Some quasitensor autoequivalences
of Drinfeld doubles of finite groups

Peter Schauenburg*

Abstract. We report on two classes of autoequivalences of the category of Yetter–Drinfeld modules over a finite group, or, equivalently the Drinfeld center of the category of representations of a finite group. Both operations are related to the r-th power operation, with r relatively prime to the exponent of the group. One is defined more generally for the group-theoretical fusion category defined by a finite group and an arbitrary subgroup, while the other seems particular to the case of Yetter–Drinfeld modules. Both autoequivalences preserve higher Frobenius–Schur indicators up to Galois conjugation, and they preserve tensor products, although neither of them can in general be endowed with the structure of a monoidal functor.

1. Introduction

Let G be a finite group of exponent e. The complex characters of G take values in the cyclotomic field $\mathbb{Q}(\zeta_e)$, where ζ_e is a primitive e-th root of unity. The Galois conjugates of a character χ, i.e. the images of χ under the operation of the Galois group of $\mathbb{Q}(\zeta_e)$, are also characters of G. This gives an operation of the group of prime residues modulo e on the characters of G, or an operation by autoequivalences on the category $\mathrm{Rep}(G)$. Another way of describing this operation is as a special case of an Adams operator, mapping χ to the character $\psi_r(\chi)$ characterized by $\psi_r(\chi)(g) = \chi(g^r)$ for r relatively prime to e. (We admit that the reference to Adams operators is something of an exaggeration, since the latter are defined for any r, and the case where r is relatively prime to e is only a very particular one.) We note that Galois conjugation of characters preserves higher Frobenius–Schur indicators and tensor products. However, as we learnded from Alexei Davydov, it does not in general, give rise to a *monoidal* autoequivalence of $\mathrm{Rep}(G)$. Thus, Galois conjugation defines what is sometimes called a quasitensor equivalence of $\mathrm{Rep}(G)$, in particular an automorphism of the Grothendieck ring of $\mathrm{Rep}(G)$.

We will study operations with somewhat similar behavior, also related to taking r-th powers (with r relatively prime to e), on the Drinfeld center of Rep(G), i.e. the category of Yetter–Drinfeld modules $^G_G\mathcal{YD}$, which can also be described as the module category of the Drinfeld double $D(G)$.

The first and rather more obvious operation, treated in Section 3, is defined more generally on the category $^\Gamma_G\mathcal{M}_G$ of $\mathbb{C}G$-bimodules graded by a finite group Γ containing G as a subgroup; the case of Yetter–Drinfeld modules is recovered by choosing $\Gamma = G \times G$ with G embedded diagonally. Simple objects in $^\Gamma_G\mathcal{M}_G$ are described "locally" by representations of certain stabilizer subgroups of the group G. The group of prime residues modulo e can thus be made to act on $^\Gamma_G\mathcal{M}_G$ by Galois conjugating the local characters of the stabilizer subgroups. By the formula for higher Frobenius–Schur indicators obtained recently in [18] (and much earlier for the case of Drinfeld doubles in [11]) it is obvious that the effect of this operation on the higher Frobenius–Schur indicators of an object is to Galois conjugate them as well. (To compare with the special case of group representations, note that higher Frobenius–Schur in the latter case are integers.) It is less obvious, though perhaps not entirely surprising, that the Galois conjugation operation also preserves tensor products in $^\Gamma_G\mathcal{M}_G$.

The second operation, which we will introduce in Section 4, is particular to the categories $^G_G\mathcal{YD}$ of Yetter–Drinfeld modules. It is defined simply by raising the degrees of all the homogeneous elements in a Yetter–Drinfeld module to the r-th power. If we describe simple Yetter–Drinfeld modules "locally" by irreducible representations of the centralizers $C_G(g)$ of elements $g \in G$, then the operation does not change the representation of $C_G(g)$ at all, but views it as associated to the element g^r instead of g. While both descriptions are very elementary, we could not see any elementary reason why the operation should behave as it does on higher Frobenius–Schur indicators (it preserves them up to Galois conjugation) or tensor products (it preserves them). We will use the Verlinde formula for multiplicities in the modular tensor category $^G_G\mathcal{YD}$, and the Bantay-type formula for indicators in modular categories to prove these facts.

We note that the two operations on $^G_G\mathcal{YD}$ thus associated in different ways to the r-th power operation in G do not usually coincide. Both give quasitensor autoequivalences that preserve higher Frobenius–Schur indicators up to Galois conjugation. Taking a suitable "quotient" of the two types of operations yields a quasitensor autoequivalence that preserves higher Frobenius–Schur indicators (instead of Galois conjugating them). However, even this "better-behaved" operation cannot be endowed, in general, with the structure of a monoidal functor. We regard it briefly in Section 5. The fact that it conserves Frobenius–Schur indicators entails, Theorem 5.5, a certain combinatorial counting identity for finite groups, based on the counting formulas in [11] involving higher Frobenius–Schur indicators.

The author's search for autoequivalences of $^G_G\mathcal{YD}$ was at first motivated by certain computational data: Keilberg [12–14] and Courter [4] have accumulated

rich results on the higher Frobenius–Schur indicators of irreducible representations of the Drinfeld doubles of certain finite groups. Both for the groups — semidirect products of cyclic groups with cyclic or dihedral groups — treated by Keilberg, and for the symmetric groups treated by Courter (with heavy computer calculations) it turns out that many different representations can have identical sets of indicators (Courter terms these representations I-equivalent), without any automorphisms of the Drinfeld doubles at hand that would map the I-equivalent representations to each other. With respect to this phenomenon, the autoequivalences studied in the present paper are both a success (they explain why certain representations of the Drinfeld doubles are I-equivalent) and a failure (the larger part of I-equivalences between different representations remains unexplained).

In Section 6 we will look at a few small examples of the two types of operations on simple Yetter–Drinfeld modules, and in Section 7 we will discuss how the second type of operation fails utterly to help explain most of the I-equivalent objects in $_{S_n}^{S_n}\mathcal{YD}$ found by Courter: On the categories $_{S_n}^{S_n}\mathcal{YD}$ the two generally different r-th power operations are identical. Here, particular thanks are due to the referee, who essentially provided the proof of Theorem 7.2, which was merely a computer-based conjecture in the first version of this paper.

2. Preliminaries

Let $e \in \mathbb{N}$ and denote by $\mathbb{Q}(\sqrt[e]{1})$ the cyclotomic field obtained by adjoining a primitive e-th root of unity ζ_e to the rationals. For r relatively prime to e we denote by σ_r the automorphism of $\mathbb{Q}(\sqrt[e]{1})$ mapping ζ_e to ζ_e^r.

Let G be a finite group, $e = \exp(G)$, and χ a complex character of G. We denote by $\psi_r(\chi)$ the character defined by $\psi_r(\chi)(g) = \chi(g^r)$ for $g \in G$. The assignment ψ_r is known as the r-th Adams operator; for general r it gives a virtual character of G, but we will only be concerned with the case (somewhat trivial in the theory of Adams operators, cf. [5, §12B]) that r is relatively prime to e. The well-known fact that $\psi_r(\chi)$ is a character in this case follows from Brauer's theorem: In fact χ is the character of a representation W which is defined over $\mathbb{Q}(\sqrt[e]{1})$; that is, with respect to a suitable basis of W the action of $g \in G$ is given by a matrix with entries in $\mathbb{Q}(\sqrt[e]{1})$. Applying the map σ_r to all the entries of all these matrices yields a conjugated representation, which we will denote by $\psi_r(W)$, and whose character is $\psi_r(\chi)$.

It is well known (and trivial to check) that $\psi_r(V \otimes W) \cong \psi_r(V) \otimes \psi_r(W)$ and $\nu_m(\psi_r(V)) = \nu_m(V)$ for all $V, W \in \operatorname{Rep}(G)$; the latter because higher Frobenius–Schur indicators of group representations are integers.

We will exhibit operators of somewhat similar behavior defined on certain group-theoretical fusion categories; to wit: One type of operator is defined on the category of graded bimodules $_G^\Gamma\mathcal{M}_G$ with Γ a finite group and $G \subset \Gamma$ a subgroup. The other

type is defined on the category of Yetter–Drinfeld modules ${}^G_G \mathcal{YD}$, equivalent in fact to ${}^\Gamma_G \mathcal{M}_G$ when we choose G embedded diagonally in $\Gamma = G \times G$. The simple objects of ${}^\Gamma_G \mathcal{M}_G$ can be described in terms of group representations. We need this description (which can be extracted from [21] and [19]) very explicitly.

The category ${}^\Gamma_G \mathcal{M}_G$ is defined as the category of $\mathbb{C}G$-bimodules over the group algebra of G, considered as an algebra in the category of $\mathbb{C}\Gamma$-comodules, that is, of Γ-graded vector spaces. Thus, an object of ${}^\Gamma_G \mathcal{M}_G$ is a Γ-graded vector space $M \in \mathrm{Vect}_\Gamma$ with a two-sided G-action compatible with the grading in the sense that $|gmh| = g|m|h$ for $g, h \in G$ and $m \in M$.

The category ${}^\Gamma_G \mathcal{M}_G$ is a fusion category. The tensor product is the tensor product of $\mathbb{C}G$-bimodules. Simple objects are parametrized by irreducible representations of the stabilizers of right cosets of G in Γ. More precisely, let $D \in G\backslash\Gamma/G$ be a double coset of G in Γ, let $d \in D$, and let $S = \mathrm{Stab}_G(dG) = G \cap (d \triangleright G)$ be the stabilizer in G of the right coset dG under the action of G on its right cosets in Γ. Then the subcategory ${}^D_G \mathcal{M}_G \subset {}^\Gamma_G \mathcal{M}_G$ defined to contain those objects the degrees of all of whose homogeneous elements lie in D is equivalent to the category $\mathrm{Rep}(S)$ of representations of S. The equivalence ${}^D_G \mathcal{M}_G \to \mathrm{Rep}(S)$ takes M to $(M_{dG})/G \cong (M/G)_{dG/G}$, the space of those vectors in the quotient of M by the right action of G whose degree lies in the right coset of d. Details are in [19, 21]. We will denote the inverse equivalence by $\mathcal{F}_d \colon \mathrm{Rep}(S) \to {}^{GdG}_G \mathcal{M}_G$. Explicitly, it can be described as follows: For $V \in \mathrm{Rep}(S)$ let $N := \mathrm{Ind}^G_S = \mathbb{C}G \underset{\mathbb{C}S}{\otimes} V$ be the induced representation of G, which we endow with a Γ/G-grading by setting $|g \otimes v| = gdG$; this grading is obviously compatible with the left action of G so that (with a hopefully obvious definition) $N \in {}^{\Gamma/G}_G \mathcal{M}$. Now we have

$$\mathcal{F}_d(V) = N \underset{\mathbb{C}[\Gamma/G]}{\square} \mathbb{C}G \tag{2.1}$$

$$= \left\{ \sum_{\gamma \in \Gamma} n_\gamma \otimes \gamma \in N \otimes \mathbb{C}G \,\middle|\, n_\gamma \in N_{\gamma G} \right\}$$

$$= \bigoplus_\gamma N_{\gamma G} \otimes \mathbb{C}\gamma G,$$

with the last sum running over a set of representatives of the right cosets of G in Γ, the grading $|n \otimes \gamma| = \gamma$, and the bimodule structure $g(n \otimes \gamma)h = gn \otimes g\gamma h$.

The functors $\mathcal{F}_d$ combine into a category equvialence

$$\bigoplus_d \mathrm{Rep}(\mathrm{Stab}_G(dG)) \xrightarrow{(\mathcal{F}_d)_d} {}^\Gamma_G \mathcal{M}_G$$

in which the sum runs over a set of representatives of the double cosets of G in Γ.

The category ${}^G_G \mathcal{YD} = {}^{\mathbb{C}G}_{\mathbb{C}G} \mathcal{YD}$ of (left-left) Yetter–Drinfeld modules over $\mathbb{C}G$ has objects the G-graded vector spaces with a left G-action compatible with the grading

in the sense that $|gv| = g|v|g^{-1}$ for $g \in G$ and $v \in V \in {}^{G}_{G}\mathcal{YD}$. The category ${}^{G}_{G}\mathcal{YD}$ is the (right) center of the category ${}^{G}\mathcal{M}$ of G-graded vector spaces: The half-braiding $c: U \otimes V \to V \otimes U$ between a graded vector space U and a Yetter–Drinfeld module V is given by $u \otimes v \mapsto |u|v \otimes u$.

Simple objects of ${}^{G}_{G}\mathcal{YD}$ are parametrized by irreducible representations of the centralizers in G of elements of G. (In fact this can be viewed as a special case of the description of graded bimodules above.). More precisely, let $g \in G$ and let $C_G(g)$ be the centralizer of g in G. Then a functor

$$\mathcal{G}_g: \operatorname{Rep}(C_G(g)) \to {}^{G}_{G}\mathcal{YD}$$

can be defined by sending $V \in \operatorname{Rep}(C_G(g))$ to the induced $\mathbb{C}G$-module $\operatorname{Ind}^{G}_{C_G(g)} V = \mathbb{C}G \underset{\mathbb{C}C_G(g)}{\otimes} V$ endowed with the grading given by $|x \otimes v| = xgx^{-1}$ for $x \in G$ and $v \in V$. In this way we obtain a category equivalence

$$\bigoplus_g \operatorname{Rep}(C_G(g)) \xrightarrow{(\mathcal{G}_g)_g} {}^{G}_{G}\mathcal{YD}.$$

The sum runs over a set of representatives of the conjugacy classes of G, and the image of the functor $\mathcal{G}_g$ consists of those Yetter–Drinfeld modules the degrees of whose homogeneous elements lie in the conjugacy class of g. The images of the functors $\mathcal{G}_g$ for conjugate elements coincide. More precisely, we have

$$\mathcal{G}_{x \triangleright g}(x \triangleright W) \cong \mathcal{G}_g(W)$$

for $W \in \operatorname{Rep}(C_G(g))$, with $x \triangleright W \in \operatorname{Rep}(x \triangleright C_G(g)) = \operatorname{Rep}(C_G(x \triangleright g))$.

The categories ${}^{\Gamma}_{G}\mathcal{M}_G$ considered in this paper are examples of fusion categories [8]. As we have already mentioned several times, we will be concerned with certain structure-preserving autoequivalences of such categories. To fix the notions, a monoidal functor or tensor functor $\mathcal{F}: \mathcal{C} \to \mathcal{D}$ consists of a functor $\mathcal{F}$ together with a natural isomorphism $\xi: \mathcal{F}(X) \otimes \mathcal{F}(Y) \to \mathcal{F}(X \otimes Y)$ which is required to fulfill a compatibility condition with the associativity constraints of $\mathcal{C}$ and $\mathcal{D}$. If this compatibility condition is *not* required, we shall speak of a quasitensor functor. Since the categories we are considering are semisimple, a k-linear equivalence $\mathcal{F}: \mathcal{C} \to \mathcal{D}$ is determined by its action on the simple objects. It can be made a quasitensor functor, provided the assignment on the level of simple objects is compatible with tensor products, that is, preserves the multiplicities of simples in tensor products of simples; in fact it is suffcient to define $\xi: \mathcal{F}(X) \otimes \mathcal{F}(Y) \to \mathcal{F}(X \otimes Y)$ when X, Y are simple; this is possible if and only if the multiplicity of $\mathcal{F}(Z)$ in $\mathcal{F}(X) \otimes \mathcal{F}(Y)$ is the same as the multiplicity of Z in $X \otimes Y$, for each simple Z; then one can extend in the obvious fashion (as required by naturality of ξ) to direct sums.

We will write $\langle M, N \rangle := \dim_{\mathbb{C}}(\operatorname{Hom}_{\mathcal{C}}(M, N))$ for objects M, N in a semisimple category.

The higher Frobenius–Schur indicators that we will consider were defined in full generality in [17]; the indicators for objects in $^G_G\mathcal{YD}$ are already defined by [15], since Yetter–Drinfeld modules are modules over the Drinfeld double, which is a semisimple Hopf algebra. For the general definition recall that a pivotal monoidal category is a monoidal category which is rigid (dual objects exist) and endowed with a monoidal natural isomorphism $V \to V^{**}$.

Definition 2.1. The m-the Frobenius–Schur indicator of the object V in the pivotal k-linear monoidal category $\mathcal{C}$ is $\nu_m(V) = \mathrm{Tr}(E_V^{(m)})$, where the endomorphism $E_V^{(m)} \in \mathrm{End}_k(\mathcal{C}(I, V^{\otimes m}))$ is defined as the composition

$$\mathcal{C}(I, V^{\otimes m}) \to \mathcal{C}(V^*, V^{\otimes(m-1)}) \to \mathcal{C}(I, V^{\otimes(m-1)} \otimes V^{**}) \cong \mathcal{C}(I, V^{\otimes m})$$

of two instances of hom-tensor adjunctions defined by duality, and the pivotal structure identifying V with its double dual.

The definition itself will play no role in the paper. We will need the formula

$$\nu_m(\mathcal{F}_d(\eta)) = \frac{1}{|G|} \sum_{\substack{x \in G \\ (dx)^m \in G}} \eta((dx)^m) \tag{2.2}$$

for the object of $^\Gamma_G\mathcal{M}_G$ associated to the double coset GdG and the character η of $\mathrm{Stab}_G(dG)$. This formula was derived in [18]; if we are only interested in the case of Drinfeld doubles, the corresponding special case that already appears in [11] is sufficient.

In [16, Thm. 7.5], a formula for Frobenius–Schur indicators in modular categories is derived, generalizing a formula used by Bantay [2] to define degree two indicators: Let (X_j) denote a set of representatives for the isomorphism classes of simples in a modular category $\mathcal{A}$. We have

$$\nu_m(X_j) = \frac{1}{\dim(\mathcal{A})} \sum_{i,k} N_{ik}^j d_i d_k \left(\frac{\omega_i}{\omega_k} \right)^m, \tag{2.3}$$

where $N_{ik}^j = \dim \mathcal{A}(X_i \otimes X_k, X_j)$ are the fusion coefficients, and ω_i are the components of the (diagonal) T-matrix, or the ribbon structure of $\mathcal{A}$; by d_j we denote the pivotal dimension of X_j, that is, the pivotal trace of the identity on X_j. In the case of interest to us, these dimensions are the ordinary dimensions of Yetter–Drinfeld modules (this defines the canonical pivotal structure of $\mathcal{A} = ^G_G\mathcal{YD}$), while $\dim(\mathcal{A}) = |G|^2$. As remarked in [16], the formula (2.3) expresses the indicators in terms of the modular data, since Verlinde's formula, see [1, Thm. 3.1.13], expresses the fusion coefficients in terms of the S-matrix:

$$N_{ik}^j = \sum_r \frac{s_{ir} s_{kr} s_{\bar{j}r}}{s_{0r}} = \frac{1}{\dim(\mathcal{A})} \sum_r \frac{S_{ir} S_{kr} S_{\bar{j}r}}{S_{0r}}, \tag{2.4}$$

where s and S are, respectively, a normalized, and the unnormalized S-matrix, that is, S_{ij} is the pivotal trace of $c_{X_j, X_i^*} c_{X_i^*, X_j}$, where c denotes the braiding of $\mathcal{A}$; In the case of interest to us, the pivotal trace is just the trace.

3. An autoequivalence of graded bimodules

In this section we define an autoequivalence of the category of graded bimodules $_G^\Gamma \mathcal{M}_G$ that generalizes the r-th Adams operator for r prime to the exponent of G in case $\Gamma = G$. (Note that $_G^G \mathcal{M}_G \cong \mathrm{Rep}(G)$ as tensor categories.)

Let Γ be a finite group and $G \subset \Gamma$ a subgroup. Let $e = \exp(G)$, and r an integer relatively prime to e. Since r is also relatively prime to the exponent of all the stabilizer subgroups of G that occur in the structure description of $_G^\Gamma \mathcal{M}_G$ given above, the characters of all those stabilizer subgroups take values in a cyclotomic field of order prime to r. Thus, we can apply the automorphism ψ_r to those characters.

Definition 3.1. Let Γ be a finite group, $G \subset \Gamma$ a subgroup, and $e = \exp(G)$. Let r be an integer relatively prime to e. Then we define an autoequivalence ψ_r of $_G^\Gamma \mathcal{M}_G$ by

$$\psi_r (\mathcal{F}_d(\eta)) := \mathcal{F}_d(\psi_r(\eta)). \tag{3.1}$$

for $d \in \Gamma$.

Remark 3.2. If Γ is abelian, then $_G^\Gamma \mathcal{M}_G \cong {}_G^{\Gamma/G} \mathcal{M} \cong {}_{\mathbb{C}[\Gamma/G]^* \otimes \mathbb{C}G} \mathcal{M}$ as monoidal categories. More precisely $\mathrm{Stab}_G(dG) = G$ for every $d \in \Gamma$, and $\mathcal{F}_d(V) = V \otimes \mathbb{C}G$ with the obvious right G-action and the diagonal left G-action, and we have canonical isomorphisms

$$\mathcal{F}_d(V) \underset{\mathbb{C}G}{\otimes} \mathcal{F}_{d'}(W) = (V \otimes \mathbb{C}G) \underset{\mathbb{C}G}{\otimes} (W \otimes \mathbb{C}G) \to (V \otimes W) \otimes \mathbb{C}G = \mathcal{F}_{dd'}(V \otimes W)$$

in $_G^\Gamma \mathcal{M}_G$ making the collection of functors $\mathcal{F}_d$ a monoidal functor with respect to the monoidal structure given by multiplication in Γ/G and the usual tensor products of representations of G.

Now the adams operator ψ_r for r relatively prime to the exponent of G is induced in this case by the automorphism ϕ_r of G given by $\phi_r(g) = g^r$. In particular the equivalence ψ_r from Theorem 3.1 is a monoidal autoequivalence of $_G^\Gamma \mathcal{M}_G$.

For general Γ we will obtain the weaker property that ψ_r preserves tensor products.

Lemma 3.3. *One can choose bases $B = (b_i)$ for each $M \in {}_G^\Gamma \mathcal{M}_G$ consisting of Γ-homogeneous elements such that, if $L_i^j(M, g), R_i^j(M, g) \in \mathbb{C}$ for $g \in G$ are defined by $g b_i = \sum L_i^j(M, g) b_j$ and $b_i g = \sum R_i^j(M, g) b_j$, then*

(1) $L_i^j(M, g) \in \mathbb{Q}(\zeta_e)$.

(2) $(R_i^j(M, g))_{i,j}$ *is a permutation matrix.*

(3) $L_i^j(\psi_r(M), g) = \sigma_r(L_i^j(M, g))$.

(4) $R_i^j(\psi_r(M), g) = R_i^j(M, g)$.

Proof. It suffices to pick such bases for each $\mathcal{F}_d(W)$, where d runs through a system of representatives for the double cosets of G in Γ, and W runs through the irreducible representations of $S = \text{Stab}_G(dG)$.

By Brauer's Theorem, we can pick a basis (b'_α) for such a representation $W \in \text{Rep}(S)$ such that the corresponding matrix representation of S has coefficients in $\mathbb{Q}(\zeta_e)$. The representation $\psi_r(W)$ can then be realized on the same underlying vector space by Galois conjugating those matrix coefficients. We will choose to realize all the representations in the same ψ_r-orbit in this way. For the induced representation $N = \text{Ind}_S^G(W) = \mathbb{C}G \underset{\mathbb{C}S}{\otimes} W$ we choose the basis $b''_{c,\alpha} = c \otimes b'_\alpha$ indexed by the index set of the basis (b'_α) along with a set of representatives of G/S. Note that $b''_{c,\alpha}$ has Γ/G-degree cdG. It is easy to check that the matrix of the left action of $g \in G$ on $\text{Ind}_S^G(W)$ has coefficients in $\mathbb{Q}(\zeta_e)$, and that the σ_r-conjugate matrix describes the action on $\text{Ind}_S^G(\psi_r(W))$. Let $\tilde{N} = N \otimes \mathbb{C}\Gamma$, with the diagonal left G-action and the right G-action on $\mathbb{C}\Gamma$. Then $\tilde{N}$ has basis $(b''_{c,\alpha} \otimes \gamma)$, indexed by c, α as above and $\gamma \in G$; with respect to this basis, the right action of G is by permutation matrices, and the left action is by the Kronecker products of the left action on N and permutation matrices. According to (2.1), a subset of the chosen basis of $\tilde{N}$ gives a basis of $\mathcal{F}_d(W) \in {}_G^\Gamma \mathcal{M}_G$, and thus the matrices of the left and right action of G on the latter behave as claimed. □

Proposition 3.4. *Let Γ be a finite group, $G \subset \Gamma$ a subgroup, $e = \exp(G)$ and r relatively prime to e.*

(1) *The functor ψ_r from Theorem 3.1 is a quasitensor autoequivalence.*

(2) *We have $\nu_m(\psi_r(M)) = \sigma_r(\nu_m(M))$ for each $M \in {}_G^\Gamma \mathcal{M}_G$.*

Proof. The second claim is obvious from the indicator formula obtained in [18]. For the first claim we have to check that the multiplicity

$$\langle M \underset{\mathbb{C}G}{\otimes} N, P \rangle = \dim \text{Hom}_{G-G}^\Gamma (M \underset{\mathbb{C}G}{\otimes} N, P) \tag{3.2}$$

for three simple objects $M, N, P \in {}_G^\Gamma \mathcal{M}_G$ does not change upon application of ψ_r to all three objects. We choose bases (b_i) for M, (c_i) for N and d_i for P for the three objects as in Theorem 3.3. The morphisms we "count" in (3.2) are described by $\mathbb{C}$-linear maps $f: M \otimes N \to P$ subject to the conditions $gf(m \otimes n) = f(gm \otimes n)$, $f(mg \otimes n) = f(m \otimes gn)$ and $f(m \otimes ng) = f(m \otimes n)g$ for all $g \in G$, where it suffices to let m, n run through bases of M and N. Write $f(b_i \otimes c_j) = \sum \phi_{ij}^k d_k$ for $\phi_{ij}^k \in \mathbb{C}$. Then the three conditions on f translate to relations between the

matrix elements ϕ_{ij}^k and the matrix elements for the left and right G-actions from Theorem 3.3, to wit:

$$\sum L_i^k(M, g)\phi_{kj}^\ell = \sum \phi_{ij}^m L_m^\ell(P, g)$$

$$\sum R_i^k(M, g)\phi_{kj}^\ell = \sum \phi_{ik}^\ell L_j^k(N, g)$$

$$\sum R_j^k(N, g)\phi_{ik}^\ell = \sum \phi_{ij}^k R_k^\ell(P, g)$$

where the respective sums run over the index occurring twice. When we apply the functor ψ_r to the three objects M, N, P, the matrices of the left actions are conjugated by σ_r, as are the (integer) matrices of the right actions. Thus, conjugating the matrix ϕ of f by σ_r as well yields a homomorphism between the conjugated objects, whence the claim. $\qquad \square$

4. An autoequivalence of Yetter–Drinfeld modules

Let G be a finite group, and r an integer relatively prime to $\exp(G)$. We consider the following autoequivalence $\widetilde{\psi}_r$ of the category $_G^G\mathcal{YD}$ of Yetter–Drinfeld modules: For $M \in {}_G^G\mathcal{YD}$ let $\widetilde{\psi}_r(M) = M$ as left $\mathbb{C}G$-module, but modify the grading by setting $\widetilde{\psi}_r(M)_{g^r} = M_g$. We will denote by $\widetilde{\psi}_r(m)$ the element $m \in M$ considered as an element of $\widetilde{\psi}_r(M)$, so that $|\widetilde{\psi}_r(m)| = |m|^r$ if m is homogeneous. The fact that $\widetilde{\psi}_r(M)$ is a Yetter–Drinfeld module is a trivial consequence of the power map's being equivariant with respect to the adjoint action. $\widetilde{\psi}_r$ is an autoequivalence since r is relatively prime to $\exp(G)$, the inverse being $\widetilde{\psi}_s$ when $rs \equiv 1 \mod \exp(G)$.

We remark that if G is abelian, then $\widetilde{\psi}_r$ is a monoidal category equivalence: Yetter–Drinfeld modules in this case are just G-graded modules with a G-action on each homogeneous component. Since taking r-th powers is an automorphism for abelian G, the functor $\widetilde{\psi}_r$ modifying the grading by this automorphism and leaving the action unchanged is monoidal.

We will denote by $(X_i)_{i \in I}$ a representative set for the isomorphism classes of simples of $_G^G\mathcal{YD}$, and define $\widetilde{\psi}_r(i)$ by $\widetilde{\psi}_r(X_i) = X_{\widetilde{\psi}_r(i)}$ for $i \in I$.

We need to examine the behavior of the modular structure of the category $_G^G\mathcal{YD}$ under the functor $\widetilde{\psi}_r$. This is rather simple for the T-matrix or ribbon structure θ of the category: The map θ_M maps m of degree g to gm of the same degree. Therefore, $\theta_{\widetilde{\psi}_r(M)} = (\theta_M)^r$. If $M = X_i$ is simple, so $\theta_M = \omega_i \, \mathrm{id}_M$ for a root of unity ω_i, we obtain $\omega_{\widetilde{\psi}_r(i)} = (\omega_i)^r = \sigma_r(\omega_i)$.

We proceed to investigate the S-matrix, denoting by c the braiding in the braided monoidal category ${}^G_G\mathcal{YD}$. Let $M, N \in {}^G_G\mathcal{YD}$. Then for $m \in M_g$ and $n \in N_h$ we have

$$
\begin{aligned}
c^2(m \otimes n) &= c(gn \otimes m) \\
&= |gn|m \otimes gn \\
&= ghg^{-1}m \otimes gn.
\end{aligned}
$$

We can endow $M \otimes N$ with a $G \times G$-grading composed of the G-gradings of M and N. Then

$$
\begin{aligned}
\deg_{G \times G} c^2(m \otimes n) &= (ghg^{-1}g(ghg^{-1})^{-1}, ghg^{-1}) \\
&= (ghg^2h^{-1}g^{-1}, ghg^{-1}).
\end{aligned}
$$

For a finite group Γ, a Γ-graded vector space V, and an endomorphism f of V write f_0 for the trivial component of f with respect to the Γ-grading of $\mathrm{End}(V)$. Then $\mathrm{Tr}(f) = \mathrm{Tr}(f_0)$.

In our example, considering the $G \times G$-grading of $M \otimes N$, we see that $c^2(m \otimes n)$ has the same degree as $m \otimes n$ if and only if g and h commute; thus $(c^2)_0(m \otimes n) = 0$ unless $(g, h) = 1$, and $(c^2)_0(m \otimes n) = hm \otimes gn$ if $(g, h) = 1$.

The last equation implies also that $(c^2)_0$ is of finite order dividing $\exp(G)$, and thus, for r relatively prime to $\exp(G)$, we have $\mathrm{tr}\left(((c^2)_0^r\right) = \sigma_r\left(\mathrm{tr}\left((c^2)_0\right)\right)$. On the other hand, $(c^2_{\widetilde{\psi}_r(M),\widetilde{\psi}_r(N)})_0 = \left(c^2_{MN}\right)_0^r$ (note that $gh = hg$ if and only if $g^r h^r = h^r g^r$). Taking these results together, we obtain $\mathrm{tr}(c^2_{\widetilde{\psi}_r M, \widetilde{\psi}_r N}) = \sigma_r(\mathrm{tr}(c^2_{MN}))$.

Theorem 4.1. *Let G be a finite group, and r an integer relatively prime to $\exp(G)$. Then $\widetilde{\psi}_r: {}^G_G\mathcal{YD} \to {}^G_G\mathcal{YD}$ is a quasitensor autoequivalence of ${}^G_G\mathcal{YD}$.*

We have $S_{\widetilde{\psi}_r(i),\widetilde{\psi}_r(j)} = \sigma_r(S_{ij})$ for simples i, j of ${}^G_G\mathcal{YD}$, and $\nu_m(\widetilde{\psi}_r(V)) = \sigma_r(\nu_m(V)) = \nu_{mr}(V)$ for $m \in \mathbb{Z}$.

Proof. We have already proved that $\widetilde{\psi}_r$ preserves the S-matrix up to Galois conjugation.

By the Verlinde formula (2.4), this implies that the multiplicities N^k_{ij} of a simple X_k in $X_i \otimes X_j$ are also preserved up to Galois conjugation, that is, $N^{\widetilde{\psi}_r(k)}_{\widetilde{\psi}_r(i),\widetilde{\psi}_r(j)} = \sigma_r(N^k_{ij})$. But the multiplicities are integers, and thus $N^{\widetilde{\psi}_r(k)}_{\widetilde{\psi}_r(i),\widetilde{\psi}_r(j)} = N^k_{ij}$.

Now $\widetilde{\psi}_r$ also preserves the T-matrix up to Galois conjugation. Thus (2.3) implies that Frobenius–Schur indicators behave under $\widetilde{\psi}_r$ as claimed above. $\square$

Remark 4.2. Iovanov, Mason, and Montgomery [9] have found examples of finite groups G such that not all the higher Frobenius–Schur indicators of simple modules of the Drinfeld double $D(G)$ are integers. For such G (the examples in [9] have order 5^6), the autoequivalences $\widetilde{\psi}_r$ do not preserve Frobenius–Schur indicators,

and thus they cannot be endowed with the structure of a *monoidal* autoequivalence of $^G_G \mathcal{YD}$.

It remains to observe how the functors $\widetilde{\psi}_r$ behave with respect to the "local" description of simples of $^G_G \mathcal{YD}$ in terms of representations of centralizers. The proof of the following result should be rather obvious:

Lemma 4.3. *Let G be a finite group, and r an integer relatively prime to $\exp(G)$. For $g \in G$ and $V \in \mathrm{Rep}(C_G(g))$ we have*

$$\widetilde{\psi}_r \left(\mathcal{G}_g \left(V \right) \right) = \mathcal{G}_{g^r} \left(V \right) \tag{4.1}$$

In other words, simple Yetter–Drinfeld modules are parametrized by an element g of G and an irreducible representation of its centralizer. Applying $\widetilde{\psi}_r$ amounts to taking the same representation, but regarding it as belonging to g^r (which has the same centralizer) instead of g.

Typically, doing explicit calculations with the category $^G_G \mathcal{YD}$, one would choose a system $\mathfrak{C}$ of representatives for the conjugacy classes in G, so that every simple is of the form $\mathcal{G}_g(V)$ for a *unique* $g \in \mathfrak{C}$ and $V \in \mathrm{Irr}(C_G(g))$. The preceding lemma, by contrast, does not follow this partition of the simples, since $g \in \mathfrak{C}$ does not imply $g^r \in \mathfrak{C}$.

Remark 4.4. Let G be a finite group, $\mathfrak{C}$ a system of representatives of the conjugacy classes of G, and r an integer relatively prime to $\exp(G)$. For $g \in \mathfrak{C}$ choose $x \in G$ such that $x \rhd g^r \in \mathfrak{C}$. Then for $V \in \mathrm{Rep}(C_G(g))$ we have

$$\widetilde{\psi}_r(\mathcal{G}_g(V)) = \mathcal{G}_{x \rhd g^r}(x \rhd V). \tag{4.2}$$

5. An autoequivalence preserving indicators

In the previous section we have constructed a quasitensor autoequivalence of $^G_G \mathcal{YD}$ preserving higher Frobenius–Schur indicators up to Galois conjugation. Another such equivalence is given by the results in Section 3, since the category of Yetter–Drinfeld modules can be viewed as a special case of the category of graded bimodules. Combining the two (rather, one of them with the inverse of the other) we obtain an quasitensor autoequivalence of $^G_G \mathcal{YD}$ that preserves higher Frobenius–Schur indicators. It is still not a *monoidal* equivalence by recent results of Davydov.

Definition 5.1. Let G be a finite group and r an integer relatively prime to the exponent $e = \exp(G)$. Define an autoequivalence $\widehat{\psi}_r$ of $^G_G \mathcal{YD}$ by $\widehat{\psi}_r = \widetilde{\psi}_s \circ \psi_r$, where $sr \equiv 1 \bmod e$.

Thus, $\widehat{\psi}_r \left(\mathcal{G}_{g^r}(V) \right) = \mathcal{G}_g(\psi_r(V))$.

Corollary 5.2. $\widehat{\psi}_r$ *is a quasitensor autoequivalence of $^G_G \mathcal{YD}$ and satisfies $\nu_m(\widehat{\psi}_r(V)) = \nu_m(V)$ for all $V \in ^G_G \mathcal{YD}$.*

Remark 5.3. $\widehat{\psi}_r$ cannot, in general, be endowed with the structure of a monoidal autoequivalence of ${}^G_G\mathcal{YD}$. In fact the restriction of $\widehat{\psi}_r$ to the monoidal subcategory $\mathrm{Rep}(G) \subset {}^G_G\mathcal{YD}$ of those Yetter–Drinfeld modules whose grading is trivial is the usual Adams operator. There are examples where these Adams operators with $r \equiv -1 \bmod e$ do not arise from a monoidal autoequivalence. The following argument is due to Alexei Davydov: First, for a simple group G, any autoequivalence of $\mathrm{Rep}(G)$ must arise from a group automorphism of G. In fact, autoequivalences of $\mathrm{Rep}(G)$ are classified by G-Bi-Galois algebras [7, 20]. But if G is simple, then by [7, Thm. 3.8 and Prop. 6.1] such G-Bi-Galois algebras are trivial as one-sided G-Galois algebras, and thus they are described by group automorphisms (cf. [20, Lem. 3.11]). Now for an automorphism of G to give rise to the Adams operator ψ_{-1}, it has to be a class-inverting automorphism (see [6]), and there exist simple groups where such automorphisms do not exist ([6] gives the Mathieu group M_{11}).

On the other hand, if G is abelian, then $\widehat{\psi}_r$ is just the autoequivalence of ${}^G_G\mathcal{YD}$ induced by the automorphism of G raising elements to the r-th power, and thus is clearly a monoidal autoequivalence. Of course we have already observed that both ψ_r and $\widetilde{\psi}_r$ separately are monoidal in this case.

Remark 5.4. Alexei Davydov pointed out that composing ψ_r with $\widetilde{\psi}_r$ (instead of using the inverse autoequivalence of one of the two) yields the Galois symmetry of ${}^G_G\mathcal{YD}$. This can be read off from the description of the latter in [3, p. 689]. Thus the two quasitensor autoequivalences can be viewed as decomposing the Galois symmetry (defined for any modular fusion category) in two factors, in the particular case of the Drinfeld double of a finite group.

Corollary 5.5. *Let G be a finite group, r an integer prime to the exponent of G, $m \in \mathbb{Z}$, and $g, y \in G$. Then*

$$|\{x \in G \,|\, x^m = (gx)^m = y\}| = |\{x \in G \,|\, x^m = (g^r x)^m = y^r\}|. \qquad (5.1)$$

Proof. According to [9, Prop. 2.3] we have

$$|\{x \in G \,|\, x^m = (gx)^m = y\}| = \sum_{\chi \in \mathrm{Irr}(C_G(g))} v_m\big(\mathcal{G}_g(\chi)\big)\,\chi(y).$$

Since ψ_r is a bijection on characters and $v_m \circ \psi_r = v_m \circ \widetilde{\psi}_r$,

$$\sum_\chi v_m\big(\mathcal{G}_g(\chi)\big)\,\chi = \sum_\chi v_m\big(\mathcal{G}_g(\psi_r(\chi))\big)\,\psi_r(\chi)$$

$$= \sum_\chi v_m\big(\mathcal{G}_{g^r}(\chi)\big)\,\psi_r(\chi),$$

and

$$\sum_\chi v_m\big(\mathcal{G}_{g^r}(\chi)\big)\,\psi_r(\chi)(y) = \sum_\chi v_m\big(\mathcal{G}_{g^r}(\chi)\big)\,\chi(y^r)$$

$$= |\{x \in G \,|\, x^m = (g^r x)^m = y^r\}|$$

finishes the proof. □

6. Some small examples

If $r = -1$, the Adams operator ψ_{-1} is complex conjugation of characters, and $\widetilde{\psi}_{-1}$ is complex conjugation of the "local" characters. It is known that higher Frobenius–Schur indicators of Drinfeld doubles are always real, so the operations ψ_{-1} and $\widetilde{\psi}_{-1}$ on ${}^G_G\mathcal{YD}$ preserve indicators, for any group G.
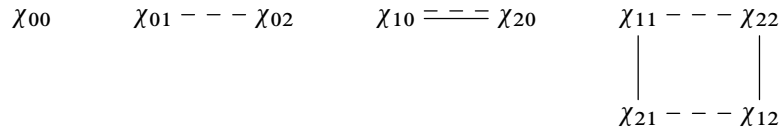
Example 6.1. Consider the semidirect product

$$G = \mathbb{Z}_3^2 \rtimes \mathbb{Z}_2$$

where $\mathbb{Z}_2$ acts on the first factor of $\mathbb{Z}_3^2$ by inversion, and trivially on the second. In other words

$$G = \langle a, b, \tau \mid a^3 = b^3 = \tau^2 = 1, \tau a = a^2 \tau, \tau b = b \tau \rangle.$$

The centralizer of $g = a$ is $C_G(a) = \langle a, b \rangle \cong C_3^2$. We denote its (linear) characters by χ_{uv} with $\chi_{uv}(a) = \zeta^u$ and $\chi_{uv}(b) = \zeta^v$, where ζ is a primitive third root of unity. By construction a^{-1} is conjugate to a by τ. Note, though, that complex conjugation of χ_{uv} affects both its indices, while conjugation by τ only affects the first. The following diagram lists a dashed line between χ and χ' if $\mathcal{G}_a(\chi') = \psi_{-1}(\mathcal{G}_a(\chi))$ and a solid line if $\mathcal{G}_a(\chi') = \widetilde{\psi}_{-1}(\mathcal{G}_a(\chi))$:

$$
\begin{array}{ccccccc}
\chi_{00} & & \chi_{01} \text{---} \chi_{02} & & \chi_{10} =\!=\!= \chi_{20} & & \chi_{11} \text{---} \chi_{22} \\
& & & & & & \;\mid \qquad\qquad \mid \\
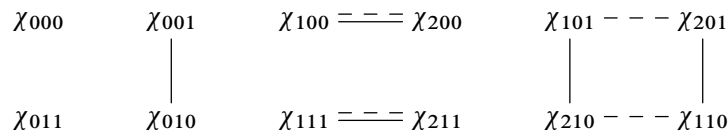& & & & & & \chi_{21} \text{---} \chi_{12}
\end{array}
$$

Thus, the nine characters of $C_G(a)$ fall in four classes of necessarily I-equivalent characters.

Example 6.2. Now consider $G = (C_3 \times C_2^2) \rtimes C_2$ with C_2 acting on C_3 by inversion, and on C_2^2 by switching factors. Thus

$$G = \langle a, b, c, \tau \mid a^3 = b^2 = \tau^2 = 1, \tau a = a^2 \tau, \tau b = c \tau, ab = ba, bc = cb \rangle.$$

We consider $g = a$ with $C_G(a) = \langle a, b, c \rangle \cong C_3 \times C_2^2$, and label its characters χ_{uvw} with $\chi_{uvw}(a) = \zeta^u$, $\chi_{uvw}(b) = (-1)^v$, and $\chi_{uvw}(c) = (-1)^w$. Again, a is conjugate to its inverse by τ. Complex conjugation of χ_{uvw} affects only its first index, while conjugation by τ affects the first and switches the other two indices. Recording as in the previous example how ψ_{-1} and $\widetilde{\psi}_{-1}$ relate the various $\mathcal{G}_a(\chi_{uvw})$ we obtain the following picture:

$$
\begin{array}{ccccccc}
\chi_{000} & & \chi_{001} & & \chi_{100} =\!=\!= \chi_{200} & & \chi_{101} \text{---} \chi_{201} \\
& & \mid & & & & \;\mid \qquad\qquad \mid \\
\chi_{011} & & \chi_{010} & & \chi_{111} =\!=\!= \chi_{211} & & \chi_{210} \text{---} \chi_{110}
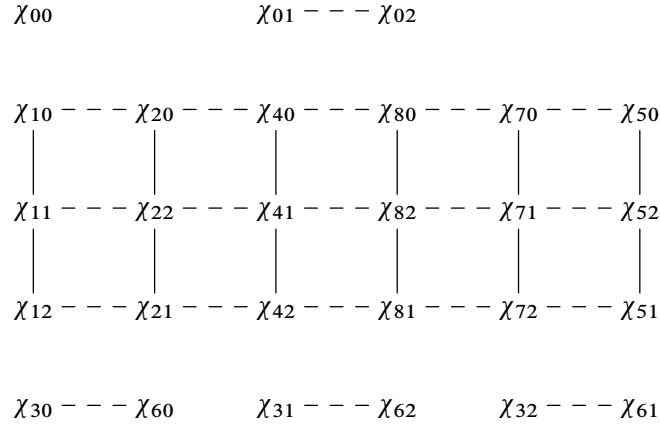\end{array}
$$

In particular, the twelve simples of $^G_G\mathcal{YD}$ in the image of $\mathcal{G}_a$ fall in six orbits under the actions of the two autoequivalences.

Example 6.3. Consider $G = \langle a, b | a^{27} = b^3 = 1, ba = a^{19}b \rangle$, that is $G \cong C_{27} \rtimes C_3$, with the action of the generator b of C_3 sending the generator a of C_{27} to a^{19}.

Consider $g = a^3b$. It is easy to check that $C_G(g) = \langle a^3, b \rangle \cong C_9 \times C_3$. We label the characters of $C_G(g)$ as χ_{uv} with $\chi_{uv}(a^3) = \zeta^u$ and $\chi_{uv}(b) = \zeta^{3v}$, where ζ is a primitive ninth root of unity.

We will study ψ_2 and $\widetilde{\psi}_2$ (note that 2 is a primitive root modulo 27). Clearly ψ_2 acts on $\mathcal{G}_g(\chi_{uv})$ by doubling both indices (modulo 9 and 3, respectively). On the other hand $\widetilde{\psi}_2$ will map the image of $\mathcal{G}_g$ to the image of $\mathcal{G}_{g^2}$, and $g^2 = a^6b^2$ is not conjugate to g. However, $g^4 = a^{12}b$ is conjugate to g: For $t = a^2b$ we have $tg^4 = a^2ba^{12}b = a^{14}b^2$ and $gt = a^3ba^2b = a^{41}b^2 = a^{14}b^2$, so $t \triangleright g^4 = g$. Note $t^{-1} = a^7b^2$. To calculate $t \triangleright \chi_{uv}$ we need $t^{-1} \triangleright a^3 = a^3$ and $t^{-1} \triangleright b = a^7b^3a^2b = a^9b$. Thus $t \triangleright \chi_{uv}(a^3) = \chi_{uv}(a^3) = \zeta^u$ and $t \triangleright \chi_{uv}(b) = \chi_{uv}(a^9b) = \zeta^{3u+3v}$. In other words $t \triangleright \chi_{uv} = \chi_{u,v+u}$. The following diagram lists the 27 characters of the centralizer. Characters connected by a dashed, resp. solid line have their images under $\mathcal{G}_g$ mapped to each other by ψ_2, resp. $\widetilde{\psi}_4$. To avoid overloading the diagram, we have not indicated the direction of the actions, nor closed the cycles of a repeated application of the two mappings.

$$
\begin{array}{llllll}
\chi_{00} & & \chi_{01} {-}{-}{-} \chi_{02} & & & \\[2ex]
\chi_{10} {-}{-}{-} \chi_{20} {-}{-}{-} \chi_{40} {-}{-}{-} \chi_{80} {-}{-}{-} \chi_{70} {-}{-}{-} \chi_{50} \\[1ex]
\;|\qquad\quad\;| \qquad\quad\;|\qquad\quad\;|\qquad\quad\;|\qquad\quad\;| \\[1ex]
\chi_{11} {-}{-}{-} \chi_{22} {-}{-}{-} \chi_{41} {-}{-}{-} \chi_{82} {-}{-}{-} \chi_{71} {-}{-}{-} \chi_{52} \\[1ex]
\;|\qquad\quad\;| \qquad\quad\;|\qquad\quad\;|\qquad\quad\;|\qquad\quad\;| \\[1ex]
\chi_{12} {-}{-}{-} \chi_{21} {-}{-}{-} \chi_{42} {-}{-}{-} \chi_{81} {-}{-}{-} \chi_{72} {-}{-}{-} \chi_{51} \\[2ex]
\chi_{30} {-}{-}{-} \chi_{60} \quad \chi_{31} {-}{-}{-} \chi_{62} \quad \chi_{32} {-}{-}{-} \chi_{61}
\end{array}
$$

Thus, the 54 simples associated to g and g^2 fall in 6 orbits under the combined actions of ψ_2 and $\widetilde{\psi}_2$; the elements in each orbit are I-equivalent.

The result for $g = b$ is substantially different, although $C_G(b) = \langle a^3, b \rangle$ as well. Since $b^4 = b$, the action of $\widetilde{\psi}_4$ on the image of $\mathcal{G}_b$ is trivial. Thus, the 54 simples associated to b and b^2 fall into 8 orbits.

Remark 6.4. In the three examples studied in this section, one can form an autoequivalence $\widehat{\psi}_r$ acting within the image of one of the functors $\mathcal{G}_g$. It so happens that in each of the examples this autoequivalence is induced by an automorphism

of the group G under consideration. In the first example this is the automorphism fixing a and τ and inverting b. In the second example it is the automorphism fixing a and τ while switching b and c. In the third example the automorphism of G sending a to a^{22} and b to $a^{18}b$ can be checked to (exist and) send a^3b to itself, a^3 to $a^{12} = (a^3)^4$ and b to $a^{18}b = (a^3)^6b$, so that its effect on $\mathcal{G}(\chi_{uv})$ is to map it to $\mathcal{G}(\chi_{4u,v+2u})$, the same as $\widetilde{\psi}_4^{-1}$ followed by ψ_4.

In particular, all the I-equivalences of characters in these cases that are accounted for by the joint operations of ψ- and $\widetilde{\psi}$-equivalences can also be accounted for by ψ-equivalences and group automorphisms, thus leaving out the somewhat more mysterious $\widetilde{\psi}$-equivalences.

We have remarked in the previous section that in general the $\widehat{\psi}_r$-autoequivalences are not in general monoidal, so cannot, *a fortiori*, come from group automorphisms. We haven't, however, worked out an example of this generality.

7. Doubles of symmetric groups

One of the author's original motivations for the study of the functors $\widetilde{\psi}_r$ was to explain an observation made by Courter [4] during a computer-aided investigation into the higher Frobenius–Schur indicators of the simple modules of Drinfeld doubles of symmetric groups. To wit, many of these simple modules fall into rather large I-equivalence classes; for larger ranks of the symmetric groups each I-equivalence classes is contained in the image of one of the functors $\mathcal{G}_g$. We will see in this section that the functors $\widetilde{\psi}_r$ *do not help in the least* to explain the abundance of I-equivalent modules.

The fact that Galois conjugate characters of $C_G(g)$ yield I-equivalent representations of $D(G)$ is (albeit not mentioned in [4]) quite obvious, but accounts for only relatively few of the I-equivalent representations. We will discuss a few examples, using the results of Courter [4] for the indicator values.

We begin with S_3, where the exponent is 6 and the unit group of $\mathbb{Z}_6$ is $\{\pm 1\}$. The characters of the centralizers $C_{S_3}(()) = S_3$ and $C_{S_3}((12)) = \langle(12)\rangle \cong C_2$ are integer-valued, hence fixed by ψ_{-1}, and clearly also by $\widetilde{\psi}_{-1}$. The two nontrivial characters of $C_{S_3}((123)) = \langle(123)\rangle \cong C_3$ are exchanged by ψ_{-1}. Courter has calculated the sequences of indicators $(\nu_1, \nu_2, \nu_3, \nu_6)$ and found $(0, 1, 1, 2)$ for all three simples in the image of $\mathcal{G}_{(123)}$ (not only the two mapped to each other by ψ_{-1}), but also for one of the objects in the image of $\mathcal{G}_0$. Also, the two objects in the image of $\mathcal{G}_{(12)}$, albeit unmoved by ψ_{-1}, share the sequence $(0, 1, 0, 3)$.

For S_4, the exponent is 12 and $(\mathbb{Z}_{12})^{\times} = \{\pm 1, \pm 5\}$. The centralizers $C_{S_4}(()) = S_4$, $C_{S_4}((12)) = \langle(12), (34)\rangle$, a Klein 4-group, and $C_{S_4}((12)(34)) \cong D_8$ have integer-valued characters unaffected by Adams operations. The two nontrivial characters of $C_{S_4}((123)) \cong C_3$ are flipped by ψ_{-1} (which agrees with ψ_5 on this centralizer of exponent 3, while $\psi_{-5} = \mathrm{id}$). The same happens for the two irreducible

characters of $C_{S_4}((1234)) \cong C_4$ that are not integer-valued (here $\psi_5 = $ id and $\psi_{-5} = \psi_{-1}$ since the centralizer has exponent 4). Again, we can look up the sequences of relevant indicators $(\nu_1, \nu_2, \nu_3, \nu_4, \nu_6, \nu_{12})$ in Courter's work. Not only two, but all four objects in the image of $\mathcal{G}_{(1234)}$ share the sequence $(0, 1, 0, 2, 3, 6)$, and this with all the objects in the image of $\mathcal{G}_{(12)}$ that are not related to them, nor among themselves, by the ψ functors. Moreover, four of the five ψ-invariant objects in the image of $\mathcal{G}_{(12)(34)}$ share the sequence $(0, 1, 1, 2, 2, 3)$ with two simples in the image of $\mathcal{G}_{()}$.

We will discuss S_5 with a little less detail: For $u \in \{(), (12), (12)(34)\}$ the characters of $C_{S_5}(u)$ are integer valued, but two of the simple objects associated to $()$ are I-equivalent; and the six simples associated to (12) fall in only two I-equivalence classes, as do the five simples associated to $(12)(34)$. Things look slightly better for $C_{S_5}((123)) \cong C_2 \times C_3$: since conjugation flips the nontrivial characters of C_3, we get four ψ-orbits. However, Courter reports only two distinct indicator sequences. Worse, $C_{S_5}((123)(45)) = C_{S_5}((123))$, but here all the six associated simples are I-equivalent. For (1234) we get four characters in three conjugacy classes, but only one I-equivalence class. Since $C_{S_5}((12345)) \cong C_5$, Galois conjugations that are not simply complex conjugation are needed for the first time to put all the four nontrivial characters in the same ψ-orbit; Courter however, reports that all five characters including the trivial one are I-equivalent.

Summing up, Galois conjugacy, or the operations ψ_r, account for some, but by far not all I-equivalences. We have seen in the previous section that the operation $\widetilde{\psi}_r$ can produce I-equivalences not accounted for by ψ_r, but unfortunately in the present case this phenomenon does not occur at all.

Let us first discuss this in the examples: For S_3, clearly $\widetilde{\psi}_{-1}$ does not affect the simples in the images of $\mathcal{G}_{()}$ and $\mathcal{G}_{(12)}$. Since $(12) \triangleright (123) = (123)^{-1}$, we have $\widetilde{\psi}_{-1}(\mathcal{G}_{(123)}(\chi)) = \mathcal{G}_{(123)}((12) \triangleright \chi)$. But since $(12) \triangleright h = h^{-1}$ for all $h \in C_{S_3}((123))$, $\widetilde{\psi}_{-1}$ also flips the two nontrivial objects in the image of $\mathcal{G}_{(123)}$. The same situation occurs for $C_{S_4}((123))$ and $C_{S_4}((1234))$, while the other centralizers in S_4 are not affected at all by the $\widetilde{\psi}$-functors; the case of S_5 is not more interesting.

For centralizers in S_9 (where we shall not discuss indicator values), a case more illustrative of the general situation occurs: For $g = (123)(456)(789)$ and $t = (12)(45)(78)$ we have $g^{-1} = t \triangleright g$, so $\widetilde{\psi}_{-1}(\mathcal{G}_g(\chi)) = \mathcal{G}_g(t \triangleright \chi)$. But conjugation by t does not generally agree with inversion on $C_{S_9}(g)$, which contains the element $h = (147)(258)(369)$ commuting with t. However, the action of t on characters is of course determined by its action on conjugacy classes, so to save the fact that $\psi_{-1} = \widetilde{\psi}_{-1}$ on $C_{S_9}(g)$ it is enough that x^{-1} is conjugate to $t \triangleright x$ within the centralizer $C_{S_9}(g)$, for all x; for example $s \triangleright (t \triangleright h) = s \triangleright h = h^{-1}$ for $s = (14)(25)(36) \in C_{S_9}(g)$.

It may seem that we were fortunate in the last example, for in general elements of a centralizer $C_g(S_n)$ that are conjugate in S_n (i.e. share the same cycle structure) need not be conjugate in the centralizer. We will see, however, that it stays generally true that the ψ and $\widetilde{\psi}$ functors agree for the case of symmetric groups.

The proof of Theorem 7.2 was kindly provided by the referee; in the first submitted version it was only a computer-based conjecture.

We note first the general fact:

Lemma 7.1. *Let G be a finite group, and r coprime to the exponent of G. Then the following are equivalent*

(1) $\psi_r = \widetilde{\psi}_r$ on $_G^G\mathcal{YD}$.

(2) (a) *For all $g \in G$, g^r is conjugate to g in G.*

(b) *If $g^r = t \rhd g$ for $g \in G$, then for all $x \in C_G(g)$ the elements x^r and $t \rhd x$ are conjugate in $C_G(g)$.*

Note that if $g^r = t \rhd g = s \rhd g$ then $s^{-1}t \in C_G(g)$, and thus $t \rhd x$ and $s \rhd x$ are conjugate in $C_G(g)$ for $x \in C_G(g)$. The condition that x^r and $t \rhd x$ are conjugate in $C_G(g)$ thus does not depend on the choice of t with $g^r = t \rhd g$.

Proof. ψ_r preserves the image of the functor $\mathcal{G}_g$, while $\widetilde{\psi}_r$ maps it to the image of the functor $\mathcal{G}_{g^r}$. If g and g^r are not conjugate, the images intersect only in the zero object.

Now assume that $g^r = t \rhd g$. Then according to Theorem 4.4, we have $\widetilde{\psi}_r(\mathcal{G}_g(\chi)) = \mathcal{G}_g(t^{-1} \rhd \chi)$ for every character χ of $C_G(g)$. On the other hand $\psi_r(\mathcal{G}_g(\chi)) = \mathcal{G}_g(\psi_r\chi)$. Thus $\psi_r = \widetilde{\psi}_r$ on the image of $\mathcal{G}_g$ if and only if for each χ we have $t^{-1} \rhd \chi = \psi_r\chi$. Now $\chi(t \rhd x) = \chi(x^r)$ for each character χ is equivalent to the condition that $t \rhd x$ and x^r be conjugate in $C_G(g)$ for each x. $\qquad\square$

Proposition 7.2. *For any r relatively prime to $\exp(S_n)$ and $V \in {}_{S_n}^{S_n}\mathcal{YD}$ we have $\psi_r(V) = \widetilde{\psi}_r(V)$.*

Proof. Note that for any $g \in S_n$ and r relatively prime to $\exp(S_n)$, the power g^r is conjugate to g since it has the same cycle structure. In choosing $t \in S_n$ with $t \rhd g = g^r$ we can arrange for t to preserve the orbits of g. Let $g = g_1 \cdots g_u$, where g_i is the product of all the cycles of one length in the cycle decomposition of g (and we include 1-cycles if there are any). The structure of the centralizer of g is well known, see for example [10, Sec. 4.1]. To wit, let $\alpha_i \subset \{1, \ldots, n\}$ be the set of letters occurring in g_i, and S_{α_i} the permutation group on those letters, then $C_{S_n}(g) \cong C_{S_{\alpha_1}}(g_1) \times \cdots \times C_{S_{\alpha_u}}(g_u)$. By our choice of t, we have $t = t_1 \cdots t_u$ with $t_i \in S_{\alpha_i}$ and $g_i^r = t_i \rhd g_i$. Thus, it is enough to consider the case where $u = 1$ and g has no fixed points (the case where $g = \mathrm{id}$ being trivial). Assume that g is the product of k cycles of length ℓ, then it is further known that $C_{S_n}(g)$ is isomorphic to the wreath product $C_\ell \wr S_k \cong C_\ell^k \rtimes S_k$ for a cyclic group C_ℓ of order ℓ.

We can assume that $g = (1, \ldots, \ell)(\ell + 1, \ldots, 2\ell) \cdots ((k-1)\ell + 1 \ldots, k\ell)$. Then the usual choice of an isomorphism $\phi: C_\ell \wr S_k \to C_{S_n}(g)$ is such that $\phi(1_{C_\ell^k}, \sigma)(a\ell + b - \ell) = \sigma(a)\ell + b - \ell$ for $b \in \{1, \ldots, \ell\}$ and $a \in \{1, \ldots, k\}$.

We can choose the element $t \in S_n$ with $g^r = t \triangleright g$ in such a way that it commutes with the image of S_k in S_n; for the choice of ϕ above, we should first choose $s \in S_\ell$ such that $s \triangleright (1, \dots, \ell) = (1, \dots, \ell)^r$, and then define $t \in S_n$ by $t(a\ell + b - \ell) = a\ell + s(b) - \ell$ for $b \in \{1, \dots, \ell\}$.

For this choice of t, we now have, for $v \in C_\ell^k$ and $\sigma \in S_k$: $t \triangleright \phi(v, \sigma) = \phi(v^r, \sigma)$. Thus we have to show that (v^r, σ) is conjugate to $(v, \sigma)^r$ in the wreath product. To do this, it is convenient to change to an additive notation, and we refer to the following lemma. $\square$

Lemma 7.3. *Consider the semidirect product $\mathbb{Z}_\ell^k \rtimes S_k$ where S_k acts on $\mathbb{Z}_\ell^k$ by permuting k-tuples. If r is relatively prime to ℓ and $\exp(S_k)$, then for all $\sigma \in S_k$ the elements (rv, σ) and $(v, \sigma)^r$ are conjugate.*

Proof. Assume $\{1, \dots, k\} = A \sqcup B$, where σ preserves A and B. Write $v = v_A + v_B$ where the components of A with index in B vanish and vice versa, and let $\sigma_{A/B}$ be the permutations induced by σ on A and B. Then $(v, \sigma) = (v_A, \sigma_A)(v_B, \sigma_B)$, a product of commuting factors, $(rv, \sigma) = (rv_A, \sigma_A)(rv_B, \sigma_B)$ and $(v, \sigma)^r = (v_A, \sigma_A)^r (v_B, \sigma_B)^r$. Thus, it is sufficient to consider the case where σ is a k-cycle. In the sequel we will freely identify permutations with automorphisms of $\mathbb{Z}_\ell^k$.

Pick τ such that $\tau \triangleright \sigma = \sigma^r$. Let $w \in \mathbb{Z}_\ell^k$. Then

$$
\begin{aligned}
(w, \tau) \triangleright (rv, \sigma) &= (w, \tau)(rv, \sigma)(-\tau^{-1} w, \tau^{-1}) \\
&= (w - (\tau \triangleright \sigma)w + \tau rv, \tau \triangleright \sigma) \\
&= ((1 - \sigma^r)w + r\tau v, \sigma^r) \\
(v, \sigma)^r &= (v + \sigma v + \sigma^2 v + \cdots + \sigma^{r-1} v, \sigma^r),
\end{aligned}
$$

so it remains to show that one can find $w \in \mathbb{Z}_\ell^k$ such that

$$
(1 - \sigma^r)w = (1 + \sigma + \cdots + \sigma^{r-1} - r\tau)v =: u.
$$

Let $p = (1 + \sigma + \cdots + \sigma^{k-1})$. Then $p\sigma = p$ since $\sigma^k = 1$, and $\tau p = p\tau$ since r is prime to k. Further, all components of pv are equal, so that $\tau pv = pv$. Thus

$$
pu = rpv - r\tau pv = 0.
$$

This in turn means that the sum of all the components of u vanishes. Since $(r, k) = 1$, σ^r is a k-cycle. Thus it remains to observe that

$$
N := \left\{ x \in \mathbb{Z}_\ell^k \,\middle|\, \sum x_i = 0 \right\} = \mathrm{Im}(1 - \kappa)
$$

for every k-cycle κ. In fact it is straightforward to check that the images of the canonical elements $e_i = (\delta_{ij})_{j \in \{1, \dots, k\}} \in \mathbb{Z}_\ell^k$ generate N. $\square$

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P. Schauenburg, Institut de Mathématiques de Bourgogne,
UMR 5584 CNRS Université Bourgogne Franche-Comté, F-21000 Dijon, France
E-mail: peter.schauenburg@u-bourgogne.fr