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Weighted noncommutative regular projective curves

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Abstract. Let $\mathcal{H}$ be a noncommutative regular projective curve over a perfect field k. We study global and local properties of the Auslander–Reiten translation τ and give an explicit description of the complete local rings, with the involvement of τ. We introduce the τ-multiplicity $e_\tau(x)$, the order of τ as a functor restricted to the tube concentrated in x. We obtain a local-global principle for the (global) skewness $s(\mathcal{H})$, defined as the square root of the dimension of the function (skew-) field over its centre. In the case of genus zero we show how the ghost group, that is, the group of automorphisms of $\mathcal{H}$ which fix all objects, is determined by the points x with $e_\tau(x) > 1$. Based on work of Witt we describe the noncommutative regular (smooth) projective curves over the real numbers; those with $s(\mathcal{H}) = 2$ we call Witt curves. In particular, we study noncommutative elliptic curves, and present an elliptic Witt curve which is a noncommutative Fourier–Mukai partner of the Klein bottle. If $\mathcal{H}$ is weighted, our main result will be formulae for the orbifold Euler characteristic, involving the weights and the τ-multiplicities. As an application we will classify the noncommutative 2-orbifolds of nonnegative Euler characteristic, that is, the real elliptic, domestic and tubular curves. Throughout, many explicit examples are discussed.

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1. Introduction

In this article we study categories $\mathcal{H}$ which have the same formal properties as categories $\operatorname{coh}(X)$ of coherent sheaves over a regular projective curve over a field k. The axioms are essentially taken from Lenzing–Reiten [56]. A similar (more general) system of axioms is formulated by Stafford–van den Bergh [80, Sec. 7.1]. Let k be a field and $\mathcal{H}$ a category satisfying the following properties (NC 1) to (NC 5); the conditions (NC 6) and (NC 7) will follow from the others.

(NC 1) $\mathcal{H}$ is small, connected, abelian, and each object in $\mathcal{H}$ is noetherian.

(NC 2) $\mathcal{H}$ is a k-category with finite dimensional Hom- and Ext-spaces.

(NC 3) There is an autoequivalence τ on $\mathcal{H}$ (called the *Auslander–Reiten translation*) such that Serre duality $\operatorname{Ext}^1_{\mathcal{H}}(X,Y)=\operatorname{D}\operatorname{Hom}_{\mathcal{H}}(Y,\tau X)$ holds, where $\operatorname{D}=\operatorname{Hom}_k(-,k)$.

(NC 4) $\mathcal{H}$ contains an object of infinite length.

It follows from Serre duality that $\mathcal{H}$ is a hereditary category, that is, $\operatorname{Ext}^n_{\mathcal{H}}$ vanishes for all $n\geq 2$. Let $\mathcal{H}_0$ be the Serre subcategory of $\mathcal{H}$ formed by the objects of finite length, and let $\mathcal{H}_+$ be the full subcategory of objects not containing a simple object. Then each indecomposable object of $\mathcal{H}$ lies in $\mathcal{H}_+$ or in $\mathcal{H}_0$. Moreover, $\mathcal{H}_0=\coprod_{x\in\mathbb{X}}\mathcal{U}_x$ (for some index set $\mathbb{X}$) where $\mathcal{U}_x$ are connected uniserial categories, called tubes. The objects in $\mathcal{U}_x$ are called (skyscraper sheaves) concentrated in x. We also write $\mathcal{H}=\operatorname{coh}(\mathbb{X})$. In order to avoid so-called degenerated cases, discussed in [56], we additionally assume:

(NC 5) $\mathbb{X}$ consists of infinitely many points.

We call $\mathcal{H}$, and also $\mathbb{X}$, a *weighted noncommutative regular projective curve* over k, if it satisfies the conditions (NC 1) to (NC 5), and we write $\mathcal{H}=\operatorname{coh}(\mathbb{X})$. We recall that, because of (NC 5), our class of noncommutative curves forms a proper subclass of those studied in [56].

Axiom (NC 5) implies, by [56, Cor. 2.4], that for each $x\in\mathbb{X}$ the number $p(x)$ of isomorphism classes of simple objects in $\mathcal{U}_x$ is finite. Also the second part of the following condition will *follow*, from the theory of hereditary orders, compare Theorem 7.11 below.

(NC 6) For all points $x\in\mathbb{X}$ we have $p(x)<\infty$, and for all except finitely many we have $p(x)=1$.

The numbers $p(x)$, with $p(x)>1$, are called the *weights* of $\mathcal{H}$. Points x with $p(x)>1$ are called *exceptional*. Thus there is a finite number of exceptional points, and of so-called *exceptional* simple sheaves S, that is, simple objects S with $\operatorname{Ext}^1(S,S)=0$. By [56, Prop. 4.9] each object in the quotient category $\mathcal{H}/\mathcal{H}_0$ is of finite length. An indecomposable object $L\in\mathcal{H}$ is called a *line bundle* if it becomes a simple object modulo $\mathcal{H}_0$. We call a line bundle $L\in\mathcal{H}$ *special*, if for each $x\in\mathbb{X}$ there is (up to isomorphism) *precisely one* simple sheaf S_x concentrated in x with $\operatorname{Ext}^1(S_x,L)\neq 0$.

If we have $p(x)=1$ for all x, then we call $\mathcal{H}$ *non-weighted* (or homogeneous [47]); this can be also expressed as follows

(NC 6′) $\operatorname{Ext}^1(S,S)\neq 0$ (equivalently: $\tau S\simeq S$) for each simple object S.

Proposition 1.1 (Reduction to the non-weighted case). *Let $\mathcal{H}$ be a weighted noncommutative regular projective curve with the exceptional points given by $x_1,\dots,x_t$, of weights $p_i=p(x_i)>1$. Choose for every $i=1,\dots,t$ one simple sheaf S_i concentrated in x_i. Let $\mathscr{S}\subseteq\mathcal{H}$ be the system*

$$\{\tau^j S_i\mid i=1,\dots,t;\ j=1,\dots,p_i-1\}.$$

(1) *The right perpendicular category* $\mathcal{H}_{nw} = \mathscr{S}^{\perp} \subseteq \mathcal{H}$ *is a full exact subcategory of* $\mathcal{H}$*, and is a non-weighted noncommutative regular projective curve.*

(2) *There is a special line bundle* L *in* $\mathcal{H}$*.*

We remark that in general there are line bundles which are not special, cf. [47, Ex. 8.5.1]. Also, even in the non-weighted cases, the group $\operatorname{Aut}(\mathcal{H})$ is in general not acting transitively on the set of line bundles.

Proof. This is similar to the proof of [52, Prop. 1]. For (2) we remark that the full exact embedding $\mathcal{H}_{nw} \subseteq \mathcal{H}$ preserves the rank. So any line bundle in $\mathcal{H}_{nw}$ gives rise to a special line bundle in $\mathcal{H}$. □

It follows, cf. Proposition 13.2, that each *weighted* noncommutative regular projective curve $\mathcal{H}$ over k is obtained from a *non-weighted* noncommutative regular projective curve $\mathcal{H}_{nw}$ over k by insertion of weights into a finite number of points of $\mathcal{H}_{nw}$, in the sense of the p-cycle construction from [53] (we refer also to [47, Sec. 6.1]). We will always consider a pair $(\mathcal{H}, L)$ with L a special line bundle, which we consider as the structure sheaf of $\mathcal{H}$. (Later we will require additional properties on L, cf. 8.1; but in the first sections this will not be needed.)

The quotient category $\widetilde{\mathcal{H}} = \mathcal{H}/\mathcal{H}_0$ is semisimple with one simple object, given by the class $\widetilde{L}$ of L (or any line bundle), thus $\widetilde{\mathcal{H}} \simeq \operatorname{mod}(k(\mathcal{H}))$ for the skew field $k(\mathcal{H}) = \operatorname{End}_{\widetilde{\mathcal{H}}}(\widetilde{L})$, which we call the *function field*. Moreover, we have $\mathcal{H}/\mathcal{H}_0 \simeq \mathcal{H}_{nw}/(\mathcal{H}_{nw})_0$, thus $k(\mathcal{H}) \simeq k(\mathcal{H}_{nw})$.

It *follows* from [8, 50], cf. Remark 3.8, that:

(NC 7) The function field $k(\mathcal{H})$ is of finite dimension over its centre $Z(k(\mathcal{H}))$, which is an algebraic function field in one variable over k.

For quotability we put this on record:

Proposition 1.2. *Each weighted noncommutative regular projective curve* $\mathcal{H}$ *over a field* k *(defined by (NC 1)–(NC 5)) satisfies (NC 6) and (NC 7) as well.*

We will later show (Theorem 7.12) that, as in the commutative case, $\mathcal{H}$, if non-weighted, is uniquely determined by its function field $k(\mathcal{H})$, moreover:

Theorem 1.3. *There is a bijection between the set of isomorphism classes of non-weighted noncommutative regular projective curves over* k *and the set of isomorphism classes of central skew field extensions of algebraic function fields in one variable over* k*.*

(This was also recently shown in [20, Thm. 6.7].) Thus the study of noncommutative regular projective curves is equivalent to the study of such skew field extensions. We call the natural number

$$s(\mathcal{H}) = [k(\mathcal{H}) : Z(k(\mathcal{H}))]^{1/2}$$

the (global) *skewness* of $\mathcal{H}$. Moreover, there we have $Z(k(\mathcal{H})) \simeq k(X)$ for a unique regular projective curve over k (we refer to 7.2). We call X the *centre curve* of $\mathcal{H}$.

Since we are mainly interested in arithmetic effects, we will mostly deal with this non-weighted case. We will then almost always omit the term "non-weighted"; instead we will use the term "weighted" for the general case, which we will treat mainly in the last chapter. In different terminology "weighted" is called or related to "orbifold" or "stacky". In order to stress this connection, we keep the term "weighted" in our general notion, although weights are a built-in feature of general noncommutative curves, cf. Proposition 1.2.

For the rest of this introduction let $\mathcal{H}$ be a noncommutative regular projective curve *over a perfect field* k, and *non-weighted* if not otherwise specified. For each point x we denote by S_x the unique simple sheaf concentrated in x.

The present paper aims for being a quite detailed introduction to noncommutative curves, working out a new approach, presenting numerous new results and discussing many explicit examples. In our approach the main focus is on the functor τ, the Auslander–Reiten translation, which is of course a global datum of the category $\mathcal{H}$. We will study local properties of this functor. In order to do this, we describe the structure of the tubes $\mathcal{U}_x$ (the full subcategories of skyscaper sheaves concentrated in one point x) explicitly. The Auslander–Reiten translation τ is acting on each $\mathcal{U}_x$, and it serves as the Auslander–Reiten translation on $\mathcal{U}_x$, which is itself a hereditary category with Serre duality. The tubes are the most basic, non-trivial examples of connected uniserial length categories. P. Gabriel [30] introduced the species of such a category. In the case of a homogeneous tube with one simple object S this species is just the D-D-bimodule $\operatorname{Ext}^1(S, S)$, where $D = \operatorname{End}(S)$. As the starting point of our local study of τ we determine these bimodules explicitly, by using results of Lenzing–Zuazua [57] on Serre duality. This is done in Section 4. In Section 5 we use this to determine the complete local rings as certain twisted power series rings.

Theorem 1.4. *For each point $x \in \mathbb{X}$ the full subcategory $\mathcal{U}_x$ of skyscraper sheaves concentrated in x is equivalent to the category of finite length modules over the skew power series ring* $\operatorname{End}(S_x)[[T, \tau^-]]$. *Here the twist τ^-, with $Tf = \tau^-(f)T$ for all $f \in \operatorname{End}(S_x)$, is given by the restriction of the inverse Auslander–Reiten translation $\tau^- : \mathcal{H} \to \mathcal{H}$ to the simple object S_x concentrated in x.*

The clou is that the twist is always given by the (inverse) Auslander–Reiten translation. From this we obtain almost at once a local-global principle of skewness, in Section 6. Namely, we get that the restriction of τ to $\mathcal{U}_x$ is of finite order, denoted by $e_\tau(x)$, which we call the τ-multiplicity in x. Then:

Theorem 1.5 (Local-global principle of skewness). *For each point $x \in \mathbb{X}$ we have*

$$s(\mathcal{H}) = e(x) \cdot e^*(x) \cdot e_\tau(x),$$

where $e(x) = [\operatorname{Ext}^1(S_x, L) : \operatorname{End}(S_x)]$, $e^*(x) = [\operatorname{End}(S_x) : Z(\operatorname{End}(S_x))]^{1/2}$.

As before, the clou is the involvement of the global functor τ. It should be noted that the multiplicities $e(x)$ were introduced in representation theory of finite dimensional algebras by Ringel [73].

In Section 7 we import a theorem of Reiten–van den Bergh [68] which states that $\mathcal{H}$ is equivalent to $\mathrm{coh}(\mathcal{A})$, the coherent $\mathcal{A}$-modules, for a sheaf $\mathcal{A}$ of maximal $\mathcal{O}_X$-orders in a central skew field over the function field $k(X)$ of the centre curve X. We will sketch the proof. In this section we also show (based on work by Artin–de Jong [7]) the already mentioned important fact that each noncommutative regular projective curve is uniquely determined by its function field. We then illustrate that many results and relations, well known in the theory of orders, follow almost automatic by our explicit constructions before. In particular, we see that the τ-multiplicities are just the ramification indices of $\mathcal{A}$, for which a similar formula is well known in certain situations [67]. Thus, our approach via τ sheds also new light on orders and ramifications.

In Section 8 we review some facts on the different and dualizing sheaves. Using a result of van den Bergh–van Geel [83] we see that the Auslander–Reiten translation lies in the Picard-shift group,

$$\tau \in \mathrm{Pic}(\mathcal{H}),$$

which is defined to be the subgroup of the automorphism (class) group $\mathrm{Aut}(\mathcal{H})$ generated by the tubular shifts σ_x in the sense of Meltzer [63] and Lenzing–de la Peña [55] (in this context agreeing with the Seidel–Thomas twists [77]). Moreover, we show that $\mathrm{Pic}(\mathcal{H})$ is essentially determined by $\mathrm{Pic}(X)$, the Picard group (of line bundles) over the centre curve X.

Theorem 1.6. *There is an exact sequence*

$$1 \to \mathrm{Pic}(X) \to \mathrm{Pic}(\mathcal{H}) \to \prod_{x \in \mathbb{X}} \mathbb{Z}/e_\tau(x)\mathbb{Z} \to 1$$

of abelian groups.

This has, for instance, the effect, if $\mathcal{H}$ is, say, elliptic and X is of genus zero (we will see such an example over the real numbers later), that then $\mathrm{Pic}(\mathcal{H})$ is finitely generated abelian of rank one.

In Section 9 we define Euler characteristic and genus of a noncommutative regular projective curve. Our definition of the genus is different and made in a more straightforward fashion than the definitions in [83] and [60]; the latter are based on [86]. Our proof of the Riemann–Roch theorem is then almost trivial. We also present a formula by Artin–de Jong [7] for the Euler characteristic, without restriction on the characteristic of the base-field. We will, in contrast to [7], normalize the Euler characteristic, so that it becomes a Morita invariant. This seems to be more natural, particularly when studying noncommutative curves (or orbifolds) over the real numbers.

We show several general results concerning the elliptic case (genus one, Euler characteristic zero). In particular, the classification of indecomposable objects is similar to Atiyah's classification of indecomposable vector bundles for elliptic curves over an algebraically closed field [10]. One major difference here is that it is possible that a noncommutative elliptic curve may have a non-trivial Fourier–Mukai partner. We will exhibit such examples later over the real numbers.

In Section 10 we treat quite detailed certain aspects of the genus zero case. This is the case which is also motivated by representation theory of finite dimensional algebras, since this case is characterized by admitting tilting objects. One of the most important techniques in representation theory is the Auslander–Reiten theory: the concepts of almost split sequences [12] and the Auslander–Reiten translation τ are the most prominent brands of this theory. Almost split sequences are strongly linked to Serre duality, see [68]. Our main focus in the present paper is on the study of the ghost group $\mathcal{G}(\mathcal{H})$, that is, the subgroup of $\mathrm{Aut}(\mathcal{H})$ given by those automorphisms fixing the structure sheaf L and all simple sheaves S_x ($x \in \mathbb{X}$). In representation theory of finite dimensional algebras it is often assumed that the base-field is algebraically closed. Then many problems and questions are already determined combinatorially (say, by working with dimension vectors instead of representations). This will typically fail over general base-fields, and the ghost group can be regarded as a measure for this failure. Good, explicit knowledge of the ghosts, the members of the ghost group, combined with the combinatorial methods, is therefore important for exploring categories of finite dimensional modules. Several of the problems posed in [47] will be solved. Concerning the ghost group our main result is the following.

Theorem 1.7. *Let $\mathcal{H}$ be of genus zero. Assume that there is a point x such that the tubular shift σ_x is efficient in the sense of [47]. Then the ghost group $\mathcal{G}(\mathcal{H})$ is finite, generated by Picard-shifts ${\sigma_x}^{-d(y)} \circ \sigma_y$, which are of order $e_\tau(y)$, where y runs through the points $y \neq x$ with $e_\tau(y) > 1$.*

Typically, x itself will be a point with $e_\tau(x) > 1$, so that then there are at most two further points $y_1,\ y_2$ with $e_\tau(y_i) > 1$, and then $\mathcal{G}(\mathcal{H}) \simeq C_{e_\tau(y_1)} \times C_{e_\tau(y_2)}$.

In order to become able to treat many interesting examples of higher genus also, we work out the whole picture of noncommutative regular projective curves over the real numbers. This is based on work by E. Witt [87] on central skew field extensions of real algebraic function fields in one variable. These skew fields correspond to noncommutative real regular projective curves, which we call (unless commutative) Witt curves. It seems that Witt's function-theoretic study [87] was never fully exploited in order to study noncommutative curves over the reals. By Witt's theorem [87] these curves correspond to Klein surfaces, with each of its ovals (= boundary components) divided into a finite number of segments, labelled with alternating signs "+" and "−" (in this context we also call them Witt surfaces if at least one "−" occurs). We prove a Riemann–Hurwitz formula for the genus (in our definition) of Witt curves. We classify all genus zero and all genus one Witt curves.

The latter will be done in Section 12 by classifying topologically the noncommutative real elliptic curves.

Theorem 1.8. *The Klein bottle has as a Fourier–Mukai partner a Witt curve given by the annulus with two differently signed ovals.*

The theorem, a non-weighted analogue of [44], describes a situation where the conclusion of a theorem of Bondal–Orlov [15] does not hold. It also shows that the recent result [59] does not extend to the non-algebraically closed base-fields.

We also show that in all elliptic cases the Auslander–Reiten translation τ has finite order, more precisely, given by 1, 2, 3, 4 or 6, depending on the specific example; of course, over the reals only 1 and 2 occur.

We end the paper with Section 13 about the weighted cases. We are convinced that separated treatments of the non-weighted and the weighted cases makes the whole theory more transparent; the focus in the non-weighted cases lies on arithmetic properties (like the multiplicities and τ-multiplicities, ghost group, etc.), and then the weighted case is of more combinatorial nature. Here our main result are formulae for the (normalized) orbifold Euler characteristic, of two types:

Theorem 1.9. *Let $\mathcal{H}$ be a weighted noncommutative regular projective curve. Let X be the centre curve, $\mathcal{H}_{nw}$ the underlying non-weighted curve. For the normalized orbifold Euler characteristic $\chi'_{\mathrm{orb}}(\mathcal{H})$ we have*

$$\begin{aligned}\chi'_{\mathrm{orb}}(\mathcal{H}) &= \chi'(X) - \frac{1}{2}\sum_x \Big(1 - \frac{1}{p(x)e_\tau(x)}\Big)[k(x):k]\\ &= \chi'(\mathcal{H}_{nw}) - \frac{1}{2}\sum_x \frac{1}{e_\tau(x)}\Big(1-\frac{1}{p(x)}\Big)[k(x):k].\end{aligned}$$

(Here, the $k(x)$ are the residue class fields over the centre curve X.) Over the real numbers this yields a formula for the Euler characteristic of noncommutative (compact) two-dimensional orbifolds, extending the formula in the classical case from Thurston's book [81]:

Corollary 1.10 (General Riemann–Hurwitz formula). *Let $\mathcal{H}$ be a noncommutative real 2-orbifold with underlying compact Riemann, Klein or Witt surface $\mathcal{H}_{nw}$. Then*

$$\chi'_{\mathrm{orb}}(\mathcal{H}) \;=\; \chi'(\mathcal{H}_{nw}) - \frac{1}{4}\cdot\sum_x\Big(1-\frac{1}{p(x)}\Big) - \frac{1}{2}\cdot\sum_y\Big(1-\frac{1}{p(y)}\Big) - \sum_z\Big(1-\frac{1}{p(z)}\Big),$$

where x runs over the ramification points, y over the other boundary points, and z over the inner points.

As an application we classify all weighted noncommutative regular projective curves $\mathcal{H}$ with $\chi'_{\mathrm{orb}}(\mathcal{H}) = 0$ over the real numbers; up to parameters there are 39 cases, 8 elliptic and 31 tubular ones. 17 have $s(\mathcal{H}) = 1$ and 22 have $s(\mathcal{H}) = 2$.

Theorem 1.11. *Each tubular curve has (fractional) Calabi–Yau dimension n/n, where n is the maximum of the numbers $p(x)e_\tau(x)$. The weight-ramification vector, given by the numbers $p(x)e_\tau(x) > 1$, each counted $[k(x) : k]$-times, is a derived invariant of a tubular curve.*

We start with Section 3 by showing several basic facts about noncommutative curves (partially extending results from [68]) like the existence of homogeneous coordinate rings (so that these curves are in particular noncommutative projective schemes in the sense of Artin–Zhang [9]). We explain two kinds of localizations, one ring-theoretic (Ore–Asano), the other categorical (Serre–Grothendieck–Gabriel), and show that both yield the same. They result in the non-complete rings R_x, associated with each point $x \in \mathbb{X}$. These rings are noncommutative Dedekind domains with a unique non-zero prime ideal, but in general not local. Their completions are (Morita-equivalent to) local rings, which we are going to describe as stated above.

We emphasize that many of our main results are in full generality, without perfectness or separability assumption. We also elaborate in detail an enlightning inseparable Example 10.13.

2. Basic concepts

Let $\mathcal{H}$ be a weighted noncommutative regular projective curve over the field k.

2.1 (Rank function). Let $\mathcal{H} \to \widetilde{\mathcal{H}} = \mathcal{H}/\mathcal{H}_0 = \operatorname{mod}(k(\mathcal{H}))$, $X \mapsto \widetilde{X}$ be the quotient functor. The $k(\mathcal{H})$-dimension on $\mathcal{H}/\mathcal{H}_0$ induces the *rank function* $\operatorname{rk}\colon \mathrm{K}_0(\mathcal{H}) \to \mathbb{Z}$ of $\mathcal{H}$. For an indecomposable object $E \in \mathcal{H}$ we have $\operatorname{rk}(E) = 0$ if $E \in \mathcal{H}_0$ and $\operatorname{rk}(E) > 0$ if $E \in \mathcal{H}_+$. In particular, an object $E \in \mathcal{H}$ has rank 0 if and only if it is of finite length. An indecomposable object L with $\operatorname{rk}(L) = 1$ is called a *line bundle*. The function field $k(\mathcal{H})$ is isomorphic to the endomorphism ring of $\widetilde{L}$ in $\widetilde{\mathcal{H}}$.

2.2 (Almost split sequences). We recall that a short exact sequence

$$\mu\colon\ 0 \to A \xrightarrow{u} B \xrightarrow{v} C \to 0 \tag{2.1}$$

in $\mathcal{H}$ is called *almost split*, [12], if it does not split, if A and C are indecomposable, and if every morphism $X \to C$, which is not a split epimorphism, factors through v. (Then also the dual factorization property holds.) Then A is, up to isomorphism, uniquely determined by C, and conversely. For every indecomposable C (resp. A) there is an almost split sequence (2.1) ending (starting) in C (in A); then $\tau C = A$ and $\tau^- A = C$, which define mutually quasiinverse autoequivalences τ, $\tau^-\colon \mathcal{H} \to \mathcal{H}$, which appear in the Serre duality. For categories of coherent sheaves τ is also known as Serre functor; we will reserve this term for the derived category of $\mathcal{H}$.

The almost split sequences are fundamental in the definition of the *Auslander–Reiten quiver* of $\mathcal{H}$: its vertices are the isomorphism classes of indecomposable objects in $\mathcal{H}$, and the arrows between classes of indecomposables are given by the so-called *irreducible* morphisms, which are the components of the maps which occur in the corresponding almost split sequences.

2.3 (Homogeneous tubes). Let $x \in \mathbb{X}$ and $\mathcal{U} = \mathcal{U}_x$ be the corresponding connected uniserial category in $\mathcal{H}_0$. Assuming $p(x) = 1$, there is up to isomorphism precisely one simple object $S = S_x$ in $\mathcal{U}$. Such categories are also called *homogeneous tubes*. For each $n \geq 1$ we denote by $S[n]$ the (up to isomorphism) unique indecomposable object in $\mathcal{U}$ of length n. (We additionally set $S[0] := 0$.) Thus we have $\mathcal{U} = \operatorname{add}(\{S[n] \mid n \geq 1\})$. We have injections $\iota_n \colon S[n] \to S[n+1]$ and surjections $\pi_n \colon S[n+1] \to S[n]$. The Auslander–Reiten translation satisfies $\tau S[n] \simeq S[n]$. We will usually identify them, $\tau S[n] = S[n]$. We then have almost split sequences $\mu_1 \colon 0 \to S \xrightarrow{\iota_1} S[2] \xrightarrow{\pi_1} S \to 0$, and for $n \geq 2$:

$$\mu_n \colon 0 \to S[n] \xrightarrow{(\pi_{n-1},\iota_n)^t} S[n-1] \oplus S[n+1] \xrightarrow{(\iota_{n-1},\pi_n)} S[n] \to 0.$$

The ι_n and π_n are the irreducible maps in $\mathcal{U}$.

2.4 (τ-multiplicity). Let $\mathcal{U}_x$ be a homogeneous tube with simple object S_x. The Auslander–Reiten translation τ restricts to an autoequivalence of $\mathcal{U}_x$. Up to isomorphism it fixes all indecomposable objects $S_x[n]$. If we consider a skeleton of $\operatorname{ind}(\mathcal{U}_x)$, we can assume that equality $\tau S_x[n] = S_x[n]$ holds for all $n \geq 1$. The action on morphisms induces, in particular, an automorphism of $D_x = \operatorname{End}(S_x)$, that is, an element τ in $\operatorname{Aut}(D_x/k)$. We define $\operatorname{Gal}(D_x/k) = \operatorname{Aut}(D_x/k)/\operatorname{Inn}(D_x/k)$, the factor group modulo inner automorphisms. By the theorem of Skolem–Noether [66, 12.6], restriction to the centre $Z(D_x)$ yields an injective homomorphism $\operatorname{Gal}(D_x/k) \to \operatorname{Gal}(Z(D_x)/k)$. We call

$$e_\tau(x) = \operatorname{ord}_{\operatorname{Gal}(D_x/k)}(\tau), \tag{2.2}$$

the order of (the class of) τ in $\operatorname{Gal}(D_x/k)$, the τ-*multiplicity* of x.

2.5 (Picard-shifts). Let $x \in \mathbb{X}$ be a point and $\mathcal{U} = \mathcal{U}_x$ be a homogeneous tube in $\mathcal{H}$. The indecomposable objects in $\mathcal{U}$ form the Auslander–Reiten component containing the simple object $S = S_x$ with support $\{x\}$. Then $\operatorname{End}(S)$ is a division algebra over k, and $\operatorname{Ext}^1(S,S)$ is one-dimensional as $\operatorname{End}(S)$-vector space. Thus S is, in the terminology of [77], a spherical object. (In [77] only the case $\operatorname{End}(S) = k$ is considered.) For every object E in $\mathcal{H}$, which has no indecomposable summand in $\mathcal{U}$, one has the S-universal extension $0 \to E \to E(x) \to E_x \to 0$ of E with $E_x = \operatorname{Ext}^1(S,E) \otimes_{\operatorname{End}(S)} S$. The assignment $E \mapsto E(x)$ induces an autoequivalence $\sigma_x \colon \mathcal{H} \to \mathcal{H}$, called the *tubular* (or *Picard-*) *shift* associated with x, coming with a natural transformation $1_{\mathcal{H}} \xrightarrow{x} \sigma_x$. We refer to [47, 55, 63]. This coincides with the notion of a *Seidel–Thomas twist*, [77]. We will later see (Lemma 8.4) that such a functor σ_x is given as the tensor product with a certain bimodule. For $n \in \mathbb{Z}$ we write

$E(nx) = {\sigma_x}^n(E)$. The assignment $E \mapsto E_x$ is also functorial. Following [53, 4.2] we call E_x the *fibre* of E; if $f \in \mathrm{Hom}(E, E')$, we call $f_x \in \mathrm{Hom}(E_x, E'_x)$ the corresponding *fibre map*.

Let $x \neq y$ be two points. It is well-known that $\sigma_x \circ \sigma_y \simeq \sigma_y \circ \sigma_x$ holds, and that the restriction of σ_x to $\mathcal{U}_y$ is isomorphic to the identity functor on $\mathcal{U}_y$. As a natural but non-trivial result we will show in Corollary 5.5, that over a perfect field σ_x acts functorially on $\mathcal{U}_x$ like the inverse Auslander–Reiten translation τ^-. We will also point out in Example 10.13 that this is not true in general over non-perfect fields. We denote by $\mathrm{Pic}(\mathcal{H})$ the *Picard-shift group*, that is, the subgroup of the automorphism (class) group $\mathrm{Aut}(\mathcal{H})$ of $\mathcal{H}$ generated by all tubular shifts σ_x ($x \in \mathbb{X}$); it is an abelian group. We call a vector bundle F a Picard-shift of a vector bundle E if there is $\sigma \in \mathrm{Pic}(\mathcal{H})$ such that $F \simeq \sigma(E)$. In particular, the group $\mathrm{Pic}(\mathcal{H})$ acts on the set of isomorphism classes of line bundles on $\mathcal{H}$. If $\mathcal{H} = \mathrm{coh}(X)$ with X commutative, then $\mathrm{Pic}(\mathcal{H})$ is isomorphic to the *Picard group* $\mathrm{Pic}(X)$, given by the set if isomorphism classes of line bundles with the tensor product. In general the action of $\mathrm{Pic}(\mathcal{H})$ on line bundles is neither transitive nor faithful. We will see that in case $\mathcal{H}$ is multiplicity free (all $e(x) = 1$), the action is transitive. The question of faithfulness is strongly linked to the study of the *ghost group* $\mathcal{G}(\mathcal{H})$, the subgroup of $\mathrm{Aut}(\mathcal{H})$, given by those σ fixing the structure sheaf L and all simple sheaves S_x. We also consider the automorphism group $\mathrm{Aut}(\mathbb{X})$, the subgroup of $\mathrm{Aut}(\mathcal{H})$ given by those σ fixing the structure sheaf L. All three, $\mathrm{Pic}(\mathcal{H})$, $\mathcal{G}(\mathcal{H})$, $\mathrm{Aut}(\mathbb{X})$, are normal subgroups of $\mathrm{Aut}(\mathcal{H})$.

For the generalization of Picard-shifts with respect to non-homogeneous tubes we refer to [55] and [47].

2.6 (Multiplicity and comultiplicity). Let L be a special line bundle so that $\mathrm{Ext}^1(S_x, L) \neq 0$ holds for all $x \in \mathbb{X}$. The dimensions

$$e(x) = [\mathrm{Ext}^1(S_x, L) \colon \mathrm{End}(S_x)] \tag{2.3}$$

are called *multiplicities*, [47,55,73]. In particular, we have the S_x-universal extension

$$0 \to L \xrightarrow{\pi_x} L(x) \to {S_x}^{e(x)} \to 0 \tag{2.4}$$

of L. The number

$$e^*(x) = [\mathrm{End}(S_x) : Z(\mathrm{End}(S_x))]^{1/2} \tag{2.5}$$

we called *comultiplicities* in [47], since (in case of genus zero) for almost all $x \in \mathbb{X}$ the product of $e(x)$ and $e^*(x)$ coincides with the skewness $s(\mathcal{H})$, [47, Cor. 2.3.5]. It was left open in [47] whether $e(x) \cdot e^*(x)$ is always a divisor of $s(\mathcal{H})$, not to speak about what the description of the cofactor could be. To answer this question, and without being restricted to the case of genus zero, was one of the main motivations for this article.

We remark that the comultiplicity, like the skewness, can be expressed in terms of polynomial identity (PI) degree.

2.7 (Orbit algebras). In (noncommutative) algebraic geometry orbit algebras are important tools for constructing homogeneous coordinate rings. We refer to the survey [80]. If E is an object in $\mathcal{H}$ and $\sigma\colon\mathcal{H}\to\mathcal{H}$ an endofunctor, then we denote by $\Pi(E,\sigma)$ the positively $\mathbb{Z}$-graded *orbit algebra* $\bigoplus_{n\geq 0}\operatorname{Hom}(E,\sigma^n E)$. The multiplication is defined on homogeneous elements $f\colon E\to\sigma^m E$, $g\colon E\to\sigma^n E$ by the rule $g*f=\sigma^m(g)\circ f\colon E\to\sigma^{m+n}E$.

The special cases we are interested in are $\Pi(L,\sigma_x)$ with L the structure sheaf and $\sigma_x\colon\mathcal{H}\to\mathcal{H}$ a Picard-shift. Then the homogeneous element π_x from (2.4) is central, [47, Lem. 1.7.1]. We denote the (homogeneous) ideal of $\Pi(L,\sigma_x)$ generated by π_x by P_x. We will later see that P_x is a homogeneous prime ideal. Whereas in [46,47] we fixed one autoequivalence σ (with additional good properties) and one coordinate algebra $\Pi(L,\sigma)$ for $\mathcal{H}$, we will in this paper for every point x make use of its "own" orbit algebra $\Pi(L,\sigma_x)$ in order to investigate the numbers $e(x)$, $e^*(x)$ and $e_\tau(x)$.

2.8 (PI-degree). We will make use (in Section 6) of some ring-theoretic tools like the polynomial identity (PI) degree. We will never use the original definition. Instead in our special situation we could take the following two properties (i) and (ii) as an equivalent definition for the PI-degree. If R is a ring (always assumed to be associative and with identity) we denote by $Z(R)$ its centre.

(i) If D is a skew field which is of finite dimension over its centre, then the PI-degree of D equals the square root of this dimension, [74, Thm. 1.5.23]. Moreover, the PI-degree of the matrix ring $\mathrm{M}_n(D)$ is n times the PI-degree of D, [74, 1.5.16].

(ii) If R is a noetherian domain, so that its quotient division ring $Q(R)$ is of finite dimension over its centre, then Posner's theorem [6, Thm. 7] tells us that the PI-degree of R equals the PI-degree of $Q(R)$.

By $\operatorname{mod}(R)$ we denote the category of finitely presented (right) R-modules, by $\operatorname{mod}_0(R)$ the full subcategory of the modules of finite length. Usually, finite length is equivalent to finite dimension over the base-field k.

We conclude the section with a motivating simple but non-trivial example, illustrating some of our results.

Example 2.9. Let $R=\mathbb{C}[X;Y,\sigma]$ be the twisted graded polynomial algebra over $k=\mathbb{R}$. Here the variables X and Y are of degree one, X central, and $Yz=\sigma(z)Y$ for all $z\in\mathbb{C}$, where $\sigma(z)=\bar{z}$ is the complex conjugation. The quotient category $\mathcal{H}=\operatorname{qgr}(R)=\operatorname{mod}^{\mathbb{Z}}(R)/\operatorname{mod}_0^{\mathbb{Z}}(R)$ is a noncommutative regular projective curve. Its function field is $k(\mathcal{H})=\mathbb{C}(t,\sigma)$, which is the quotient division ring (of degree zero fractions) of R. The centre of R is $\mathbb{R}[X,Y^2]$, the centre of $k(\mathcal{H})$ is $\mathbb{R}(t^2)$. For the skewness we obtain $s(\mathcal{H})=2$. In this example we have an explicit description for all the points of $\mathbb{X}$. These correspond bijectively to the homogeneous prime elements in R (up to scalars). With the exception of Y, all prime elements belong to the centre; they are listed in Table 1. For a point x we write D_x for the endomorphism ring of the corresponding simple object S_x. Then $e^*(x) = [D_x : Z(D_x)]^{1/2}$. The number $e(x)$ coincides with the number of irreducible factors of the corresponding prime element. It is shown in [47, Cor. 5.4.4] (and will be again shown in this paper in a broader context) that the Auslander–Reiten translation τ is given as the product $\tau = {\sigma_x}^{-1}{\sigma_y}^{-1}$ of two (inverse) Picard-shifts, where (from now on) x and y are the points corresponding to the primes X and Y, respectively. It follows readily that $e_\tau(p) = 1$ for all points $p \neq x, y$. Moreover, the ghost group $\mathcal{G}(\mathcal{H})$ is shown to be of order 2, generated by $\gamma = \sigma_x{\sigma_y}^{-1}$.

prime/point x	D_x	$e(x)$	$e^*(x)$	$e_\tau(x)$	$D_x[[T,\tau^-]]$
X, Y	$\mathbb{C}$	1	1	2	$\mathbb{C}[[T,\sigma]]$
$(Y-\sqrt{\alpha}X)(Y+\sqrt{\alpha}X), \alpha > 0$	$\mathbb{R}$	2	1	1	$\mathbb{R}[[T]]$
$Y^2 - \alpha X^2, \alpha < 0$	$\mathbb{H}$	1	2	1	$\mathbb{H}[[T]]$
$(Y^2 - zX^2)(Y^2 - \bar{z}X^2), z \in \mathbb{C} \setminus \mathbb{R}$	$\mathbb{C}$	2	1	1	$\mathbb{C}[[T]]$

Table 1. $k(\mathcal{H}) = \mathbb{C}(t, \sigma)$

It can be seen directly (though not trivially, cf. [47, Cor. 5.4.3]) that $e_\tau(x) = 2 = e_\tau(y)$. Of course, having already computed $e(x)$ and $e^*(x)$ (and the same for y) it will generally follow from Theorem 1.5.

This example is a special case of the treatment of genus zero curves in Section 10, and it is also a special case of what we call a Witt curve, treated in Section 11. These are obtained by Klein surfaces (=certain quotients of compact Riemann surfaces) together with a so-called $\pm$-configuration. These were studied by Witt in his seminal paper [87]. We will show in Corollary 7.17 that the τ-multiplicities coincide with the ramification indices of the function (skew) field. That in the present example x and y are the only ramification points (having index 2) then also follows from Witt's work.

We sketch another way for computing $e_\tau(x)$: we can localize R with respect to the homogeneous prime ideal $P = RX$, considering only fractions of degree zero, denoting this ring by R_x. This is (in this special case!) a local ring whose maximal (left and right) ideal J is generated by $\pi = XY^{-1}$, which satisfies $\pi z = \bar{z}\pi$ for all $z \in \mathbb{C}$. The J-adic completion then is easily seen to be $\widehat{R}_x = \mathbb{C}[[\pi, \sigma]] \simeq \mathbb{C}[[T, \sigma]]$ (as indicated in the table). The centre is given by $\mathbb{R}[[\pi^2]] = \mathbb{R}[[T^2]]$. In the language of valuations, π is a uniformizer for the completion, π^2 a uniformizer of the centre. We see readily that the ramification index $e_{\mathrm{ra}}(x)$ of x, defined by $\widehat{R}_x\pi^2 = \widehat{J}^{e_{\mathrm{ra}}(x)}$, equals 2. Since the tube to x is given by $\mathcal{U}_x = \mathrm{mod}_0(\widehat{R}_x)$ it is not difficult to see (cf. Theorem 7.21) that the twist σ induces the Picard-shift $\sigma_{x|\mathcal{U}_x}$, restricted to the tube $\mathcal{U}_x$. Therefore τ acts like ${\sigma_x}^{-1}$ as functor on $\mathcal{U}_x$ with order 2, that is, $e_\tau(x) = 2$. (A similar argument holds for y, considering $\pi^{-1} = YX^{-1}$.) We will use the notion of the *different* in Sec. 8, defined as the Weil divisor $\Delta = \sum_p (e_{\mathrm{ra}}(p) - 1) \cdot p$. In the present example it follows from the preceding computations that $\Delta = 1x + 1y$.

3. Homogeneous coordinate rings and localizations

We assume that $(\mathcal{H}, L)$ is a (non-weighted) noncommutative regular projective curve over the field k. In this section we show that, via the Serre construction, $\mathcal{H}$ is a noncommutative noetherian projective scheme in the sense of Artin–Zhang [9], and accordingly $\mathbb{X}$ a projective spectrum. Moreover, via localization we study rings locally at a point $x \in \mathbb{X}$.

Lemma 3.1. *Each vector bundle has a line bundle filtration.*

Proof. We refer to [56, Prop. 1.6]. □

Lemma 3.2. *Let* $0 \to L \xrightarrow{\pi} L(x) \to S^e \to 0$ *be the* S*-universal sequence of* L *with* $S = S_x$ *and* $e = e(x)$*. For* $n \geq 1$ *we have the exact sequence*

$$0 \to L \xrightarrow{\pi^n} L(nx) \to S[n]^e \to 0.$$

Proof. By induction on n. For $n = 1$ the assertion is trivial. Let $n > 1$. Write $\mu\colon 0 \to L \xrightarrow{\pi^n} L(nx) \longrightarrow E \to 0$. By induction hypothesis, from the snake lemma we obtain that E appears as the middle term of a short exact sequence

$$0 \to S^e \to E \to S[n-1]^e \to 0. \tag{3.1}$$

Write $E = E_1 \oplus \cdots \oplus E_m$ with $E_i = S[\ell_i]$ indecomposable. By uniseriality we have $\mathrm{Soc}(E_i) = S$. This yields $S^e = \mathrm{Soc}(S^e) \subseteq \mathrm{Soc}(E) = S^m$, and thus $m \geq e$. On the other hand, assume that $m > e$. Let $u_i\colon S \to S[\ell_i] = E_i \xrightarrow{j_i} E$ a monomorphism. By the definition of $e = e(x)$, there are $f_1, \ldots, f_m$ in $\mathrm{End}(S)$, not all of them zero, such that $0 = \sum_{i=1}^m \mu \cdot u_i f_i = \mu \cdot \big(\sum_{i=1}^m u_i f_i\big)$. Denoting $\sum_{i=1}^m u_i f_i$ by $0 \neq h\colon S \to E$, the short exact sequence $\mu \cdot h$ splits, and we obtain, that S embeds into $L(nx)$, which gives a contradiction. We conclude $m = e$. Let $R = \mathrm{End}(S[\infty])$ be the complete local ring with maximal ideal $\mathfrak{m}$ such that $\mathcal{U} = \mathrm{mod}_0(R)$. Since S^e is annihilated by $\mathfrak{m}$ and $S[n-1]^e$ by $\mathfrak{m}^{n-1}$, we deduce from sequence (3.1) that E is annihilated by $\mathfrak{m}^n$, and thus all $\ell_i \leq n$. Since the length of E is $n \cdot e$, we get $\ell_i = n$ for all i. This completes the proof of the lemma. □

Lemma 3.3. *Let* E *be an indecomposable vector bundle and* S *be a simple sheaf. Then* $\mathrm{Hom}(E, S) \neq 0$.

Proof. Using connectedness of $\mathcal{H}$ this is shown like in [55, (S11)] or [68, Cor. IV.1.8]. □

Lemma 3.4. *Let L and L' be line bundles, and let $x \in \mathbb{X}$ be a point. Then* $\operatorname{Hom}(L(-nx), L') \neq 0$ *for* $n \gg 0$.

Proof. By the preceding lemma we have an exact sequence $0 \to L(-nx) \to L \to S[n]^e \to 0$ for each $n \geq 0$. Applying $\operatorname{Hom}(-, L')$ gives $0 \to \operatorname{Hom}(L, L') \to \operatorname{Hom}(L(-nx), L') \to \operatorname{Ext}^1(S[n]^e, L') \to \operatorname{Ext}^1(L, L')$. By Lemma 3.3 we have $d := \dim_k \operatorname{Hom}(L', S) > 0$, and thus $\dim_k \operatorname{Ext}^1(S[n]^e, L') = dne \gg 0$ for $n \gg 0$. From this follows the claim. □

Lemma 3.5. *For each $x \in \mathbb{X}$ the pair (L, σ_x) is ample in the sense of [9]. Accordingly,*

$$\mathcal{H} \simeq \frac{\operatorname{mod}^{\mathbb{Z}}(\Pi(L, \sigma_x))}{\operatorname{mod}_0^{\mathbb{Z}}(\Pi(L, \sigma_x))}. \tag{3.2}$$

In particular, a noncommutative regular projective curve $\mathcal{H}$ is a noncommutative projective scheme in the sense of Artin–Zhang [9].

Proof. (Compare the proof of [68, Lem. IV.4.1]) We have the inverse system $\cdots \to L(-2x) \to L(-x) \to L$ of subobjects with zero intersection. By [68, Lem. IV.1.3]) there is a line bundle $L' \subseteq L$ such that $\operatorname{Ext}^1(U, L) = 0$ for all subobjects (line bundles) $U \subseteq L'$. Moreover, for $n \gg 0$ we have $L(-nx) \subseteq L'$, and we conclude $\operatorname{Ext}^1(L(-nx), L) = 0$.

Let $E \in \mathcal{H}$. Let $F \subseteq E$ be the largest subobject such there is an epimorphism $G := \oplus_{i=1}^t L(-\alpha_i x) \to F$, and let $C = E/F$. We assume that $C \neq 0$, and will show that this yields a contradiction. If C is of finite length, then it follows from Lemma 3.2 that a finite direct sum of copies of L maps onto C. Thus we can assume that C is a vector bundle, and it suffices to assume that C is a line bundle. We have an exact sequence $0 \to K \to G \to F \to 0$ with G a finite direct sum of σ_x-shifts of L. By the preceding paragraph there is n_0 such that $\operatorname{Ext}^1(L(-nx), G) = 0$, and then $\operatorname{Ext}^1(L(-nx), F) = 0$ for all $n \geq n_0$.

By Lemma 3.4 we have a non-trivial morphism $L(-mx) \to C$ for some $m \geq n_0$. Since $\operatorname{Ext}^1(L(-mx), F) = 0$, this lifts to a non-trivial morphism $L(-mx) \to E$, giving a contradiction. □

Lemma 3.6. *The homogeneous ideal P_x in $\Pi(L, \sigma_x)$ generated by π_x is prime.*

Proof. This follows like in [47, Thm. 1.2.3]. We only need to show that for $n \gg 0$ sufficiently large we have $\operatorname{Hom}(L, \tau L(-nx)) = 0$, as in [47, Lem. 1.2.2]. To this end, by Lemma 3.4 for $n \gg 0$ there is a non-zero morphism $g \colon \tau L(-nx) \to L$. We assume that there is a non-zero morphism $f \colon L \to \tau L(-nx)$. Both, f and g, are monomorphisms, and $g \circ f \colon L \to L$ is an isomorphism, thus g is an isomorphism. Enlarging n further, we see that there is $m > 0$ such that L and $L(mx)$ are isomorphic. But then, repeating the argument just given, also $(\pi_x)^m$ would be an isomorphism. But this is not true by Lemma 3.2, giving a contradiction. Thus $\mathrm{Hom}(L, \tau L(-nx)) = 0$. □

Lemma 3.7. *For each $x \in \mathbb{X}$ the ring $\Pi(L, \sigma_x)$ is a graded noetherian domain which has a central prime element π_x of degree one, and the quotient division ring of degree-zero fractions $s^{-1}r$ (with r, s homogeneous of the same degree, $s \neq 0$) is the function field $k(\mathcal{H})$.*

Proof. Noetherianness follows from the proof of [47, Prop. 1.4.4] also in this more general setting (right-noetherianness is also shown in [9, Thm. 4.5]). Since non-zero morphisms between line bundles are monomorphisms, the orbit algebra $R = \Pi(L, \sigma_x)$ is a graded domain. By the preceding lemma the homogeneous element π_x is central and prime. The assertion about the function field follows like in [68, Lem. IV.4.1 Step 4]. (We remark that like in [68, Lem. IV.4.1 Step 3] the Gelfand–Kirillov dimension of the finitely graded (in the sense of [8]) k-algebra R is two, and then [8, Thm. 0.1] implies that (NC 7) holds.) □

Remark 3.8. Assume that $\mathcal{H}$ satisfies, more generally, conditions (NC 1) to (NC 5), and let L be a line bundle. Then similar statements of most of the preceding results remain true, with similar proofs. More precisely, for σ a suitable product of Picard-shifts, we get an ample pair (L, σ), and $R = \Pi(L, \sigma)$ is a projective coordinate algebra for $\mathcal{H}$ of Gelfand–Kirillov dimension two, and the zero component of the graded quotient division ring of R is the function field $k(\mathcal{H})$; we refer to [68, Lem. IV.4.1]. Then [8, Thm. 0.1] implies (NC 7).

Lemma 3.9. *Let $x \in \mathbb{X}$ be of multiplicity $e(x)$ and with simple sheaf S_x.*

(1) *For a non-zero homogeneous element $s \in \Pi(L, \sigma_x)$ the following conditions are equivalent:*
   - *$s \in \mathcal{C}(P_x)$, that is, s is regular modulo P_x.*
   - *The cokernel of s lies in $\coprod_{y \neq x} \mathcal{U}_y$.*
   - *The fibre map $s_x \in \mathrm{End}({S_x}^{e(x)})$ is an isomorphism.*

(2) *The set $\mathcal{C}(P_x)$ is a denominator set.*

(3) *For the graded localization $R_x^{\mathrm{gr}} = \Pi(L, \sigma_x)_{\mathcal{C}(P_x)}$ the graded Jacobson radical is generated by the central element $\pi_x 1^{-1}$ and is the only non-zero graded prime ideal.*

(4) *As graded rings, $R_x^{\mathrm{gr}} / \mathrm{rad}^{\mathrm{gr}}(R_x^{\mathrm{gr}}) \simeq \mathrm{M}_{e(x)}\big(\mathrm{END}(S_x)\big)$, where $\mathrm{END}(S_x)$ is the graded skew field $\bigoplus_{n \in \mathbb{Z}} \mathrm{Hom}(S_x, S_x(n))$.*

Proof. (1) The equivalence of the three conditions is shown in [47, Lem. 2.2.1].

(2) Since $\Pi(L, \sigma_x)$ is graded noetherian (left and right) and π_x is a central, this follows from a graded version of [79, Thm. 4.3].

(3), (4) (For analogous ungraded statements we refer to [62, Thm. 4.3.18] and [34, Lem. 14.18].) Localizing the universal exact sequence

$$0 \to L \xrightarrow{\pi_x} L(x) \longrightarrow {S_x}^{e(x)} \to 0$$

we get, like in [47, Prop. 2.2.8], a short exact sequence

$$0 \to R_x^{\mathrm{gr}} \xrightarrow{\cdot \pi_x} R_x^{\mathrm{gr}}(x) \longrightarrow {S_x}^{e(x)} \to 0$$

of graded R_x^{gr}-modules, where S_x is simple. As graded rings thus $R_x^{\mathrm{gr}}/(\pi_x 1^{-1}) \simeq \mathrm{M}_{e(x)}^{\mathrm{gr}}(\mathrm{END}(S_x))$. The graded Jacobson radical $\mathrm{rad}^{\mathrm{gr}}(R_x^{\mathrm{gr}})$ is the principal ideal generated by $\pi_x 1^{-1}$: clearly, $1 - \pi_x r \in \mathcal{C}(P_x)$ for each r, so that $\pi_x 1^{-1}$ lies in the radical. The canonical surjective ring homomorphism $R_x^{\mathrm{gr}}/(\pi_x 1^{-1}) \to R_x^{\mathrm{gr}}/\,\mathrm{rad}^{\mathrm{gr}}(R_x^{\mathrm{gr}})$ is an isomorphism, by simplicity of the graded ring on the left hand side. □

We denote by R_x the degree-zero component of the localization $\Pi(L, \sigma_x)_{\mathcal{C}(P_x)}$.

Proposition 3.10. *Let $\mathbb{X}$ be a noncommutative regular projective curve over the field k. Let $x \in \mathbb{X}$ be a point. There is an isomorphism*

$$R_x/\,\mathrm{rad}(R_x) \simeq \mathrm{M}_{e(x)}\big(\mathrm{End}(S_x)\big)$$

of rings.

Proof. Since $\mathrm{rad}(R_x)$ is the degree zero part of $\mathrm{rad}^{\mathrm{gr}}(R_x^{\mathrm{gr}})$, the assertion follows from the preceding lemma. □

3.11. For $x \in \mathbb{X}$ we denote by $\mathcal{H}_x = \mathcal{H}/\langle \coprod_{y \neq x} \mathcal{U}_y \rangle$ the quotient category, modulo a Serre subcategory, where all tubes except $\mathcal{U}_x$ are "removed", and by $p_x\colon \mathcal{H} \to \mathcal{H}_x$ the quotient functor.

Lemma 3.12. *The object $L_x = p_x(L)$ is an indecomposable projective generator of $\mathcal{H}_x$. Accordingly, for the ring $V_x = \mathrm{End}_{\mathcal{H}_x}(L_x)$ we have $\mathcal{H}_x \simeq \mathrm{mod}(V_x) = \mathrm{mod}_+(V_x) \vee \mathrm{mod}_0(V_x)$, with $\mathrm{mod}_0(V_x) \simeq \mathcal{U}_x$ the finite length modules and $\mathrm{mod}_+(V_x)$ the finitely generated torsionfree modules.*

Proof. Let $y \in \mathbb{X}$ be another point, $y \neq x$. Then π_y induces an isomorphism $L_x(-y) \simeq L_x$. Using ampleness of the pair (L, σ_y) we see that L_x is a generator for $\mathcal{H}_x$. It is easy to see that for each exact sequence $\eta\colon 0 \to A \to B \to L \to 0$ in $\mathcal{H}$ the exact sequence $p_x(\eta)$ in $\mathcal{H}_x$ splits, showing that L_x is a projective object in $\mathcal{H}_x$. From this it follows that $\mathrm{Hom}_{\mathcal{H}_x}(L_x, -)\colon \mathcal{H}_x \to \mathrm{mod}(V_x)$ is an equivalence. It is easy to see that p_x induces an injective homomorphism $V_x \to k(\mathcal{H})$ of rings, and thus V_x is a domain. We infer that L_x is indecomposable. □

Proposition 3.13. *There is an isomorphism of rings $R_x \simeq V_x$.*

Proof. By using the definition of morphisms in the quotient category we see easily $R_x \subseteq V_x$. Let $0 \to L' \xrightarrow{s} L \to C \to 0$ be an exact sequence in $\mathcal{H}$ with L' a line bundle and $C \in \coprod_{y \neq x} \mathcal{U}_y$. By ampleness of (L, σ_x) there is an epimorphism $f = (f_1, \ldots, f_n) \colon \bigoplus_{i=1}^{n} L(-\alpha_i x) \to L'$ (with $\alpha_i \geq 1$). If we assume that each $C_i = \operatorname{Coker}(f_i)$ has a non-zero summand in $\mathcal{U}_x$, then there is an epimorphism $C_i \to S_x$, and thus we can write $f_i = \pi_x \circ f_i'$. But then $f = \pi_x \circ f'$ is not surjective, giving a contradiction. Thus there is i such that $C_i \in \coprod_{y \neq x} \mathcal{U}_y$. We conclude that there is $f \colon L(-\alpha x) \to L'$ such that $s \circ f \colon L(-\alpha x) \to L$ is a non-zero homogeneous element in $\Pi(L, \sigma_x)$ with $\operatorname{Coker}(sf) \in \coprod_{y \neq x} \mathcal{U}_y$, that is, $sf \in \mathcal{C}(P_x)$. Thus we can write $rs^{-1} = (rf)(sf)^{-1}$, from which we infer the converse inclusion. □

Corollary 3.14. *For each $x \in \mathbb{X}$ we have $\mathcal{U}_x \simeq \operatorname{mod}_0(R_x)$.* □

Corollary 3.15. *Each ring R_x is a noncommutative Dedekind domain with unique non-zero prime ideal given by* $\operatorname{rad}(R_x)$.

Proof. R_x is right hereditary since $\operatorname{mod}(R_x) \simeq \mathcal{H}_x$ is hereditary. Since R_x is noetherian, by [78, Cor. 3] it is also left hereditary. It follows from Proposition 3.10 that the radical $J = \operatorname{rad}(R_x)$ is the only (two-sided) maximal ideal. As in [62, 4.3.20] one shows $\bigcap_{n \geq 0} J^n = 0$. Thus, if $r \in R_x$, $r \neq 0$, then there is $v(r) = n$ with $r \in J^n$ but $r \notin J^{n+1}$. Let I be a non-zero idempotent ideal in R_x. Let $0 \neq r \in I$ with $v(r)$ minimal. From the condition $I = I^2$ we get $v(r) = 0$, and then $I = R_x$ since J is maximal. Thus R_x is Dedekind by [62, 5.6.3]. By [62, 5.2.9] each non-zero ideal is of the form J^n, and it follows that J is the only non-zero prime ideal. □

We also consider the category $\vec{\mathcal{H}} = \operatorname{Qcoh} \mathbb{X} = \frac{\operatorname{Mod}^{\mathbb{Z}}(\Pi(L,\sigma_x))}{\operatorname{Mod}_0^{\mathbb{Z}}(\Pi(L,\sigma_x))}$ of quasicoherent sheaves, where Mod_0 denotes the localizing Serre subcategory of torsion (that is, locally finite length) graded modules. This is a hereditary, locally noetherian Grothendieck category. In this we can consider the *Prüfer sheaf* $S_x[\infty]$ for $x \in \mathbb{X}$, which is the union $\bigcup_{n \geq 1} S_x[n]$, that is, the direct limit of the direct system $(S_x[n], \iota_n)$, and thus is a quasicoherent torsion sheaf. We now have the main result of this section.

Proposition 3.16. *For the* $\operatorname{rad}(R_x)$*-adic completion of R_x we have*

$$\widehat{R}_x \simeq \operatorname{M}_{e(x)}\big(\operatorname{End}(S_x[\infty])\big).$$

Proof. By [49, Thm. 21.31] the completion $\widehat{R}_x$ is semiperfect, it satisfies, with $D_x = \operatorname{End}(S_x)$,

$$\widehat{R}_x / \operatorname{rad}(\widehat{R}_x) \simeq R_x / \operatorname{rad}(R_x) \simeq \operatorname{M}_{e(x)}(D_x), \tag{3.3}$$

and from [49, Thm. 23.10] it follows that $\widehat{R}_x \simeq \operatorname{M}_{e(x)}(\widehat{E}_x)$ for a complete local ring $\widehat{E}_x$. Moreover, for the categories of finite length modules we have

$$\operatorname{mod}_0(\widehat{E}_x) \simeq \operatorname{mod}_0(\widehat{R}_x) \simeq \operatorname{mod}_0(R_x) \simeq \mathcal{U}_x.$$

The result follows now, since the complete local ring $\mathrm{End}(S_x[\infty])$ is uniquely determined such that $\mathrm{mod}_0(\mathrm{End}(S_x[\infty])) \simeq \mathcal{U}_x$, by [31, IV. Prop. 13]. □

4. Serre duality and the bimodule of a homogeneous tube

In [30] P. Gabriel defined the *species* of a uniserial category $\mathcal{U}$. In the most basic situation, when there is (up to isomorphism) only one simple object S in $\mathcal{U}$, like in the case of a homogeneous tube, then this species is just the bimodule ${}_{\mathrm{End}(S)}\mathrm{Ext}^1(S,S)_{\mathrm{End}(S)}$. In order to describe this bimodule more precisely, we derive from [57] some general facts about Serre duality.

We call a k-bilinear map $\langle -|-\rangle\colon V \times W \to k$ a *perfect pairing*, if for each non-zero $x \in V$ there exists $y \in W$ with $\langle x|y\rangle \neq 0$, and if for each non-zero $y \in W$ there is $x \in V$ with $\langle x|y\rangle \neq 0$. Let $\mathcal{H}$ be a noncommutative regular projective curve over the field k. For each indecomposable object $X \in \mathcal{H}$ we fix an almost split sequence $\mu_X\colon 0 \to \tau X \to E \to X \to 0$ and a k-linear map $\kappa_X\colon \mathrm{Ext}^1(X,\tau X) \to k$ with $\kappa_X(\mu_X) \neq 0$. Similarly, for $Y \in \mathcal{H}$ indecomposable and an almost split sequence $\mu_{\tau^- Y}\colon 0 \to Y \to F \to \tau^- Y \to 0$ we fix $\kappa_{\tau^- Y}\colon \mathrm{Ext}^1(\tau^- Y, Y) \to k$ with $\kappa_{\tau^- Y}(\mu_{\tau^- Y}) \neq 0$. Then

$$\langle -|-\rangle\colon \mathrm{Ext}^1(X,Y) \times \mathrm{Hom}(\tau^- Y, X) \to k,\ (\eta, f) \mapsto \kappa_{\tau^- Y}(\eta \cdot f)$$

is a perfect pairing, and similarly so is

$$\langle -|-\rangle\colon \mathrm{Hom}(Y,\tau X) \times \mathrm{Ext}^1(X,Y) \to k,\ (g,\eta) \mapsto \kappa_X(g \cdot \eta).$$

From these perfect pairings we obtain Serre duality

$$\mathrm{Hom}(Y,\tau X) \xrightarrow{\psi_{XY}} \mathrm{D}\,\mathrm{Ext}^1(X,Y) \xleftarrow{\phi_{XY}} \mathrm{Hom}(\tau^- Y, X), \tag{4.1}$$

where $\psi_{XY}\colon f \mapsto \langle f|-\rangle$ and $\phi_{XY}\colon g \mapsto \langle -|g\rangle$ are isomorphisms, natural in X and Y.

Proposition 4.1. *Let $X \in \mathcal{H}$ be indecomposable such that* $\mathrm{End}(X)$ *is a skew field. Denote by $\mu\colon 0 \to \tau X \xrightarrow{u} E \xrightarrow{v} X \to 0$ the almost split sequence ending in X. For all $f \in \mathrm{End}(X)$ we have*

$$\tau(f) \cdot \mu = \mu \cdot f.$$

Proof. The isomorphism ψ_{XY} from (4.1) is natural in X and Y and thus, in particular, an isomorphism of $\mathrm{End}(X) - \mathrm{End}(Y)$-bimodules. Then we have the following rules:

$$\langle f|g\eta\rangle = \langle fg|\eta\rangle, \quad \langle f\eta|g\rangle = \langle f|\eta g\rangle,$$
$$\langle \eta|fg\rangle = \langle \eta f|g\rangle, \quad \langle f|\eta g\rangle = \langle \tau(g) f|\eta\rangle$$

(compare [57, (3.2)]). The last equality is just $\operatorname{End}(X)$-linearity. Moreover, by definition of the $\operatorname{End}(X) - \operatorname{End}(Y)$-bimodule structure on $\mathrm{D}\operatorname{Ext}^1(X, Y)$ we have $\langle g|\eta f\rangle = f \cdot \langle g|\eta\rangle$ for all $f \in \operatorname{End}(X)$. Let now $Y = \tau X$ and $\mu \in \operatorname{Ext}^1(X, \tau X)$ be the almost split sequence. Since $D = \operatorname{End}(X) \simeq \operatorname{End}(\tau X)$ is a skew field, $M = \operatorname{Ext}^1(X, \tau X)$ is a onedimensional $D - D$-bimodule, in particular $D\mu = M = \mu D$. Thus for each $f \in D = \operatorname{End}(X)$ there is a unique $f' \in D = \operatorname{End}(\tau X)$ such that $f'\mu = \mu f$. We have to show that $f' = \tau(f)$. Let $\eta \in \operatorname{Ext}^1(X, \tau X)$. Then there is an $h \in \operatorname{End}(X)$ such that $\eta = \mu \cdot h$. First we have

$$\langle f'|\mu\rangle = \langle 1|f'\mu\rangle = \langle 1|\mu f\rangle = \langle \mu|f\rangle = \langle \tau(f)|\mu\rangle,$$

and then

$$\langle f'|\eta\rangle = \langle f'|\mu \cdot h\rangle = h \cdot \langle f'|\mu\rangle = h \cdot \langle \tau(f)|\mu\rangle = \langle \tau(f)|\mu \cdot h\rangle = \langle \tau(f)|\eta\rangle.$$

Since $\langle -|-\rangle$ is a perfect pairing we conclude $f' = \tau(f)$, finishing the proof. □

Corollary 4.2. *Let* $\operatorname{End}(X)$ *be a skew field. Let* $f \in \operatorname{End}(X)$ *such that there is a commutative diagram*

$$\begin{array}{lccccccccc}
\mu: & 0 & \longrightarrow & \tau X & \xrightarrow{u} & E & \xrightarrow{v} & X & \longrightarrow & 0 \\
 & & & \downarrow{\scriptstyle f'} & & \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle f} & & \\
\mu: & 0 & \longrightarrow & \tau X & \xrightarrow{u} & E & \xrightarrow{v} & X & \longrightarrow & 0.
\end{array}$$

Then $f' = \tau(f)$ *holds.*

Proof. We show that from the assumptions $f' \cdot \mu = \mu \cdot f$ follows. By the preceding proposition then $f' \cdot \mu = \tau(f) \cdot \mu$, thus $(f' - \tau(f)) \cdot \mu = 0$; since $\mu \neq 0$ and $\operatorname{End}(\tau X)$ is a skew field, this yields $f' = \tau(f)$.

If $f \neq 0$, then f is an isomorphism, and since μ does not split, $f' \neq 0$ follows easily. Dually, if $f = 0$, then also $f' = 0$ holds. Thus we may assume that f, and then also f' and g are isomorphisms. One computes that

$$f' \cdot \mu \colon 0 \to \tau X \xrightarrow{u} E \xrightarrow{f^{-1}v} X \to 0$$

and

$$\mu \cdot f \colon 0 \to \tau X \xrightarrow{uf'^{-1}} E \xrightarrow{v} X \to 0.$$

Then we have the commutative exact diagram:

$$\begin{array}{lccccccccc}
f' \cdot \mu: & 0 & \longrightarrow & \tau X & \xrightarrow{u} & E & \xrightarrow{f^{-1}v} & X & \longrightarrow & 0 \\
 & & & \| & & \downarrow{\scriptstyle g^{-1}} & & \| & & \\
\mu \cdot f: & 0 & \longrightarrow & \tau X & \xrightarrow{uf'^{-1}} & E & \xrightarrow{v} & X & \longrightarrow & 0,
\end{array}$$

thus $f' \cdot \mu = \mu \cdot f$ as claimed. □

As a special case we get the following description of the bimodule of a homogeneous tube.

Corollary 4.3. *Let $\mathcal{U}$ be a homogeneous tube in $\mathcal{H}$ with simple object $S = \tau S$, almost split sequence $\mu\colon 0 \to S \to S[2] \to S \to 0$ and division algebra $D = \mathrm{End}(S)$. Then the bimodule of $\mathcal{U}$, that is, the D-D-bimodule $E = \mathrm{Ext}^1(S,S)$, is given by $E = D\cdot\mu = \mu\cdot D$ with relations $\mu\cdot d = \tau(d)\cdot\mu$ (for all $d \in D$), where $\tau \in \mathrm{Aut}(D/k)$ is induced by the Auslander–Reiten translation.* □

5. Tubes and their complete local rings

Let k be a field. If D is a division algebra over k and $\sigma \in \mathrm{Aut}(D/k)$, then we denote by $D[[T,\sigma]]$ the ring of formal power series $\sum_{n\geq 0} a_n T^n$ over D, subject to the relation $Ta = \sigma(a)T$ for all $a \in D$. Such rings occur naturally in the study of generalized uniserial algebras over a perfect field, cf. [42].

Let $(R,\mathfrak{m})$ be a (not necessarily commutative) local ring with Jacobson radical $\mathfrak{m}$. We write $\mathrm{gr}(R) = \bigoplus_{n\geq 0}\mathfrak{m}^n/\mathfrak{m}^{n+1}$. This is a graded local ring, with graded Jacobson radical given by $\mathrm{gr}_+(R) = \bigoplus_{n\geq 1}\mathfrak{m}^n/\mathfrak{m}^{n+1}$. Its $\mathrm{gr}_+(R)$-adic completion is given by $\widehat{\mathrm{gr}}(R) = \prod_{n\geq 0}\mathfrak{m}^n/\mathfrak{m}^{n+1}$, with multiplication given by the Cauchy product.

Proposition 5.1. *Let k be a field. Let $\mathcal{U}$ be a (homogeneous) tube over a noncommutative regular projective curve over k with simple object S and $D = \mathrm{End}(S)$ the endomorphism skew field. Let $\tau \in \mathrm{Gal}(D/k)$ be the automorphism (modulo inner) induced by the Auslander–Reiten translation τ. Let $S[\infty]$ be the corresponding Prüfer sheaf and $R = \mathrm{End}(S[\infty])$ its endomorphism ring.*

Then R is a complete local domain with maximal ideal $\mathfrak{m} = R\pi = \pi R$, where π is a surjective endomorphism of $S[\infty]$ having kernel S. Each one-sided ideal is two-sided, and if non-zero then of the form $\mathfrak{m}^n = R\pi^n = \pi^n R$. Moreover, $\mathcal{U} \simeq \mathrm{mod}_0(R)$, and there are isomorphisms

$$\mathrm{gr}(R) \simeq D[T,\tau^-] \quad \textit{(of graded rings) and} \quad \widehat{\mathrm{gr}}(R) \simeq D[[T,\tau^-]].$$

Proof. (1) With the notations from 2.3, the Prüfer object $S[\infty]$ is the direct limit of the direct system $(S[n],\iota_n)$. The direct limit closure $\vec{\mathcal{U}}$ of $\mathcal{U}$ in $\vec{\mathcal{H}}$ is a hereditary locally finite Grothendieck category in which $S[\infty]$ is an indecomposable injective cogenerator. Its endomorphism ring R is the inverse limit of the inverse system of rings $(\mathrm{End}(S[n],p_n)$, where the p_n are the surjective restriction maps.

It is well known (we refer to [5, 30, 71, 73], and [84, Prop. 4.10]) that R is a complete local domain with $\mathcal{U} \simeq \mathrm{mod}_0(R)$, having the properties stated in the proposition. The two isomorphisms of rings remain to show. Since $\mathcal{U}$ is hereditary, $\widehat{\mathrm{gr}}(R)$ is, by [30, 8.5], isomorphic to the complete tensor algebra [30, 7.5] Ω of the species of $\mathcal{U}$, which is given by the D-D-bimodule $E = \mathrm{Ext}^1(S,S)$.

(2) We now determine the complete tensor algebra of the bimodule E. Let $\mu \in E$ denote the almost split sequence $0 \to S \to S[2] \to S \to 0$. We have $E = D\mu = \mu D$, and from Proposition 4.1 we get $\mu \cdot d = \tau(d) \cdot \mu$ for each $d \in D$. For each natural number n there is a canonical isomorphism ${}_D D^n_D \otimes {}_D E_D \simeq ({}_D D_D \otimes {}_D E_D)^n \simeq {}_D E^n_D$, where E^n as left D-module is isomorphic to D^n, and the right D-module structure on E^n is given by τ-twist: $(x_1, \dots, x_n) \cdot d = (x_1\tau(d), \dots, x_n\tau(d))$.

We denote by $\mathcal{U}'$ the category of *small representations* [30] of the species given by the division ring D and one loop labelled by the D-D-bimodule E. The indecomposable of length n is given by $S'[n]$, which is the representation $D^n \xrightarrow{g} E^n$, with g nilpotent right D-linear and given by the indecomposable Jordan matrix

$$J_n = J_n(0) = \begin{pmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 1 \\ 0 & 0 & 0 & \dots & 0 \end{pmatrix}$$

to the eigenvalue 0. That is, we have

$$g(\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}) = J_n \cdot \begin{pmatrix} \tau(x_1) \\ \vdots \\ \tau(x_n) \end{pmatrix}.$$

If

$$\begin{array}{ccc} D^n & \xrightarrow{g} & E^n \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle f'} \\ D^n & \xrightarrow{g} & E^n \end{array}$$

is an endomorphism of $S'[n]$, where f and f' are right D-linear maps given by the same $n \times n$-matrix $A = (a_{ij})$ with entries in D, then for the matrices $A \cdot J_n = J_n \cdot A^\tau$ holds, where $A^\tau = (\tau(a_{ij}))$. Similar relations hold for morphisms of representations. This yields that we can A write as

$$A = a_0 \cdot I_n + a_1 \cdot J_n + a_2 \cdot (J_n)^2 + \dots + a_{n-1} \cdot (J_n)^{n-1} \tag{5.1}$$

with unique $a_0, \dots, a_{n-1} \in D$, and where $a \cdot (J_n)^\ell$ is given by replacing the side-diagonal given by $1, 1, \dots, 1$ by the elements $a, \tau^-(a), \dots, \tau^{-(n-\ell-1)}(a)$. Clearly $(a \cdot I_n) \cdot (b \cdot I_n) = (ab) \cdot I_n$ holds, and $D \cdot I_n$ forms a subalgebra of $\operatorname{End}(S'[n])$ isomorphic to D. Moreover,

$$(J_n)^\ell \cdot (a \cdot I_n) = \tau^{-\ell}(a) \cdot (J_n)^\ell \tag{5.2}$$

holds, and the Jacobson radical of $\mathrm{End}(S'[n])$ is generated, as a left ideal and as a right ideal, by the map with matrix J_n.

We denote by $\iota'_n \colon S'[n] \to S'[n+1]$ and $\pi'_n \colon S'[n+1] \to S'[n]$ the morphisms given by the matrices

$$\begin{pmatrix} I_n \\ \hline 0 \end{pmatrix} \quad \text{and} \quad (0 \mid I_n),$$

respectively. This yields that $\iota'_n \circ \pi'_n$ is given by multiplication with J_{n+1}, and $\pi'_n \circ \iota'_n$ by J_n. If f' is the restriction of the endomorphism f to $S'[n-1]$, that is, $\iota'_{n-1} \circ f' = f \circ \iota'_{n-1}$ holds, and if A' denotes the corresponding $(n-1) \times (n-1)$-matrix, then equation (5.1) yields

$$A' = a_0 \cdot I_{n-1} + a_1 \cdot J_{n-1} + a_2 \cdot (J_{n-1})^2 + \cdots + a_{n-2} \cdot (J_{n-1})^{n-2}.$$

Thus, sending f to its restriction f', gives a, clearly surjective, homomorphism $p'_n \colon \mathrm{End}(S'[n]) \to \mathrm{End}(S'[n-1])$ of k-algebras.

Starting with the almost split sequence μ'_1 with end term $S' = S'[1]$ we have the following direct system of short exact sequences

$$\begin{array}{lccccccccc}
\mu'_1\colon & 0 & \longrightarrow & S' & \xrightarrow{\iota'_1} & S'[2] & \xrightarrow{\pi'_1} & S' & \longrightarrow & 0 \\
 & & & \| & & \downarrow{\scriptstyle \iota'_2} & & \downarrow{\scriptstyle \iota'_1} & & \\
\mu'_2\colon & 0 & \longrightarrow & S' & \longrightarrow & S'[3] & \xrightarrow{\pi'_2} & S'[2] & \longrightarrow & 0 \\
 & & & \| & & \downarrow{\scriptstyle \iota'_3} & & \downarrow{\scriptstyle \iota'_2} & & \\
\mu'_3\colon & 0 & \longrightarrow & S' & \longrightarrow & S'[4] & \xrightarrow{\pi'_3} & S'[3] & \longrightarrow & 0 \\
 & & & \| & & \downarrow{\scriptstyle \iota'_4} & & \downarrow{\scriptstyle \iota'_3} & & \\
\vdots & & & \vdots & & \vdots & & \vdots & &
\end{array}$$

and its direct limit

$$\mu'_\infty\colon \qquad 0 \to S' \to S'[\infty] \xrightarrow{\pi'} S'[\infty] \to 0.$$

The ring $R' = \mathrm{End}(S'[\infty])$ is isomorphic to the inverse limit of the $\mathrm{End}(S'[n])$ (with respect to the inverse system given by the p'_n). As in (1), R' is a complete local ring with maximal ideal $\mathfrak{m}' = R'\pi' = \pi' R'$, and with $\mathrm{mod}_0(R') \simeq \mathcal{U}'$. Thus there is an automorphism $\sigma \in \mathrm{Aut}(R'/k)$ with $\pi' f = \sigma(f)\pi'$ for all $f \in R'$.

Each $f \in \mathrm{End}(S'[\infty])$ has a unique expression $f = (f_1, f_2, f_3, \dots)$ with $f_n \in \mathrm{End}(S'[n])$ and $f_n = f_{|S'[n]}$ the restriction of f to $S'[n]$ for each n. The restriction of π' to $S'[n]$ is given by $\iota'_{n-1} \circ \pi'_{n-1}$, hence by the matrix J_n. We conclude that f has a unique expression as formal power series

$$f = \sum_{n=0}^{\infty} a_n \pi'^n.$$

From (5.2) we deduce

$$\pi' a = \tau^-(a)\pi'$$

for all $a \in D$. Thus $R' = D[[\pi', \tau^-]] \simeq D[[T, \tau^-]]$. On the other hand, by [30, 7.5] the complete tensor algebra Ω of E is a complete local ring also satisfying $\mathcal{U}' \simeq \operatorname{mod}_0(\Omega)$. It follows, by [31, IV. Prop. 13], that $R' \simeq \Omega \simeq \widehat{\operatorname{gr}} R$. Finally this yields $\operatorname{gr}(R) \simeq D[T, \tau^-]$ as graded rings. □

In the separable (perfect) case, we can apply the Wedderburn–Malcev theorem [66, Thm. 11.6] in order to get the following.

Proposition 5.2. *With the notations of the preceding proposition, assume that D/k is separable (that is, $Z(D)/k$ is a separable field extension). Then $R \simeq \widehat{\operatorname{gr}}(R)$.*

In different words, R is obtained as the complete tensor algebra of the species of the tube $\mathcal{U}$, so that $\mathcal{U}$ can be recovered from its species. Without the separability assumption the statement is wrong in general, cf. Example 10.13.

Proof. The proof is based on [30, 8.4]. For $n \geq 1$ let B_n be the finite dimensional k-algebra $\operatorname{End}(S[n]) \simeq R/\mathfrak{m}^n$, where R is the endomorphism ring of $S[\infty]$ with maximal ideal $\mathfrak{m}$. The Wedderburn–Malcev theorem implies that the projection $B_n \to B_n/\operatorname{rad}(B_n) \simeq D$ splits. Thus $B_n = D_n \oplus \operatorname{rad}(B_n)$, with a subalgebra D_n of B_n isomorphic to D. Then B_n becomes a D-D-bimodule, and $\operatorname{rad}(B_n)$ contains a subbimodule which is isomorphic to $V_n = \operatorname{rad}(B_n)/\operatorname{rad}^2(B_n)$. Thus there is a surjective homomorphism from the tensor algebra of V_n, and then also from the complete tensor algebra Ω of the species of $\mathcal{U}$, onto B_n. We get an isomorphism $\Omega/\operatorname{rad}^n(\Omega) \simeq B_n$. A more detailed analysis shows that this can be done inductively in such a way that we obtain an isomorphism of inverse systems of rings. Taking inverse limits we get $\Omega \simeq R$. This finishes the proof. □

If $\mathcal{U} = \mathcal{U}_x$ satisfies this separability condition, we call x (resp. $\mathcal{U}$) a *separable* point (tube). If k is a perfect field, then all points are separable.

Theorem 5.3. *Let k be a field. Let $\mathcal{U}$ be a* separable *tube over a noncommutative regular projective curve over k with simple object S and $D = \operatorname{End}(S)$ the endomorphism skew field. Let $\tau \in \operatorname{Gal}(D/k)$ be the automorphism induced by the Auslander–Reiten translation τ. Let $S[\infty]$ be the corresponding Prüfer sheaf. Then*

$$\operatorname{End}(S[\infty]) \simeq D[[T, \tau^-]].$$

In particular, $\mathcal{U} \simeq \operatorname{mod}_0\big(D[[T, \tau^-]]\big)$. □

We set

$$\operatorname{Aut}_\tau(D/k) = \{\sigma \in \operatorname{Aut}(D/k) \mid \sigma\tau = \tau\sigma\},$$
$$\operatorname{Inn}_\tau(D/k) = \{\iota_u \in \operatorname{Inn}(D/k) \mid u \in \operatorname{Fix}(\tau)\},$$

and finally $\operatorname{Gal}_\tau(D/k) = \operatorname{Aut}_\tau(D/k)/\operatorname{Inn}_\tau(D/k)$. Clearly $\operatorname{Inn}_\tau(D/k) = \operatorname{Inn}(D/k) \cap \operatorname{Aut}_\tau(D/k)$, so that $\operatorname{Gal}_\tau(D/k)$ can be regarded as a subgroup of $\operatorname{Gal}(D/k)$. Trivially $\tau \in \operatorname{Aut}_\tau(D/k)$ holds, so that the order of τ in $\operatorname{Gal}(D/k)$ is the same as the order of τ in $\operatorname{Gal}_\tau(D/k)$.

Corollary 5.4. *Let x be a separable point, $\mathcal{U} = \mathcal{U}_x$ and $D = \operatorname{End}(S_x)$. Then* $\operatorname{Aut}(\mathcal{U}/k) \simeq \operatorname{Gal}_\tau(D/k)$.

Proof. We have $\operatorname{gr}(R) \simeq D[T, \tau^-]$ and $\mathcal{U} \simeq \operatorname{mod}_0^{\mathbb{Z}}(\operatorname{gr}(R))/s^{\mathbb{Z}}$, the orbit category with respect to the degree shift s; in different words, this is the category of finite dimensional $\operatorname{gr}(R)$-modules which are annihilated by some power of T. A graded automorphism of $\operatorname{gr}(R)$ is uniquely determined by its action on degrees zero and one, and is thus of the form

$$\sum a_i T^i \mapsto \sum f(a_i) N_i(b) T^i,$$

with $f \in \operatorname{Aut}(D/k)$ and $b \in D^\times$, satisfying $f\tau^-(a) \cdot b = b \cdot \tau^- f(a)$ for all $a \in D$. Here $N_i(b)$ is defined as $b \cdot \tau^-(b) \ldots \tau^{-(i-1)}(b)$. We define the group of graded inner automorphisms of $\operatorname{gr}(R)$, denoted by $\overline{\operatorname{Inn}}(\operatorname{gr}(R))$, generated by automorphisms of the form ι_u, $r \mapsto u^{-1}ru$ ($u \in D^\times$), and by automorphisms induced by $T^n \mapsto N_n(b)T^n$ (with $b \in Z(D)^\times$). Each graded automorphism $\sigma = (f, b)$ of $\operatorname{gr}(R)$ induces an autoequivalence F^σ on $\mathcal{U}$, and $F^\sigma \simeq 1_{\mathcal{U}}$ if and only if σ is a graded inner automorphism. We refer to [47, Prop. 3.2.3] for a similar statement. On the other hand, each automorphism of $\mathcal{U}$ is uniquely determined by its action on the bimodule $\operatorname{Ext}^1(S, S)$, and thus on $R/\mathfrak{m} = D$ and $\mathfrak{m}/\mathfrak{m}^2$, and thus induces a graded automorphism of $\operatorname{gr}(R) = D\langle \mathfrak{m}/\mathfrak{m}^2 \rangle$.

Considering the skeleton of $\mathcal{U}$ and requiring that automorphisms are the identity on objects (e.g. equality $\tau S = S$), the automorphism F^σ on S commutes with τ on S, which follows from the diagram in Corollary 4.2. We thus can assume that $f \in \operatorname{Aut}_\tau(D/k)$. Then also $b \in Z(D)^\times$. We write $\operatorname{Aut}_\tau(\operatorname{gr}(R))$ for the subgroup of the automorphisms with these properties, and $\overline{\operatorname{Inn}}_\tau(\operatorname{gr}(R)) = \overline{\operatorname{Inn}}(\operatorname{gr}(R)) \cap \operatorname{Aut}_\tau(\operatorname{gr}(R))$. We conclude $\operatorname{Aut}(\mathcal{U}/k) \simeq \operatorname{Aut}_\tau(\operatorname{gr}(R))/\overline{\operatorname{Inn}}_\tau(\operatorname{gr}(R)) \simeq \operatorname{Gal}_\tau(D/k)$, finishing the proof. □

Corollary 5.5. *Let x be a point.*

(1) *The functors τ^- and σ_x, restricted to the simple S_x, yield the same elements in* $\operatorname{Gal}(\operatorname{End}(S_x)/k)$.

(2) *Let x be separable. Then the functors τ^- and σ_x, restricted to $\mathcal{U}_x$, are isomorphic.*

The separability assumption in (2) is essential, cf. Example 10.13.

Proof. (1) We write $S = S_x$ and $D = \operatorname{End}(S)$. By [47, 0.4.2] there is a natural isomorphism $u\colon \sigma_x S \xrightarrow{\sim} \operatorname{Ext}^1(S,S) \otimes_D S$, and for $f \in D$ the endomorphism $\sigma_x(f)$ corresponds to $\eta \otimes s \mapsto f\eta \otimes s$. Let $\eta \in \operatorname{Ext}^1(S,S)$. There is $d \in D$ with $\eta = \mu \cdot d$, where μ is the almost split sequence starting and ending in S. For $f \in Z(D)$ we have $f\eta \otimes s = \mu\tau^-(f)d \otimes s = \mu d \otimes \tau^-(f)(s) = \eta \otimes \tau^-(f)(s)$. We conclude $\sigma_x(f) = u^{-1}\tau^-(f)u = \tau^-(f)$ for all $f \in Z(D)$. Thus the restrictions of σ_x and τ^- to $Z(D)$ yield the same element in $\operatorname{Gal}(Z(D)/k)$. By the Skolem–Noether theorem the restrictions of σ_x and τ^- to D yield the same element in $\operatorname{Gal}(D/k)$.

(2) This follows from (1) together with the preceding corollary. □

Definition 5.6. Let $\mathbb{X}$ be a noncommutative regular projective curve over a field. We call a point $x \in \mathbb{X}$ a *separation point*, if it is separable and $e_\tau(x) > 1$ holds.

Corollary 5.7. *Let $U \subseteq \mathbb{X}$ be a subset such that* $\operatorname{Pic}(\mathcal{H})$ *is generated by σ_x ($x \in U$). Then U contains all separation points.* □

6. Local-global principle of skewness

For a point x of a noncommutative regular projective curve over an arbitrary field we write $e^{**}(x)$ for the PI-degree of $\operatorname{End}(S_x[\infty])$.

Theorem 6.1 (General skewness principle). *Let $\mathcal{H}$ be a noncommutative regular projective curve over an arbitrary field k. For all points $x \in \mathbb{X}$ the following hold:*

(1) $e(x) \cdot e^{**}(x) = s(\mathcal{H})$.

(2) $e^*(x)$ *divides* $e^{**}(x)$.

Proof. (1) By Proposition 3.16 the PI-degree of $\widehat{R}_x$ is $e(x) \cdot e^{**}(x)$. By [17, Thm.13] the ring R_x and its completion $\widehat{R}_x$ have the same PI-degree. The PI-degree of R_x coincides with the PI-degree of its quotient division ring $k(\mathcal{H})$, which is $s(\mathcal{H})$. Thus we get the equation.

(2) By a theorem of Bergman–Small (see [74, Thm. 1.10.70]), applied to the surjective ring homomorphism $R_x \to R_x/\operatorname{rad}(R_x) \simeq \mathrm{M}_{e(x)}(D_x)$, the PI-degree of the factor, which is $e(x) \cdot e^*(x)$, divides the PI-degree of R_x, which is $s(\mathcal{H})$. Together with (1) we get that $e^*(x)$ divides $e^{**}(x)$. □

Lemma 6.2. *Let x be a point with associated skew field $D_x = \operatorname{End}(S_x)$. Denote by*

$$\widehat{D}_x = D_x((T, \tau^-)) \tag{6.1}$$

the skew Laurent power series ring over D_x in the variable T. It is a skew field of dimension $e^*(x)^2 \cdot e_\tau(x)^2$ over its centre. Moreover, it is v_x-complete, where the *valuation* v_x *is given by* $v_x(\sum_m^{\infty} a_i T^i) = (1/2)^{\ell}$*, with* ℓ *the infimum of indices* i *with* $a_i \neq 0$.

Proof. Let $r = e_\tau(x)$ and $\sigma^{-r}(d) = u^{-1}du$ for some $u \in \mathrm{Fix}(\tau)^{\times}$. By [66, 19.7], the centre of $D_x((T,\tau^-))$ is given by

$$\widehat{K}_x = K_x((uT^r)) \quad \text{with } K_x = Z(D_x) \cap \mathrm{Fix}(\tau^-). \tag{6.2}$$

From this the assertion about the centre follows. Completeness is shown in [66, 19.7]. □

Proposition 6.3. *Let* x *be a separable point.*

(1) *We have* $e^{**}(x) = e^*(x) \cdot e_\tau(x)$.

(2) $e_\tau(x)$ *coincides with*

  (i) *the order of* $\tau \in \mathrm{Aut}(\mathcal{U}_x/k)$*, the group of (isomorphism classes of) autoequivalences on the tube* $\mathcal{U}_x$*;*

  (ii) *the order of the cyclic group* $\mathrm{Gal}(Z(D_x)/K_x)$*, generated by* τ.

(3) *If* $\phi \in \mathrm{Aut}(\mathcal{H})$*, and* $\phi(S_x) = S_y$*, then* $e_\tau(x) = e_\tau(y)$.

Proof. (1) We have $\mathrm{End}(S_x[\infty]) \simeq D_x[[T,\tau^-]]$. By Posner's theorem (see [6, Thm. 7]) the PI-degree of $D_x[[T,\tau^-]]$ coincides with the PI-degree of its quotient division ring, which is $D_x((T,\tau^-))$. The assertion follows from the preceding lemma.

(2) (i) follows from Corollary 5.4, (ii) from the preceding lemma.

(3) It follows that y is also separable, and the equality of τ-multiplicites is obtained from (2) (i). □

We will see in the next section, that (ii) just means that $e_\tau(x)$ coincides with the ramification index of x with respect to the maximal order $\mathcal{A}$ associated with $\mathcal{H}$.

Example 6.4. If $k = \mathbb{R}$ and the tube $\mathcal{U}$ is either of the form $\mathrm{mod}_0\,\mathbb{C}[[T]]$, or $\mathrm{mod}_0\mathbb{C}[[T,\tau^-]]$ with τ^- of order two, then in both cases $\mathrm{Aut}(\mathcal{U}/k) \simeq \mathrm{Gal}(\mathbb{C}/\mathbb{R}) = C_2$, generated by complex conjugation. In the first case τ acts trivially on $\mathcal{U}$, in the second it generates $\mathrm{Aut}(\mathcal{U}/k)$.

The following local-global principle is the main result on skewness.

Theorem 6.5 (Local-global principle of skewness). *Let* $\mathcal{H}$ *be a noncommutative regular projective curve over a field. Then for each separable point* $x \in \mathbb{X}$ *the formula*

$$e(x) \cdot e^*(x) \cdot e_\tau(x) = s(\mathcal{H})$$

holds.

For a perfect field, we obtain Theorem 1.5.

Proof. Follows from Theorem 6.1 (1) and Proposition 6.3 (1). □

The preceding theorems give answers to [47, Probl. 2.3.11 + 4.3.10]. It follows, by the way, that for each separable point the multiplicity $e(x)$ does not depend on the line bundle L used in its definition, since $e(x) = \frac{s(\mathcal{H})}{e^*(x)e_\tau(x)}$.

7. Maximal orders and ramifications

Following [68, 69], we will use in this section an alternative description of noncommutative curves in terms of hereditary and maximal orders. Here our main result is that the τ-multiplicities $e_\tau(x)$ coincide with the ramification indices of the underlying maximal order $\mathcal{A}$. We will temporarily, in Theorem 7.11, also permit weighted curves. This will allow to characterize the non-weighted situation in terms of orders. Namely, the weights $p(x)$ correspond to the local types of the, in general, hereditary order $\mathcal{A}$, which measure the deviation of $\mathcal{A}_x$ from being maximal. For excellent expositions on orders we refer to [11, 18, 19, 37, 67, 75], and the unpublished [7].

7.1. By a (commutative) *curve* we mean a one-dimensional scheme over k, which we always assume to be integral, separated and of finite type over k. A curve X is *regular* (or *non-singular*) if all local rings $\mathcal{O}_{X,x}$ are regular, equivalently, discrete valuation domains; in particular they are hereditary. We remark that if k is a perfect field, regularity is equivalent to smoothness; cf. [27, I.5.3.2].

7.2 (The centre curve). Let $(\mathcal{H}, L)$ be a noncommutative regular projective curve over the field k with point set $\mathbb{X}$ and function field $D = k(\mathcal{H})$. Let $K = Z(k(\mathcal{H}))$ be the centre of D. There is a unique (commutative) regular complete curve $X = C_K$ with function field $k(X) = K$ and whose points are in bijective correspondence with the discrete valuations of K/k; we refer to [35, Prop. (7.4.18)], also [27, I.5.3.7]. By Chow's lemma (we refer to [38, Ex. II.4.10] and [35, 5.6]) there is a (irreducible) *projective* curve X' and a surjective, birational morphism $\pi\colon X' \to X$ over k (in particular: $X' = \operatorname{Proj}(S')$ where the commutative graded ring S' is generated in degrees 0 and 1). By [36, Cor. (4.4.9)] we have that π is even an isomorphism. In particular, X itself is projective over k. We call X the *centre curve* of $\mathcal{H}$ (or $\mathbb{X}$). If $\mathcal{O} = \mathcal{O}_X$ is the structure sheaf of X, we denote by $(\mathcal{O}_x, \mathfrak{m}_x)$ the local rings ($x \in X$) and by $k(x) = \mathcal{O}_x/\mathfrak{m}_x$ the residue class fields.

Example 7.3. Let $k = \mathbb{R}$ be the field of real numbers and $R = \mathbb{C}[X; Y, \sigma]$ the twisted polynomial algebra, graded by total degree, where X is central and $Yz = \sigma(z)Y$ for each $z \in \mathbb{C}$, with $\sigma(z) = \bar{z}$ the complex conjugation. Then $\mathcal{H} = \operatorname{mod}^{\mathbb{Z}}(R)/\operatorname{mod}^{\mathbb{Z}}_0(R)$ is a noncommutative regular projective curve (we refer to [47] for more details). The function field is $\mathbb{C}(T, \sigma)$, its centre given by $\mathbb{R}(T^2)$. The centre of R is $S = \mathbb{R}[X, Y^2]$, and $\operatorname{Proj}(S)$ is the centre curve of $\mathcal{H}$. It is isomorphic to the projective spectrum of $S' = \mathbb{R}[X, Y]$, graded by total degree, having function field $\mathbb{R}(T)$, which as $\mathbb{R}$-algebra is isomorphic to $\mathbb{R}(T^2)$. Thus the centre curve of $\mathcal{H}$ is isomorphic to the projective line $\mathbb{P}^1(\mathbb{R})$.

7.4 (The categorical centre). We also have the centre of $\mathcal{H}$ in the categorical sense, namely $Z(\mathcal{H}) = \mathrm{End}(1_{\mathcal{H}})$, the ring of natural endotransformations of the identity functor $1_{\mathcal{H}}$.

Lemma 7.5. *The categorical centre $Z(\mathcal{H})$ is a field, of finite dimension over k. For each line bundle L', the assignment $\alpha \mapsto \alpha_{L'}$ yields a k-monomorphism from $Z(\mathcal{H})$ into* $\mathrm{End}(L')$.

Therefore we can usually assume without loss of generality that k is the centre of $\mathcal{H}$.

Proof. We proceed as in [55, (S 19)]. If α is a non-zero element in the centre, then $\alpha_{L'}$ is non-zero for each line bundle L': if otherwise $\alpha_{L'} = 0$, then it follows that also $\alpha_{L'(nx)} = 0$ for all $x \in \mathbb{X}$ and $n \in \mathbb{Z}$. Using ampleness (cf. Lemma 3.5 and Remark 3.8) we the get easily $\alpha = 0$, contradiction. Since $\mathrm{End}(L')$ is a skew field, $\alpha_{L'}$ is an isomorphism. Using line bundle filtrations and the fact that each simple object is the cokernel of a monomorphism between line bundles, we obtain that α_F is an isomorphism for each object $F \in \mathcal{H}$. Thus α is invertible. Finally, for all $\alpha, \beta \in Z(\mathcal{H})$ we clearly have $\alpha_{L'}\beta_{L'} = \beta_{L'}\alpha_{L'}$, and hence $Z(\mathcal{H})$ is commutative. □

Proposition 7.6. *Let $(\mathcal{H}, L)$ be a noncommutative regular projective curve.*

(1) *For each $x \in \mathbb{X}$ the graded ring $\Pi(L, \sigma_x)$ is finitely generated as module over its centre.*

(2) *If $s(\mathcal{H}) = 1$, then $\Pi(L, \sigma_x)$ is commutative.*

Proof. Like in [47, Prop. 4.3.3] we have a graded inclusion $\Pi(L, \sigma_x) \subseteq k(\mathcal{H})[T]$, where T is a central variable. From this, (2) follows immediately, and (1) follows with [50] and [8, Thm. 0.1(ii)]. □

Corollary 7.7. *Let $(\mathcal{H}, L)$ be a noncommutative regular projective curve over k with $s(\mathcal{H}) = 1$. Then there is a (commutative) regular projective curve X over k such that $\mathcal{H} \simeq \mathrm{coh}(X)$, and the points of $\mathbb{X}$ are in bijective correspondence with the closed points of X.*

Proof. Let S be the commutative graded ring $\Pi(L, \sigma_x)$ for some $x \in \mathbb{X}$ and $X = \mathrm{Proj}(S)$. □

Corollary 7.8. *Let k be an algebraically closed field. Then $\mathcal{H}$ is a noncommutative regular projective curve over k if and only if $\mathcal{H}$ is equivalent to the category* $\mathrm{coh}(X)$ *of coherent sheaves over a (commutative) regular (=smooth) projective curve X.*

Proof. By Tsen's theorem [82] we have $s(\mathcal{H}) = 1$. □

7.9 (The centre curve in the weighted case). We assume that $\mathcal{H}$ satisfies (NC 1) to (NC 5). By Remark 3.8 also (NC 7) holds. Thus the centre of the function field $k(\mathcal{H})$ is of the form $k(X)$, for a unique regular projective curve X, which we also call the *centre curve* of $\mathcal{H}$ in this weighted case. Similarly, $R = \Pi(L, \sigma)$, with σ a suitable product of Picard-shifts, is module-finite over its centre, by the same arguments given in Proposition 7.6. Part (2) of the next theorem below will show that also (NC 6) is satisfied.

7.10 (Orders over the centre curve). Let X be the centre curve with function field $K = k(X)$. Let A be a finite dimensional central simple K-algebra. As in [7] we call a torsionfree, coherent $\mathcal{O}_X$-algebra $\mathcal{A}$ an *$\mathcal{O}_X$-order* in A, if the generic fibre of $\mathcal{A}$ is isomorphic to A, or equivalently, if $\mathcal{A} \otimes_{\mathcal{O}_X} K \simeq A$. An order $\mathcal{A}$ is called *maximal* if it is not contained properly in another order. Then all stalks $\mathcal{A}_x = \mathcal{A} \otimes \mathcal{O}_x$ are maximal $\mathcal{O}_x$-orders in A. An order $\mathcal{A}$ is called *hereditary*, if all stalks $\mathcal{A}_x$ are hereditary $\mathcal{O}_x$-orders in A. Each maximal order is hereditary. The $\mathcal{O}_X$-order $\mathcal{A}$ is called an *Azumaya algebra* of degree n, if $\mathcal{A}$ is locally-free of rank n^2, and if for each $x \in X$ the geometric fibre $\mathcal{A}(x) = \mathcal{A}_x \otimes_{\mathcal{O}_x} k(x) = \mathcal{A}_x / \operatorname{rad}(\mathcal{A}_x)$ is a full matrix algebra with centre $k(x)$. Equivalently (by [7, Prop. 1.9.2]): For each x we have $[\mathcal{A}(x) : k(x)] = n^2$. Azumaya algebras over $\mathcal{O}_X$ are maximal orders (by [7, Prop. 1.8.2]).

We now have the following fundamental description of noncommutative regular projective curves, essentially due to Reiten–van den Bergh [68, Prop. III.2.3].

Theorem 7.11. *Let k be a field.*

(1) *For a k-category $\mathcal{H}$ the following two conditions are equivalent:*

   (a) *$\mathcal{H}$ is a weighted noncommutative regular projective curve over k.*

   (b) *There is a (commutative) regular projective curve X over k, a (finite dimensional) central simple $k(X)$-algebra A and a torsionfree coherent sheaf $\mathcal{A}$ of hereditary $\mathcal{O} = \mathcal{O}_X$-orders in A such that $\mathcal{H} \simeq \operatorname{coh}(\mathcal{A})$, the category of coherent $\mathcal{A}$-modules.*

(2) *If the equivalent conditions in (1) hold, then X is the centre curve. Accordingly, the points of $\mathbb{X}$ correspond bijectively to the closed points of X, and for each $x \in \mathbb{X}$ its weight $p(x)$ is the local type (in the sense of [67, p. 369]) of the hereditary $\mathcal{O}$-order $\mathcal{A}$ at x. Accordingly, $p(x) > 1$ if and only if $\mathcal{A}_x$ is not maximal, and there is only a finite number of such points x.*

(3) *In (1) we have that $\mathcal{H}$ is non-weighted if and only if $\mathcal{A}$ is a maximal $\mathcal{O}$-order in A.*

Proof. (1) This is shown like in [68, Prop. III.2.3]. For the fact that the centre of a hereditary order is a Dedekind domain, we refer to [37, Thm. 2.6]. By [83] the category $\operatorname{coh}(\mathcal{A})$ has Serre duality. We recall the construction of A and $\mathcal{A}$ if $\mathcal{H}$ is given. Let X be the underlying centre curve. Let R be a positively $\mathbb{Z}$-graded coordinate algebra of $\mathcal{H}$, module-finite over its centre S. Let $x_1, \dots, x_t$ be a set of homogeneous generators of S over the field S_0. Let n be the least common multiple of their degrees. Then

$$T = \begin{pmatrix} R & R(1) & \dots & R(n-1) \\ R(-1) & R & \dots & R(n-2) \\ \vdots & \vdots & \ddots & \vdots \\ R(-n+1) & R(-n+2) & \dots & R \end{pmatrix}$$

is graded Morita-equivalent to R and strongly $\mathbb{Z}$-graded; thus $\operatorname{mod}^{\mathbb{Z}}(T) \simeq \operatorname{mod}(T_0)$. Let $\mathcal{T}$ be the corresponding sheaf of graded rings. We set $A = \mathrm{M}_n(k(\mathcal{H}))$, which is of finite dimension over its centre $k(X)$, and $\mathcal{A} = \mathcal{T}_0 \subseteq A$, equipped canonically with the structure of an $\mathcal{O}_X$-module.

(2) The assertion is clear from the structure of hereditary orders [18, 19], we refer also to [67, Ch. 9], and the Auslander–Goldman criterion [11, Thm. 2.3] for maximality. (This in particular shows that (NC 6) follows from (NC 1) to (NC 5).)

(3) This follows from (2), since by [67, (40.8)] the order $\mathcal{A}$ is maximal if and only if all $\mathcal{A}_x$ are maximal. □

We switch back to the non-weighted case. The next theorem extends [35, Prop. (7.4.18)] to this noncommutative setting, and it gives a positive answer to [47, Probl. 4.3.9], even in this much more general context. It is an easy consequence of well known results in the theory of maximal orders. It was also shown recently in [20, Thm. 6.7].

Theorem 7.12. *Two noncommutative regular projective curves $\mathcal{H}$ and $\mathcal{H}'$ over a field k are isomorphic (that is, they are equivalent as k-categories) if and only if their function fields $k(\mathcal{H})$ and $k(\mathcal{H}')$ are isomorphic.*

Proof. If $\mathcal{H} \simeq \mathcal{H}'$, then $\mathcal{H}_0 \simeq \mathcal{H}'_0$ and $\mathcal{H}/\mathcal{H}_0 \simeq \mathcal{H}'/\mathcal{H}'_0$, and consequently $k(\mathcal{H})$ and $k(\mathcal{H}')$ are isomorphic. Assume conversely, that the function fields $k(\mathcal{H})$ and $k(\mathcal{H}')$ are isomorphic and have the common centre $K = k(X)$. By parts (1) and (3) of the preceding theorem, there are maximal orders $\mathcal{A}$ and $\mathcal{A}'$ in Morita-equivalent central simple K-algebras A and A', respectively, such that $\mathcal{H} \simeq \operatorname{coh}(\mathcal{A})$ and $\mathcal{H}' \simeq \operatorname{coh}(\mathcal{A}')$. Since X is a normal curve, by [7, Prop. 1.9.1 (ii)] (for an affine version we refer to [67, Cor. (21.7)]; for a similar result on hereditary orders over a smooth curve we refer to [23, Thm. 7.6]) it follows, that $\mathcal{A}$ and $\mathcal{A}'$ are Morita-equivalent, that is (by definition), $\operatorname{Qcoh}(\mathcal{A}) \simeq \operatorname{Qcoh}(\mathcal{A}')$. Then clearly $\operatorname{coh}(\mathcal{A}) \simeq \operatorname{coh}(\mathcal{A}')$, and thus $\mathcal{H} \simeq \mathcal{H}'$ follows. □

Since maximal $\mathcal{O}_X$-orders over a regular projective curve X in a central simple $k(X)$-algebra always exist, by [7, Prop. 1.8.2], we have even more:

Corollary 7.13. *The assignments*

$$\mathcal{H} \mapsto k(\mathcal{H}) \quad \text{and} \quad A \mapsto \mathrm{coh}(\mathcal{A}),$$

where $\mathcal{A}$ is a maximal order in A (whose centre is of the form $k(X)$), induce mutually inverse bijections between the sets of

- *noncommutative regular projective curves over k, up to equivalence of categories; and*
- *algebraic function skew fields of one variable over k, up to isomorphism.* □

Let $\mathcal{H} = \mathrm{coh}(\mathcal{A})$ be a noncommutative regular projective curve with a maximal order $\mathcal{A}$ in $A = \mathrm{M}_n(k(\mathcal{H}))$ as above. Let $K = k(X)$ be the centre, as above. Let $x \in \mathbb{X}$ be separable. We write $D_x = \mathrm{End}(S_x)$ and denote by $\widehat{E}_x = \mathrm{End}(S_x[\infty])$ the endomorphism ring of the corresponding Prüfer sheaf, which is a complete local domain, the maximal ideal generated by π_x. By Proposition 3.16 we have $\widehat{R}_x \simeq \mathrm{M}_{e(x)}(\widehat{E}_x)$. If $\widehat{K}_x$ denotes the quotient field of the $\mathfrak{m}_x$-adic completion $\widehat{\mathcal{O}}_x$ of $\mathcal{O}_x$, then

$$A \otimes_K \widehat{K}_x \simeq \mathrm{M}_n\big(k(\mathcal{H}) \otimes_K \widehat{K}_x\big) \simeq \mathrm{M}_{n \cdot e(x)}(\widehat{D}_x), \tag{7.1}$$

with $\widehat{D}_x$ a skew field (unique up to isomorphism) with centre $\widehat{K}_x$; compare Proposition 7.16 below. Analogously to the global situation we make the following local definition.

Definition 7.14. We call the number

$$s(x) = [\widehat{D}_x : \widehat{K}_x]^{1/2} \tag{7.2}$$

the *local skewness* at x.

By (7.1) we get the following relationship between global and local skewness

$$s(\mathcal{H}) = e(x) \cdot s(x), \tag{7.3}$$

and with the skewness principle we have

$$s(x) = e^*(x) \cdot e_\tau(x). \tag{7.4}$$

The following results make the situation quite explicit.

Lemma 7.15. (1) *$\mathcal{O}_x$ is the centre of R_x.*

(2) *Let $\mathscr{S}_x$ be the multiplicative set $\mathcal{O}_x \setminus \{0\}$. The central localization $\mathscr{S}_x^{-1} R_x$ is equal to the function field $k(\mathcal{H})$.*

Proof. (1) We have $\mathrm{mod}(\mathcal{A}_x) \simeq \mathcal{H}_x \simeq \mathrm{mod}(R_x)$. Hence $\mathcal{A}_x$ and R_x have same centres. So it is sufficient to show that $\mathcal{O}_x$ is the centre of $\mathcal{A}_x = \mathcal{A} \otimes_{\mathcal{O}} \mathcal{O}_x$. Let $U \subseteq X$ be an affine open subset with $x \in U$. Then $\mathcal{O}(U)$ is a Dedekind domain with quotient field K, and $\mathcal{A}(U)$ is a maximal $\mathcal{O}(U)$-order in A, whose centre $\mathcal{A}(U) \cap K$ is $\mathcal{O}(U)$, since $\mathcal{O}(U)$ is integrally closed. Now, localization at x is compatible with the centres [74, Prop. 1.7.4] and yields the result.

(2) $k(\mathcal{H})$ is the quotient division ring of R_x. By [74, Thm. 1.7.9] the ring $\mathscr{S}_x^{-1}R_x$ is a skew field with centre K. From $R_x \subseteq \mathscr{S}_x^{-1}R_x \subseteq k(\mathcal{H})$ the result follows. □

Proposition 7.16. *Let x be separable. The skew field $\widehat{D}_x$ and its centre $\widehat{K}_x$ from (7.1) agree with the (skew) Laurent power series rings in (6.1) and (6.2), respectively. Moreover, $k(x) = K_x := Z(D_x) \cap \mathrm{Fix}(\tau)$.*

Proof. Write $r = e_\tau(x)$. Using the preceding lemma, we apply $\mathscr{S}_x^{-1}$ (that is, central localization) to the isomorphism $R_x \otimes_{\mathcal{O}_x} \widehat{\mathcal{O}}_x \simeq \mathrm{M}_{e(x)}(\widehat{E}_x)$. By Hom-tensor properties of the localization [16, II.§2.7], we obtain the isomorphism $Q(R_x) \otimes_K \mathscr{S}_x^{-1}\widehat{\mathcal{O}}_x \simeq \mathrm{M}_{e(x)}(\mathscr{S}_x^{-1}\widehat{E}_x)$, where $Q(-)$ stands for quotient division ring. Moreover, since $\widehat{\mathcal{O}}_x$ and $\widehat{E}_x$ are (skew) power series rings in one variable by Theorem 5.3, clearly $\mathscr{S}_x^{-1}\widehat{\mathcal{O}}_x = Q(\widehat{\mathcal{O}}_x)$ and $\mathscr{S}_x^{-1}\widehat{E}_x = Q(\widehat{E}_x)$ hold, since in each case the uniformizer becomes invertible, so that we get the corresponding (skew) Laurent series rings. We conclude $\widehat{D}_x \simeq Q(\widehat{E}_x) \simeq D_x((T, \tau^-))$. Moreover, for the centre we deduce $k(x)((T)) \simeq \widehat{K}_x \simeq K_x((uT^r))$, from which also $K_x = k(x)$ follows. □

We can now derive well-known identities for well-studied local invariants, and also their relationship with the τ-multiplicity. As usual, define the

$$\textit{inertial degree} \quad f_{\mathrm{in}}(x) = [\widehat{D}_x / \operatorname{rad}(\widehat{D}_x) : k(x)] = [D_x : k(x)] \tag{7.5}$$

and the

$$\textit{ramification index} \quad e_{\mathrm{ra}}(x) = [\Gamma_{\widehat{D}_x} : \Gamma_{\widehat{K}_x}] \tag{7.6}$$

(the index of the discrete value group $\Gamma_{\widehat{K}_x}$ in $\Gamma_{\widehat{D}_x}$) of the skew field part of the completion of A in x. If $e_{\mathrm{ra}}(x) > 1$, then x is called a *ramification point* of A. By Proposition 7.16 it is easy to see that the ramification index coincides with the number e such that $\mathfrak{m}_x \widehat{E}_x = \operatorname{rad}(\widehat{E}_x)^e$, and also with the number

$$e'(x) = [Z(D_x) : k(x)]. \tag{7.7}$$

Corollary 7.17. *Let x be a separable point. Then*

$$e_\tau(x) = e_{\mathrm{ra}}(x).$$

The assumption is essential, cf. Example 10.13, and Theorem 7.21.

Proof. Follows from Propositions 6.3 and 7.16. □

Corollary 7.18. *Let $\mathcal{H} = \mathrm{coh}(\mathcal{A})$ be a noncommutative regular projective curve over the perfect field k. The separation points of $\mathcal{H}$ are just the ramification points of $\mathcal{A}$, and there are only finitely many of them.*

Proof. It is well known in the theory of maximal orders that the number of ramification points is finite, see e.g. [67, p. 372]. In Proposition 8.8 below a standard argument for this will be given. □

We also conclude

Corollary 7.19. *If x is a separable point, then*

$$e_{\mathrm{ra}}(x) \cdot f_{\mathrm{in}}(x) = s(x)^2.$$

Proof. Use

$$f_{\mathrm{in}}(x) = [D_x : k(x)] = [D_x : Z(D_x)] \cdot [Z(D_x) : k(x)] = e^*(x)^2 \cdot e_{\mathrm{ra}}(x). \quad \square$$

We call $\mathbb{X}$ (or $\mathcal{H}$) *unramified* (resp. τ*-unramified*) if $e_{\mathrm{ra}}(x) = 1$ (resp. $e_\tau(x) = 1$) for all $x \in \mathbb{X}$; if k is perfect both notions agree. We call it *multiplicity free* if $e(x) = 1$ for all $x \in \mathbb{X}$.

Corollary 7.20. *Let k be a perfect field.*

(1) *We have*

$$\tau = \prod_{x \in \mathbb{X}} {\sigma_x}^{e_\tau(x)-1} \qquad \text{on } \mathcal{H}_0.$$

(2) *$\mathbb{X}$ is unramified if and only if $\tau_{|\mathcal{H}_0} \simeq 1_{\mathcal{H}_0}$.*

Proof. (1) follows from Corollary 5.5. (2) is then clear by the definition of $e_\tau(x)$. □

The following general result expresses $e_{\mathrm{ra}}(x)$ as the local order of a certain functor, namely of σ_x on $\mathcal{U}_x$.

Theorem 7.21. *Let $\mathcal{H}$ be a noncommutative regular projective curve over a field k. Let $\mathcal{U} = \mathcal{U}_x$ be a tube and $V = (V, \pi, \sigma)$ be the associated complete discrete valuation domain [5] with $\mathcal{U} = \mathrm{mod}_0(V)$ from Proposition 5.1, where $V\pi = \pi V$ is the maximal ideal and $\sigma\colon V \to V$ the automorphism given by $\pi r = \sigma(r)\pi$. Then the Picard-shift functor σ_x, restricted to $\mathcal{U}$, is induced by σ. Its order in $\mathrm{Aut}(\mathcal{U}/k)$ equals the order of σ in $\mathrm{Aut}(V/k)$ modulo inner automorphisms, and equals the ramification index $e_{\mathrm{ra}}(x)$.*

Proof. Given x, we form the orbit algebra $\Pi(L, \sigma_x)$, which has a central prime element π_x of degree one. By [47, Thm. 3.1.2] multiplication with π_x yields the natural transformation $1_{\mathcal{H}} \xrightarrow{x} \sigma_x$. Extending σ_x to the direct limit closure of $\mathcal{U}$ (so working in the category $\mathrm{Qcoh}(\mathcal{A})$ of quasicoherent $\mathcal{A}$-modules), we see that the natural sequence in [47, 0.4.2(5)] for the injective object $S[\infty]$ becomes $0 \to S \to S[\infty] \xrightarrow{\pi} S[\infty] \to 0$. We conclude that σ_x on $\mathcal{U}$ is induced by the automorphism σ. The statement about the orders follows from the observation that π^n is central up to a unit if and only if σ^n is inner. □

8. Dualizing sheaf and the Picard-shift group

8.1 (Structure sheaf). Let $\mathcal{H} = \mathrm{coh}(\mathcal{A})$ with $\mathcal{A}$ a maximal order in $k(\mathcal{H})$, and with centre curve X and $\mathcal{O} = \mathcal{O}_X$. We will now specify our structure sheaf $L \in \mathcal{H}$, namely

$$L_{\mathcal{A}} = \mathcal{A}_{\mathcal{A}}. \tag{8.1}$$

Hence $\mathcal{A} \simeq \mathcal{E}nd_{\mathcal{A}}(L)$ holds. *From now on we will always assume this.* We remark that L is locally-free of finite rank, both over $\mathcal{A}$ and over $\mathcal{O}$.

8.2. We denote by $\mathrm{Pic}(\mathcal{A})$ the group of all isomorphism classes of invertible $\mathcal{A}$-$\mathcal{A}$-bimodules in $\mathrm{coh}(\mathcal{A})$, with multiplication given by the tensor product over $\mathcal{A}$. The neutral element is the class of $\mathcal{A}$. Each invertible bimodule ${}_{\mathcal{A}}\mathcal{M}_{\mathcal{A}}$ gives rise to the (exact) autoequivalence $t_{\mathcal{M}} = - \otimes_{\mathcal{A}} \mathcal{M}$ of $\mathcal{H}$, and $t_{\mathcal{M}} \simeq 1_{\mathcal{H}}$ if and only if $\mathcal{M} \simeq \mathcal{A}$ as bimodules.

8.3 (Divisors). Let $\delta = \sum_{x \in \mathbb{X}} \delta_x \cdot x \in \mathbb{Z}^{(\mathbb{X})}$ be a (Weil) divisor. In (2.4) we defined the line bundle $L(x) = \sigma_x(L)$, which extends canonically to $L(\delta) = \prod_{x \in \mathbb{X}} {\sigma_x}^{\delta_x}(L)$. This definition of the line bundle $L(\delta)$ is clearly dual to the definition in [85, p. 34]. Moreover, $\mathcal{A}(x)_{\mathcal{A}} = L(x)$ defines an $\mathcal{A}$-$\mathcal{A}$-bimodule, with left action induced by left multiplication in $\mathcal{A}$.

For $x \in \mathbb{X}$ let $t_x\colon \mathrm{coh}(\mathcal{A}) \to \mathrm{coh}(\mathcal{A})$ denote the functor $- \otimes_{\mathcal{A}} \mathcal{A}(x)$, that is, $t_x = t_{\mathcal{A}(x)}$.

Lemma 8.4. *For all x the functors σ_x and $t_x\colon \mathcal{H} \to \mathcal{H}$ are isomorphic.*

Proof. We proceed as in the proof of the theorem of Eilenberg–Watts, [14, Thm. II.(2.3)]. We have, as right $\mathcal{A}$-modules, $t_x(L) \simeq L \otimes_{\mathcal{A}} \mathcal{A}(x) \simeq \mathcal{A}(x) = L(x) = \sigma_x(L)$. Both functors are autoequivalences, and exact ($\mathcal{A}$ is a locally-free $\mathcal{O}$-module). $\sigma_x(L)$ is a right $\mathcal{A}$-module; it can be made into a bimodule in the canonical way, and this bimodule agrees with $\mathcal{A}(x)$. For each $E \in \mathrm{coh}(\mathcal{A})$ we have a natural morphism $f_E\colon E \simeq \mathcal{H}om_{\mathcal{A}}(\mathcal{A}, E) \to \mathcal{H}om_{\mathcal{A}}(\mathcal{A}(x), \sigma_x(E))$ induced by σ_x; for this we remark, that this can be indeed defined locally, since σ_x fixes all tubes, and is therefore compatible with localizations in the sense of 3.11; then one can imitate the (more general) proof of [40, Thm. 19.5.4]. Under the natural isomorphisms $\mathrm{Hom}_{\mathcal{A}}(E, \mathcal{H}om_{\mathcal{A}}(\mathcal{A}(x), \sigma_x(E))) \simeq \mathrm{Hom}_{\mathcal{A}}(E \otimes_{\mathcal{A}} \mathcal{A}(x), \sigma_x(E))$ it corresponds to a natural morphism $g_E\colon t_x(E) \to \sigma_x(E)$, thus we have a natural transformation $g\colon t_x \to \sigma_x$. This is an isomorphism on $L_{\mathcal{A}}$, which is locally a progenerator for $\mathrm{coh}(\mathcal{A})$. Since both functors also preserve finite coproducts, it follows that g_E is an isomorphism for every object $E \in \mathrm{coh}(\mathcal{A})$. □

We recall from Theorem 7.11 (2) that there is a bijection between the closed points of the centre curve X and the points of $\mathbb{X}$. By abuse of notation we use the same symbol x for $x \in X$ and the corresponding point $x \in \mathbb{X}$.

Theorem 8.5. *Let $\mathcal{H}$ be a noncommutative regular projective curve over the field k. Let X be the centre curve. Then there is an exact sequence*

$$1 \to \operatorname{Pic}(\operatorname{coh}(X)) \xrightarrow{\iota} \operatorname{Pic}(\mathcal{H}) \xrightarrow{\phi} \prod_x \mathbb{Z}/e_{\mathrm{ra}}(x)\mathbb{Z} \to 1 \tag{8.2}$$

of abelian groups. Here, $\phi(\sigma) = \sigma_{|\mathcal{H}_0}$ and ι sends a Picard-shift s_x of $\operatorname{coh}(X)$*, for a point $x \in X$, to ${\sigma_x}^{e_{\mathrm{ra}}(x)}$, for the corresponding point $x \in \mathbb{X}$. Moreover,* $\operatorname{Pic}(\operatorname{coh}(X)) \simeq \operatorname{Pic}(X)$*, the Picard group of isomorphism classes of line bundles in* $\operatorname{coh}(X)$ *with the tensor product.*

Proof. Since σ_x on $\mathcal{U}_x$ has order $e_{\mathrm{ra}}(x)$, it is clear that ϕ induces a surjective homomorphism as indicated, and its kernel is given by $\langle {\sigma_x}^{e_{\mathrm{ra}}(x)} \mid x \in \mathbb{X}\rangle$. We have to show that $s_x \mapsto {\sigma_x}^{e_{\mathrm{ra}}(x)}$ yields an isomorphism between $\operatorname{Pic}(\operatorname{coh}(X))$ and this kernel. Surjectivity is clear. For well-definedness and injectivity we have to show that a word in the s_x is trivial if and only if the corresponding word in the ${\sigma_x}^{e_{\mathrm{ra}}(x)}$ is trivial. Let $x \in X$ and $\mathcal{A}_x = \mathcal{A} \otimes_{\mathcal{O}} \mathcal{O}_x$. It follows from Corollary 3.15 (its proof) and Lemma 7.15 that $\mathfrak{m}_x\mathcal{A}_x = \operatorname{rad}(\mathcal{A}_x)^e$ for some natural number e. Forming completions we see that $e = e_{\mathrm{ra}}(x)$. From this we deduce that

$$\mathcal{O}(x) \otimes_{\mathcal{O}} \mathcal{A} \simeq \mathcal{A}(e_{\mathrm{ra}}(x) \cdot x)$$

as $\mathcal{A}$-$\mathcal{A}$-bimodules. For $\delta = \sum_{x \in X} \delta_x \cdot x$ define $\overline{\delta} = \sum_{x \in \mathbb{X}} e_{\mathrm{ra}}(x)\delta_x \cdot x$. Thus $\mathcal{O}(\delta) \otimes_{\mathcal{O}} \mathcal{A} \simeq \mathcal{A}(\overline{\delta})$. We hence have that $\mathcal{O}(\delta) \simeq \mathcal{O}$ implies $\mathcal{A}(\overline{\delta}) \simeq \mathcal{A}$. Moreover, $s_x \simeq - \otimes_{\mathcal{O}} \mathcal{O}(x)$ and $\sigma_x \simeq - \otimes_{\mathcal{A}} \mathcal{A}(x)$, by the lemma. So $s_x \mapsto {\sigma_x}^{e_{\mathrm{ra}}(x)}$ gives a well-defined homomorphism. For an $\mathcal{A}$-$\mathcal{A}$-bimodule M we define, as in [67, (37.28)], the $\mathcal{O}$-$\mathcal{O}$-subbimodule $M^{\mathcal{A}}$ of M, locally, by consisting of all $x \in M$ such that $\alpha x = x\alpha$ holds for all $\alpha \in \mathcal{A}$. By Lemma 7.15 (1), we have $\mathcal{A}^{\mathcal{A}} = \mathcal{O}$. Then $(\mathcal{O}(\delta) \otimes_{\mathcal{O}} \mathcal{A})^{\mathcal{A}} = \mathcal{O}(\delta)$. If now $\mathcal{A}(\overline{\delta}) \simeq \mathcal{A}$ as bimodules, then we obtain $\mathcal{O}(\delta) = (\mathcal{O}(\delta) \otimes_{\mathcal{O}} \mathcal{A})^{\mathcal{A}} \simeq \mathcal{A}^{\mathcal{A}} = \mathcal{O}$. Thus our map is also injective.

The last statement about $\operatorname{Pic}(X)$ is, since X is commutative, easy to show. □

8.6 (The $\mathcal{O}$-dual). For $E \in \mathcal{H}$ let $E^{\vee} = \mathscr{H}om_{\mathcal{O}}(E, \mathcal{O})$ denote the $\mathcal{O}$-dual of E.

The different. By [83] and well known Hom-tensor relations

$$\boldsymbol{\omega}_{\mathcal{A}} := \mathscr{H}om_{\mathcal{O}}(\mathcal{A}, \boldsymbol{\omega}_X) \simeq \boldsymbol{\omega}_X \otimes_{\mathcal{O}} \mathcal{A}^{\vee} \simeq \varphi^* \boldsymbol{\omega}_X \otimes_{\mathcal{A}} \mathcal{A}(\Delta) \tag{8.3}$$

is the dualizing sheaf for $\operatorname{coh}(\mathcal{A})$, with $\varphi^*\colon \operatorname{coh}(X) \to \operatorname{coh}(\mathcal{A})$ the functor $- \otimes_{\mathcal{O}} \mathcal{A}$ and Δ a divisor, which is called the *different*.

The "difference" between $\tau_{|\mathcal{H}_0}$ and $\tau = \boldsymbol{\omega}_{\mathcal{A}} \otimes -$, globally on $\mathcal{H}$, is given by the Auslander–Reiten translation on the centre $\operatorname{coh}(X)$. Moreover, it follows from Theorem 8.5, that on $\mathcal{H}_0$ the functor $\varphi^* \boldsymbol{\omega}_X \otimes_{\mathcal{A}} -$ is the identity. We have $\boldsymbol{\omega}_X = \mathcal{O}_X(\gamma)$, where γ is the canonical divisor on X (we refer to [4, VIII. Prop. 1.13]). We then get $\boldsymbol{\omega}_{\mathcal{A}} = \mathcal{A}\big(\overline{\gamma} + \Delta\big)$. Formulated in terms of Picard-shifts we obtain the following.

Theorem 8.7. *Let $\mathcal{H}$ be a noncommutative regular projective curve over a field k. Then*

(1) $\tau \in \mathrm{Pic}(\mathcal{H})$.

(2) *Each $e_\tau(x)$ divides $e_{\mathrm{ra}}(x)$.*

It is shown in Example 10.13 that in general $e_\tau(x) \neq e_{\mathrm{ra}}(x)$.

Proof. (1) Follows from the preceding discussions. (2) follows from (1) and Theorem 7.21. More precisely we have that the order of τ in $\mathrm{Gal}(\mathrm{End}(S_x)/k)$ divides the order of τ in $\mathrm{Aut}(\mathcal{U}_x/k)$, which divides the order of σ_x in $\mathrm{Aut}(\mathcal{U}_x/k)$. □

In order to have "good" ramifications, we assume now that either k is perfect, or that the characteristic of k does not divide the skewness $s(\mathcal{H})$ (cf. [7, 1.3.9], [75, Ch. 5, Sec. 6]). The divisor $\Delta = \sum_x (e_{\mathrm{ra}}(x) - 1) \cdot x$ is called the *different* of $\mathcal{H}$. It is, locally in x, induced by the exact sequence

$$0 \to L \xrightarrow{{\pi_x}^{e_{\mathrm{ra}}(x)-1}} L((e_{\mathrm{ra}}(x)-1)x) \longrightarrow S_x[e_{\mathrm{ra}}(x)-1]^{e(x)} \to 0 \tag{8.4}$$

from Lemma 3.2. It follows then that the cokernel C of the injective (reduced) trace map $\mathcal{A} \to \mathcal{A}^\vee$ locally in x has k-dimension

$$\dim_k C_x = e_{\mathrm{ra}}(x)(e_{\mathrm{ra}}(x)-1)e(x)^2 e^*(x)^2[k(x):k] \overset{(*)}{=} s(\mathcal{H})^2\big(1-\frac{1}{e_\tau(x)}\big)[k(x):k], \tag{8.5}$$

with equation $(*)$ holding in the separable case. From this we immediately get:

Proposition 8.8. *There are only finitely many separation points.* □

Theorem 8.9. *Let $\mathcal{H}$ be a noncommutative regular projective curve over a field k which is perfect or of characteristic prime to $s(\mathcal{H})$. Let $\gamma = \sum_x \gamma_x \cdot x$ be the canonical divisor of the centre curve X. For $\overline{\gamma} = \sum_{x \in \mathbb{X}} \gamma_x e_{\mathrm{ra}}(x) \cdot x$ we write $\sigma^{|\overline{\gamma}|}$ for the corresponding Picard-shift. Then*

$$\tau = \sigma^{|\overline{\gamma}|} \cdot \prod_x {\sigma_x}^{e_{\mathrm{ra}}(x)-1} = \prod_x {\sigma_x}^{e_{\mathrm{ra}}(x)(\gamma_x+1)-1}. \tag{8.6}$$

9. The genus and the Euler characteristic

We recall that if k is algebraically closed, e.g. $k = \mathbb{C}$, then there is the well-known relation $\chi(X) = 2(1 - g(X))$ between the Euler characteristic $\chi(X)$ and the genus $g(X) = \dim_k \mathrm{Ext}^1(\mathcal{O}, \mathcal{O})$ of the regular projective curve (or compact Riemann surface) X. Let $(\mathcal{H}, L)$ be a noncommutative regular projective curve over the field k. We set $\kappa = \dim_k \mathrm{End}(L)$. The *Euler form* is defined by

$$\langle E, F \rangle = \dim_k \mathrm{Hom}(E, F) - \dim_k \mathrm{Ext}^1(E, F)$$

for objects $E, F \in \mathcal{H}$. We call

$$g(\mathcal{H}) = \dim_{\operatorname{End}(L)} \operatorname{Ext}^1(L, L)$$

the *genus* of $\mathcal{H}$ and

$$\chi'(\mathcal{H}) = \frac{1}{s(\mathcal{H})^2} \cdot \langle L, L\rangle = \frac{\kappa}{s(\mathcal{H})^2} \cdot (1 - g(\mathcal{H})) \tag{9.1}$$

the *normalized Euler characteristic* of $\mathcal{H}$ (over k). Note that this definition depends on the base-field k, which can be by-passed by assuming that k is the centre of $\mathcal{H}$. If $\mathcal{A}$ is Azumaya of degree s over $\mathcal{O}$, for instance, a maximal order in $M_s(\mathcal{O})$, then $\chi(\mathcal{A}) = s^2 \cdot \chi(\mathcal{O})$; compare [7, 4.1.1 + 4.1.5], with $\chi(\mathcal{A}) := \langle \mathcal{A}, \mathcal{A}\rangle$. Thus one can regard the normalized Euler characteristic χ' to be invariant under Morita-equivalence. Note that in case k is algebraically closed, we have $\chi(\mathcal{H}) = 2\chi'(\mathcal{H})$. In case $k = \mathbb{R}$ and $s(\mathcal{H}) = 1$ the definition of χ' agrees with the topological definition of the Euler characteristic for the underlying manifold; we refer to further discussion in 11.17. With this we have

$$g(\mathcal{H}) = 0 \;\Leftrightarrow\; \chi'(\mathcal{H}) > 0 \quad \text{and} \quad g(\mathcal{H}) = 1 \;\Leftrightarrow\; \chi'(\mathcal{H}) = 0.$$

For $F \in \mathcal{H}$ we define

$$\deg(F) = \frac{1}{\kappa\varepsilon} \cdot \langle L, F\rangle - \frac{1}{\kappa\varepsilon} \cdot \langle L, L\rangle \cdot \operatorname{rk}(F), \tag{9.2}$$

where $\varepsilon \geq 1$ is the natural number such that the resulting linear form $\deg\colon K_0(\mathcal{H}) \to \mathbb{Z}$ becomes surjective. We obtain

Proposition 9.1 (Riemann–Roch formula). *Let $(\mathcal{H}, L)$ be a noncommutative regular projective curve over a field. For all $E, F \in \mathcal{H}$ we have*

$$\frac{1}{\kappa} \cdot \langle E, F\rangle = (1 - g(\mathcal{H})) \cdot \operatorname{rk}(E) \cdot \operatorname{rk}(F) + \varepsilon \cdot \begin{vmatrix} \operatorname{rk}(E) & \operatorname{rk}(F) \\ \deg(E) & \deg(F) \end{vmatrix}.$$

Proof. First we remark that deg is additive on short exact sequences, and that $\langle -, -\rangle$ and the right hand side of the formula induce bilinear maps $K_0(\mathcal{H}) \times K_0(\mathcal{H}) \to \mathbb{Z}$. If $E = L$, then the formula is just the definition of the degree. If $E, F \in \mathcal{H}_0$, then both sides are zero, by the structure of $\mathcal{H}_0$ and the τ-invariance of each object in $\mathcal{H}_0$. This yields, if L' is a line bundle and S is simple, then $\langle L', S\rangle = \langle L, S\rangle$ (considering $[L] - [L']$). Then, if one of E or F belongs to $\mathcal{H}_0$, it is easy to see that the formula holds, by using line bundle filtrations. If $\sigma \in \operatorname{Aut}(\mathcal{H})$ is point-fixing, then both sides remain equal, if replacing the pair (E, F) by $(\sigma E, \sigma F)$ (for all $E, F \in \mathcal{H}$). If L' is a line bundle, by Lemma 3.4 we have an exact sequence $0 \to L(-nx) \to L' \to C \to 0$ (x any point, $n \gg 0$, $C \in \mathcal{U}_x$). From this the formula holds for $E = L'$ and $F \in \mathcal{H}$. Using line bundle filtrations, the formula holds for every $E \in \mathcal{H}_+$ and $F \in \mathcal{H}$, and then generally. □

In particular, if $\boldsymbol{\omega} = \boldsymbol{\omega}_{\mathcal{A}} = \tau L$ denotes the dualizing sheaf in $\mathcal{H}$, then

$$\deg(\boldsymbol{\omega}) = -\frac{2}{\kappa\varepsilon} \cdot \langle L, L \rangle = \frac{2}{\varepsilon} \cdot (g(\mathcal{H}) - 1). \tag{9.3}$$

Moreover, if $\delta = \sum_x \delta_x \cdot x \in \mathbb{Z}^{(\mathbb{X})}$ is a divisor, and if we define $\ell(\delta) = [\operatorname{Hom}(L, L(\delta)) : \operatorname{End}(L)]$, then (by specializing to $E = L$, $F = L(\delta)$)

$$\ell(\delta) = 1 - g(\mathcal{H}) + \deg \delta + \ell(\omega - \delta),$$

with $L(\omega) = \boldsymbol{\omega}$, which is the Riemann–Roch in more classical form. Here, $\deg \delta := \varepsilon \deg(L(\delta)) = \frac{1}{\kappa} \sum_x \delta_x \cdot [\mathcal{A}_x / \operatorname{rad}(\mathcal{A}_x) : k]$.

Lemma 9.2. *Let $\mathcal{H}$ be a noncommutative regular projective curve over the perfect field k. For each $x \in \mathbb{X}$ we have*

$$\deg(S_x) = \frac{s(\mathcal{H})}{\kappa\varepsilon} \cdot e^*(x) \cdot [k(x) : k]. \tag{9.4}$$

Proof. Follows easily from the skewness principle. □

For a Picard-shift $\sigma \in \operatorname{Pic}(\mathcal{H})$ we call $\deg(\sigma(L))$ the *degree* of σ, and we denote by $\operatorname{Pic}_0(\mathcal{H})$ the subgroup of degree zero Picard-shifts; similarly for $\operatorname{coh}(X)$ and $\mathcal{O}$, where we use the symbols $\deg_X$, κ_X and ε_X. From the lemma we easily get the following.

Proposition 9.3. *For the injective homomorphism $\iota \colon \operatorname{Pic}(\operatorname{coh}(X)) \to \operatorname{Pic}(\mathcal{H})$ in (8.2) we have in the perfect case*

$$\deg(\iota(s)) = s(\mathcal{H})^2 \cdot \frac{\kappa_X \varepsilon_X}{\kappa\varepsilon} \cdot \deg_X(s).$$

In particular, $\operatorname{Pic}_0(X)$ *can be regarded as a subgroup of* $\operatorname{Pic}_0(\mathcal{H})$. □

Let $\mathcal{H} = \operatorname{coh}(\mathcal{A})$ be a noncommutative regular projective curve with centre curve X and with $\mathcal{A}$ a maximal $\mathcal{O}_X$-order in a central simple $k(X)$-algebra. Following [7] we call $\mathbf{e} = (e_1, \ldots, e_n)$ the *ramification vector* of $\mathcal{A}$ if $e_1, \ldots, e_n$ are all ramification indices > 1, and moreover, for each ramification point x of $\mathcal{A}$ its ramification index $e_{\mathrm{ra}}(x)$ appears precisely $[k(x) : k]$ times in $\mathbf{e}$. If for all ramification points $x_1, \ldots, x_t$ (pairwise different) the numbers $f_i = [k(x_i) : k]$ are given, then we will also write more precisely $\mathbf{e} = ({e_1}^{f_1}, \ldots, {e_t}^{f_t})$ and call it the ramification *sequence*.

Proposition 9.4 (Artin–de Jong [7, Lemma 4.1.5]). *Let $\mathcal{H} = \operatorname{coh}(\mathcal{A})$ be a noncommutative regular projective curve over a perfect field k with centre curve X. Let $\mathcal{A}$ be a maximal $\mathcal{O}_X$-order in $k(\mathcal{H})$ and $\mathbf{e} = ({e_1}^{f_1}, \ldots, {e_n}^{f_n})$ its ramification sequence. Then we have for the normalized Euler characteristic*

$$\chi'(\mathcal{H}) = \chi'(X) - \frac{1}{2} \sum_{i=1}^{n} f_i \cdot \Big(1 - \frac{1}{e_i}\Big). \tag{9.5}$$

Proof. Write $s = s(\mathcal{H})$. We consider the exact sequence $0 \to \mathcal{A} \to \mathcal{A}^\vee \to C \to 0$ in $\mathrm{coh}(\mathcal{A})$. It is also exact in $\mathrm{coh}(\mathcal{O})$. With the degree $\deg_X$ over X we obtain, using $\deg_X(\mathcal{A}) = -\deg_X(\mathcal{A}^\vee)$, that

$$\deg_X(\mathcal{A}) = -\frac{1}{2}\deg_X(C) = -\frac{s^2}{2\kappa_X\varepsilon_X}\sum_x (1 - 1/e_\tau(x))[k(x):k],$$

by (8.5). We have $\mathrm{rk}_X(\mathcal{A}) = s^2$. By the Riemann–Roch, over X, we get

$$\langle \mathcal{O}, \mathcal{A}\rangle_X = s^2\chi'(X) - \frac{s^2}{2}\sum_x (1 - 1/e_\tau(x))[k(x):k].$$

Finally, by flatness $\langle \mathcal{A}, \mathcal{A}\rangle = \langle \mathcal{O}, \mathcal{A}\rangle_X$, and division by s^2 gives the claim. □

Proposition 9.5. *Let $\mathcal{H}$ be a noncommutative regular projective curve. If $g(\mathcal{H}) > 1$, then all Auslander–Reiten components of $\mathcal{H}_+ = \mathrm{vect}(\mathbb{X})$ have as underlying graph $\mathbb{Z}A_\infty$, and the category $\mathcal{H}$ is wild.*

Proof. This follows from [56, Prop. 4.7]. □

Noncommutative elliptic curves.

Definition 9.6. We call a (non-weighted) noncommutative regular projective curve of genus one a *(noncommutative) elliptic curve*.

For elliptic curves there is the following analogue of Atiyah's classification of vector bundles over an elliptic curve over an algebraically closed field [10]:

Theorem 9.7. *Let $\mathcal{H} = \mathrm{coh}(\mathbb{X})$ be a noncommutative elliptic curve. Then the following holds:*

(1) *Each indecomposable object E in $\mathcal{H}$ is semistable of slope $\mu(E) = \frac{\deg(E)}{\mathrm{rk}(E)}$ in $\widehat{\mathbb{Q}} := \mathbb{Q} \cup \{\infty\}$ and satisfies $\mathrm{Ext}^1(E,E) \neq 0$.*

(2) *For each α the subcategory $\mathbf{t}_\alpha$ of semistable objects of slope α is non-trivial and forms a tubular family, again parametrized by a noncommutative elliptic curve $\mathbb{X}_\alpha$, so that $\mathcal{H}' = \mathrm{coh}(\mathbb{X}_\alpha)$ is derived-equivalent to $\mathcal{H}$.*

We remark that one major difference to Atiyah's result is, that in general $\mathcal{H}'$ may be *not* isomorphic to $\mathcal{H}$. We will give an example later.

Proof. (1) Semistability follows like in [32, Prop. 5.5]. The condition $\mathrm{Ext}^1(E,E) \neq 0$ is a direct consequence of the Riemann–Roch formula.

(2) The proof of [47, Prop. 8.1.6] works also in this situation, with a slight modification: Let $\alpha \in \widehat{\mathbb{Q}}$. It is sufficient to show that $\mathbf{t}_\alpha \neq 0$. In the bounded derived category $\mathcal{D}^b(\mathcal{H})$ we form the *interval category* $\mathcal{H}' = \mathcal{H}\langle\alpha\rangle$, the additive closure of $\bigcup_{\gamma>\alpha}\mathbf{t}_\gamma[-1] \cup \bigcup_{\beta\leq\alpha}\mathbf{t}_\beta$. Let $\mathcal{H}'_0$ be its subcategory of objects of finite

length. By [56, 4.9 + 5.2], $\mathcal{H}'$ is again noetherian with Serre duality. Since $\mathcal{H}' \neq 0$ is noetherian, there are simple objects. Let $S \in \mathcal{H}'$ be simple. If $\mathbf{t}_\alpha = 0$, then S has a slope $\beta < \alpha$. The Riemann–Roch formula in this elliptic case implies

$$\mu(E) < \mu(F) \quad \Rightarrow \quad \operatorname{Hom}(E, F) \neq 0$$

for all indecomposable objects E and F. We conclude, that another simple object S' has the same slope β, and then $\mathcal{H}'_0 = \mathbf{t}_\beta$. Then there is no indecomposable object in $\mathcal{H}'$ of slope $\gamma > \beta$. But we find such an object by Picard-shifting a line bundle L' in $\mathcal{H}'$ sufficiently far to the left, and then applying suspension [1]. This contradiction shows, that $\mathbf{t}_\alpha \neq 0$. □

9.8 (Fourier–Mukai partners). Let $\mathcal{H}$ and $\mathcal{H}'$ be noncommutative regular projective curves over k. We call $\mathcal{H}'$ a *Fourier–Mukai partner* of $\mathcal{H}$, if there is an exact equivalence $\mathcal{D}^b(\mathcal{H}) \to \mathcal{D}^b(\mathcal{H}')$. We are mainly interested in this notion when at the same time the categories $\mathcal{H}$ and $\mathcal{H}'$ are not equivalent.

We recall that, since $\mathcal{H}$ is hereditary, $\mathcal{D}^b(\mathcal{H})$ is the *repetitive category* of $\mathcal{H}$, that is, $\mathcal{D}^b(\mathcal{H}) = \bigvee_{n\in\mathbb{Z}} \mathcal{H}[n]$, and for $E, F \in \mathcal{H}$ we have $\operatorname{Hom}_{\mathcal{D}^b(\mathcal{H})}(E[m], F[n]) = \operatorname{Ext}^{n-m}_{\mathcal{H}}(E, F)$. In the elliptic case, $\mathcal{H} = \operatorname{coh}(\mathbb{X}) = \bigvee_{\beta\in\widehat{\mathbb{Q}}} \mathbf{t}_\beta$, and $\mathbb{X}$ parametrizes the tubular family $\mathbf{t}_\infty$. If $\alpha \in \widehat{\mathbb{Q}}$, we have seen that $\mathcal{H}\langle\alpha\rangle \simeq \operatorname{coh}(\mathbb{X}_\alpha)$, where $\mathbb{X}_\alpha$ is the elliptic curve parametrizing the tubular family $\mathbf{t}_\alpha$. From the explicit description in Theorem 9.7 we readily obtain $\mathcal{D}^b(\mathcal{H}) = \mathcal{D}^b(\mathcal{H}\langle\alpha\rangle)$.

From this it follows easily, that if $\mathcal{H}$ is elliptic and $\mathcal{D}^b(\mathcal{H}) \to \mathcal{D}^b(\mathcal{H}')$ an exact equivalence, then $\mathcal{H}'$ is equivalent to $\mathcal{H}\langle\alpha\rangle$ for some α.

Examples 9.9. (1) Each (commutative) regular projective curve of genus one is elliptic in our sense; we do not require the existence of a k-rational base-point. In particular, the Klein bottle 12.1 is a real elliptic curve without $\mathbb{R}$-rational (= real) points (it is a Klein surface without boundary).

(2) Let X be a regular projective curve over its perfect centre k, let $K = k(X)$ its function field and D a finite dimensional central division K-algebra. Let $\mathcal{A}$ be a maximal $\mathcal{O}_X$-order in D and $\mathbf{e}$ its ramification vector. Let $\mathcal{H} = \operatorname{coh}(\mathcal{A})$.

(a) We have $\chi'(\mathcal{H}) > 0$ if and only if X has genus zero and $\mathbf{e} = (e)$, (e_1, e_2), $(2, 2, e)$, $(2, 3, 3)$, $(2, 3, 4)$ or $(2, 3, 5)$.

(b) $\mathcal{H}$ is elliptic, if either X has genus one and $\mathcal{A}$ is Azumaya, or X has genus zero and $\mathcal{A}$ has ramification vector $\mathbf{e} = (2, 3, 6)$, $(2, 4, 4)$, $(3, 3, 3)$ or $(2, 2, 2, 2)$. This follows directly from (9.5). Moreover, if $g(X) = 0$, then we conclude from (8.6) that the order of τ in $\operatorname{Aut}(\mathcal{H})$ is given by the least common multiple (= maximum) n of the τ-multiplicities, and $n = 2$, 3, 4, or 6. If, on the other hand, $g(X) = 1$, then $\boldsymbol{\omega}_X = \mathcal{O}(\omega)$ has degree zero, and from the Riemann–Roch formula we get $\ell(\omega) = g(X) = 1$, and thus $\boldsymbol{\omega}_X = \mathcal{O}$. Moreover, $\mathcal{A}$ is unramified (that is, Azumaya), thus we obtain $\boldsymbol{\omega}_{\mathcal{A}} = \boldsymbol{\omega}_X \otimes_{\mathcal{O}} \mathcal{A} = \mathcal{A}$. Thus $\tau = 1$.

Theorem 9.10. *Let $\mathcal{H}$ be elliptic over a perfect field. Then the order of τ is* 1, 2, 3, 4, *or* 6*, given by the maximum of the τ-multiplicities. The ramification vector is a derived invariant of an elliptic curve.*

Proof. The order of τ is a derived invariant, and the maximum of the ramification indices determines the ramification vector, which is one of ( ), $(2, 3, 6)$, $(2, 4, 4)$, $(3, 3, 3)$ or $(2, 2, 2, 2)$. □

10. Genus zero: ghosts and ramifications

Noncommutative projective curves of genus zero are important in the representation theory of finite dimensional algebras, in particular (but not only) for the tame algebras. For details we refer to [47] and [46]. We recall that genus zero means that $\operatorname{Ext}^1(L, L) = 0$, or equivalently, the existence of a tilting bundle $T \in \mathcal{H}$. The noncommutative regular genus zero curves $\mathcal{H}$ over a field k correspond to the so-called tame bimodules ${}_F M_G$, where F and G are finite dimensional division algebras over k, and M is an F-G-bimodule on which k is acting centrally and $\dim_F M \cdot \dim M_G = 4$. The corresponding bimodule algebra $\Lambda = \left(\begin{smallmatrix} G & 0 \\ M & F \end{smallmatrix}\right)$ is a finite dimensional tame hereditary k-algebra, whose category $\operatorname{mod}(\Lambda)$ of finite dimensional right modules is derived equivalent to $\mathcal{H}$. The function field $k(\mathcal{H})$ is isomorphic to the endomorphism ring of the unique generic Λ-module. We refer to [13, 26, 29, 72] as references for the representation theory of tame bimodules and tame hereditary algebras. We define the *numerical type* ε of $\mathbb{X}$ by $\varepsilon = 1$, if $(\dim_F M, \dim M_G) = (2, 2)$, and $\varepsilon = 2$, if this dimension pair is given by $(1, 4)$ or $(4, 1)$. (ε coincides with the previously defined normalizing factor of the degree.) We further set

$$f(x) = \frac{1}{\varepsilon} \cdot [\operatorname{Ext}^1(S_x, L) : \operatorname{End}(L)] = \deg(S_x). \tag{10.1}$$

A point $x \in \mathbb{X}$ is called *rational*, if $f(x) = 1$. In the genus zero case, rational points always exist, [55, Prop. 4.1]. It is shown in [47] that there is a so-called efficient automorphism $\sigma \in \operatorname{Aut}(\mathcal{H})$. The orbit algebra $R = \Pi(L, \sigma)$ serves as a homogeneous coordinate algebra for $\mathcal{H}$. We recall the definition from [47, Def. 1.1.3]: an automorphism $\sigma \colon \mathcal{H} \to \mathcal{H}$ is called *efficient* if it is point-fixing (that is, $\sigma(\mathcal{U}_x) = \mathcal{U}_x$ for all x), if the degree of σL is positive, and if there is no point-fixing automorphism such that the degree of the image of L is positive and smaller. As a consequence $\Pi(L, \sigma)$ is shown to be graded factorial (a graded version of the notion of a unique factorization ring in [24]), the points $x \in \mathbb{X}$ in correspondence with the homogeneous height one prime ideals, each generated by a normal element (called *prime*) π_x.

Example 10.1 (Commutative case). Let $\mathcal{H} = \operatorname{coh}(X)$ be a commutative regular projective curve of genus zero with centre k. If the characteristic is different from 2, then either $k(\mathcal{H}) = k(T)$, the rational function field over k in one variable (case $\varepsilon = 1$), or $k(\mathcal{H}) = k(U, V)/(-aU^2 - bV^2 + ab)$, with $a, b \in k^\times$ such that $-aY^2 - bZ^2 + abX^2$ is anisotropic over k (case $\varepsilon = 2$). (If k is additionally perfect, this also holds in characteristic 2, where in case $\varepsilon = 2$ the quadratic equation is slightly different; we refer to [46, Thm. 6.2].) For this classical result we refer to [46, Thm. 6.1]; there also the characteristic 2 case is treated. In all cases (also in characteristic 2), we have $\mathrm{End}(L) = k$ (for all line bundles) and $\chi'(X) = 1$.

Lemma 10.2. *Let $\mathcal{H}$ have centre curve X with $g(X) = 0$. Let $\sigma \in \mathrm{Pic}(\mathcal{H})$ be of degree zero and with $\sigma_{|\mathcal{H}_0} = 1_{\mathcal{H}_0}$. Then $\sigma = 1_{\mathcal{H}}$.*

Proof. By the assumptions it follows with Theorem 8.5 that σ lies in $\mathrm{Pic}(\mathrm{coh}(X))$, and is there of degree zero. Since $g(X) = 0$, this means $\sigma(\mathcal{O}) \simeq \mathcal{O}$. Since X is commutative, we get $\sigma \simeq 1_{\mathrm{coh}(X)}$ from [46, Cor. 3.3], and then also $\sigma \simeq 1_{\mathcal{H}}$ with Theorem 8.5. □

Theorem 8.9 yields

Proposition 10.3. *Let k be a perfect field. Let $\mathcal{H}$ be a noncommutative regular projective curve over k. We assume that its centre curve X is of genus zero, of numerical type ε. Then*

$$\tau = \sigma_{x_0}{}^{-2/\varepsilon} \cdot \prod_x \sigma_x{}^{e_\tau(x)-1}$$

for any point $x_0 \in \mathbb{X}$ which is rational in X and not ramification. □

The following result follows from Example 9.9 (2)(a). We state it here explicitly because of its influence to representation theory of finite dimensional algebras.

Theorem 10.4. *Let $\mathcal{H}$ be a noncommutative regular projective curve of genus zero over a perfect field. There are at most three separation points.* □

Ghosts and ramifications. We recall from [47], that the ghost group $\mathcal{G} = \mathcal{G}(\mathcal{H})$ is defined as the subgroup of the automorphism (class) group $\mathrm{Aut}(\mathcal{H})$ defined by those elements, called ghosts, which fix (up to isomorphism) the structure sheaf L and all simple sheaves S_x ($x \in \mathbb{X}$); it follows then, in this genus zero case, that all objects in $\mathcal{H}$ are fixed. In [46, Cor. 3.3] we have shown that commutativity (= multiplicity freeness) implies that the ghost group is trivial. As we shall see now, this last property follows already when $\mathbb{X}$ is unramified, but there are further cases with one ramification point. We remark that Proposition 10.3 is in particular applicable in case $\mathcal{H}$ is of genus zero. We will not use it in the following (since it will not simplify our proofs). Instead, we will recover its validity in the examples treated below.

Proposition 10.5. *Let $\mathcal{H}$ be a noncommutative regular projective curve of genus zero over the perfect field k, which is (without loss of generality) the centre of $\mathcal{H}$.*

We assume additionally that the characteristic of k is different from 2. Let X be the centre curve. The following are equivalent:

(i) *On $\mathcal{H}_0$ the functor τ is isomorphic to the identity functor.*

(ii) *There is a skew field D with centre k and $[D : k] = s(\mathcal{H})^2$ such that $k(\mathcal{H}) \simeq D \otimes_k k(X)$, a constant extension.*

(iii) *There is a skew field D with centre k and $[D : k] = s(\mathcal{H})^2$ such that $k(\mathcal{H})$ is either isomorphic to $D(T)$, with T central, or to a skew field of the form $D(U, V)/(-aU^2 - bV^2 + ab)$ with central variables U, V and non-zero elements $a, b \in k$ such that the quadratic form $-aY^2 - bZ^2 + abX^2$ is anisotropic over k.*

(iv) *There is a rational point x with $e(x) = 1$, $e_\tau(x) = 1$ and* $\operatorname{Pic}(\mathcal{H}) = \langle\sigma_x\rangle$.

When this holds, then the ghost group is trivial, $\mathcal{G}(\mathcal{H}) = 1$, and we have $\tau = {\sigma_x}^{-2/\varepsilon}$.

Proof. Condition (i) is equivalent to say that $\mathcal{H}$ is unramified, by Corollary 7.20. By a result of van den Bergh and van Geel [85, Prop. 2.2] this is equivalent to (ii); we also refer to [88]. By [46, Thm. 6.2] condition (iii) is just a more explicit reformulation of (ii).

(iii)⇒(iv) We can rewrite (iii) (or (ii)) in terms of coordinate algebras. There is a rational and multiplicity free point x such that $R = \Pi(L, \sigma_x)$ is either isomorphic to $D[X, Y]$ or to $D[X, Y, Z]/(-aY^2 - bZ^2 + abX^2)$ with central variables X, Y and Z of degree one, and x corresponds to the variable X. The centre $Z(R)$ is given by $k[X, Y]$ and $k[X, Y, Z]/(-aY^2 - bZ^2 + abX^2)$, respectively. From this we infer $e_\tau(x) = e_{\mathrm{ra}}(x) = 1$. If γ is a graded automorphism of R fixing all prime elements, then it is easy to see that its restriction to the centre has the form $\gamma(s) = a^{|s|} \cdot s$ for some $a \in k^\times$ (independent from s), where $|s|$ denotes the degree of the homogeneous element s. From this we easily deduce $\mathcal{G} = 1$ with [46, Thm. 3.1]. We also have $\operatorname{Pic}(\mathcal{H}) = \langle\sigma_x\rangle$ and $\tau^- = {\sigma_x}^{2/\varepsilon}$.

(iv)⇒(i) The conditions $e(x) = 1 = f(x)$ imply that σ_x is an efficient automorphism, and we get $\tau^- = {\sigma_x}^{2/\varepsilon}$. Since $\operatorname{Pic}(\mathcal{H}) = \langle\sigma_x\rangle$, all points different from x are τ-unramified. Since also $e_\tau(x) = 1$ by assumption, (i) follows. □

Condition (iv) allows us to relate the preceding proposition to the next theorem and, in particular, to its corollary.

Theorem 10.6. *Let $(\mathcal{H}, L)$ be a noncommutative regular projective curve of genus zero over the perfect field k. Assume that there is an efficient tubular shift σ_x, and let $R = \Pi(L, \sigma_x)$. For a normal homogeneous element $r \neq 0$ in R define the graded algebra automorphism γ_r by $rs = \gamma_r(s)r$ for all $s \in R$, and denote by γ_r^* the induced element in* $\operatorname{Aut}(\mathcal{H})$.

(1) *Each ghost γ is induced by a graded algebra automorphism of the form γ_r where r is homogeneous normal: $\gamma \simeq \gamma_r^*$.*

(2) *For each point* $y \neq x$ *the automorphism*

$$\gamma^*_{\pi_y} = \sigma_x{}^{-d(y)} \circ \sigma_y$$

is a ghost of order $e_\tau(y)$*. Moreover,* $\pi_y{}^{e_\tau(y)}$ *is central in* R *up to multiplication with a unit in* R_0.

(3) *We have that the ghost group* $\mathcal{G}(\mathcal{H}) = \langle \gamma^*_{\pi_y} \mid y \neq x,\ e_\tau(y) > 1 \rangle$ *is finite abelian and coincides with the subgroup* $\operatorname{Pic}_0(\mathcal{H}) \subseteq \operatorname{Pic}(\mathcal{H})$ *of Picard-shifts of degree zero. Moreover,* $\operatorname{Pic}(\mathcal{H}) \simeq \langle \sigma_x \rangle \times \mathcal{G}(\mathcal{H})$ *is finitely generated abelian of rank one, and we have* $\tau \in \operatorname{Pic}(\mathcal{H})$.

Proof. (1) By [46, Thm. 3.1] we have $\gamma = \beta^*$ for a graded, prime fixing algebra automorphism β of R; there are units $u_y \in R_0$ with $\beta(\pi_y) = u_y \pi_y$ for all $y \in \mathbb{X}$. Since π_x is central, and since each central element is a product of prime elements, we get $\beta(s) = u^{|s|} \cdot s$ for all $s \in S = Z(R)$, with $u = u_x$ central. Thus, the automorphism $\beta' = \beta \circ \varphi_u{}^{-1}$ of R is the identity on S. Since $\beta^* \simeq \beta'^*$, we can assume that β gives the identity on S. The induced automorphism $\overline{\beta}$ of $k(\mathcal{H})$ is the identity on its centre $k(X)$, thus $\overline{\beta}$ is inner by the Skolem–Noether theorem. Since $k(\mathcal{H})$ is obtained from R by central localization, there is $r \in R$ homogeneous with $\beta(s) = rsr^{-1}$ for all $s \in R$. The relation $\beta(s)r = rs$ shows that r is normal, and $\beta = \gamma_r$.

(2), (3) Moreover, by [47, Thm. 3.2.8] we have $\gamma^*_{\pi_y} = \sigma_x{}^{-d(y)} \circ \sigma_y \in \operatorname{Pic}(\mathcal{H})$. We also obtain that $\gamma^*_{\pi_y} \in \mathcal{G}(\mathcal{H})$ acts on $\mathcal{U}_y$ like σ_y, and thus the order of $\gamma^*_{\pi_y}$ is $\geq e_\tau(x)$. Let n be the least common multiple of $e_\tau(y)$ and $e_\tau(x)/h$, where h is the greatest common divisor of $e_\tau(x)$ and $d(y)$. Then n is the smallest natural number such that $\beta^* = \left(\sigma_x{}^{-d(y)}\sigma_y\right)^n$ is the identity functor on $\mathcal{H}_0$. We have $\beta(s) = asa^{-1}$ for a normal element a, say of degree m. We can assume that a does not have a central divisor of degree ≥ 1. Assume that there is a point p such that the prime π_p is a divisor of a. The element $a\pi_x{}^{-m}$ lies in the radical of the localization R_p. It is then easy to see that β^* cannot be isomorphic to the identity functor on the factor module $R_p/(\pi_p \pi_x{}^{-d(p)}) \simeq S_p{}^{e(p)}$, giving a contradiction. It follows, that a does not have any prime divisor, and so $a \in R_0$ is a unit. Thus β is an inner automorphism of R, and thus $\beta^* \simeq 1_{\mathcal{H}}$.

Since each normal element is a product of prime elements, we obtain $\mathcal{G}(\mathcal{H}) = \langle \gamma^*_{\pi_y} \mid y \neq x \rangle \subseteq \operatorname{Pic}(\mathcal{H})$. Since σ_x is efficient, $\tau \circ \sigma_x{}^{2/\varepsilon\ell}$ is a ghost and thus an element of $\operatorname{Pic}(\mathcal{H})$. If $e_\tau(x) = 1$, then we obtain $\mathcal{G}(\mathcal{H}) = \langle \gamma^*_{\pi_y} \mid e_\tau(y) > 1 \rangle$ is finite. Assume now $e_\tau(x) > 1$, and let $y \neq x$ be another point. Calculations using (9.4) show

$$e_\tau(y) \cdot d(y) = \frac{[k(y):k]}{[k(x):k]} \cdot e_\tau(x).$$

Since π_x is of degree one in the centre, it is clear that the fraction is an integer. We obtain $\gamma^*_{\pi_y}$ has order $e_\tau(y)$. In particular, $\gamma^*_{\pi_y} \simeq 1_{\mathcal{H}}$ unless y is a ramification point. Thus, again $\mathcal{G}(\mathcal{H}) = \langle \gamma^*_{\pi_y} \mid y \neq x,\ e_\tau(y) > 1 \rangle$ is finite. □

Corollary 10.7. *Let $\mathcal{H}$ be a noncommutative regular projective curve of genus zero over the perfect field k.*

(1) *Assume that $\mathcal{G}(\mathcal{H}) = 1$. Then there is a point $x \in \mathbb{X}$ with* $\mathrm{Pic}(\mathcal{H}) = \langle\sigma_x\rangle$. *Moreover, either*

(a) *$\mathbb{X}$ is unramified and $e(x) = 1 = f(x)$ holds; or*

(b) *x is the unique ramification point.*

(2) *Assume that there is an efficient tubular shift σ_x with* $\mathrm{Pic}(\mathcal{H}) = \langle\sigma_x\rangle$. *Then the ghost group is trivial, $\mathcal{G}(\mathcal{H}) = 1$.*

Proof. (1) Let σ be an efficient automorphism. Since $\mathcal{G} = 1$, we have $\sigma_x = \sigma^{d(x)}$, and $\sigma_x = \sigma_y$ if and only if $x = y$. If $\mathbb{X}$ is unramified, then there exists x with $e(x) = 1 = f(x)$ and $\mathrm{Pic}(\mathcal{H}) = \langle\sigma_x\rangle$ by the preceding proposition. Assume that x is such that $e_\tau(x) > 1$. Since σ_x does not lie in the subgroup of $\mathrm{Pic}(\mathcal{H})$ generated by all σ_y, with $y \neq x$, by Corollary 5.7, we have that all $d(y)$ are multiples of $d(x)$, that is, $d(y) = a(y) \cdot d(x)$ with $a(y) \geq 1$. Since σ_y, for $y \neq x$, is the identity on $\mathcal{U}_x$, we see that $e_\tau(x)$ divides $a(y)$. We conclude that σ_x generates $\mathrm{Pic}(\mathcal{H})$, and x is the only ramification point.

(2) This follows directly as a special case from the preceding theorem. □

Examples. We illustrate the theory in some genus zero examples over a perfect field.

Example 10.8 (Finite fields). Let k be a finite field and $\mathcal{H}$ over k of genus zero. Then there is an efficient tubular shift $\sigma = \sigma_x$. Then $\tau\sigma^{2/\varepsilon}$ is a ghost. Moreover, for all points p we have $[k(p) : k] = \frac{\kappa\varepsilon}{s(\mathcal{H})} \cdot f(p)$, by (9.4). Without loss of generality we can assume that k is the centre of $\mathcal{H}$. There are two possible cases, which describe *all* genus zero cases over a finite field:

(1) $\varepsilon = 1$. [46, Prop. 4.1]. Then $M = M(K,\alpha) = K \oplus K$ where K acts canonically from the left and by $(x, y) \cdot z = (xz, y\alpha(z))$ from the right, and $\alpha\colon K \to K$ is a k-automorphism. Since k is the centre of M, it is the fixed field of α, so that $\mathrm{Gal}(K/k)$ is cyclic, generated by α. Let $n = [K : k]$. Then $s(\mathcal{H}) = n$. The case $n = 1$ is the Kronecker/projective line over k. So assume $n \geq 2$. We have that $R = \Pi(L, \sigma_x)$ is isomorphic to $K[X; Y, \alpha]$. The two points x and y (corresponding to X and Y) are the only rational points of multiplicity 1, and the only rational points p such that $e_\tau(p) > 1$; moreover, $e_\tau(p) = s(\mathcal{H})$; compare [47, Cor. 5.4.2]. Since also $\tau = {\sigma_x}^{-1}{\sigma_y}^{-1}$, these are the only separation points. The ramification sequence is $\mathbf{e} = (n^1, n^1)$. The ghost group is cyclic of order n, generated by ${\sigma_x}^{-1}\sigma_y$. Moreover, $\tau = {\sigma_x}^{-1}{\sigma_y}^{-1}$.

(2) $\varepsilon = 2$. Then $M = {}_kK_K$, where $[K : k] = 4$. Indeed, a priori we have $M = {}_{F^\alpha}K_K$ with F/k a finite field extension and $M_K = K_K$, with $[K : F] = 4$, and $\alpha \in \mathrm{Gal}(K/k)$; the left F-structure is given by $f \cdot x = \alpha(f)x$. It is easy

to see that for all α we obtain isomorphic bimodules (in the sense that the induced hereditary bimodule k-algebras are isomorphic, compare [47, 5.1.3]). Thus we can assume that $M = {}_F K_K$ is equipped with the canonical structure induced by the subring $F \subseteq K$. Then F is the centre (in the sense of [47, 0.5.5]) of the bimodule M; thus, we can assume $F = k$.

There is a unique intermediate field F of degree two over k, which is of the form $F = k(\alpha)$. This defines, by [46, Lem. 2.6], the simple regular representation

$$S_x = (k^2 \otimes K \overset{(1,\alpha)}{\longrightarrow} K)$$

with $f(x) = 1$ and $\mathrm{End}(S_x) = F$, hence $e(x) = 1$. By uniqueness of F, we have that x is the only rational point p with $e(p) = 1$. Hence $e_\tau(x) = 2$ and $e_\tau(p) = 1$ for all other rational points. Since $[k(p) : k] = f(p)$ for all p and $\mathrm{End}(L) = k$ is the field of constants, we deduce then from (9.5) (since $g(\mathcal{H}) = 0$) that besides x there is precisely one additional ramification point p, and this must satisfy $f(p) = 2$ and $e(p) = 1$. Thus the ramification sequence is $\mathbf{e} = (2^1, 2^2)$. The ghost group $\mathcal{G}$ is cyclic of order 2, generated by ${\sigma_x}^{-2}\sigma_p$. We obtain $\tau = \sigma_x {\sigma_p}^{-1}$. The automorphism group $\mathrm{Aut}(\mathbb{X})$ is cyclic of order 4.

We now additionally assume $\mathrm{char}(k) \neq 2$. By [46, Thm. 5.1],

$$\Pi(L, \sigma_x) \simeq k\langle X, Y, Z\rangle / \begin{pmatrix} XY - YX,\ XZ - ZX, \\ YZ + ZY + a_1X^2,\ Z^2 + c_0Y^2 - a_0X^2 \end{pmatrix}$$

for certain $a_0,\ a_1,\ c_0 \in k$. The point x corresponds to the prime element given by the class of X in R. The second ramification point p corresponds to a prime element π_p in R of degree 2, and which is also irreducible, that is, not a product of two elements of degree 1. It follows that (up to multiplication with a non-zero element from k) there is precisely one such prime.

Example 10.9 (Non-simple tame bimodule). Let k be perfect and M a non-simple tame bimodule with centre k. It follows from [66, Prop. 11.5] and [39, Thm. 1.1.21] that M is of the form $F \oplus F$ with canonical F-action from the left, and the right F-action given by $(a, b)f = (af, b\alpha(f))$ for some k-automorphism α of F, and F is a skew field over k of finite dimension. Let n be the order of α considered as element in $\mathrm{Gal}(F/k)$. There is a rational point x with $e(x) = 1$, and hence σ_x is efficient. We have $\Pi(L, \sigma_x) \simeq F[X; Y, \alpha]$, the twisted polynomial ring graded by total degree. The prime elements are given by $\pi_x = X$, $\pi_y = Y$, and some further polynomials lying in the centre, thus in the variables X^n and Y^n. It follows that the points x and y are the only ramification points. The ramification sequence is $\mathbf{e} = (n^1, n^1)$. The ghost group $\mathcal{G}$ is cyclic of order n, generated by ${\sigma_x}^{-1}\sigma_y$. Moreover, we obtain $\tau = {\sigma_x}^{-1}{\sigma_y}^{-1}$.

Example 10.10. Let $M = {}_{\mathbb{Q}}\mathbb{Q}(\sqrt{2}, \sqrt{3})_{\mathbb{Q}(\sqrt{2},\sqrt{3})}$.

$$\Pi(L,\sigma_x) \simeq k\langle X,Y,Z\rangle / \begin{pmatrix} XY - YX,\ XZ - ZX, \\ YZ + ZY,\ Z^2 + 2Y^2 - 3X^2 \end{pmatrix}.$$

Here, the three rational points x, y, z (corresponding to the variables X, Y, Z, respectively) satisfy $e(p) = 1$, and thus $e_\tau(p) = 2 = s(\mathcal{H})$. Moreover, by [46, Prop. 7.1] we have

$$\tau = \sigma_x\ {\sigma_y}^{-1}{\sigma_z}^{-1},$$

by which it follows again, that x, y, z are the only separation points. The ramification sequence is $\mathbf{e} = (2^1, 2^1, 2^1)$. The ghost group is the Klein four group generated by ${\sigma_x}^{-1}\sigma_y$ and ${\sigma_x}^{-1}\sigma_z$. Moreover, $\tau = \sigma_x{\sigma_y}^{-1}{\sigma_z}^{-1}$.

Example 10.11. We consider the simple $(2,2)$-bimodule from [47, Ex. 5.7.3] with skewness $s(\mathcal{H}) = 4$: Let $k = \mathbb{Q}$ and $F = \left(\frac{-1,-1}{\mathbb{Q}}\right)$ be the skew field of quaternions over $\mathbb{Q}$ on generators $\mathbf{i}, \mathbf{j}$ with relations $\mathbf{i}^2 = -1 = \mathbf{j}^2$, $\mathbf{ij} = -\mathbf{ji}$, $K = \mathbb{Q}(\sqrt{-3}, \sqrt{2})$ and M be the bimodule ${}_K(K \oplus K)_F$ with the canonical K-action, and where the F-action on M is defined by

$$(x,y)\cdot\mathbf{i} = \frac{1}{\sqrt{-3}}(\sqrt{2}x + y, x - \sqrt{2}y),\ (x,y)\cdot\mathbf{j} = (y,-x)$$

for all x, $y \in K$. By [47, Prop. 5.7.5], each rational point x satisfies $e(x) = 2$ or $e(x) = 4$ (and both cases occur), and moreover $e^*(x) = 1$; those with $e(x) = 2$ are separation with $e(x)\cdot e^*(x) = 2$ and $e_\tau(x) = 2$. Actually, since the tame bimodule M is linked to the tame bimodule in the preceding example via a derived equivalence in the tubular case [47, Prop. 8.3.1] we will deduce from Example 13.25 below that the ramification sequence is given by $\mathbf{e} = (2^1, 2^1, 2^1)$. Accordingly, the ghost group is the Klein four group (but not trivial, [47, Prop. 5.7.4]).

Remark 10.12. All the preceding examples are *ruled* ($\mathbf{e} = (n,n)$) or *half-ruled* ($\mathbf{e} = (2,2,2)$), in the terminology of [7], see also [7, Prop. 4.2.4].

An inseparable example. We conclude this section with a detailed analysis of an inseparable example which nicely illustrates that (and why) for many of the preceding results the separability assumption was indispensable.

Example 10.13. Let $\mathbb{F}_2$ be any field of characteristic 2 and $k = \mathbb{F}_2(t)$ the rational function field in one variable over $\mathbb{F}_2$. Let $K = k(u)/(u^2 - t)$ and the k-derivation $\delta\colon K \to K$ be given by $\delta(u) = 1$, and $\delta_{|k} = 0$. We have $\delta^2 = 0$ and $\mathrm{Gal}(K/k) = 1$. Let M be the tame K-K-bimodule $M = M(1,\delta) = K \oplus K$ with canonical left K-action and right K-action given by $(a,b)f = (af + b\delta(f), bf)$. Since M is a non-simple bimodule, the corresponding curve $\mathcal{H}$ of genus zero admits a point x with $e(x) = 1 = f(x)$, and the orbit algebra $\Pi(L,\sigma_x)$ is isomorphic, as graded ring, to the differential polynomial ring $R = K[X; Y, 1, \delta]$. Here Y and the central variable X have degree 1, and we have the relations $Yf = \delta(f)X + fY$ $(f \in K)$. The point x above is associated with the prime element X, cf. [47, Prop. 1.7.3]. It is easily shown that the centre is given by $k[X, Y^2]$. We conclude that $s(\mathcal{H}) = 2$.

(1) We consider the tube $\mathcal{U} = \mathcal{U}_x$ associated with x. For the simple $S = S_x$ we have $\mathrm{End}(S) \simeq K$, in particular $e^*(x) = 1$. We have $V := \widehat{R}_x \simeq \mathrm{End}(S[\infty])$, the complete local ring with $\mathcal{U} \simeq \mathrm{mod}_0(V)$. From Theorem 6.1 we conclude that V has PI-degree $e^{**}(x) = 2$. By [72, 7.4] the K-K-bimodule $\mathrm{Ext}^1(S, S)$ is isomorphic to $M/N \simeq {}_K K_K$, the canonical one-dimensional K-K-bimodule, where N is the subbimodule $K \oplus 0$ of M. From this we get $\widehat{\mathrm{gr}}(V) \simeq K[[T]]$, which is commutative, and thus $\widehat{\mathrm{gr}}(V) \not\simeq V$. Since $\mathrm{Gal}(\mathrm{End}(S)/k) = 1$, for the τ-multiplicity (2.2) we have $e_\tau(x) = 1$. We thus see that for the inseparable point x the conclusions of Propositions 6.3 and 5.2, and Theorems 6.5 and 5.3 do not hold.

(2) We shall determine the algebra $V = \widehat{R}_x$ explicitly. Since K is a subalgebra of R_x, it is also a subalgebra of the completion V. Let $\pi \in V$ be the generator of the Jacobson radical which is given by the element XY^{-1} (cf. the proof of [47, Thm. 2.2.10]). Since $V/(\pi) \simeq K$ and $\mathrm{Gal}(K/k) = 1$, we get a decomposition $V = K \oplus (\pi)$ of K-K-bimodules. Using $\delta(a) \in k$ the relations in R induce the relations

$$\pi a = a\pi + \delta(a)\pi^2 \qquad (a \in K).$$

The decomposition above also yields the decomposition $V\pi = K\pi \oplus (\pi^2)$ of left K-modules, but $K\pi$ is not a K-K-bimodule. By the universal property of power series rings, we get a k-algebra homomorphism $\phi \colon K[[t^{-1}, \delta]] \to V$ sending t^{-1} to π, where $K[[t^{-1}, \delta]]$ denotes the (pseudo-) differential power series ring defined in [39, Thm. 1.11.8]; it is a local domain with Jacobson radical $J = (t^{-1})$ and $\bigcap_{n\geq 1} J^n = 0$. By the above decompositions ϕ is surjective. It is also injective, since the kernel is a completely prime ideal. Thus ϕ is an isomorphism. We hence write

$$V = K[[\pi, \delta]].$$

Moreover, since $\pi^2 a = a\pi^2$ for all $a \in K$, the power series ring $K[[T]] \simeq \widehat{\mathrm{gr}}(V)$ is isomorphic to the subring $K[[\pi^2]]$ of V. The centre of V is the subring $k[[\pi^2]]$. For the ramification index (of the exponential valuations [67, (13.1)]) we have $e_{\mathrm{ra}}(x) = 2$.

(3) There is an automorphism σ of V given by $\pi r = \sigma(r)\pi$ (for $r \in R$). We have $\sigma(\pi) = \pi$ and $\sigma(f) = f + \delta(f)\pi$ $(f \in K)$, and we see that σ has order 2, and since the centre is $k[[\pi^2]]$ it is easily seen to be not inner. (The property $\sigma(K) \not\subseteq K$ makes the difference to the separable case.) By [47, Thm. 3.1.2] multiplication with X yields the natural transformation $1_{\mathcal{H}} \xrightarrow{x} \sigma_x$. Extending σ_x to the direct limit closure of $\mathcal{U}_x$, we see that the natural sequence in [47, 0.4.2(5)] for the injective object $S[\infty]$ becomes $0 \to S \to S[\infty] \xrightarrow{\pi} S[\infty] \to 0$. We conclude that σ_x on $\mathcal{U}$ is induced by the automorphism σ and is thus of order 2; it acts non-trivially e.g. on $\mathrm{End}(S[n])$ for $n \geq 2$.

(4) Since $\tau^-(L) \simeq {\sigma_x}^2(L)$, the composition $\tau \circ {\sigma_x}^2$, on $\mathcal{H}$, is an element of the ghost group. Using [46, Thm. 3.1], computing the graded automorphisms α of R which are prime fixing (and hence preserve kX and elements of the centre $k[X, Y^2]$), it is easy to see that $\mathcal{G}(\mathcal{H}) = 1$. (Indeed, since $\alpha(X) \in kX$, we can assume $\alpha(X) = X$. Exploiting $\alpha(Y^2) \in kY^2$ and $\alpha(Yb) = \alpha(Y)b$ for all $b \in K$, we obtain $\alpha(Y) = Y + aX$ with $a \in K$ satisfying $a^2 = \delta(a)$. If $a \neq 0$, then $a^{-1}Ya = Y + aX$, so that α is inner.) We conclude that globally $\tau^- = {\sigma_x}^2$ holds. (We remark that here τ is not given by formula (8.6).) This shows that τ^- acts, unlike σ_x, as the identity functor on $\mathcal{U}_x$. Thus we see that Corollary 5.5 (2) and Corollary 7.17 do not extend to inseparable points. Moreover, Theorem 6.5 does not hold for such a point even if $e_\tau(x)$ is replaced by the order of τ in $\mathrm{Aut}(\mathcal{U}_x/k)$. We also infer that $\mathcal{H}$ is τ-unramified but not unramified.

11. The real case: Witt curves

11.1 (Real smooth projective curves and Klein surfaces). If k is algebraically closed, then by Corollary 7.8 each noncommutative regular projective curve over k is actually commutative. For $k = \mathbb{C}$ the field of complex numbers it is well-known that the three concepts regular (=smooth) projective curves X over $\mathbb{C}$, algebraic function fields K in one variable over $\mathbb{C}$, and compact Riemann surfaces S are equivalent/dual to each other; here K is the field of meromorphic functions on S (which are the holomorphic functions $\alpha: S \to \mathbb{S}^2$ to the Riemann sphere) and also the function field $k(X)$. Over the field $k = \mathbb{R}$ of real numbers there are similar correspondences, where the Riemann surfaces are replaced by the Klein surfaces $\mathcal{K}$, [3,65]. Each (compact) Klein surface $\mathcal{K}$ is of the form S/σ, where S is a compact Riemann surface and $\sigma: S \to S$ an antiholomorphic involution, [65, Thm. 1.1]; the Riemann surface S is also called the complex double of $\mathcal{K}$. It should be noted that in such a case $\chi_{\mathrm{top}}(S) = 2 \cdot \chi_{\mathrm{top}}(\mathcal{K})$ holds [3, 1.6.9]; since $k = \mathbb{R}$, we also have $\chi'(Y) = 2 - 2g = \chi_{\mathrm{top}}(S)$ and, by [1, Thm. 1.1], $\chi'(X) = 1 - g = \chi_{\mathrm{top}}(\mathcal{K})$, where Y and X are the corresponding *real* regular projective curves, respectively, and $g = g(S) = g(\mathcal{K})$; here, χ' is the normalized Euler characteristic as defined in (9.1), and χ_{top} the usual Euler characteristic for surfaces defined topologically via triangulations. The real points on $\mathcal{K}$ (if any) form the boundary $\partial\mathcal{K}$. By Harnack's theorem $\partial\mathcal{K}$ has at most $g(\mathcal{K}) + 1$ components, called *ovals*, since they are homeomorphic to a circle $\mathbb{S}^1$. The ovals are given by the set S^σ of fixed points of σ. By a theorem of Weichold [65, p. 56], every $\mathcal{K}$ is, topologically, uniquely determined by a triple (g, t, s), where $g = g(\mathcal{K}) = g(S)$ is the genus of the Riemann surface S, t is the number of ovals, and $s = 0$ if $S \setminus S^\sigma$ is connected, and $s = 1$ otherwise. Moreover, precisely the triples (g, t, s) with $s = 0$ and $t \leq g$, or $s = 1$, $t \equiv g + 1 \pmod 2$ and $1 \leq t \leq g + 1$ occur.

11.2 (Noncommutative function fields and configurations on Klein surfaces). The field $k(\mathcal{K})$ of meromorphic functions $\alpha: \mathcal{K} \to \mathbb{S}^2$ on a Klein surface $\mathcal{K}$ is an algebraic function field in one variable over $\mathbb{R}$, and each such real function field is of this form. (For the precise definition of a meromorphic function on a Klein surface we refer to [3, Ch. 1, §3].) Finite dimensional central skew fields over the meromorphic function field $K = k(\mathcal{K})$ of a Klein surface $\mathcal{K}$ are, if non-trivial, quaternion skew fields of the form $\left(\frac{\alpha,-1}{K}\right)$, where $0 \neq \alpha\colon \mathcal{K} \to \mathbb{S}^2$ is a meromorphic function. This follows easily from Tsen's theorem, [87]. We recall that such an algebra is of dimension four over K on generators $\mathbf{i}$, $\mathbf{j}$ and relation $\mathbf{i}^2 = -1$, $\mathbf{j}^2 = \alpha$ and $\mathbf{ji} = -\mathbf{ij}$. Such a function α is real-valued on the boundary $\delta\mathcal{K}$, that is, on each oval. On each of the ovals α might have an even number (or zero) *sign-changes*, that is, zeros or poles of odd order. Thus α determines on $\mathcal{K}$ what we call a $\pm$-*configuration*, which is given by

- an even number (≥ 0) of points on each oval, called *segmentation points*;
- each open *segment* between segmentation points, and each oval without segmentation points, labelled by either the sign $+$ or the sign $-$, in an alternating way (changing the sign at each segmentation point).

We call it *clean* if there are no segmentation points at all. The $\pm$-configuration is *induced* by the function α, if the $\pm$-configuration reflects the sign-behaviour of α on the ovals; a segment then has a $+$, if α is non-negative on this segment (we write $\alpha(x) > 0$, locally), and has a $-$, if α is non-positive on this segment ($\alpha(x) < 0$). We recall from [87] that $\alpha \neq 0$ is called *positive definite* if it never becomes negative on any oval. ([87, I.] says that then α is of the form $\alpha = \beta^2 + \gamma^2$.) Moreover, α is *definite*, if there is no sign-change on any oval. So, $\left(\frac{\alpha,-1}{K}\right)$ does not split (it is a skew field) if and only if α is not positive definite.

Example 11.3. Let $\mathbb{D}$ be the compact unit disc. We have $K = k(\mathbb{D}) = \mathbb{R}(t)$, the rational function field over $\mathbb{R}$ in one variable. We consider three different $\pm$-configurations on $\mathbb{D}$.

(a) Let $\alpha = -1\colon \mathbb{D} \to \mathbb{S}^2$ be the function with constant value -1. This gives the clean $\pm$-configuration as shown in Figure 1. The associated quaternion skew field $\left(\frac{-1,-1}{K}\right)$ is the function field $\mathbb{H}(t)$ in one (central) variable t over $\mathbb{H}$.

(b) Let $\alpha\colon \mathbb{D} \to \mathbb{S}^2$, $z \mapsto z$ be the canonical identification of $\mathbb{D}$ with the "northern" half ball, the "equator" defining $\mathbb{R} \cup \{\infty\}$. On the unique oval α has sign-changes in $z = 0$ (zero of order 1) and $z = \infty$ (pole of order 1). These two segmentation points yield the two segments of the oval, the negative real numbers marked by $-$, the positive real points marked by $+$. The quaternion skew field $\left(\frac{\alpha,-1}{K}\right)$ is isomorphic to $\mathbb{C}(u,\sigma)$, the skew function field ($uz = \sigma(z)u$ for all $z \in \mathbb{C}$, with σ the complex conjugation), in the variable $u = t^{1/2}$.

(c) Let similarly α be a meromorphic function associated with the element $t(t-1)(t+1)$ in $\mathbb{R}(t)$ (cf. [3, Thm. 1.4.6]). This gives rise to four segmentation points $z = 0,\ 1,\ -1,\ \infty$, and the skew function field $\mathbb{C}(u,t)/(u^2 - t(t-1)(t+1), u\mathbf{i} + \mathbf{i}u)$.

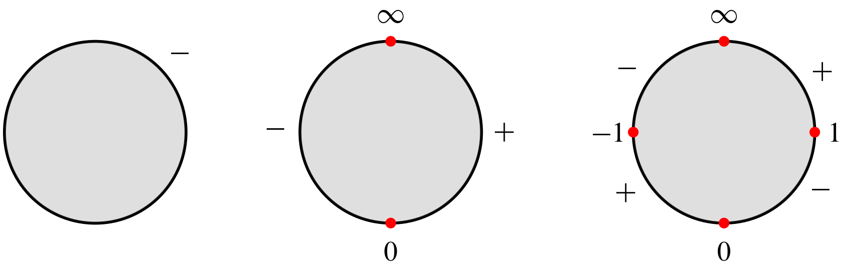

Figure 1. The disc $\mathbb{D}$ with different $\pm$-configurations. Left: (a), middle: (b), right: (c)

Witt's theorem [87], here formulated in our language, shows that all noncommutative real algebraic function fields in one variable are obtained by Klein surfaces with $\pm$-configurations.

Theorem 11.4 (Witt; cf. [3, Thm. 2.4.5]). *Let $\mathcal{K}$ be a Klein surface whose boundary is given by t ovals. We assume $t \geq 1$. Let K be the field of meromorphic functions on $\mathcal{K}$.*

(1) *Every (clean) $\pm$-configuration on $\mathcal{K}$ is induced by a (definite) meromorphic function $\alpha: \mathcal{K} \to \mathbb{S}^2$. [87, II. + III.]*

(2) *The resulting quaternion algebra $A = \left(\frac{\alpha,-1}{K}\right)$ is uniquely determined already by the $\pm$-configuration. It is a skew field if and only if α is not positive definite. It is unramified if and only if α is definite. Otherwise its ramification points are just the segmentation points. [87, p. 10]*

(3) *Each finite dimensional central skew field extension of K is obtained in this way. [87, III.′]* □

11.5 (Local data). Witt [87, p. 10] described (function-theoretically) also the local data. We assume that $\alpha \neq 0$ on $\mathcal{K}$ is *not* positive definite. That is, there is an oval on which α becomes negative. For convenience we already use the notions S_x and $e(x)$ in each concluding statement ("Thus…"), which will get a proper meaning only below when we define the notion of a Witt curve.

- If x is inner then $\left(\frac{\alpha,-1}{K}\right)_x$ splits. Thus $\operatorname{End}(S_x) = \mathbb{C}$ and $e(x) = 2$.

If x is a boundary point ($x = x^\sigma$), then

- If $\alpha(x) > 0$ (that is, α does not change its positive sign in a neighbourhood of x) then $\left(\frac{\alpha,-1}{K}\right)_x$ splits. Thus $\operatorname{End}(S_x) = \mathbb{R}$ and $e(x) = 2$.
- If $\alpha(x) < 0$ (that is, α does not change its negative sign) then $\left(\frac{\alpha,-1}{K}\right)_x$ does not split, and x is inert in $\left(\frac{\alpha,-1}{K}\right)$. Thus $\operatorname{End}(S_x) = \mathbb{H}$ and $e(x) = 1$.
- Otherwise (if α changes the sign in x, that is, x is segmentation) $\left(\frac{\alpha,-1}{K}\right)_x$ does not split. Thus $\operatorname{End}(S_x) = \mathbb{C}$ and $e(x) = 1$.

Thus the interior of a connected closed segment of an oval, whose endpoints are segmentation points, is “coloured” *real*, in case α is nonnegative on this segment, and *quaternion*, in case α is negative on this segment. From now on we will use these colourings of segments by $\mathbb{R}$ and $\mathbb{H}$, instead of $+$ and $-$, respectively, cf. Figure 2 below. (Inner points and segmentation points are always complex.) If $\alpha \neq 0$ is *not* positive definite then we call $\alpha: \mathcal{K} \to \mathbb{S}^2$ a *Witt function* (and $\mathcal{K}$ with the induced $\pm$-configuration, formally, a *Witt surface*); we will give $(\mathcal{K}, \alpha)$ a canonical structure of a noncommutative regular projective curve below. Some of the following considerations are reformulations of results of Section 7.

Let $\mathcal{K}$ be a Klein surface with function field $K = k(\mathcal{K})$ and α a Witt function. Let $A = \left(\frac{\alpha,-1}{K}\right)$ be the corresponding quaternion skew field. Let $A(x) = A_x \otimes_{\mathcal{O}_x} k(x) = A_x/\mathfrak{m}_x A_x$ with $k(x) = \mathcal{O}_x/\mathfrak{m}_x$ be the geometric fibre. If $\alpha > 0$, then $A(x) \simeq \mathrm{M}_2(\mathbb{R})$ is split on an oval O, if $\alpha < 0$, then $A(x) \simeq \mathbb{H}$ on O. The segmentation points are just the ramification points of A. There is the injective homomorphism

$$\beta: \mathrm{Br}(\mathcal{K}) \to \mathrm{Br}(K) \tag{11.1}$$

of Brauer groups. Here, $\mathrm{Br}(\mathcal{K})$ consists of (classes of) Azumaya algebras $\mathcal{A}$. The homomorphism β sends the class of $\mathcal{A}$ to the class of $\mathcal{A}_\xi$, where ξ is the generic point. In the image of β are precisely those $A = \left(\frac{\alpha,-1}{K}\right)$, which are unramified on $\mathcal{K}$. (We refer to [28].) In other words, if A is unramified on $\mathcal{K}$, then it can be equipped with the structure of a unique Azumaya algebra $\mathcal{A}$. In [21, Thm. 1.3.7] (also [22]) it is shown that the category $\mathcal{H} = \mathrm{coh}(\mathcal{A})$ of coherent $\mathcal{A}$-modules is equivalent to the category $\mathrm{coh}(\mathcal{K}, \alpha)$ of α-twisted coherent sheaves on $\mathcal{K}$.

Proposition 11.6. *Assume the Witt function α is definite. Let $\mathcal{A}$ be the corresponding Azumaya algebra. The category $\mathcal{H} = \mathrm{coh}(\mathcal{A})$ of coherent $\mathcal{A}$-modules is a noncommutative regular projective curve with $s(\mathcal{H}) = 2$.*

Proof. Since α is not positive definite, by Witt's theorem [87] the quaternion algebra $A = \left(\frac{\alpha,-1}{K}\right)$ does not split. Since α is definite, A is unramified. The assertion follows from Theorem 7.11. □

We now treat the general (ramified) situation. Denote by $U = \mathcal{K} \setminus \{x_1, \dots, x_n\}$ the Zariski-open unramified locus. The given quaternion algebra $A = \left(\frac{\alpha,-1}{K}\right)$ defines an Azumaya algebra $\mathcal{A}_U$ in $\mathrm{Br}(U)$.

Theorem 11.7. *Each Witt function $\alpha: \mathcal{K} \to \mathbb{S}^2$ gives rise to a noncommutative regular projective curve $\mathcal{H}$ of skewness $s(\mathcal{H}) = 2$, and with the following properties:*

(1) *The centre curve is $\mathcal{K}$.*

(2) *The function field is $k(\mathcal{H}) = \left(\frac{\alpha,-1}{k(\mathcal{K})}\right)$.*

(3) *Up to an equivalence of categories, $\mathcal{H}$ is uniquely determined by $\mathcal{K}$ and the coloured segments of the ovals.*

Proof. Let $\mathcal{K}$ and α be given. Denote by $\mathcal{O} = \mathcal{O}_{\mathcal{K}}$ the structure sheaf of $\mathcal{K}$. With $K = k(\mathcal{K})$ let A be the quaternion skew field $\left(\frac{\alpha,-1}{K}\right)$ over K. Let $U \subseteq \mathcal{K}$ be the unramified locus and $j\colon U \to \mathcal{K}$ the inclusion. The function field of U is K. By [7, Cor. 1.9.6] there exists an Azumaya $\mathcal{O}_U$-algebra $\mathcal{A}'$ in A. By [7, Prop. 1.8.1] there is an $\mathcal{O}$-order $\mathcal{B}$ in A with $j^*\mathcal{B} = \mathcal{A}'$. By [7, Prop. 1.8.2] there is a maximal $\mathcal{O}$-order $\mathcal{A}$ in A containing $\mathcal{B}$. Then, by Theorem 7.11, $\mathcal{H} = \operatorname{coh}(\mathcal{A})$ is a noncommutative regular projective curve over $\mathbb{R}$. Moreover, conditions (1) and (2) are clearly satisfied.

(3) This follows from part (3) of Witt's theorem above, and Theorem 7.12. □

Definition 11.8. Let $(\mathcal{K}, \alpha)$ be a Klein surface with a Witt function $\alpha\colon \mathcal{K} \to \mathbb{S}^2$. (Recall that this means that α is not positive definite.) We call the noncommutative regular projective curve constructed in the preceding theorem *Witt curve* (associated with $(\mathcal{K}, \alpha)$).

Theorem 11.9. *Each noncommutative regular projective curve $\mathcal{H}$ over $k = \mathbb{R}$ with $s(\mathcal{H}) > 1$ is a Witt curve.*

Proof. The centre curve of $\mathcal{H}$ is a real regular projective curve, thus a Klein surface $\mathcal{K}$. The function field $k(\mathcal{H})$ is a skew field of quaternions over $K = k(\mathcal{K})$, thus of the form $\left(\frac{\alpha,-1}{K}\right)$. By the uniqueness part of Theorem 11.7 then $\mathcal{H}$ is the Witt curve associated with $(\mathcal{K}, \alpha)$. □

We will call *commutative* real regular projective curves with centre $\mathbb{R}$ (thus corresponding to the Klein surfaces) also *Klein curves*, and will use the letter X instead of $\mathcal{K}$.

The main result for the τ-multiplicities for Witt curves is the following.

Theorem 11.10. *Let $\mathcal{H}$ be a Witt curve. Then $e_\tau(x) = 2$ if and only if x is a segmentation point. In other words: precisely for the segmentation points τ is acting non-trivially — by complex conjugation — on the corresponding tubes.*

Proof. This follows from Corollary 7.18. □

11.11. Let $\mathcal{H}$ be a Witt curve. Table 2 summarizes some local data.

point x	$e(x)$	$e^*(x)$	$e_\tau(x)$	$k(x)$	$Z(D_x)$	D_x	$\widehat{D}_x$
inner	2	1	1	$\mathbb{C}$	$\mathbb{C}$	$\mathbb{C}$	$\mathbb{C}((T))$
real	2	1	1	$\mathbb{R}$	$\mathbb{R}$	$\mathbb{R}$	$\mathbb{R}(T))$
quaternion	1	2	1	$\mathbb{R}$	$\mathbb{R}$	$\mathbb{H}$	$\mathbb{H}((T))$
segmentation	1	1	2	$\mathbb{R}$	$\mathbb{C}$	$\mathbb{C}$	$\mathbb{C}((T,\sigma))$

Table 2. Local data of a Witt curve

A Klein surface has the constants field $\mathbb{R}$. We will now determine the field of constants $\mathrm{End}(L)$ of a Witt curve $(\mathcal{H}, L)$. This terminology is justified since it follows from properties of maximal orders that the endomorphism ring of any line bundle can be embedded into the endomorphism ring of L.

Lemma 11.12 (Field of constants). *Let $(\mathcal{H}, L)$ be a Witt curve. Let r be the number of completely real coloured ovals, and $n = 2m$ the total number of segmentation points. Then*

$$\mathrm{End}(L) \simeq \begin{cases} \mathbb{C} & \text{if } m > 0 \text{ or } r > 0, \\ \mathbb{H} & \text{if } m = 0 \text{ and } r = 0. \end{cases}$$

For the normalization factor ε from (9.2) we have $\varepsilon = 1$.

Proof. Let $K = k(X)$ be the function field of the centre curve, so $k(\mathcal{H}) = \left(\frac{\alpha, -1}{K}\right)$. Scalar extension by $\mathbb{C}$, denoted by the overline symbol, then gives $\overline{\mathcal{A}} \otimes \overline{K} = \mathrm{M}_2(\overline{K})$. Assume first that $m = 0$. Then $\overline{\mathcal{A}}$ is Azumaya, and by [7, Cor. 1.7.6] there is a locally-free $\overline{\mathcal{O}}$-module $\mathcal{E}$ of rank 2 such that $\overline{\mathcal{A}} \simeq \mathcal{E}nd_{\overline{\mathcal{O}}}(\mathcal{E})$. Then $\mathrm{End}(L) = \mathrm{End}(\mathcal{A}) = \mathbb{R}$ is not possible, since otherwise $\mathrm{End}_{\overline{\mathcal{A}}}(\overline{\mathcal{A}}) \simeq \mathbb{C}$, which is not the case by the sentence before. This also holds in case $m > 0$: then $\overline{\mathcal{A}}$ is not Azumaya, instead it is weighted by an even number of the weight 2, and the endomorphism ring of the structure sheaf is not changed by insertion of weights. So in any case $\mathrm{End}(L) = \mathbb{C}$ or $\mathbb{H}$.

Thus $\kappa = \dim_k \mathrm{End}(L) \geq 2$, and we have $\kappa\varepsilon = 2$ in cases $m > 0$ or $r > 0$ and $\kappa\varepsilon = 4$ in the cases $m = 0$, $r = 0$: this follows from the existence of a simple object of degree 1 (by the definition of ε) and the formula (9.4).

Thus it remains to show that in case $m = 0$, $r = 0$ we have $\kappa = 4$. In this case $\mathcal{A}$ is Azumaya with each of its ovals coloured quaternion. Then $k(\mathcal{H}) = k(X) \otimes \mathbb{H}$ and $\mathcal{A} = \mathcal{O} \otimes \mathbb{H}$, hence $\mathrm{End}(\mathcal{A}) = \mathbb{H}$ follows. □

Proposition 11.13 (Hurwitz genus formula for Witt curves). *Let $\mathcal{H}$ be a Witt curve with $n = 2m$ segmentation points and underlying Klein curve X, and let r be the number of completely real coloured ovals. Then*

$$g(\mathcal{H}) = \begin{cases} 2g(X) - 1 + m & \text{if } m > 0 \text{ or } r > 0, \\ g(X) & \text{if } m = 0 \text{ and } r = 0. \end{cases} \tag{11.2}$$

Proof. By formula (9.5) we have for the normalized Euler characteristics $\chi'(\mathcal{H}) = \chi'(X) - n/4$. With $\kappa = \dim_k \mathrm{End}(L)$ we obtain $\kappa(1 - g(\mathcal{H})) = 4\chi'(\mathcal{H}) = 4\chi'(X) - n = 4(1 - g(X)) - n$, thus

$$g(\mathcal{H}) = \frac{4}{\kappa} g(X) - \frac{4}{\kappa} + 1 + \frac{2m}{\kappa}. \tag{11.3}$$

Now the assertion follows with Lemma 11.12. □

Remark 11.14. (1) Our definition of the genus differs from the definition of the genus of function skew fields in [86], since we always have $g(\mathcal{H}) \geq 0$. The analogues formula obtained in [60] is $4g(X) - 3 + n$, which we would get for $\kappa = 1$. For example, the function fields $\mathbb{H}(T)$ and $\mathbb{C}(T, \sigma)$ have genus 0 by our definition, but genus -3 by the other definitions. But the conditions $g \leq 0$ and $g = 1$ are equivalent for both definitions.

(2) Let $\mathcal{K}$ be a Klein surface with a $\pm$-configuration induced by α. We form a double cover $\pi: \mathcal{K}' \to \mathcal{K}$ of Klein surfaces as in the proof of Witt's theorem in [3, Thm. 2.4.5]: $\mathcal{K}'$ is obtained by gluing two copies of $\mathcal{K}$ together at the closures of the segments, or complete ovals, where $\alpha(x) < 0$. The boundary of $\mathcal{K}'$ is then given by two copies of the segments (or ovals), where $\alpha(x) > 0$, and each pair of these segments (if not an entire oval) is bounded by two segmentation points of $\mathcal{K}$. The connected components are then ovals, each of which contains precisely zero or two of the segmentation points from $\mathcal{K}$. Setting $\alpha' = \pi^*(\alpha) = \alpha \circ \pi$, then α' is positive on the segments. Thus the former segmentation points are not longer segmentation points on $\mathcal{K}'$. By construction $k(\mathcal{K}') = k(\mathcal{K})(\sqrt{\alpha})$. Since $\mathcal{K}'$ is a Riemann surface only in case $m = 0$ and $r = 0$, formula (11.2) (for $\mathcal{H}$) coincides with the Hurwitz equation for the covering $\pi: \mathcal{K}' \to \mathcal{K}$ in [61, Thm. 2].

Corollary 11.15. *Let $n > 0$ be the number of segmentation points. Then:*

- $g(\mathcal{H}) = 0$ *if and only if* $g(X) = 0$ *(hence $\mathcal{K}$ is the compact disc) and* $n = 2$.
- $g(\mathcal{H}) = 1$ *if and only if* $g(X) = 0$ *and* $n = 4$.

Example 11.16 (Genus zero). There are two Witt curves of genus zero; we describe the corresponding Witt surfaces, Figure 2.

(1) The compact disc $\mathbb{D}_{\mathbb{H}}$, the boundary coloured quaternion. This is the projective spectrum of the graded polynomial ring $\mathbb{H}[X, Y]$ with central variables X, Y of degree 1. The function field is given by $\mathbb{H}(T)$.

(2) The compact disc $\mathbb{D}_{2,2}$ with two segmentation points on its boundary. This is the projective spectrum of the graded skew-polynomial algebra $\mathbb{C}[X; Y, \sigma]$. Accordingly, the function field is $\mathbb{C}(T, \sigma)$.

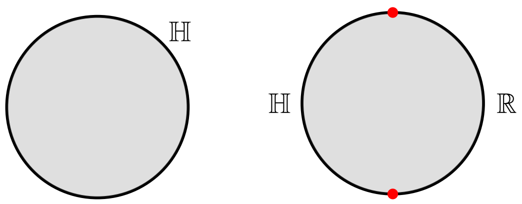

Figure 2. The Witt curves with $g(\mathcal{H}) = 0$: $\mathbb{D}_{\mathbb{H}}$ and $\mathbb{D}_{2,2}$

11.17 (Euler characteristic of a Witt curve). We now come back to the question how to normalize the Euler characteristic "correctly".

If $\mathcal{K}$ is a Klein surface and S the corresponding complex double then, since $\mathcal{K}$ is a $\mathbb{Z}_2$-quotient of S, we have $\chi(S) = 2 \cdot \chi(\mathcal{K})$. This can also be expressed in the following way: if $\mathcal{H}$ is a real regular projective curve, then

$$\chi(\mathcal{H}) = \chi(\mathcal{H} \otimes \mathbb{C})/2. \tag{11.4}$$

We assume that this should also hold for the Euler characteristic of Witt curves $\mathcal{H}$. Here, $\mathcal{H} \otimes \mathbb{C}$ (tensor product over $k = \mathbb{R}$) is the karoubian closure of the category with the same objects as in $\mathcal{H}$, and where Hom-spaces are tensored with $\mathbb{C}$ and then considered "modulo Morita-equivalence" $\sim_M$. Two examples: (1) $\mathbb{D}_{\mathbb{H}}$. At the boundary, $\mathbb{H}$ becomes $\mathbb{H} \otimes \mathbb{C} = \mathrm{M}_2(\mathbb{C}) \sim_M \mathbb{C}$. In the inner, $\mathbb{C}$ becomes $\mathbb{C} \otimes \mathbb{C} = \mathbb{C} \times \mathbb{C}$, two copies of $\mathbb{C}$. We hence get two discs, all points complex, and the boundaries identified. This gives the Riemann sphere. Alternatively: $\mathbb{H}[X, Y] \otimes \mathbb{C} = \mathrm{M}_2(\mathbb{C})[X, Y] \sim_M \mathbb{C}[X, Y]$. (2) $\mathbb{D}_{2,2}$. Here we get, modulo $\sim_M$ two copies of discs with all points complex, the boundaries identified; the two ramification points (their simples having endomorphism ring $\mathbb{C}$) are "doubled", and become weighted by 2. So we have the weighted projective line over $\mathbb{C}$ with weight sequence $(2, 2)$. The examples suggest that $\mathcal{H}$ is somehow a $\mathbb{Z}_2$-quotient of $\mathcal{H} \otimes \mathbb{C}$ (justifying (11.4)), but this is yet not well understood. In general, if $\mathcal{H}$ has $2n$ ramification points, and if $\mathcal{K}$ is the underlying Klein surface, S the complex double, then $\mathcal{H} \otimes \mathbb{C}$ is S weighted with the $2n$-sequence $(2, 2, \dots, 2)$. Thus,

$$\chi(\mathcal{H} \otimes \mathbb{C}) = \chi(S) - \sum_{i=1}^{2n}(1 - 1/2) = \chi(S) - n.$$

Moreover, if $\chi(\mathcal{H}) := \langle L, L \rangle$, then by (9.5),

$$\chi(\mathcal{H}) = s(\mathcal{H})^2\big(\chi(\mathcal{K}) - 1/2\sum_{i=1}^{2n}(1 - 1/2)\big) = s(\mathcal{H})^2/2 \cdot \chi(\mathcal{H} \otimes \mathbb{C}).$$

Therefore, if (11.4) should hold, we have to replace $\chi(\mathcal{H})$ by the normalized $\chi'(\mathcal{H})$.

Because of these considerations we regard χ' as the correct Euler characteristic for a Witt curve.

12. Real elliptic curves

In the following examples we treat all real elliptic curves, that is, the Klein and Witt surfaces with $g(\mathcal{H}) = 1$. Here, we classify them only topologically, not up to isomorphism, where one has to add a real parameter to each topological case.

Example 12.1 (Commutative real elliptic curves). There are (up to parameters) three real elliptic curves with $s(\mathcal{H}) = 1$, with corresponding Klein surfaces given by the annulus $\mathbb{A}$, the Klein bottle $\mathbb{K}$ and the Möbius band $\mathbb{M}$. We refer to the book [2].

Example 12.2 (Elliptic Witt curves). As only real elliptic curves with $s(\mathcal{H}) > 1$ we have (up to parameters) the corresponding Witt surfaces: the annuli $\mathbb{A}_{\mathbb{R},\mathbb{H}}$ and $\mathbb{A}_{\mathbb{H},\mathbb{H}}$, where one and both ovals, respectively, are coloured quaternion, the Möbius band $\mathbb{M}_{\mathbb{H}}$, with quaternion coloured boundary, and the compact disc $\mathbb{D}_{2,2,2,2}$ with four segmentation points on the boundary, Figure 3; there is a moduli parameter $\lambda > 0$ involved, so the general case is not as symmetric as in the figure.

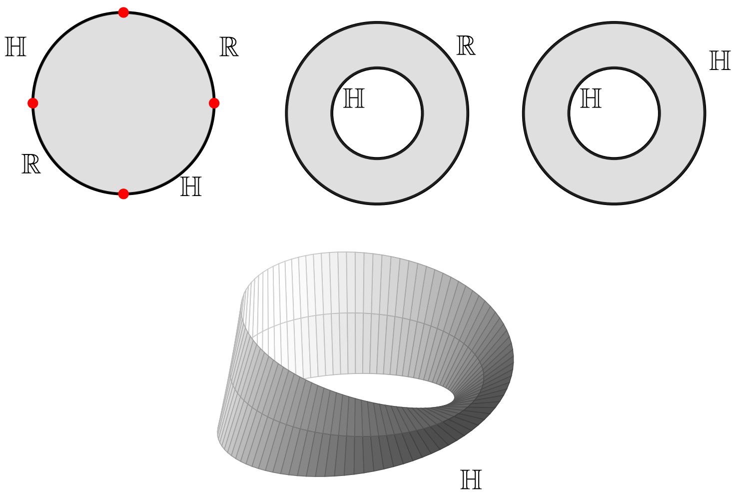

Figure 3. The elliptic Witt curves: $\mathbb{D}_{2,2,2,2}$, $\mathbb{A}_{\mathbb{R},\mathbb{H}}$, $\mathbb{A}_{\mathbb{H},\mathbb{H}}$, $\mathbb{M}_{\mathbb{H}}$

Lemma 12.3. *For the real elliptic curves* $(\mathcal{H}, L)$ *Table 3 describes the endomorphism ring of the structure sheaf* L*, the endomorphism ring of a certain simple sheaf* $S = S_x$ *of degree* 1*, the multiplicity* $e(x)$*, the number* ε*, and the number of orbits in* $\widehat{\mathbb{Q}}$ *of the action of* $\operatorname{Aut}(\mathcal{D}^b(\mathcal{H}))$ *on the slopes.*

	$\mathbb{A}$	$\mathbb{M}$	$\mathbb{K}$	$\mathbb{A}_{\mathbb{R},\mathbb{H}}$	$\mathbb{A}_{\mathbb{H},\mathbb{H}}$	$\mathbb{M}_{\mathbb{H}}$	$\mathbb{D}_{2,2,2,2}$
$\operatorname{End}(L)$	$\mathbb{R}$	$\mathbb{R}$	$\mathbb{R}$	$\mathbb{C}$	$\mathbb{H}$	$\mathbb{H}$	$\mathbb{C}$
$\operatorname{End}(S)$	$\mathbb{R}$	$\mathbb{R}$	$\mathbb{C}$	$\mathbb{R}$	$\mathbb{H}$	$\mathbb{H}$	$\mathbb{C}$
$e(x)$	1	1	1	2	1	1	1
ε	1	1	2	1	1	1	1
# orbits	1	1	2	2	1	1	1

Table 3. Data of real elliptic curves

For $\mathbb{K}$ the two orbits are given by those fractions a/b (with a, b coprime) with b even or odd, respectively, in case $\mathbb{A}_{\mathbb{R},\mathbb{H}}$ with a even or odd, respectively; in both cases the slopes 0 and ∞ belong to different orbits.

Proof. We get $\mathrm{End}(L)$ and the value of ε from Lemma 11.12. For $\mathbb{K}$ we have $\varepsilon = 2$ since there is no boundary. There exists a simple S as claimed. The objects L and S define tubular mutations $\sigma_L, \sigma_S \colon \mathcal{D}^b(\mathcal{H}) \to \mathcal{D}^b(\mathcal{H})$, respectively, which on $\mathrm{K}_0(\mathcal{H})$ act as follows (with $\mathbf{a} := [L]$, $\mathbf{s} := [S]$ and $\kappa = \dim_{\mathbb{R}} \mathrm{End}(L)$):

$$\sigma_L(\mathbf{y}) = \mathbf{y} \pm \frac{\langle \mathbf{a}, \mathbf{y} \rangle}{\kappa} \mathbf{a}, \quad \sigma_S(\mathbf{y}) = \mathbf{y} \pm \frac{\langle \mathbf{s}, \mathbf{y} \rangle}{|\mathrm{End}(S)|} \mathbf{s}. \tag{12.1}$$

With $\deg(\mathbf{y}) = \frac{1}{\kappa\varepsilon} \langle \mathbf{a}, \mathbf{y} \rangle$ and $\mathrm{rk}(\mathbf{y}) = \frac{1}{\kappa\varepsilon \deg(S)} \langle \mathbf{y}, \mathbf{s} \rangle$ we obtain the induced actions on the slopes

$$q \mapsto \frac{q}{1 \pm \varepsilon q} \quad \text{and} \quad q \mapsto q \pm \deg(S) e(x) = q \pm e(x) \quad \text{(resp.)}.$$

The claimed numbers and shapes of orbits follow from [45, Lem. 6.1]. □

It was already observed in [54] that the Klein bottle must have a Fourier–Mukai partner different from a Klein bottle. The first part of the following statement is a non-weighted analogue of [44].

Theorem 12.4. (1) *The Klein bottle $\mathbb{K}$ (with any parameter) has as a Fourier–Mukai partner a Witt curve given by the annulus $\mathbb{A}_{\mathbb{R},\mathbb{H}}$ with two differently coloured ovals (with a suitable parameter).*

(2) *If $\mathcal{H}$ is a noncommutative real elliptic curve, which is neither a Klein bottle, nor an annulus $\mathbb{A}_{\mathbb{R},\mathbb{H}}$, then each Fourier–Mukai partner of $\mathcal{H}$ is isomorphic to $\mathcal{H}$ itself.*

Proof. (1) A Klein bottle $\mathbb{K}$ has no boundary. Thus all simple sheaves have endomorphism ring $\mathbb{C}$, the complex numbers. On the other hand, the structure sheaf $\mathcal{O}_{\mathbb{K}}$ (which is stable and of slope 0) has endomorphism ring $\mathbb{R}$. Thus by Theorem 9.7 the subcategory of semistable bundles of slope 0 is parametrized by a noncommutative projective curve $\mathcal{H}$ of $g(\mathcal{H}) = 1$ with $\mathcal{H} \not\simeq \mathbb{K}$, and $\mathcal{H}$ is derived-equivalent to $\mathbb{K}$. There must be a simple sheaf S in $\mathcal{H}$ with $\mathrm{End}(S) \simeq \mathbb{R}$. By Table 3 the only possibility is then $\mathcal{H} = \mathbb{A}_{\mathbb{R},\mathbb{H}}$ (with a suitable parameter).

(2) This follows from Table 3. □

Corollary 12.5. *The skewness, and thus the function field, of a noncommutative regular projective curve is in general* not *a derived invariant.*

Proposition 12.6. *Let $\mathcal{H}$ be an elliptic Witt curve.*

(1) *If $\mathcal{H}$ is unramified, then $\tau = 1_{\mathcal{H}}$. Moreover, $\mathrm{Pic}_0(\mathcal{H})$ is not finitely generated.*

(2) *Otherwise, that is, if $\mathcal{H}$ is given by $\mathbb{D}_{2,2,2,2}$ (with some parameter), then* $\tau = ({\sigma_{x_1}} \circ {\sigma_{y_1}}^{-1}) \circ ({\sigma_{x_2}} \circ {\sigma_{y_2}}^{-1})$ *(where $x_1,\ y_1,\ x_2,\ y_2$ are the four ramification points) is of order* 2*. Moreover, in this case the Picard-shift group* $\mathrm{Pic}(\mathcal{H})$ *is finitely generated abelian of rank one, and* $\mathrm{Pic}_0(\mathcal{H})$ *is finite.*

Proof. In all cases, the result on the order of τ is given by Theorem 9.10. For the Picard-shift group we get in the Azumaya cases $\mathrm{Pic}(\mathcal{H}) \simeq \mathrm{Pic}(X)$ by Theorem 8.5, with X the underlying Klein curve, which is elliptic. It is well-known that $\mathrm{Pic}_0(X)$ is not finitely generated in this case (we refer to [1, Thm. 5.7]). In the ramified case $\mathbb{D}_{2,2,2,2}$ the centre curve is $X = \mathbb{P}^1(\mathbb{R})$, of genus zero, and $\mathrm{Pic}(\mathbb{P}^1(\mathbb{R})) \simeq \mathbb{Z}$, and the last claim follows from Theorem 8.5. Actually, $\mathrm{Pic}_0(\mathcal{H}) \simeq \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$, generated by ${\sigma_{x_1}} \circ {\sigma_{y_1}}^{-1}$, ${\sigma_{y_1}} \circ {\sigma_{x_2}}^{-1}$ and ${\sigma_{x_2}} \circ {\sigma_{y_2}}^{-1}$. □

Remark 12.7 (Calabi–Yau). Let $\mathcal{T}$ be a triangulated k-category with finite dimensional Hom-spaces and with Serre duality $\mathrm{Hom}_{\mathcal{T}}(X,Y) = \mathrm{D}\,\mathrm{Hom}_{\mathcal{T}}(Y, SX)$, where S is an exact autoequivalence of $\mathcal{T}$, the (triangulated) *Serre functor*. If $S^m \simeq [n]$, the n-th suspension functor (with $m \geq 1$ minimal), then $\mathcal{T}$ is called *triangulated Calabi–Yau of (fractional) dimension* $\frac{n}{m}$ (we refer to [41]). If $\mathcal{T} = \mathcal{D}^b(\mathcal{H})$, with $\mathcal{H}$ a noncommutative regular projective curve with Auslander–Reiten translation τ, then $S = \tau \circ [1]$ is the Serre functor of $\mathcal{T}$. If $g(\mathcal{H}) = 1$, then the functor τ is of finite order p, and then $S^p \simeq [p]$, that is, $\mathcal{T}$ is Calabi–Yau of dimension $\frac{p}{p}$.

The preceding discussion shows that the derived category of the elliptic Witt curve $\mathbb{D}_{2,2,2,2}$ has Calabi–Yau dimension $\frac{2}{2}$; all the others have dimension $\frac{1}{1}$.

Proposition 12.8. *Let $(\mathcal{H}, L) = \mathrm{coh}(\mathbb{X})$ be the Witt curve $\mathbb{X} = \mathbb{D}_{2,2,2,2}$ for some parameter. The stable bundles of degree* 0 *are parametrized by* $\mathbb{X}$*. The line bundles of degree* 0 *are in bijection with the non-quaternion boundary points; the structure sheaf L corresponds to one of the ramification points.*

Proof. The composition of tubular mutations $\sigma = \sigma_S \circ \sigma_L$ acts on slopes $\mu = \binom{\deg}{\mathrm{rk}}$ like the matrix $\left(\begin{smallmatrix} 0 & -1 \\ 1 & -1 \end{smallmatrix}\right)$, sends ∞ to 0 and preserves endomorphism rings. If $E \in \mathcal{H}_0$, then $\mathrm{rk}(\sigma(E)) = \deg(E)$. Thus the degree 0 line bundles correspond to the simples sheaves of degree 1. Moreover, L is one of them with $\mathrm{End}(L) = \mathbb{C}$. □

13. Weighted curves, and noncommutative 2-orbifolds

We conclude the article by treating the weighted case. We will first show that each weighted noncommutative regular projective curve $\mathcal{H}$ arises from a non-weighted one, $\mathcal{H}_{nw}$, by insertion of weights $p(x) > 1$ in a finite number of points x. This insertion of weights is described, in abstract terms, by the p-cycle construction [53]; the inverse technique, reducing weights, is the perpendicular calculus [33], cf. Proposition 1.1. Data like the function field $k(\mathcal{H})$, the skewness $s(\mathcal{H})$, the multiplicities $e(x)$, the endomorphism rings $\mathrm{End}(S_x)$ of simples, and the ghost group $\mathcal{G}(\mathcal{H})$ remain unchanged by these processes. The change of the Picard-shift group $\mathrm{Pic}(\mathcal{H})$ is easy to describe: one adjoints the $p(x)$-th root of the tubular shift σ_x, for each weight point x. We refer to [47, Ch. 6]. We also remark that there is an analogues result to Lemma 8.4.

13.1 (p-cycle construction). Let $\mathcal{H} = \mathrm{coh}(\mathcal{A})$ be a weighted noncommutative regular projective curve of a field k, with $\mathcal{A}$ a hereditary order in a full matrix algebra A over $k(\mathcal{H})$. Let x be a point such that the tube $\mathcal{U}_x$ is homogeneous. Let $\sigma_x\colon \mathcal{H} \to \mathcal{H}$ be the Picard-shift with respect to x, with natural transformation $1_{\mathcal{H}} \xrightarrow{x} \sigma_x$. Let $p > 1$ be a "weight". Following [53] the category $\mathcal{H}(\substack{p\\x})$ of *p-cycles in x* has *objects* $E = (E_i, e_i)_{i\in\mathbb{Z}}$, where $E_i \in \mathcal{H}$ and $e_i \in \mathrm{Hom}(E_i, E_{i+1})$ such that $E_{i+p} = \sigma_x(E_i)$ and the composition $e_{i+p-1} \circ \cdots \circ e_{i+1} \circ e_i$ is the natural map $x_{E_i}\colon E_i \to E_i(x)$ for each i. If $F = (F_i, f_i)_{i\in\mathbb{Z}}$ is another p-cycle in x, then a *morphism* between E and F is a tuple $(h_i)_{i\in\mathbb{Z}}$ with morphisms $h_i \in \mathrm{Hom}(E_i, F_i)$ satisfying $h_{i+1}e_i = f_i h_i$ and $h_{i+p} = \sigma_x(h_i)$ for all i. The category of p-cycles in x over $\mathcal{H}$ is, like $\mathcal{H}$ itself, abelian, noetherian, hereditary and does not contain non-zero projectives or injectives, [53, Thm. 4.3].

Let $\mathcal{A}(p, x)$ be the (hereditary) order in $\mathrm{M}_p(A)$ given by

$$\begin{pmatrix} \mathcal{A} & \mathcal{A} & \dots & \mathcal{A} \\ \mathcal{A}(-x) & \mathcal{A} & \dots & \mathcal{A} \\ \vdots & & \ddots & \vdots \\ \mathcal{A}(-x) & \mathcal{A}(-x) & \dots & \mathcal{A} \end{pmatrix}.$$

There is an equivalence [58, Prop. 6.1]

$$\mathrm{coh}\big(\mathcal{A}(p, x)\big) \simeq \mathcal{H}\begin{pmatrix} p \\ x \end{pmatrix},$$

and we have then $p(x) = p$. This can be iterated:

Proposition 13.2. (1) *Let $\mathcal{H}_{nw} = \mathrm{coh}(\mathcal{A})$ be a (non-weighted) noncommutative regular projective curve over k. Let $x_1, \dots, x_t$ be distinct points, and let $p_1, \dots, p_t > 1$. Then*

$$\mathcal{H} = \mathrm{coh}\Big(\bigotimes_{i=1}^{t} \mathcal{A}(p_i, x_i)\Big) \simeq \mathcal{H}_{nw}\begin{pmatrix} p_1, \dots, p_t \\ x_1, \dots, x_t \end{pmatrix}$$

is a weighted noncommutative regular projective curve with weight points $x_1, \dots, x_t$, having weights $p(x_i) = p_i$.

(2) *Each weighted noncommutative regular projective curve $\mathcal{H}$ is obtained in this way from its underlying non-weighted curve $\mathcal{H}_{nw}$ (cf. Proposition 1.1).*

Proof. (1) As in [58, Prop. 6.1 + 6.5]. (2) Clearly $\mathcal{H}$ and $\mathcal{H}_{nw}\binom{p_1,\dots,p_t}{x_1,\dots,x_t}$ have the same underlying non-weighted curve, namely $\mathcal{H}_{nw}$. In particular they have the same centre curve and the same function field, and also the same weight function $p\colon X \to \mathbb{N}$. The statement then follows from [20, Thm. 6.7]. □

Concerning the τ-multiplicities the following observation is fundamental.

Proposition 13.3. *Let $\mathcal{H}$ be a noncommutative regular projective curve over a field k and x a separable point. Let $p > 1$ be a weight and let $\overline{\mathcal{H}}$ be a weighted noncommutative curve arising from $\mathcal{H}$ by insertion of the weight p into x. Let $\overline{\mathcal{U}}_x$ be the corresponding tube of rank p. Then the order of $\tau_{\overline{\mathcal{H}}}$ in* $\operatorname{Aut}(\overline{\mathcal{U}}_x/k)$ *is $e_\tau(x)\cdot p$.*

Proof. Working with p-cycles in x one sees easily that $(\tau_{\overline{\mathcal{H}}})^p$ acts on $\operatorname{End}(S_x)$ like σ_x^{-1} and hence like $\tau_{\mathcal{H}}$. □

Proposition 13.4 (The complete local rings). *Let x be a point of weight $p = p(x)$. Then $\mathcal{U}_x \simeq \operatorname{mod}_0(H_p(\widehat{R}_x))$, with $\widehat{R}_x$ the complete local ring as in Proposition 5.1 (and for separable x the skew power series ring as in Theorem 5.3), and*

$$H_p(\widehat{R}_x) = \begin{pmatrix} \widehat{R}_x & \widehat{R}_x & \dots & \widehat{R}_x \\ P_x & \widehat{R}_x & \dots & \widehat{R}_x \\ \vdots & & \ddots & \vdots \\ P_x & P_x & \dots & \widehat{R}_x \end{pmatrix}$$

of size $p \times p$ and with $P_x = \operatorname{rad}(\widehat{R}_x)$, generated by π_x. The ring $H_p(\widehat{R}_x)$ is a semiperfect bounded hereditary noetherian prime ring, whose radical is generated as left and right ideal by the element

$$\overline{\pi}_x = \begin{pmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & & & \ddots & \vdots \\ 0 & & & & 1 \\ \pi_x & 0 & \dots & & 0 \end{pmatrix},$$

which satisfies ${\overline{\pi}_x}^p = \pi_x$.

Proof. This follows as in [73, 4.4]. □

Let $\mathcal{A}$ be a hereditary $\mathcal{O}$-order with $\mathcal{H} = \operatorname{coh}(\mathcal{A})$, by Theorem 7.11. By $\bar{p}$ we will always denote the least common multiple of the weights. We can assume that $\mathcal{A}$ is a hereditary $\mathcal{O}$-order in $\mathrm{M}_{\bar{p}}(k(\mathcal{H}))$. (By [20, Rem. 6.8] even the maximum of the weights can be chosen as the matrix size.) Moreover, we can and will always assume that the structure sheaf L is a special line bundle, corresponding to the structure sheaf (8.1) of $\mathcal{H}_{nw}$ via Proposition 1.1. For a point x we denote by S_x the (up to isomorphism) unique simple sheaf concentrated in x with $\mathrm{Hom}(L, S_x) \neq 0$. For a point $x \in \mathbb{X}$ let $\sigma_x(L) = L(x)$, and then the bimodule $\mathcal{A}(x)$ is defined as in the unweighted case. One also shows that the functors σ_x and $-\otimes_{\mathcal{A}}\mathcal{A}(x)$ are isomorphic. From the preceding results we get $\mathfrak{m}_x H_{p(x)}(\widehat{R}_x) = (\overline{\pi}_x)^{p(x)e_{\mathrm{ra}}(x)}$ and

$$\mathcal{O}(x)\otimes_{\mathcal{O}} \mathcal{A} \simeq \mathcal{A}(p(x)e_{\mathrm{ra}}(x)\cdot x). \tag{13.1}$$

With this one obtains the more general, weighted version of Theorem 8.5. We note that there is a formal similarity to [67, (40.9)].

Theorem 13.5. *Let $\mathcal{H}$ be a weighted noncommutative regular projective curve over a field k. Let X be the (non-weighted) centre curve. Then there is an exact sequence*

$$1 \to \mathrm{Pic}(X) \xrightarrow{\iota} \mathrm{Pic}(\mathcal{H}) \xrightarrow{\phi} \prod_x \mathbb{Z}/p(x)e_{\mathrm{ra}}(x)\mathbb{Z} \to 1 \tag{13.2}$$

of abelian groups. Here, $\phi(\sigma) = \sigma_{|\mathcal{H}_0}$ and ι sends a (class of a) line bundle $\mathcal{O}(x)$ of $\mathrm{coh}(X)$*, for a point $x \in X$, to ${\sigma_x}^{p(x)e_{\mathrm{ra}}(x)}$, for the corresponding point $x \in \mathbb{X}$.* □

Theorem 13.6. *Let $(\mathcal{H}, L)$ be a weighted noncommutative regular projective curve over a perfect field k. Let $\gamma = \sum_{x\in X} \gamma_x \cdot x$ be the canonical divisor of the (non-weighted) centre curve X. For $\overline{\gamma} = \sum_{x\in\mathbb{X}} \gamma_x p(x)e_\tau(x)\cdot x$ we write $\sigma^{|\overline{\gamma}|}$ for the corresponding Picard-shift. Then*

$$\tau \;=\; \sigma^{|\overline{\gamma}|}\cdot \prod_x {\sigma_x}^{p(x)e_{\mathrm{ra}}(x)-1} \;=\; \prod_x {\sigma_x}^{p(x)e_{\mathrm{ra}}(x)(\gamma_x+1)-1} \quad \in \mathrm{Pic}(\mathcal{H}). \tag{13.3}$$

Proof. We just remark that here the different is given by $\Delta = \sum_x (p(x)e_{\mathrm{ra}}(x)-1)\cdot x$, which can be seen as in the unweighted case, and that the dualizing sheaf $\boldsymbol{\omega}_{\mathcal{A}}$ also here is given by $\mathscr{H}om_{\mathcal{O}}(\mathcal{A}, \boldsymbol{\omega}_X)$, see [68, III.2], and (8.3) also holds here. □

Let $(\mathcal{H}, L)$ be a weighted noncommutative regular projective curve over k of skewness $s = s(\mathcal{H})$. Let $\kappa = \dim_k \mathrm{End}(L)$ and ε as defined before (for the underlying non-weighted curve $\mathcal{H}_{nw}$). We define the *average Euler form* and the *(orbifold) degree*

$$\langle\!\langle E, F\rangle\!\rangle = \sum_{j=0}^{\bar{p}-1}\langle \tau^j E, F\rangle, \quad \deg(F) = \frac{1}{\kappa\varepsilon}\cdot\langle\!\langle L, F\rangle\!\rangle - \frac{1}{\kappa\varepsilon}\cdot\langle\!\langle L, L\rangle\!\rangle\cdot \mathrm{rk}(F)$$

and the *normalized orbifold Euler characteristic* $\chi'_{\mathrm{orb}}(\mathcal{H})$ and the *orbifold genus* $g_{\mathrm{orb}}(\mathcal{H})$ by the equations

$$\chi'_{\mathrm{orb}}(\mathcal{H}) := \frac{1}{\bar{p}^2 s^2}\cdot\langle\!\langle L, L\rangle\!\rangle \overset{(*)}{=} -\frac{\kappa\varepsilon}{2\bar{p}s^2}\cdot \deg(\tau L) =: \frac{\kappa}{\bar{p}s^2}\cdot(1 - g_{\mathrm{orb}}(\mathcal{H})).$$

The equality $(*)$ is given by the following.

Lemma 13.7. *We have* $\tau L = L \otimes_{\mathcal{A}} \boldsymbol{\omega}_{\mathcal{A}}$ *and* $\deg \tau L = -\frac{2}{\bar{p}\kappa\varepsilon}\langle\!\langle L, L\rangle\!\rangle$.

Proof. The first equality is clear. For the second we show like in [56, Prop. 3.2] that the difference $\deg \tau^n L - \deg \tau^{n-1} L$ does not depend on n, and as in [51, Lem. 9.1] the claim then follows. □

Like in the unweighted case we obtain (compare [51, Thm. 9.2]):

Theorem 13.8 (Riemann–Roch formula). *Let* $\mathcal{H}$ *be a weighted noncommutative regular projective curve over the field* k. *Then*

$$\frac{1}{\kappa\bar{p}} \cdot \langle\!\langle E, F\rangle\!\rangle = (1 - g_{\mathrm{orb}}(\mathcal{H})) \cdot \mathrm{rk}(E) \cdot \mathrm{rk}(F) + \frac{\varepsilon}{\bar{p}} \cdot \begin{vmatrix} \mathrm{rk}(E) & \mathrm{rk}(F) \\ \deg(E) & \deg(F) \end{vmatrix}$$

holds for all $E,\ F \in \mathcal{H}$. □

Lemma 13.9. *Let* $(\mathcal{H}, L)$ *be a weighted noncommutative regular projective curve over the perfect field* k. *For each* $x \in \mathbb{X}$ *we have*

$$\deg(S_x) = \frac{\bar{p} \cdot s(\mathcal{H})}{p(x) \cdot \kappa \cdot \varepsilon} \cdot e^*(x) \cdot [k(x) : k].$$

Proof. In general ($\mathcal{H}$ regular over any field) we have (with $e'(x)$ as in (7.7))

$$\deg(S_x) = \frac{\bar{p}}{p(x)\kappa\varepsilon} \dim_k \mathrm{Hom}(L, S_x) = \frac{\bar{p}}{p(x)\kappa\varepsilon} e(x)e^*(x)^2 e'(x)[k(x) : k]. \tag{13.4}$$

In the perfect case we know $e'(x) = e_\tau(x)$ and $e(x)e^*(x)e_\tau(x) = s(\mathcal{H})$. □

Theorem 13.10 (Noncommutative Riemann–Hurwitz formulae). *Let* k *be a perfect field and* $\mathcal{H}$ *a weighted noncommutative regular projective curve over* k. *Let* X *be the centre curve,* $\mathcal{H}_{nw}$ *the underlying non-weighted curve. For the normalized orbifold Euler characteristic* $\chi'_{\mathrm{orb}}(\mathcal{H})$ *we have*

$$\chi'_{\mathrm{orb}}(\mathcal{H}) = \chi'(X) - \frac{1}{2}\sum_x \Big(1 - \frac{1}{p(x)e_\tau(x)}\Big)[k(x) : k] \tag{13.5}$$

$$= \chi'(\mathcal{H}_{nw}) - \frac{1}{2}\sum_x \frac{1}{e_\tau(x)}\Big(1 - \frac{1}{p(x)}\Big)[k(x) : k]. \tag{13.6}$$

If we assume k *to be the centre of* $\mathcal{H}$, *then* $\chi'_{\mathrm{orb}}(\mathcal{H})$ *is an invariant of* $\mathcal{H}$.

Proof. We set $s = s(\mathcal{H})$ and denote by $\mathbf{a} = [L]$ and $\mathbf{s_x} = [S_x]$ the classes in the Grothendieck group. We denote the Coxeter transformation on $\mathrm{K}_0(\mathcal{H})$ also by τ. We further set $\mathbf{w_x} = \sum_{j=0}^{p(x)-1} \tau^j \mathbf{s_x}$. By (13.3) we have

$$\tau\mathbf{a} = \mathbf{a} + \sum_x \gamma_x p(x)e_\tau(x)e(x)\mathbf{s_x} + \sum_x (p(x)e_\tau(x) - 1)e(x)\mathbf{s_x}.$$

Then

$$\tau^{\bar{p}}\mathbf{a}-\mathbf{a}=\sum_x \frac{\bar{p}}{p(x)}\gamma_x p(x)e_\tau(x)e(x)\mathbf{w_x}+\sum_x \frac{\bar{p}}{p(x)}(p(x)e_\tau(x)-1)e(x)\mathbf{w_x}.$$

By Riemann–Roch and the preceding lemma,

$$\langle \mathbf{a},\mathbf{w_x}\rangle=\langle \mathbf{a},\mathbf{s_x}\rangle=\frac{p(x)}{\bar{p}}\langle\!\langle \mathbf{a},\mathbf{s_x}\rangle\!\rangle=\frac{p(x)\kappa\varepsilon}{\bar{p}}\deg \mathbf{s_x}=se^*(x)[k(x):k].$$

We obtain

$$\langle \mathbf{a},\tau^{\bar{p}}\mathbf{a}-\mathbf{a}\rangle=s^2\bar{p}\sum_x \gamma_x[k(x):k]+s^2\bar{p}\sum_x\Big(1-\frac{1}{p(x)e_\tau(x)}\Big)[k(x):k].$$

Similarly we compute $\langle \tau^j\mathbf{a},\tau^{\bar{p}}\mathbf{a}-\mathbf{a}\rangle$. With this we get, as in [51, Lem. 9.1],

$$\langle\!\langle \mathbf{a},\mathbf{a}\rangle\!\rangle=-\frac{1}{2}\bar{p}^2s^2\sum_x \gamma_x[k(x):k]-\frac{1}{2}\bar{p}^2s^2\sum_x\Big(1-\frac{1}{p(x)e_\tau(x)}\Big)[k(x):k].$$

Since $\sum_x \gamma_x[k(x):k]=[\mathrm{End}(\mathcal{O}):k]\varepsilon_X \deg_X(\boldsymbol{\omega}_X)=-2\chi'(X)$ we obtain

$$\langle\!\langle \mathbf{a},\mathbf{a}\rangle\!\rangle=\bar{p}^2s^2\chi'(X)-\frac{1}{2}\bar{p}^2s^2\sum_x\Big(1-\frac{1}{p(x)e_\tau(x)}\Big)[k(x):k].$$

Division by $\bar{p}^2s^2$ yields the first equation.

Then, the second follows with (9.5), (9.4) and using the equation $1-\frac{1}{pe}=(1-\frac{1}{e})+\frac{1}{e}(1-\frac{1}{p})$. □

Remark 13.11. (1) Let k be algebraically closed. As in the unweighted cases, the orbifold Euler characteristic $\chi_{\mathrm{orb}}(\mathcal{H})$ satisfies $\chi_{\mathrm{orb}}(\mathcal{H})=2\chi'_{\mathrm{orb}}(\mathcal{H})$. Since moreover $s(\mathcal{H})=1$ and $\mathcal{H}_{nw}=\mathrm{coh}(X)$, equations (13.5) and (13.6) yield

$$\chi_{\mathrm{orb}}(\mathcal{H})=\chi(X)-\sum_x\Big(1-\frac{1}{p(x)}\Big).$$

In case $k=\mathbb{C}$ and $\mathcal{H}$ a weighted complex regular projective curve, or a complex 2-orbifold, then $\chi_{\mathrm{orb}}(\mathcal{H})=\chi'_{\mathrm{orb}}(\mathcal{H})$, if the values are computed over the field $\mathbb{R}$ of real numbers.

(2) The factor $[k(x):k]$ (a datum of the centre curve) equals $\frac{\kappa\cdot\varepsilon\cdot\deg(S_x)}{e^*(x)\cdot s(\mathcal{H})}$, with the degree of S_x in $\mathcal{H}_{nw}$ (a datum of the underlying non-weighted curve).

(3) In the preceding theorem, we made the assumption that k is perfect since we used the skewness equation, Theorem 6.5. Of course, we only need that the involved points are separable. But also in full generality we may have a “nice” formula. In (13.8) below we have a still compact formula, in a special case, but over any field.

Weighted Klein and Witt curves. The following formula, obtained from (13.6), can be regarded as the extension of the Riemann–Hurwitz formula [81, Thm. 13.3.4] in Thurston's book (we also refer to [76]) to *noncommutative real* 2*-orbifolds*.

Corollary 13.12. *Let $\mathcal{H}$ be a weighted noncommutative regular projective curve over $k = \mathbb{R}$. Then*

$$\chi'_{\mathrm{orb}}(\mathcal{H}) = \chi'(\mathcal{H}_{nw}) - \frac{1}{4} \cdot \sum_x \Big(1 - \frac{1}{p(x)}\Big) - \frac{1}{2} \cdot \sum_y \Big(1 - \frac{1}{p(y)}\Big) - \sum_z \Big(1 - \frac{1}{p(z)}\Big),$$

where x runs over the ramification points, y over the other boundary points, and z over the inner points.

We remark that in case $s(\mathcal{H}) = 1$ we have $\chi'(\mathcal{H}_{nw}) = \chi_{\mathrm{top}}(S)$, where S is the underlying Klein or Riemann surface.

Multiplicity freeness and line bundles. The following fact on line bundles may be of general interest. In case of genus zero it was first shown in [43], compare [25, Prop. 7.3.5].

Proposition 13.13. *Let $(\mathcal{H}, L)$ be a weighted noncommutative regular projective curve over a field k. If $\mathcal{H}$ is multiplicity free, then each line bundle is a Picard-shift of the structure sheaf L. That is,* $\mathrm{Pic}(\mathcal{H})$ *acts transitively on the set of isomorphism classes of line bundles.*

Proof. Let L' be a line bundle. Let $x \in \mathbb{X}$ be any point. For $n \gg 0$ we have a short exact sequence $0 \to L \to L'(nx) \to E \to 0$ with E of finite length; this follows by a weighted version of Lemma 3.4. Applying (a weighted version of) Lemma 3.2 (with $e = 1$) to each indecomposable summand of E gives the result. □

The statement is not true in general, if $\mathcal{H}$ has multiplicities. In [47] examples of genus zero (even non-weighted) are given, where there are two line bundles having non-isomorphic endomorphism rings.

Tilting objects. The following well-known (see [55]) fact shows why genus zero (in the non-orbifold sense) curves are important in the representation theory of finite dimensional algebras.

Theorem 13.14. *Let $\mathcal{H}$ be a weighted noncommutative regular projective over the field k, with underlying non-weighted curve $\mathcal{H}_{nw}$. The following are equivalent:*

(1) $g(\mathcal{H}_{nw}) = 0$.

(2) *$\mathcal{H}$ admits a tilting object T.*

If this is the case, then there is even a tilting bundle T in $\mathcal{H}$ such that its endomorphism ring is a canonical algebra in the sense of Ringel–Crawley-Boevey [70].

These curves were called exceptional curves in [47, 53], and noncommutative curves of genus zero in [47]. Special cases are the weighted projective lines introduced by Geigle–Lenzing [32]. If k is algebraically closed, these notions coincide.

If $\mathcal{H}_{nw}$ is of genus zero, then so is the centre curve.

Corollary 13.15 (of (13.3)). *Let k be a perfect field. We assume that the centre curve X is of genus zero, of numerical type ε. Then*

$$\tau \;=\; {\sigma_{x_0}}^{-2/\varepsilon} \cdot \prod_x {\sigma_x}^{p(x)e_\tau(x)-1} \tag{13.7}$$

for any point $x_0 \in \mathbb{X}$ which is rational in X and neither ramification nor weight. □

In the following special case, the formulae for orbifold Euler characteristic and genus are the well-known ones. These can be simply obtained as special cases from (13.6), or they can be proved directly (compare [51, Thm. 9.2]), even over any field.

Corollary 13.16. *Let $\mathcal{H}$ be a weighted noncommutative regular projective curve over a field k. Assume that the non-weighted curve $\mathcal{H}_{nw}$ is of genus zero. Then*

$$\chi'_{\mathrm{orb}}(\mathcal{H}) = \frac{\kappa}{s(\mathcal{H})^2} - \frac{\kappa\varepsilon}{2s(\mathcal{H})^2}\sum_x e(x)f(x)\Big(1-\frac{1}{p(x)}\Big) \tag{13.8}$$

and

$$g_{\mathrm{orb}}(\mathcal{H}) = 1 + \frac{\varepsilon\bar{p}}{2}\Big(\sum_x e(x)f(x)\Big(1-\frac{1}{p(x)}\Big) - \frac{2}{\varepsilon}\Big).$$

Proof. In case x is separable, then (invoking (9.1) and (10.1))

$$e(x)f(x) = \frac{s(\mathcal{H})^2}{\kappa\varepsilon}\frac{1}{e_\tau(x)}\cdot[k(x):k], \tag{13.9}$$

and the result follows from (13.6). □

Negative orbifold Euler characteristic. The orbifold Euler characteristic is strongly linked to the Gorenstein parameter in singularity theory; we refer to [48]. In that more general context, the case of negative orbifold Euler characteristic is also called the *anti-Fano* case, or *of general type*. This situation is the most complicated, in terms of complexity of the category $\mathcal{H}$.

Proposition 13.17 ([56, Prop. 4.7]). *Let $\mathcal{H}$ be a weighted noncommutative regular projective curve over a field. If $\chi'_{\mathrm{orb}}(\mathcal{H}) < 0$, then each Auslander–Reiten component in $\mathcal{H}_+ = \mathrm{vect}(\mathbb{X})$ is of type $\mathbb{Z}A_\infty$, and $\mathcal{H}$ is of wild representation type.* □

The weighted noncommutative regular projective curves of nonnegative (orbifold) Euler characteristic are the (non-weighted) elliptic curves, the domestic and the tubular curves, which we will consider now.

Domestic curves. This case is also called the *Fano* case.

Definition 13.18. We call a weighted noncommutative regular projective curve *domestic*, if $\chi'_{\mathrm{orb}}(\mathcal{H}) > 0$ (equivalently: $g_{\mathrm{orb}}(\mathcal{H}) < 1$).

Theorem 13.19. (1) *Let $\mathcal{H}$ be domestic. Then $\mathcal{H}$ admits a tilting bundle, and each indecomposable vector bundle is stable and exceptional. The endomorphism rings of tilting bundles (sheaves) are just the (almost) concealed canonical algebras of tame-domestic type.*

(2) *Let k be perfect. A weighted noncommutative regular projective curve with centre k is domestic if and only if the centre curve X is of genus zero and the* weight-ramification vector *of all numbers $p(x)e_{\tau}(x) > 1$, each counted $[k(x):k]$-times, is $(\,)$, (p), (p,q), $(2,2,n)$, $(2,3,3)$, $(2,3,4)$ or $(2,3,5)$.*

Proof. (1) follows from [56, Thm. 6.1 + 6.6]. (2) follows easily from Theorem 13.10. □

Corollary 13.20 (The real domestic zoo). *Let $k = \mathbb{R}$ be the field of real numbers. There are the following* 38 *(families of) weighted noncommutative regular projective curves of positive orbifold Euler characteristic:*

- *Non-weighted ($\bar{p} = 1$). With centre $\mathbb{R}$: the Klein curves $\mathbb{D}$ and $\mathbb{S}^2/\pm$ (sometimes called the real projective plane), the Witt curves $\mathbb{D}_{\mathbb{H}}$ and $\mathbb{D}_{2,2}$. With centre $\mathbb{C}$: the Riemann sphere $\mathbb{S}^2$.*
- *Weighted ($\bar{p} > 1$). With centre $\mathbb{R}$: the* 27 *(families of) curves shown in the tables [47, Appendix A]. With centre $\mathbb{C}$: the weighted projective lines of weight types (p), (p,q), $(2,2,n)$, $(2,3,3)$, $(2,3,4)$ and $(2,3,5)$.*

There are no parameters [47, Prop. A.1.1]. The commutative cases are just the elliptic *and* bad 2*-orbifolds listed in [81, Thm. 13.3.6].* □

Tubular curves. The weighted noncommutative regular projective curves of orbifold Euler characteristic zero (also called the *Calabi–Yau* case) are the noncommutative elliptic curves (non-weighted, $\bar{p} = 1$) and the tubular curves ($\bar{p} > 1$).

Definition 13.21. We call a weighted noncommutative regular projective curve *tubular*, if $\bar{p} > 1$ and $\chi'_{\mathrm{orb}}(\mathcal{H}) = 0$ (equivalently: $g_{\mathrm{orb}}(\mathcal{H}) = 1$).

Theorem 13.22. (1) *If $\mathcal{H} = \mathrm{coh}(\mathbb{X})$ is tubular, then $\mathcal{H}$ admits a tilting bundle, and each indecomposable coherent sheaf is semistable. The endomorphism rings of tilting sheaves are just the tubular algebras. Moreover, for each $\alpha \in \widehat{\mathbb{Q}}$ the full category of semistable sheaves of slope α is a tubular family, again parametrized by a tubular curve $\mathbb{X}'$, with $\mathcal{H}' = \mathrm{coh}(\mathbb{X}')$ derived equivalent to $\mathcal{H}$.*

(2) *Let k be perfect. A weighted noncommutative regular projective curve with $\bar{p} > 1$ and centre k is tubular if and only if the centre curve X is of genus zero and the weight-ramification vector of all numbers $p(x)e_\tau(x) > 1$, each counted $[k(x):k]$-times, is $(2,3,6)$, $(2,4,4)$, $(3,3,3)$ or $(2,2,2,2)$.*

(3) *If $\mathcal{H}$ is tubular over a perfect field, then the order of τ in $\mathrm{Aut}(\mathcal{H})$ is the maximum of the numbers $p(x)e_\tau(x)$.*

(4) *Let k be perfect. The weight sequence and the weight-ramification vector of a tubular curve, with centre k, are derived invariants. The ramification vector is not a derived invariant.*

Proof. (1) is well known. We refer to [47, Ch. 8], also [56, Thm. 5.3].

(2) follows easily from Theorem 13.10, and

(3) from Proposition 13.3 and (13.3).

(4) It is well known that the weight sequence is even a K-theoretic invariant (this follows from [51, Prop. 7.8], also [45]). Clearly the fractional Calabi–Yau dimension by its very definition is a derived invariant, and thus so is the maximum of the numbers $p(x)e_\tau(x)$. Since for all possible weight-ramification vectors, $(2,3,6)$, $(2,4,4)$, $(3,3,3)$ or $(2,2,2,2)$, this maximum is different, it follows that the weight-ramification vector is uniquely determined, in its derived class, by the weight sequence. That the ramification vector is not a derived invariant follows from the real example in [44], see Example 13.24 (c) below. □

Corollary 13.23 (The real tubular zoo). *Let $k = \mathbb{R}$ be the field of real numbers. There are (up to parameters) 39 real weighted noncommutative regular projective curves of orbifold Euler characteristic zero:*

- *Non-weighted ($\bar{p} = 1$). 8 elliptic curves. With centre $\mathbb{R}$: the Klein bottle $\mathbb{K}$, the Möbius band $\mathbb{M}$ (the oval coloured real or quaternion), the annulus $\mathbb{A}$ (there are three possibilities to colour the two ovals). The disc $\mathbb{D}_{2,2,2,2}$ with four segmentation points. With centre $\mathbb{C}$: the torus $\mathbb{T}$.*
- *Weighted ($\bar{p} > 1$). 31 tubular curves. Those 27 with centre $\mathbb{R}$ are shown in the tables [47, Appendix A]; with centre $\mathbb{C}$ there are the tubular weighted projective lines of the 4 weight types $(2,4,4)$, $(2,3,6)$, $(3,3,3)$ and $(2,2,2,2)$.*

17 of these have $s(\mathcal{H}) = 1$ (these are the parabolic *(or* flat*, that is, of curvature zero) 2-orbifolds shown in [81, Thm. 13.3.6], and they correspond to the 17 wallpaper patterns [64, App. A]), and 22 have $s(\mathcal{H}) = 2$. Moreover, all these cases are fractional Calabi–Yau of dimension n/n with n the maximum of the numbers $p(x)e_\tau(x)$, and thus $n = 1, 2, 3, 4$ or 6.* □

The preceding discussion can be regarded as a classification of noncommutative 2-orbifolds of nonnegative Euler characteristic.

Example 13.24. We exhibit two tubular examples, (a) and (b) below, over the field $k = \mathbb{R}$ of real numbers. In both cases the underlying non-weighted curve $\mathcal{H}_{nw}$ is the Witt curve $\mathbb{D}_{2,2}$ in Figure 2. The weight-points z are drawn as large grey spots, their weights $p(z)$ are indicated besides. In both cases, one of the weight-points z is also a segmentation point, the other weighted point is not.

(a) Weight sequence $(3, 3)$. (See Figure 4.) The least common multiple is $\bar{p} = 3$. The second weight-point is real. (The case when the second weight-point is quaternion is similar.) The weight-ramification vector is $(2, 3, 6)$. It follows, that the Calabi–Yau dimension is $\frac{6}{6}$ (and not $\frac{\bar{p}}{\bar{p}} = \frac{3}{3}$).

(b) Same situation, but with weight sequence $(2, 4)$. (See Figure 4.) Here, $\bar{p} = 4$. The weight-ramification vector is $(2, 4, 4)$. Therefore the Calabi–Yau dimension is $\frac{\bar{p}}{\bar{p}} = \frac{4}{4}$.

(c) In a third real example we have a situation of two derived-equivalent tubular curves. The weighted real projective plane $\mathcal{H}$, given by $\mathbb{S}^2/\pm$ with weight sequence $(2, 2)$ (for certain weight points x_1, x_2), is derived-equivalent to $\mathcal{H}'$, given by the disc $\mathbb{D}_{2,2}$ with two weights 2, one on the real coloured boundary, the other on the quaternion coloured boundary. (See Figure 4.) In both cases, the weight sequence is $(2, 2)$ and the weight-ramification vector $(2, 2, 2, 2)$; in case $\mathcal{H}$ each weight appears twice, since $[k(x_i) : k] = 2$; in case $\mathcal{H}'$ the weight sequence $(2, 2)$ is complemented "disjointly" by the ramification indices. The Calabi–Yau dimension is $\frac{2}{2}$. In $\mathcal{H}_0$ there are precisely 2 tubes, on which τ has order 2, in $\mathcal{H}'_0$ there are precisely 4 such tubes.

For further, similar examples we refer to [47, Table A.5].

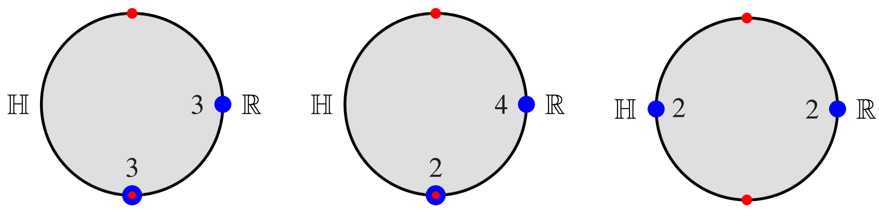

Figure 4. Some tubular cases. Left: (a), middle: (b), right: (c)

Example 13.25. (1) In [47, Prop. 8.3.1] we discussed an example of a triple of tubular curves over the field $k = \mathbb{Q}$ of rational numbers, each with weight sequence (2), which are Fourier–Mukai partners. One derives from Theorem 13.22, or computes directly using (13.9), that in each case the weight-ramification vector is given by $(2, 2, 2, 2)$. Since one of these three tubular curves arises by insertion of weights from the curve in Example 10.11, that curve has three ramification points.

(2) In [47, Prop. 8.4.1] we discussed an example of a tubular curve over the field $\mathbb{Q}(\mathbf{i})$, where $\mathbf{i} = \sqrt{-1}$, also with weight sequence (2). Its Grothendieck group

is isometric-isomorphic to the Grothendieck group of the curves from part (1). By invoking Example 10.9 we see that here the weight-ramification vector is given by $(2, 4, 4)$, in contrast to part (1).

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