

5.2. Certain pointwise inner actions on amalgamated free product C*-algebras.

Throughout this section, A denotes a unital, separable C*-algebra and $\alpha : \mathbb{Z}^2 \curvearrowright A$ a pointwise inner action whose associated commutator $u(\alpha)$ has full spectrum. Let $u, v \in A$ be unitaries satisfying $\alpha_1 = \mathrm{Ad}(u)$ and $\alpha_2 = \mathrm{Ad}(v)$.

Let B be a unital, separable C*-algebra whose K-groups both do not vanish. Also assume that there exists a central unitary $w \in \mathcal{U}(\mathcal{Z}(B))$, some $n \in \mathbb{N}$, and a projection $p \in \mathcal{P}_n(B)$ such that

$$\kappa(p, w^{(n)}) \neq k[w] \in K_1(B) \quad \text{for all } k \in \mathbb{Z}. \tag{5.2}$$

Observe that w must have full spectrum.

Consider the two injective $*$-homomorphisms

$$i_1 : \mathcal{C}(\mathbb{T}) \hookrightarrow A,\ i_1(z) := u(\alpha) \quad \text{and} \quad i_2 : \mathcal{C}(\mathbb{T}) \hookrightarrow B,\ i_2(z) := w,$$

and form the amalgamated free product $C := A *_{\mathcal{C}(\mathbb{T})} B$, see [5, Section 10.11.11]. There are natural unital $*$-homomorphisms $j_1 : A \to C$ and $j_2 : B \to C$, which are also injective by [4, Theorem 3.1]. Since $u(\alpha) = w$ is central in C, the action on A extends to a pointwise inner $\mathbb{Z}^2$-action on C, which we also denote by α. The associated obstruction homomorphism $d_*(\alpha) : K_*(C) \to K_{*+1}(C)$ satisfies

$$d_0(\alpha)([p]) = \kappa(p, u(\alpha)^{(n)}) = \kappa(p, w^{(n)}) \in K_1(C).$$

Lemma 5.8. *We have that* $d_0(\alpha)([p]) = \kappa(p, w^{(n)}) \neq 0 \in K_1(C)$.

Proof. By [22, Theorem 6.3], there exists a six-term exact sequence

$$\begin{array}{ccccc} K_0(\mathcal{C}(\mathbb{T})) & \xrightarrow{K_0(i_1)\oplus K_0(i_2)} & K_0(A)\oplus K_0(B) & \xrightarrow{K_0(j_1)-K_0(j_2)} & K_0(C) \\ \uparrow & & & & \downarrow \\ K_1(C) & \xleftarrow{K_1(j_1)-K_1(j_2)} & K_1(A)\oplus K_1(B) & \xleftarrow{K_1(i_1)\oplus K_1(i_2)} & K_1(\mathcal{C}(\mathbb{T})) \end{array} \tag{5.3}$$

Since

$$(K_1(j_1) - K_1(j_2))(0 \oplus -\kappa(p, w^{(n)})) = \kappa(p, w^{(n)}),$$

it suffices to check that $0 \oplus -\kappa(p, w^{(n)}) \notin \operatorname{im}(K_1(i_1) \oplus K_1(i_2))$. We have that $[u(\alpha)] = 0 \in K_1(A)$, and therefore

$$\operatorname{im}(K_1(i_1) \oplus K_1(i_2)) = 0 \oplus \mathbb{Z}[w].$$

By assumption, $\kappa(p, w^{(n)}) \neq k[w]$ for all $k \in \mathbb{Z}$, and the proof is complete. □

As the conditions on A and B are very mild, Lemma 5.8 applies in many situations. We would like to discuss one example, which is of particular interest. Take $A := C^*(H_3)$ and equip it with the natural action α defined in the last subsection. Let $B := \mathcal{C}(\mathbb{T}) \oplus \mathcal{C}(\mathbb{T})$ and set $w := z \oplus z$ and $p := 1 \oplus 0$. Observe that these elements satisfy (5.2). Define the amalgamated free product $C_1 := A *_{\mathcal{C}(\mathbb{T})} B$ and consider the pointwise inner action on C_1 induced by $\alpha : \mathbb{Z}^2 \curvearrowright A$, which we also denote by α. By Lemma 5.8, $\kappa(p, w) \neq 0 \in K_1(C_1)$.

The C*-dynamical system $(C_1, \alpha, \mathbb{Z}^2)$ is universal for K_1-obstructions for pointwise inner $\mathbb{Z}^2$-actions in the following sense. For every unital C*-algebra D, any pointwise inner action $\gamma : \mathbb{Z}^2 \curvearrowright D$ with $\gamma_1 = \operatorname{Ad}(\tilde{u})$ and $\gamma_2 = \operatorname{Ad}(\tilde{v})$, and every projection $\tilde{p} \in D$, there is a unital and equivariant $*$-homomorphism

$$\varphi : (C_1, \alpha, \mathbb{Z}^2) \to (D, \gamma, \mathbb{Z}^2), \quad u \mapsto \tilde{u},\ v \mapsto \tilde{v},\ p \mapsto \tilde{p}.$$

By naturality of the obstruction homomorphism (Proposition 3.2),

$$K_1(\varphi)(d_0(\alpha)([p])) = d_0(\gamma)([\tilde{p}]).$$

Therefore, we can think of $d_0(\alpha)([p]) = \kappa(p, u(\alpha)) \in K_1(C_1)$ as the universal K_1-obstruction for pointwise inner $\mathbb{Z}^2$-actions.

The universal property of $(C_1, \alpha, \mathbb{Z}^2)$ also yields that $\kappa(p, u(\alpha)) \in K_1(C_1)$ has infinite order and induces a split-injection $\mathbb{Z}\,[\kappa(p, u(\alpha))] \hookrightarrow K_1(C_1)$. To see this, consider the C*-dynamical system $(A \otimes \mathcal{O}^\infty, \alpha \otimes \mathrm{id}, \mathbb{Z}^2)$, where $\mathcal{O}^\infty$ is the (unique) UCT Kirchberg algebra with $K_0(\mathcal{O}^\infty) = 0$ and $K_1(\mathcal{O}^\infty) \cong \mathbb{Z}$. The proof of Theorem 5.6 and the fact that $A \otimes \mathcal{O}^\infty$ is properly infinite show that there exists a projection $q \in A \otimes \mathcal{O}^\infty$ such that the cyclic subgroup generated by

$$0 \neq d_0(\alpha \otimes \mathrm{id})([q]) \in K_1(A \otimes \mathcal{O}^\infty)$$

sits inside $K_1(A \otimes \mathcal{O}^\infty) \cong \mathbb{Z}^3$ as a direct summand. Hence, the universal property of $(C_1, \alpha, \mathbb{Z}^2)$ applied to $(A \otimes \mathcal{O}^\infty, \alpha \otimes \mathrm{id}, \mathbb{Z}^2)$ and q yields the desired result.

Proposition 5.9. *The canonical embedding $j_1 : A \hookrightarrow C_1$ is split-injective in K-theory and induces the following decompositions:*

$$K_0(C_1) \cong K_0(A) \oplus \mathbb{Z}[p] \quad \text{and} \quad K_1(C_1) \cong K_1(A) \oplus \mathbb{Z}[\kappa(p, u(\alpha))].$$

In particular, $K_0(C_1) \cong K_1(C_1) \cong \mathbb{Z}^4$.

Proof. Direct computation shows that

$$K_*(i_1) \oplus K_*(i_2) : K_*(C(\mathbb{T})) \to K_*(A) \oplus K_*(B)$$

is split-injective. Hence the six-term exact sequence (5.3) associated with the amalgamated free product $C_1 = A *_{C(\mathbb{T})} B$ reduces to a split-extension

$$0 \longrightarrow K_*(C(\mathbb{T})) \xrightarrow{K_*(i_1) \oplus K_*(i_2)} K_*(A) \oplus K_*(B) \xrightarrow{K_*(j_1) - K_*(j_2)} K_*(C_1) \longrightarrow 0. \tag{5.4}$$

Consequently, $K_*(C_1)$ is torsion-free. As $K_0(A) \cong K_1(A) \cong \mathbb{Z}^3$ by Proposition 5.5, we conclude that $K_0(C_1) \cong K_1(C_1) \cong \mathbb{Z}^4$.

The universal property of the amalgamated free product yields a homomorphism $\varphi : C_1 \to A$ satisfying $\varphi \circ j_1 = \mathrm{id}_A$, $(\varphi \circ j_2)(p) = 1$, and $(\varphi \circ j_2)(w) = u(\alpha)$. Obviously, φ is surjective with splitting $j_1 : A \hookrightarrow C_1$. This shows that $K_*(j_1)$ is split-injective. Moreover, we get that $[p] \in K_0(C_1)$ has infinite order and induces a split-injection $\mathbb{Z}[p] \hookrightarrow K_0(C_1)$. Since we already know that the analogous statement for $\kappa(p, u(\alpha)) \in K_1(C_1)$ is true as well, it remains to show that $[p]$ and $\kappa(p, u(\alpha))$ both do not lie in $K_*(j_1)(K_*(A)) \subseteq K_*(C_1)$.

Suppose that there is some $g \in K_0(A)$ with $K_0(j_1)(g) = [p]$. As

$$(K_0(j_1) - K_0(j_2))(0 \oplus -[1 \oplus 0]) = [p],$$

exactness of (5.4) yields the existence of some $k \in \mathbb{Z}$ satisfying

$$k([1] \oplus [1 \oplus 1]) + g \oplus 0 = 0 \oplus -[1 \oplus 0].$$

This is a contradiction, and thus $[p] \notin K_0(j_1)(K_0(A))$. A similar argument yields $\kappa(p, u(\alpha)) \notin K_1(j_1)(K_1(A))$ using that

$$(K_1(j_1) - K_1(j_2))(0 \oplus -[z \oplus 0]) = \kappa(p, u(\alpha)) \in K_1(C_1). \qquad \square$$

Theorem 5.10. *The obstruction homomorphism associated with* $\alpha : \mathbb{Z}^2 \curvearrowright C_1$ *is non-trivial. Moreover, the* K*-theory of* $C_1 \rtimes_\alpha \mathbb{Z}^2$ *satisfies*

$$K_0(C_1 \rtimes_\alpha \mathbb{Z}^2) \cong K_1(C_1 \rtimes_\alpha \mathbb{Z}^2) \cong \mathbb{Z}^{13}.$$

In particular, $K_*(C_1 \rtimes_\alpha \mathbb{Z}^2) \ncong K_*(C_1 \otimes \mathcal{C}(\mathbb{T}^2))$.

Proof. We have that

$$d_0(\alpha)([p]) = \kappa(p, u(\alpha)) \quad \text{and} \quad d_1(\alpha)(\kappa(p, u(\alpha))) = 0,$$

where the second equality is a consequence of Proposition 5.4. As $K_*(j_1) : K_*(A) \to K_*(C_1)$ is split-injective by Proposition 5.9, naturality of the obstruction homomorphism yields that on $K_*(A) \subset K_*(C_1)$, $d_*(\alpha) : K_*(C_1) \to K_{*+1}(C_1)$ coincides with the obstruction homomorphism associated with $(A, \alpha, \mathbb{Z}^2)$. The proof of Theorem 5.6 therefore yields that

$$\operatorname{coker}(d_0(\alpha)) \cong \ker(d_0(\alpha)) \cong \mathbb{Z}^3 \quad \text{and} \quad \operatorname{coker}(d_1(\alpha)) \cong \ker(d_1(\alpha)) \cong \mathbb{Z}^2.$$

As in the proof of Theorem 5.6, the statement now follows from the Pimsner–Voiculescu sequence associated with $(C_1 \rtimes_{\alpha_1} \mathbb{Z}, \check{\alpha}_2, \mathbb{Z})$. $\square$

Let us present another instance of a C*-dynamical with non-trivial obstruction homomorphism arising from the above construction. Whereas $(C_1, \alpha, \mathbb{Z}^2)$ is interesting for its universal property, the next C*-dynamical system is minimal concerning the K-groups of the underlying C*-algebra.

Theorem 5.11. *There exists a unital, separable* C**-algebra* C *with* $K_0(C) \cong K_1(C) \cong \mathbb{Z}$*, which admits a pointwise inner* $\mathbb{Z}^2$*-action* α *that is pointwise homotopic to the trivial action inside* $\operatorname{Inn}(A)$ *and whose associated obstruction homomorphism is non-trivial. The* K*-theory of the associated crossed product is given by*

$$K_0(C \rtimes_\alpha \mathbb{Z}^2) \cong K_1(C \rtimes_\alpha \mathbb{Z}^2) \cong \mathbb{Z}^3.$$

In particular, $K_*(C \rtimes_\alpha \mathbb{Z}^2) \ncong K_*(C \otimes \mathcal{C}(\mathbb{T}^2))$.

Proof. Define $A := C^*(H_3) \otimes \mathcal{O}_2$ and note that this C*-algebra has trivial K-theory, see [7, Theorem 2.3]. Kirchberg's absorption theorem [15] implies that $A \cong A \otimes \mathcal{Z}$, where $\mathcal{Z}$ denotes the Jiang-Su algebra. Hence A is K_1-injective by [14, Corollary 2.10]. The unitaries $u \otimes 1$ and $v \otimes 1 \in A$ are therefore homotopic to $1 \in \mathcal{U}(A)$. By identifying $u(\alpha) \otimes 1 \in A$ with $z \oplus z \in \mathcal{C}(\mathbb{T}) \oplus \mathcal{C}(\mathbb{T})$, we can form the amalgamated free product

$$C := A *_{\mathcal{C}(\mathbb{T})} (\mathcal{C}(\mathbb{T}) \oplus \mathcal{C}(\mathbb{T})).$$

Consider the pointwise inner $\mathbb{Z}^2$-action α on C induced by $\mathrm{Ad}(u \otimes 1)$ and $\mathrm{Ad}(v \otimes 1)$, which is obviously pointwise homotopic to the trivial action inside $\mathrm{Inn}(A)$. A similar calculation as in the proof of Proposition 5.9 shows that $K_0(C) = \mathbb{Z}[p]$ and $K_1(C) = \mathbb{Z}\,[\kappa(p, u(\alpha))]$, where $p := 1 \oplus 0$. Moreover, $d_0(\alpha)([p]) = \kappa(p, u(\alpha))$ and $d_1(\alpha) = 0$. The result now follows from the Pimsner–Voiculescu sequence for $(C \rtimes_{\alpha_1} \mathbb{Z}, \check{\alpha}_2, \mathbb{Z})$. □

Next, we present an analogous construction yielding pointwise inner $\mathbb{Z}^2$-actions α on amalgamated free product C*-algebras with non-trivial obstruction homomorphisms $d_1(\alpha)$. Let B be a unital, separable C*-algebra whose K-groups both do not vanish. Assume further that there is a central unitary $w \in \mathcal{U}(\mathcal{Z}(B))$ with full spectrum and a unitary $x \in \mathcal{U}_n(B)$ such that

$$\kappa(w^{(n)}, x) \neq k[1] \in K_0(B) \quad \text{for all } k \in \mathbb{Z}. \tag{5.5}$$

The two injective $*$-homomorphisms

$$i_1 : \mathcal{C}(\mathbb{T}) \hookrightarrow A,\ i_1(z) := u(\alpha) \quad \text{and} \quad i_2 : \mathcal{C}(\mathbb{T}) \hookrightarrow B,\ i_2(z) := w,$$

give rise to an amalgamated free product $C := A *_{\mathcal{C}(\mathbb{T})} B$. Observe that $u(\alpha) = w$ is a central unitary in C, and hence α extends to a pointwise inner action on C, which we also denote by α. Then $d_1(\alpha) : K_1(C) \to K_0(C)$ satisfies

$$d_1(\alpha)([x]) = \kappa(u(\alpha)^{(n)}, x) = \kappa(w^{(n)}, x) \in K_0(C).$$

Lemma 5.12. *It holds that* $d_1(\alpha)([x]) = \kappa(w^{(n)}, x) \neq 0 \in K_0(C)$.

Proof. The proof is very similar to the one of Lemma 5.8. □

Remark 5.13. If $[1] \in K_0(A)$ has infinite order, then Lemma 5.12 remains true if we replace (5.5) by the condition that $\kappa(w^{(n)}, x) \neq 0 \in K_0(B)$.

There exists a C*-dynamical system $(C_0, \alpha, \mathbb{Z}^2)$ which is universal for K_0-obstructions for pointwise inner $\mathbb{Z}^2$-actions. To define it, let again $A := C^*(H_3)$ and equip it with the natural $\mathbb{Z}^2$-action α from the last subsection. Moreover, let $B := \mathcal{C}(\mathbb{T}^2)$ and set $w := z_1$ and $x := z_2$. As $\kappa(w, x) = \mathfrak{b} \in K_0(B)$, (5.5) is clearly satisfied. Form the amalgamated free product $C_0 := A *_{\mathcal{C}(\mathbb{T})} \mathcal{C}(\mathbb{T}^2)$, which carries the induced pointwise inner $\mathbb{Z}^2$-action α.

Universality of this system expresses in the following property. Given a unital C*-algebra D, a pointwise inner action $\gamma : \mathbb{Z}^2 \curvearrowright D$ with $\gamma_1 = \mathrm{Ad}(\tilde{u})$ and $\gamma_2 = \mathrm{Ad}(\tilde{v})$ and a unitary $\tilde{x} \in \mathcal{U}(D)$, there is a unital and equivariant $*$-homomorphism

$$\varphi : (C_0, \alpha, \mathbb{Z}^2) \to (D, \gamma, \mathbb{Z}^2), \quad u \mapsto \tilde{u},\ v \mapsto \tilde{v},\ x \mapsto \tilde{x}.$$

By the naturality of the obstruction homomorphism (Proposition 3.2),

$$K_0(\varphi)(d_1(\alpha)([x])) = d_1(\gamma)([\tilde{x}]).$$

In this way, $d_1(\alpha)([x]) = \kappa(u(\alpha), x) \in K_0(C_0)$ can be considered as the universal K_0-obstruction for pointwise inner $\mathbb{Z}^2$-actions.

Note that the proof of Theorem 5.6 shows that $\kappa(u(\alpha), x) \in K_0(C_0)$ has infinite order and that $\mathbb{Z}[\kappa(u(\alpha), x)] \hookrightarrow K_0(C_0)$ is split-injective.

Proposition 5.14. *The canonical embedding $j_1 : A \hookrightarrow C_0$ is split-injective in K-theory and induces the following decompositions:*

$$K_0(C_0) \cong K_0(A) \oplus \mathbb{Z}[\kappa(u(\alpha), x)] \quad \textit{and} \quad K_1(C_0) \cong K_1(A) \oplus \mathbb{Z}[x].$$

In particular, $K_0(C_0) \cong K_1(C_0) \cong \mathbb{Z}^4$.

Proof. The proof is similar to the one of Proposition 5.9. □

Theorem 5.15. *The obstruction homomorphism associated with $\alpha : \mathbb{Z}^2 \curvearrowright C_0$ is non-trivial. Moreover, the K-theory of $C_0 \rtimes_\alpha \mathbb{Z}^2$ satisfies*

$$K_0(C_0 \rtimes_\alpha \mathbb{Z}^2) \cong K_1(C_0 \rtimes_\alpha \mathbb{Z}^2) \cong \mathbb{Z}^{13}.$$

In particular, $K_*(C_0 \rtimes_\alpha \mathbb{Z}^2) \ncong K_*(C_0 \otimes \mathcal{C}(\mathbb{T}^2))$.

Proof. It holds that

$$d_1(\alpha)([x]) = \kappa(u(\alpha), x) \quad \text{and} \quad d_0(\alpha)(\kappa(u(\alpha), x)) = 0,$$

where the second equality is a consequence of Proposition 5.4. As $K_*(j_1) : K_*(A) \to K_*(C_0)$ is split-injective by Proposition 5.14, naturality of the obstruction homomorphism yields that on $K_*(A) \subset K_*(C_0)$, $d_*(\alpha) : K_*(C_0) \to K_{*+1}(C_0)$ coincides with the obstruction homomorphism associated with $(A, \alpha, \mathbb{Z}^2)$. The proof of Theorem 5.6 therefore yields

$$d_0(\alpha) = 0 \quad \text{and} \quad \ker(d_1(\alpha)) \cong \operatorname{coker}(d_1(\alpha)) \cong \mathbb{Z}.$$

Finally, we proceed as in the proof of Theorem 5.6, and consider the Pimsner–Voiculescu sequence for $(C_0 \rtimes_{\alpha_1} \mathbb{Z}, \check{\alpha}_2, \mathbb{Z})$. □

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Received 27 April, 2015

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