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Étale twists in noncommutative algebraic geometry and the twisted Brauer space

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Abstract. This paper studies étale twists of derived categories of schemes and associative algebras. A general method, based on a new construction called the twisted Brauer space, is given for classifying étale twists, and a complete classification is carried out for genus 0 curves, quadrics, and noncommutative projective spaces. A partial classification is given for curves of higher genus. The techniques build upon my recent work with David Gepner on the Brauer groups of commutative ring spectra.

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1. Introduction

1.1. An example. The purpose of this paper is to create a formalism for answering questions of the following kind. Suppose that X is a variety over a field k. How can one classify the k-linear derived categories D such that $\mathrm{D}_{\overline{k}} \simeq \mathrm{D}^b(X_{\overline{k}})$? For the purposes of the following example, please take this problem at face value and believe that there is a good notion of such "derived categories" D together with a way to tensor with $\overline{k}$. This will all be explained later in the introduction and in the rest of the paper.

Allow me to begin the paper with a motivating example. Let $\mathrm{Br}^{\mathbb{P}^1}(\mathbb{R})$ denote the set of (derived equivalence classes of) $\mathbb{R}$-linear derived categories D such that $\mathrm{D}_{\mathbb{C}} \simeq \mathrm{D}^b(\mathbb{P}^1_{\mathbb{C}})$. Thus, $\mathrm{Br}^{\mathbb{P}^1}(\mathbb{R})$ classifies derived categories that are étale locally equivalent to the derived category of $\mathbb{P}^1$. Its objects can be viewed as noncommutative étale twists of the projective line. I call $\mathrm{Br}^{\mathbb{P}^1}(\mathbb{R})$ the $\mathbb{P}^1$-twisted Brauer set of $\mathbb{R}$. It is a pointed set, where the point is the category $\mathrm{D}^b(\mathbb{P}^1_{\mathbb{R}})$.

Consider the real path algebra $\mathbb{R}Q$, where Q is the quiver $\bullet \rightrightarrows \bullet$. It is a result of Beĭlinson [2] that $\mathrm{D}^b(\mathbb{P}^1_{\mathbb{R}})$ and $\mathrm{D}^b(\mathbb{R}Q)$ are equivalent as $\mathbb{R}$-linear triangulated categories.

There are two obvious ways to construct elements in $\mathrm{Br}^{\mathbb{P}^1}(\mathbb{R})$. First, one can tensor any given element with the quaternion algebra $\mathbb{H}$. For instance, $\mathbb{H} \otimes_{\mathbb{R}} \mathbb{R}Q$ gives another $\mathbb{R}$-algebra which becomes Morita equivalent to $\mathbb{C}Q$ over $\mathbb{C}$. Indeed, $\mathbb{C} \otimes_{\mathbb{R}} \mathbb{R}Q \cong \mathbb{C}Q$, while $\mathbb{C} \otimes_{\mathbb{R}} (\mathbb{H} \otimes_{\mathbb{R}} \mathbb{R}Q) \cong \mathrm{M}_2(\mathbb{C}) \otimes_{\mathbb{C}} \mathbb{C}Q \cong \mathrm{M}_2(\mathbb{C}Q)$. Thus, $\mathrm{D}^b(\mathbb{H} \otimes_{\mathbb{R}} \mathbb{R}Q)$ is an element of $\mathrm{Br}^{\mathbb{P}^1}(\mathbb{R})$. This derived category has a more geometric interpretation: it is the derived category $\mathrm{D}^b(\mathbb{P}^1_{\mathbb{R}}, \alpha)$ of α-twisted coherent sheaves on $\mathbb{P}^1$, where α is the class in $\mathrm{Br}(\mathbb{P}^1_{\mathbb{R}})$ pulled back from $\mathbb{H}$. Equivalently, $\mathrm{D}^b(\mathbb{P}^1_{\mathbb{R}}, \alpha)$ is the derived category of quaternionic vector bundles on $\mathbb{P}^1_{\mathbb{R}}$. Since $\mathrm{Br}(\mathbb{R}) = \mathbb{Z}/2 \cdot \mathbb{H}$, there are no further iterations of the construction.

The algebras $\mathbb{R}Q$ and $\mathbb{H} \otimes_{\mathbb{R}} \mathbb{R}Q$ represent distinct elements in the pointed set $\mathrm{Br}^{\mathbb{P}^1}(\mathbb{R})$. The easiest way to see this is via algebraic K-theory. The K-theory of $\mathbb{P}^1$ (or equivalently of $\mathbb{R}Q$) is $\mathrm{K}_*(\mathbb{R}) \oplus \mathrm{K}_*(\mathbb{R})$ by Quillen's computation [31, Theorem 8.2.1], while the K-theory of $\mathbb{H} \otimes_{\mathbb{R}} \mathbb{R}Q$ is $\mathrm{K}_*(\mathbb{H}) \oplus \mathrm{K}_*(\mathbb{H})$. The torsion part of $\mathrm{K}_1(\mathbb{R}) \cong \mathbb{R}^\times$ is $\mathbb{Z}/2$, while the torsion part of $\mathrm{K}_1(\mathbb{H}) \cong \mathbb{H}^\times/[\mathbb{H}^\times, \mathbb{H}^\times]$ is 0, where $[\mathbb{H}^\times, \mathbb{H}^\times]$ is the commutator subgroup of $\mathbb{H}^\times$. The point is that the reduced norm $\mathrm{K}_1(\mathbb{H}) \to \mathrm{K}_1(\mathbb{R})$ is injective by the theorem of Wang [42]. But, clearly, $-1 \in \mathbb{R}^*$ cannot be the reduced norm of a quaternion. Thus, $\mathbb{H} \otimes_{\mathbb{R}} \mathbb{R}Q$ and $\mathbb{R}Q$ are not derived Morita equivalent.

The second obvious way to construct elements in $\mathrm{Br}^{\mathbb{P}^1}(\mathbb{R})$ is to look at another variety over $\mathrm{Spec}\,\mathbb{R}$ that becomes isomorphic to $\mathbb{P}^1$ over $\mathrm{Spec}\,\mathbb{C}$. Up to isomorphism, there is only one such variety, which is the genus 0 curve C cut out by $x^2+y^2+z^2 = 0$ in $\mathbb{P}^2_{\mathbb{R}}$. Since this curve does not have an $\mathbb{R}$-point, it is not the projective line, but it becomes isomorphic to $\mathbb{P}^1$ over $\mathbb{C}$. Thus $\mathrm{D}^b(C)$ represents another point of $\mathrm{Br}^{\mathbb{P}^1}(\mathbb{R})$. Interestingly, in this case, considering α-twisted sheaves gives nothing new. Because C is the Severi–Brauer variety of $\mathbb{H}$, the pullback of $\mathbb{H}$ to C has zero Brauer class. Thus, $\mathrm{D}^b(C, \alpha) \simeq \mathrm{D}^b(C)$. To see that $\mathrm{D}^b(C)$ is distinct from either of the module categories from the previous paragraph, note that its K-theory is isomorphic to $\mathrm{K}_*(\mathbb{R}) \oplus \mathrm{K}_*(\mathbb{H})$ by Quillen's computation of the K-theory of Severi–Brauer varieties [31, Theorem 8.4.1], and this is different from either of the other K-theories, by consideration of torsion in degree 1.

Thus, there are at least 3 elements of $\mathrm{Br}^{\mathbb{P}^1}(\mathbb{R})$, and there is an action on these elements by $\mathrm{Br}(\mathbb{R})$, which is described above. The main point of this paper is to develop methods that will allow a precise formulation of the problems of the type posed in the example, and to give a computational tool for solving these problems, which I apply in many cases. In particular, in Section 3.3 this computational tool will be used to show that there are no other elements in $\mathrm{Br}^{\mathbb{P}^1}(\mathbb{R})$ besides those described already.

Every element of $\mathrm{Br}^{\mathbb{P}^1}(\mathbb{R})$ is represented by a category of modules over an associative algebra. This has already been remarked upon for $\mathrm{D}^b(\mathbb{P}^1_{\mathbb{R}})$ and $\mathrm{D}^b(\mathbb{P}^1_{\mathbb{R}}, \alpha)$. For the genus 0 curve C, there is an equivalence $\mathrm{D}^b(C) \simeq \mathrm{D}^b(A)$, where A is the path algebra of the modulated quiver $\mathbb{R} : \bullet \rightrightarrows \bullet : \mathbb{H}$. Modulated quivers were used to classify finite dimensional hereditary algebras of finite representation type. For details, see Dlab–Ringel [13], where they are called species.

1.2. Overview. The noncommutative algebraic geometry of the title is what Ginzburg has called noncommutative algebraic geometry "in the large," where one replaces schemes with derived categories of sheaves and isomorphisms with derived Morita equivalences. This form of noncommutative algebraic geometry, which began with the work of Beĭlinson [2], has been reinforced by ideas originating in string theory, where two varieties with equivalent derived categories should describe the same physical theory. The mathematical theory has been pursued by Bondal, Ginzburg, Kontsevich, Orlov, Rosenberg, and van den Bergh to name just a few. See [4–6, 15, 23, 41]. Thus, if A is an associative algebra, the derived category of A-modules $\mathrm{D}(A)$ is viewed as a geometric object. Noncommutative algebraic geometry in the large is distinct from both noncommutative algebraic geometry in the small and derived algebraic geometry. The former is about noncommutative deformations of commutative rings and is modeled on the coordinate algebras that arise in quantum mechanics. Derived algebraic geometry on the other hand replaces ordinary commutative rings with "derived" commutative rings, which are either simplicial commutative rings, commutative dg algebras, or commutative ring spectra.

This viewpoint is motivic in the sense that many classical motivic invariants, such as Hochschild homology and K-theory, depend only on the derived category.

To formulate this kind of geometry correctly requires a more flexible framework than simply triangulated categories. Thus, $\mathrm{D}(A)$ is replaced by Mod_A, the stable ∞-category of right A-modules, and the triangulated category of complexes of $\mathcal{O}_X$-modules is replaced by Mod_X, a stable ∞-categorical model for $\mathrm{D}_{\mathrm{qc}}(X)$. Another option would be to use dg enhancements or A_∞-categories. Stable ∞-categories include all of these examples, and have the added benefit that, for instance, one can do geometry over the sphere spectrum.

Since I am interested in developing a theory that works over the sphere, my commutative rings will be connective commutative ($\mathbb{E}_\infty$-)ring spectra, and my associative rings will be A_∞-ring spectra. The reader will lose little in thinking of ordinary commutative rings and associative dg algebras. But, in any case, a module over a ring A, even an ordinary associative ring, means an A_∞-module. So, over an ordinary associative ring, modules are really complexes of ordinary A-modules.

Recall that a compact object in a stable ∞-category M is an object x such that the mapping space functor $\mathrm{map}_{\mathrm{M}}(x, -)$ commutes with filtered colimits. This is the appropriate generalization of compactness in triangulated categories having all coproducts, where a compact object x is one where taking maps out commutes with coproducts. Compact objects are the cornerstone of noncommutative algebraic geometry. When X is a quasi-compact and quasi-separated scheme, the compact objects of Mod_X are precisely the perfect complexes, which are complexes of

$\mathcal{O}_X$-modules locally quasi-isomorphic to bounded complexes of finitely generated vector bundles.

The compact objects are fundamental to derived Morita theory. If A and B are two associative algebras, then to give an equivalence $F : \mathrm{Mod}_A \tilde{\to} \mathrm{Mod}_B$ is to give a compact right B-module $F(A)$ such that $F(A)$ generates Mod_B and $\mathrm{End}_B(F(A)) \simeq A$. Note that I will use derived equivalence for any equivalence between stable ∞-categories, and not for a triangulated equivalence $\mathrm{D}(A) \simeq \mathrm{D}(B)$, although once a functor $F : \mathrm{Mod}_A \to \mathrm{Mod}_B$ is given, the property of it being an equivalence can be detected on the homotopy categories.

When X is a reasonable scheme (quasi-compact and quasi-separated), Bondal and van den Bergh [4] showed that there is a single perfect complex E that generates the entire derived category $\mathrm{D}_{\mathrm{qc}}(X)$. Thus, derived Morita theory says that at the level of ∞-categories, there is an equivalence $\mathrm{Mod}_X \simeq \mathrm{Mod}_A$, where $A = \mathrm{End}_{\mathrm{Mod}_X}(E)^{\mathrm{op}}$ is the derived endomorphism algebra spectrum of E. The example of Beĭlinson's, that $\mathrm{D}^b(\mathbb{P}^1) \simeq \mathrm{D}^b(\mathbb{R}Q)$, from the previous section is an especially nice example of this phenomenon. In particular, the algebra A is typically truly an A_∞-algebra, and is not derived Morita equivalent to any ordinary associative algebra. Bondal and ven den Bergh's theorem justifies the term noncommutative algebraic geometry. Almost *every* derived category that arises in ordinary algebraic geometry is the module category for an A_∞-algebra, or is built from such a category.

Therefore, from the perspective of noncommutative algebraic geometry, derived categories of algebras are a natural generalization of derived categories of schemes. Thus, the first question to ask is when two algebras or schemes give rise to the same noncommutative geometric object. For algebras, the answer, abstractly, is the subject of derived Morita theory, which goes back to Cline–Parshall–Scott [11], Happel [19], and Rickard [32], and has been developed by many people for use in the study of finite-dimensional associative algebras and in block theory for modular representation theory. In the dg setting, Keller [22] and Toën [38] have worked out the theory very nicely. For ring spectra, the theory follows from work of Schwede and Shipley [35]. The problem of when two varieties X and Y are derived equivalent has been the subject of a great deal of research by Bondal, Bridgeland, Huybrechts, Kawamata, Orlov, Stellari, van den Bergh, and many, many others. For a comprehensive introduction to the subject and the literature, see [20].

Now that there is an excellent categorical framework for studying derived equivalences, and since the work of many authors has provided a clear picture of when to expect derived equivalences, the follow-up question I want to ask in this paper is: is it possible to classify when two algebras A and B, say over a field k, represent the same geometric object over $\overline{k}$? In fact, in general, it is better to ask for a finite separable extension l/k such that A_l and B_l are derived Morita equivalent. The analogous question for potentially infinite or inseparable extensions is considered in a special case in Section 4.

Problem 1.1. *Let A be an A_∞-algebra over k. Classify, up to derived equivalence over k, all A_∞-algebras B such that $A \otimes_k k^{\mathrm{sep}}$ and $B \otimes_k k^{\mathrm{sep}}$ are derived Morita equivalent.*

When $\mathrm{Mod}_A \simeq \mathrm{Mod}_X$, the algebras B should be viewed as noncommutative étale twists of X. The rest of this paper develops a tool, the twisted Brauer space, to solve the problem. In various concrete examples, the twisted Brauer space will turn out to encode interesting geometric and arithmetic information about X. Perhaps the central thesis is that while this problem would be intractable using triangulated categories, by using stable ∞-categories one is able to give a precise answer which moreover accords with our intuition: twists are classified by 1-cocycles in automorphisms. There is a subtlety, which is that in this setting the automorphisms really form a topological space, and so twists are classified by 1-cocycles in a sheaf of spaces. That this can be made precise is a triumph of the work of Lurie, Toën, and others on ∞-categories.

One might ask to classify more generally all stable ∞-categories M such that $\mathrm{M}_{k^{\mathrm{sep}}} \simeq \mathrm{Mod}_{A \otimes_k k^{\mathrm{sep}}}$. An important structural theorem due to Toën [39] in the simplicial commutative setting and Antieau–Gepner [1] in the $\mathbb{E}_\infty$-setting shows that these classification problems are the same: any such M is already a module category for some k-algebra B.

Note that parts of the problem of classifying étale twists have already been studied. For instance, if two schemes X and Y become isomorphic over k^{sep}, then Mod_Y is a twisted form of Mod_X. Thus, the cohomology set $\mathrm{H}^1_{\text{ét}}(\mathrm{Spec}\, k, \mathbf{Aut}_X)$, which classifies étale twists of X as a scheme, contributes to the answer of the problem. If two varieties X and Y are derived equivalent, then étale twists of each of Mod_X and Mod_Y give different interpretations for the answer.

Besides the case of schemes, another version of this problem is very well known, although possibly in a different guise. Suppose one attempts to find ordinary k-algebras A such that $A \otimes_k \overline{k}$ is Morita equivalent to $\overline{k}$. Then, every such algebra A is Morita equivalent to a central simple division algebra D over k. So, the Brauer group $\mathrm{Br}(k)$ classifies these algebras. This remains true in the derived world: every A_∞-algebra A such that $A \otimes_k \overline{k}$ is derived Morita equivalent to $\overline{k}$ is derived Morita equivalent to a central division algebra over k. This result is due to Toën [39]. The Brauer group again has a cohomological interpretation: it is $\mathrm{H}^2_{\text{ét}}(k, \mathbb{G}_m)$.

Of course, there is no reason to settle for classifying algebras over k. One can also attempt to classify algebras over a scheme X. So, consider the problem of classifying sheaves of quasi-coherent A_∞algebras A over X such that there is an étale cover $p : U \to X$ where $\mathcal{M}\mathrm{od}_{p^*A} \simeq \mathcal{M}\mathrm{od}_U$, where this is an equivalence of U-stacks of module categories. The derived Brauer group of X is obtained by taking all such algebras and taking the quotient by derived Morita equivalence of X-stacks. It turns out that the derived Brauer group is computable with cohomological methods. When X is an ordinary scheme, the derived Brauer group is $\mathrm{H}^2_{\text{ét}}(X, \mathbb{G}_m) \times \mathrm{H}^1_{\text{ét}}(X, \mathbb{Z})$.

If I only cared about ordinary algebras, there would be a problem at this point: for some quasi-compact and quasi-separated schemes, not every derived Brauer class is the class of an ordinary algebra (see [1, Section 7.5]).

My point in the previous paragraph is simply that in order to obtain a cohomological classification, which might be amenable to computation, of Azumaya algebras, it is important to allow A_∞-algebras.

The examples of étale twists of schemes and of the Brauer group show that the solution to Problem 1.1 should be very interesting, and that it should be in some way cohomological. As in the example of the previous section, it is frequently easy to construct some examples, but showing that they are exhaustive is much more difficult, and this is why cohomological methods are important. Such methods are already required to show, for instance, that $\mathrm{Br}(\mathbb{Z}) = 0$ (see [16]).

1.3. The twisted Brauer space. Let me describe the main tool of this paper in a special case. Let R be a commutative ring (or a connective commutative ring spectrum), and let A be an R-algebra (hence, an A_∞-ring).

Theorem 1.2. *There is a sheaf of spaces* $\mathbf{Br}^A$ *on the étale site of* $\operatorname{Spec} R$ *with homotopy sheaves*

$$\pi_i^s \mathbf{Br}^A \cong \begin{cases} 0 & \text{if } i = 0, \\ \mathbf{Aut}_{\mathrm{Mod}_A} & \text{if } i = 1, \\ \mathbf{HH}_R^0(A)^\times & \text{if } i = 2, \\ \mathbf{HH}_R^{2-i}(A) & \text{if } i \geq 3, \end{cases}$$

where $\mathbf{HH}_R^*(A)$ *is the Hochschild cohomology sheaf of* A *over* $\operatorname{Spec} R$. *There is a fringed spectral sequence*

$$\mathrm{E}_2^{p,q} = \mathrm{H}^p_{\text{ét}}(\operatorname{Spec} R, \pi_q^s \mathbf{Br}^A) \Rightarrow \pi_{q-p} \mathbf{Br}^A(R),$$

which converges completely (in the sense of fringed spectral sequences) when either A *is a smooth and proper* R*-algebra or* A *and* R *are ordinary rings. The set* $\mathrm{Br}^A(R) = \pi_0 \mathbf{Br}^A(R)$ *solves Problem 1.1. Namely, every* R*-algebra* B *such that* B *is étale locally derived Morita equivalent to* A *determines a point of the space* $\mathbf{Br}^A(R)$ *and conversely. Two points* B_0 *and* B_1 *are connected by a path if and only if* $\mathrm{Mod}_{B_0} \simeq \mathrm{Mod}_{B_1}$. *Moreover, there is an action of the derived Brauer group* $\mathrm{Br}(R)$ *on* $\mathrm{Br}^A(R)$. *If* Z *is a derived Azumaya* R*-algebra, then* $[Z] \cdot [B] = [Z \otimes_R B]$.

The twisted Brauer space and the spectral sequence are generalizations of the Brauer space and spectral sequence developed in Antieau–Gepner [1]. Besides having a computational tool to compute twists, the twisted Brauer space together with its action of the (untwisted) Brauer space carries a large amount of arithmetic information. For instance, the stabilizer of Mod_C in $\mathbf{Br}^{\mathbb{P}^1}(k)$, where C is a smooth projective genus 0 curve has enough information to determine over which fields C has rational points.

When R is a connective ring spectrum, the Brauer space $\mathbf{Br}(R)$ is a 2-fold delooping of the units spectrum $R^{\times}$. When A is an R-algebra the space $\mathbf{Br}^A(R)$ should be viewed as a 2-fold delooping of the spectrum of units in $\mathrm{HH}^*_R(A)$. Note that this is exactly the correct amount of delooping. Since A is an A_∞-algebra, which is the same as being an $\mathbb{E}_1$-algebra, its Hochschild cohomology $\mathrm{HH}^*_R(A)$ is an $\mathbb{E}_2$-algebra by Deligne's conjecture, which has been proven by many authors; see [27, Section 6.1.4]. So, the spectrum of units is a 2-fold loopspace. The twisted Brauer space construction has the same formal properties of $\mathbf{Br}(R)$. For example, the group $\mathrm{Aut}_{\mathrm{Mod}_A}$ is the derived Picard group of A; that is, it is the group of invertible (complexes of) (A, A)-bimodules. The derived Picard group was introduced by Rickard [33], and has been studied extensively, by Miyachi–Yekutieli [29] and Rouquier–Zimmermann [34].

When R is an ordinary commutative ring and A is an ordinary associative R-algebra, or X is an ordinary R-scheme, then $\mathrm{HH}^{2-i}_R(A) = 0$ (resp. $\mathrm{HH}^{2-i}_R(X) = 0$) for $i \geq 3$, since one can create projective (resp. locally free) resolutions.

The spectral sequence is used to show that $\mathrm{Br}^C(\mathbb{R})$ does indeed have exactly three elements, to classify noncommutative étale twists of curves and quadric hypersurfaces, and to classify twists of a certain path algebra, which corresponds to noncommutative projective space. These last twists lead to noncommutative Severi–Brauer varieties.

Let me explain briefly two of these examples.

Given an elliptic curve E/k, there are three interesting groups that act on $\mathrm{D}^b(E)$. The first is the automorphism group of E as a variety, which is an extension of the automorphism group of E as an elliptic curve (a finite group) by E acting on itself acting via translation. The twists by this action are homogeneous spaces for twists of E as an elliptic curve. The curve E also acts on $\mathrm{D}^b(E)$ by viewing it as the moduli space of line bundles of degree 0 over E. The action is then given by tensoring with line bundles. Twists by this action lead to the twisted derived categories $\mathrm{D}^b(E, \alpha)$ for $\alpha \in \mathrm{Br}(E)$. This makes sense as every such class α is killed by passage to the algebraic closure of k.

But, there is a final group acting on $\mathrm{D}^b(E)$, which is $\widetilde{\mathrm{SL}}_2(\mathbb{Z})$, an extension of $\mathrm{SL}_2(\mathbb{Z})$ by $\mathbb{Z}$. It follows that modular representations in $\mathrm{SL}_2(\mathbb{Z})$ give rise to twists of $\mathrm{D}^b(E)$. Unlike in the other two cases, this action does not preserve the natural t-structure on $\mathrm{D}^b(E)$, and hence the twists are truly derived. The interesting point is that *every* twist of $\mathrm{D}^b(E)$ is "built out of" four things: central simple algebras over k, homogeneous spaces over twists of E as an elliptic curve, the abelian categories mentioned above, and the derived categories associated to modular representations.

The quiver Ω_n consists of two points a and b and n arrows from a to b. Kontsevich and Rosenberg showed that the path algebra $k\Omega_n$ is derived equivalent to the derived category of coherent sheaves on noncommutative projective space $\mathbb{NP}^{n-1}$. For $n \geq 3$, $\mathbb{NP}^{n-1}$ and $\mathbb{P}^{n-1}$ are not derived equivalent, so these spaces are new from the perspective of noncommutative algebraic geometry above. However, Miyachi and

Yekutieli [29, Corollary 0.4] computed the automorphisms of the derived category of $k\Omega_n$, showing that it is an extension of $\mathbf{PGL}_n(k)$. Using their calculation, the work below shows that there is one twist of $\mathrm{D}^b(k\Omega_n)$ for each classical Severi–Brauer variety over k. Thus, the twists of $k\Omega_n$ are noncommutative Severi–Brauer varieties.

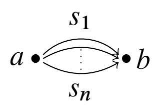

Figure 1. The quiver Ω_n.

By restricting attention to simplicial commutative rings, as for instance used by Toën [39] and Toën–Vaquié [40], it is possible to use the fppf topology instead of the étale topology. The theory below carries over without change to the simplicial setting.

In Section 2, the necessary background is reviewed and the definition and first properties of the twisted Brauer space are studied. The spectral sequence that computes the homotopy of the twisted Brauer space is constructed in Section 3. This is used to give a complete description of $\mathrm{Br}^{\mathbb{P}^1}(\mathbb{R})$. In Section 4, the problem of when it is enough to check derived Morita equivalence over $\overline{k}$ is considered. Several examples are studied in Section 5.

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2. The twisted Brauer space

2.1. Derived Morita theory. Recall from the introduction that if A is an A_∞-algebra, then Mod_A denotes the stable ∞-category of right A-modules. This is a large ∞-category: it is complete and cocomplete. The subcategory $\mathrm{Mod}_A^{\mathrm{c}}$ is the small stable ∞-category of compact A-modules. For a scheme X, Mod_X denotes the stable ∞-category of complexes of $\mathcal{O}_X$-modules with quasi-coherent cohomology sheaves. In this case, $\mathrm{Mod}_X^{\mathrm{c}}$, the subcategory of compact objects, is the same as the ∞-category of perfect complexes on X, at least when X is quasi-compact and quasi-separated. See [4].

The ∞-categories Mod_A and Mod_X, besides being stable, are also presentable ∞-categories, which is equivalent to saying that their homotopy categories have all coproducts, are locally small, and are κ-compactly generated for some regular cardinal κ (see [27, Corollary 1.4.4.2]. This fact follows from Lurie [27] in the case of Mod_A, and from [4] when X is quasi-compact and quasi-separated, because in that case $\mathrm{Mod}_X \simeq \mathrm{Mod}_A$ for an appropriate choice of A. Presentability ensures that Mod_A and Mod_X have all small limits and colimits and that they can be described by a set of generators in a reasonable way. For details, see [25, Chapter 5]. If A is an R-algebra, where R is a commutative ring spectrum, then Mod_A is enriched over R, in the sense that the mapping spectra in Mod_A are naturally R-modules. By an R-linear category, I will mean a stable presentable categories and whose morphisms are R-linear functors that have right adjoints.

There are two points of derived Morita theory to bear in mind for the paper below. First, if A and B are R-algebras, then any R-linear functor $F : \mathrm{Mod}_A \to \mathrm{Mod}_B$ in Cat_R is determined by the $A^{\mathrm{op}} \otimes B$-module $F(A)$. Moreover, there are natural equivalences

$$\mathrm{Fun}_R^{\mathrm{L}}(\mathrm{Mod}_A, \mathrm{Mod}_B) \simeq \mathrm{Mod}_{A^{\mathrm{op}}} \otimes_{\mathrm{Mod}_R} \mathrm{Mod}_B \simeq \mathrm{Mod}_{A^{\mathrm{op}} \otimes_R B},$$

where $\mathrm{Fun}_R^{\mathrm{L}}(-,-)$ denotes the functor ∞-category of left adjoint R-linear functors.

The second point is that if E is a compact generator of any R-linear stable ∞-category M, then the mapping spectrum out of E induces an equivalence

$$\mathrm{Map}_{\mathrm{M}}(E,-) : \mathrm{Mod}_A \to \mathrm{Mod}_{\mathrm{End}_R(E)^{\mathrm{op}}}$$

by Schwede–Shipley [35]. The converse is also true. In particular, the result of Bondal and van den Bergh says that there is a compact generator of Mod_X when X is quasi-compact and quasi-separated, so $\mathrm{Mod}_X \simeq \mathrm{Mod}_A$ for some A_∞-algebra A.

These results should be compared in two directions to more familiar facts. First, they are essentially a translation into the world of stable ∞-categories of facts that are true for abelian categories of modules, from which the appellation Morita originated. Second, for a scheme, functors $\mathrm{Mod}_X \to \mathrm{Mod}_Y$ are determined by complexes on $X \times Y$. For fully faithful functors, Orlov proved this result for functors of the derived categories $\mathrm{D}^b(X) \to \mathrm{D}^b(Y)$. A the level of ∞-categorical models, it is due to Ben-Zvi, Francis, and Nadler [3], while Toën proves it for dg models [38].

As a last point in this section of background, if M is an R-linear category, and if S is a commutative R-algebra, then one can base-change M up to S via $\mathrm{M}_S = \mathrm{Mod}_S \otimes_{\mathrm{Mod}_R} \mathrm{M}$.

2.2. The definition. Let R be a connective commutative ring spectrum, and let $\mathrm{Shv}_R^{\text{ét}}$ be the big étale topos over $\mathrm{Spec}\, R$. If S is a connective commutative R-algebra, an S-linear category M is said to satisfy étale hyperdescent if for every connective commutative S-algebra T and every étale hypercover $T \to U^\bullet$ of T, the induced morphism

$$\mathrm{M}_T \to \lim_{\Delta} \mathrm{M}_{U^\bullet}$$

is an equivalence. There is a stack of large ∞-categories $\mathcal{C}\mathrm{at}^{\mathrm{desc}}$ over Spec R that classifies linear categories with étale hyperdescent and left adjoint functors between them [26, Theorem 7.5]. Write **Pr** for the underlying sheaf of spaces. For details, see Antieau–Gepner [1, Section 6].

Suppose now that Z is in $\mathrm{Shv}_R^{\text{ét}}$, and let $\alpha : Z \to \mathbf{Pr}$ be a map of sheaves. The corresponding linear category with descent, or, equivalently, stack of linear categories, will be denoted $\mathcal{M}\mathrm{od}^{\alpha}$. The ∞-category of sections over $f : \mathrm{Spec}\, S \to Z$ is the S-linear category $\mathrm{Mod}_S^{f\circ\alpha}$ classified by $f\circ\alpha$ by Yoneda's lemma. By definition, the ∞-category Mod_X^{α} of sections over a sheaf X over Z is

$$\mathrm{Mod}_X^{\alpha} = \lim_{f:\mathrm{Spec}\, S\to X} \mathrm{Mod}_S^{f\circ\alpha}.$$

For instance, let $\mathcal{O} : \mathrm{Spec}\, R \to \mathbf{Pr}$ send Spec S to Mod_S. Then, $\mathrm{Mod}_X^{\mathcal{O}}$ is the stable ∞-category of quasi-coherent $\mathcal{O}_X$-modules. The properties of this construction of sheaves have been studied extensively in [3, 26], and [1].

For an object $f : X \to Z$ of $\mathrm{Shv}_Z^{\text{ét}} = \left(\mathrm{Shv}_R^{\text{ét}}\right)_{/Z}$, there is a pullback stack $f^*\alpha$. Say that a stack of linear categories $\beta : X \to \mathbf{Pr}$ over X is étale locally equivalent to $f^*\alpha$ if there is an étale cover $p : U \to X$ such that $p^*\beta \simeq p^*f^*\alpha$ as stacks of linear categories over U. There is a subspace $\mathbf{Br}^{\alpha}(X)$ of $\mathbf{Pr}(X)$ of stacks of linear categories that are étale locally equivalent to $f^*\alpha$.

Lemma 2.1. *The presheaf* $\mathbf{Br}^{\alpha}$ *on* $\mathrm{Shv}_Z^{\text{ét}}$ *is an étale sheaf.*

Proof. The presheaf is the same as the sheafification of the point α in $\mathbf{Pr}|_Z$. □

Definition 2.2. The sheaf of spaces $\mathbf{Br}^{\alpha}$ is called the α-twisted Brauer sheaf. For a sheaf X, $\mathbf{Br}^{\alpha}(X)$ is the α-twisted Brauer space of X. The pointed set $\pi_0\mathbf{Br}^{\alpha}(X)$ is the α-twisted Brauer set of X, and it will be written $\mathrm{Br}^{\alpha}(X)$ in the sequel.

To summarize in a fast and loose way in a familiar setting, if X is a k-variety, where k is a field, and if A is an ordinary associative k-algebra, then the twisted Brauer set $\mathrm{Br}^{A}(X)$ classifies sheaves quasi-coherent dg algebras $\mathcal{B}$ that are étale locally derived Morita equivalent on X to $\mathcal{O}_X \otimes_k A$. This is fast because it has yet to be observed that elements of $\mathrm{Br}^{A}(X)$ actually correspond to algebras, although this is true; see the next section. The only looseness in this description is that the étale-local Morita equivalence is an equivalence of the *stacks* of modules. See Remark 2.7 at the end of the section.

For example, if $\mathcal{O}$ classifies the stack of quasi-coherent modules over Z, then $\mathbf{Br}^{\mathcal{O}} = \mathbf{Br}$, the Brauer sheaf studied in [1].

Example 2.3. Suppose that A is an associative S-algebra. Then, the stack $\mathcal{M}\mathrm{od}^{A}$ is the stack of linear categories whose ∞-category of sections over a connective commutative S-algebra T is $\mathrm{Mod}_{T\otimes_S A}$. In this case, the twisted Brauer sheaf is denoted $\mathbf{Br}^{A}$.

Example 2.4. Suppose that X is a scheme over Spec S. Then, $\mathcal{M}\mathrm{od}^X$ is the stack of linear categories over Spec S whose category of sections over T is Mod_{X_T}, where $X_T = X \times_{\mathrm{Spec}\, S} \mathrm{Spec}\, T$. Here Mod_{X_T} is the stable T-linear ∞-category with homotopy category equivalent to $\mathrm{D}_{\mathrm{qc}}(X_T)$, the derived category of complexes of $\mathcal{O}_{X_T}$-modules with quasi-coherent cohomology. This slightly unusual notation is meant to emphasize that $\mathcal{M}\mathrm{od}^X$ is viewed not as a stack over X but over Spec S. The associated twisted Brauer sheaf is $\mathbf{Br}^X$. Note that if X is quasi-compact and quasi-separated, then by the results of [4], this is a special case of the previous example. When $X \to \mathrm{Spec}\, S$ is smooth, then the elements of $\mathbf{Br}^X$ may viewed as étale twists of $\mathrm{D}^b(X)$. In general, they should be viewed as either twists of Mod_X or Perf_X.

It is not clear that, in general, $\mathbf{Br}^\alpha$ is a sheaf of small spaces. However, in most cases of interest, and all cases considered in this paper, it is. To prove this, we need a lemma first, which will be of use later in the paper for computing twisted Brauer spaces.

Lemma 2.5. *The sheaf* $\mathbf{Br}^\alpha$ *is equivalent to the classifying sheaf of the sheaf of autoequivalences of the stack* α*.*

Proof. By definition, any two points of $\mathbf{Br}^\alpha(X)$ are étale locally connected. It follows that the homotopy sheaf $\pi_0^s \mathbf{Br}^\alpha$ is just a point. There is an obvious morphism $\mathbf{Baut}(\alpha) \to \mathbf{Br}^\alpha$. So, it suffices to compute the homotopy sheaves of the loopspace $\Omega \mathbf{Br}^\alpha$ at the point α. But, these are just the equivalences from the stack α to α, as desired. □

We say that $\alpha : Z \to \mathbf{Pr}$ classifies a stack of compactly generated linear categories if Mod_S^α is compactly generated for every $\mathrm{Spec}\, S \to Z$ and every connective commutative R-algebra S. Note that this hypothesis does not imply that, for instance, Mod_Z^α is compactly generated. However, if Z is a quasi-compact and quasi-separated derived scheme, then the methods of Lurie [26, Section 6] can be used to show that Mod_Z^α is compactly generated.

Proposition 2.6. *Suppose that* α *classifies a stack of compactly generated linear categories over a sheaf* Z*. Then,* $\mathbf{Br}^\alpha$ *is a sheaf of small spaces.*

Proof. By the previous lemma, it is enough check that $\mathbf{aut}(\alpha)$ is a sheaf of small spaces, which we can check on affines $\mathrm{Spec}\, S \to Z$, and by hypothesis Mod_S^α is compactly generated. The space of sections over $\mathrm{Spec}\, S \to Z$ is a subspace of the ∞-category of functors $\mathrm{Mod}_S^\alpha \to \mathrm{Mod}_S^\alpha$ that preserve the subcategory of compact objects $\mathrm{Mod}_S^{\alpha,c}$. Thus, it is a subspace of functors $\mathrm{Mod}_S^{\alpha,c} \to \mathrm{Mod}_S^{\alpha,c}$, where $\mathrm{Mod}_S^{\alpha,c}$ is the full subcategory of Mod_S^α of compact objects. Write $-\alpha$ for the opposite linear category. Then, $\mathbf{aut}(\alpha)(S)$ is a subspace of $\mathrm{Mod}_S^{-\alpha,c} \otimes \mathrm{Mod}_S^{\alpha,c}$, which is a small idempotent complete stable ∞-category. It follows that $\mathbf{aut}(\alpha)$ is a sheaf of small spaces. □

Remark 2.7. It is worth noting that there is a subtlety required when defining the Brauer group via Morita equivalences. If X is a scheme with an automorphism σ such that $\sigma^*\alpha \neq \alpha$ for some Brauer class α, then it is clear that the categories of α-twisted coherent sheaves Mod_X^{α} and $\mathrm{Mod}_X^{\sigma^*\alpha}$ are equivalent. So, the Brauer group is a finer invariant than this sort of derived equivalence. I learned of this issue from Căldăraru's thesis [9, Example 1.3.16]. However, the underlying stacks $\mathcal{M}\mathrm{od}^{\alpha}$ and $\mathcal{M}\mathrm{od}^{\sigma^*\alpha}$ are *not* equivalent over X. The Brauer group classifies Azumaya algebras up to the Morita equivalence classes of the stacks of their modules. This perspective is implicit in [39] and [1]. The hypothesis $\mathrm{Mod}_X^{\alpha} \simeq \mathrm{Mod}_X^{\sigma^*\alpha}$ is rather strange from the perspective of stacks: it is like saying that two coherent sheaves have isomorphic vector spaces of global sections. The correct definition of stacky Morita equivalence is built into $\mathbf{Br}^{\alpha}(X)$.

2.3. Algebras and the twisted Brauer space. If $\alpha : Z \to \mathbf{Pr}$ is a stack of linear categories, and if $f : X \to Z$ is a map of sheaves, then an object x of Mod_X^{α} is perfect if its restriction to Mod_T^{α} for every $\mathrm{Spec}\, T \to X$ compact. A set of perfect objects Γ perfectly generates Mod_X^{α} if its restriction to every affine is a set of compact generators. It globally generates if it perfectly generates, if the objects are compact, and if it generates Mod_X^{α}.

Lurie has shown in [26, Theorem 6.1] that if Mod_S^{α} is an S-linear category with descent that is étale locally compactly generated, then Mod_S^{α} is globally generated. In [1, Theorem 6.17], Gepner and I showed that if Mod_S^{α} is étale locally compactly generated by a *single* compact object, then Mod_S^{α} is globally generated by a single compact object. Toën proved similar theorems for simplicial commutative rings in [39], although with somewhat different methods.

Proposition 2.8. *Suppose that* $\alpha : Z \to \mathbf{Pr}$ *classifies a stack of linear categories, and fix a morphism* $f : X \to Z$*, where* X *is a quasi-compact and quasi-separated derived scheme. Then, if* Mod_X^{α} *is globally generated, so is* Mod_X^{β} *for every* $\beta \in \mathbf{Br}^{\alpha}(X)$*. Similarly, if* Mod_X^{α} *is globally generated by a single object, then so is* Mod_X^{β} *for every* $\beta \in \mathbf{Br}^{\alpha}(X)$*.*

Proof. These statements follow from Lurie [26, Theorem 6.1] and Antieau–Gepner [1, Theorem 6.17], respectively. For instance, if Mod_X^{α} is globally generated by a single object, then, using the étale local equivalence of α and β over X, it follows that Mod_X^{β} is étale locally compactly generated by a single compact object. Now, apply [1, Theorem 6.17]. □

As a corollary, in the case of generation by a single object, the stacks $\mathcal{M}\mathrm{od}^{\beta}$ are stacks of modules for a quasi-coherent algebra. This is a useful thing to know, as it can make it easier to compute Hochschild cohomology and other invariants of the categories.

Corollary 2.9. *Let $\mathcal{A} : Z \to \mathbf{Pr}$ classify the stack of $\mathcal{A}$-modules for a quasi-coherent algebra $\mathcal{A}$ over Z. If $p : X \to Z$ is a quasi-compact and quasi-separated derived scheme, then for every $\beta \in \mathbf{Br}^{\mathcal{A}}(X)$, the stack $\mathcal{M}\mathrm{od}^{\beta}$ is equivalent to the stack of modules $\mathcal{M}\mathrm{od}^{\mathcal{B}}$ for a quasi-coherent $\mathcal{O}_X$-algebra $\mathcal{B}$.*

The corollary is a twisted and derived form of the $\mathrm{Br} = \mathrm{Br}'$ question of Grothendieck, which asks if every cohomological Brauer class comes from an Azumaya algebra. This turns out to be false if one only considers ordinary Azumaya algebras over an ordinary scheme. It is necessary in certain cases to allow derived Azumaya algebras as well. On the other hand, if X has an ample line bundle, then a theorem of Gabber (see de Jong [12]) showed that $\mathrm{Br}(X) = \mathrm{H}^2_{\text{ét}}(X, \mathbb{G}_m)_{\text{tors}}$. If in addition X is regular and noetherian, then the computations of [1, Section 7] show that every derived Azumaya algebra on X is Morita equivalent to an ordinary Azumaya algebra.

Question 2.10. Suppose that R is a regular, noetherian ring, and let A be an ordinary associative R-algebra. Is every element $\beta \in \mathbf{Br}^{A}(\operatorname{Spec} R)$ derived Morita equivalent to an ordinary R-algebra B?

Although this seems like a difficult question in general, this paper gives a positive answer for quadric hypersurfaces and noncommutative projective spaces.

2.4. The action of the Brauer group and α-twisted sheaves.

There is an action of the Brauer space $\mathbf{Br}$ on $\mathbf{Br}^{\alpha}$ for any α. Indeed, if $\beta : X \to \mathbf{Pr}$ is étale locally equivalent to α, and if $\gamma : X \to \mathbf{Pr}$ is étale locally equivalent to $\mathcal{O} : X \to \mathbf{Pr}$, then the tensor product $\gamma \otimes \beta$ is étale locally equivalent to α, since, if $\operatorname{Spec} S \to X$ is a map from an affine on which β is equivalent to α and γ is equivalent to $\mathcal{O}$, one sees that

$$\mathrm{Mod}_S^{\gamma} \otimes_{\mathrm{Mod}_S} \mathrm{Mod}_S^{\beta} \simeq \mathrm{Mod}_S \otimes_{\mathrm{Mod}_S} \mathrm{Mod}_S^{\alpha} \simeq \mathrm{Mod}_S^{\alpha}.$$

A special case of this action has already gained a great deal of attention under a different guise, namely as derived categories of twisted sheaves (see [9] or [24]). Suppose that X is an ordinary scheme and that $\alpha \in \mathrm{Br}'(X) = \mathrm{H}^2(X, \mathbb{G}_m)_{\text{tors}}$. One can represent α as a 2-cocycle (α_{ijk}) over some étale cover $\{U_i\}_{i \in I}$ of X. An α-twisted coherent sheaf consists of a coherent $\mathcal{O}_{U_i}$-module $\mathcal{F}_i$ for each i and an isomorphism $\theta_{ij} : \mathcal{F}_i|_{U_{ij}} \to \mathcal{F}_j|_{U_{ij}}$ such that

$$\theta_{ki} \circ \theta_{jk} \circ \theta_{ij}$$

is multiplication by α_{ijk} on $\mathcal{F}_i|_{U_{ijk}}$. The α-twisted coherent sheaves form an abelian category and so one can speak of complexes of α-twisted coherent sheaves and obtain a derived category $\mathrm{D}^b(X, \alpha)$.

More generally, one can consider the stable ∞-category Mod_X^{α} of complexes of α-twisted $\mathcal{O}_X$-modules with quasi-coherent (α-twisted) cohomology sheaves. The subcategory of compact objects may be identified with the category of complexes of α-twisted perfect complexes.

If $\alpha \in \mathrm{Br}'(X)$, write $\mathcal{M}\mathrm{od}^{X,\alpha}$ for the stack (over Spec R) of α-twisted sheaves on X. If X is regular and noetherian, $\mathrm{D}^b(X, \alpha) \simeq \mathrm{Ho}(\mathrm{Mod}_R^{X,\alpha,\mathrm{perf}})$, the homotopy category of the stable ∞-category of α-twisted perfect complexes on X. These are the compact objects in $\mathrm{Mod}^{X,\alpha}$.

Let X be a scheme over an ordinary commutative ring S. For an arbitrary $\alpha \in \mathrm{Br}'(X)$, it will not be the case that $\mathcal{M}\mathrm{od}^{X,\alpha}$ will be étale locally equivalent to $\mathcal{M}\mathrm{od}^{X}$ over Spec S. For instance, if X is a K3 surface over an algebraically closed field k, then $\mathrm{D}^b(X)$ is generically not equivalent to $\mathrm{D}^b(X, \alpha)$ for any $\alpha \neq 0$ in $\mathrm{Br}'(X)$. The stacks of linear categories $\mathcal{M}\mathrm{od}^{X}$ and $\mathcal{M}\mathrm{od}^{X,\alpha}$ are in general only étale locally equivalent on X. However, if $\alpha \in \mathrm{Br}(S)$, we can pull-back via the structure morphism $p : X \to \mathrm{Spec}\, S$ to obtain $p^*\alpha$. Then, $\mathcal{M}\mathrm{od}^{X,p^*\alpha}$ is étale locally Morita equivalent to $\mathcal{M}\mathrm{od}^{X}$ over Spec S.

Proposition 2.11. *The action of* $\alpha \in \mathrm{Br}(S)$ *on* $\mathrm{Br}^X(S)$ *sends* $\mathcal{M}\mathrm{od}^{X}$ *to* $\mathcal{M}\mathrm{od}^{X,p^*\alpha}$.

Proof. Write $\pi : X \to \mathrm{Spec}\, S$. By definition, $\alpha \cdot \mathcal{M}\mathrm{od}^{X}$ is the stack that sends $f : \mathrm{Spec}\, T \to \mathrm{Spec}\, S$ to the T-linear category

$$\mathrm{Mod}_T^{\alpha} \otimes_{\mathrm{Mod}_T} \mathrm{Mod}_T^{X} \simeq \mathrm{Mod}_T^{\alpha} \otimes_{\mathrm{Mod}_T} \mathrm{Mod}_{X_T}.$$

On the other hand, $\mathcal{M}\mathrm{od}^{X,p^*\alpha}$ is the stack that sends f to the T-linear category $\mathrm{Mod}_{X_T}^{p^*\alpha}$. There is a natural map from $\mathrm{Mod}_T^{\alpha} \otimes_{\mathrm{Mod}_T} \mathrm{Mod}_{X_T}$ to $\mathrm{Mod}_{X_T}^{p^*\alpha}$, which is an equivalence étale locally on Spec T. It follows that it is already an equivalence by descent. Taking $T = S$, the proposition follows. □

The action will be given a cohomological interpretation at the end of Section 3.2.

Corollary 2.12. *If the pullback* $p^*\alpha$ *is zero in* $\mathrm{Br}(X)$*, then* α *stabilizes* $\mathcal{M}\mathrm{od}^{X}$.

I conjecture that the converse is true. The conjecture will be verified in various cases throughout the paper, including for smooth projective varieties X over a field with ω_X ample or anti-ample.

Conjecture 2.13 (Stabilizer conjecture). *If* $\alpha \in \mathrm{Br}(S)$ *stabilizes* $\mathcal{M}\mathrm{od}^{X}$*, then* $\alpha \in \ker(\mathrm{Br}(S) \to \mathrm{Br}(X))$.

To conclude the section, I include a more formal structural remark. If A, B, C, and D are S-algebras, and if C is étale locally Morita equivalent to A and D is étale locally Morita equivalent to B, then $C \otimes_S D$ is étale locally Morita equivalent to $A \otimes_S D$. Thus, there are natural products $\mathbf{Br}(-; A) \times \mathbf{Br}(-; B) \to \mathbf{Br}(-; A \otimes_S B)$ of sheaves of spaces over Spec S. The Brauer sheaf $\mathbf{Br}$ is an $\mathbb{E}_\infty$-algebra object in $\mathrm{Shv}_S^{\text{ét}}$ by [1, Corollary 7.5], which means that it is a sheaf of group-like $\mathbb{E}_\infty$-spaces.

Proposition 2.14. *If A is an S-algebra, then $\mathbf{Br}^A$ is a module for the $\mathbb{E}_\infty$-algebra object $\mathbf{Br}$, and thus can be viewed as an element of* $\mathrm{Mod}_{\mathbf{Br}}(\mathrm{Shv}_S^{\text{ét}})$. *There is a natural equivalence*

$$\mathbf{Br}^A \otimes_{\mathbf{Br}} \mathbf{Br}^B \simeq \mathbf{Br}^{A \otimes_S B}.$$

Proof. The claim that $\mathbf{Br}^A$ is a module for $\mathbf{Br}$ follows from the symmetric monoidal structure on the sheaf $\mathbf{Pr}$. To prove the second claim, it is enough to note that if T is a connective commutative S-algebra and C is a T-algebra that is étale locally equivalent to $A \otimes_S B \otimes_S T$, then C is étale locally equivalent to the tensor product of an algebra in $\mathbf{Br}^A(T)$ and an algebra in $\mathbf{Br}^B(T)$. □

3. The descent spectral sequence

3.1. Fringed spectral sequences. To consider carefully what happens in the fringed spectral sequences that appear when doing descent spectral sequences, it is useful to first consider the long exact sequence of homotopy groups associated to a fibration $p : X \to Y$ of pointed spaces. Let $b \in Y$ be the basepoint, and let $f \in F$, where $F = p^{-1}\{b\}$. Then, there is a sequence of homotopy groups and pointed homotopy sets

$$\begin{aligned} \to \cdots \pi_2(Y, b) \to \pi_1(F, f) \to \pi_1(X, f) \to \pi_1(Y, b) \\ \to \pi_0(F, f) \to \pi_0(X, f) \to \pi_0(Y, b), \end{aligned}$$

where $\pi_0(-, f)$ is the set of path components pointed by f. This sequence is exact in the following sense:

- at any place $\pi_i(F, f)$, $\pi_i(X, f)$, or $\pi_i(Y, b)$ where $i > 0$, it is exact in the usual sense that ker = im;
- the image of $\pi_2(Y, b)$ is in the center of $\pi_1(F, f)$;
- there is an action of $\pi_1(Y, b)$ on $\pi_0(F, f)$ such that two elements of $\pi_0(F, f)$ agree in $\pi_0(X, f)$ if and only if they are in the same orbit;
- the map $\pi_1(Y, b) \to \pi_0(F, f)$ induces a bijection between $\pi_1(Y, f)/\pi_1(X, f)$ and the orbit of the point f in $\pi_0(F, f)$;
- a point $g \in \pi_0(X, f)$ goes to b in $\pi_0(Y, b)$ if and only if it is in the image of $\pi_0(F, f) \to \pi_0(X, f)$.

The main information that this sequence does not see is the fact that the fibers of $\pi_0(X, f) \to \pi_0(Y, b)$ can vary widely over different points of $\pi_0(Y, b)$ and can be empty, so that in particular $\pi_0(X, f) \to \pi_0(Y, b)$ might not be surjective.

Now, let $\cdots \to X_n \to X_{n-1} \to \cdots X_0 \to *$ be a sequence of fibrations of pointed spaces, where X_n is pointed by f_n. Let f be the point of $X = \lim X_n$ that is the inverse limit of the points f_n. Write F_n for the homotopy fiber of $X_n \to X_{n-1}$

over f_{n-1}. Bousfield and Kan [7, Section IX.4] created a spectral sequence that converges conditionally to $\pi_* X$ by rolling up all of the fibration sequences $F_n \to X_n \to X_{n-1}$ into a generalized triple, generalized in the sense that some terms are not abelian groups. Without going into many details, there is a fringed spectral sequence

$$\mathrm{E}_2^{s,t} = \pi_{t-s}(F_t, f_t) \Rightarrow \pi_{t-s} X,$$

fringed in the sense that $\mathrm{E}_r^{s,s}$ is just a pointed set, and $\mathrm{E}_r^{s,s+1}$ is a possibly non-abelian group.

The differential d_r in the E_r-page of this spectral sequence has degree $(r, r-1)$. (Note that Bousfield and Kan index the spectral sequence differently, beginning instead with $\mathrm{E}_1^{s,t} = \pi_{t-s} F_s$.) When $t - s > 0$, the $\mathrm{E}_{r+1}^{s,t}$ term is computed in the usual way from $\mathrm{E}_r^{s,t}$, as cycles modulo boundaries. When $t - s = 0$, there is not only a differential with target $\mathrm{E}_r^{s,t}$, but the source $\mathrm{E}_r^{s-r,t-r+1}$ acts on $\mathrm{E}_r^{s,t}$, and $\mathrm{E}_{r+1}^{s,t}$ is the orbit space of this action. The meaning of convergence is clear when $t-s > 0$. When $t - s = 0$, there is a filtration of $\pi_0 X$ as a pointed set. This means that there is a sequence of inclusions of pointed sets

$$* \subseteq \cdots \subseteq F_{s+1}\pi_0 X \subseteq F_s \pi_0 X \subseteq \cdots \subseteq F_0 \pi_0 X = \pi_0 X$$

and the successive quotients $F_s\pi_0 X / F_{s+1}\pi_0 X$ are bijective to $\mathrm{E}_\infty^{s,s}$ as pointed sets. The filtration on $\pi_i X$ has the same indexing. Namely, there is a decreasing filtration $F_s \pi_i X$ and $F_s\pi_i X / F_{s+1}\pi_i X \cong \mathrm{E}_\infty^{s,s+i}$, when the spectral sequence converges.

The reader is warned that the convergence of this spectral sequence is in general only conditional. However, the spectral sequence will converge completely in all cases considered in this paper (for $t - s > 0$). For a discussion of convergence of these spectral sequence, see [7, Section IX.5]. It makes sense only for the terms abutting to $\pi_i X$ where $i > 0$, where it coincides with the usual notion of convergence. In general, more work is needed to get a handle on $\pi_0 X$, which is the case of greatest interest in this paper. However, the complete convergence for $t - s > 0$ will often give crucial information for understanding what happens for $\pi_0 X$.

The descent spectral sequence, sometimes called the Brown–Gersten spectral sequence [8], associated to a sheaf of spaces on a topos is a special case of the spectral sequence associated to a tower of fibrations. Let F be a sheaf of pointed spaces, which is to say an object of $\mathrm{Shv}_R^{\text{ét}}$. The construction below works for any object in any ∞-topos. However, convergence is a more delicate question, closely related to notion of hypercompleteness discussed in [25, Section 6.5]. The Postnikov tower of F as a sheaf is obtained via the truncations of F

$$F \to \cdots \tau_{\leq n} F \to \tau_{\leq n-1} \to \cdots \tau_{\leq 0} F \to *,$$

and the fiber of $\tau_{\leq n} F \to \tau_{\leq n-1} F$ is the Eilenberg–MacLane sheaf $K(\pi_n^s F, n)$, which has homotopy sheaves $\pi_n^s F$ in degree n and 0 (or a point) elsewhere. Let X

be another sheaf in $\mathrm{Shv}_R^{\text{ét}}$. Then, the sequence

$$\cdots \to \mathrm{map}(X, \tau_{\leq n} F) \to \mathrm{map}(X, \tau_{\leq n-1} F) \to \cdots,$$

is a tower of fibrations which, in good cases, and all cases in this paper, has inverse limit the space $\mathrm{map}(X, F)$. The spectral associated to this tower has the form

$$\mathrm{E}_2^{p,q} = \pi_{q-p}\mathrm{map}(X, K(\pi_q^s F, q)) \Rightarrow \pi_{q-p}\mathrm{map}(X, F).$$

Since $K(\pi_q^s F, q)$ is an infinite loop space (at least for $q > 1$),

$$\pi_{q-p}\mathrm{map}(X, K(\pi_q^s F, q)) \simeq \pi_0\mathrm{map}(X, K(\pi_q^s F, p)).$$

Suppose for a moment that C is a small category with a Grothendieck topology and that X is an object of $\mathrm{Shv}_{\mathrm{N}C}$ and A is a sheaf of abelian groups on C. Then, Lurie shows [25, Remark 7.2.2.17] that

$$\pi_0\mathrm{map}(X, K(A, n)) \simeq \mathrm{H}^n(X, A),$$

where $\mathrm{H}^n(X, A)$ denotes the usual cohomology group of X with coefficients in A. Since the small étale site over a connective commutative R-algebra S is equivalent to the nerve of the small étale site over $\pi_0 S$, it follows that if $X = \mathrm{Spec}\, S$, then the groups

$$\begin{aligned}\pi_{q-p}\mathrm{map}(\mathrm{Spec}\, S, K(\pi_q^s F, q)) &\simeq \pi_0\mathrm{map}(\mathrm{Spec}\, S, K(\pi_q^s F, p)) \\ &\simeq \mathrm{H}^p_{\text{ét}}(\mathrm{Spec}\, \pi_0 S, \pi_q^s F).\end{aligned}$$

This has the following generalization to schemes.

Proposition 3.1. *Let R be an ordinary commutative ring, and let X be an ordinary R-scheme, viewed as an object of the ∞-topos $\mathrm{Shv}_{\mathrm{H}R}^{\text{ét}}$. If A is an abelian group object in the underlying discrete topos, then*

$$\pi_0\mathrm{map}(X, K(A, n)) \cong \mathrm{H}^n_{\text{ét}}(X, A),$$

where $\mathrm{H}^n_{\text{ét}}(X, A)$ denotes the usual étale cohomology group of X with coefficients in A.

Proof. The Eilenberg–MacLane sheaf is hypercomplete, so one can compute the group $\pi_0\mathrm{map}(X, K(A, n))$ with a suitably nice étale hypercover of X that will also compute the group $\mathrm{H}^n_{\text{ét}}(X, A)$. Assuming that this hypercover consists of disjoint unions of affine schemes, the observation above that the statement is true for affine schemes shows that the proposition is true by comparing the Čech complexes. □

3.2. The spectral sequence. Here and in the rest of the paper I will abuse notation and use Mod_A and $\mathcal{M}\mathrm{od}^A$ interchangeably. There is no danger of confusion or error, since Mod_A is an R-linear category with descent, so that $\mathcal{M}\mathrm{od}^A$ can be constructed from Mod_A, and vice versa.

In this paper, the main objects of interest are $\mathbf{Br}^X$, where X is a smooth proper scheme over an ordinary commutative ring R, which is a special case of $\mathbf{Br}^A$ where A is a smooth and proper R-algebra. The strategy for actually computing $\mathbf{Br}^X(R)$ is to determine the sheaf of spaces $\mathbf{aut}_{\mathrm{Mod}_X}$, use the fact that $\mathbf{Br}^X$ is the classifying sheaf of $\mathbf{aut}_{\mathrm{Mod}_X}$, and use the descent spectral sequence.

Proposition 3.2. *Let A be an R-algebra. Then, the homotopy sheaves of $\mathbf{aut}_{\mathrm{Mod}_A}$ are*

$$\pi_i^s \mathbf{aut}_{\mathrm{Mod}_A} \cong \begin{cases} \mathbf{Aut}_{\mathrm{Mod}_A} & \text{if } i = 0, \\ \mathbf{HH}^0(A)^\times & \text{if } i = 1, \\ \mathbf{HH}^{1-i}(A) & \text{if } i \geq 2, \end{cases}$$

where $\mathbf{Aut}_{\mathrm{Mod}_A}$ is the sheaf of groups with sections over S the group $\mathrm{Aut}_{\mathrm{Mod}_{A\otimes_R S}}$, and $\mathbf{HH}^*(A)$ is the Hochschild cohomology sheaf of A, which sends S to $\mathrm{HH}^*_S(A \otimes_R S)$.

Proof. The description of $\pi_0^s \mathbf{aut}_{\mathrm{Mod}_A}$ is by definition. Since this is a sheaf of group-like $\mathbb{E}_1$-spaces, the higher homotopy sheaves are independent of the basepoint chosen. The canonical basepoint is the identity functor id, and it suffices to compute the homotopy sheaves of the loopsheaf $\Omega_{\mathrm{id}} \mathbf{aut}_{\mathrm{Mod}_A}$. Thus, one wants to compute the space of automorphisms of $\mathrm{id} : \mathrm{Mod}_A \to \mathrm{Mod}_A$ as a functor. This is nothing other than the space of automorphisms of A as an $A^{\mathrm{op}} \otimes_R A$-module, which is precisely the space of units in the Hochschild cohomology algebra of A. The Hochschild cohomology algebra $\mathbf{HH}^*(A)$ is a sheaf over $\mathrm{Spec}\, R$ because the category $\mathrm{Mod}_{A^{\mathrm{op}} \otimes_R A}$ satisfies étale hyperdescent. □

This is a sheafy version of [38, Corollary 1.6]. When X is quasi-compact and quasi-separated, $\mathrm{Mod}_X \simeq \mathrm{Mod}_A$ for some A, so that the proposition also applies to the automorphism sheaf of Mod_X.

When X is smooth, proper, and geometrically connected over R, $\mathbf{HH}^0(X)^\times \cong \mathbb{G}_m$, since $\mathrm{HH}^0_R(X) \cong \mathrm{H}^0(X, \mathcal{O}_X)$ by the Hodge spectral sequence for Hochschild cohomology [37]. Moreover, if R is an ordinary ring, and if A is an ordinary R-algebra or X is an ordinary scheme, then the negative Hochschild cohomology groups vanish, since projective resolutions exist.

The next theorem gives the main computational tool for determining $\mathrm{Br}^A(R)$. Throughout, when writing $\mathbf{Br}^A$, it is assumed that Mod_A is chosen as the global basepoint of the sheaf.

Theorem 3.3. *There is a fringed spectral sequence*

$$\mathrm{E}_2^{p,q} = \begin{cases} \mathrm{H}^p_{\text{ét}}(R, \pi_q^s \mathbf{Br}^A) & \textit{if } q - p \geq 0, \\ 0 & \textit{otherwise} \end{cases}$$
$$\Rightarrow \pi_{q-p}\mathbf{Br}^A(R),$$

where

$$\pi_i^s \mathbf{Br}^A \cong \begin{cases} 0 & \textit{if } i = 0, \\ \mathbf{Aut}_{\mathrm{Mod}_A} & \textit{if } i = 1, \\ \mathbf{HH}^0(A)^\times & \textit{if } i = 2, \\ \mathbf{HH}^{2-i}(A) & \textit{if } i \geq 3, \end{cases}$$

which converges completely if A is smooth and proper or if R and A are ordinary rings.

Proof. The spectral sequence is nothing more than the descent spectral sequence of the previous section. The first statement about convergence follows because if A is smooth and proper, the Hochschild cohomology of A vanishes in sufficiently high degrees, so that the spectral sequence collapses after some finite stage. The second statement follows because, if R and A are ordinary, $\mathrm{HH}^{2-i}(A) = 0$ for $i \geq 3$. □

The theorem is especially strong when R and A are ordinary rings, or when R is an ordinary ring and one considers $\mathbf{Br}^X$ for a smooth, proper, geometrically connected R-scheme X. In either of these cases the homotopy sheaves of the twisted Brauer sheaf are concentrated in two degrees, 1 and 2. For instance,

$$\pi_i^s \mathbf{Br}^X \cong \begin{cases} 0 & \text{if } i = 0, \\ \mathbf{Aut}_{\mathrm{Mod}_X} & \text{if } i = 1, \\ \mathbb{G}_m & \text{if } i = 2, \\ 0 & \text{if } i \geq 3. \end{cases}$$

This means that the sheaf $\mathbf{Br}^X$ is an extension of Eilenberg–MacLane sheaves

$$K(\mathbb{G}_m, 2) \to \mathbf{Br}^X \to K(\mathbf{Aut}_{\mathrm{Mod}_X}, 1).$$

Since $\mathbb{G}_m$ is a sheaf of abelian groups, $K(\mathbb{G}_m, 2)$ is an infinite loop space in $\mathrm{Shv}_R^{\text{ét}}$. This implies that the sequence above can be delooped, and $\mathbf{Br}^X$ can be identified as the fiber in the sequence

$$\mathbf{Br}^X \to K(\mathbf{Aut}_{\mathrm{Mod}_X}, 1) \to K(\mathbb{G}_m, 3).$$

Then, taking global sections, there is a fiber sequence

$$\mathbf{Br}^X(R) \to \mathrm{map}(\mathrm{Spec}\, R, K(\mathbf{Aut}_{\mathrm{Mod}_X}, 1)) \to \mathrm{map}(\mathrm{Spec}\, R, K(\mathbb{G}_m, 3)).$$

We can point the spaces in this sequence by choosing the point Mod_X of $\pi_0\mathbf{Br}^X(R)$. The spectral sequence degenerates into the long exact sequence of homotopy groups associated to this fibration. In particular, there is an isomorphism $\pi_2\mathbf{Br}^X(R) \cong \mathbb{G}_m(R)$ and an exact sequence

$$\begin{aligned} 0 &\to \mathrm{H}^1_{\acute{e}t}(\mathrm{Spec}\, R, \mathbb{G}_m) \to \pi_1\mathbf{Br}^X(R) \to \mathbf{Aut}_{\mathrm{Mod}_X}(R) \\ &\to \mathrm{H}^2_{\acute{e}t}(\mathrm{Spec}\, R, \mathbb{G}_m) \to \pi_0\mathbf{Br}^X(R) \to \mathrm{H}^1_{\acute{e}t}(\mathrm{Spec}\, R, \mathbf{Aut}_{\mathrm{Mod}_X}) \to \mathrm{H}^3_{\acute{e}t}(\mathrm{Spec}\, R, \mathbb{G}_m). \end{aligned}$$

The meaning of exact here is just as in the beginning of the previous section. In particular, there is an action of $\mathrm{H}^2_{\acute{e}t}(\mathrm{Spec}\, R, \mathbb{G}_m)$ on $\pi_0\mathbf{Br}^X(R)$, and the fibers of $\pi_0\mathbf{Br}^X(R) \to \mathrm{H}^1_{\acute{e}t}(\mathrm{Spec}\, R, \mathbf{Aut}_{\mathrm{Mod}_X})$ are precisely the orbits of this action. The quotient $\mathrm{H}^2_{\acute{e}t}(\mathrm{Spec}\, R, \mathbb{G}_m)/\mathbf{Aut}_{\mathrm{Mod}_X}(R)$ is in bijection with the orbit of Mod_X in $\pi_0\mathbf{Br}^X(R)$.

The kernel of $\mathbf{Aut}_{\mathrm{Mod}_X}(R) \to \mathrm{H}^2_{\acute{e}t}(\mathrm{Spec}\, R, \mathbb{G}_m)$ consists of those elements that come from actual autoequivalences of Mod_X.

An element of $\mathrm{H}^1_{\acute{e}t}(\mathrm{Spec}\, R, \mathbf{Aut}_{\mathrm{Mod}_X})$ maps to 0 in $\mathrm{H}^3_{\acute{e}t}(\mathrm{Spec}\, R, \mathbb{G}_m)$ if and only if it can be lifted to $\pi_0\mathbf{Br}^X(R)$. The class in $\mathrm{H}^3_{\acute{e}t}(\mathrm{Spec}\, R, \mathbb{G}_m)$ represents the obstruction to lifting a cohomology class in $\mathrm{H}^1_{\acute{e}t}(\mathrm{Spec}\, R, \mathbf{Aut}_{\mathrm{Mod}_X})$ to an actual collection of gluing data to obtain a twisted form of the stack $\mathcal{M}\mathrm{od}_X$. We will see that these obstructions frequently vanish. This occurs when the gluing data can be made to act on an object with less homotopical information, such as a scheme, as opposed to the stable ∞-categories appearing in $\mathcal{M}\mathrm{od}_X$.

3.3. The example of the introduction. Recall that $\mathrm{Mod}_{\mathbb{P}^1_{\mathbb{R}}} \simeq \mathrm{Mod}_{\mathbb{R}Q}$ where Q is quiver $\bullet \rightrightarrows \bullet$. Since $\mathbb{P}^1$ is Fano, the computation of Bondal and Orlov [5] shows that $\mathbf{Aut}_{\mathrm{Mod}_{\mathbb{P}^1}} \cong \mathbb{Z} \times (\mathbb{Z} \rtimes \mathbf{PGL}_2)$. Thus, by the vanishing of negative Hochschild cohomology for ordinary schemes, the homotopy sheaves of $\mathbf{Br}^{\mathbb{P}^1}$ are

$$\pi_i^s \mathbf{Br}^{\mathbb{P}^1} = \begin{cases} 0 & \text{if } i = 0, \\ \mathbb{Z} \times (\mathbb{Z} \times \mathbf{PGL}_2) & \text{if } i = 1, \\ \mathbb{G}_m & \text{if } i = 2, \\ 0 & \text{otherwise,} \end{cases}$$

where the degree 1 term splits because $\mathbf{PGL}_2$ acts trivially on $\mathrm{Pic}(\mathbb{P}^1) = \mathbb{Z}$.

In the descent spectral sequence for $\mathbf{Br}^{\mathbb{P}^1}$ there is only one possible non-zero differential, which is $d_2 : \mathbb{Z} \times (\mathbb{Z} \times \mathbf{PGL}_2(\mathbb{R})) \to \mathrm{H}^2_{\acute{e}t}(\mathrm{Spec}\, \mathbb{R}, \mathbb{G}_m)$. But, it is clear that this is zero, because $\mathbb{Z} \times (\mathbb{Z} \times \mathbf{PGL}_2(\mathbb{R}))$ survives to the E_∞-page to act as automorphisms of $\mathrm{Mod}_{\mathbb{P}^1}$. Since $\mathrm{H}^1_{\acute{e}t}(\mathrm{Spec}\, k, \mathbb{Z}) = 0$ for any field k, there is an exact sequence of pointed sets

$$0 \to \mathrm{Br}(\mathbb{R}) \to \mathrm{Br}^{\mathbb{P}^1}(\mathbb{R}) \to \mathrm{H}^1_{\acute{e}t}(\mathrm{Spec}\, \mathbb{R}, \mathbf{PGL}_2) \to *.$$

This sequence is not split, because the action of $\mathrm{Br}(\mathbb{R})$ on the non-trivial point of $\mathrm{H}^1_{\text{ét}}(\operatorname{Spec}\mathbb{R}, \mathbf{PGL}_2)$ is trivial. However, since $\mathrm{H}^1_{\text{ét}}(\operatorname{Spec}\mathbb{R}, \mathbf{PGL}_2)$ is the set of isomorphism classes of smooth projective genus 0 curves over $\mathbb{R}$, the map $\mathrm{Br}^{\mathbb{P}^1}(\mathbb{R}) \to \mathrm{H}^1_{\text{ét}}(\operatorname{Spec}\mathbb{R}, \mathbf{PGL}_2)$ is indeed surjective, and the set $\mathrm{Br}^{\mathbb{P}^1}(\mathbb{R})$ consists of categories of twisted sheaves on genus 0 curves.

We can compute the higher homotopy of $\mathbf{Br}^{\mathbb{P}^1}(\mathbb{R})$ at the point $\mathrm{Mod}_{\mathbb{P}^1}$. From the spectral sequence,

$$\pi_i \mathbf{Br}^{\mathbb{P}^1}(\mathbb{R}) \cong \begin{cases} \mathbb{Z} \times (\mathbb{Z} \times \mathbf{PGL}_2(\mathbb{R})) & \text{if } i = 1, \\ \mathbb{R}^* & \text{if } i = 2, \\ 0 & \text{if } i \geq 3. \end{cases}$$

Note that the fundamental group (at the point $\mathrm{Mod}_{\mathbb{P}^1_{\mathbb{R}}}$ is the automorphism group of $\mathrm{Mod}_{\mathbb{P}^1_{\mathbb{R}}}$. The first $\mathbb{Z}$ is just translation, while the second corresponds to tensoring with $\mathcal{O}(1)$. The group π_2 is the group of invertible natural transformations between automorphisms.

The reader might be disturbed by an apparent asymmetry in the computation above. Namely, what would happen if we did the calculation instead at the point Mod_C where C is again the curve $x^2 + y^2 + z^2 = 0$ in $\mathbb{P}^2$ over $\mathbb{R}$? In this case,

$$\pi_i^s \mathbf{Br}^C = \begin{cases} 0 & \text{if } i = 0, \\ \mathbb{Z} \times (\mathrm{R}\, p_*^1 \mathbb{G}_{m,C} \times \mathbf{Aut}_C) & \text{if } i = 1, \\ \mathbb{G}_m & \text{if } i = 2, \\ 0 & \text{otherwise.} \end{cases}$$

Here, $\mathbf{Aut}_C$ is a form of $\mathbf{PGL}_2$ over $\mathbb{R}$, and $p : C \to \operatorname{Spec}\mathbb{R}$ is the structure map, so $\mathrm{R}\, p_*^1 \mathbb{G}_{m,C}$ is the relative Picard sheaf. Then, by considering the Leray spectral sequence for the sheaf $\mathbb{G}_{m,C}$ and the map p, it is easy to see that $\mathrm{Pic}(C) \to \Gamma(\operatorname{Spec}\mathbb{R}, \mathrm{R}\, p_*^1 \mathbb{G}_{m,C})$ has cokernel equal to $\mathbb{Z}/2$. Since we know that the Brauer group acts trivially, it follows that there is a non-zero differential in the descent spectral sequence, and we obtain a filtration of pointed sets

$$0 \to \mathrm{Br}^C(\mathbb{R}) \to \mathrm{H}^1_{\text{ét}}(\operatorname{Spec}\mathbb{R}, \mathbf{Aut}_C) \to *.$$

The difference between this computation and that for $\mathbb{P}^1$ is simply because of the dependence of the fiber on the basepoint for fibrations $X \to Y$, as discussed in Section 3.1.

Thus, $\mathrm{Mod}_{\mathbb{P}^1}$, $\mathrm{Mod}^{\mathbb{H}}_{\mathbb{P}^1}$, and Mod_C are the only 3 elements of $\mathrm{Br}^{\mathbb{P}^1}$, which gives a positive answer to Question 2.10.

Theorem 3.4. *Suppose that A is an $\mathbb{R}$-algebra such that $\mathbb{C} \otimes_{\mathbb{R}} A$ is derived Morita equivalent to $\mathbb{P}^1$. Then, A is derived Morita equivalent over $\mathbb{R}$ to an ordinary $\mathbb{R}$-algebra, either $\mathbb{R}Q$, $\mathbb{H}Q$, or the modulated quiver algebra associated to C.*

3.4. The stabilizer conjecture in the canonical (anti-)ample case. The next theorem verifies the stabilizer conjecture in many cases.

Theorem 3.5. *Let X be a smooth, projective, and geometrically connected variety over a field k. Suppose that the canonical line bundle ω_X is either ample or anti-ample. Then, the stabilizer conjecture holds for* Mod_X. *That is, the kernel of* $\mathrm{Br}(k) \to \mathrm{Br}(X)$ *is the same as the fiber of* $\mathrm{Br}(k) \to \mathrm{Br}^X(k)$.

Proof. Consider the split exact sequence of sheaves of groups

$$0 \to \mathbb{Z} \times \mathbf{Pic}_{X/k} \to \mathbf{Aut}_{\mathrm{Mod}_X} \to \mathbf{Aut}_X \to 0$$

given by the theorem of Bondal and Orlov [5]. The end of Section 3.2 provides an exact sequence

$$\pi_1 \mathbf{Br}^X(k) \to \mathbf{Aut}_{\mathrm{Mod}_X}(k) \to \mathrm{H}^2_{\text{ét}}(k, \mathbb{G}_m) \to \pi_0 \mathbf{Br}^X(k).$$

Thus, it suffices to show that the image of $\mathbf{Aut}_{\mathrm{Mod}_X}(k) \to \mathrm{H}^2_{\text{ét}}(k, \mathbb{G}_m)$ is precisely $\ker(\mathrm{Br}(k) \to \mathrm{Br}(X))$. By examining the exact sequence of sheaves above, it is clear that the only sections of $\mathbf{Aut}_{\mathrm{Mod}_X}$ over $\operatorname{Spec} k$ that might not lift to automorphisms of Mod_X come from elements of $\mathbf{Pic}_{X/k}(k)$ that do not lift to $\mathrm{Pic}(X)$. But, the cokernel of $\mathrm{Pic}(X) \to \mathbf{Pic}_{X/k}(k)$ injects into $\mathrm{Br}(k)$ as the kernel of $\mathrm{Br}(k) \to \mathrm{Br}(X)$ by the Leray spectral sequence. This completes the proof. □

4. Lifting Morita equivalences

Let k be a field, and let A be a k-algebra. Up to this point, only algebras B that become derived Morita equivalent to A after a finite separable extension l/k have been considered.

Question 4.1. When is it the case that if A and B are k-algebras that are derived Morita equivalent over $\overline{k}$, then they are derived Morita equivalent over a finite separable extension of k.

This is a question about the smoothness of the stack of derived Morita equivalences between A and B. It is possible to solve it using the techniques of [1] when A is a smooth finite-dimensional hereditary k-algebra. Recall that A is hereditary if it has global dimension 1, and that A is smooth if it has finite projective dimension over $A^{\mathrm{op}} \otimes_R A$.

Theorem 4.2. *Let A be a smooth finite-dimensional hereditary k-algebra. Then, if B is derived Morita equivalent to A over $\overline{k}$, it is derived Morita equivalent to A over some finite separable extension l/k.*

Proof. Let $\mathbf{Mor}_{A\to B}$ the sheaf of derived Morita equivalences from A to B. Then, by hypothesis, $\mathbf{Mor}_{A\to B} \to \operatorname{Spec} k$ is surjective on geometric points. Let M be a Morita equivalence over a field l. I can assume given M that it is in fact a self-equivalence $\mathrm{Mod}_A \to \mathrm{Mod}_A$, and even the identity, viewed as a perfect complex of $\mathrm{Mod}_{A^{\mathrm{op}}\otimes_k A}$. But then, by [1, Corollary 5.9], the cotangent complex of $\mathbf{Mor}_{A\to B}$ at M is equivalent to

$$\Sigma^{-1}\,\mathrm{End}_{A^{\mathrm{op}}\otimes_k A}(A)^\vee.$$

The conditions on A ensure that $\mathrm{End}_{A^{\mathrm{op}}\otimes_k A}(A)$ has homology (and hence, in this case, Tor-amplitude) contained in degrees $[-1, 0]$. Since the base is a field, the dual $\mathrm{End}_{A^{\mathrm{op}}\otimes_k A}(A)^\vee$ has Tor-amplitude contained in $[0, 1]$. Thus, the cotangent complex has Tor-amplitude contained in degrees $[-1, 0]$. Therefore, the sheaf of derived Morita equivalences is smooth when A is a smooth finite-dimensional hereditary hereditary k-algebra. By Theorem [1, Theorem 4.47], it follows that there are étale local sections of $\mathbf{Mor}_{A\to B} \to \operatorname{Spec} k$, as desired. □

The theorem applies in particular to all path algebras. There is also a global version of the theorem, which has the same proof.

Scholium 4.3. *Suppose that X is a regular noetherian scheme and that $\mathcal{A}$ is a perfect sheaf of coherent algebras on X such that $\mathcal{A}_{k(x)}$ is smooth and hereditary for each point x of X. If $\mathcal{B}$ is another perfect sheaf of coherent algebras on X, and if*

$$\mathrm{Mod}_{\mathcal{B}\otimes_{\mathcal{O}_X} k(\bar{x})} \simeq \mathrm{Mod}_{\mathcal{A}\otimes_{\mathcal{O}_X} k(\bar{x})}$$

for each geometric point $\bar{x}$ of X, then there is an étale cover $U \to X$ such that $\mathrm{Mod}_{\mathcal{B}\otimes_{\mathcal{O}_X}\mathcal{O}_U} \simeq \mathrm{Mod}_{\mathcal{A}\otimes_{\mathcal{O}_X}\mathcal{O}_U}$.

5. Examples

The purpose of this section is to give a taste of the computational power of the spectral sequence rather than to give a complete treatment. However, complete computations are obtained for genus 0 curves, quadric hypersurfaces, and twists of the quiver Ω_n. For curves of higher genus, only the outline of the theory is exposed, a more detailed treatment being left to future work.

The spectral sequence makes it possible to describe representatives for all of the elements of $\pi_0\mathbf{Br}^A(k)$ or $\pi_0\mathbf{Br}^X(k)$ in many cases. However, it is a much more difficult question to decide when two representatives determine the same point in the set of connected components. There are two reasons for this difficulty. First, it is in general a subtle problem to determine the stabilizer of a point under the action of the Brauer group. In some good cases, such as genus 0 curves or quadrics, this is possible. But, for curves of higher genus, for example, it is much harder to determine the stabilizer. The second problem is that many sequences involve short exact sequences in nonabelian cohomology, where exactness is only certain over basepoints.

Recall that I will abuse notation and write Mod_A and $\mathcal{M}\mathrm{od}^A$ interchangeably.

5.1. Genus 0 curves. The reader can easily use the arguments in the introduction and Section 3.3 to compute $\mathrm{Br}^{\mathbb{P}^1}(k)$ for any field k. There is a sequence

$$0 \to \mathrm{Br}(k) \to \mathrm{Br}^{\mathbb{P}^1}(k) \to \mathrm{H}^1_{\text{ét}}(\operatorname{Spec} k, \mathbf{PGL}_2) \to *,$$

which is exact in the following senses. There is an action of $\mathrm{Br}(k)$ on $\mathrm{Br}^{\mathbb{P}^1}(k)$, which is faithful. Moreover, the action of $\mathrm{Br}(k)$ on the point $\mathrm{Mod}_{\mathbb{P}^1}$ is free. The map $\mathrm{Br}^{\mathbb{P}^1}(k) \to \mathrm{H}^1_{\text{ét}}(\operatorname{Spec} k, \mathbf{PGL}_2)$ is surjective, and two elements of $\mathrm{Br}^{\mathbb{P}^1}(k)$ lay over the same genus 0 curve C in $\mathrm{H}^1_{\text{ét}}(\operatorname{Spec} k, \mathbf{PGL}_2)$ if and only they are in the same $\mathrm{Br}(k)$-orbit. A remark is in order about surjectivity, as the end of Section 3.2 implies that there is in general an obstruction. However, it vanishes in this case as $\mathrm{H}^1_{\text{ét}}(\operatorname{Spec} k, \mathbf{PGL}_2)$ is the set of isomorphism of smooth projective genus 0 curves over k. A class of $\mathrm{Br}^{\mathbb{P}^1}(k)$ mapping to $C \in \mathrm{H}^1_{\text{ét}}(\operatorname{Spec} k, \mathbf{PGL}_2)$ can be constructed explicitly be taking Mod_C.

The interesting question is to determine the stabilizers of the action of $\mathrm{Br}(k)$ on Mod_C for a genus 0 curve without any k-points. The curve C is the Severi–Brauer variety of a unique degree 2 central division algebra D over k. By Amitsur's theorem [14, Theorem 5.4.1], the kernel of $\mathrm{Br}(k) \to \mathrm{Br}(k(C))$ is exactly the cyclic subgroup generated by $[D]$. Since $\mathrm{Br}(C) \to \mathrm{Br}(k(C))$ is injective, Theorem 3.5 says that the stabilizer is precisely $([D]) \subseteq \mathrm{Br}(k)$. It follows that the orbit of Mod_C in $\mathrm{Br}^{\mathbb{P}^1}(k)$ is in bijection with $\mathrm{Br}(k)/([D])$.

In summary, the noncommutative étale twists of $\mathbb{P}^1$ are all determined by a genus 0 curve C and a Brauer class $\alpha \in \mathrm{Br}(k)$. The ∞-category of modules over this noncommutative twist is Mod^{α}_C, the ∞-category of α-twisted sheaves on C. By using modulated quivers (for which, see [13]), all twists are derived Morita equivalent to ordinary k-algebras, which answers Question 2.10 for the path algebra of $\bullet \rightrightarrows \bullet$.

5.2. Genus 1 curves and modular representations. Let E be an elliptic curve over k. A group isomorphism $E \times E \to E \times E$ can be given by a matrix

$$f = \begin{pmatrix} f_1 & f_2 \\ f_3 & f_4 \end{pmatrix}$$

of group isomorphisms $f_i : E \to E$. By using an isomorphism $E \cong \hat{E}$, where $\hat{E}$ is the dual of E, one obtains

$$\tilde{f} = \begin{pmatrix} \hat{f}_4 & -\hat{f}_2 \\ -\hat{f}_3 & \hat{f}_1 \end{pmatrix}.$$

Let $U(E)$ be the subgroup of group automorphisms $f : E \times E \to E \times E$ such that $\tilde{f} = f^{-1}$. Then, by Orlov [30], there is an exact sequence

$$0 \to \mathbb{Z} \times E \times \hat{E} \to \mathbf{Aut}_{\mathrm{Mod}_E} \to U(E) \to 0.$$

From this, one can describe the elements of $\mathrm{Br}^E(k)$.

I will consider a special case, when $U(E) \cong \mathrm{SL}_2(\mathbb{Z})$, which happens for a non-CM elliptic curve. In this case, the sequence reduces to

$$0 \to \mathbb{Z} \times E \times \hat{E} \to \mathbf{Aut}_{\mathrm{Mod}_E} \to \mathrm{SL}_2(\mathbb{Z}) \to 0. \tag{5.1}$$

Let $\widetilde{\mathrm{SL}}_2(\mathbb{Z})$ be the group generated by x, y, and t with relations $(xy)^3 = t$, $y^4 = t^2$, $xt = tx$, and $yt = ty$. Then, the quotient of $\widetilde{\mathrm{SL}}_2(\mathbb{Z})$ by the central subgroup (t) is isomorphic to $\mathrm{SL}_2(\mathbb{Z})$. Moreover, there is a homomorphism $\widetilde{\mathrm{SL}}_2(\mathbb{Z}) \to \mathbf{Aut}_{\mathrm{Mod}_E}$ whose composition with $\mathbf{Aut}_{\mathrm{Mod}_E} \to \mathrm{SL}_2(\mathbb{Z})$ is the surjection above. The element t maps to the translation functor. See [20, Section 9.5]. Since $\mathbb{Z} \cong (t) \subseteq \widetilde{\mathrm{SL}}(\mathbb{Z})$ is a central subgroup, it follows from [36, Proposition 42] that

$$\mathrm{H}^1_{\text{ét}}(\operatorname{Spec} k, \widetilde{\mathrm{SL}}_2(\mathbb{Z})) \to \mathrm{H}^1_{\text{ét}}(\operatorname{Spec} k, \mathrm{SL}_2(\mathbb{Z}))$$

is a bijection of pointed sets. Combining this fact with the exact sequence (5.1), one easily proves the following lemma.

Lemma 5.1. *The natural map* $\mathrm{H}^1_{\text{ét}}(\operatorname{Spec} k, \mathbf{Aut}_{\mathrm{Mod}_E}) \to \mathrm{H}^1_{\text{ét}}(\operatorname{Spec} k, \mathrm{SL}_2(\mathbb{Z}))$ *is surjective.*

Now, the descent spectral sequence for $\mathbf{Br}^E(k)$ yields an exact sequence

$$0 \to \mathrm{Br}(k) \to \mathrm{Br}^E(k) \to \mathrm{H}^1_{\text{ét}}(\operatorname{Spec} k, \mathbf{Aut}_{\mathrm{Mod}_E}) \to \mathrm{H}^3_{\text{ét}}(\operatorname{Spec} k, \mathbb{G}_m).$$

(The exactness on the left follows from the stabilizer conjecture for E, which can be proved by adapting the proof of the canonical (anti-)ample case to the non-CM elliptic curve E by using the explicit description of the sheaf of derived autoequivalences of Mod_E.) Let $v : \mathrm{Br}^E(k) \to \mathrm{H}^1_{\text{ét}}(\operatorname{Spec} k, \mathbf{Aut}_{\mathrm{Mod}_E}) \to \mathrm{H}^1_{\text{ét}}(\operatorname{Spec} k, \mathrm{SL}_2(\mathbb{Z}))$. Since $\mathrm{SL}_2(\mathbb{Z})$ is the constant étale sheaf, there is an equivalence

$$\mathrm{H}^1_{\text{ét}}(\operatorname{Spec} k, \mathrm{SL}_2(\mathbb{Z})) \cong \mathrm{Hom}_{\mathrm{cont}}(\mathrm{Gal}(k), \mathrm{SL}_2(\mathbb{Z})),$$

where $\mathrm{Hom}_{\mathrm{cont}}$ denotes continuous group homomorphisms.

Proposition 5.2. *To every twisted form* M *of* Mod_E *there is a canonical modular representation of* $\mathrm{Gal}(k)$.

Now, suppose that the v-invariant of M is trivial. Then, using the exact sequence

$$\mathrm{H}^1_{\text{ét}}(\operatorname{Spec} k, E \times \hat{E}) \to \mathrm{H}^1_{\text{ét}}(\operatorname{Spec} k, \mathbf{Aut}_{\mathrm{Mod}_E}) \to \mathrm{H}^1_{\text{ét}}(\operatorname{Spec} k, \mathrm{SL}_2(\mathbb{Z})),$$

it follows that the $\mathrm{Br}(k)$-orbit of M corresponds to a class of $\mathrm{H}^1_{\text{ét}}(\operatorname{Spec} k, E \times E)$. The two copies of E are not equal. One is E acting on itself via translations, the other is $\hat{E}$ acting on Mod_E by tensoring with line bundles. The set $\mathrm{H}^1_{\text{ét}}(\operatorname{Spec} k, E)$ contributes the categories Mod_C for homogeneous spaces of E, while $\mathrm{H}^1_{\text{ét}}(\operatorname{Spec} k, \hat{E})$ contributes the categories Mod^{α}_E, where $\alpha \in \mathrm{Br}(E)/\mathrm{Br}(k) \subseteq \mathrm{H}^1_{\text{ét}}(\operatorname{Spec} k, \hat{E})$, which fits into the exact sequence $0 \to \mathrm{Br}(k) \to \mathrm{Br}(E) \to \mathrm{H}^1_{\text{ét}}(k, \hat{E}) \to \mathrm{H}^3_{\text{ét}}(k, \mathbb{G}_m)$ coming from the Leray spectral sequence.

Consider twists $\mathrm{M} \simeq \mathrm{Mod}_C$ where C is a homogeneous space for E. It is impossible at the moment to give a full treatment of the stabilizer of Mod_C in $\mathrm{Br}(k)$. The same arguments used to prove the stabilizer conjecture in the canonical (anti-)ample case can be used for a non-CM elliptic curve as well, which shows that the stabilizer is exactly the kernel of $\mathrm{Br}(k) \to \mathrm{Br}(k(C))$. Until recently, very little was known about this kernel when C is a curve of genus higher than 0. This has changed with the work of [10, 17, 18]. In [10], the authors study this problem, and show that for homogeneous spaces of curves over numbers fields or local fields, the kernel can be computed algorithmically. They describe, for instance, a homogeneous space C for

$$y^2 + xy + y = x^3 = x^2 - 10x - 10$$

over $\mathbb{Q}$ where the kernel, and hence stabilizer group, is isomorphic to $\mathbb{Z}/4 \times \mathbb{Z}/2$. They also show that for a homogeneous space over a local field or number field, the stabilizer is finite [10, Proposition 4.11]. Over larger fields, they give an example to show that the stabilizer need not be finite in general. In the case of genus 0 curves, the stabilizer is not only finite, but in all cases has order at most 2.

5.3. Genus $g \geq 2$ curves. Let C be a smooth projective curve over k having genus $g \geq 2$. Then, ω_C is ample, so that the automorphism group of Mod_C can be computed by Bondal and Orlov:

$$\mathrm{Aut}_{\mathrm{Mod}_C} \cong \mathbb{Z} \times (\mathrm{Pic}(C) \rtimes \mathrm{Aut}(C)).$$

Since there are no $\mathbb{Z}$-torsors over $\operatorname{Spec} k$, there is an exact sequence

$$\mathrm{H}^1_{\text{ét}}(\operatorname{Spec} k, \mathbf{Pic}^0_{C/k}) \to \mathrm{H}^1_{\text{ét}}(\operatorname{Spec} k, \mathbf{Aut}_{\mathrm{Mod}_C}) \to \mathrm{H}^1_{\text{ét}}(\operatorname{Spec} k, \mathbf{Aut}_C) \to *,$$

where $\mathbf{Pic}^0_{C/k}$ is the Jacobian variety of C, and where the map is surjective on the right since the surjection $\mathbf{Aut}_{\mathrm{Mod}_C} \to \mathbf{Aut}_C$ splits. There is again a sequence

$$\mathrm{Br}(k) \to \mathrm{Br}^C(k) \to \mathrm{H}^1_{\text{ét}}(\operatorname{Spec} k, \mathbf{Pic}^0_{C/k} \rtimes \mathbf{Aut}_C) \to \mathrm{H}^3_{\text{ét}}(\operatorname{Spec} k, \mathbb{G}_m),$$

with the same exactness properties as the sequence above for $\mathbb{P}^1$. The kernel on the left is precisely the kernel of $\mathrm{Br}(k) \to \mathrm{Br}(C)$ by Theorem 3.5.

Proposition 5.3. *The twists* M *of* Mod_C *are the categories* Mod^{α}_D *for D a twisted form of C and $\alpha \in \mathrm{Br}_{\mathrm{sep}}(D)$, where* $\mathrm{Br}_{\mathrm{sep}}(D) = \ker(\mathrm{Br}(D) \to \mathrm{Br}(D_{k^{\mathrm{sep}}}))$.

Proof. If k is separable, then $\mathrm{Br}_{\mathrm{sep}}(D) = \mathrm{Br}(D)$. Since we can change the basepoint of $\mathrm{Br}^C(k)$ to Mod_D for any twisted form D of C, it is enough to treat the classes in the fiber of

$$\mathrm{Br}^C(k) \to \mathrm{H}^1_{\text{ét}}(\operatorname{Spec} k, \mathbf{Aut}_{\mathrm{Mod}_C}) \to \mathrm{H}^1_{\text{ét}}(\operatorname{Spec} k, \mathbf{Aut}_C).$$

Write F for the set of these points. Then, F can be described by the exact sequence

$$\mathrm{Br}(k) \to F \to \mathrm{H}^1_{\text{ét}}(\operatorname{Spec} k, \mathbf{Pic}^0_{C/k}) \to \mathrm{H}^3_{\text{ét}}(\operatorname{Spec} k, \mathbb{G}_m),$$

where the map $\mathrm{H}^1_{\text{ét}}(\operatorname{Spec} k, \mathbf{Pic}^0_{C/k}) \to \mathrm{H}^3_{\text{ét}}(\operatorname{Spec} k, \mathbb{G}_m)$ is induced from the sequence above. The Leray spectral sequence and the fact that $R^2 p_* \mathbb{G}_{m,C} = 0$, where $p : C \to \operatorname{Spec} k$, shows that there is also an exact sequence

$$\mathrm{Br}(k) \to \mathrm{Br}_{\mathrm{sep}}(C) \to \mathrm{H}^1_{\text{ét}}(\operatorname{Spec} k, \mathbf{Pic}^0_{C/k}) \to \mathrm{H}^3_{\text{ét}}(\operatorname{Spec} k, \mathbb{G}_m).$$

The map $\mathrm{Br}_{\mathrm{sep}}(C) \to F$ given by sending $\alpha \in \mathrm{Br}_{\mathrm{sep}}(C)$ to Mod^α_C induces a map of these sequences, from which it follows that $\mathrm{Br}_{\mathrm{sep}}(C)$ surjects onto F, which proves the proposition. □

The stabilizer of Mod_C is again the kernel of $\mathrm{Br}(k) \to \mathrm{Br}(C) = \mathrm{Br}(k(C))$. As far as I know, when the genus is $g \geq 2$, almost nothing is known about the stabilizer groups, except for the fact that it vanishes if C has a k-point. In that case, the map $\mathrm{H}^1_{\text{ét}}(\operatorname{Spec} k, \mathbf{Pic}^0_{C/k}) \to \mathrm{H}^3_{\text{ét}}(\operatorname{Spec} k, \mathbb{G}_m)$ is identically zero, because $\mathrm{H}^3_{\text{ét}}(\operatorname{Spec} k, \mathbb{G}_m) \to \mathrm{H}^3_{\text{ét}}(C, \mathbb{G}_m)$ is injective (a k-point defines a section).

5.4. Quadric hypersurfaces. Assume for simplicity that the characteristic of k is not 2.

Consider the quadratic form $q = x_0^2 + \cdots + x_{2n-1}^2 - x_{2n}^2$ on k^{2n+1}. Let $X = X(q)$ be the quadric hypersurface in $\mathbb{P}^{2n}$ cut out by q. I want to study $\mathrm{Br}^X(k)$. As X is Fano and $\mathrm{Pic}(X) = \mathbb{Z}$, the theorem of Bondal and Orlov [5] says that $\mathrm{Aut}_{\mathrm{Mod}_X} = \mathbb{Z} \times (\mathbb{Z} \times \mathrm{Aut}(X))$. Therefore, every element of $\mathrm{Br}^X(k)$ is Mod^α_Y for a twisted form Y of X and a Brauer class $\alpha \in \mathrm{Br}(k)$. Every such Y is determined by another nondegenerate quadratic form p on k^{2n+1}. The interesting question is then what is the stabilizer of Mod_Y for such a twist Y; in other words, by Theorem 3.5, what is the kernel of $\mathrm{Br}(k) \to \mathrm{Br}(k(Y))$? This is in fact a classical question. Let $C_0(p)$ denote the even Clifford algebra of p, which is a central simple algebra. To compute the kernel of $\mathrm{Br}(k) \to \mathrm{Br}(Y)$, it is enough to compute the kernel of $\mathrm{Br}(k) \to \mathrm{Br}(k(Y))$, since Y is smooth. A division algebra D is in the kernel if and only if the index of $D \otimes_k k(Y)$ is 1. But, this index was computed to be

$$\gcd\{\mathrm{ind}(D), 2^{n-1}\mathrm{ind}(D \otimes_k C_0(p))\},$$

by, for instance, Merkurjev–Panin–Wadsworth [28]. Because $C_0(p)$ has degree a power of 2, the kernel must be 2-primary. Therefore, if $n > 1$, the kernel is always 0.

The case $n = 1$ was already handled in the case of genus 0 curves. The following theorem summarizes the situation. Note that the statement about surjectivity follows for the same reason as for genus 0 curves; namely, concrete models can be constructed by taking a twist $Y \in \mathrm{H}^1_{\text{ét}}(\operatorname{Spec} k, \mathbf{PSO}(q))$ and then considering $\operatorname{Mod}_Y \in \operatorname{Br}^X(k)$.

Theorem 5.4. *Suppose that $n > 1$ and that X is the quadric hypersurface in $\mathbb{P}^{2n}$ considered above. Then, there is a sequence*

$$0 \to \operatorname{Br}(k) \to \operatorname{Br}^X(k) \to \mathrm{H}^1_{\text{ét}}(\operatorname{Spec} k, \mathbf{PSO}(q)) \to *,$$

which is exact in the sense that the action of $\operatorname{Br}(k)$ on $\operatorname{Br}^X(k)$ is free, and two elements of $\operatorname{Br}^X(k)$ map to the same element of $\mathrm{H}^1_{\text{ét}}(\operatorname{Spec} k, \mathbf{PSO}(q))$ if and only if they are in the same $\operatorname{Br}(k)$-orbit.

Now, consider $q = x_0^2 + \cdots + x_{2n-2}^2 - x_{2n-1}^2$ on k^{2n}, and let $X = X(q)$ be the quadric hypersurface in $\mathbb{P}^{2n-1}$ cut out by q. Again, in this case every class of $\operatorname{Br}^X(k)$ is $\operatorname{Mod}_Y^\alpha$ where Y is a twist of X (an involution variety) and $\alpha \in \operatorname{Br}(k)$. To consider the stabilizer of $\operatorname{Br}(k)$ on Y, it suffices to compute the index of $D \otimes_k k(Y)$ as above. By [28], this is

$$\operatorname{ind}(D \otimes_k k(Y)) = \gcd\{\operatorname{ind}(D), 2^{n-2}\operatorname{ind}(D \otimes_k C(p)\},$$

where $C(p)$ is the full Clifford algebra of p. The Clifford algebra $C(p)$ has index a 2-power, so that the kernel is once again 2-primary. Therefore, if $n > 2$, the kernel vanishes. When $n = 2$, the stabilizer of the quadric hypersurface Y in $\operatorname{Br}^X(k)$ is generated by the central simple algebra $C(p)$.

Theorem 5.5. *Suppose that $n \geq 1$, and let X be the quadric hypersurface in $\mathbb{P}^{2n-1}$ considered above. Then, there is a sequence*

$$0 \to \operatorname{Br}(k) \to \operatorname{Br}^X(k) \to \mathrm{H}^1_{\text{ét}}(\operatorname{Spec} k, \mathbf{PSO}(q)) \to *,$$

which is exact in the sense that the action of $\operatorname{Br}(k)$ on $\operatorname{Br}^X(k)$ is faithful, and two elements of $\operatorname{Br}^X(k)$ map to the same element of $\mathrm{H}^1_{\text{ét}}(\operatorname{Spec} k, \mathbf{PSO}(q))$ if and only if they are in the same $\operatorname{Br}(k)$-orbit. Moreover, if $n > 2$, the action is free. If $n = 2$, then the stabilizer of the quadric surface Y associated to a nondegenerate quadratic form p is generated by the Clifford algebra $C(p)$.

Remark 5.6. Using the work of Merkurjev, Panin, and Wadsworth, the same game can be played for the twisted flag varieties associated to any classical semisimple adjoint linear algebraic group.

If X is a quadric hypersurface, then $\operatorname{Mod}_X \simeq \operatorname{Mod}_A$ for an ordinary associative algebra A, as follows from a theorem of Kapranov [21]. The classification theorem says that every object of $\operatorname{Br}^X(k)$ is equivalent to Mod_B for some ordinary k-algebra B, giving another positive answer to Question 2.10.

5.5. Noncommutative Severi–Brauer varieties. Kontsevich and Rosenberg [23] introduced a noncommutative space $\mathbb{NP}^{n-1}$ that represents the functor which takes an associative algebra A to the set of quotients of A^n that are locally isomorphic to A in a flat topology on associative rings. They thus called it the noncommutative projective space. They identified its derived category with the derived category of finite representations of the quiver Ω_n. Thus, I consider $\mathrm{Mod}_{k\Omega_n}$ as the model for noncommutative projective space. Except when $n = 2$, this is *not* the derived category of $\mathbb{P}^{n-1}$. Nevertheless, in [29], Miyachi and Yekutieli showed another similarity between $\mathbb{NP}^{n-1}$ and $\mathbb{P}^{n-1}$ by computing the group of equivalences of $\mathrm{Mod}_{k\Omega_n}$ and showing that it is $\mathbb{Z} \times (\mathbb{Z} \rtimes \mathbf{PGL}_n(k))$. It follows that for every PGL_n-torsor P over k, there is a well-defined twisted form of $\mathrm{Mod}_{k\Omega_n}$, which I will denote M^P. But, the PGL_n-torsors are in one-to-one correspondence with Severi–Brauer varieties. So, M^P is a noncommutative twist of the Severi–Brauer variety P.

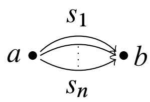

Figure 2. The quiver Ω_n.

Theorem 5.7. *There is a bijection between the* $\mathrm{Br}(k)$*-sets* $\mathrm{Br}^{\mathbb{P}^n}(k)$ *and* $\mathrm{Br}^{\mathbb{NP}^n}(k)$.

Once again, these can be described using path algebras for modulated quivers, so there is a positive answer to Question 2.10.

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