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The Gauss–Manin connection for the cyclic homology of smooth deformations, and noncommutative tori

Allan Yashinski*

Abstract. Given a smooth deformation of topological algebras, we define Getzler's Gauss–Manin connection on the periodic cyclic homology of the corresponding smooth field of algebras. Basic properties are investigated including the interaction with the Chern–Connes pairing with K-theory. We use the Gauss–Manin connection to prove a rigidity result for periodic cyclic cohomology of Banach algebras with finite weak bidimension. Then we illustrate the Gauss–Manin connection for the deformation of noncommutative tori. We use the Gauss–Manin connection to identify the periodic cyclic homology of a noncommutative torus with that of the commutative torus via a parallel translation isomorphism. We explicitly calculate the parallel translation maps and use them to describe the behavior of the Chern–Connes pairing under this deformation.

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1. Introduction

Our motivating problem is to understand the behavior of periodic cyclic homology under deformation of the algebra structure. Given a family of algebras $\{A_t\}_{t\in J}$ parametrized by a real number t, we would like to identify conditions under which we can conclude

$$HP_\bullet(A_t) \cong HP_\bullet(A_s), \quad \forall t, s \in J.$$

The types of algebras we consider will be topological algebras, and the deformations will have a smooth dependence on t.

In the setting of formal deformations, Getzler constructed a connection on the periodic cyclic complex of a deformation [13]. His connection, called the *Gauss–Manin connection*, commutes with the boundary map on the periodic cyclic complex

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and descends to a flat connection on the periodic cyclic homology of the deformation. We shall adapt Getzler's connection to our setting of smooth deformations and investigate its properties.

For a real interval $J \subseteq \mathbb{R}$, we consider a smoothly varying family $\{m_t\}_{t\in J}$ of jointly continuous associative multiplications on a complete locally convex vector space X. For each $t \in J$, we have a locally convex algebra A_t whose underlying space is X and whose multiplication is given by m_t. These algebras can be collected to form the algebra A_J of smooth sections of the bundle of algebras over J whose fiber at $t \in J$ is A_t, where the multiplication in A_J is defined fiberwise. Then A_J is an algebra over $C^\infty(J)$, the space of smooth complex-valued functions defined on the parameter space J.

The complex of interest to us is the periodic cyclic complex of A_J over the ground ring $C^\infty(J)$. This can be thought of as the space of smooth sections of the bundle of chain complexes over J whose fiber at $t \in J$ is the periodic cyclic complex of A_t. It is on this complex that we shall define Getzler's Gauss–Manin connection ∇_{GM}. The connection ∇_{GM} commutes with the boundary map and thus descends to a connection on the $C^\infty(J)$-linear periodic cyclic homology $HP_\bullet(A_J)$.

A fundamental issue for us is to determine when we can perform parallel translation with respect to ∇_{GM} at the level of periodic cyclic homology. Indeed, doing so would provide isomorphisms $HP_\bullet(A_t) \cong HP_\bullet(A_s)$ between the periodic cyclic homology groups of any two algebras in the deformation. Of course, the striking degree of generality for which ∇_{GM} exists indicates that the connection will not always be integrable. The goal then is to identify properties of a deformation that allow for parallel translation.

An important feature of cyclic cohomology is that it provides numerical invariants of K-theory classes through the Chern–Connes pairing

$$\langle \cdot, \cdot \rangle : HP^i(A) \times K_i(A) \to \mathbb{C}, \quad i = 0, 1,$$

between periodic cyclic cohomology and algebraic K-theory. The Gauss–Manin connection is compatible with this pairing in the following sense. Given a smoothly varying family $\{\varphi_t \in C^{\text{even}}(A_t)\}_{t\in J}$ of even cocycles and a smoothly varying family of idempotents $\{P_t \in M_N(A_t)\}_{t\in J}$ in matrix algebras, we have

$$\frac{d}{dt}\langle [\varphi_t], [P_t] \rangle = \langle \nabla^{GM}[\varphi_t], [P_t] \rangle,$$

where ∇^{GM} denotes the dual Gauss–Manin connection on periodic cyclic cohomology. A similar result holds for the pairing of odd cocycles and invertibles, representing classes in $K_1(A_t)$. Thus being able to compute with ∇^{GM} gives insight into how the numerical invariants arising from this pairing are changing with the parameter t.

Our main abstract result is a rigidity result for periodic cyclic cohomology of Banach algebras of finite weak bidimension. The weak bidimension $\mathrm{db}_w\, A$ of a

Banach algebra A is the smallest integer n such that the Hochschild cohomology $H^{n+1}(A, M^*)$ vanishes for all Banach A-bimodules M. A Banach algebra A is called amenable if $\mathrm{db}_w A = 0$, and more generally is called $(n+1)$-amenable if $\mathrm{db}_w A = n$. Amenable Banach algebras were defined by Johnson [18], and higher dimensional versions of amenability were studied in [9,24,27,33]. We prove that the Gauss–Manin connection is integrable for small enough deformations of a Banach algebra of finite weak bidimension. Consequently, periodic cyclic cohomology is preserved under such deformations.

We then illustrate the Gauss–Manin connection in the nontrivial deformation of noncommutative tori, a well-studied example in noncommutative geometry [30]. The smooth noncommutative n-torus $\mathcal{A}_\Theta$ can be naturally viewed as a deformation of $C^\infty(\mathbb{T}^n)$, the algebra of smooth complex-valued functions on the n-torus. The cyclic cohomology of $\mathcal{A}_\Theta$ is well-known, as it was computed directly by Connes in the $n = 2$ case [4] and by Nest in the general case [25]. We prove the integrability of the Gauss–Manin connection for this deformation and thus obtain a parallel translation isomorphism $HP_\bullet(\mathcal{A}_\Theta) \cong HP_\bullet(C^\infty(\mathbb{T}^n))$. Since the latter can be computed in terms of de Rham cohomology [4], this provides a deformation theoretic computation of $HP_\bullet(\mathcal{A}_\Theta)$.

We also explicitly describe the dual Gauss–Manin connection ∇^{GM} in terms of a natural basis for $HP^\bullet(\mathcal{A}_\Theta)$. This allows us to compute the derivatives of the Chern–Connes pairing, as discussed above. We show that the Chern–Connes pairings for the noncommutative torus can be expressed as a polynomial function in the parameters whose coefficients are Chern–Connes pairings for the commutative torus $C^\infty(\mathbb{T}^n)$. For example, given a smoothly varying (in Θ) family of idempotents $\{P_\Theta \in M_N(\mathcal{A}_\Theta)\}$, the Chern–Connes pairings with P_Θ are determined by, and can be computed from, the characteristic classes of the smooth vector bundle associated to $P_0 \in M_N(C^\infty(\mathbb{T}^n))$. Related formulas were found by Elliott [10], who computed the Chern character of noncommutative tori in the sense of [2].

Similar work on the Gauss–Manin connection was carried out independently by Yamashita [36].

The outline of the paper is as follows. In §2, we fix notation and recall necessary background material regarding locally convex topological vector spaces, Hochschild and cyclic homology, operations on the cyclic complex and Getzler's Cartan homotopy formula. In §3, we set up the general framework in which we will study smooth deformations of algebras and chain complexes. Our main techniques for studying deformations use connections and parallel translation, in particular we give a criterion for triviality of these deformations in terms of integrable connections. In §4, we define Getzler's Gauss–Manin connection in our setting of smooth deformations, and prove some of its basic properties. The rigidity theorem for cyclic cohomology of Banach algebras of finite weak bidimension is proved in §5. The remainder of the paper is on the application to noncommutative tori. In §6, we prove the integrability of the Gauss–Manin connection for the deformation of noncommutative tori and obtain a parallel translation isomorphism in periodic cyclic homology. In §7, we explicitly describe the operator ∇^{GM} and its parallel translation for noncommutative tori. This is used to describe the deformation of the Chern–Connes pairing. For the sake of streamlining the presentation, the proofs of some statements have been postponed to an appendix.

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2. Preliminaries

2.1. Locally convex algebras and modules. See [34] for background in the theory of locally convex topological vector spaces. More details concerning topological tensor products can be found in [14,15,34].

We shall work in the category LCTVS of complete, Hausdorff locally convex topological vector spaces over $\mathbb{C}$ and continuous linear maps. All bilinear/multilinear maps we work with will be assumed to be jointly continuous. For example, by a *locally convex algebra*, we mean a space $A \in$ LCTVS with a jointly continuous associative multiplication. Similarly, a *locally convex module* over a locally convex algebra has a jointly continuous module action.

Dealing with joint continuity leads naturally to projective tensor products. If R is a unital commutative locally convex algebra and M and N are locally convex R-modules, then $M \widehat{\otimes}_R N$ denotes the completed projective tensor product over R, which is universal for jointly continuous R-bilinear maps from $M \times N$ into locally convex R-modules, see [15, Chapter II]. Given $X, Y \in$ LCTVS, we shall simply write $X \widehat{\otimes} Y$ for $X \widehat{\otimes}_{\mathbb{C}} Y$.

Given $X, Y \in$ LCTVS, we equip the space $\mathrm{Hom}(X, Y)$ of continuous linear maps from X to Y with the topology of uniform convergence on bounded subsets of X. It is a Hausdorff locally convex topological vector space, but it may not be complete. However, it is for many nice cases, for example if X is a Fréchet space or an LF-space. The *strong dual* of X is $X^* = \mathrm{Hom}(X, \mathbb{C})$. We remark that the strong dual of a Banach space is a Banach space, but the strong dual of a Fréchet space is never a Fréchet space, unless the original space is actually a Banach space.

Given two locally convex R-modules M and N, we topologize $\mathrm{Hom}_R(M, N)$ as a subspace of $\mathrm{Hom}_{\mathbb{C}}(M, N)$. When $N = R$, we obtain the topological R-linear dual

$$M^\star := \mathrm{Hom}_R(M, R).$$

We shall use the notation $M^\bigstar$ to distinguish from M^*, which will always mean the usual $\mathbb{C}$-linear topological dual space.

A locally convex R-module is *free* if it is isomorphic to $R\widehat{\otimes}_{\mathbb{C}}X$ for some $X \in$ LCTVS, where the module structure is given by multiplication in the R factor. The free module $R\widehat{\otimes}X$ has the usual universal property that any continuous $\mathbb{C}$-linear map $\widetilde{F}$ from X into a locally convex R-module M induces a unique continuous R-linear map $F : R\widehat{\otimes}X \to M$ given by

$$F(r \otimes x) = r\widetilde{F}(x).$$

This establishes a linear isomorphism

$$\operatorname{Hom}_R(R\widehat{\otimes}X, M) \cong \operatorname{Hom}(X, M),$$

which is a topological isomorphism for nice spaces, see Proposition A.1.

By a *locally convex cochain complex*, we mean a collection of spaces $\{\mathcal{C}^n\}_{n\in\mathbb{Z}}$ in LCTVS and continuous linear maps $\{d^n : \mathcal{C}^n \to \mathcal{C}^{n+1}\}_{n\in\mathbb{Z}}$ such that $d^{n+1} \circ d^n = 0$. We'll use the notation $Z^n(\mathcal{C}) = \ker d^n$ for cocycles and $B^n(\mathcal{C}) = \operatorname{im} d^{n-1}$ for coboundaries. The cohomology is $H^n(\mathcal{C}) = Z^n(\mathcal{C})/B^n(\mathcal{C})$, which may not be Hausdorff or complete. If each $\mathcal{C}^n$ is a locally convex R-module and the coboundary maps are R-linear, then $\mathcal{C}^\bullet$ is a *locally convex cochain complex of R-modules*. In this case, the cohomology spaces are R-modules.

2.2. Hochschild and cyclic homology for locally convex algebras. Additional details regarding cyclic and Hochschild homology can be found in [4,23].

Let R be a unital commutative locally convex algebra and let A be a (possibly nonunital) locally convex R-algebra. Let A_+ denote the R-linear unitization of the algebra A. As an R-module, $A_+ = A \oplus R$. We can, and will, form the unitization in the case where A is already unital. We shall let $e = (0, 1) \in A_+$ denote the unit of A_+, to avoid possible confusion with the original unit of A, if it exists.

For $n \geq 0$, the space of Hochschild n-chains is defined to be

$$C_n(A) = \begin{cases} A, & n = 0, \\ A_+\widehat{\otimes}_R A^{\widehat{\otimes}_R n}, & n \geq 1. \end{cases}$$

The maps $b : C_n(A) \to C_{n-1}(A)$ and $B : C_n(A) \to C_{n+1}(A)$ are given by

$$\begin{aligned} b(a_0 \otimes \cdots \otimes a_n) &= \sum_{j=0}^{n-1}(-1)^j a_0 \otimes \cdots \otimes a_{j-1} \otimes a_j a_{j+1} \otimes a_{j+2} \otimes \cdots \otimes a_n \\ &\qquad + (-1)^n a_n a_0 \otimes a_1 \otimes \cdots \otimes a_{n-1}, \\ B(a_0 \otimes \cdots \otimes a_n) &= \sum_{j=0}^{n}(-1)^{jn} e \otimes a_j \otimes \cdots \otimes a_n \otimes a_0 \otimes \cdots \otimes a_{j-1}, \end{aligned}$$

$B(e \otimes a_1 \otimes \cdots \otimes a_n) = 0,$

and satisfy $b^2 = B^2 = bB + Bb = 0$. The homology of the complex $(C_\bullet(A), b)$ is the Hochschild homology of A, denoted $HH_\bullet(A)$ or $HH^R_\bullet(A)$ to emphasize R.

The periodic cyclic chain complex is the $\mathbb{Z}/2$-graded complex

$$C_{\mathrm{per}}(A) = C_{\mathrm{even}}(A) \oplus C_{\mathrm{odd}}(A) = \left(\prod_{n=0}^{\infty} C_{2n}(A)\right) \bigoplus \left(\prod_{n=0}^{\infty} C_{2n+1}(A)\right)$$

equipped with the differential $b + B$. Its homology groups are the even and odd periodic cyclic homology groups of A, and are denoted $HP_0(A)$ and $HP_1(A)$ respectively.

To obtain Hochschild cohomology $HH^\bullet(A)$ and periodic cyclic cohomology $HP^\bullet(A)$ of A, we take the cohomology of the continuous R-linear dual complexes

$$C^n(A) = C_n(A)^\bigstar, \quad C^{\mathrm{per}}(A) = C_{\mathrm{per}}(A)^\bigstar.$$

Then we have a canonical pairing

$$\langle \cdot, \cdot \rangle : C^{\mathrm{per}}(A) \times C_{\mathrm{per}}(A) \to R,$$

which descends to an R-bilinear map

$$\langle \cdot, \cdot \rangle : HP^\bullet(A) \times HP_\bullet(A) \to R.$$

We need another variant of Hochschild cohomology. Let

$$C^k(A, A) = \mathrm{Hom}_R(A^{\widehat{\otimes}_R k}, A)$$

with the coboundary map $\delta : C^k(A, A) \to C^{k+1}(A, A)$ given by

$$\delta D(a_1, \ldots, a_{k+1}) = D(a_1, \ldots, a_k)a_{k+1} + (-1)^{k+1} a_1 D(a_2, \ldots, a_{k+1}) + \sum_{j=1}^{k} (-1)^{k-j+1} D(a_1, \ldots, a_{j-1}, a_j a_{j+1}, a_{j+2}, \ldots, a_k).$$

This is a cochain complex, and its cohomology $H^\bullet(A, A)$ is the Hochschild cohomology of A (with coefficients in A). A relevant example is that $D \in C^1(A, A)$ satisfies $\delta D = 0$ if and only if D is a derivation. There is much additional structure on $C^\bullet(A, A)$, including a cup product and a Lie bracket, called the Gerstenhaber bracket [11].

2.3. Chern–Connes pairing. The Chern–Connes character, see [23, Ch. 8], is a group homomorphism

$$\mathrm{ch} : K_i(A) \to HP^R_i(A), \quad i = 0, 1,$$

where $K_i(A)$ is the algebraic K-theory group of A. The natural pairing between cyclic homology and cyclic cohomology gives the Chern–Connes pairing

$$\langle\cdot,\cdot\rangle : HP^i_R(A) \times K_i(A) \to R,$$
$$\langle[\varphi],[P]\rangle = \langle[\varphi],[\mathrm{ch}\, P]\rangle, \quad \langle[\psi],[U]\rangle = \langle[\psi],[\mathrm{ch}\, U]\rangle,$$

where P is an idempotent and U is an invertible in a matrix algebra $M_N(A)$.

2.4. Operations on the cyclic complex. The Cartan homotopy formula that follows was first observed by Rinehart in [32] in the case where D is a derivation, and later in full generality by Getzler in [13], see also [26,35]. An elegant and conceptual proof of the Cartan homotopy formula can be found in [20]. Our conventions vary slightly from [13], and are like those in [35].

Given $D \in C^k(A,A)$, the *Lie derivative* is the operator

$$L_D : C_n(A) \to C_{n-k+1}(A)$$

of degree $-(k-1)$ given by

$$L_D(a_0 \otimes \cdots \otimes a_n) = \sum_{i=0}^{n-k+1} (-1)^{i(k-1)} a_0 \otimes \cdots \otimes D(a_i,\ldots,a_{i+k-1}) \otimes \cdots \otimes a_n$$
$$+ \sum_{i=1}^{k-1} (-1)^{in} D(a_{n-i+1},\ldots,a_n,a_0,\ldots,a_{k-1-i}) \otimes a_{k-i} \otimes \cdots \otimes a_{n-i}.$$

In this formula and others below, it is understood that the result is 0 whenever D has the adjoined unit e as one of its arguments. A simple, but relevant, example is that if $D \in C^1(A,A)$, then

$$L_D(a_0 \otimes \cdots \otimes a_n) = \sum_{j=0}^{n} a_0 \otimes \cdots \otimes a_{j-1} \otimes D(a_j) \otimes a_{j+1} \otimes \cdots \otimes a_n.$$

These operators satisfy the following identities, where brackets denote graded commutators,

$$[L_D, L_E] = L_{[D,E]}, \quad [b, L_D] = L_{\delta D}, \quad [B, L_D] = 0.$$

There is also a contraction $I_D = \iota_D + S_D$ where

$$\iota_D : C_n(A) \to C_{n-k}(A), \quad S_D : C_n(A) \to C_{n-k+2}(A)$$

are given by

$$\iota_D(a_0 \otimes \cdots \otimes a_n) = (-1)^{k-1} a_0 D(a_1, \ldots, a_k) \otimes a_{k+1} \otimes \cdots \otimes a_n,$$

$$S_D(a_0 \otimes \cdots \otimes a_n) = \sum_{i=1}^{n-k+1} \sum_{j=0}^{n-i-k+1} (-1)^{i(k-1)+j(n-k+1)} e \otimes a_{n-j+1} \otimes \cdots$$

$$\cdots \otimes a_n \otimes a_0 \otimes \cdots \otimes a_{i-1} \otimes D(a_i, \ldots, a_{i+k-1}) \otimes a_{i+k} \otimes \cdots \otimes a_{n-j},$$

$$S_D(e \otimes a_1 \otimes \cdots \otimes a_n) = 0.$$

Theorem 2.1 (Cartan homotopy formula [13]). *For any $D \in C^\bullet(A, A)$,*

$$[b + B, I_D] = L_D - I_{\delta D}.$$

The Lie derivative and contraction operations of the previous section have multiple generalizations, see e.g. [13] or [35]. We shall need just one of these. For $X, Y \in C^1(A, A)$, define the operators $L\{X, Y\}$ and $I\{X, Y\}$ on $C_\bullet(A)$ by

$$L\{X, Y\}(a_0 \otimes \cdots \otimes a_n) = \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} a_0 \otimes \cdots \otimes X(a_i) \otimes \cdots \otimes Y(a_j) \otimes \cdots \otimes a_n$$

$$+ \sum_{i=1}^{n} Y(a_0) \otimes a_1 \otimes \cdots \otimes X(a_i) \otimes \cdots \otimes a_n.$$

and

$$I\{X, Y\}(a_0 \otimes \cdots \otimes a_n) = \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} \sum_{m=0}^{n-j} (-1)^{nm} e \otimes a_{n-m+1} \otimes \cdots$$

$$\cdots \otimes a_n \otimes a_0 \otimes \cdots \otimes X(a_i) \otimes \cdots \otimes Y(a_j) \otimes \cdots \otimes a_{n-m}$$

if $a_0 \in A$ and

$$I\{X, Y\}(e \otimes a_1 \otimes \cdots \otimes a_n) = 0.$$

The following formulas appear in [13], with slightly different conventions.

Theorem 2.2. *If X and Y are derivations, then*

(i) $[b + B, I\{X, Y\}] = L\{X, Y\} + I_{X \smile Y} - I_Y I_X$.

(ii) $[b + B, L\{X, Y\}] = -L_{X \smile Y} + L_Y I_X - I_Y L_X$.

Notice that the second formula follows from the first by applying the commutator with $b + B$ and using the Cartan homotopy formula. The cup product is given by

$$(X \smile Y)(a_1, a_2) = -X(a_1) Y(a_2).$$

Example 2.3. Consider a nonnegatively graded algebra $A = \bigoplus_{n=0}^{\infty} A_n$. Let $D : A \to A$ be the algebra derivation defined by $D(a) = n \cdot a$ for all $a \in A_n$. The complex $C_{\text{per}}(A)$ decomposes into eigenspaces for L_D, depending on the total degree of a tensor. However, L_D acts by zero on $HP_\bullet(A)$ by Theorem 2.1 since $\delta D = 0$. Thus the nontrivial part of the homology is contained entirely in the 0-eigenspace for L_D, which coincides with $C_{\text{per}}(A_0)$. In this way, we see the inclusion $A_0 \to A$ induces an isomorphism $HP_\bullet(A_0) \cong HP_\bullet(A)$.

3. Smooth deformation theory

Let $J \subseteq \mathbb{R}$ be a nonempty open interval, which will serve as a parameter space. Loosely speaking, our general approach to deformation theory is as follows. Given a family of objects $\{E_t\}_{t\in J}$ that depend smoothly on t, we form a bundle E over J whose fiber at t is E_t. The object of interest is then the space of smooth sections of the bundle E. If each E_t has an underlying vector space structure (for example, if we are dealing with algebras, chain complexes, Lie algebras, etc), then the space of smooth sections is a locally convex $C^\infty(J)$-module. The space of sections will generally inherit new structure by considering any additional structure $\{E_t\}_{t\in J}$ had fiberwise.

We discuss some relevant properties of such modules below related to single variable calculus in a locally convex space. An extensive treatment of calculus in infinite-dimensional settings can be found in [22].

3.1. $C^\infty(J)$-modules.

Let $X \in$ LCTVS and consider the space $C^\infty(J, X)$ of infinitely differentiable functions on J with values in X, equipped with its usual topology of uniform convergence of functions and all their derivatives on compact subsets of J. Then $C^\infty(J) := C^\infty(J, \mathbb{C})$ is a nuclear Fréchet algebra and $C^\infty(J, X)$ is a locally convex module over $C^\infty(J)$ using pointwise scalar multiplication. As X is complete,

$$C^\infty(J, X) \cong C^\infty(J)\widehat{\otimes} X,$$

see e.g. [34, Theorem 44.1]. In other words, $C^\infty(J, X)$ is a free module. There are continuous linear “evaluation maps”

$$\epsilon_t : C^\infty(J, X) \to X, \quad \epsilon_t(x) = x(t).$$

Let us turn to morphisms of such modules. By properties of free modules, a continuous $C^\infty(J)$-linear map $F : C^\infty(J, X) \to C^\infty(J, Y)$ comes from a linear map $\widetilde{F} : X \to C^\infty(J, Y)$ and consequently a family of maps $\{F_t : X \to Y\}_{t\in J}$ defined by $F_t = \epsilon_t \circ \widetilde{F}$. We shall refer to a collection of maps $\{F_t : X \to Y\}_{t\in J}$ arising in this way as a *smooth homotopy of linear maps*. The map F is determined by the F_t, and these maps clearly are such that $t \mapsto F_t(x)$ is smooth for each $x \in X$.

Conversely if X is barreled (Fréchet spaces are examples of barreled spaces), then a family $\{F_t : X \to Y\}_{t\in J}$ of continuous linear maps for which $t \mapsto F_t(x)$ is smooth for each $x \in X$ induces a continuous $C^\infty(J)$-linear map

$$F : C^\infty(J, X) \to C^\infty(J, Y), \quad F(x)(t) = F_t(x(t)),$$

hence $\{F_t : X \to Y\}_{t\in J}$ is a smooth homotopy of maps (Proposition A.2).

The family of maps $\{F_t : X \to Y\}_{t\in J}$ arising from a $C^\infty(J)$-module map can also be viewed as a path in $\mathrm{Hom}(X, Y)$, which is smooth (Proposition A.4). This association

$$\mathrm{Hom}_{C^\infty(J)}\left(C^\infty(J, X), C^\infty(J, Y)\right) \to C^\infty\left(J, \mathrm{Hom}(X, Y)\right)$$

is a topological isomorphism if either

(i) X and Y are Banach spaces, or

(ii) X is a nuclear Fréchet space and $Y \in$ LCTVS,

see Proposition A.6. By considering the case $Y = \mathbb{C}$, we see that if X is either a Banach space or a nuclear Fréchet space, then

$$C^\infty(J, X)^\bigstar \cong C^\infty(J, X^*).$$

3.2. Connections and parallel translation. Since we are only dealing with one-parameter deformations, we shall only treat connections on $C^\infty(J)$-modules where the interval J represents the parameter space. As there is only one direction to differentiate in, a connection is determined by its covariant derivative. In what follows, we shall identify the two notions, and will commonly refer to covariant differential operators as connections.

Definition 3.1. A *connection* on a locally convex $C^\infty(J)$-module M is a continuous $\mathbb{C}$-linear map $\nabla : M \to M$ such that

$$\nabla(f \cdot m) = f' \cdot m + f \cdot \nabla m$$

for all $f \in C^\infty(J)$ and $m \in M$.

It is immediate that the difference of two connections is a continuous $C^\infty(J)$-linear map. Since the operator $\frac{d}{dt}$ is an example of a connection on the free module $C^\infty(J, X)$, we see that every connection ∇ on $C^\infty(J, X)$ is of the form

$$\nabla = \frac{d}{dt} - F$$

for some continuous $C^\infty(J)$-linear endomorphism $F : C^\infty(J, X) \to C^\infty(J, X)$.

An element in the kernel of a connection ∇ will be called a *parallel section* for ∇. A continuous $C^\infty(J)$-linear map

$$F : (M, \nabla_M) \to (N, \nabla_N)$$

between two modules with connections is *parallel* if $F \circ \nabla_M = \nabla_N \circ F$. A parallel map sends parallel sections to parallel sections.

If (M, ∇_M) and (N, ∇_N) are $C^\infty(J)$-modules with connections, then the operator $\nabla_M \otimes 1 + 1 \otimes \nabla_N$ is a connection on $M \widehat{\otimes}_{C^\infty(J)} N$. Additionally, the operator $\nabla_M^\star$ on $M^\star = \mathrm{Hom}_{C^\infty(J)}(M, C^\infty(J))$ given by

$$(\nabla_M^\star \varphi)(m) = \frac{d}{dt}\varphi(m) - \varphi(\nabla_M m)$$

is a connection. The definition of $\nabla_M^\star$ ensures that the canonical pairing

$$\langle \cdot, \cdot \rangle : M^\star \otimes_{C^\infty(J)} M \to C^\infty(J)$$

is a parallel map, where we consider the ground ring $C^\infty(J)$ with the connection $\frac{d}{dt}$. In other words,

$$\frac{d}{dt}\langle \varphi, m\rangle = \langle \nabla_M^\star \varphi, m\rangle + \langle \varphi, \nabla_M m\rangle.$$

Definition 3.2. A connection ∇ on $M = C^\infty(J, X)$ is *integrable* if there is a parallel isomorphism

$$F : (M, \nabla) \to \left(C^\infty(J, X), \frac{d}{dt}\right).$$

We shall express this condition in terms of parallel translation. We will think of M as sections of the trivial bundle whose fiber over $t \in J$ is $M_t \cong X$. Parallel translation relies on the existence and uniqueness of a solution $m \in M$ to the initial value problem

$$\nabla m = 0, \quad m(s) = x$$

for any given $s \in J$ and $x \in M_s$. In this case, the parallel translation operator

$$P_{s,t}^\nabla : M_s \to M_t$$

is the linear map defined by $P_{s,t}^\nabla(x) = m(t)$, where m is the unique solution to the above initial value problem. Then $P_{s,t}^\nabla$ is a linear isomorphism with inverse $P_{t,s}^\nabla$.

Parallel translation can always be done with respect to an integrable connection, because it obviously can be done with respect to $\frac{d}{dt}$. In the case where X is barreled, then a connection ∇ is integrable if and only if

(i) For every $s \in J$ and $x \in M_s$, there is a unique $m \in M$ such that

$$\nabla m = 0, \quad m(s) = x.$$

(ii) Each $P_{s,t}^\nabla : M_s \to M_t$ is continuous, and for each fixed $x \in X$, the map $(s,t) \mapsto P_{s,t}^\nabla(x)$ is smooth (i.e. all mixed partial derivatives exist);

see Proposition A.9.

If $F : (M, \nabla_M) \to (N, \nabla_N)$ is a parallel map between $C^\infty(J)$-modules with integrable connections, then it is straightforward to verify that the diagram

$$\begin{array}{ccc} M_s & \xrightarrow{F_s} & N_s \\ {\scriptstyle P^{\nabla_M}_{s,t}}\downarrow & & \downarrow{\scriptstyle P^{\nabla_N}_{s,t}} \\ M_t & \xrightarrow{F_t} & N_t \end{array}$$

commutes. From this we see that if $F : (M, \nabla) \to (C^\infty(J, X), \frac{d}{dt})$ is a parallel isomorphism, then $P^{\nabla}_{s,t} = F_t^{-1} \circ F_s$.

If ∇_M and ∇_N are integrable connections on M and N, then $\nabla_{\widehat{\otimes}} := \nabla_M \otimes 1 + 1 \otimes \nabla_n$ is integrable on $M \widehat{\otimes}_{C^\infty(J)} N$, and

$$P^{\nabla_{\widehat{\otimes}}}_{s,t} = P^{\nabla_M}_{s,t} \otimes P^{\nabla_N}_{s,t} : M_s \widehat{\otimes} N_s \to M_t \widehat{\otimes} N_t.$$

If the fiber X is either a Banach space or a nuclear Fréchet space (so that $C^\infty(J, X)^\bigstar \cong C^\infty(J, X^*)$), then the dual connection $\nabla^\bigstar_M$ is integrable on $M^\bigstar$ and

$$P^{\nabla^\bigstar_M}_{s,t} = \left(P^{\nabla}_{t,s}\right)^* : M_s^* \to M_t^*.$$

Let us consider the problem of parallel translation for a connection ∇ on $M = C^\infty(J, X)$. We have $\nabla = \frac{d}{dt} - F$ for some endomorphism $F : C^\infty(J, X) \to C^\infty(J, X)$ with corresponding maps $\{F_t : X \to X\}_{t \in J}$. To parallel translate, we must solve the first order linear ODE

$$x'(t) = F_t(x(t)), \quad x(s) = x_0$$

given any $s \in J$ and $x_0 \in X$. By iterating the fundamental theorem of calculus inductively, we see that any solution $x(t)$ satisfies

$$\begin{aligned} x(t) = x_0 &+ \sum_{n=1}^{N} \int_s^t \int_s^{u_1} \cdots \int_s^{u_{n-1}} (F_{u_1} \circ \cdots \circ F_{u_n})(x_0) du_n \ldots du_1 \\ &+ \int_s^t \int_s^{u_1} \cdots \int_s^{u_N} (F_{u_1} \circ \cdots \circ F_{u_{N+1}})(x(u_{N+1})) du_{N+1} du_N \ldots du_1 \end{aligned}$$

for any N. If the last term can be shown to converge to 0 in $C^\infty(J, X)$ as $N \to \infty$, then any solution $x(t)$ has the form

$$x(t) = x_0 + \sum_{n=1}^{\infty} \int_s^t \int_s^{u_1} \cdots \int_s^{u_{n-1}} (F_{u_1} \circ \cdots \circ F_{u_n})(x_0) du_n \ldots du_1.$$

This gives uniqueness of solutions. If this series can be shown to converge, we obtain existence of solutions. It is straightforward to show both of these in the case where X

is a Banach space. The fundamental theorem of calculus ensures that the solution depends smoothly on both t and s. These are well-known results from the theory of first order linear ODE's on a Banach space, which we restate in our language.

Theorem 3.3. *If X is a Banach space, then every connection on $C^\infty(J, X)$ is integrable.*

Once we start considering other classes of locally convex vector spaces, e.g. Fréchet spaces, the existence and uniqueness theorem for solutions to linear ODE's is false. One cannot guarantee that the above series defining the solution will converge.

3.3. Deformations of algebras. Let $X \in$ LCTVS and let J denote an open interval of real numbers.

Definition 3.4. A *smooth one-parameter deformation of algebras* is a smooth homotopy of continuous linear maps $\{m_t : X \widehat{\otimes} X \to X\}_{t\in J}$ for which each m_t is associative.

So for each $t \in J$, we have a locally convex algebra $A_t := (X, m_t)$ whose underlying space is X. By definition, there is a $C^\infty(J)$-linear map

$$m : C^\infty(J, X \widehat{\otimes} X) \to C^\infty(J, X)$$

associated to the maps $\{m_t\}_{t\in J}$. Letting $A_J = C^\infty(J, X)$, then m can be viewed as an associative multiplication

$$m : A_J \widehat{\otimes}_{C^\infty(J)} A_J \to A_J,$$

making A_J into a locally convex $C^\infty(J)$-algebra, which we shall refer to as the *algebra of sections* of the deformation $\{A_t\}_{t\in J}$. Explicitly, the multiplication in A_J is given by

$$(a_1 a_2)(t) = m_t(a_1(t), a_2(t))$$

for all $a_1, a_2 \in A_J$. The evaluation maps $\epsilon_t : A_J \to A_t$ are algebra homomorphisms.

If X is Fréchet, then our definition of a smooth deformation is equivalent to a smooth path in $\mathrm{Hom}(X \widehat{\otimes} X, X)$ whose image lies in the set of associative products. For a Fréchet space X, a set $\{m_t : X \widehat{\otimes} X \to X\}_{t\in J}$ of continuous multiplications is a smooth deformation if and only if the map $t \mapsto m_t(x_1, x_2)$ is smooth for each fixed $x_1, x_2 \in X$, see Corollary A.3.

A *morphism* between the deformations $\{A_t\}_{t\in J}$ and $\{B_t\}_{t\in J}$ is a continuous $C^\infty(J)$-linear algebra homomorphism $F : A_J \to B_J$. Thus a morphism is equivalent to a smooth homotopy $\{F_t : A_t \to B_t\}_{t\in J}$, for which each F_t is an algebra map. A deformation is *constant* if the products $\{m_t\}$ do not depend on t. A deformation is called *trivial* if it is isomorphic to a constant deformation. Thus $\{A_t\}_{t\in J}$ is trivial if and only if there is a locally convex algebra B such that $A_J \cong C^\infty(J, B)$ as algebras. We can characterize triviality of a smooth deformation of algebras in terms of connections.

Proposition 3.5. *The deformation* $\{A_t\}_{t\in J}$ *is trivial if and only if* A_J *admits an integrable connection that is a derivation with respect to the algebra structure. In this case, the parallel translation maps* $P^{\nabla}_{s,t} : A_s \to A_t$ *are isomorphisms of locally convex algebras.*

Proof. Notice that $\frac{d}{dt}$ is an integrable connection and a derivation on a constant deformation, so every trivial deformation possesses such a connection.

Conversely, suppose A_J has an integrable connection ∇ that is a derivation. That ∇ is a derivation is equivalent to the multiplication map

$$m : (A_J \widehat{\otimes}_{C^\infty(J)} A_J, \nabla \otimes 1 + 1 \otimes \nabla) \to (A_J, \nabla)$$

being a parallel map. Since parallel maps commute with parallel translation, we obtain

$$P^{\nabla}_{s,t} \circ m_s = m_t \circ (P^{\nabla}_{s,t} \otimes P^{\nabla}_{s,t}),$$

which shows that $P^{\nabla}_{s,t}$ is an algebra isomorphism. Let

$$F : (A_J, \nabla) \to \left(C^\infty(J, X), \frac{d}{dt}\right)$$

be a parallel isomorphism. Fix $s \in J$ and observe that

$$F^{-1} \circ F_s : C^\infty(J, A_s) \to A_J, \quad (F^{-1} \circ F_s)(a)(t) = F_t^{-1}(F_s(a(t))) = P^{\nabla}_{s,t}(a(t))$$

is an isomorphism of deformations, where $C^\infty(J, A_s)$ is the algebra of sections of the constant deformation with fiber A_s. □

Thus it is important to determine if a deformation has a connection that is a derivation. In analogy with the work of Gerstenhaber on formal deformations [12], the obstruction to this is cohomological.

Given any connection ∇ on A_J, define the bilinear map E by

$$\nabla(a_1 a_2) = \nabla(a_1)a_2 + a_1\nabla(a_2) - E(a_1, a_2).$$

So E is the defect of ∇ from being a derivation, and in fact $E = \delta\nabla$, where δ is the Hochschild coboundary. It follows that $\delta E = 0$. Using the Leibniz rule for ∇, one can check that E is a $C^\infty(J)$-bilinear map. So E defines a cohomology class $[E] \in H^2_{C^\infty(J)}(A_J, A_J)$. Notice that ∇ is only $\mathbb{C}$-linear and not $C^\infty(J)$-linear. Thus, we may have $[E] \neq 0$.

Proposition 3.6. *The cohomology class* $[E] \in H^2_{C^\infty(J)}(A_J, A_J)$ *is independent of the choice of connection. Moreover,* $[E] = 0$ *if and only if* A_J *possesses a connection that is a derivation.*

Proof. Let ∇ and ∇' be two connections with corresponding cocycles E and E'. Since $\nabla' = \nabla - F$ for some $F \in C^1_{C^\infty(J)}(A_J, A_J)$, we have

$$E' = \delta(\nabla - F) = E - \delta F,$$

which shows that $[E] = [E']$.

If ∇ is a connection that is a derivation, then $E = \delta\nabla = 0$. Conversely, if ∇ is any connection on A_J and $[E] = 0$, then $E = \delta F$ for some $F \in C^1_{C^\infty(J)}(A_J, A_J)$. Hence $\delta(\nabla - F) = 0$, so the connection $\nabla - F$ is a derivation. □

From this, we see that the cohomology class $[E]$ provides an obstruction to the triviality of a deformation. Even if this obstruction vanishes, there is still an analytic obstruction in that the corresponding connection may not be integrable. These two issues are common to the smooth deformation theory of other types of structures as well, e.g. cochain complexes (see below) or A_∞-algebras [37]. The class $[E]$ will also play an important role in the Gauss–Manin connection.

3.4. Deformations of cochain complexes. By a *smooth one-parameter deformation of cochain complexes*, we mean a collection $\{X^n\}_{n\in\mathbb{Z}}$ of spaces in LCTVS together with a smooth homotopy of continuous linear maps $\{d_t^n : X^n \to X^{n+1}\}_{t\in J}$ for each n such that $d_t^{n+1} \circ d_t^n = 0$ for all $t \in J$. For each $t \in J$, we have a locally convex cochain complex

$$(\mathcal{C}_t^\bullet, d_t) := \Big(\cdots \xrightarrow{d_t} X^{n-1} \xrightarrow{d_t} X^n \xrightarrow{d_t} X^{n+1} \xrightarrow{d_t} \cdots \Big)$$

built on the same underlying family of spaces. Let $\mathcal{C}_J^n = C^\infty(J, X^n)$ and let $d : \mathcal{C}_J^n \to \mathcal{C}_J^{n+1}$ be the continuous $C^\infty(J)$-linear map associated to the collection $\{d_t\}_{t\in J}$. We obtain a cochain complex

$$(\mathcal{C}_J^\bullet, d) := \Big(\cdots \xrightarrow{d} \mathcal{C}_J^{n-1} \xrightarrow{d} \mathcal{C}_J^n \xrightarrow{d} \mathcal{C}_J^{n+1} \xrightarrow{d} \cdots \Big)$$

of locally convex $C^\infty(J)$-modules. We'll call $\mathcal{C}_J^\bullet$ the *complex of sections* of the deformation. The cohomology $H^\bullet(\mathcal{C}_J)$ is a $C^\infty(J)$-module, and the evaluation chain maps $\epsilon_t : \mathcal{C}_J^\bullet \to \mathcal{C}_t^\bullet$ induce maps on cohomology

$$(\epsilon_t)_* : H^\bullet(\mathcal{C}_J) \to H^\bullet(\mathcal{C}_t).$$

The cohomology module $H^\bullet(\mathcal{C}_J)$ may be quite pathological. As a space, it may not be Hausdorff, and as a $C^\infty(J)$-module, it may not be free.

By a morphism of two deformations, we mean a continuous $C^\infty(J)$-linear (degree 0) chain map between their respective complexes of sections. We'll call a deformation *trivial* if it is isomorphic to a constant deformation. Under reasonable assumptions, the cohomology of a trivial deformation is a trivial bundle.

Proposition 3.7. *Suppose $(\mathcal{C}^\bullet, d)$ is a cochain complex of Fréchet spaces such that the cohomology $H^\bullet(\mathcal{C})$ is Hausdorff. Let $\mathcal{C}_J^\bullet = C^\infty(J, \mathcal{C}^\bullet)$ be the complex of sections of the constant deformation with fiber $\mathcal{C}^\bullet$. Then*

$$H^\bullet(\mathcal{C}_J) \cong C^\infty\left(J, H^\bullet(\mathcal{C})\right)$$

as locally convex $C^\infty(J)$-modules.

Proof. Notice that requiring $H^n(\mathcal{C}) = Z^n(\mathcal{C})/B^n(\mathcal{C})$ to be Hausdorff is equivalent to requiring the space of coboundaries $B^n(\mathcal{C})$ to be closed. In this case, both $B^n(\mathcal{C})$ and $H^n(\mathcal{C})$ are Fréchet spaces for all n.

Notice that $Z^n(\mathcal{C}_J) = C^\infty(J, Z^n(\mathcal{C}))$, but a priori we only have $B^n(\mathcal{C}_J) \subseteq C^\infty(J, B^n(\mathcal{C}))$. However, since $d : \mathcal{C}^n \to B^{n+1}(\mathcal{C})$ is a surjection of Fréchet spaces, it follows from [34, Proposition 43.9] that

$$1 \otimes d : C^\infty(J)\widehat{\otimes}\mathcal{C}^n \to C^\infty(J)\widehat{\otimes}B^{n+1}(\mathcal{C})$$

is surjective as well. That is, $B^n(\mathcal{C}_J) = C^\infty(J, B^n(\mathcal{C}))$ for all n. Thus,

$$\begin{aligned} H^n(\mathcal{C}_J) &= Z^n(\mathcal{C}_J)/B^n(\mathcal{C}_J) \\ &= C^\infty(J, Z^n(\mathcal{C}))/C^\infty(J, B^n(\mathcal{C})) \cong C^\infty(J, H^n(\mathcal{C})), \end{aligned}$$

where the last isomorphism is from Proposition A.1. □

Example 3.8. If $\{A_t\}_{t\in J}$ is a smooth deformation of algebras, then

$$\left\{(C_{\text{per}}(A_t), b_t + B)\right\}_{t\in J}$$

is a smooth deformation of chain complexes. Notice that the Hochschild boundary b_t depends on the multiplication of A_t, whereas the operator B does not. Since the completed projective tensor product commutes with direct products [17, Theorem 15.4.1], the complex of sections of $\{C_{\text{per}}(A_t)\}_{t\in J}$ is naturally identified with the periodic cyclic complex $C_{\text{per}}^{C^\infty(J)}(A_J)$. One can also consider the complexes associated to the various other homology/cohomology theories discussed above.

As in the algebra case, we can characterize triviality of a deformation of chain complexes in terms of connections. The proofs here are analogous to those in the algebra case.

Proposition 3.9. *A smooth deformation of cochain complexes $\{\mathcal{C}_t^\bullet\}_{t\in J}$ is trivial if and only if the complex of sections $\mathcal{C}_J^\bullet$ admits an integrable connection that is a chain map. For such a connection ∇, the parallel translation map $P_{s,t}^\nabla : \mathcal{C}_s^\bullet \to \mathcal{C}_t^\bullet$ is an isomorphism of locally convex cochain complexes for all $s, t \in J$. In particular, the parallel translation maps induce isomorphisms*

$$(P_{s,t}^\nabla)_* : H^\bullet(\mathcal{C}_s) \to H^\bullet(\mathcal{C}_t).$$

The obstruction to the existence of such a connection is again cohomological. Let ∇ be any connection on $\mathcal{C}_J^\bullet$, and consider the map

$$G = [d, \nabla] : \mathcal{C}_J^\bullet \to \mathcal{C}_J^{\bullet+1},$$

which is the defect of d from being a ∇-parallel map (equivalently, the defect of ∇ from being a chain map). It follows that G is $C^\infty(J)$-linear and $[d, G] = 0$, so G is a cocycle in the endomorphism complex $\mathrm{End}_{C^\infty(J)}(\mathcal{C}_J)$.

Proposition 3.10. *The cohomology class $[G] \in H^1(\mathrm{End}_{C^\infty(J)}(\mathcal{C}_J))$ is independent of the choice of connection ∇. Moreover, $[G] = 0$ if and only if $\mathcal{C}_J^\bullet$ admits a connection that is a chain map.*

Suppose $\mathcal{C}_J^\bullet$ is equipped with a connection ∇ that is a chain map. Our main goal is to identify the cohomology groups $H^\bullet(\mathcal{C}_s) \cong H^\bullet(\mathcal{C}_t)$ of different fibers via parallel translation. By Proposition 3.9, this happens when ∇ is integrable. In this case, the cochain complexes themselves are fiberwise isomorphic, and this may be too strong of a condition to be useful in practice.

As ∇ is a chain map, it induces a connection ∇_* on the $C^\infty(J)$-module $H^\bullet(\mathcal{C}_J)$. The homology module may not be free, complete, or even Hausdorff, so we should be careful about what we mean by integrability of ∇_*. Nonetheless, it makes sense to inquire about the existence and uniqueness of a solution $[c] \in H^\bullet(\mathcal{C}_J)$ to the cohomological differential equation

$$\nabla_*[c] = 0, \quad [c(s)] = [c_0]$$

with initial value $[c_0] \in H^\bullet(\mathcal{C}_s)$. If we have existence and uniqueness, we can construct parallel translation operators

$$P_{s,t}^{\nabla_*} : H^\bullet(\mathcal{C}_s) \to H^\bullet(\mathcal{C}_t),$$

which are linear isomorphisms.

4. The Gauss–Manin connection

4.1. Gauss–Manin connection in periodic cyclic homology. In this section, we'll construct Getzler's Gauss–Manin connection in our setting of smooth deformations. Let A_J denote the algebra of sections of a smooth one-parameter deformation of locally convex algebras $\{A_t\}_{t\in J}$. Unless specified otherwise, all chain groups and homology groups associated to A_J that follow are over the ground ring $C^\infty(J)$.

Consider the deformation of chain complexes $\{(C_{\mathrm{per}}(A_t), b_t + B)\}_{t\in J}$. As in Example 3.8, we can identify its complex of sections with $(C_{\mathrm{per}}(A_J), b + B)$. We would like to show, under favorable circumstances, that this deformation of complexes is trivial at the level of homology. To that end, we'd like to construct a connection on $C_{\mathrm{per}}(A_J)$ that is a chain map. As described in Proposition 3.10, this is a problem in cohomology. To start, let ∇ be any connection on A_J, and let $E = \delta\nabla$ as in Proposition 3.6. Then ∇ extends to a connection on the periodic cyclic complex $C_{\text{per}}(A_J)$, which is given by the Lie derivative L_∇. From Proposition 3.10, $C_{\text{per}}(A_J)$ has a connection that is a chain map if and only if the class of

$$G := [b + B, L_\nabla] = L_E$$

vanishes in $H^1(\text{End}(C_{\text{per}}(A_J)))$, i.e. L_E is chain homotopic to zero via a $C^\infty(J)$-linear homotopy operator. But the Cartan Homotopy formula

$$[b + B, I_E] = L_E$$

of Theorem 2.1 implies exactly this. Notice that I_E is $C^\infty(J)$-linear because E is $C^\infty(J)$-linear. We conclude that the *Gauss–Manin connection*

$$\nabla_{GM} = L_\nabla - I_E$$

is a connection on $C_{\text{per}}(A_J)$ and a chain map. Amazingly, the cohomological obstruction to the existence of such a connection vanishes for any deformation $\{A_t\}_{t\in J}$.

Proposition 4.1. *The Gauss–Manin connection ∇_{GM} commutes with the differential $b + B$ and hence induces a connection on the $C^\infty(J)$-module $HP_\bullet(A_J)$. Moreover, the induced connection on $HP_\bullet(A_J)$ is independent of the choice of connection ∇ on A_J.*

Proof. We have already established the first claim. For another connection ∇', let

$$\nabla'_{GM} = L_{\nabla'} - I_{E'}$$

be the corresponding Gauss–Manin connection. Then

$$\nabla' - \nabla = F, \qquad E' - E = \delta F$$

for some $C^\infty(J)$-linear map $F : A_J \to A_J$. Thus,

$$\nabla'_{GM} - \nabla_{GM} = L_F - I_{\delta F} = [b + B, I_F],$$

by Theorem 2.1. Thus the Gauss–Manin connection is unique up to chain homotopy. □

Corollary 4.2. *If A admits a connection ∇ which is also a derivation, then the Gauss–Manin connection on $HP_\bullet(A)$ is given by*

$$\nabla_{GM}[\omega] = [L_\nabla \omega].$$

As a trivial example, we see that the Gauss–Manin connection associated to a constant deformation is just the usual differentiation $\frac{d}{dt}$.

The Gauss–Manin connection is a canonical choice of a connection on $HP_\bullet(A_J)$. It is natural in the sense that morphisms of deformations induce parallel maps at the level of periodic cyclic homology.

Proposition 4.3 (Naturality of ∇_{GM}). *Let A_J and B_J denote the algebras of sections of two deformations over the same parameter space J, and let $F : A_J \to B_J$ be a morphism of deformations. Then the following diagram commutes.*

$$\begin{array}{ccc} HP_\bullet(A_J) & \xrightarrow{F_*} & HP_\bullet(B_J) \\ \downarrow{\scriptstyle \nabla_{GM}} & & \downarrow{\scriptstyle \nabla_{GM}} \\ HP_\bullet(A_J) & \xrightarrow{F_*} & HP_\bullet(B_J) \end{array}$$

Proof. Let ∇^A and ∇^B denote connections on A_J and B_J with respective cocycles E^A and E^B, and let $F_* : C_{\rm per}(A_J) \to C_{\rm per}(B_J)$ be the induced map of complexes. For

$$h = F_* I_{\nabla^A} - I_{\nabla^B} F_*,$$

we have

$$\begin{aligned} [b + B, h] &= F_*[b + B, I_{\nabla^A}] - [b + B, I_{\nabla^B}]F_* \\ &= F_*(L_{\nabla^A} - I_{E^A}) - (L_{\nabla^B} - I_{E^B})F_* \\ &= F_*\nabla^A_{GM} - \nabla^B_{GM} F_*. \end{aligned}$$

This shows that the diagram commutes up to chain homotopy. The problem is that I_{∇^A} and I_{∇^B} are not well-defined operators on the complexes $C_{\rm per}(A_J)$ and $C_{\rm per}(B_J)$ respectively (over $C^\infty(J)$), because ∇^A and ∇^B are not $C^\infty(J)$-linear operators. However, one can show that thanks to the Leibniz rule, h descends to a map of quotient complexes such that the following diagram

$$\begin{array}{ccc} C^{\mathbb{C}}_{\rm per}(A_J) & \xrightarrow{h} & C^{\mathbb{C}}_{\rm per}(B_J) \\ \downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle \pi} \\ C^{C^\infty(J)}_{\rm per}(A_J) & \xrightarrow{\bar h} & C^{C^\infty(J)}_{\rm per}(B_J) \end{array}$$

commutes, and consequently $[b + B, \bar h] = F_*\nabla^A_{GM} - \nabla^B_{GM} F_*$ as desired. □

As an application of Proposition 4.3, we obtain the differentiable homotopy invariance property of periodic cyclic homology by considering a morphism $\{F_t : A \to B\}_{t\in J}$ between two constant deformations. For both deformations, $\nabla_{GM} = \frac{d}{dt}$.

4.2. Dual Gauss–Manin connection.

We define ∇^{GM} on $C^{\rm per}(A_J)$ to be the dual connection of ∇_{GM}. In terms of the canonical pairing,

$$\langle \nabla^{GM}\varphi, \omega\rangle = \frac{d}{dt}\langle \varphi, \omega\rangle - \langle \varphi, \nabla_{GM}\omega\rangle.$$

It is straightforward to verify that ∇^{GM} commutes with $b + B$ and therefore induces a connection on $HP^\bullet(A_J)$. The connections ∇_{GM} and ∇^{GM} satisfy

$$\frac{d}{dt}\langle[\varphi],[\omega]\rangle = \langle\nabla^{GM}[\varphi],[\omega]\rangle + \langle[\varphi],\nabla_{GM}[\omega]\rangle,$$

for all $[\varphi] \in HP^\bullet(A_J)$ and $[\omega] \in HP_\bullet(A_J)$.

4.3. Interaction with the Chern character. The algebra A_J can be viewed as an algebra over $\mathbb{C}$ or $C^\infty(J)$, and there is a natural morphism of complexes

$$\pi : C^{\mathbb{C}}_{\text{per}}(A_J) \to C^{C^\infty(J)}_{\text{per}}(A_J).$$

Proposition 4.4. *If $\omega \in C^{C^\infty(J)}_{\text{per}}(A_J)$ is a cycle that lifts to a cycle $\tilde{\omega} \in C^{\mathbb{C}}_{\text{per}}(A_J)$, then $\nabla_{GM}[\omega] = 0$ in $HP^{C^\infty(J)}_\bullet(A_J)$.*

Proof. Let $\nabla^{\mathbb{C}}_{GM} = L_\nabla - I_E$, viewed as a linear operator on $C^{\mathbb{C}}_{\text{per}}(A_J)$. By Theorem 2.1,

$$\nabla^{\mathbb{C}}_{GM} = L_\nabla - I_{\delta\nabla} = [b + B, I_\nabla]$$

and so $\nabla^{\mathbb{C}}_{GM}$ is the zero operator on $HP^{\mathbb{C}}_\bullet(A_J)$. Thus, at the level of homology, we have

$$\nabla_{GM} \circ \pi = \pi \circ \nabla^{\mathbb{C}}_{GM} = 0$$

where $\pi : HP^{\mathbb{C}}_\bullet(A_J) \to HP^{C^\infty(J)}_\bullet(A_J)$ is the map induced by the quotient map. By hypothesis, $[\omega]$ is in the image of π. □

Note that the homotopy used in the previous proof does not imply that ∇_{GM} is zero on $HP^{C^\infty(J)}_\bullet(A_J)$. The reason is that the operator I_∇ is not a well-defined operator on the quotient complex $C^{C^\infty(J)}_{\text{per}}(A_J)$.

Theorem 4.5. *If $P \in M_N(A_J)$ is an idempotent and $U \in M_N(A_J)$ is an invertible, then*

$$\nabla_{GM}[\operatorname{ch} P] = 0, \qquad \nabla_{GM}[\operatorname{ch} U] = 0$$

in $HP_\bullet(A_J)$.

Proof. This is immediate from the previous proposition because the cycle $\operatorname{ch} P \in C^{\mathbb{C}}_{\text{per}}(A_J)$ is a lift of the cycle $\operatorname{ch} P \in C^{C^\infty(J)}_{\text{per}}(A_J)$, and similarly for $\operatorname{ch} U$. □

Combining this with the identity

$$\frac{d}{dt}\langle[\varphi],[\omega]\rangle = \langle\nabla^{GM}[\varphi],[\omega]\rangle + \langle[\varphi],\nabla_{GM}[\omega]\rangle,$$

we obtain the following differentiation formula for the pairing between K-theory and periodic cyclic cohomology.

Corollary 4.6. *If $P \in M_N(A_J)$ is an idempotent and $U \in M_N(A_J)$ is an invertible, then*

$$\frac{d}{dt}\langle[\varphi],[P]\rangle = \langle\nabla^{GM}[\varphi],[P]\rangle, \quad \frac{d}{dt}\langle[\varphi],[U]\rangle = \langle\nabla^{GM}[\varphi],[U]\rangle.$$

Remark 4.7. Proposition 4.3 can be used to give another proof that

$$\nabla_{GM}[\mathrm{ch}\, P] = 0$$

when $P \in A_J$ is an idempotent. Indeed, an idempotent in A_J is equivalent to a morphism of deformations

$$\{F_t : \mathbb{C} \to A_t\}_{t\in J}$$

from the constant deformation with fiber $\mathbb{C}$. The induced algebra map

$$F : C^\infty(J,\mathbb{C}) \to A_J$$

sends 1 to P. Applying Proposition 4.3, we see

$$\nabla_{GM}[\mathrm{ch}\, P] = \nabla_{GM} F_*[\mathrm{ch}\, 1] = F_*\frac{d}{dt}[\mathrm{ch}\, 1] = 0.$$

4.4. Integrating ∇_{GM}. The very fact that ∇_{GM} exists for all smooth one-parameter deformations implies that the problem of proving ∇_{GM} is integrable cannot be attacked with methods that are too general. Indeed, one cannot expect periodic cyclic homology to be rigid for all deformations, there are plenty of finite dimensional examples for which it is not.

Example 4.8. For $t \in \mathbb{R}$, let A_t be the two-dimensional algebra generated by an element x and the unit 1 subject to the relation $x^2 = t \cdot 1$. Then $A_t \cong \mathbb{C} \oplus \mathbb{C}$ as an algebra when $t \neq 0$, and A_0 is the exterior algebra on a one dimensional vector space. Consequently,

$$HP_0(A_t) \cong \begin{cases} \mathbb{C} \oplus \mathbb{C}, & t \neq 0, \\ \mathbb{C}, & t = 0. \end{cases}$$

From the point of view of differential equations, a major issue is that the periodic cyclic complex is never a Banach space. Even in the case where A is a Banach algebra, e.g. finite dimensional, the chain groups $C_n(A)$ are also Banach spaces, but the periodic cyclic complex

$$C_{\mathrm{per}}(A) = \prod_{n=0}^{\infty} C_n(A)$$

is a Fréchet space, as it is a countable product of Banach spaces. The operator ∇_{GM} contains the degree -2 term $\iota_E : C_n(A) \to C_{n-2}(A)$. Thus unless $E = 0$, one cannot reduce the problem to the individual Banach space factors, as the differential equations are hopelessly coupled together.

One instance in which ∇_{GM} is clearly integrable is when the deformation $\{A_t\}_{t\in J}$ is trivial. Indeed, if A_J has an integrable connection ∇ that is a derivation, then $\nabla_{GM} = L_\nabla$ is integrable on $C_{\mathrm{per}}(A_J)$, and $P^{\nabla_{GM}}_{s,t} : C_{\mathrm{per}}(A_s) \to C_{\mathrm{per}}(A_t)$ is the map of complexes induced by the algebra isomorphism $P^{\nabla}_{s,t} : A_s \to A_t$. While this is not surprising, it is interesting to note is that if we consider another connection ∇' on A_J, the corresponding Gauss–Manin connection ∇'_{GM} on $C_{\mathrm{per}}(A_J)$ need not be integrable, and in general seems unlikely to be so. However the induced connection $\left(\nabla'_{GM}\right)_*$ on $HP_\bullet(A_J)$ is necessarily integrable by the uniqueness of the Gauss–Manin connection up to chain homotopy.

A general strategy for dealing with ∇_{GM} is to replace $C_{\mathrm{per}}(A_J)$ with a more manageable chain equivalent complex which computes $HP_\bullet(A_J)$. For Banach algebras of finite weak bidimension, we use a retract of $C_{\mathrm{per}}(A_J)$ whose fibers are Banach spaces. For smooth noncommutative tori, we use a smaller complex on which it is easier to describe the operator ∇_{GM}.

5. $HP^\bullet$-rigidity for Banach algebras of finite weak bidimension

In this section, we will use the Gauss–Manin connection to prove the rigidity of periodic cyclic cohomology for Banach algebras of finite weak bidimension.

5.1. A rigidity lemma.

The following is our main technical lemma for rigidity in the setting of Banach algebras.

Lemma 5.1. *Let $\{(\mathcal{C}^\bullet_t, d_t)\}_{t\in J}$ be a smooth deformation of cochain complexes of Banach spaces, and suppose the complex $\mathcal{C}^\bullet_0$ has a continuous linear contracting homotopy in degree n*

$$\mathcal{C}_0^{n-1} \underset{h}{\overset{d_0}{\rightleftarrows}} \mathcal{C}_0^{n} \underset{h}{\overset{d_0}{\rightleftarrows}} \mathcal{C}_0^{n+1}, \qquad d_0h + hd_0 = 1.$$

Then there is a subinterval $J' = (-\epsilon, \epsilon)$ such that

$$Z^n(\mathcal{C}_{J'}) \cong C^\infty(J', Z^n(\mathcal{C}_0)), \quad B^n(\mathcal{C}_{J'}) \cong C^\infty(J', B^n(\mathcal{C}_0)), \quad H^n(\mathcal{C}_{J'}) = 0.$$

It can be shown using the homological perturbation lemma that each $\mathcal{C}^\bullet_t$ has a contracting homotopy in degree n for small enough t, see [7]. However the additional result that the space $Z^n(\mathcal{C}_{J'})$ is a free module will be crucial to our proof of Theorem 5.8 below.

Proof. By assumption, there are split short exact sequences

$$0 \longrightarrow Z^{n-1}(\mathcal{C}_0) \longrightarrow \mathcal{C}_0^{n-1} \underset{h}{\overset{d_0}{\rightleftarrows}} B^n(\mathcal{C}_0) \longrightarrow 0,$$

$$0 \longrightarrow Z^{n}(\mathcal{C}_0) \longrightarrow \mathcal{C}_0^{n} \underset{h}{\overset{d_0}{\rightleftarrows}} B^{n+1}(\mathcal{C}_0) \longrightarrow 0,$$

and $B^n(\mathcal{C}_0) = Z^n(\mathcal{C}_0)$. So the cocycles are complemented in the space of cochains, that is, there are closed subspaces $W^{n-1} = h(B^n(\mathcal{C}_0))$ and $W^n = h(B^{n+1}(\mathcal{C}_0))$ for which

$$\mathcal{C}_0^{n-1} = Z^{n-1}(\mathcal{C}_0) \oplus W^{n-1}, \quad \mathcal{C}_0^n = Z^n(\mathcal{C}_0) \oplus W^n.$$

Let $\pi : \mathcal{C}_0^n \to Z^n(\mathcal{C}_0) = B^n(\mathcal{C}_0)$ be the projection. Then π induces a continuous $C^\infty(J)$-linear map $\pi : \mathcal{C}_J^n \to C^\infty(J, Z^n(\mathcal{C}_0))$, which restricts to the maps

$$\pi : Z^n(\mathcal{C}_J) \to C^\infty(J, Z^n(\mathcal{C}_0)), \quad \pi : B^n(\mathcal{C}_J) \to C^\infty(J, B^n(\mathcal{C}_0)).$$

We claim these are topological isomorphisms for a small enough interval J' containing 0. We'll prove this by showing π is injective on cocycles and surjective on coboundaries. The results then follow from the commutative diagram

$$\begin{array}{ccc} B^n(\mathcal{C}_{J'}) & \hookrightarrow & Z^n(\mathcal{C}_{J'}) \\ \downarrow \pi & & \downarrow \pi \\ C^\infty(J', B^n(\mathcal{C}_0)) & = & C^\infty(J', Z^n(\mathcal{C}_0)) \end{array}$$

and the open mapping theorem.

Consider the family of maps $\pi \circ d_t$ restricted to W^{n-1}. When $t = 0$,

$$\pi \circ d_0 : W^{n-1} \to B^n(\mathcal{C}_0)$$

is a topological isomorphism of Banach spaces. Since the invertibles form an open subset of $\mathrm{Hom}(W^{n-1}, B^n(\mathcal{C}_0))$, there is some $\epsilon > 0$ for which $\pi \circ d_t : W^{n-1} \to B^n(\mathcal{C}_0)$ is a topological isomorphism for all $t \in J' := (-\epsilon, \epsilon)$. From Corollary A.5, the induced $C^\infty(J')$-linear map

$$\pi \circ d : C^\infty(J', W^{n-1}) \to C^\infty(J', B^n(\mathcal{C}_0))$$

is an isomorphism, and in particular it is surjective. It follows that $\pi : B^n(\mathcal{C}_{J'}) \to C^\infty(J', B^n(\mathcal{C}_0))$ is surjective.

Now consider the map $d_0 : W^n \to B^{n+1}(\mathcal{C}_0)$, which is a topological isomorphism of Banach spaces. In particular it is bounded below, so that

$$\|d_0(w)\| \geq C\|w\|, \quad \forall w \in W^n$$

for some constant $C > 0$. Since $t \mapsto d_t$ is norm continuous, the maps $d_t : W^n \to \mathcal{C}_t^{n+1}$ are bounded below for t in a small enough interval J'. In particular they are injective. Let's show $\pi : Z^n(\mathcal{C}_t) \to Z^n(\mathcal{C}_0)$ is injective for all $t \in J'$. Consider an element $z \in Z^n(\mathcal{C}_t)$. As vector spaces, $\mathcal{C}_t^n = \mathcal{C}_0^n = Z^n(\mathcal{C}_0) \oplus W^n$, so we can write

$$z = z_0 + w, \quad z_0 \in Z^n(\mathcal{C}_0),\ w \in W^n.$$

If $z \in \ker \pi$, then $z_0 = 0$. Since $z \in Z^n(\mathcal{C}_t)$, we have $0 = d_t(z) = d_t(w)$. Since d_t is injective on W^n, $w = 0$ and so $z = 0$. This shows $\pi : Z^n(\mathcal{C}_t) \to Z^n(\mathcal{C}_0)$ is injective for all $t \in J'$, and consequently $\pi : Z^n(\mathcal{C}_{J'}) \to C^\infty(J', Z^n(\mathcal{C}_0))$ is injective. □

We have set up enough theory to quickly prove a version of a rigidity theorem of Raeburn and Taylor for Banach algebras [28]. The original proof used a certain "inverse function theorem". We'll call an algebra A *(smoothly) rigid* if every smooth deformation $\{A_t\}_{t\in J}$ with $A_0 = A$ is trivial on some interval $J' \subset J$ containing 0.

Theorem 5.2 ([28]). *Let A be a Banach algebra whose Hochschild cochain complex has a continuous linear contracting homotopy in degree* 2

$$C^1(A,A) \underset{h}{\overset{\delta}{\rightleftarrows}} C^2(A,A) \underset{h}{\overset{\delta}{\rightleftarrows}} C^3(A,A), \qquad \delta h + h\delta = 1,$$

so that $H^2(A,A) = 0$. Then A is rigid.

Proof. Given a smooth deformation $\{A_t\}_{t\in J}$ with $A_0 = A$, consider the deformation of cochain complexes $\{C^\bullet(A_t, A_t)\}_{t\in J}$. Using Proposition A.6, its complex of sections naturally identifies with the Hochschild complex $C^\bullet_{C^\infty(J)}(A_J, A_J)$. By Lemma 5.1, $H^2_{C^\infty(J')}(A_{J'}, A_{J'}) = 0$ for some subinterval $J' \subset J$ containing 0. So by Proposition 3.6, $A_{J'}$ has a connection that is a derivation, and it is integrable because the underlying space is a Banach space. This shows $\{A_t\}_{t\in J'}$ is trivial. □

5.2. Homological bidimension. Here we recall the notion of (weak) homological bidimension for a Banach algebra, and we prove that these dimensions are upper semicontinuous for smooth deformations of Banach algebras.

Let A be a (possibly nonunital) Banach algebra, and let $A^e = A_+ \widehat{\otimes} A_+^{\mathrm{op}}$ be its topological enveloping algebra. The algebra A^e is designed so that there is a one-to-one correspondence between locally convex A-bimodules and locally convex unital left A^e-modules. Here, we shall only discuss modules whose underlying space is a Banach space. The continuous Hochschild cohomology of A with coefficients in a Banach A-bimodule M is defined as

$$H^\bullet(A, M) := \mathrm{Ext}^\bullet_{A^e}(A_+, M).$$

See [15] for a discussion of derived functors in the context of locally convex algebras and modules. The Hochschild cohomology $H^\bullet(A, A)$ coincides with our previous notation, and $HH^\bullet(A) = H^\bullet(A, A^*)$. The bimodule structure on A^* comes from a general construction: given any A-bimodule M, the topological dual $M^* = \mathrm{Hom}(M, \mathbb{C})$ is an A-bimodule via

$$(a \cdot \varphi \cdot b)(m) = \varphi(bma), \quad \forall \varphi \in M^*.$$

By considering the topological bar resolution $B_\bullet(A)$, which is a projective resolution of A_+ by A^e-modules, we obtain the *standard complex*

$$C^n(A, M) = \mathrm{Hom}_{A^e}(B_n(A), M) \cong \mathrm{Hom}(A^{\widehat{\otimes} n}, M),$$

with differential

$$(\delta D)(a_1, \dots, a_{n+1}) = a_1 D(a_2, \dots, a_{n+1}) + (-1)^{n+1} D(a_1, \dots, a_n) a_{n+1} + \sum_{j=1}^{n} (-1)^j D(a_1, \dots, a_j a_{j+1}, \dots, a_{n+1}),$$

whose cohomology is $H^\bullet(A, M)$, see [15, Section III.4.2].

The *homological bidimension* of a Banach algebra A is

$$\mathrm{db}\, A = \inf \{ n \mid H^{n+1}(A, M) = 0 \text{ for all Banach } A\text{-bimodules } M \}$$

and the *weak homological bidimension* of A is

$$\mathrm{db}_w\, A = \inf \{ n \mid H^{n+1}(A, M^*) = 0 \text{ for all Banach } A\text{-bimodules } M \}.$$

Clearly, $\mathrm{db}_w\, A \le \mathrm{db}\, A$. It is a fact that if $H^{n+1}(A, M) = 0$ (resp., $H^{n+1}(A, M^*) = 0$) for all M, then $H^m(A, M) = 0$ (resp., $H^m(A, M^*) = 0$) for all M and all $m \ge n+1$ [15, Theorem III.5.4] (resp., [33]). A Banach algebra A is called *amenable* if $\mathrm{db}_w\, A = 0$. As an example, Johnson proved that the convolution algebra $L^1(G)$ of a locally compact group is amenable if and only if the group G is amenable [18, Theorem 2.5]. A Banach algebra A for which $\mathrm{db}_w\, A = n$ is also called $(n+1)$-amenable.

As in [4], we shall consider the universal differential graded algebra $(\Omega^\bullet A, d)$ associated to A. We'll use the topological version, constructed using completed projective tensor products. Explicitly, $\Omega^0 A \cong A = C_0(A)$ and

$$\Omega^n A \cong A_+ \widehat{\otimes} A^{\widehat{\otimes} n} = C_n(A),$$

under the identification

$$a_0 da_1 da_2 \dots da_n \longleftrightarrow (a_0, a_1, a_2, \dots, a_n).$$

Then $\Omega^n A$ is a Banach A-bimodule with the left action

$$a \cdot (a_0 da_1 \dots da_n) = (aa_0) da_1 \dots da_n.$$

The right action is determined by the relation

$$(da_1)a_2 = d(a_1 a_2) - a_1(da_2).$$

The map

$$d^{\otimes n} : A^{\widehat{\otimes} n} \to \Omega^n A, \quad (a_1, \dots, a_n) \mapsto da_1 \cdot \dots \cdot da_n$$

is a Hochschild n-cocycle in the standard complex with coefficients in the bimodule $\Omega^n A$. In fact, $d^{\otimes n}$ is the universal Hochschild n-cocycle in the sense that any Hochschild cocycle $D : A^{\widehat{\otimes} n} \to M$ into a Banach A-bimodule factors through a unique A-bimodule map $F : \Omega^n A \to M$, determined by

$$F(da_1 \dots da_n) = D(a_1, \dots, a_n),$$

as in [8]. Thus the cohomology class $[D] \in H^n(A, M)$ is the image of $[d^{\otimes n}]$ under the map

$$H^n(A, \Omega^n A) \to H^n(A, M)$$

induced by F. It follows that

$$\operatorname{db} A \le n \quad \text{if and only if } H^{n+1}(A, \Omega^{n+1} A) = 0.$$

Let's now consider cocycles with values in dual Banach modules. Compose the universal n-cocycle $d^{\otimes n}$ with the canonical embedding into the double dual to obtain an n-cocycle

$$d^{\otimes n} : A^{\widehat{\otimes} n} \to (\Omega^n A)^{**}.$$

Given any standard n-cocycle $D : A^{\widehat{\otimes} n} \to M^*$, consider the bimodule map

$$F : \Omega^n A \to M^*$$

induced by the universal property of $\Omega^n A$. Define a bimodule map

$$G : M \to (\Omega^n A)^*, \quad G(m)(\omega) = F(\omega)(m).$$

Then the dual map

$$G^* : (\Omega^n A)^{**} \to M^*$$

is an A-bimodule map that satisfies $G^* \circ d^{\otimes n} = D$. It follows that

$$\operatorname{db}_w A \le n \quad \text{if and only if } H^{n+1}(A, (\Omega^{n+1} A)^{**}) = 0.$$

Now suppose $\{A_t\}_{t \in J}$ is a smooth deformation of Banach algebras and let A_J be its algebra of sections. One can form the space of abstract n-forms $\Omega^n A_J$ over the ground ring $C^\infty(J)$ by taking the projective tensor products over $C^\infty(J)$. Notice that the spaces $\{\Omega^n A_t\}_{t\in J}$ are all canonically isomorphic as Banach spaces, and $\Omega^n A_J$ is isomorphic, as a $C^\infty(J)$-module, to the space of smooth functions from J into the underlying Banach space of $\Omega^n A_t$. There is a universal $C^\infty(J)$-linear cocycle

$$d^{\otimes n} : A_J^{\widehat{\otimes}_{C^\infty(J)} n} \to \Omega^n A_J, \quad d^{\otimes n}(a_1, \dots, a_n) = da_1 \dots da_n.$$

Using Proposition A.6, the complex

$$C^\bullet_{C^\infty(J)}(A_J, \Omega^n A_J) = \mathrm{Hom}_{C^\infty(J)}\left(A_J^{\widehat{\otimes}_{C^\infty(J)}\bullet}, \Omega^n A_J\right)$$

is isomorphic to the complex of sections of the deformation $\{C^\bullet(A_t, \Omega^n A_t)\}_{t\in J}$. Moreover, evaluation at $t \in J$ induces a chain map

$$\epsilon_t : C^\bullet_{C^\infty(J)}(A_J, \Omega^n A_J) \to C^\bullet(A_t, \Omega^n A_t)$$

which maps the universal cocycle for A_J to the universal cocycle for A_t. Thus if $H^{n+1}_{C^\infty(J)}(A_J, \Omega^{n+1} A_J) = 0$, we have $H^{n+1}(A_t, \Omega^{n+1} A_t) = 0$ for all $t \in J$, and consequently $\mathrm{db}\, A_t \le n$ for all $t \in J$.

We can also consider the $C^\infty(J)$-linear double dual module $(\Omega^n A_J)^{\star\star}$ and the cocycle

$$d^{\otimes n} : A_J^{\otimes_{C^\infty(J)} n} \to (\Omega^n A_J)^{\star\star}$$

obtained by composing the universal cocycle with the canonical embedding into the double dual. Using Proposition A.6 and Corollary A.7, the Hochschild complex $C^\bullet_{C^\infty(J)}(A_J, (\Omega^n A_J)^{\star\star})$ identifies with the complex of sections of the deformation $\{C^\bullet(A_t, (\Omega^n A_t)^{**})\}_{t\in J}$. By considering evaluation at $t \in J$, we see that if $[d^{\otimes n}] = 0$ in $H^n_{C^\infty(J)}(A_J, (\Omega^n A_J)^{\star\star})$, then $[d^{\otimes n}] = 0$ in $H^n(A_t, (\Omega^n A_t)^{**})$ for all $t \in J$. We have proved the following proposition.

Proposition 5.3. *Let $\{A_t\}_{t\in J}$ be a deformation of Banach algebras.*

(i) *If $H^{n+1}_{C^\infty(J)}(A_J, \Omega^n A_J) = 0$, then $\mathrm{db}\, A_t \le n$ for all $t \in J$.*

(ii) *If $H^{n+1}_{C^\infty(J)}(A_J, (\Omega^n A_J)^{\star\star}) = 0$, then $\mathrm{db}_w\, A_t \le n$ for all $t \in J$.*

Lemma 5.4. *Let A be a Banach algebra and M be a Banach A-bimodule.*

(i) *If $\mathrm{db}\, A \le n$, then the standard complex $C^\bullet(A, M)$ has a contracting homotopy in degree $n+1$*

$$C^n(A, M) \underset{h}{\overset{\delta}{\rightleftarrows}} C^{n+1}(A, M) \underset{h}{\overset{\delta}{\rightleftarrows}} C^{n+2}(A, M), \quad \delta h + h\delta = 1.$$

(ii) *If $\mathrm{db}_w\, A \le n$, then the standard complex $C^\bullet(A, M^*)$ has a contracting homotopy in degree $n+1$*

$$C^n(A, M^*) \underset{h}{\overset{\delta}{\rightleftarrows}} C^{n+1}(A, M^*) \underset{h}{\overset{\delta}{\rightleftarrows}} C^{n+2}(A, M^*), \quad \delta h + h\delta = 1.$$

Proof. If $\operatorname{db} A \leq n$, then A_+ has a projective resolution of length n [15, Theorem III.5.4]. By the "Comparison theorem" [15, Theorem III.2.3], a projective resolution is unique up to chain homotopy equivalence in the category of complexes of Banach A-bimodules. By applying the functor $\operatorname{Hom}_{A^e}(\cdot, M)$, it follows that the standard complex $C^\bullet(A, M)$ has the required homotopy.

If $\operatorname{db}_w A \leq n$, then A_+ has a flat resolution of length n [33, Theorem 1], that is, a resolution by Banach A-bimodules which are flat as left A^e-modules. Since the dual of a flat module is injective [15, Theorem VII.1.14], it follows that A^* has an injective resolution of length n. Using the Comparison theorem for injective resolutions, the dual $B_\bullet(A)^*$ of the bar resolution has a contracting homotopy in degree $n+1$ consisting of A-bimodule maps. After applying the functor $\operatorname{Hom}_{A^e}(M, \cdot)$, we see the complex $\operatorname{Hom}_{A^e}(M, B_\bullet(A)^*)$ has a contracting homotopy in degree $n+1$. However there is a natural isomorphism of complexes

$$\operatorname{Hom}_{A^e}(M, B_\bullet(A)^*) \cong \operatorname{Hom}_{A^e}(B_\bullet(A), M^*) = C^\bullet(A, M^*),$$

which gives the result, see [15, Proposition III.4.13]. □

Corollary 5.5. *Let $\{A_t\}_{t\in J}$ be a smooth deformation of Banach algebras. Then the functions*

$$t \mapsto \operatorname{db} A_t \quad \text{and} \quad t \mapsto \operatorname{db}_w A_t$$

are upper semi-continuous.

Proof. Simply combine the previous two results with Lemma 5.1. □

Example 5.6. Let G be a connected semisimple Lie group with maximal compact subgroup K. Let $\mathfrak{g}$ and $\mathfrak{k}$ denote their respective Lie algebras. In [16], a smooth deformation $\{G_t\}$ of Lie groups is constructed in such a way that $G_0 = K \ltimes \mathfrak{g}/\mathfrak{k}$ and $G_t \cong G$ for all $t \neq 0$. The group G_0 is amenable, but G may not be, e.g. $G = SL(2, \mathbb{R})$. So Corollary 5.5 and Johnson's theorem on amenability of $L^1(G)$ imply that there is no corresponding smooth deformation $\{L^1(G_t)\}$ of Banach algebras.

5.3. Contractions and retractions. When the universal $(n+1)$-cocycle is a coboundary, one can construct a contracting homotopy in the Hochschild complex in a uniform way. As described in [19], if $\varphi : A^{\widehat{\otimes} n} \to \Omega^{n+1}A$ satisfies $\delta\varphi = d^{\otimes(n+1)}$, then

$$\alpha : \Omega^k A \to \Omega^{k+1} A, \quad \alpha(a_0 da_1 \ldots da_k) = a_0 \varphi(a_1, \ldots, a_n) da_{n+1} \ldots da_k$$

defines a contracting homotopy of the Hochschild chain complex $(C_\bullet(A), b)$ in degrees $k \geq n+1$. The transpose of α gives a contracting homotopy in degree $k \geq n+1$ for the Hochschild cochain complex $(C^\bullet(A), b)$.

Khalkhali showed in [19] that if the cocycle $d^{\otimes(n+1)} : A^{\widehat{\otimes}(n+1)} \to (\Omega^{n+1}A)^{**}$ is a coboundary, then one can construct a contracting homotopy of $(C^\bullet(A), b)$ in degrees $k \geq n+1$ in a similar way. Given a cochain $\varphi : A^{\widehat{\otimes} n} \to (\Omega^{n+1}A)^{**}$ such that $\delta\varphi = d^{\otimes(n+1)}$, define

$$\alpha : (\Omega^{k+1}A)^* \to (\Omega^k A)^*, \quad k \geq n$$

by

$$(\alpha f)(a_0 da_1 \dots da_k) = [a_0 \cdot \varphi(a_1, \dots, a_n)]\,(f_{da_{n+1}\dots da_k}),$$

where $f \in (\Omega^{k+1}A)^*$ and $f_{da_{n+1}\dots da_k} \in (\Omega^{n+1}A)^*$ is given by

$$f_{da_{n+1}\dots da_k}(\omega) = f(\omega da_{n+1} \dots da_k).$$

Then $b\alpha + \alpha b = 1$ in $C^k(A)$ when $k \geq n+1$.

Given such a contracting homotopy $\alpha : C^{k+1}(A) \to C^k(A)$, Khalkhali constructed a retract of the periodic cyclic cochain complex supported in only finitely many degrees [19], which we now describe. Let N be such that α is a contracting homotopy in all degrees above $2N$. Let

$$C_0^{\text{even}}(A) = \left(\bigoplus_{k=0}^{N-1} C^{2k}(A)\right) \bigoplus \ker\left\{b : C^{2N}(A) \to C^{2N+1}(A)\right\}$$

and

$$C_0^{\text{odd}}(A) = \bigoplus_{k=0}^{N-1} C^{2k+1}(A).$$

The $\mathbb{Z}/2$-graded complex $C_0^{\text{per}}(A) = C_0^{\text{even}}(A) \oplus C_0^{\text{odd}}(A)$ has differential $b + B$. Then $C_0^{\text{per}}(A)$ is a subcomplex of $C^{\text{per}}(A)$, and in fact is a deformation retract. That is, there is a chain map $R : C^{\text{per}}(A) \to C_0^{\text{per}}(A)$ such that $RI = \text{id}$ and IR is chain homotopic to id, where $I : C_0^{\text{per}}(A) \to C^{\text{per}}(A)$ is the inclusion. Thus, the cohomology of $C_0^{\text{per}}(A)$ is $HP^\bullet(A)$. The key feature is that $C_0^{\text{per}}(A)$ is a complex of Banach spaces. We won't need the explicit form of the retraction R, but we remark that it depends heavily on the contracting homotopy α.

All of the above can be carried out for the algebra of sections A_J of a smooth deformation $\{A_t\}_{t\in J}$ of Banach algebras, where everything is considered over the ground ring $C^\infty(J)$. If $\varphi : A_J^{\widehat{\otimes} n} \to \Omega^{n+1}A_J$ satisfies $\delta\varphi = d^{\otimes(n+1)}$, then

$$\alpha : \Omega^k A_J \to \Omega^{k+1} A_J, \quad \alpha(a_0 da_1 \dots da_k) = a_0\varphi(a_1, \dots, a_n) da_{n+1} \dots da_k$$

defines a contracting homotopy of the Hochschild chain complex $(C_\bullet(A_J), b)$ in degrees $k \geq n+1$. Its dual

$$\alpha^\star : C^{k+1}(A_J) \to C^k(A_J)$$

is a contraction for the Hochschild cochain complex.

If $\varphi : A_J^{\otimes n} \to (\Omega^{n+1} A_J)^{\star\star}$ satisfies $\delta\varphi = d^{\otimes(n+1)}$, then Khalkhali's contracting homotopy

$$\alpha : (\Omega^{k+1} A_J)^{\star} \to (\Omega^k A_J)^{\star}, \quad k \geq n$$

can be defined by the same formula as above, and $b\alpha + \alpha b = 1$ in $C^k(A_J)$ for $k \geq n+1$. Moreover, given an α which is a contracting homotopy in degrees above $2N$, we can define

$$C_0^{\text{even}}(A_J) = \left(\bigoplus_{k=0}^{N-1} C^{2k}(A_J) \right) \bigoplus \ker \left\{ b : C^{2N}(A_J) \to C^{2N+1}(A_J) \right\}$$

and

$$C_0^{\text{odd}}(A_J) = \bigoplus_{k=0}^{N-1} C^{2k+1}(A_J).$$

As in the $\mathbb{C}$-linear case, there is an inclusion $I : C_0^{\text{per}}(A_J) \to C^{\text{per}}(A_J)$ and a retraction $R : C^{\text{per}}(A_J) \to C_0^{\text{per}}(A_J)$ which are $C^\infty(J)$-linear chain maps such that $RI = \text{id}$ and IR is chain homotopic to id. The retraction R is built in the same way as the $\mathbb{C}$-linear case using the homotopy α.

Definition 5.7. We'll say that a locally convex algebra A is *$HP^\bullet$-rigid* if whenever $\{A_t\}_{t \in J}$ is a smooth deformation with $A_0 = A$, then there some subinterval $J' \subseteq J$ containing 0 for which $HP^\bullet(A_t) \cong HP^\bullet(A_0)$ for all $t \in J'$.

We now give our main application of the Gauss–Manin connection.

Theorem 5.8. *Let A be a Banach algebra such that* $\text{db}_w A < \infty$. *Then A is $HP^\bullet$-rigid.*

Proof. Let $\{A_t\}_{t \in J}$ be a smooth deformation with $A_0 = A$. Suppose $\text{db}_w A_0 = n$. As described above, the Hochschild complex $C^\bullet_{C^\infty(J)}(A_J, (\Omega^{n+1} A_J)^{\star\star})$ identifies with the complex of sections of the deformation $\{C^\bullet(A_t, (\Omega^{n+1} A_t)^{**})\}_{t \in J}$. From Lemmas 5.4 and 5.1, $H^{n+1}_{C^\infty(J')}(A_{J'}, (\Omega^{n+1} A_{J'})^{\star\star}) = 0$ for a subinterval $J' \subseteq J$ containing 0. So there is a $\varphi : A_{J'}^{\widehat{\otimes} n} \to (\Omega^{n+1} A_{J'})^{\star\star}$ with $\delta\varphi = d^{\otimes(n+1)}$. As described above, we can construct from this the deformation retract $C_0^{\text{per}}(A_{J'})$ of $C^{\text{per}}(A_{J'})$ for a suitable N. A priori, the space of cocycles $\ker\{b : C^{2N}(A_{J'}) \to C^{2N+1}(A_{J'})\}$ may not be a free $C^\infty(J')$-module. However, the conclusion of Lemma 5.1 guarantees that it is, and moreover $C_0^{\text{per}}(A_{J'})$ is the complex of sections of $\{C_0^{\text{per}}(A_t)\}_{t \in J'}$.

We can now transfer the Gauss–Manin connection to $C_0^{\text{per}}(A_{J'})$. Let

$$I : C_0^{\text{per}}(A_{J'}) \to C^{\text{per}}(A_{J'}), \quad R : C^{\text{per}}(A_{J'}) \to C_0^{\text{per}}(A_{J'})$$

be the inclusion and retraction, which are continuous $C^\infty(J')$-linear chain maps. Define $\widetilde{\nabla} = R \circ \nabla_{GM} \circ I$ on $C_0^{\text{per}}(A_{J'})$. Then $\widetilde{\nabla}$ is a chain map and it is a connection because $RI = \mathrm{id}$. Since the underlying spaces $\{C_0^{\mathrm{per}}(A_t)\}$ are all the same Banach space, the connection $\widetilde{\nabla}$ is integrable, and the result follows from Proposition 3.9. □

Let $HE^\bullet(A)$ denote the entire cyclic cohomology of A, see [3]. As Khalkhali showed, the canonical inclusion $HP^\bullet(A) \to HE^\bullet(A)$ is an isomorphism for Banach algebras of finite weak bidimension [19]. We immediately obtain the following.

Corollary 5.9. *Let A be a Banach algebra such that $\mathrm{db}_w\, A < \infty$. Then A is $HE^\bullet$-rigid.*

Example 5.10. We'll show how our theorem can be used to give a proof of a variation of a theorem of Block on the cyclic homology of filtered algebras [1]. In Block's setting we have an increasing filtration of an algebra A,

$$F_0 \subset F_1 \subset F_2 \subset \cdots$$

where $A = \bigcup_n F_n$ and $F_n \cdot F_m \subset F_{n+m}$. Letting $B = \mathrm{gr}(A)$ be the associated graded algebra, his result is that if $HH_n(B) = 0$ for all large enough n, then the inclusion $F_0 \to A$ induces an isomorphism $HP_\bullet(F_0) \cong HP_\bullet(A)$. Let's consider the situation of a finite filtration of a Banach algebra

$$F_0 \subset F_1 \subset \cdots \subset F_N = A,$$

and suppose there exist closed subspaces $B_k \subset A$ for which each $F_n \cong \bigoplus_{k=0}^n B_k$ as Banach spaces. Then we can identify the associated graded algebra $\mathrm{gr}(A) \cong \bigoplus_{k=0}^N B_k$ with A as Banach spaces. The multiplication in $\mathrm{gr}(A)$ is such that $B_n \cdot B_m \subset B_{n+m}$. Given $a \in B_n$ and $b \in B_m$, the product in the filtered algebra A can be written as

$$ab = \sum_{k=0}^{n+m} \pi_{n,m}^k(a,b)$$

for uniquely defined operators

$$\pi_{n,m}^k : B_n \widehat{\otimes} B_m \to B_{n+m-k}.$$

Given $t \in \mathbb{R}$, we can define a new associative product m_t on A by

$$m_t(a,b) = \sum_{k=0}^{n+m} t^k \pi_{n,m}^k(a,b), \quad a \in B_n, b \in B_m.$$

This clearly gives a smooth deformation $\{A_t\}_{t\in\mathbb{R}}$ of Banach algebras, as the products depend polynomially on t. We have $A_1 = A$, $A_0 = \mathrm{gr}(A)$, and $A_t \cong A$ for all $t \neq 0$. If $\mathrm{db}_w\, \mathrm{gr}(A) < \infty$, then the Gauss–Manin connection is integrable for this deformation, and $HP^\bullet(A) \cong HP^\bullet(\mathrm{gr}(A))$. View the inclusions $\{F_0 \to A_t\}_{t\in J}$ as a morphism of deformations out of the constant deformation. From Proposition 4.3, this morphism induces a ∇^{GM}-parallel map. Since $HP^\bullet(\mathrm{gr}(A)) \to HP^\bullet(F_0)$ is an isomorphism (Example 2.3), it follows that $HP^\bullet(A) \to HP^\bullet(F_0)$ is an isomorphism.

Notice that Example 4.8 is such a deformation of a filtered algebra A_1 into its associated graded algebra A_0. However $\mathrm{db}_w A_0 = \infty$, as one can show that $HH^n(A_0) \cong \mathbb{C}$ for all $n > 0$.

6. Integrability of ∇_{GM} for noncommutative tori

6.1. Smooth noncommutative tori. Given an $n \times n$ skew-symmetric real-valued matrix Θ, the *noncommutative torus* A_Θ is the universal C^*-algebra generated by n unitaries $u_1, \dots, u_n$ such that

$$u_j u_k = e^{2\pi i \theta_{jk}} u_k u_j,$$

where $\Theta = (\theta_{jk})$, see [30]. In the case $\Theta = 0$, all of the generating unitaries commute, and we have $A_0 \cong C(\mathbb{T}^n)$, the algebra of continuous complex-valued functions on the n-torus. It is for this reason why the algebra A_Θ has earned its name, as it can be philosophically viewed as functions on some "noncommutative torus" in the spirit of Alain Connes' noncommutative geometry [5].

It is helpful to think of an element in $x \in A_\Theta$ formally as a "Fourier series"

$$x = \sum_{\alpha \in \mathbb{Z}^n} x_\alpha u^\alpha,$$

where $\alpha = (\alpha_1, \dots, \alpha_n)$ is a multi-index, $u^\alpha = u_1^{\alpha_1} u_2^{\alpha_2} \dots u_n^{\alpha_n}$, and $x_\alpha \in \mathbb{C}$. The *smooth noncommutative torus* is the subalgebra $\mathcal{A}_\Theta$ of elements whose "Fourier coefficients" are of rapid decay. As a topological vector space, we identify $\mathcal{A}_\Theta$ with the Schwartz space $\mathcal{S}(\mathbb{Z}^n)$. An element $x = (x_\alpha)_{\alpha \in \mathbb{Z}^n}$ satisfies

$$p_k(x) := \sum_{\alpha \in \mathbb{Z}^n} (1 + |\alpha|)^k |x_\alpha| < \infty$$

for each positive integer k, where $|\alpha| = |\alpha_1| + \dots + |\alpha_n|$. Under the locally convex topology defined by the collection $\{p_k\}$ of seminorms, $\mathcal{A}_\Theta$ is a nuclear Fréchet algebra. The multiplication is given by the twisted convolution product

$$(xy)_\alpha = \sum_{\beta \in \mathbb{Z}^n} e^{2\pi i B_\Theta(\alpha - \beta, \beta)} x_{\alpha - \beta} y_\beta,$$

where

$$B_\Theta(\alpha, \beta) = \sum_{j > k} \alpha_j \beta_k \theta_{jk}$$

is the associated $\mathbb{R}$-valued group 2-cocycle on $\mathbb{Z}^n$.

The algebra $\mathcal{A}_\Theta$ possesses n canonical continuous derivations

$$\delta_1, \dots, \delta_n : \mathcal{A}_\Theta \to \mathcal{A}_\Theta, \quad (\delta_j(x))_\alpha = 2\pi i \alpha_j \cdot x_\alpha.$$

In the case $\theta = 0$, the derivations $\delta_1, \ldots, \delta_n$ correspond to the usual n partial differentiation operators on $\mathbb{T}^n$. There is also a continuous trace $\tau : \mathcal{A}_\Theta \to \mathbb{C}$ given by $\tau(x) = x_0$, which corresponds to integration with respect to the normalized Haar measure in the case $\Theta = 0$. Notice that $\tau \circ \delta_j = 0$ for all j.

The smooth noncommutative torus $\mathcal{A}_\Theta$ can be viewed as a smooth one-parameter deformation of $C^\infty(\mathbb{T}^n) \cong \mathcal{A}_0$ in the following way. For each $t \in J = \mathbb{R}$, let $A_t = \mathcal{A}_{t\Theta}$. The product in A_t is given by

$$m_t(x, y)_\alpha = \sum_{\beta \in \mathbb{Z}^n} e^{2\pi i B_\Theta(\alpha-\beta,\beta)t} x_{\alpha-\beta} y_\beta.$$

Proposition 6.1. *Given an $n \times n$ skew-symmetric real matrix Θ, the deformation $\{\mathcal{A}_{t\Theta}\}_{t\in\mathbb{R}}$ is smooth, and for x, y in the underlying space $\mathcal{S}(\mathbb{Z}^n)$,*

$$\frac{d}{dt} m_t(x, y) = \frac{1}{2\pi i} \sum_{j>k} \theta_{jk} \cdot m_t(\delta_j(x), \delta_k(y)).$$

Proof. We first derive an estimate that we will need. Taylor's formula for a function f of a real variable is

$$f(t) = f(0) + f'(0)t + \int_0^t f''(u)(t-u)du.$$

Applying this to $f(t) = e^{2\pi i a t}$ for a fixed $a \in \mathbb{R}$, we obtain the estimate

$$\left| \frac{e^{2\pi i a t} - 1}{t} - 2\pi i a \right| \le 2\pi^2 a^2 |t|.$$

To prove the deformation is smooth, it suffices to prove $t \mapsto m_t(x, y)$ is smooth for each $x, y \in \mathcal{S}(\mathbb{Z}^n)$ because $\mathcal{S}(\mathbb{Z}^n)$ is Fréchet, see Corollary A.3. For fixed x, y, and t, let

$$z_h = \frac{m_{t+h}(x, y) - m_t(x, y)}{h} - \frac{1}{2\pi i} \sum_{j>k} \theta_{jk} \cdot m_t(\delta_j(x), \delta_k(y)),$$

and we shall show $z_h \to 0$ in $\mathcal{S}(\mathbb{Z}^n)$. Then

$$(z_h)_\alpha = \sum_{\beta \in \mathbb{Z}^n} e^{2\pi i B_\Theta(\alpha-\beta,\beta)t} \left(\frac{e^{2\pi i B_\Theta(\alpha-\beta,\beta)h} - 1}{h} - 2\pi i B_\Theta(\alpha-\beta, \beta) \right) x_{\alpha-\beta} y_\beta.$$

Using the above estimate, for a nonnegative integer r we obtain

$$\begin{aligned} p_r(z_h) &= \sum_{\alpha \in \mathbb{Z}^n} (1 + |\alpha|)^r |(z_h)_\alpha| \\ &\le \sum_{\alpha \in \mathbb{Z}^n} \sum_{\beta \in \mathbb{Z}^n} (1 + |\alpha|)^r 2\pi^2 B_\Theta(\alpha-\beta, \beta)^2 |h| |x_{\alpha-\beta}| |y_\beta|. \end{aligned}$$

Consider the map $D_\Theta : \mathcal{S}(\mathbb{Z}^n) \otimes \mathcal{S}(\mathbb{Z}^n) \to \mathcal{S}(\mathbb{Z}^n) \otimes \mathcal{S}(\mathbb{Z}^n)$ given by

$$x \otimes y \mapsto \frac{1}{2\pi i} \sum_{j>k} \theta_{jk} \cdot \delta_j(x) \otimes \delta_k(y).$$

Notice that

$$m_t(D_\Theta^2(x \otimes y))_\alpha = (2\pi i)^2 \sum_{\beta \in \mathbb{Z}^n} e^{2\pi i B_\Theta(\alpha-\beta,\beta)} B_\Theta(\alpha-\beta,\beta)^2 x_{\alpha-\beta} y_\beta.$$

So the last inequality can be rewritten as

$$p_r(z_h) \le \frac{|h|}{2} p_r(m_t(D_\Theta^2(x \otimes y))).$$

Thus, $p_r(z_h) \to 0$ as $h \to 0$ because $p_r(m_t(D_\Theta^2(x \otimes y)))$ is a finite quantity, independent of h. This proves the formula

$$\frac{d}{dt} m_t(x,y) = \frac{1}{2\pi i} \sum_{j>k} \theta_{jk} \cdot m_t(\delta_j(x), \delta_k(y)).$$

By induction, it follows from this formula that $t \mapsto m_t(x,y)$ is infinitely differentiable. □

Let A_J be the algebra of sections of the smooth deformation $\{\mathcal{A}_{t\Theta}\}_{t \in J}$. Notice that the derivations $\delta_1, \dots, \delta_n$ on the fibers $\mathcal{A}_{t\Theta}$ induce $C^\infty(J)$-linear derivations on A_J, which we also denote $\delta_1, \dots, \delta_n$. Let $\nabla = \frac{d}{dt}$ be the canonical connection on A_J. We conclude that the cocycle $E = \delta\nabla$, as in Proposition 3.6, is given by

$$E = \frac{1}{2\pi i} \sum_{j>k} \theta_{jk} \cdot \delta_j \smile \delta_k.$$

Indeed, both sides are $C^\infty(J)$-bilinear and continuous, and they agree on pairs of constant elements (x,y) by Proposition 6.1. Thus they are equal because the $C^\infty(J)$-linear span of the constants is dense in $A_J \widehat{\otimes}_{C^\infty(J)} A_J$. So we have the identity

$$\nabla(a_1 a_2) = \nabla(a_1)a_2 + a_1\nabla(a_2) + \frac{1}{2\pi i} \sum_{j>k} \theta_{jk} \delta_j(a_1)\delta_k(a_2), \quad a_1, a_2 \in A_J.$$

One can show $[E] \neq 0$ in $H^2_{C^\infty(J)}(A_J, A_J)$, from which it follows that this deformation is nontrivial. In fact, $[E_t] \neq 0 \in H^2(\mathcal{A}_{t\Theta}, \mathcal{A}_{t\Theta})$, which implies the deformation is locally nontrivial at each fiber (though this is well known).

6.2. The $\mathfrak{g}$-invariant cyclic complex. In this section, we consider a locally convex R-algebra A with an action of an abelian Lie algebra $\mathfrak{g}$ by derivations. Our main example is the noncommutative n-torus $\mathcal{A}_\Theta$ and $\mathfrak{g} = \mathrm{Span}\{\delta_1, \dots, \delta_n\}$ and $R = \mathbb{C}$. We also consider the algebra of sections A_J of a noncommutative torus deformation $\{\mathcal{A}_{t\Theta}\}_{t \in J}$, where $\mathfrak{g}$ is the $\mathbb{C}$-linear span of $\{\delta_1, \dots, \delta_n\}$, and $R = C^\infty(J)$. The results of this section do not depend on any additional structure of these examples.

Suppose that $\mathfrak{g} \subset C^1_R(A, A)$ is a complex abelian Lie subalgebra of R-linear derivations on a locally convex R-algebra A. Then $\mathfrak{g}$ also acts on $C^R_\bullet(A)$ by Lie derivatives. Define the $\mathfrak{g}$-*invariant Hochschild chain group* $C^{\mathfrak{g}}_\bullet(A)$ to be the space of coinvariants of this action, that is

$$C^{\mathfrak{g}}_\bullet(A) = C^R_\bullet(A)/\mathfrak{g} \cdot C^R_\bullet(A).$$

We shall make the assumption that $\mathfrak{g} \cdot C^R_\bullet(A)$ is closed submodule, as this holds in the examples which are of interest to us. Thus, $C^{\mathfrak{g}}_\bullet(A)$ is Hausdorff. That $X \in \mathfrak{g}$ is a derivation on A implies that $\delta X = 0$, and consequently the operators b and B descend to operators on $C^{\mathfrak{g}}_\bullet(A)$. One can define the $\mathfrak{g}$-invariant periodic cyclic complex $C^{\mathfrak{g}}_{\mathrm{per}}(A)$ as the product of $\mathfrak{g}$-invariant Hochschild chain groups, and its homology is the $\mathfrak{g}$-*invariant periodic cyclic homology* $HP^{\mathfrak{g}}_\bullet(A)$.

Let $C^\bullet_{\mathfrak{g}}(A, A)$ denote the space of all Hochschild cochains D for which $[X, D] = 0$ for all $X \in \mathfrak{g}$. Since $\mathfrak{g}$ is abelian, we have $\mathfrak{g} \subseteq C^1_{\mathfrak{g}}(A, A)$. If $D \in C^\bullet_{\mathfrak{g}}(A, A)$, then the formula

$$0 = \delta[X, D] = [\delta X, D] + [X, \delta D]$$

shows that $C^\bullet_{\mathfrak{g}}(A, A)$ is a subcomplex because $\delta X = 0$.

Proposition 6.2. *For any $D \in C^\bullet_{\mathfrak{g}}(A, A)$ and $X, Y \in C^1_{\mathfrak{g}}(A, A)$, the operators $L_D, I_D, L\{X, Y\}$ and $I\{X, Y\}$ are well-defined on the $\mathfrak{g}$-invariant complex $C^{\mathfrak{g}}_\bullet(A)$.*

Proof. If $Z \in C^1(A, A)$ is a derivation, then one can verify directly that

$$[L_X, I_D] = I_{[X,D]}, \quad [L_X, L_D] = L_{[X,D]},$$
$$[L_Z, I\{X, Y\}] = I\{[Z, X], Y\} + I\{X, [Z, Y]\},$$

and

$$[L_Z, L\{X, Y\}] = L\{[Z, X], Y\} + L\{X, [Z, Y]\},$$

from which the proposition follows. □

One of the benefits of working in the $\mathfrak{g}$-invariant complex is that the contraction I_X is now a chain map when $X \in \mathfrak{g}$. Indeed,

$$[b + B, I_X] = L_X = 0$$

in $C^{\mathfrak{g}}_\bullet(A)$. These contraction operators obey the following algebra as operators on homology.

Theorem 6.3. *There is an algebra map* $\chi : \Lambda^\bullet \mathfrak{g} \to \mathrm{End}(HP_\bullet^{\mathfrak{g}}(A))$ *given by*

$$\chi(X_1 \wedge X_2 \wedge \cdots \wedge X_k) = I_{X_1} I_{X_2} \ldots I_{X_k}.$$

Proof. First notice that $X \mapsto I_X$ is a linear mapping. Next, we shall show that $I_X I_X$ is chain homotopic to zero. On $C_\bullet^{\mathfrak{g}}(A)$, observe that

$$0 = L_X L_X = L_{X^2} + 2L\{X, X\}$$

where X^2 denotes the composition of X with itself. Thus,

$$L\{X, X\} = -L_{\frac{1}{2}X^2}.$$

Next, notice that

$$\delta\left(\frac{1}{2}X^2\right) = X \smile X.$$

By Theorems 2.2 and 2.1,

$$\begin{aligned} -[b + B, I\{X, X\}] &= -L\{X, X\} - I_{X \smile X} + I_X I_X \\ &= L_{\frac{1}{2}X^2} - I_{\delta(\frac{1}{2}X^2)} + I_X I_X \\ &= [b + B, I_{\frac{1}{2}X^2}] + I_X I_X, \end{aligned}$$

which proves that $I_X I_X$ is chain homotopic to zero. By the universal property of the exterior algebra, the map χ exists as asserted. □

There are some additional simplifications regarding the operator $L\{X, Y\}$ once we pass to $C_\bullet^{\mathfrak{g}}(A)$.

Proposition 6.4. *For* $X, Y \in \mathfrak{g}$, *the operator* $L\{X, Y\}$ *satisfies*

$$L\{X, Y\}(a_0 \otimes \cdots \otimes a_n) = \sum_{i=0}^{n-1} \sum_{j=i+1}^{n} a_0 \otimes \cdots \otimes X(a_i) \otimes \cdots \otimes Y(a_j) \otimes \cdots \otimes a_n,$$

$$[b + B, L\{X, Y\}] = -L_{X \smile Y}$$

on $C_\bullet^{\mathfrak{g}}(A)$.

Proof. Notice that

$$\begin{aligned} &L_Y(X(a_0), a_1, \ldots, a_n) - L_X(Y(a_0), a_1, \ldots, a_n) \\ &= \sum_{j=1}^{n} X(a_0) \otimes \cdots \otimes Y(a_j) \otimes \cdots \otimes a_n - \sum_{i=1}^{n} Y(a_0) \otimes \cdots \otimes X(a_i) \otimes \cdots \otimes a_n, \end{aligned}$$

using the fact that $[X, Y] = 0$. So

$$L\{X,Y\}(a_0\otimes\cdots\otimes a_n)+L_Y(X(a_0)\otimes a_1\otimes\cdots\otimes a_n)-L_X(Y(a_0)\otimes a_1\otimes\cdots\otimes a_n) = \sum_{i=0}^{n-1}\sum_{j=i+1}^{n} a_0\otimes\cdots\otimes X(a_i)\otimes\cdots\otimes Y(a_j)\otimes\cdots\otimes a_n$$

gives the desired conclusion. The formula

$$[b + B, L\{X, Y\}] = -L_{X\smile Y}$$

is immediate from Theorem 2.2 because $L_X = L_Y = 0$ on $C^{\mathfrak{g}}_\bullet(A)$. □

6.3. Connections on the $\mathfrak{g}$-invariant complex. The following situation is modeled on what we see with the noncommutative tori deformation. Suppose A_J is the algebra of sections of a smooth deformation of locally convex algebras $\{A_t\}_{t\in J}$. Suppose $\mathfrak{g}$ is an abelian Lie algebra of $C^\infty(J)$-linear derivations on A_J. Suppose ∇ is a $\mathfrak{g}$-invariant connection on A_J for which

$$E := \delta\nabla = \sum_{i=1}^{r} X_i \smile Y_i,$$

where $X_i, Y_i \in \mathfrak{g}$. This means that

$$\nabla(a_1a_2) = \nabla(a_1)a_2 + a_1\nabla(a_2) + \sum_{i=1}^{r} X_i(a_1)Y_i(a_2), \quad a_1, a_2 \in A_J.$$

More generally, Rieffel's deformations by actions of $\mathbb{R}^d$ are deformations of this type [31]. Here the cocycle E is a type of "Poisson bracket" built from the action, though the underlying algebras may not be commutative.

Proposition 6.5. *In the above situation, the Gauss–Manin connection*

$$\nabla_{GM} = L_\nabla - I_E$$

descends to the $\mathfrak{g}$-invariant complex $C^{\mathfrak{g}}_\bullet(A_J)$ and therefore to a connection on the $\mathfrak{g}$-invariant periodic cyclic homology $HP^{\mathfrak{g}}_\bullet(A_J)$.

Proof. Since ∇ is $\mathfrak{g}$-invariant, so is $E = \delta\nabla$ because $\mathfrak{g}$-invariant cochains form a subcomplex. The statement follows from Proposition 6.2. □

Our main reason for working with the $\mathfrak{g}$-invariant complex is that we can define another connection on $HP^{\mathfrak{g}}_\bullet(A_J)$ which is easier to work with than ∇_{GM}. Recall that the connection L_∇ satisfies

$$[b + B, L_\nabla] = L_E = \sum_{i=1}^{r} L_{X_i\smile Y_i}.$$

Thus, by Proposition 6.4,

$$\widetilde{\nabla} = L_\nabla + \sum_{i=1}^{r} L\{X_i, Y_i\}$$

is a connection on $C^{\mathfrak{g}}_{\mathrm{per}}(A_J)$ that commutes with $b+B$ and therefore descends to a connection on $HP^{\mathfrak{g}}_\bullet(A_J)$. We emphasize that $\widetilde{\nabla}$ does not commute with $b+B$ on the ordinary periodic cyclic complex $C_{\mathrm{per}}(A_J)$.

Remark 6.6. There is a conceptual explanation for the form of this connection $\widetilde{\nabla}$. Let $\mathcal{H}$ be the Hopf algebra whose underlying algebra is the symmetric algebra $S(\mathfrak{g} \oplus \mathrm{Span}\{\nabla\})$. The coproduct $\Delta : \mathcal{H} \to \mathcal{H} \otimes \mathcal{H}$ is the unique algebra map that satisfies

$$\Delta(X_i) = X_i \otimes 1 + 1 \otimes X_i, \quad \Delta(Y_i) = Y_i \otimes 1 + 1 \otimes Y_i,$$
$$\Delta(\nabla) = \nabla \otimes 1 + 1 \otimes \nabla + \sum_{i=1}^{r} X_i \otimes Y_i.$$

These are the defining relations of the Hopf algebra of polynomial functions on the $(2r+1)$-dimensional Heisenberg group. As an algebra, $\mathcal{H}$ acts on A_J in the obvious way, and this action is a Hopf action in the sense that for all $h \in \mathcal{H}$,

$$h(a_1 a_2) = \sum h_{(1)}(a_1) h_{(2)}(a_2),$$

where $\Delta(h) = \sum h_{(1)} \otimes h_{(2)}$. When a Hopf algebra $\mathcal{H}$ acts (say, on the left) on two spaces V and W, the *diagonal action* of $\mathcal{H}$ on $V \otimes W$ is

$$h(v \otimes w) = \sum h_{(1)}(v) \otimes h_{(2)}(w).$$

The connection $\widetilde{\nabla}$ on $C^{\mathfrak{g}}_n(A_J)$ is none other than the diagonal action of ∇ on $A_J^{\widehat{\otimes}(n+1)}$ after passing to the quotient.

Proposition 6.7. *As operators on $HP^{\mathfrak{g}}_\bullet(A_J)$,*

$$\nabla_{GM} = \widetilde{\nabla} + \sum_{i=1}^{r} \chi(X_i \wedge Y_i).$$

Proof. We have

$$\begin{aligned}\nabla_{GM} - \widetilde{\nabla} &= -I_E - \sum_{i=1}^{r} L\{X_i, Y_i\} \\ &= -\sum_{i=1}^{r} \big(I_{X_i \smile Y_i} + L\{X_i, Y_i\}\big) \\ &= -\sum_{i=1}^{r} \big([b + B, I\{X_i, Y_i\}] + I_{Y_i} I_{X_i}\big) \\ &= -\sum_{i=1}^{r} [b + B, I\{X_i, Y_i\}] - \sum_{i=1}^{r} \chi(Y_i \wedge X_i),\end{aligned}$$

using Theorem 2.2. So at the level of homology,

$$\nabla_{GM} = \widetilde{\nabla} + \sum_{i=1}^{r} \chi(X_i \wedge Y_i). \qquad \square$$

Let us use the notation $\Omega = \sum_{i=1}^{r} X_i \wedge Y_i \in \Lambda^2\mathfrak{g}$, so that

$$\nabla_{GM} = \widetilde{\nabla} + \chi(\Omega)$$

as operators on $HP_\bullet^{\mathfrak{g}}(A_J)$. This shows that the two connections differ by a nilpotent operator. As a result, we expect one to be integrable if and only if the other is.

6.4. Integrating $\widetilde{\nabla}$ for noncommutative tori. In this section, we specialize to the noncommutative tori deformation $\{\mathcal{A}_{t\Theta}\}_{t\in J}$ for a given Θ with $J = \mathbb{R}$, A_J the algebra of sections and $\mathfrak{g} = \mathrm{Span}\{\delta_1, \dots, \delta_n\}$. The connection $\nabla = \frac{d}{dt}$ is $\mathfrak{g}$-invariant, and

$$E = \delta\nabla = \frac{1}{2\pi i} \sum_{j>k} \theta_{jk} \cdot \delta_j \smile \delta_k.$$

As in the previous section, we can define the connection

$$\widetilde{\nabla} = L_\nabla + \frac{1}{2\pi i} \sum_{j>k} \theta_{jk} \cdot L\{\delta_j, \delta_k\}$$

on the $\mathfrak{g}$-invariant complex $C_\bullet^{\mathfrak{g}}(A_J)$ which descends to a connection on $HP_\bullet^{\mathfrak{g}}(A_J)$. The connection on homology is a nilpotent perturbation of ∇_{GM}.

We shall now show that we are not losing anything in passing to $\mathfrak{g}$-invariant cyclic homology, in that the canonical map $HP_\bullet(\mathcal{A}_\Theta) \to HP_\bullet^{\mathfrak{g}}(\mathcal{A}_\Theta)$ is an isomorphism.

Theorem 6.8. *The canonical map* $C_{\mathrm{per}}(\mathcal{A}_\Theta) \to C_{\mathrm{per}}^{\mathfrak{g}}(\mathcal{A}_\Theta)$ *induces an isomorphism* $HP_\bullet(\mathcal{A}_\Theta) \cong HP_\bullet^{\mathfrak{g}}(\mathcal{A}_\Theta)$.

Proof. The space $\mathcal{A}_\Theta = \mathcal{S}(\mathbb{Z}^n)$ is a topological direct sum graded by $\mathbb{Z}^n$. For each $1 \le j \le n$, the jth degree of a homogenous element $a = u_1^{\alpha_1} \cdot \dots \cdot u_n^{\alpha_n}$ is $\deg_j a = \alpha_j$. The complex $C_{\text{per}}(\mathcal{A}_\Theta)$ is also graded by $\mathbb{Z}^n$. If $\omega = a_0 \otimes \cdots \otimes a_m$ is an elementary tensor of homogeneous elements $a_i \in \mathcal{A}_\Theta$, then define

$$\deg_j \omega = \sum_{i=0}^{m} \deg_j a_i.$$

The Lie derivative L_{δ_j} has the property that

$$L_{\delta_j}\omega = (2\pi i \cdot \deg_j \omega)\omega$$

for all homogeneous chains. From this, we see that $\mathfrak{g} \cdot C_{\text{per}}(\mathcal{A}_\Theta)$ is a closed direct summand of $C_{\text{per}}(\mathcal{A}_\Theta)$ which is complemented by $\bigcap_{j=1}^{n} \ker L_{\delta_j}$. This complement is just the subspace of homogeneous elements whose $\mathbb{Z}^n$-grading is $(0, 0, \dots, 0)$. By the Cartan homotopy formula (Theorem 2.1), $L_{\delta_j} = 0$ as an operator on $HP_\bullet(\mathcal{A}_\Theta)$. It follows that the summand $\mathfrak{g} \cdot C_{\text{per}}(\mathcal{A}_\Theta)$ is acyclic, and so the quotient map

$$C_{\text{per}}(\mathcal{A}_\Theta) \to C_{\text{per}}^{\mathfrak{g}}(\mathcal{A}_\Theta)$$

is a quasi-isomorphism. ☐

The above proof can be carried out because the action of $\mathfrak{g}$ by derivations on $\mathcal{A}_\Theta$ is the infinitesimal of an action of the Lie group $\mathbb{T}^n$ by algebra automorphisms. Thus $\mathcal{A}_\Theta$ decomposes as a topological direct sum of eigenspaces indexed by $\mathbb{Z}^n$ for the action of $\mathfrak{g}$. Note that the same proof shows that $C_{\text{per}}(A_J) \to C_{\text{per}}^{\mathfrak{g}}(A_J)$ is a quasi-isomorphism for the algebra of sections A_J of the deformation.

Theorem 6.9. *For the algebra of sections A_J of the noncommutative tori deformation, the connection $\widetilde{\nabla}$ is integrable on $C_{\text{per}}(A_J)$ and consequently on $C_{\text{per}}^{\mathfrak{g}}(A_J)$. Thus, $\{C_{\text{per}}^{\mathfrak{g}}(A_{t\Theta})\}_{t\in J}$ is a trivial deformation of chain complexes.*

Proof. It suffices to prove $\widetilde{\nabla}$ is integrable on $C_{\text{per}}(A_J)$. The quotient $C_{\text{per}}^{\mathfrak{g}}(A_J)$ is obtained from $C_{\text{per}}(A_J)$ as a quotient by a closed $\widetilde{\nabla}$-invariant direct summand, so integrability of $\widetilde{\nabla}$ on $C_{\text{per}}^{\mathfrak{g}}(A_J)$ will follow immediately.

Since $\widetilde{\nabla}$ restricts to a connection on $C_m(A_J)$ for each m, it suffices to prove $\widetilde{\nabla}$ is integrable on $C_m(A_J)$. Given $m+1$ multi-indices $\alpha^0, \dots, \alpha^m \in \mathbb{Z}^n$, we shall use the notation

$$u^{\overline{\alpha}} = u^{\alpha^0} \otimes u^{\alpha^1} \otimes \cdots \otimes u^{\alpha^m} \in C_m(\mathcal{A}_\Theta).$$

An element $\omega \in C_m(\mathcal{A}_\Theta)$ has the form

$$\omega = \sum_{\overline{\alpha}} c_{\overline{\alpha}} u^{\overline{\alpha}}.$$

Since $\mathcal{S}(\mathbb{Z}^n)^{\widehat{\otimes}(m+1)} \cong \mathcal{S}(\mathbb{Z}^{n(m+1)})$, the coefficients $c_{\overline{\alpha}}$ must be of rapid decay. Notice that $L\{\delta_j, \delta_k\}$ is diagonal with respect to the basis $\{u^{\overline{\alpha}}\}$, and

$$\frac{1}{2\pi i} L\{\delta_j, \delta_k\} u^{\overline{\alpha}} = 2\pi i \cdot R(\overline{\alpha}) u^{\overline{\alpha}}$$

for some real-valued polynomial $R(\overline{\alpha})$ in the multi-indices of $\overline{\alpha}$. An element $\sigma = \sum_{\overline{\alpha}} f_{\overline{\alpha}} u^{\overline{\alpha}} \in C_m(A_J)$ with $f_{\overline{\alpha}} \in C^\infty(J)$ satisfies $\widetilde{\nabla}(\sigma) = 0$ if and only if

$$f'_{\overline{\alpha}} + 2\pi i \cdot R(\overline{\alpha}) f_{\overline{\alpha}} = 0$$

for each $\overline{\alpha}$. Given an initial condition $f_{\overline{\alpha}}(s) = c_{\overline{\alpha}}$, the unique solution is given by

$$f_{\overline{\alpha}}(t) = c_{\overline{\alpha}} \exp(-2\pi i \cdot R(\overline{\alpha})(t - s)).$$

Notice that $|f_{\overline{\alpha}}(t)| = |c_{\overline{\alpha}}|$. So for each $t \in J$, the coefficients $\{f_{\overline{\alpha}}(t)\}$ satisfy the same decay conditions as $\{c_{\overline{\alpha}}\}$. Therefore if $\omega = \sum_{\overline{\alpha}} c_{\overline{\alpha}} u^{\overline{\alpha}} \in C_m(\mathcal{A}_{s\Theta})$, then $\rho(t) = \sum_{\overline{\alpha}} f_{\overline{\alpha}}(t) u^{\overline{\alpha}} \in C_m(\mathcal{A}_{t\Theta})$ for each $t \in J$. Moreover, since $R(\overline{\alpha})$ is a polynomial function, all derivatives of ρ will satisfy the rapid decay condition also, and so ρ is smooth as a function of t. The solution ρ is also smooth as we vary the initial parameter s, so $\widetilde{\nabla}$ is integrable by Proposition A.9. □

Corollary 6.10. *For any $n \times n$ skew-symmetric matrix Θ, there is a parallel translation isomorphism*

$$HP_\bullet(C^\infty(\mathbb{T}^n)) \cong HP_\bullet(\mathcal{A}_\Theta).$$

Consequently,

$$HP_0(\mathcal{A}_\Theta) \cong \mathbb{C}^{2^{n-1}}, \quad HP_1(\mathcal{A}_\Theta) \cong \mathbb{C}^{2^{n-1}}.$$

Proof. Since $\widetilde{\nabla}$ commutes with $b + B$ on $C^{\mathfrak{g}}_{\text{per}}(A_J)$, its parallel translation operators are isomorphisms of chain complexes. Thus we have

$$HP_\bullet(C^\infty(\mathbb{T}^n)) \xrightarrow{\cong} HP^{\mathfrak{g}}_\bullet(C^\infty(\mathbb{T}^n)) \xrightarrow{P_{01}^{\widetilde{\nabla}}} HP^{\mathfrak{g}}_\bullet(\mathcal{A}_\Theta) \xrightarrow{\cong} HP_\bullet(\mathcal{A}_\Theta).$$

As shown in [4], if M is a compact smooth manifold, then

$$HP_\bullet(C^\infty(M)) \cong \bigoplus_k H^{\bullet+2k}_{dR}(M, \mathbb{C}),$$

where $H^\bullet_{dR}(M, \mathbb{C})$ is the complex-valued de Rham cohomology of M. Now, $H^m_{dR}(\mathbb{T}^n, \mathbb{C})$ is a vector space of dimension $\binom{n}{m}$, and this gives the result. □

Corollary 6.11. *For the algebra of sections A_J of the noncommutative tori deformation, the Gauss–Manin connection is integrable on $HP_\bullet(A_J)$.*

Proof. Using Proposition 3.7, we have an isomorphism of $C^\infty(J)$-modules

$$HP_\bullet(A_J) \cong HP^{\mathfrak{g}}_\bullet(A_J) \cong C^\infty(J, HP^{\mathfrak{g}}_\bullet(\mathcal{A}_0)) \cong C^\infty(J, HP_\bullet(C^\infty(\mathbb{T}^n))$$

because $\widetilde{\nabla}$ is integrable on the complex $C^{\mathfrak{g}}_{\mathrm{per}}(A_J)$. Here we see that ∇_{GM} is a connection on a finite rank trivial bundle, so it must be integrable. □

Analogous results can be obtained for periodic cyclic cohomology by duality. For example, we can consider

$$\begin{aligned} C^{\mathrm{per}}_{\mathfrak{g}}(\mathcal{A}_\Theta) &= C^{\mathfrak{g}}_{\mathrm{per}}(\mathcal{A}_\Theta)^* \\ &= \left\{\varphi \in C^{\mathrm{per}}(\mathcal{A}_\Theta) : \varphi(L_{\delta_j}\omega) = 0,\ \forall \omega \in C_{\mathrm{per}}(\mathcal{A}_\Theta),\ \forall j\right\}. \end{aligned}$$

This is the space of all cochains which are supported on chains whose $\mathbb{Z}^n$-grading is $(0, 0, \dots, 0)$. The inclusion $C^{\mathrm{per}}_{\mathfrak{g}}(\mathcal{A}_\Theta) \to C^{\mathrm{per}}(\mathcal{A}_\Theta)$ is the transpose of the quotient map from Theorem 6.8, and also is a quasi-isomorphism. Since the underlying space of each fiber of $C^{\mathfrak{g}}_{\mathrm{per}}(A_J)$ is a nuclear Fréchet space, it follows that the dual connection $\widetilde{\nabla}^\star$ is automatically integrable on $C^{\mathrm{per}}_{\mathfrak{g}}(A_J) = C^{\mathfrak{g}}_{\mathrm{per}}(A_J)^\star$, though this could also be proved directly. So we have parallel translation isomorphisms $HP^\bullet(C^\infty(\mathbb{T}^n)) \cong HP^\bullet(\mathcal{A}_\Theta)$ for periodic cyclic cohomology as well.

We have proved the rigidity of periodic cyclic homology/cohomology for the deformation of noncommutative tori. It is interesting to note that the Hochschild homology/cohomology and (non periodic) cyclic homology/cohomology are very far from rigid in this deformation. As an example, $HH^0(A) = HC^0(A)$ is the space of all traces on the algebra A. Now in the simplest case where $n = 2$ and

$$\Theta = \begin{pmatrix} 0 & -\theta \\ \theta & 0 \end{pmatrix},$$

it is well known that there is a unique (normalized) trace on $\mathcal{A}_\Theta$ when $\theta \notin \mathbb{Q}$ and an infinite dimensional space of traces when $\theta \in \mathbb{Q}$. For example, every linear functional on the commutative algebra $\mathcal{A}_0 \cong C^\infty(\mathbb{T}^n)$ is a trace, and thus $HH^0(C^\infty(\mathbb{T}^n)) = C^\infty(\mathbb{T}^n)^*$ is the space of distributions on $\mathbb{T}^n$. Moreover, Connes showed in [4] that in the case $\theta \notin \mathbb{Q}$, $HH^1(\mathcal{A}_\Theta)$ and $HH^2(\mathcal{A}_\Theta)$ are either finite dimensional, or infinite dimensional and non-Hausdorff depending on the diophantine properties of θ. Looking back, we conclude that there are no integrable connections on $C^\bullet(A)$ that commute with b, as such a connection would imply rigidity of Hochschild cohomology.

However, our connection $\widetilde{\nabla}$ does commute with b on the invariant complex $C^\bullet_{\mathfrak{g}}(A)$. This shows that the invariant Hochschild cohomology $HH^\bullet_{\mathfrak{g}}(\mathcal{A}_\Theta)$ is independent of Θ. For example, there is exactly one (normalized) $\mathfrak{g}$-invariant trace on $C^\infty(\mathbb{T}^n)$, and that corresponds to integration with respect to the only (normalized) translation invariant measure. Thus $HH^0_{\mathfrak{g}}(C^\infty(\mathbb{T}^n)) = HH^0_{\mathfrak{g}}(\mathcal{A}_\Theta) = \mathbb{C}$. Consequently, the canonical map $HH^\bullet_{\mathfrak{g}}(\mathcal{A}_\Theta) \to HH^\bullet(\mathcal{A}_\Theta)$ is not, in general, an isomorphism.

The argument presented here should work with other variants of cyclic cohomology. One example is Connes' entire cyclic cohomology which is constructed by allowing for infinite cochains $(\varphi_n) \in \prod_n C^n(A)$ satisfying a certain growth condition [3]. The Lie derivative and contraction operators extend to the entire cochain complex. One can introduce the connection $\widetilde{\nabla}$ on the $\mathfrak{g}$-invariant entire cyclic cochain complex, and it is likely integrable, though we haven't checked the analytic details.

7. Calculations with ∇^{GM} and the Chern–Connes pairing

We have shown that ∇_{GM} is integrable for noncommutative tori in a rather indirect way by proving integrability of the auxiliary connection $\widetilde{\nabla}$. However, it is still useful to understand ∇_{GM} here because it is canonical and has good properties with respect to the Chern–Connes pairing. Here, we shall calculate with the dual connection ∇^{GM} to determine the parallel translation maps, as well as the deformation of the Chern–Connes pairing.

7.1. Cyclic cocycles, characteristic maps, and cup products. By a *cyclic k-cocycle* we mean an element $\varphi \in C^k(A)$ such that

$$b\varphi = 0, \quad \varphi(e, a_1, \ldots, a_k) = 0, \quad \varphi(a_k, a_0, \ldots, a_{k-1}) = (-1)^k \varphi(a_0, a_1, \ldots, a_k).$$

A cyclic cocycle φ automatically satisfies $B\varphi = 0$, and so φ gives a cohomology class in $HP^\bullet(A)$. Recall that elements in the first slot of a cochain can be in the unitization A_+. Below we will use the notation $\widetilde{a}_0$ for elements of the unitization A_+ and just a_0 for elements of A.

We return to the general setting of an algebra A equipped with an abelian Lie algebra $\mathfrak{g}$ of derivations, and a $\mathfrak{g}$-invariant trace. Define the *characteristic map* $\gamma : \Lambda^\bullet \mathfrak{g} \to C^\bullet(A)$ by

$$\gamma(X_1 \wedge \cdots \wedge X_k)(\widetilde{a}_0, \ldots, a_k) = \frac{1}{k!} \sum_{\sigma \in \mathbb{S}_k} (-1)^\sigma \tau\big(\widetilde{a}_0 X_{\sigma(1)}(a_1) X_{\sigma(2)}(a_2) \ldots X_{\sigma(k)}(a_k)\big).$$

Proposition 7.1. *The functional $\gamma(X_1 \wedge \cdots \wedge X_k)$ is a $\mathfrak{g}$-invariant cyclic k-cocycle.*

Remark 7.2. The map γ is a simple case of the Connes–Moscovici characteristic map in Hopf cyclic cohomology [6]. In their work, $\mathcal{H}$ is a Hopf algebra equipped with some extra structure called a modular pair, and A is an algebra equipped with a Hopf action of $\mathcal{H}$. Assuming A possesses a compatible trace, they construct a map

$$\gamma : HP^\bullet_{\mathrm{Hopf}}(\mathcal{H}) \to HP^\bullet(A)$$

from the Hopf periodic cyclic cohomology of $\mathcal{H}$ to the ordinary periodic cyclic cohomology of A. In our situation, $\mathcal{H} = \mathcal{U}(\mathfrak{g})$, the universal enveloping algebra of $\mathfrak{g}$. As was shown in [6],

$$HP^{\bullet}_{\mathrm{Hopf}}(\mathcal{U}(\mathfrak{g})) \cong \bigoplus_{k=\bullet \bmod 2} H_k^{\mathrm{Lie}}(\mathfrak{g}, \mathbb{C}),$$

where $H_k^{\mathrm{Lie}}(\mathfrak{g}, \mathbb{C})$ is the Lie algebra homology of $\mathfrak{g}$ with coefficients in the trivial $\mathfrak{g}$-module $\mathbb{C}$. As $\mathfrak{g}$ is abelian, there is an isomorphism

$$H_k^{\mathrm{Lie}}(\mathfrak{g}, \mathbb{C}) \cong \Lambda^k(\mathfrak{g}).$$

The obtained characteristic map

$$\gamma : \Lambda^{\bullet}(\mathfrak{g}) \to HP^{\bullet}(A)$$

is the map defined above.

Lemma 7.3. *Let $X_1, \dots, X_n$ be derivations on an algebra A, and let τ be a trace on A. There exists $\psi \in C^{n-1}(A)$ such that $B\psi = 0$ and*

$$\begin{aligned}\tau\big(\widetilde{a}_0 X_1(a_1) \dots X_n(a_n)\big) &= \frac{1}{n}\sum_{j=1}^{n}(-1)^{(j-1)(n+1)}\tau\big(\widetilde{a}_0 X_j(a_1)\dots X_n(a_{n-j+1})X_1(a_{n-j+2})\dots X_{j-1}(a_n)\big)\\ &\quad + (b\psi)(\widetilde{a}_0, \dots, a_n).\end{aligned}$$

Proof. Given any n derivations $Y_1, \dots, Y_n$, the cochain $\varphi \in C^{n-1}(A)$ given by

$$\varphi(a_0, \dots, a_{n-1}) = \tau(Y_1(a_0)Y_2(a_1)\dots Y_n(a_{n-1})), \quad \varphi(e, a_1, \dots, a_{n-1}) = 0,$$

satisfies

$$(b\varphi)(\widetilde{a}_0, \dots, a_n) = \tau\big(\widetilde{a}_0 Y_1(a_1)\dots Y_n(a_n) + (-1)^n \widetilde{a}_0 Y_2(a_1)\dots Y_n(a_{n-1})Y_1(a_n)\big)$$

and $B\varphi = 0$. It follows that

$$\begin{aligned}\psi(a_0, \dots, a_{n-1}) &= \frac{1}{n}\sum_{j=1}^{n-1}(-1)^{(j-1)(n+1)}(n-j)\\ &\qquad\cdot \tau\big(X_j(a_0)X_{j+1}(a_1)\dots X_{j-1}(a_n)\big),\\ \psi(e, a_1, \dots, a_{n-1}) &= 0\end{aligned}$$

satisfies the conclusions of the lemma. □

Recall that for any $Z \in \mathfrak{g}$, the contraction I_Z is a chain map on the invariant complex $C^{\mathfrak{g}}_{\mathrm{per}}(A)$. We shall use the same notation I_Z to denote its transpose on $C_{\mathfrak{g}}^{\mathrm{per}}(A)$. We now compute this operator on the image of the characteristic map.

Proposition 7.4. *For any $Z \in \mathfrak{g}$ and $\omega \in \Lambda^\bullet\mathfrak{g}$,*

$$I_Z[\gamma(\omega)] = [\gamma(Z \wedge \omega)]$$

in $HP^\bullet_{\mathfrak{g}}(A)$.

Proof. Let $\varphi = \gamma(X_1 \wedge \cdots \wedge X_k)$. Since $\varphi(e, a_1, \ldots, a_k) = 0$, we immediately have $S_Z\varphi = 0$. Thus, $I_Z\varphi = \iota_Z\varphi$, and

$$\begin{aligned}
(\iota_Z\varphi)(a_0, \ldots, a_{k+1}) \\
&= \varphi\big(a_0 Z(a_1), a_2, \ldots a_{k+1}\big) \\
&= \frac{1}{k!}\sum_{\sigma\in\mathbb{S}_k}(-1)^\sigma\tau\big(a_0 Z(a_1)X_{\sigma(1)}(a_2)X_{\sigma(2)}(a_3)\ldots X_{\sigma(k)}(a_{k+1})\big) \\
&= \gamma(Z \wedge X_1 \wedge \cdots \wedge X_k)(a_0, \ldots, a_{k+1}) + (b\psi)(a_0, \ldots, a_{k+1})
\end{aligned}$$

for some ψ with $B\psi = 0$ by applying the previous lemma to each term in the sum. Hence, $I_Z\gamma(X_1 \wedge \cdots \wedge X_k) = \gamma(Z \wedge X_1 \wedge \cdots \wedge X_k) + (b + B)\psi$. □

As in the homology case (Theorem 6.3) there is an algebra map

$$\chi : \Lambda^\bullet(\mathfrak{g}) \to \operatorname{End}\big(HP^\bullet_{\mathfrak{g}}(A)\big), \quad \chi(X_1 \wedge \cdots \wedge X_k) = I_{X_1}I_{X_2}\ldots I_{X_k}.$$

Corollary 7.5. *For any $\omega \in \Lambda^\bullet\mathfrak{g}$,*

$$[\gamma(\omega)] = \chi(\omega)[\tau]$$

in $HP^\bullet_{\mathfrak{g}}(A)$.

Remark 7.6. A generalization of the Connes–Moscovici characteristic map was constructed in [21]. A special case of this construction is a cup product

$$\smile : HP^p_{\mathrm{Hopf}}(\mathcal{H}) \otimes HP^q_{\mathcal{H}}(A) \to HP^{p+q}(A),$$

where $HP^\bullet_{\mathcal{H}}(A)$ is the periodic cyclic cohomology of A built out of cochains which are invariant in some sense with respect to an action of $\mathcal{H}$. In the Connes–Moscovici picture, the properties of the trace τ ensures that it gives a cohomology class in $HP^\bullet_{\mathcal{H}}(A)$, and

$$[\omega] \smile [\tau] = \gamma[\omega]$$

for all $[\omega] \in HP^\bullet_{\mathrm{Hopf}}(\mathcal{H})$. In our situation where $\mathcal{H} = \mathcal{U}(\mathfrak{g})$, we have that $HP^\bullet_{\mathcal{H}}(A) = HP^\bullet_{\mathfrak{g}}(A)$ and the cup product is a map

$$\smile : \Lambda^p\mathfrak{g} \otimes HP^q_{\mathfrak{g}}(A) \to HP^{p+q}(A).$$

Our map $\chi : \Lambda^\bullet(A) \to \operatorname{End}(HP^\bullet_{\mathfrak{g}}(A))$ followed by the inclusion $HP^\bullet_{\mathfrak{g}}(A) \to HP^\bullet(A)$ coincides with this cup product.

7.2. ∇^{GM}-derivatives of characteristic cocycles. Now consider the situation of Section 6.3. Suppose A_J is the algebra of sections of a deformation, $\mathfrak{g}$ is an abelian Lie algebra of derivations on A_J, ∇ is a $\mathfrak{g}$-invariant connection on A_J satisfying

$$\nabla(a_1a_2) = \nabla(a_1)a_2 + a_1\nabla(a_2) + \sum_{i=1}^{r} X_i(a_1)Y_i(a_2), \quad a_1, a_2 \in A_J$$

for $X_i, Y_i \in \mathfrak{g}$. In addition we assume A_J has a $\mathfrak{g}$-invariant ∇-parallel trace τ. That is, there is a trace $\tau : A_J \to C^\infty(J)$ such that $\tau \circ Z = 0$ for all $Z \in \mathfrak{g}$ and $\tau \circ \nabla = \frac{d}{dt} \circ \tau$. The dual of the connection

$$\widetilde{\nabla} = L_\nabla + \sum_{i=1}^{r} L\{X_i, Y_i\}$$

is given by

$$\widetilde{\nabla}^\star \varphi = \frac{d}{dt} \circ \varphi - \varphi \circ L_\nabla - \sum_{i=1}^{r} \varphi \circ L\{X_i, Y_i\}$$

on $C_{\mathfrak{g}}^{\text{per}}(A_J)$.

Proposition 7.7. *For any $\omega \in \Lambda^\bullet \mathfrak{g}$, we have $\widetilde{\nabla}^\star \gamma(\omega) = 0$.*

Proof. Letting $\varphi = \gamma(Z_1 \wedge \cdots \wedge Z_m)$, one can show that

$$\frac{d}{dt} \circ \varphi = \varphi \circ L_\nabla + \sum_{i=1}^{r} \varphi \circ L\{X_i, Y_i\}$$

by using $\frac{d}{dt} \circ \tau = \tau \circ \nabla$, the identity

$$\nabla(a_0 \ldots a_m) = \sum_{j=0}^{k} a_0 \ldots \nabla(a_j) \ldots a_m + \sum_{i=1}^{r} \sum_{j<k} a_0 \ldots X_i(a_j) \ldots Y_i(a_k) \ldots a_m,$$

and the fact that all derivations commute with each other and ∇. □

As before, let $\Omega = \sum_{i=1}^{r} X_i \wedge Y_i \in \Lambda^2 \mathfrak{g}$. Recall from Proposition 6.7 that

$$\nabla_{GM} = \widetilde{\nabla} + \chi(\Omega)$$

as operators on $HP_\bullet^{\mathfrak{g}}(A_J)$. Dualizing gives

$$\nabla^{GM}[\varphi] = \widetilde{\nabla}^\star[\varphi] - \sum_{i=1}^{r} \left[\varphi \circ \chi(X_i \wedge Y_i)\right]$$

and further

$$\big[\varphi \circ \chi(X_i \wedge Y_i)\big] = \big[\varphi \circ (I_{X_i} I_{Y_i})\big] = I_{Y_i} I_{X_i}[\varphi] = -\chi(X_i \wedge Y_i)[\varphi].$$

Consequently,

$$\nabla^{GM} = \widetilde{\nabla}^{\star} + \chi(\Omega)$$

as operators on $HP^{\bullet}_{\mathfrak{g}}(A_J)$. Combining the previous results, we immediately obtain the following.

Theorem 7.8. *In the above situation, for any $\omega \in \Lambda^{\bullet}\mathfrak{g}$,*

$$\nabla^{GM}\big[\gamma(\omega)\big] = \big[\gamma(\Omega \wedge \omega)\big]$$

in $HP^{\bullet}_{\mathfrak{g}}(A_J)$.

It is worth mentioning that this result is independent of the integrability of $\widetilde{\nabla}^{\star}$ or ∇^{GM}. Using this theorem, we can explicitly describe ∇^{GM}-parallel classes through a given characteristic cocycle.

Corollary 7.9. *Let $\omega \in \Lambda^{\bullet}\mathfrak{g}$ and view $\gamma(\omega) \in HP^{\bullet}_{\mathfrak{g}}(A_s)$. Then the cocycle $\varphi \in HP^{\bullet}_{\mathfrak{g}}(A_J)$ given by*

$$\varphi = \sum_{p=0}^{\lfloor \dim \mathfrak{g}/2 \rfloor} (-1)^p \frac{(t-s)^p}{p!} \gamma(\Omega^{\wedge p} \wedge \omega)$$

is a ∇^{GM}-parallel section through $\gamma(\omega) \in HP^{\bullet}_{\mathfrak{g}}(A_s)$.

Also from Theorem 7.8, we obtain the following result about the deformation of the Chern–Connes character.

Corollary 7.10. *Let $\omega \in \Lambda^{\bullet}\mathfrak{g}$.*

(i) *For any idempotent $P \in M_N(A_J)$,*

$$\frac{d}{dt}\langle[\gamma(\omega)],[P]\rangle = \langle[\gamma(\Omega \wedge \omega)],[P]\rangle.$$

(ii) *For any invertible $U \in M_N(A_J)$,*

$$\frac{d}{dt}\langle[\gamma(\omega)],[U]\rangle = \langle[\gamma(\Omega \wedge \omega)],[U]\rangle.$$

We can be a little more explicit with these formulas using the form of the cycles $\operatorname{ch} P$ and $\operatorname{ch} U$. First notice that $M_N(A_J) \cong M_N(\mathbb{C}) \otimes A_J$ is the algebra of sections of a deformation that has the same properties as A_J. Namely we have an action of an abelian Lie algebra $\mathfrak{g}$ generated by the derivations $\mathrm{id} \otimes X_i$ and $\mathrm{id} \otimes Y_i$. There is a $\mathfrak{g}$-invariant connection $\mathrm{id} \otimes \nabla$ as well as a parallel $\mathfrak{g}$-invariant trace $\mathrm{tr} \otimes \tau$.

Here $\mathrm{tr} : M_N(\mathbb{C}) \to \mathbb{C}$ is the usual matrix trace. Notationally, we will refer to the above objects as simply X_i, Y_i, ∇, and τ respectively. The connection ∇ satisfies

$$\nabla(a_1a_2) = \nabla(a_1)a_2 + a_1\nabla(a_2) + \sum_{i=1}^{r} X_i(a_1)Y_i(a_2), \quad a_1, a_2 \in M_N(A_J),$$

so all the previous results apply to this deformation. Notice we have a commutative diagram

$$\begin{array}{ccc} \Lambda^\bullet\mathfrak{g} & \xrightarrow{\ \gamma\ } & C^{\mathrm{per}}(A_J) \\ & \searrow^{\gamma} & \downarrow T^\star \\ & & C^{\mathrm{per}}(M_N(A_J)) \end{array}$$

where $T^\star$ is the transpose of the generalized trace $T : C_{\mathrm{per}}(M_N(A_J)) \to C_{\mathrm{per}}(A_J)$. Given a projection $M_N(A_J)$ and a cocycle $\varphi \in C^{\mathrm{per}}_{C^\infty(J)}(A_J)$, we will view the Chern–Connes pairing as a pairing between a cocycle and a cycle on $M_N(A_J)$:

$$\langle[\varphi], [P]\rangle = \langle T^\star\varphi, \mathrm{ch}\, P\rangle.$$

So if $\omega \in \Lambda^{2m}\mathfrak{g}$, then by using the explicit form of $\mathrm{ch}\, P \in C^{C^\infty(J)}_{\mathrm{even}}(M_N(A_J))$,

$$\langle[\gamma(\omega)], [P]\rangle = (-1)^m \frac{(2m)!}{m!}\gamma(\omega)(P, P, \dots, P),$$

where $\gamma(\omega)$ is viewed as a cocycle in $C^{\mathrm{even}}_{C^\infty(J)}(M_N(A_J))$, and there are $2m + 1$ appearances of P on the right hand side. Similarly, if $U \in M_N(A_J)$ is an invertible and $\omega \in \Lambda^{2m+1}\mathfrak{g}$, then

$$\langle[\gamma(\omega)], [U]\rangle = (-1)^m m!\gamma(\omega)(U^{-1}, U, \dots, U^{-1}, U).$$

Combining these with Corollary 7.10, we obtain the following result.

Theorem 7.11. *In the above situation,*

(i) *If $\omega \in \Lambda^{2m}\mathfrak{g}$, and $P \in M_N(A_J)$ is an idempotent,*

$$\frac{d}{dt}\gamma(\omega)(P, P, \dots, P) = -(4m + 2)\gamma(\Omega \wedge \omega)(P, P, \dots, P).$$

(ii) *If $\omega \in \Lambda^{2m+1}\mathfrak{g}$, and $U \in M_N(A_J)$ is invertible,*

$$\frac{d}{dt}\gamma(\omega)(U^{-1}, U, \dots, U^{-1}, U) = -(m+1)\gamma(\Omega \wedge \omega)(U^{-1}, U, \dots, U^{-1}, U).$$

7.3. Noncommutative tori. Here, we shall apply our results to the noncommutative tori deformation.

Theorem 7.12. *For every Θ, the map $\gamma : \Lambda^\bullet \mathfrak{g} \to HP^\bullet_{\mathfrak{g}}(A_\Theta)$ is an isomorphism of $\mathbb{Z}/2$-graded spaces.*

Proof. By Theorem 6.9 and Proposition 7.7, is suffices to prove this for $\mathcal{A}_0 \cong C^\infty(\mathbb{T}^n)$. Let $s_1, \dots, s_n$ be the coordinates in $\mathbb{T}^n = \mathbb{R}^n/\mathbb{Z}^n$. Choosing a subset of coordinates $s_{i_1}, \dots, s_{i_m}$ determines a subtorus T of dimension m. All such subtori are in bijection with a generating set of homology classes of $\mathbb{T}^n$. The $\mathfrak{g}$-invariant de Rham cycle corresponding to T is given by taking the average of the integral of a differential form over all subtori parallel to T. The cochain in $C^m(C^\infty(\mathbb{T}^n))$ corresponding to this cycle is

$$\varphi_T(f_0, \dots, f_m) = \int_{\mathbb{T}^n/T} \left(\int_{gT} f_0 df_1 \wedge \cdots \wedge df_m \right) dgT,$$

and one can show $\varphi_T = \gamma(\delta_{i_1} \wedge \cdots \wedge \delta_{i_m})$ up to a scalar multiple. □

Hence the characteristic cocycles form a basis for $HP^\bullet(\mathcal{A}_\Theta)$. So we can use Corollary 7.9 to explicitly describe the parallel translation of ∇^{GM}. Let's do this for the $n = 2$ case. Here, the noncommutative torus is determined by a single real parameter $\theta := \theta_{21}$, and we shall denote the algebra by $\mathcal{A}_\theta$. In the case $\theta \notin \mathbb{Q}$, $\mathcal{A}_\theta$ is also known as (the smooth version of) the irrational rotation algebra. We shall consider $\{\mathcal{A}_\theta\}_{\theta \in J}$ as a smooth one-parameter deformation, where $J \subset \mathbb{R}$ is an open interval containing 0. Here, $\mathfrak{g} = \operatorname{Span}\{\delta_1, \delta_2\}$ and $\Omega = \frac{1}{2\pi i}\delta_2 \wedge \delta_1$. Let τ_2 be the cyclic 2-cocycle $\tau_2 = \frac{1}{\pi i}\gamma(\delta_1 \wedge \delta_2)$, which is given explicitly by

$$\tau_2(a_0, a_1, a_2) = \frac{1}{2\pi i}\tau\big(a_0\delta_1(a_1)\delta_2(a_2) - a_0\delta_2(a_1)\delta_1(a_2)\big).$$

Then from Theorem 7.8,

$$\nabla^{GM}[\tau] = \frac{1}{2\pi i}[\gamma(\delta_2 \wedge \delta_1) = -\frac{1}{2}[\tau_2],$$
$$\nabla^{GM}[\tau_2] = \frac{1}{\pi i(2\pi i)}[\gamma(\delta_2 \wedge \delta_1 \wedge \delta_1 \wedge \delta_2] = 0.$$

We see that $[\tau_2] \in HP^\bullet(A_J)$ is a ∇^{GM}-parallel section, and $[\tau] + \frac{\theta}{2}[\tau_2]$ is a ∇^{GM}-parallel section through $[\tau]$ at the fiber $\theta = 0$.

Now consider the odd classes

$$\tau_1^1 = \frac{1}{2\pi i}\gamma(\delta_1), \quad \tau_1^2 = \frac{1}{2\pi i}\gamma(\delta_2).$$

Then we have

$$\nabla^{GM}\big[\tau_1^j\big] = \frac{1}{(2\pi i)^2}\big[\gamma(\delta_2 \wedge \delta_1 \wedge \delta_j)\big] = 0$$

for $j = 1, 2$. So we can completely describe the parallel translation of ∇^{GM}.

Theorem 7.13. *Let $\theta \in \mathbb{R}$, then $P_{0,\theta}^{\nabla^{GM}} : HP^\bullet(\mathcal{A}_0) \to HP^\bullet(\mathcal{A}_\theta)$ is given by*

$$P_{0,\theta}^{\nabla^{GM}} : [\tau] \mapsto [\tau] + \frac{\theta}{2}[\tau_2], \quad [\tau_2] \mapsto [\tau_2], \quad \left[\tau_1^j\right] \mapsto \left[\tau_1^j\right].$$

It is interesting to notice that this parallel translation gives a nontrivial automorphism

$$P_{0,1}^{\nabla^{GM}} : HP^\bullet(C^\infty(\mathbb{T}^n)) \to HP^\bullet(\mathcal{A}_1) = HP^\bullet(C^\infty(\mathbb{T}^n)).$$

Now let's consider the Chern–Connes pairing. Using Theorem 7.11, we see that for any idempotent $P \in M_N(A_J)$,

$$\frac{d}{d\theta}\tau(P) = \tau_2(P, P, P)$$

and

$$\frac{d^2}{d\theta^2}\tau(P) = \frac{d}{d\theta}\tau_2(P, P, P) = 0.$$

Thus

$$\tau(P(\theta)) = \tau(P(0)) + \tau_2(P(0), P(0), P(0)) \cdot \theta.$$

Now the idempotent $P(0) \in M_N(\mathcal{A}_0) \cong M_N(C^\infty(\mathbb{T}^2))$ corresponds to a smooth vector bundle over $\mathbb{T}^2$ and the value $\tau(P(0))$ is the dimension of this bundle. The number $\tau_2(P(0), P(0), P(0))$ is the first Chern number of the bundle, which is an integer. So P satisfies

$$\tau(P) = C + D\theta$$

for integers C and D, which are determined by the topological information of $P(0)$.

Starting with a vector bundle represented by $P(0) \in M_N(C^\infty(\mathbb{T}^2))$, one can always extend it to a smooth family of idempotents $P(\theta) \in M_N(C^\infty(\mathcal{A}_\theta))$ for small enough θ using a functional calculus argument. Our results imply that the trace $\tau(P(\theta))$ is determined by, and can be computed from, the characteristic classes of the vector bundle $P(0)$. The same is true for other Chern–Connes pairings. For example, the value $\tau_2(P(\theta), P(\theta), P(\theta))$ is constant, hence an integer. This integrality was also explained with an index formula in [2]. Similarly, the pairing of an invertible with τ_1^j is integral.

These results also show that A_θ has a K-theory class with trace θ, at least for small enough θ, and so suggest that one may be able to find an idempotent in $\mathcal{A}_\theta$ with trace θ. Of course it is well-known now that such idempotents exist [29]. Given such an idempotent $P_\theta \in \mathcal{A}_\theta$, one could try to extend it to $\theta = 0$ through an idempotent $P \in A_J$. However this is impossible because $P(0)$ would necessarily satisfy $\tau_2(P(0), P(0), P(0)) \neq 0$, and the only idempotents in $\mathcal{A}_0$, namely 0 and 1, do not. However, one can extend P_θ to an idempotent in A_J provided J doesn't contain integers. One can also find an idempotent $P \in M_2(A_J)$ of trace $1+\theta$, but only if $J \subseteq (-1,1)$.

Two other interesting situations to consider are $J = \mathbb{R}$ or $J = \mathbb{T}$. For both cases, the trace of any idempotent $P \in M_N(A_J)$ must be constant and integral, and $\tau_2(P,P,P) = 0$. If $\tau_2(P,P,P) \neq 0$, then $\tau(P)$ would be negative somewhere in the $J = \mathbb{R}$ case, and $\tau(P)$ wouldn't be continuous in the $J = \mathbb{T}$ case.

Analogous results can be worked out for higher dimensional noncommutative tori. For the deformation $\{\mathcal{A}_{t\Theta}\}_{t\in J}$, the element $\Omega \in \Lambda^2\mathfrak{g}$ is

$$\Omega = \frac{1}{2\pi i}\sum_{j>k} \theta_{jk}\cdot \delta_j \wedge \delta_k.$$

One can explicitly compute ∇^{GM} and its parallel translation operators using Theorem 7.8. The deformation of the Chern–Connes pairing can be determined from Theorem 7.11. As functions of t, the pairings can be higher degree polynomials, depending on the size of n. This was already observed in Elliott's Chern character computation [10]. The most interesting case would be to choose Θ so that Ω is nondegenerate in the sense that $\Omega^{\wedge\lfloor n/2\rfloor} \neq 0$.

A. Omitted proofs

A.1. Free modules. For more information about nuclearity (in the sense of Grothendieck), see [14] or [34].

Proposition A.1.

(i) *If* $X, Y \in$ LCTVS, *then*

$$(R\widehat{\otimes}_{\mathbb{C}} X)\widehat{\otimes}_R(R\widehat{\otimes}_{\mathbb{C}} Y) \cong R\widehat{\otimes}_{\mathbb{C}}(X\widehat{\otimes}_{\mathbb{C}} Y).$$

(ii) *If R is a nuclear Fréchet algebra, X is a Fréchet space, and Y is a closed subspace of X, then*

$$(R\widehat{\otimes} X)/(R\widehat{\otimes} Y) \cong R\widehat{\otimes}(X/Y)$$

(iii) *If X and R are Fréchet spaces, one of which is nuclear, then the linear isomorphism*

$$\mathrm{Hom}_R(R\widehat{\otimes} X, M) \cong \mathrm{Hom}(X, M)$$

is a topological isomorphism.

Proof. (i) The proof is straightforward using universal properties.

(ii) Nuclearity of R implies that $R\widehat{\otimes} Y$ is a closed subspace of $R\widehat{\otimes} X$ [34, Proposition 43.7]. Since all spaces are Fréchet, all quotients appearing are complete.

One then induces mutually inverse isomorphisms using the universal properties of completed projective tensor products and quotients.

(iii) The linear isomorphisms

$$\Phi : \mathrm{Hom}(X, M) \to \mathrm{Hom}_R(R\widehat{\otimes}X, M), \quad \Psi : \mathrm{Hom}_R(R\widehat{\otimes}X, M) \to \mathrm{Hom}(X, M)$$

are given by

$$\Phi(F) = \mu(1 \otimes F), \quad \Psi(G)(x) = G(1 \otimes x),$$

where $\mu : R\widehat{\otimes}M \to M$ is the module action. Continuity of Ψ follows from the fact that if $B \subset X$ is bounded, then $1 \otimes B \subset R\widehat{\otimes}X$ is bounded.

The continuity of Φ is more subtle. The map Φ factors as

$$\Phi : \mathrm{Hom}(X, M) \xrightarrow{\Phi_1} \mathrm{Hom}_R(R\widehat{\otimes}X, R\widehat{\otimes}M) \xrightarrow{\Phi_2} \mathrm{Hom}_R(R\widehat{\otimes}X, M),$$

where $\Phi_1(F) = 1 \otimes F$ and Φ_2 is composition with the module action μ. Continuity of Φ_2 follows from continuity of μ. To show that Φ_1 is continuous, we need to relate the bounded subsets of $R\widehat{\otimes}X$ to the bounded subsets of R and X. This is related to the difficult "problème des topologies" of Grothendieck [14]. If either R or X are nuclear, then for every bounded subset $D \subset R\widehat{\otimes}X$, there are bounded subsets $A \subset R$, $B \subset X$ such that D is contained in the closed convex hull of

$$A \otimes B = \{r \otimes x \mid r \in A, x \in B\},$$

see [17, Theorem 21.5.8]. Continuity of Φ_1 follows from this fact. □

A.2. $C^\infty(J)$-modules. For the results below, the Banach–Steinhaus theorem (uniform boundedness principle) [34, Theorem 33.1] is the main advantage of considering barreled spaces.

Proposition A.2. *If X is barreled, then $\{F_t : X \to Y\}_{t\in J}$ is a smooth homotopy of continuous linear maps if and only if the map*

$$t \mapsto F_t(x)$$

is smooth for every $x \in X$.

Proof. Suppose $t \mapsto F_t(x)$ is smooth for each $x \in X$, and let $F_t^{(n)}(x)$ denote the nth derivative of this map. The linear map $F_t^{(n)} : X \to X$ is in fact continuous for each t. Using induction, this follows from the Banach–Steinhaus theorem because $F_t^{(n)}(x)$ is a pointwise limit of continuous linear maps by its very definition.

We must show that the map $\widetilde{F} : X \to C^\infty(J, Y)$ defined by

$$\widetilde{F}(x)(t) = F_t(x)$$

is continuous. For any compact $K \subset J$, any n, and any $x \in X$, the set

$$\left\{F_t^{(n)}(x) \mid t \in K\right\}$$

is compact in Y, hence bounded, because the map $t \mapsto F_t^{(n)}(x)$ is continuous. By the Banach–Steinhaus theorem, the set $\{F_t^{(n)} : X \to Y\}_{t \in K}$ is equicontinuous. Thus for any continuous seminorm q on Y, there exists a continuous seminorm p on X such that

$$q\big(F_t^{(n)}(x)\big) \le p(x), \quad \forall x \in X, \ \forall t \in K.$$

Consequently,

$$\sup_{t \in K} q\big(F_t^{(n)}(x)\big) \le p(x), \quad \forall x \in X.$$

The expression on the left, which depends on K, n, and q, is one of the defining seminorms of $C^\infty(J, Y)$ applied to $\widetilde{F}(x)$. The topology of $C^\infty(J, Y)$ is generated by all such seminorms as K, n, and q vary. This shows that $\widetilde{F}$ is continuous. □

Corollary A.3. *If X is a Fréchet space, then a set of continuous associative multiplications $\{m_t : X \widehat{\otimes} X \to X\}_{t \in J}$ is a smooth one-parameter deformation if and only if the map*

$$t \mapsto m_t(x_1, x_2)$$

is smooth for each fixed $x_1, x_2 \in X$.

Proof. The forward direction is trivial. Conversely, if $t \mapsto m_t(x_1, x_2)$ is smooth for each fixed $x_1, x_2 \in X$, then the map

$$m : X \times X \to C^\infty(J, X), \quad m(x_1, x_2)(t) = m_t(x_1, x_2)$$

is separately continuous by Proposition A.2. Since X is Fréchet, it follows that m is jointly continuous and so induces a continuous linear map

$$m : X \widehat{\otimes} X \to C^\infty(J, X).$$ □

Proposition A.4. *Consider a collection $\{F_t : X \to Y\}_{t \in J}$ of continuous linear maps and the corresponding map $F : J \to \mathrm{Hom}(X, Y)$.*

(i) *If $\{F_t\}_{t \in J}$ is a smooth homotopy, then $F : J \to \mathrm{Hom}(X, Y)$ is a smooth curve.*

(ii) *If the curve $F : J \to \mathrm{Hom}(X, Y)$ is smooth, then $F : J \to \mathrm{Hom}_\sigma(X, Y)$ is smooth (with respect to the topology of pointwise convergence).*

(iii) *If X is barreled and the curve $F : J \to \mathrm{Hom}_\sigma(X, Y)$ is smooth, then $\{F_t\}_{t \in J}$ is a smooth homotopy.*

Proof. The second statement is trivial and the third is a restatement of Proposition A.2. For the first statement, we shall prove F is differentiable, and the proof that F is n times differentiable follows by replacing F with $F^{(n-1)}$. Fix $t \in J$, $\epsilon > 0$, and a continuous seminorm q on Y. By continuity of $\widetilde{F}$, there is a seminorm p on X such that

$$\sup_{u\in[t-\epsilon,t+\epsilon]} q\big(F_u''(x)\big) \leq p(x), \quad \forall x \in X.$$

From Taylor's formula

$$F_{t+h}(x) = F_t(x) + F_t'(x)h + \int_t^{t+h} F_u''(x)(t+h-u)du,$$

we see

$$q\left(\frac{F_{t+h}(x) - F_t(x)}{h} - F_t'(x)\right) \leq \sup_{u\in[t,t+h]} q\big(F_u''(x)\big)\frac{h}{2} \leq p(x)\frac{h}{2}$$

for $|h| < \epsilon$. Given a bounded subset $A \subset X$, consider the seminorm on $\mathrm{Hom}(X, Y)$

$$q_A(G) = \sup_{x\in A} q(G(x)), \quad \forall G \in \mathrm{Hom}(X, Y).$$

Then we have shown

$$q_A\left(\frac{F_{t+h} - F_t}{h} - F_t'\right) \leq C_A\frac{h}{2},$$

where $C_A = \sup_{x\in A} p(x) < \infty$. Since the right side goes to 0 as $h \to 0$, this proves $\frac{d}{dt}F_t = F_t'$ in the topology of $\mathrm{Hom}(X, Y)$. □

Corollary A.5. *Let X be a Banach space and let $\{F_t : X \to X\}_{t\in J}$ be a smooth homotopy of continuous linear maps such that each F_t is bijective. Then $\{F_t^{-1}\}_{t\in J}$ is a smooth homotopy as well. Consequently, the map $F : C^\infty(J, X) \to C^\infty(J, X)$ induced by $\{F_t\}_{t\in J}$ is a topological isomorphism of $C^\infty(J)$-modules.*

Proof. That each F_t^{-1} is continuous follows from the open mapping theorem. It is well known that the inversion map on the set of invertibles of the Banach algebra $\mathrm{Hom}(X, X)$ is differentiable. If we view $\{F_t\}_{t\in J}$ as a differentiable path in $\mathrm{Hom}(X, X)$, then it follows from the chain rule that the path corresponding to $\{F_t^{-1}\}_{t\in J}$ is differentiable. From Proposition A.4, $\{F_t^{-1}\}_{t\in J}$ is a smooth homotopy. The induced endomorphism of $C^\infty(J, X)$ is clearly inverse to F. □

There are continuous inclusions

$$\mathrm{Hom}_{C^\infty(J)}\big(C^\infty(J, X), C^\infty(J, Y)\big) \to C^\infty\big(J, \mathrm{Hom}(X, Y)\big) \to C^\infty\big(J, \mathrm{Hom}_\sigma(X, Y)\big),$$

which are linear, but not necessarily topological, isomorphisms when X is barreled.

Proposition A.6. *The canonical map*

$$\mathrm{Hom}_{C^\infty(J)}\big(C^\infty(J,X), C^\infty(J,Y)\big) \to C^\infty\big(J, \mathrm{Hom}(X,Y)\big)$$

is a topological isomorphism if either

(i) *X and Y are Banach spaces, or*

(ii) *X is a nuclear Fréchet space and $Y \in$ LCTVS.*

Proof. If X and Y are Banach spaces, then we claim that the domain and codomain are both Fréchet spaces. Then the result follows from the open mapping theorem. Since $\mathrm{Hom}(X,Y)$ is a Banach space, $C^\infty(J, \mathrm{Hom}(X,Y))$ is a Fréchet space. Proposition A.1 gives a topological isomorphism

$$\mathrm{Hom}_{C^\infty(J)}\big(C^\infty(J,X), C^\infty(J,Y)\big) \cong \mathrm{Hom}\big(X, C^\infty(J,Y)\big),$$

and the topology of the latter is generated by a countable family of seminorms.

If X is a nuclear Fréchet space, then there is a topological isomorphism

$$X^* \widehat{\otimes} Z \cong \mathrm{Hom}(X,Z)$$

for any $Z \in$ LCTVS, see [34, Proposition 50.5]. Using this and Proposition A.1, we have topological isomorphisms

$$\begin{aligned}\mathrm{Hom}_{C^\infty(J)}\big(C^\infty(J,X), C^\infty(J,Y)\big) &\cong \mathrm{Hom}\big(X, C^\infty(J,Y)\big)\\ &\cong X^* \widehat{\otimes} C^\infty(J) \widehat{\otimes} Y\\ &\cong C^\infty(J) \widehat{\otimes}\, \mathrm{Hom}(X,Y)\\ &\cong C^\infty\big(J, \mathrm{Hom}(X,Y)\big).\end{aligned}$$

□

Corollary A.7. *If X is either a Banach space or a nuclear Fréchet space, then*

$$C^\infty(J,X)^\bigstar \cong C^\infty(J, X^*).$$

A.3. Integrability and parallel translation.

Proposition A.8. *A connection ∇ on $M = C^\infty(J,X)$ is integrable if and only if the following two conditions hold:*

(i) *For every $s \in J$ and $x \in M_s$, there is a unique $m \in M$ such that*

$$\nabla m = 0, \quad m(s) = x.$$

(ii) *The linear map $P^\nabla : X \to C^\infty(J \times J, X)$ given by*

$$P^\nabla(x)(s,t) = P^\nabla_{s,t}(x)$$

is well-defined and continuous.

Proof. Notice that the connection $\frac{d}{dt}$ on $C^\infty(J, X)$ satisfies both conditions. Moreover, both conditions are preserved by parallel isomorphism. So an integrable connection ∇ satisfies (i) and (ii).

Conversely, suppose ∇ satisfies (i) and (ii) and fix a value $s \in J$. By condition (ii), the linear maps

$$\widetilde{F} : X \to C^\infty(J, M_s), \quad \widetilde{F}(x)(t) = P^\nabla_{t,s}(x)$$
$$\widetilde{G} : M_s \to M, \quad \widetilde{G}(x)(t) = P^\nabla_{s,t}(x)$$

are continuous, and induce mutually inverse $C^\infty(J)$-linear isomorphisms

$$F : M \to C^\infty(J, M_s), \quad G : C^\infty(J, M_s) \to M$$

by the universal property of free modules. We'll show that

$$G : \left(C^\infty(J, M_s), \frac{d}{dt}\right) \to (M, \nabla)$$

is parallel. By $C^\infty(J)$-linearity, the Leibniz rule, and continuity, it suffices to check

$$G \circ \frac{d}{dt} = \nabla \circ G$$

for elements of the form $1 \otimes x \in C^\infty(J)\widehat{\otimes} M_s$. But this follows immediately by definition of parallel translation. That $F = G^{-1}$ is parallel follows automatically. □

The second condition in the theorem can be weakened if X is barreled.

Proposition A.9. *If $X \in$ LCTVS is barreled, then a connection ∇ on $M = C^\infty(J, X)$ is integrable if and only if the following two conditions hold:*

(i) *For every $s \in J$ and $x \in M_s$, there is a unique $m \in M$ such that*

$$\nabla m = 0, \quad m(s) = x.$$

(ii) *Each $P^\nabla_{s,t} : M_s \to M_t$ is continuous, and for each fixed $x \in X$, the map $(s, t) \mapsto P^\nabla_{s,t}(x)$ is smooth (i.e. all mixed partial derivatives exist).*

Proof. Mimic the proof of Proposition A.2 to show that

$$P^\nabla : X \to C^\infty(J \times J, X), \quad P^\nabla(x)(s, t) = P^\nabla_{s,t}(x)$$

is continuous. □

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A. Yashinski, Department of Mathematics, University of Maryland,
College Park, MD 20742-4015, USA
E-mail: ayashins@math.umd.edu