Spectral theory of von Neumann algebra valued differential operators over non-compact manifolds

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Abstract. We provide criteria for self-adjointness and τ-Fredholmness of first and second order differential operators acting on sections of infinite dimensional bundles, whose fibers are modules of finite type over a von Neumann algebra A endowed with a trace τ. We extend the Callias-type index to operators acting on sections of such bundles and show that this index is stable under compact perturbations.

1. Introduction

In this paper we extend some results about the spectral properties of first and second order differential operators on complete Riemannian manifolds (see [6,10,16,18,24, 27,29,30]) to operators acting on sections of certain infinite dimensional bundles, called A-Hilbert bundles of finite type, cf. Section 3.3. As a main example we consider lifts of operators acting on a finite dimensional vector bundle over a complete Riemannian manifold M to a Galois cover $\widetilde{M}$ of M.

Let A be a von Neumann algebra with a finite, faithful, normal trace τ. Let E^+ and E^- be A-Hilbert bundles of finite type over a complete Riemannian manifold M, cf. Section 3.3. In particular, this means that the fibers of $E^\pm$ are Hilbert spaces endowed with an action of A. We consider a first order differential operator $D\colon C_c^\infty(M, E^+) \to C_c^\infty(M, E^-)$ (here C_c^∞ denotes the space of smooth compactly supported sections). We also fix an A-linear bundle map $V : E^+ \to E^-$.

We give a criterion for self-adjointness of the generalized Schrödinger operator $H_V := D^*D + V$. Then we show that if there exists a constant $C > 0$ such that $V(x) > C$ for all x outside of a compact subset of M, then the *von Neumann spectral counting function* $N_\tau(\lambda; H_V)$ of H_V is finite for all $\lambda < C$. For operators acting on finite dimensional bundles this result is obtained in [1].

One of the motivations for our study is an attempt to extend Atiayh's L^2-index theory for covering manifolds, [3, 28], to coverings of non-compact manifolds. As a first application to index theory we develop in Section 7 a Callias-type index theory, [2, 5, 13, 15], for operators acting on A-Hilbert bundles over non-compact manifolds.

More specifically, given a bundle map $F : E^+ \to E^-$ we provide a criterion for τ-Fredholmness of the operator $D_F := D + F$. The τ-Fredholmness implies that the kernel of D_F and D_F^* have finite τ-dimension, where D_F^* is the formal adjoint of D_F. Hence, we can define the τ-index $\mathrm{ind}_\tau D_F$. We prove that this index is stable under compact perturbations of F.

Another possible application of our analysis, which will be discussed elsewhere, is to the equivariant index theory of operators acting on a non-compact manifold with a proper action of a Lie group G. A significant progress was achieved recently in developing the equivariant index theory of operators acting on sections of a finite dimensional bundle over such manifolds, [7–9, 19, 21–23, 26]. These results find applications in the study of geometric quantization, representations of reductive groups and other areas. This paper provides the analytic tools needed to extend these results to operators acting on A-Hilbert bundles.

The paper is organized as follows. In Section 2 we formulate our main results. In Section 3 we recall the basic properties of von Neumann algebras and A-Hilbert modules. In Sections 4 and 5 we prove criteria of self-adjointness for first and second order operators respectively. In Section 6 we prove criteria for finiteness of the spectral counting function $N_\tau(\lambda; H_V)$ and for Fredholmness of the operator $D_F := D + F$. In Section 7 we introduce the Callias-type τ-index and prove its stability.

2. The main results

In this section we formulate the main results of the paper.

2.1. An operator of order one. Let A be a von Neumann algebra with a finite, faithful, normal trace $\tau : A \to \mathbb{C}$. Let $E = E^+ \oplus E^-$ be a $\mathbb{Z}_2$-graded A-Hilbert bundle of finite type over a complete Riemannian manifold M, cf. Section 3.3. In particular, this means that the fibers of $E^\pm$ are Hilbert spaces endowed with an action of A. We denote the Riemannian metric on M by g^{TM}.

Consider a first order differential operator $D : C_c^\infty(M, E^+) \to C_c^\infty(M, E^-)$ (here C_c^∞ denotes the space of smooth compactly supported sections). We assume that the principal symbol $\sigma(D)$ of D is injective so that D is elliptic.

Assumption. Throughout the paper we assume that there exists a constant $c > 0$ such that

$$0 < \|\sigma(D)(x, \xi)\| \leq c\,|\xi|, \quad \text{for all } x \in M, \xi \in T_x^*M \setminus \{0\}. \qquad (2.1)$$

Here $|\xi|$ denotes the length of ξ defined by the Riemannian metric on M, $\sigma(D)(x, \xi) : E_x^+ \to E_x^-$ is the leading symbol of D, and $\|\sigma(D)(x, \xi)\|$ is its operator norm.

An interesting class of examples of operators satisfying (2.1) is given by Dirac-type operators on M paired with a connection on an A-Hilbert bundle, cf. [28, Section 7.4].

2.2. Self-adjointness of a first order operator. Let $d\mu$ be a smooth measure on M. We don't assume that $d\mu$ is the measure defined by the Riemannian metric g^{TM}. Let $L^2(M, E^\pm)$ denote the space of square-integrable sections of $E^\pm$. We also set $E := E^+ \oplus E^-$. Then

$$L^2(M, E) = L^2(M, E^+) \oplus L^2(M, E^-).$$

Let D^* denote the formal adjoint of D. Consider the operator

$$\mathcal{D} := \begin{pmatrix} 0 & D^* \\ D & 0 \end{pmatrix}. \qquad (2.2)$$

Then $\mathcal{D}$ is an unbounded operator on $L^2(M, E)$.

Our first result is the following extension of [18, Th. 1.17]:

Theorem 2.1. *Suppose that $D \colon C_c^\infty(M, E^+) \to C_c^\infty(M, E^-)$ satisfies (2.1) and the Riemannian metric g^{TM} is complete. Then $\mathcal{D}$ is essentially self-adjoint with initial domain $C_c^\infty(M, E) = C_c^\infty(M, E^+) \oplus C_c^\infty(M, E^-)$.*

The proof is given is Section 4.

2.3. A Schrödinger-type operator. Consider the Schrödinger-type operator

$$H_V = D^*D + V, \qquad (2.3)$$

where $V(x) : E_x^+ \to E_x^+$ is an A-linear self-adjoint bundle endomorphism.

We view H_V as an unbounded operator on $L^2(M, E^+)$ with initial domain $C_c^\infty(M, E^+)$ and give a sufficient condition for self-adjointness of this operator. For simplicity we assume that the potential V is a measurable section which belongs to L^∞_{loc}.[1]

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[1] Much more general potentials were considered in [10]. It would be interesting to extend the results we present in this paper to the type of potentials considered in [10].

Our next result is the following extension of [10, Th. 2.7] to A-Hilbert bundles:

Theorem 2.2. *Suppose there exists a function* $q \colon M \to \mathbb{R}$ *such that*

(1) $V(x) \geq -q(x) \operatorname{Id}$ *for all* $x \in M$;

(2) $q \geq 1$ *and* $q^{-1/2}$ *is globally Lipschitz, i.e. there exists a constant* $L > 0$ *such that, for every* $x_1, x_2 \in M$,

$$|q^{-1/2}(x_1) - q^{-1/2}(x_2)| \leq Ld(x_1, x_2), \tag{2.4}$$

where d *is the distance induced by the metric* g^{TM};

(3) *the metric* $g := q^{-1} g^{TM}$ *on* M *is complete.*

Then H_V *is essentially self-adjoint on* $C_c^\infty(M, E^+)$.

The proof is given in Section 5.

Remark 2.3. We say that a curve $\gamma \colon [a, \infty) \to M$ *goes to infinity* if for any compact set $K \subset M$ there exists $t_K > 0$ such that $\gamma(t) \notin K$, for all $t \geq t_K$. Condition (3) of the theorem is equivalent to the condition that the integral $\int_\gamma \frac{ds}{\sqrt{q}} = \infty$ for every going to infinity curve γ.

Corollary 2.4. *If* $V(x)$ *is bounded below, i.e. there exists a constant* $b > 0$ *such that* $V(x) \geq -b$, *then the operator* H_V *is essentially self-adjoint.*

In the rest of this paper we denote by H_V both the operator and its self-adjoint closure.

2.4. The spectral counting function of a Schrödinger-type operator.

The trace $\tau : A \to \mathbb{C}$ on A extends to a (possibly infinite) trace Tr_τ on the space of bounded A-linear operators acting on $L^2(M, E^\pm)$.

For each $\lambda \in \mathbb{R}$ let P_λ denote the orthogonal projection onto the spectral subspace of the operator H_V corresponding to the ray $(-\infty, \lambda]$ and define the *spectral counting function*

$$N_\tau(\lambda; H_V) := \operatorname{Tr}_\tau P_\lambda. \tag{2.5}$$

Theorem 2.5. *Suppose there exist a compact* $K \subset M$ *and a constant* C *such that*

$$V(x) \geq C, \quad \text{for all } x \notin K. \tag{2.6}$$

Then

$$N_\tau(\lambda; H_V) < \infty, \quad \text{for all } \lambda < C. \tag{2.7}$$

Remark 2.6. Inequality (2.6) implies that $V(x)$ is bounded below and, hence, the operator H_V is self-adjoint by Corollary 2.4.

2.5. τ-Fredholmness. We first recall the notion of Fredholmness in the von Neumann setting, cf. [11, Section 3]. Let H_1 and H_2 be A-Hilbert spaces and $T : \mathrm{Dom}(T) \subset H_1 \to H_2$ be a closed A-operator.

Definition 2.7. We say that the operator T is τ-*Fredholm* if the following two conditions are satisfied:

(i) $\mathrm{Ker}\, T$ has finite τ-dimension;

(ii) there exists an A-Hilbert subspace L of H_2 such that $L \subset \mathrm{im}\, T$ and $\dim_\tau L^\perp < \infty$.

Remark 2.8. If H_1 and H_2 are Hilbert spaces over $\mathbb{C}$, an operator $T : H_1 \to H_2$ is called Fredholm if its kernel and cokernel are finite dimensional. Equivalently, this means that the kernels of T and T^* are finite dimensional and the image of T is closed. The image of a τ-Fredholm operator does not need to be closed. Moreover, the image of T is not in general an A-Hilbert space and the τ-dimension of the cokernel of T is not defined. Because of this we need to replace the condition of finite-dimensionality of the cokernel by Condition (ii) of Definition 2.7.

Consider the operator

$$\mathcal{T} = \begin{pmatrix} 0 & T^* \\ T & 0 \end{pmatrix} : H_1 \oplus H_2 \longrightarrow H_1 \oplus H_2. \tag{2.8}$$

Lemma 2.9. *The operator T is τ-Fredholm if and only if there exists $\lambda > 0$ such that*

$$N_\tau(\lambda; \mathcal{T}^2) < \infty. \tag{2.9}$$

The lemma is proven in Section 6.5.

Remark 2.10. The inequality (2.9) is equivalent to

$$N_\tau(\lambda; T^*T) < \infty \quad \text{and} \quad N_\tau(\lambda; TT^*) < \infty.$$

Remark 2.11. The condition (2.9) is often taken as a definition of τ-Fredholmness, cf. for example [20, §2.1.1].

2.6. τ-Fredholmness of a Callias-type operator. Let D be as in Section 2.1 and let $F : E^+ \to E^-$ be an A-linear bundle map. We set

$$\mathcal{F} := \begin{pmatrix} 0 & F^* \\ F & 0 \end{pmatrix}$$

and consider the operator

$$\mathcal{D}_\mathcal{F} := \mathcal{D} + \mathcal{F} = \begin{pmatrix} 0 & D^* + F^* \\ D + F & 0 \end{pmatrix}. \tag{2.10}$$

This operator satisfies the conditions of Theorem 2.1 and, hence, is essentially self-adjoint. Its self-adjoint closure is denoted by the same symbol $\mathcal{D}_\mathcal{F}$.

Let

$$\{\mathcal{D}, \mathcal{F}\} := \mathcal{D} \circ \mathcal{F} + \mathcal{F} \circ \mathcal{D} = \begin{pmatrix} D^*F + F^*D & 0 \\ 0 & DF^* + FD^* \end{pmatrix}$$

denote the *anti-commutator* of $\mathcal{D}$ and $\mathcal{F}$. We also set $D_F := D + F$.

Definition 2.12. We say that the operators D_F and $\mathcal{D}_\mathcal{F}$ are of *Callias-type* if

(1) the leading symbol $\sigma(\mathcal{D})$ anti-commutes with $\mathcal{F}$ so that the anti-commutator $\{\mathcal{D}, \mathcal{F}\}$ is an A-linear bundle endomorphism of $E = E^+ \oplus E^-$;

(2) there exist a compact set $K \subset M$ and $\epsilon > 0$ such that

$$\mathcal{F}^2(x) \geq \|\{\mathcal{D}, \mathcal{F}\}(x)\| + \epsilon \tag{2.11}$$

for all $x \in M \setminus K$.

Here $\|\{\mathcal{D}, \mathcal{F}\}(x)\|$ denotes the norm of the A-linear map $\{\mathcal{D}, \mathcal{F}\}(x) : E_x \to E_x$.

In Section 6 we get the following corollary of Theorem 2.5:

Theorem 2.13. *A Callias-type operator D_F is τ-Fredholm.*

2.7. A Callias-type index. Suppose now that D_F is an operator of Callias-type. By Theorem 2.13, $\dim_\tau \operatorname{Ker} D_F$ and $\dim_\tau \operatorname{Ker} D_F^*$ are finite. Hence, we can define the τ-index of D_F by

$$\operatorname{ind}_\tau D_F := \dim_\tau \operatorname{Ker} D_F - \dim_\tau \operatorname{Ker} D_F^*. \tag{2.12}$$

For operators acting on sections of finite-dimensional bundles (i.e. when the von Neumann algebra $A = \mathbb{C}$) this index was introduced by C. Callias [15] and was extended and studied in many papers including [2, 5, 13]. We refer to (2.12) as the *Callias-type τ-index* of the pair (D, F).

In Section 7 we prove the following stability property of the Callias-type τ-index:

Theorem 2.14. *Let F_0 and F_1 be two A-linear bundle maps $E^+ \to E^-$ satisfying conditions* (1) *and* (2) *of Definition 2.12. If there exists a compact set $K \subset M$ such that $F_0(x) = F_1(x)$ for all $x \notin K$, then*

$$\operatorname{ind}_\tau D_{F_0} = \operatorname{ind}_\tau D_{F_1}. \tag{2.13}$$

3. A-Hilbert bundles

In this section we recall some basic properties of von Neumann algebras, cf. [4], and recall the definitions of A-Hilbert space and A-Hilbert bundle, cf. [28].

3.1. A Hilbert space completion. Let A be a von Neumann algebra endowed with a finite, faithful, normal trace $\tau : A \to \mathbb{C}$, cf. [4, III.2.5]. We normalize the trace so that $\tau(\mathrm{Id}) = 1$.

We define on A the inner product

$$\langle a, b \rangle_\tau := \tau(ab^*), \quad a, b \in A, \tag{3.1}$$

and denote by $l^2(A)$ the Hilbert space completion of A with respect to this inner product. Notice that $l^2(A)$ is an Hilbert space endowed with an A-module structure.

Example 3.1. Suppose Γ is a discrete group and denote by $l^2(\Gamma)$ the Hilbert space of complex valued square summable functions on Γ. The *right regular representation* of Γ is the unitary representation $R : \Gamma \to \mathcal{B}(l^2(\Gamma))$ defined by

$$(R_g u)(h) = u(hg), \quad h, g \in \Gamma; \ u \in l^2(\Gamma).$$

The smallest subalgebra of $\mathcal{B}(l^2(\Gamma))$ which is weakly closed and contains all the operators R_g $(g \in \Gamma)$ is called the *von Neumann algebra* of Γ and is denoted by $\mathcal{N}\Gamma$.

On $\mathcal{N}\Gamma$ we have the canonical faithful positive trace τ defined by:

$$\tau(f) = \langle f(\delta_e), \delta_e \rangle_{l^2(\Gamma)}, \quad f \in \mathcal{N}\Gamma, \tag{3.2}$$

where $\delta_e \in l^2(\Gamma)$ is by definition the characteristic function of the unit element.

The completion $l^2(\mathcal{N}\Gamma)$ of $\mathcal{N}\Gamma$ with respect to this inner product is canonically isomorphic to $l^2(\Gamma)$, cf. [28, Example 7.11].

3.2. A-Hilbert spaces. Let A be a von Neumnn algebra endowed with a trace τ satisfying the same properties as in section 3.1. Suppose A acts on a Hilbert space H. The action of A on H is said to be *compatible* with the inner product if

$$\langle xa^*, y \rangle = \langle x, ya \rangle, \quad a \in A; \ x, y \in H. \tag{3.3}$$

Notice, that the action of $\mathcal{N}\Gamma$ on $l^2(\Gamma)$ in Example 3.1 is compatible with the inner product on $l^2(\Gamma)$.

Definition 3.2. An *A-Hilbert space* is a Hilbert space H endowed with a compatible A-module structure such that there exists a separable Hilbert space V_H and an isometric A-linear embedding

$$i : H \ \hookrightarrow \ l^2(A) \otimes V_H.$$

We say that H is *an A-Hilbert space of finite type* if V_H can be chosen finite dimensional.

Notice, that since the action of A is compatible with the scalar product, the orthogonal complement $i(H)^\perp$ to $i(H)$ in $l^2(A) \otimes V_H$ is also an A-module. In other words H is a projective A-module, cf. [4].

3.3. A-Hilbert bundles.

Definition 3.3. An *A-Hilbert bundle E* on a manifold M is a locally trivial bundle of A-Hilbert spaces, the transition functions being A-Hilbert space isomorphisms.

If the fibers are A-Hilbert spaces of finite type, the bundle is called an *A-Hilbert bundle of finite type*.

We denote by $C_c^\infty(M, E)$ the space of smooth compactly supported sections of E.

If M is endowed with a smooth measure $d\mu$ we define the L^2-scalar product

$$(s_1, s_2)_2 := \int_M \langle s_1(x), s_2(x) \rangle d\mu, \quad s_1, s_2 \in C_c^\infty(M, E),$$

and by $L^2(M, E)$ the completion of $C_c^\infty(M, E)$ with respect to this scalar product.

We set $|s(x)| := \langle s(x), s(x) \rangle^{1/2}$ and denote by

$$\|s\| := \left(\int_M |s|^2 d\mu \right)^{1/2}$$

the L^2-norm of s.

Example 3.4. Let M be a smooth compact manifold and $\pi \colon \widetilde{M} \to M$ be the Galois cover of M with deck transformation group Γ. We let Γ act on the Hilbert space $l^2(\Gamma)$ by left and right convolution.

Let $A = \mathcal{N}\Gamma$ denote the von Neumann algebra of Γ. The right action of Γ extends to a right action of A on $l^2(\Gamma)$ commuting with the left Γ-action. Therefore, $E := \widetilde{M} \times_\Gamma l^2(\Gamma)$ is an A-Hilbert bundle of finite type on M.

Notice that the space $L^2(M, E)$ coincides with the space $L^2(\widetilde{M})$ of square-integrable functions on $\widetilde{M}$.

4. Self-adjointness of first order differential operators

In this section we prove Theorem 2.1.

Since the operator $\mathcal{D}$ is formally self-adjoint, to show that it is essentially self-adjoint we need to prove that its maximal and minimal extensions coincide. Since the domain $\mathrm{Dom}(\mathcal{D}_{\min})$ of the minimal extension is closed in the operator norm of $\mathcal{D}$, it is enough to show that for every $s \in \mathrm{Dom}(\mathcal{D}_{\max})$ there exists a sequence $\{s_k\}$ in $\mathrm{Dom}(\mathcal{D}_{\min})$ such that

$$\lim_{k \to \infty} s_k = s, \quad \lim_{k \to \infty} \mathcal{D}s_k = \mathcal{D}s, \tag{4.1}$$

where the limits are in L^2-norm topology.

4.1. The minimal extension. Recall that the Sobolev spaces of sections of an A-vector bundle were defined by Miščenko and Fomenko [17]. In particular, if $\Omega \subset M$ is an open set with compact closure, the Sobolev space $H_0^s(\Omega, E)$ is defined as the closure in Sobolev norm of the space $C_0^\infty(\Omega, E)$ of smooth sections, having compact support in Ω. If T is a differential operator of order k, then T extends to a bounded operator

$$T : H_0^k(\Omega, E) \to L^2(M, E).$$

Recall that the minimal domain $\mathrm{Dom}(T_{\min})$ is the closure of $C_c^\infty(M, E)$ with respect to the graph norm of T. We conclude that

$$H_0^k(\Omega, E) \subset \mathrm{Dom}(T_{\min}) \qquad (4.2)$$

for any open set Ω whose closure is compact.

4.2. The maximal extension. Recall that the domain $\mathrm{Dom}(T_{\max})$ of the maximal extension consists of all sections $s \in L^2(M, E)$ such that $Ts \in L^2(M, E)$, where Ts is understood in distributional sense.

Since $\mathcal{D}$ is a first order elliptic operator, $\mathrm{Dom}(\mathcal{D}_{\max}) \subset H^1_{\mathrm{loc}}(M, E)$. It follows now from (4.2), that for any Lipschitz function $\phi \in C_c^0(M)$ and any $s \in \mathrm{Dom}(\mathcal{D}_{\max})$

$$\phi s \in \mathrm{Dom}(\mathcal{D}_{\min}). \qquad (4.3)$$

4.3. Proof of Theorem 2.1. By Lemma 8.9 of [10] there exists a sequence $\{\phi_k\}$ of Lipschitz functions with compact support on M such that

$$
\begin{aligned}
&\text{(i)} \quad 0 \leq \phi_k \leq 1; \\
&\text{(ii)} \quad |d\phi_k| \leq \frac{1}{k}; \\
&\text{(iii)} \quad \lim_{k \to \infty} \phi_k(x) = 1, \quad \text{for all } x \in M.
\end{aligned} \qquad (4.4)
$$

For any $s \in \mathrm{Dom}(\mathcal{D}_{\max})$ set $s_k = \phi_k s$. Then $s_k \in \mathrm{Dom}(\mathcal{D}_{\min})$ by (4.3) and $\lim_{k \to \infty} s_k = s$. It remains to show that $\mathcal{D}s_k$ converges to $\mathcal{D}s$ in the L^2-norm.

We have:

$$\mathcal{D}s_k = \phi_k \mathcal{D}s + [\mathcal{D}, \phi_k]s. \qquad (4.5)$$

Notice that

$$[\mathcal{D}, \phi_k](x) = -i\,\sigma(\mathcal{D})\big(x, d\phi_k(x)\big)$$

is a bundle map. From (4.4) and (2.1) we conclude that

$$\big\|[\mathcal{D}, \phi_k]s\big\| = \big\|\sigma(\mathcal{D})(x, d\phi_k)s\big\| \leq \frac{c}{k} \cdot \|s\|.$$

Hence, $\lim_{k\to\infty}[\mathcal{D}, \phi_k] = 0$. Since $\phi_k \mathcal{D}s \to \mathcal{D}s$ in L^2-norm we obtain

$$\lim_{k\to\infty} \mathcal{D}s_k = \mathcal{D}s.$$

The essential self-adjointness of the operator $\mathcal{D}$ is proved. $\square$

5. Self-adjointness of a Schrödinger-type operator

In this section we prove Theorem 2.2. We use the notation of Section 2.3.

We denote by $H_{V,0}$ the restriction of the operator (2.3) to the space $C_c^\infty(M, E^+)$ of smooth compactly supported sections of E^+. Let $H_{V,0}^*$ denote the operator adjoint to $H_{V,0}$ and let $\mathrm{Dom}(H_{V,0}^*)$ denote its domain.

Since the operator $H_{V,0}$ is symmetric, to show that its closure is self-adjoint it is enough to prove that

$$(H_V s_1, s_2) = (s_1, H_V s_2), \quad s_1, s_2 \in \mathrm{Dom}(H_{V,0}^*). \tag{5.1}$$

To prove (5.1) we need some information about the behavior of sections from $\mathrm{Dom}(H_{V,0}^*)$ at infinity. This information is provided by the following

Proposition 5.1. *Suppose $q : M \to \mathbb{R}$ is a function satisfying the assumptions of Theorem 2.2. If $s \in \mathrm{Dom}(H_{V,0}^*)$, then $q^{-1/2}Ds$ is square integrable and*

$$\|q^{-1/2}Ds\|^2 \le 2\big((1 + 2L^2)\|s\|^2 + \|s\|\,\|H_V s\|\big), \tag{5.2}$$

where L is the Lipschitz constant introduced in (2.4).

Remark 5.2. For the Schrödinger operator on scalar valued functions on $\mathbb{R}^n$ an analogous lemma was established in [27]. The proof was adapted in [24, 25] to the case of a Riemannian manifold and to differential forms in [6]. The case of a general operator D and a singular potential V was considered in [10].

5.1. Regularity of sections from Dom($H_{V,0}^*$). The theory of elliptic (pseudo)-differential operators on A-Hilbert bundles of finite type was developed in [14] (see also [17] for a similar theory for bundles of finitely generated A-modules). In particular, the Sobolev spaces of sections of such bundles are introduced in these papers and it is shown that any $s \in \mathrm{Dom}(H_{V,0}^*)$ belongs to the Sobolev space H_{loc}^2. Hence,

$$Ds \in L_{\mathrm{loc}}^2(M, E^-), \ Vs \in L_{\mathrm{loc}}^2(M, E^+), \quad \text{for any } s \in \mathrm{Dom}(H_{V,0}^*). \tag{5.3}$$

The new information provided by Proposition 5.1 is about the rate of decay of Ds at infinity.

Remark 5.3. The equation (5.3) is the only place in the proof of Theorem 2.2 where we use the fact that the fibers of $E^{\pm}$ are modules of finite type. The rest of the proof of Theorem 2.2 follows the lines of [10, §9] with almost no changes. It is even simpler, since in [10] much more singular potentials are considered.

Set $\widehat{D} = -i\sigma(D)$. Then

$$D(\phi s) = \widehat{D}(d\phi)s + \phi Ds.$$

Note that $\widehat{D^*} = -(\widehat{D})^*$.

5.2. Proof of Proposition 5.1. Let ψ be a Lipschitz function with compact support such that a $0 \leq \psi \leq q^{-1/2} \leq 1$. Set

$$C = \operatorname*{ess\,sup}_{x \in M} \|\widehat{D}(d\psi)\|.$$

Using (5.3) we obtain

$$
\begin{aligned}
\|\psi Ds\|^2 = (D^*(\psi^2 Ds), s) &= (\psi^2 D^* Ds, s) + 2(\psi \widehat{D^*}(d\psi)Ds, s) \\
&= \operatorname{Re}(\psi^2 D^* Ds, s) + 2\operatorname{Re}(\psi \widehat{D^*}(d\psi)Ds, s) \\
&\leq \operatorname{Re}(\psi^2 D^* Ds, s) + 2C\|\psi Ds\|\|s\| \\
&= \operatorname{Re}(\psi^2 H_V s, s) - (\psi^2 Vs, s) + 2C\|\psi Ds\|\,\|s\|.
\end{aligned}
\tag{5.4}
$$

Since $V \geq -q\,\mathrm{Id}$, $q \geq 1$ and $q\psi^2 \leq 1$, we have

$$(\psi^2 Vs, s) = (V\psi s, \psi s) \geq -(q\psi s, \psi s) \geq -\|s\|^2.
\tag{5.5}$$

Using the inequality $ab \leq \frac{1}{2}a^2 + \frac{1}{2}b^2$ we obtain

$$2C\|\psi Ds\|\,\|s\| \leq \frac{1}{2}\|\psi Ds\|^2 + 2C^2\|s\|^2.
\tag{5.6}$$

Combining (5.4), (5.5), and (5.6) and using that $\psi^2 \leq q^{-1} \leq 1$ we get

$$\|\psi Ds\|^2 \leq \|H_V s\|\,\|s\| + \frac{1}{2}\|\psi Ds\|^2 + (1 + 2C)\|s\|^2,$$

and

$$\|\psi Ds\|^2 \leq 2\big((1 + 2C^2)\|s\|^2 + \|s\|\,\|H_V s\|\big).
\tag{5.7}$$

To prove (5.2) we now make a special choice of the function ψ. Let ϕ_k be as in (4.4) and set $\psi_k := \phi_k \cdot q^{-1/2}$. Then $0 \leq \psi_k \leq q^{-1/2}$, and

$$|d\psi_k| \leq |d\phi_k| \cdot q^{-1/2} + \phi_k |dq^{-1/2}|.$$

Therefore, $|d\psi_k| \leq \frac{1}{k} + L$. Since $\psi_k(x) \to q^{-1/2}(x)$ as $k \to \infty$, the dominated convergence theorem applied to (5.7) with $\psi = \psi_k$ immediately implies (5.2). $\square$

5.3. Proof of Theorem 2.2. Let $s_1, s_2 \in \mathrm{Dom}(H_{V,0}^*)$. Then

$$(\phi s_1, D^* D s_2) = (D(\phi s_1), D s_2) = (\widehat{D}(d\phi)s_1, D s_2) + (\phi D s_1, D s_2).$$

Similarly,

$$(D^* D s_1, \phi s_2) = (D s_1, \widehat{D}(d\phi)s_2) + (\phi D s_1, D s_2)$$

Hence,

$$(\phi s_1, H_V s_2) - (H_V s_1, \phi s_2) = (\widehat{D}(d\phi)s_1, D s_2) - (D s_1, \widehat{D}(d\phi)s_2).$$

By (2.1),

$$\operatorname*{ess\,sup}_{x \in M} |\widehat{D}(d\phi)| \leq c \operatorname*{ess\,sup}_{x \in M} |d\phi(x)|,$$

Therefore,

$$\begin{aligned}
|(\phi s_1, H_V s_2) &- (H_V s_1, \phi s_2)| \\
&\leq c \operatorname*{ess\,sup}_{x \in M}\big(|d\phi| q^{1/2}\big) \cdot \big(\|s_1\|\|q^{-1/2} D s_2\| + \|s_2\|\|q^{-1/2} D s_1\|\big). \quad (5.8)
\end{aligned}$$

Consider a metric $g := q^{-1} g^{TM}$. By condition (iii) of Theorem 2.2 this metric is complete. By Lemma 8.9 of [10] there exists a sequence $\{\phi_k\}$ of Lipschitz functions such that

(1) $0 \leq \phi_k \leq 1$ and $|d\phi_k|_g \leq \frac{1}{k}$;

(2) $\lim_{k\to\infty} \phi_k(x) = 1$, for all $x \in M$.

Since $|d\phi_k|_g = q^{1/2}|d\phi_k|$, we conclude

$$\operatorname*{ess\,sup}_{x \in M}(|d\phi_k| q^{1/2}(x)) \leq \frac{1}{k}.$$

Using (5.8), we obtain

$$\begin{aligned}
|(\phi_k u, H_V v) &- (H_V u, \phi_k v)| \\
&\leq \frac{c}{k}\big(\|s_1\|\|q^{-1/2} D s_2\| + \|q^{-1/2} D s_1\|\|s_2\|\big) \to 0, \quad \text{as } k \to \infty,
\end{aligned}$$

where the convergence to 0 of the RHS of this inequality follows from Proposition 5.1. On the other side, by the dominated convergence theorem we have

$$(\phi_k s_1, H_V s_2) - (H_V s_1, \phi_k s_2) \longrightarrow (s_1, H_V s_2) - (H_V s_1, s_2)$$

as $k \to \infty$. Thus $(H_V s_1, s_2) = (s_1, H_V s_2)$, for all $s_1, s_2 \in \mathrm{Dom}(H_{V,0}^*)$. Therefore, H_V is essentially self-adjoint. $\qquad\square$

6. Fredholmness

In this section we prove Theorems 2.5 and 2.13.

6.1. Variational principle. We make use of a variational principle in the von Neumann setting, stated in the following:

Lemma 6.1. *Let H be an A-Hilbert space and T be a self-adjoint operator commuting with the action of A. Then, for every $\lambda \in \mathbb{R}$,*

$$N_\tau(\lambda; T) = \sup_L \dim_\tau L, \tag{6.1}$$

where L varies among the A-Hilbert subspaces of H with $L \subset \mathrm{Dom}(T)$ and satisfying:

$$(Tu, u) \le \lambda (u, u), \quad u \in L.$$

This lemma is well-known. For the proof we refer to [31, Lemma 2.4] where the case $A = \mathcal{N}\Gamma$ is treated (the case of a general finite von Neumann algebra A follows with minor modifications).

6.2. Restriction to a compact subset. Note, first, that for any A-linear self-adjoint operator T and any l, $\lambda \in \mathbb{R}$ we have:

$$N_\tau(\lambda + l; T + l) = N_\tau(\lambda; T). \tag{6.2}$$

Therefore, by replacing $V(x)$ with $V(x) + l$ in Theorem 2.5, we can assume that $C, \lambda > 0$ in (2.6) and (2.7) and $V(x) \ge 0$ for all $x \in M$.

Let the compact set $K \subset M$ be as in (2.6). For any $\lambda > 0$ consider the set

$$M_\lambda := \{x \in M : V(x) \ge \lambda\}.$$

Then

$$\Omega_\lambda := M \setminus M_\lambda$$

is an open set. We denote its closure by $\bar{\Omega}_\lambda$. We now assume that $0 < \lambda < C$ and choose λ_1 such that $0 < \lambda < \lambda_1 < C$. Then

$$\bar{\Omega}_\lambda \subset \Omega_{\lambda_1}, \quad \bar{\Omega}_{\lambda_1} \subset \Omega_C, \quad \bar{\Omega}_C \subset K.$$

Let $\phi : M \to [0, 1]$ be a smooth function such that

$$\phi|_{\Omega_{\lambda_1}} \equiv 1, \quad \phi|_{M \setminus \Omega_C} \equiv 0.$$

Define the A-linear restriction map $\rho : L^2(M, E) \to L^2(\Omega_C, E|_{\Omega_C})$ by the formula

$$\rho(s) := \phi s. \tag{6.3}$$

Lemma 6.2. *Let $L \subset \mathrm{Dom}(H_V)$ be an A-Hilbert subspace of $L^2(M, E)$ satisfying*

$$(H_V s, s) \leq \lambda (s, s), \quad s \in L. \tag{6.4}$$

Then ρ is injective when restricted to L and $\rho(L)$ is a closed A-invariant subspace of $L^2(\Omega_\lambda, E|_{\Omega_\lambda})$.

Proof. The potential V satisfies the inequality $V(x) \geq 0$ for any $x \in M$. Hence, the operator H_V satisfies the conditions of Theorem 2.2 with $q = \mathrm{const}$. It follows from Proposition 5.1 that for any $s \in \mathrm{Dom}(H_V)$ we have $Ds \in L^2(M, E)$. Hence,

$$(H_V s, s) = \|Ds\|^2 + (Vs, s) \geq (Vs, s). \tag{6.5}$$

In particular, $(Vs, s) < \infty$.

For any $s \in L$ using (6.5) we obtain

$$\lambda \|s\|^2 \geq (H_V s, s) \geq (Vs, s) \geq \lambda_1 \int_{M \setminus \Omega_{\lambda_1}} |s(x)|^2 \, d\mu(x)$$

and

$$\lambda \int_{\Omega_{\lambda_1}} |s(x)|^2 \, d\mu(x) \geq (\lambda_1 - \lambda) \int_{M \setminus \Omega_{\lambda_1}} |s(x)|^2 \, d\mu(x).$$

Hence,

$$\|\rho(s)\|^2 \geq \int_{\Omega_{\lambda_1}} |s(x)|^2 \, d\mu(x) \geq \frac{\lambda_1 - \lambda}{\lambda} \int_{M \setminus \Omega_{\lambda_1}} |s(x)|^2 \, d\mu(x).$$

Therefore,

$$\begin{aligned}
\|s\|^2 &= \int_{\Omega_{\lambda_1}} |s(x)|^2 \, d\mu(x) + \int_{M \setminus \Omega_{\lambda_1}} |s(x)|^2 \, d\mu(x) \\
&\leq \|\rho(s)\|^2 + \frac{\lambda}{\lambda_1 - \lambda} \|\rho(s)\|^2 = \left(\frac{\lambda_1}{\lambda_1 - \lambda} \right) \|\rho(s)\|^2.
\end{aligned} \tag{6.6}$$

This inequality together with the fact that ρ is a bounded A-equivariant map proves the lemma. $\qquad \square$

6.3. Extension to a closed manifold. Choose a closed manifold $\widehat{M}$ containing Ω_C as an open subset. Let $\widehat{E} = \widehat{E}^+ \oplus \widehat{E}^-$ be a graded A-Hilbert bundle of finite type over $\widehat{M}$ extending $E|_{\Omega_C}$. Let $\widehat{D} : C^\infty(\widehat{E}^+) \to C^\infty(\widehat{E}^-)$ and $\widehat{V} : \widehat{E}^+ \to \widehat{E}^+$ be a first order elliptic differential operator and a positive bundle map which agree with D and V on Ω_C. Set

$$H_{\widehat{V}} := \widehat{D}^* \widehat{D} + \widehat{V}.$$

Then the restrictions of H_V and $H_{\widehat{V}}$ to Ω_C coincide.

We view $\rho(L) \subset L^2(\Omega_C, E^+|_{\Omega_C})$ as a subspace of $L^2(\widehat{M}, \widehat{E}^+)$.

Lemma 6.3. *Under the assumptions of Lemma 6.2, there exists a constant $R > 0$ such that for any $u \in \rho(L)$ we have*

$$\left(H_{\widehat{V}}u, u\right) \leq R(u, u). \tag{6.7}$$

Proof. Set

$$a := \max_{x \in M} \|\sigma(D)(x, d\phi(x))\|.$$

Notice that the maximum exists, since ϕ has compact support.

For $u \in \rho(L)$ there exists $s \in L$ such that $u = \rho(s) = \phi s$. Then

$$\begin{aligned}
\|\widehat{D}u\|^2 &= \|D(\phi s)\|^2 = \left(\|\phi D s\| + \|[D, \phi]s\|\right)^2 \\
&\leq \left(\|Ds\| + a\|s\|\right)^2 \leq 2\|Ds\|^2 + 2a^2\|s\|^2.
\end{aligned} \tag{6.8}$$

Also, since $V(x) > 0$ for all $x \in M$ we conclude

$$(\widehat{V}u, u) = (\phi^2 V s, s) \leq (Vs, s). \tag{6.9}$$

By using (6.8) and (6.9) we obtain

$$\begin{aligned}
\left(H_{\widehat{V}}u, u\right) &= \|\widehat{D}u\|^2 + (\widehat{V}u, u) \leq 2\|Ds\|^2 + 2a^2\|s\|^2 + (Vs, s) \\
&\leq 2\left(\|Ds\|^2 + (Vs, s)\right) + 2a^2\|s\|^2 = 2(H_V s, s) + 2a^2\|s\|^2.
\end{aligned}$$

Using (6.4) and (6.6) we now conclude

$$\left(H_{\widehat{V}}u, u\right) \leq 2(\lambda + a^2)\|s\|^2 \leq 2(\lambda + a^2)\left(\frac{\lambda_1}{\lambda_1 - \lambda}\right)\|u\|^2.$$

Hence, (6.7) holds with $R = 2(\lambda + a^2)\left(\frac{\lambda_1}{\lambda_1 - \lambda}\right)$. □

6.4. Proof of Theorem 2.5. Since $\widehat{V} \geq 0$, we have

$$N_\tau(\lambda; H_{\widehat{V}}) \leq N_\tau(\lambda; \widehat{D}^*\widehat{D}), \quad \lambda \in \mathbb{R}. \tag{6.10}$$

It is shown in [14, §2.4] that the spectral counting function $N_\tau(\lambda; \widehat{D}^*\widehat{D})$ is finite. Hence, it follows from (6.10) and Lemmas 6.1 and 6.3 that

$$\dim_\tau \rho(L) \leq N_\tau(R; H_{\widehat{V}}) < \infty.$$

From Lemma 6.2 and the open mapping theorem we deduce that the map $\rho|_L : L \to \rho(L)$ is an isomorphism of Hilbert A-spaces. By [20, Theorem 1.12 (2)]

$$\dim_\tau L = \dim_\tau \rho(L) \leq N_\tau(R; H_{\widehat{V}}). \tag{6.11}$$

Hence, by Lemma 6.1 we get

$$N_\tau(\lambda; H_V) \leq N_\tau(R; H_{\widehat{V}}) < \infty.$$

Theorem 2.5 is proven. □

6.5. Proof of Lemma 2.9. Consider the bounded operator

$$\Phi(\mathcal{T}) := \mathcal{T}(I + \mathcal{T}^2)^{-\frac{1}{2}}, \tag{6.12}$$

where $\mathcal{T}$ is the operator defined in (2.8). Observe that the operator $\Phi(\mathcal{T})$ is τ-Fredholm if and only if T is τ-Fredholm, and

$$N_\tau(\lambda; \mathcal{T}^2) = N_\tau \left(\frac{\lambda}{1 + \lambda}; \Phi(\mathcal{T})^2 \right).$$

Thus Lemma 2.9 is a direct consequence of the following:

Lemma 6.4. *Let S be a bounded self-adjoint A-linear operator on an A-Hilbert space H. Then S is τ-Fredholm if and only if there exists $\lambda > 0$ such that*

$$N_\tau(\lambda; S^2) < \infty. \tag{6.13}$$

Proof. Suppose $N_\tau(\lambda; S^2) < \infty$ for some $\lambda > 0$. We need to show that S is τ-Fredholm in the sense of Definition 2.7.

Since the function $N_\tau(\cdot; S^2)$ is nondecreasing, we have:

$$\dim_\tau \operatorname{Ker} S = N_\tau(0; S^2) \leq N_\tau(\lambda; S^2) < \infty.$$

Thus the condition (i) of Definition 2.7 is satisfied.

Set $L := \operatorname{im}(I - P_\lambda(S^2))$, where $P_\lambda(S^2)$ denotes the orthogonal projection onto the spectral subspace of the operator S^2 corresponding to $[0, \lambda]$. Then L is A-invariant, $L \subseteq \operatorname{im} S^2 \subseteq \operatorname{im} S$ and

$$\dim_\tau L^\perp = \dim_\tau(P_\lambda(S)) = N_\tau(\lambda; S^2) < \infty.$$

Hence the condition (ii) of Definition 2.7 is also satisfied.

Suppose now that S is τ-Fredholm and let L be an A-Hilbert subspace of H such that $L \subseteq \operatorname{im} S$ and $\dim_\tau L^\perp < \infty$.

The map

$$S_1 := S|_{(\operatorname{Ker} S)^\perp} : (\operatorname{Ker} S)^\perp \to H \tag{6.14}$$

is one-to-one. Set

$$L_1 := S_1^{-1}(L) \subset (\operatorname{Ker} S)^\perp.$$

Then L_1 is a closed A-invariant subspace of H and $S : L_1 \to L$ is a bijection. It follows from the Open Mapping Theorem that there exists $\epsilon > 0$ such that

$$\|Su\| > \epsilon \|u\|, \quad \text{for any } u \in L_1. \tag{6.15}$$

We finish the proof of the lemma by showing that any $\lambda < \epsilon^2$ satisfies (6.13). Recall that $P_\lambda(S^2)$ denotes the orthogonal projection on the spectral subspace of S^2 corresponding to the interval $[0, \lambda]$. Thus

$$\|Su\| \leq \sqrt{\lambda} \|u\| < \epsilon \|u\|, \quad \text{for any } u \in \operatorname{im} P_\lambda(S^2). \tag{6.16}$$

From (6.15) and (6.16) we now conclude that im $P_\lambda(S^2) \cap L_1 = \{0\}$. Hence, it follows from [20, Theorem 1.12] that

$$N_\tau(\lambda; S^2) \le \dim_\tau L_1^\perp. \tag{6.17}$$

To finish the proof of the lemma it is now enough to show that $\dim_\tau L_1^\perp < \infty$.

Let L_2 be the orthogonal complement of L_1 in $(\mathrm{Ker}\, S)^\perp$. Then

$$\dim_\tau L_1^\perp = \dim_\tau L_2 + \dim_\tau \mathrm{Ker}\, S. \tag{6.18}$$

Since S is τ-Fredholm, $\dim_\tau \mathrm{Ker}\, S$ is finite and it suffices to prove that $\dim_\tau L_2$ is finite.

Notice that, by (6.15), $S(L_2)$ is a closed subspace of H and $S : L_2 \to S(L_2)$ is a topological isomorphism. Hence,

$$\dim_\tau S(L_2) = \dim_\tau L_2, \tag{6.19}$$

cf. [20, Lemma 1.13]. Since the map (6.14) is one-to-one, we conclude that

$$S(L_2) \cap L = S_1(L_2) \cap L = \{0\}.$$

Let P_L be the orthogonal projection onto the subspace L. Then

$$I - P_L : S(L_2) \to L^\perp$$

is a one-to one map. Therefore, by [20, Theorem 1.12(2)],

$$\dim_\tau S(L_2) \le \dim_\tau L^\perp < \infty.$$

From (6.17), (6.18), and (6.19) we now conclude that $N_\tau(\lambda; S^2) < \infty$. $\square$

6.6. Proof of Theorem 2.13. We are now ready to prove τ-Fredholmness of a Callias-type operator $D + F$. The operator (2.10) satisfies the conditions of Theorem 2.1 and, hence, is essentially self-adjoint. Hence

$$N_\tau(\lambda; \mathcal{D}_\mathcal{F}) = N_\tau(\lambda^2; \mathcal{D}_\mathcal{F}^2). \tag{6.20}$$

Moreover

$$\mathcal{D}_\mathcal{F}^2 = \mathcal{D}^2 + \{\mathcal{D}, \mathcal{F}\} + \mathcal{F}^2 = \mathcal{D}^2 + V, \tag{6.21}$$

where $V = \{\mathcal{D}, \mathcal{F}\} + \mathcal{F}^2$. By the Callias condition (2.11), $V(x) > \epsilon$ for all $x \in M \setminus K$. Hence, it follows from Theorem 2.5 that $N_\tau(\epsilon; \mathcal{D}_\mathcal{F}^2)$ is finite. Theorem 2.13 follows now from Lemma 2.9. $\square$

7. Stability of the Callias-type τ-index

In this section we prove Theorem 2.14.

7.1. Continuous perturbations. We start with an abstract result. Let H_1 and H_2 be A-Hilbert spaces and suppose that $T : H_1 \to H_2$ is a closed A-linear operator. We denote by $\mathrm{Dom}(T)$ the domain of T considered as a Hilbert space with the graph scalar product

$$(x, y)_{\mathrm{Dom}(T)} := (x, y)_{H_1} + (Dx, Dy)_{H_2}.$$

Then $T : \mathrm{Dom}(T) \to H_2$ is a bounded A-linear operator.

Assume in addition that T is τ-Fredholm, cf. Definition 2.7.

Definition 7.1. A one parameter family $\{T_t\}_{t\in[0,1]}$ of τ-Fredholm operators $T_t : H_1 \to H_2$ is called a *continuous perturbation of T with fixed domain* if $T_0 = T$, $\mathrm{Dom}(T_t) = \mathrm{Dom}(T)$ for all $t \in [0, 1]$, and the induced family $T_t : \mathrm{Dom}(T) \to H_2$ is continuous in operator norm (as a family of maps between the Hilbert spaces $\mathrm{Dom}(T)$ and H_2).

The next lemma shows that the τ-index is stable with respect to continuous perturbations with fixed domain.

Lemma 7.2. *Let $\{T_t\}_{t\in[0,1]}$ be a continuous perturbation of T with fixed domain. Then*

$$\mathrm{ind}_\tau T_t = \mathrm{ind}_\tau T, \quad \text{for all } t \in [0, 1]. \tag{7.1}$$

Proof. In general, note that the operator

$$(I + T^*T)^{\frac{1}{2}} : \mathrm{Dom}(T) \longrightarrow H_1$$

is an isometric isomorphism of A-Hilbert spaces and consider the family of operators

$$\Phi(T_t) := T_t(I + T^*T)^{-\frac{1}{2}} : H_1 \longrightarrow H_2.$$

The operators $\Phi(T_t)$ form a continuous family of bounded τ-Fredholm operators and

$$\mathrm{ind}_\tau \Phi(T_t) = \mathrm{ind}_\tau T_t.$$

Hence, it is enough to prove the lemma for the case when $\mathrm{Dom}(T) = H_1$ and T_t is a continuous family of bounded operators. In this case the lemma is proven in [12, Theorem 4]. □

7.2. Proof of Theorem 2.14. For $0 \le t \le 1$, we set:

$$D_t := D + F_0 + t(F_1 - F_0).$$

Since the A-endomorphism $F_1 - F_0$ vanishes outside of the compact set K, all D_t satisfy the condition (2.11) and, hence, are Callias-type operators. It follows from Theorem 2.13 that all the operators D_t are τ-Fredholm.

Since $F_1 - F_0 : L^2(M, E^+) \to L^2(M, E^-)$ is a bounded A-operator, the domain of D_t is independent of t. Finally, we have:

$$\|D_s - D_t\| = |s - t| \, \|F_1 - F_0\|, \quad \text{for all } s, t \in [0, 1].$$

Thus the family $\{D_t\}$ is continuous in operator norm. Theorem 2.13 follows now from Lemma 7.2. $\qquad\square$

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