J. Noncommut. Geom. 11 (2017), 757–789
DOI 10.4171/JNCG/11-2-10
Journal of Noncommutative Geometry

About the convolution of distributions on groupoids

Jean-Marie Lescure,* Dominique Manchon and Stéphane Vassout*

Abstract. We review the properties of transversality of distributions with respect to submersions. This allows us to construct a convolution product for a large class of distributions on Lie groupoids. We get a unital involutive algebra $\mathcal{E}'_{r,s}(G,\Omega^{1/2})$ enlarging the convolution algebra $C_c^\infty(G,\Omega^{1/2})$ associated with any Lie groupoid G. We prove that G-operators are convolution operators by transversal distributions. We also investigate the microlocal aspects of the convolution product. We give sufficient conditions on wave front sets to compute the convolution product and we show that the wave front set of the convolution product of two distributions is essentially the product of their wave front sets in the symplectic groupoid T^*G of Coste–Dazord–Weinstein. This also leads to a subalgebra $\mathcal{E}'_a(G,\Omega^{1/2})$ of $\mathcal{E}'_{r,s}(G,\Omega^{1/2})$ which contains for instance the algebra of pseudodifferential G-operators and a class of Fourier integral G-operators which will be the central theme of a forthcoming paper.

Mathematics Subject Classification (2010). 46F10, 22A22, 58H05.

Keywords. Convolution of distributions, Lie groupoids, wave front set.

1. Introduction

The motivation of this paper is twofold. Firstly, we wish to study the convolution of distributions on a Lie groupoid and its relationship with the action of the so-called G-operators. Secondly, we would like to set up a neat framework in order to investigate in a future work the notions of Lagrangian distributions and Fourier integral operators on a groupoid.

The notion of C^∞ longitudinal family of pseudodifferential operators in the framework of groupoids appeared in the fundamental work of Alain Connes [2] in the case of the holonomy groupoid of a foliation and was then extended by several authors [19,20,27] to arbitrary Lie groupoids. Also, in the works of Monthubert [17], these families are considered from the point of view of distributions on the whole groupoid, so that the action of the corresponding pseudodifferential operators on C^∞ functions is given by a convolution product. Here, we carry on this idea by exploring the correspondence between C^∞ longitudinal families of distributions and single

*The first and third authors are supported by ANR Grant ANR-14-CE25-0012-01 SINGSTAR.

distributions on the whole underlying manifold of the groupoid and by studying the convolution product of distributions on groupoids. This is achieved at two levels.

The first level is based on the notion of transversality of distributions with respect to a submersion $\pi : M \to B$ [15]. It appears that the space $\mathcal{D}'_\pi(M)$ of such distributions is isomorphic to the space of C^∞ family of distributions in the fibers of π. Also, in the spirit of the Schwartz kernel Theorem suitably stated on the total space of a submersion, the space $\mathcal{D}'_\pi(M)$ coincides with the space of continuous $C^\infty(B)$-linear maps between $C^\infty_c(M)$ and $C^\infty_c(B)$. Furthermore, operations such as push-forwards and fibered-products of distributions behave well on transversal distributions and these operations allow to define the convolution product of distributions on groupoids, as soon as these distributions satisfy some transversality assumptions with respect to source or target maps. Distributions on a groupoid which are transversal both to the source and target maps are called bi-transversal and they give rise to an involutive unital algebra $\mathcal{E}'_{r,s}(G, \Omega^{1/2})$ for the convolution product. Then, one has the necessary tools to prove that G-operators on a groupoid are in 1 to 1 correspondence with transversal distributions acting by convolution and that bi-transversal distributions are in 1 to 1 correspondence with adjointable G-operators.

The second level is a microlocal refinement of the first one and consists in using the wave front set of distributions. A basic observation, due to Coste, Dazord and Weinstein [4], is that the cotangent manifold T^*G of any Lie groupoid G carries a non trivial structure of symplectic groupoid over the dual of the Lie algebroid A^*G, this structure being intimately related to the multiplication of G and then to the convolution on $C^\infty_c(G, \Omega^{1/2})$. This groupoid combined with the classical calculus of wave front sets developed by Hörmander brings in natural conditions on wave front sets of distributions on a groupoid allowing to define their convolution product and to compute the corresponding wave front set using the law of T^*G. The main consequence of this approach is that the space of compactly supported *admissible* distributions:

$$\mathcal{E}'_a(G, \Omega^{1/2}) = \big\{u \in \mathcal{E}'(G, \Omega^{1/2})\ ;\ \mathrm{WF}(u) \cap \ker s_\Gamma = \mathrm{WF}(u) \cap \ker r_\Gamma = \emptyset\big\},$$

where s_Γ, r_Γ denotes the source and target maps of $T^*G \rightrightarrows A^*G$, is a unital involutive sub-algebra of $(\mathcal{E}'_{r,s}(G, \Omega^{1/2}), *)$ and that

$$\mathrm{WF}(u * v) \subset \mathrm{WF}(u) * \mathrm{WF}(v), \quad \forall u, v \in \mathcal{E}'_a(G, \Omega^{1/2}),$$

where $*$ is the multiplication in the Coste–Dazord–Weinstein groupoid T^*G. We would like to add that the corresponding formula of Hormander for the wave front set of composition of kernels [12,13] makes the above formula quite predictable. Indeed, given a manifold X, the composition of kernels corresponds to convolution in the pair groupoid $X \times X$ and the composition law that Hörmander defines on $T^*(X \times X)$ to compute wave front sets of composition of kernels is precisely the multiplication map of the Coste–Dazord–Weinstein symplectic groupoid $T^*(X \times X)$.

The distributions belonging to $\mathcal{E}'_a(G,\Omega^{1/2})$ are said to have a bi-transversal wave front set. Actually, this second approach of the convolution product of distributions, based on the groupoid T^*G and Hörmander's techniques, works under assumptions on the wave front sets of distributions weaker than bi-transversality, and we shall briefly develop this point too. However, the algebra $\mathcal{E}'_a(G,\Omega^{1/2})$ is already large enough for the applications that we have in mind. For instance, pseudodifferential G-operators are admissible:

$$\Psi_c(G) \subset \mathcal{E}'_a(G,\Omega^{1/2}).$$

More importantly, if $\Lambda \subset T^*G\setminus 0$ is a homogeneous Lagrangian submanifold of T^*G which is also bi-transversal as a subset of T^*G, then Lagrangian distributions [14] subordinated to Λ are admissible:

$$I^*(G,\Lambda,\Omega^{1/2}) \subset \mathcal{E}'_a(G,\Omega^{1/2})$$

and in particular they give rise to G-operators. This will be the starting point of a second paper.

The present paper is organized as follows. In Section 2, we revisit the Schwartz kernel Theorem in the framework of submersions. Then the notion of distributions transversal with respect to a submersion is recalled, we give some examples and we study natural operations available on them. In Section 3, we apply the results of Section 2 to the case of groupoids. We then define the convolution product of transversal distributions and obtain the unital algebra $\mathcal{E}'_{r,s}(G,\Omega^{1/2})$ of bi-transversal distributions. In Section 4, we link the notion of G-operators with the one of transversal distributions and we obtain a 1 to 1 correspondence between the space of adjointable compactly supported G-operators and $\mathcal{E}'_{r,s}(G,\Omega^{1/2})$. In Section 5, we use both the Hörmander's results about wave front sets of distributions and the symplectic groupoid structure on T^*G to identify an important subalgebra of $\mathcal{E}'_{r,s}(G,\Omega^{1/2})$, namely $\mathcal{E}'_a(G,\Omega^{1/2})$ the subspace of distributions with bi-transversal wave front sets, onto which wave front sets behave particulary well with respect to the convolution product.

Finally, we recall in Section A the definition of the Coste–Dazord–Weinstein groupoid [4] and add some explanations and comments.

The authors would like to mention that the subject of convolution of transversal distributions is also studied in an independent work by E. Van Erp and R. Yuncken [10].

Acknowledgements. We are happy to thank Claire Debord, Georges Skandalis and Robert Yuncken for many enlightening discussions. Also, the present version of our article has greatly benefited from the remarks addressed by the referees and we would like to warmly thank them.

2. Distributions, submersions, transversality

2.1. Schwartz kernel theorem for submersions. To handle distributions on groupoids, it is useful to study distributions in the total space of a submersion. The notion of transversality we shall recall is borrowed from [15] and it extends the condition of semi-regularity given in [24, p. 532].

For any manifold M and real number α, the bundle of α-densities is denoted by Ω^α_M. The space $\mathscr{D}'(M,\Omega^\alpha_M)$ (resp. $\mathscr{E}'(M,\Omega^\alpha_M)$) is the topological dual of the space $C^\infty_c(M,\Omega^{1-\alpha}_M)$ (resp. $C^\infty(M,\Omega^{1-\alpha}_M)$). With the convention chosen, we have canonical topological embeddings

$$C^\infty(M,\Omega^\alpha) \hookrightarrow \mathscr{D}'(M,\Omega^\alpha)$$

and we abbreviate $\mathscr{D}'(M) = \mathscr{D}'(M,\Omega^0_M)$, $\Omega_M = \Omega^1_M$.

Distributions spaces are endowed with the strong topology. The space of continuous linear maps between two locally convex vector spaces E, F is denoted by $\mathscr{L}(E,F)$ and endowed with the topology of uniform convergence on bounded subsets. If E, F are modules over an algebra A, the subspace of continuous A-linear maps between E and F is denoted by $\mathscr{L}_A(E,F)$ and considered as a topological subspace of $\mathscr{L}(E,F)$.

We are going to reformulate the Schwartz kernel Theorem for distributions in the total space of a submersion $\pi : M \longrightarrow B$ between C^∞-manifolds. To do this, we begin with the product case $\pi = \mathrm{pr}_1 : X\times Y \longrightarrow X$ where $X \subset \mathbb{R}^{n_X}$ and $Y \subset \mathbb{R}^{n_Y}$ denote open subsets.

The Schwartz kernel theorem then asserts that the map

$$\mathscr{D}'(X\times Y) \ni u \longmapsto \left(f \longmapsto u_f(x) = \int_Y u(x,y)f(y)dy \right) \in \mathscr{L}(C^\infty_c(Y), \mathscr{D}'(X)) \tag{2.1}$$

where the integral is understood in the distribution sense, is a topological isomorphism.

We shall now give another form to the previous isomorphism, directly in the general case of a submersion. In the case of a product this will turn to be:

$$\mathscr{L}(C^\infty_c(Y),\mathscr{D}'(X)) \simeq \mathscr{D}'(X\times Y) \simeq \mathscr{L}_{C^\infty(X)}(C^\infty_c(X\times Y),\mathscr{E}'(X)) \tag{2.2}$$

In the general situation of a submersion $\pi : M \longrightarrow B$, to any $f \in C^\infty_c(M,\Omega_M)$, one can associate a distribution $\pi_*(uf)$ on B defined for any $g \in C^\infty_c(B,\Omega_B)$ by

$$\langle \pi_*(uf), g\rangle = \langle uf, g\circ\pi\rangle = \langle u, f.g\circ\pi\rangle. \tag{2.3}$$

One can view naturally $C^\infty_c(M,\Omega_M)$ as a $C^\infty(B)$-module by using π: for $f \in C^\infty_c(M,\Omega_M)$ and $g \in C^\infty(B,\Omega_B)$, one defines $f.g$ on M by $(f.g)(m) = f(m)g(\pi(m))$.

We have:

Theorem 2.1 (Schwartz kernel theorem for submersions). *The map*

$$\begin{aligned}\pi_* : \mathcal{D}'(M) &\longrightarrow \mathcal{L}_{C^\infty(B)}(C_c^\infty(M,\Omega_M),\mathcal{E}'(B,\Omega_B))\\ u &\longmapsto \pi_*(u\cdot)\end{aligned}$$

is a topological isomorphism (where $\mathcal{L}_{C^\infty(B)}(C_c^\infty(M,\Omega_M),\mathcal{E}'(B,\Omega_B))$ is considered as a subspace of $\mathcal{L}(C_c^\infty(M,\Omega_M),\mathcal{D}'(B,\Omega_B))$).

Proof. In fact, by $C^\infty(B)$-linearity, one has the following equality of spaces

$$\mathcal{L}_{C^\infty(B)}(C_c^\infty(M,\Omega_M),\mathcal{E}'(B,\Omega_B)) = \mathcal{L}_{C^\infty(B)}(C_c^\infty(M,\Omega_M),\mathcal{D}'(B,\Omega_B)).$$

Indeed, let $f \in C_c^\infty(M,\Omega_M)$, take $\psi \in C_c^\infty(B)$ such that ψ is identically one on the compact $\pi(\mathrm{supp}(f))$.Then one has, for any $U \in \mathcal{L}_{C^\infty(B)}(C_c^\infty(M,\Omega_M),\mathcal{D}'(B,\Omega_B))$ that $U(f) = (U(f.\psi) = \psi U(f)$, which shows that the support of the distribution $U(f)$ is compact.

Now let C be a bounded subset of $C_c^\infty(M,\Omega_M)$ and D be a bounded subset of $C_c^\infty(B)$. Then $C.D = \{f.g \;;\; f \in C, g \in D\}$ is still a bounded subset of $C_c^\infty(M,\Omega_M)$. The continuity of π_* follows. Conversely, we define $I : \mathcal{L}_{C^\infty(B)}(C_c^\infty(M,\Omega_M),\mathcal{D}'(B,\Omega_B)) \to \mathcal{D}'(M)$ by

$$\langle I(T), f\rangle = \langle T(f),\psi\rangle \quad f \in C_c^\infty(M,\Omega_M),\ \psi \in C_c^\infty(B),\ f\psi = f. \tag{2.4}$$

As T is $C^\infty(B)$-linear, the left hand side does not depend on the choice of ψ such that $f\psi = f$, hence the definition of $I(T)$ as a linear form on $C_c^\infty(M,\Omega_M)$ is consistant. Moreover, if E is a bounded subset of $C_c^\infty(M,\Omega_M)$ then there exists a compact subset $K \subset M$ such that $f \in E$ implies $\mathrm{supp}(f) \subset K$. Fixing $\psi \in C_c^\infty(B)$ such that $\psi = 1$ on K yields that $I(T)$ is a distribution for any T and the continuity of the map I. The relations $\pi_* \circ I = \mathrm{Id}$ and $I \circ \pi_* = \mathrm{Id}$ are obvious. □

Remark 2.2. Playing with supports, we also get

$$\mathcal{E}'(M) \simeq \mathcal{L}_{C^\infty(B)}(C^\infty(M,\Omega_M),\mathcal{E}'(B,\Omega_B)).$$

2.2. Transversal distributions.

Definition 2.3 (Androulidakis–Skandalis [15]). A distribution $u \in \mathcal{D}'(M)$ is transversal to π if $\pi_*(u.f) \in C_c^\infty(B,\Omega_B)$ for any $f \in C_c^\infty(M,\Omega_M)$. We note $\mathcal{D}'_\pi(M)$ the space of π-transversal distributions. We also set $\mathcal{E}'_\pi(M) = \mathcal{D}'_\pi(M) \cap \mathcal{E}'(M)$

Observe that if u is π-transversal, it follows from the closed graph theorem for LF-spaces [21, Cor. 1.2.20, p. 22] that $\pi_*(u\cdot) \in \mathcal{L}(C_c^\infty(M,\Omega_M), C_c^\infty(B,\Omega_B))$.

This gives:

Proposition 2.4. *Denoting by π_* the isomorphism in Theorem 2.1, we have*

$$\pi_*(\mathcal{D}'_\pi(M)) = \mathcal{L}_{C^\infty(B)}(C^\infty_c(M,\Omega_M), C^\infty_c(B,\Omega_B)). \tag{2.5}$$

Remark 2.5. Similarly,

$$\pi_*(\mathcal{E}'_\pi(M)) = \mathcal{L}_{C^\infty(B)}(C^\infty(M,\Omega_M), C^\infty_c(B,\Omega_B)), \tag{2.6}$$

In both cases, the inverse of the map π_* is given by

$$\langle \pi_*^{-1}(T), f\rangle = \int_B T(f)\,, \quad f \in C^\infty_c(M,\Omega_M). \tag{2.7}$$

When $\pi : X \times Y \to X,\ (x,y) \to x$, the π-transversal distributions are exactly the distributions semi-regular with respect to x, in the former terminology of [24, p. 532].

Actually, transversal distributions are nothing else but C^∞ families of distributions in the fibers of π. In the product case $\pi : X \times Y \to X, (x,y) \mapsto x$, we are talking about the space $C^\infty(X,\mathcal{D}'(Y))$ of C^∞ functions on X taking values in the topological vector space $\mathcal{D}'(Y)$, which is isomorphic [24, p. 532] to $\mathcal{L}(C^\infty_c(Y), C^\infty(X))$. Since $\mathcal{D}'(Y)$ is a Montel space, the classical argument using Banach–Steinhaus theorem shows the useful equivalence

$$u_n \longrightarrow u \text{ in } C^\infty(X,\mathcal{D}'(Y)) \;\Leftrightarrow\; \forall f \in C^\infty_c(Y),\ \langle u_n, f\rangle \longrightarrow \langle u, f\rangle \text{ in } C^\infty(X). \tag{2.8}$$

This space is generalized as follows for general submersions.

Definition 2.6. A family $u = (u_x)_{x\in B}$ of distributions in the fibers of π is C^∞ if for any local trivialization of π

$$U \subset M,\ X \subset B,\ \kappa : U \xrightarrow{\simeq} X \times Y,\ \pi|_U = \pi_X \circ \kappa,$$

we have $\kappa_*(u|_U) \in C^\infty(X,\mathcal{D}'(Y))$. The space of C^∞ families is noted $C^\infty_\pi(B,\mathcal{D}'(M))$.

The space $C^\infty_{\pi,\mathrm{cpct}}(B,\mathcal{E}'(M))$ is defined accordingly, where the subscript cpct means that there exists a fixed compact K of M such that the support of every $u_b \in \mathcal{E}'(\pi^{-1}(b))$ is contained in K.

Using a covering of M by local trivializations and a partition of unity, we use the topology of $C^\infty(X,\mathcal{D}'(Y))$ to build on $C^\infty_\pi(B,\mathcal{D}'(M))$ a complete Hausdorff locally convex vector space structure.

Also, (2.8) becomes

$$u_n \longrightarrow u \text{ in } C^\infty_\pi(B,\mathcal{D}'(M)) \;\Leftrightarrow\; \forall f \in C^\infty_c(M,\Omega_M),\quad \langle u_n, f\rangle \longrightarrow \langle u, f\rangle \text{ in } C^\infty_c(B,\Omega_B). \tag{2.9}$$

Then:

Proposition 2.7. *Using on $\mathcal{D}'_\pi(M)$ the topology given by (2.5), the map*

$$C^\infty_\pi(B, \mathcal{D}'(M)) \xrightarrow{J} \mathcal{D}'_\pi(M) \qquad (2.10)$$
$$u \longmapsto \left(f \mapsto \int_B \langle u_x, f(x,\cdot)\rangle \right)$$

is a topological isomorphism.

Proof. Using a partition of unity, we can suppose that we are in the product case, that is $M = X \times Y \underset{\pi}{\to} X = B$. Using the identifications

$$C^\infty(X, \mathcal{D}'(Y)) \simeq \mathcal{L}(C^\infty_c(Y), C^\infty(X))$$

and

$$\mathcal{D}'_\pi(X \times Y) \simeq \mathcal{L}_{C^\infty(X)}(C^\infty_c(X \times Y), C^\infty(X)),$$

the map J is given by

$$J(U)(f)(x) = \langle u_x, f(x,\cdot)\rangle,$$
$$U \in \mathcal{L}(C^\infty_c(Y), C^\infty(X)),\ f \in C^\infty_c(X \times Y),\ x \in X.$$

Conversely, let us define $\mathcal{L}_{C^\infty(X)}(C^\infty_c(X \times Y), C^\infty(X)) \xrightarrow{E} \mathcal{L}(C^\infty_c(Y), C^\infty(X))$ by

$$E(T)(f)(x) = T(\widetilde{f}_x)(x) \qquad (2.11)$$

where $f \in C^\infty_c(Y)$ and $\widetilde{f}_x \in C^\infty_c(X \times Y)$ is any map such that $\widetilde{f}_x(x,y) = f(y)$. Note that if we have a map $f \in C^\infty_c(X \times Y)$ such that $f_{|x_0 \times Y} = 0$ then by Taylor formula, one can find maps $\phi_i \in C^\infty_c(X)$ with $1 \le i \le \dim(X)$ such that $\phi_i(x_0) = 0$ and $f(x,y) = \sum \phi_i(x) g_i(x,y)$ with $g_i \in C^\infty_c(X \times Y)$. This proves, using $C^\infty(X)$-linearity, that there is no ambiguity in the definition of E. Observe that for any $f \in C^\infty_c(X \times Y)$ and any $U \in \mathcal{L}(C^\infty_c(Y), C^\infty(X))$, the map $x \mapsto J(U)(f)(x)$ is smooth by [13, Theorem 2.1.3]. As J is the restriction of the isomorphism (2.2) to the subspace $\mathcal{L}_{C^\infty(X)}(C^\infty_c(X \times Y), C^\infty(X))$, we already know that $J(U) \in \mathcal{L}(C^\infty_c(Y), \mathcal{D}'(X))$ and we can use the closed graph theorem again to show that the image of J is in $\mathcal{L}_{C^\infty(X)}(C^\infty_c(X \times Y), C^\infty(X))$. The same holds for E and we have to show that for any $T \in \mathcal{L}(C^\infty_c(Y), C^\infty(X))$ and any $f \in C^\infty_c(Y)$ the map $x \mapsto E(T)(f)(x)$ is smooth. As this is local in x, it is enough to check it on any relatively compact open $\Omega \in X$. Take $\chi \in C^\infty_c(X)$ such that χ is identically 1 on Ω gives for any $x \in \Omega$, $E(T(f))(x) = T(\chi f)(x)$ which shows the result. It is easy to check that $E = J^{-1}$ and that the topologies given by uniform convergence on bounded subsets on $\mathcal{L}(C^\infty_c(Y), C^\infty(X))$ and $\mathcal{L}_{C^\infty(X)}(C^\infty_c(X \times Y), C^\infty(X))$ coincide through the bijection J. $\square$

Remark 2.8. We similarly get: $C^\infty_{\pi,\mathrm{cpct}}(B,\mathcal{E}'(M)) \simeq \mathcal{E}'_\pi(M)$. If finite dimensional real vector bundles E over M and F over B are given, we obtain canonical embeddings

$$\begin{aligned}\mathcal{D}'_\pi(M,E) &\hookrightarrow \mathcal{D}'_\pi(M, E\otimes \mathrm{End}(\pi^*F))\\ &\simeq \mathcal{L}_{C^\infty(B)}(C^\infty_c(M,\Omega_M\otimes E^*\otimes\pi^*F), C^\infty_c(B,\Omega_B\otimes F))\end{aligned} \tag{2.12}$$

and

$$\begin{aligned}C^\infty_\pi(B,\mathcal{D}'(M,E)) &\hookrightarrow C^\infty_\pi(B,\mathcal{D}'(M,E\otimes\mathrm{End}(\pi^*F)))\\ &\simeq \mathcal{D}'_\pi(M,E\otimes\mathrm{End}(\pi^*F)).\end{aligned} \tag{2.13}$$

2.3. Examples of transversal distributions. Obviously, if $\pi : M \to M$ is the identity map then $\mathcal{D}'_\pi(M) = C^\infty(M)$ and if π maps M to a point then $\mathcal{D}'_\pi(M) = \mathcal{D}'(M)$.

The wave front set [13, Chapter 8] is a powerful tool to analyse the singularities of a distribution. It can be thought of as the set of directed points in $T^*M\setminus 0$, around which the Fourier transform is not rapidly decreasing. Using wave front set is a convenient way to check the transversality of distributions with respect to a given submersion $\pi : M \to B$, and it thus gives access to more interesting examples. Indeed,

Proposition 2.9. *Let $W \subset T^*M\setminus 0$ be a closed cone and $\mathcal{D}'_W(M) = \{u \in \mathcal{D}'(M)\ ;\ \mathrm{WF}(u)\subset W\}$. If $W\cap(\ker d\pi)^\perp = \emptyset$, then*

$$\mathcal{D}'_W(M)\subset\mathcal{D}'_\pi(M).$$

Proof. We apply the formula (3.6) of [11, p. 328]:

$$\begin{aligned}\mathrm{WF}(\pi_*(u.f)) \subset (d\pi)_*(\mathrm{WF}(u.f)) &\subset (d\pi)_*(\mathrm{WF}(u))\\ &= \big\{(x,\xi)\ ;\ x=\pi(m)\ ,(m,{}^t d\pi_m(\xi))\in\mathrm{WF}(u)\big\}.\end{aligned}$$

Since $(\ker d\pi)^\perp = \{(m,\zeta)\ ;\ \zeta\in\mathrm{Im}({}^t d\pi_m)\}$, we obtain $\mathrm{WF}(\pi_*(u.f)) = \emptyset$, and thus $\pi_*(u.f)$ is smooth. □

For instance, consider a section of π, that is a submanifold $X\subset M$ such that $\pi : X\to B$ is a diffeomorphism onto an open subset of B. Let $\omega\in\Omega_X$ be any C^∞ density and define $l_\omega\in\mathcal{D}'(M,\Omega_M)$ by

$$\langle l_\omega, f\rangle = \int_X f\omega. \tag{2.14}$$

Then $l_\omega\in\mathcal{D}'_\pi(M,\Omega_M)$, for $\mathrm{WF}(l_\omega)\subset N^*(X)$ (see [13, Example 8.2.5]) and $N^*(X)\cap(\ker d\pi)^\perp = X\times\{0\}$. Alternatively, it is easy to check that $\pi_*(l_\omega.f)$ is given by the C^∞ density $\pi_*(\omega f|_X)$. Of course, for any differential operator P on M, we still have $Pl_\omega\in\mathcal{D}'_\pi(M,\Omega_M)$, for $\mathrm{WF}(Pu)\subset\mathrm{WF}(u)$ for any distribution u.

Actually, this gives all instances of transversal distributions supported within a section. Indeed, let $u \in \mathcal{E}'_\pi(M, \Omega_M)$ such that $\operatorname{supp}(u) \subset X$. There is no restriction to work in a local trivialization, that is to assume $\pi : M = X \times \mathbb{R}^n \to X$, $(x, y) \mapsto x$ and identify $X \simeq X \times \{0\}$. By [13, Theorem 2.3.5], we have

$$\langle u, \phi \rangle = \sum_{|\alpha| \leq k} \langle u_\alpha, (\partial_y^\alpha \phi)(\cdot, 0) \rangle, \quad \forall \phi \in C_c^\infty(X \times \mathbb{R}^n) \tag{2.15}$$

where k is the order of u and $u_\alpha \in \mathcal{D}'(X)$ has order $k - |\alpha|$. It follows that

$$C^\infty(X) \ni \pi_*(fu) = \sum_{|\alpha| \leq k} (\partial_y^\alpha f)(\cdot, 0).u_\alpha, \quad \forall f \in C^\infty(X \times \mathbb{R}^n). \tag{2.16}$$

Selecting $f = y^\alpha$ shows that u_α is C^∞. We have proved:

Proposition 2.10. *Let $u \in \mathcal{E}'(M, \Omega_M)$ such that $\operatorname{supp}(u) \subset X$, X being a section of π. Then $u \in \mathcal{E}'_\pi(M, \Omega_M)$ if and only if u is a finite sum of distributions obtained by differentiation along the fibers of π of distributions of the kind* (2.14).

Remark 2.11. $u \in \mathcal{D}'_\pi(M)$ does not imply $\operatorname{WF}(u) \cap \ker d\pi^\perp = \emptyset$. Indeed, consider $\pi : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$, $(x, y) \mapsto x$ and define $u \in C^\infty(\mathbb{R}, \mathcal{D}'(\mathbb{R}))$ by

$$\langle u, \phi \rangle(x) = \sqrt{2\pi} \int \chi(\eta) |\eta| e^{-\eta^2 x^2/2} \hat{\phi}(-\eta) d\eta \tag{2.17}$$

where χ is C^∞, $\chi(\eta) = 1$ if $|\eta| \geq 1$ and $\chi(\eta) = 0$ if $|\eta| \leq 1/2$. Since $\hat{u}(\xi, \eta) = \chi(\eta) e^{-\xi^2/(2\eta^2)}$ we conclude $\operatorname{WF}(u) \cap (\ker d\pi)^\perp \neq \emptyset$ [13, Section 8.1].

It is not obvious to us how to characterize transversal distributions whose wave front set avoids $(\ker d\pi)^\perp$. Nevertheless, a sufficient condition can be worked out locally, that is in local trivializations of π, and this is the content of the next lemma.

Lemma 2.12. *Let $v \in \mathcal{D}'_{\pi_X}(X \times Y)$ and assume that there exist constants $d \in \mathbb{N}$ and $\delta \in [0, 1)$ such that for any compact subset K of Y and multi-index $\beta \in \mathbb{N}^{n_X}$, one can find a constant $C_{K\beta}$ such that*

$$|\langle \partial^\beta v_x, f \rangle| \leq C_{K\beta} \|f\|_{K, d+\delta|\beta|}, \quad \forall f \in C_c^\infty(Y),\ x \in X. \tag{2.18}$$

Here, we have set $\|f\|_{K, d+\delta|\beta|} = \sum_{|\alpha| \leq d+\delta|\beta|} \sup_K |\partial^\alpha f|$. Then we have $\operatorname{WF}(v) \subset (\ker d\pi_Y)^\perp$. In particular, $\operatorname{WF}(v) \cap (\ker d\pi_X)^\perp = \emptyset$.

Remark 2.13. Distributions in Proposition 2.10 satisfy the assumption of the lemma with $\delta = 0$.

Proof of the lemma. Let us fix $(x_0, y_0, \xi_0, \eta_0) \notin (\ker d\pi_Y)^\perp$, that is, $\xi_0 \neq 0$ and assume that $|(\xi_0, \eta_0)| = 1$. We work below in a conic neighborhood Γ of $(x_0, y_0, \xi_0, \eta_0)$ such that for all $(x, y, \xi, \eta) \in \Gamma$ with $|(\xi, \eta)| = 1$, we have $|\xi_j| \geq |\xi_{0j}|/2$ for some fixed j such that $\xi_{0j} \neq 0$.

Let $(x, y, \xi, \eta) \in \Gamma$ be such that $|(\xi, \eta)| = 1$ and $\varphi(x, y)$ be supported in a compact neighborhood $K \times L$ of (x_0, y_0) in $X \times Y$. Denoting $\varphi_x = \varphi(x, \cdot)$, we have for any $N > 0$

$$\begin{aligned} &|\langle v, \varphi e^{-it\langle(\cdot,\cdot),(\xi,\eta)\rangle}\rangle| \\ &\quad = \left| \int \langle v_x\,,\, \varphi_x e^{-it\langle(x,\cdot)\,,\,(\xi,\eta)\rangle}\rangle dx \right| = \left| \int \widehat{\varphi_x v_x}(t\eta) e^{-it\langle x\,,\,\xi\rangle} dx \right| \\ &\quad \leq C.\Big(\sum_{|\alpha|\leq N} \sup_{x\in L} |\partial_x^\alpha \widehat{\varphi_x v_x}(t\eta)||\xi|^{|\alpha|-2N} \Big) t^{-N} \quad \text{by [13, Theorem 7.7.1]}. \end{aligned} \tag{2.19}$$

Moreover, since $v : x \mapsto v_x$ is C^∞, we have

$$\partial_{x_j}^N \widehat{\varphi_x v_x}(t\eta) = \partial_{x_j}^N \langle \varphi_x v_x, e^{-it\langle\cdot\,,\,\eta\rangle}\rangle = \langle \partial_{x_j}^N \varphi_x v_x, e^{-it\langle\cdot\,,\,\eta\rangle}\rangle = \widehat{\partial_{x_j}^N \varphi_x v_x}(t\eta).$$

We note $K_\epsilon = \{y + z;\ y \in K, |z| < \epsilon\}$ for any $\epsilon > 0$ and let $\chi_\epsilon \in C_c^\infty(K_\epsilon)$ be such that $\chi_\epsilon = 1$ on $K_{\epsilon/2}$. If $H(\eta)$ denotes the supporting function of K [13, 4.3.1], we get using the assumption (2.18) and the proof of the Paley–Wiener–Schwartz Theorem in [13, 7.3.1]

$$\begin{aligned} |\widehat{\partial_{x_j}^N \varphi_x v_x}(\eta)| &= |\partial_{x_j}^N \varphi_x v_x(\chi_\epsilon e^{-i\langle\cdot\,,\,\eta\rangle})| \\ &\leq C_{K_\epsilon N} \sum_{|\beta|\leq d+\delta N} \sup |\partial^\beta(\chi_\epsilon e^{-i\langle\cdot\,,\,\eta\rangle})| \\ &\leq C_{K_\epsilon N}.C.e^{H(0)} \sum_{|\beta|\leq d+\delta N} \epsilon^{-|\beta|}(1+|\eta|)^{d+\delta N-|\beta|}. \end{aligned}$$

With $\epsilon = 1/(1+|\eta|)$ and using the inequalities $C_{K_\epsilon N} \leq C_{K_{\epsilon'} N}$ if $\epsilon < \epsilon'$, we obtain

$$|\widehat{\partial_{x_j}^N \varphi_x v_x}(\eta)| \leq C_{K_1 N}.C.(1+|\eta|)^{d+\delta N} \leq C'_{KN}(1+|\eta|)^{d+\delta N}. \tag{2.20}$$

Using uniform estimates $|\xi| \geq c_1 > 0$ and $(1+|t\eta|) \leq c_2 t$ for $(\xi, \eta) \in \Gamma$, $|(\xi, \eta)| = 1$ and the estimate (2.20) applied to (2.19), we get

$$|\langle v, \varphi e^{-it\langle(\cdot,\cdot),(\xi,\eta)\rangle}\rangle| \leq C.t^{d+(\delta-1)N}$$

since $\delta - 1 < 0$, we conclude that $(x_0, y_0, \xi_0, \eta_0) \notin \mathrm{WF}(v)$. □

2.4. Operations on transversal distributions.

Proposition 2.14. *Let $\rho : Z \to M$ and $\pi : M \to B$ be submersions. There is a separately continuous and bilinear map:*

$$\bullet : \mathcal{D}'_\pi(M) \times \mathcal{D}'_\rho(Z) \longrightarrow \mathcal{D}'_{\pi\circ\rho}(Z)$$

which extends the map $(u,v) \mapsto (u\circ\rho)v$ when u and v are C^∞ maps respectively on M and Z. In particular, the pull back of distributions on M by the submersion ρ restricts to a continuous map

$$\rho^* : \mathcal{D}'_\pi(M) \longrightarrow \mathcal{D}'_{\pi\circ\rho}(Z).$$

Proof. Let $(u,v) \in \mathcal{D}'_\pi(M)\times\mathcal{D}'_\rho(Z)$. Using the isomorphism (2.5), we will denote

$$V = \rho_*(v.) \in \mathcal{L}(C^\infty_c(Z,\Omega_Z), C^\infty_c(M,\Omega_M))$$

and
$$U = \pi_*(u.) \in \mathcal{L}(C^\infty_c(M,\Omega_M), C^\infty_c(B,\Omega_B)).$$

As $U\circ V \in \mathcal{L}_{C^\infty(B)}(C^\infty_c(Z,\Omega_Z), C^\infty_c(B,\Omega_B))$ this precisely defines a distribution in $\mathcal{D}'_{\pi\circ\rho}(Z)$, which we will denote $u\bullet v = (\pi\circ\rho)_*^{-1}(U\circ V)$. Observe that if $u \in C^\infty(M)$ and $v \in C^\infty(Z)$, then for $f \in C^\infty_c(Z,\Omega_Z)$, we have

$$U\circ V(f)(b) = \int_{\pi(m)=b} u(m)\left(\int_{\rho(z)=m} v(z)f(z)\right) = \int_{\pi\circ\rho(z)=b} (u\circ\rho)(z)v(z)f(z),$$

which shows that in this case $u\bullet v = (u\circ\rho)v$. The bilinearity and separate continuity of $(u,v)\mapsto u\bullet v$ follow from the bilinearity and separate continuity of the composition of continous linear maps. The latter precisely means that, for any locally convex spaces E, F, G, the map

$$\begin{aligned} \mathcal{L}(E,F)\times\mathcal{L}(F,G) &\longrightarrow \mathcal{L}(E,G)\\ (S,T) &\longmapsto T\circ S \end{aligned}$$

is separately continuous, where the three spaces of continuous linear maps are again provided with the topology of uniform convergence on bounded subsets.

Finally, taking $v = 1$, we get a continuous map

$$\begin{aligned} \mathcal{D}'_\pi(M) &\longrightarrow \mathcal{D}'_{\pi\circ\rho}(Z)\\ u &\longmapsto u\bullet 1 \end{aligned}$$

which clearly coincides with the pull back of the distribution u by the submersion ρ:

$$\langle u\bullet 1, \varphi\rangle = \int_M\int_{\rho(z)=m}\varphi(z).u(m) = \langle\rho^*u,\varphi\rangle, \quad \varphi\in C^\infty_c(Z,\Omega_Z). \qquad \square$$

Proposition 2.15. *Let $\pi : M \longrightarrow B$ be a submersion, $f : N \longrightarrow B$ a C^∞ map and consider their fibered product:*

$$\begin{array}{ccc} M \underset{B}{\times} N & \xrightarrow{f^*(\pi)} & N \\ \Big\downarrow{\scriptstyle \pi^*(f)} & & \Big\downarrow{\scriptstyle f} \\ M & \xrightarrow{\pi} & B. \end{array} \tag{2.21}$$

Then the pull-back $(\pi^*(f))^* : C^\infty(M) \longrightarrow C^\infty(M \underset{B}{\times} N)$ extends to a continuous map

$$(\pi^*(f))^* : \mathcal{D}'_\pi(M) \longrightarrow \mathcal{D}'_{f^*(\pi)}(M \underset{B}{\times} N). \tag{2.22}$$

In particular, if C is a submanifold of B, if we set $N = \pi^{-1}(C)$ and $f = \pi_C : N \to B$ is the restriction of π to $\pi^{-1}(C)$, we get that the restriction map gives a continuous map

$$\mathrm{Rest}_C : \mathcal{D}'_\pi(M) \longrightarrow \mathcal{D}'_{\pi_C}(\pi^{-1}(C)).$$

Proof. We identify transversal distributions with C^∞ families and we can work locally, that is we assume that $\pi : X \times Y \to X$, with X, Y open subsets in euclidean spaces. Then locally $M \underset{B}{\times} N$ is of the form $Y \times Z$, and the map $f^*(\pi)$ is given by the projection $Y \times Z \to Z$. If $u \in C^\infty(X, \mathcal{D}'(Y))$ then $(\pi^*(f))^*(u) \in C^\infty(Z, \mathcal{D}'(Y))$ is given by the family

$$Z \ni z \longmapsto u_{f(z)} \in \mathcal{D}'(Y).$$

The statement follows. □

Remark 2.16. (1) The assertion of the previous proposition holds for commutative square of submersions

$$\begin{array}{ccc} M & \xrightarrow{p_2} & M_2 \\ \Big\downarrow{\scriptstyle p_1} & & \Big\downarrow{\scriptstyle \pi_2} \\ M_1 & \xrightarrow{\pi_1} & B \end{array} \tag{2.23}$$

such that any point of M, $\ker d(\pi_1 \circ p_1) = \ker dp_1 + \ker dp_2$ or, equivalently, such that $p_1 : p_2^{-1}(m_2) \to \pi_1^{-1}(b)$, $b = \pi_2(m_2)$, is a submersion for any $m_2 \in M_2$.

(2) Finite dimensional vector bundles can be added in the statements of the two previous propositions. We have omitted them to lighten the notations.

When a finite set $\mathcal{J}$ of submersions is given on M, we introduce

$$\mathcal{D}'_{\mathcal{J}}(M) = \bigcap_{\rho \in \mathcal{J}} \mathcal{D}'_\rho(M) \subset \mathcal{D}'(M). \tag{2.24}$$

The space $\mathcal{D}'_{\mathcal{J}}(M, E)$ is given the topology generated by the union of the topologies induced by each $\mathcal{D}'_\rho(M)$, $\rho \in \mathcal{J}$. We adopt similar convention for the space $\mathcal{E}'_{\mathcal{J}}(M)$. The previous proposition is now used to define fibered product of distributions.

Proposition 2.17. *Let $\pi_i : M_i \longrightarrow B$, $i = 1, 2$ be two submersions and consider their fibered product*

$$\begin{array}{ccc} M_1 \underset{B}{\times} M_2 & \xrightarrow{\mathrm{pr}_2} & M_2 \\ \Big\downarrow{\scriptstyle \mathrm{pr}_1} & & \Big\downarrow{\scriptstyle \pi_2} \\ M_1 & \xrightarrow{\pi_1} & B. \end{array} \tag{2.25}$$

Set $\pi = \pi_i \circ \mathrm{pr}_i$ and consider extra submersions $\sigma_1 : M_1 \longrightarrow A$, $\sigma_2 : M_2 \longrightarrow C$.

$$\begin{array}{ccccc} M_1 \underset{B}{\times} M_2 & \xrightarrow{\mathrm{pr}_2} & M_2 & \xrightarrow{\sigma_2} & C \\ \Big\downarrow{\scriptstyle \mathrm{pr}_1} & \searrow^{\pi} & \Big\downarrow{\scriptstyle \pi_2} & & \\ M_1 & \xrightarrow{\pi_1} & B & & \\ \Big\downarrow{\scriptstyle \sigma_1} & & & & \\ A & & & & \end{array}$$

The fibered product of C^∞ functions $(f_1, f_2) \longmapsto f_1 \otimes f_2|_{M_1 \underset{B}{\times} M_2}$ extends uniquely to separately continuous bilinear maps

$$\begin{aligned} \mathcal{D}'_{\pi_1}(M_1) \times \mathcal{D}'_{\sigma_2}(M_2) &\longrightarrow \mathcal{D}'_{\sigma_2 \circ \mathrm{pr}_2}(M_1 \underset{B}{\times} M_2) \\ (u_1, u_2) &\longmapsto u_1 \underset{\pi_1}{\times} u_2 \end{aligned}$$

and

$$\begin{aligned} \mathcal{D}'_{\sigma_1}(M_1) \times \mathcal{D}'_{\pi_2}(M_2) &\longrightarrow \mathcal{D}'_{\sigma_1 \circ \mathrm{pr}_1}(M_1 \underset{B}{\times} M_2) \\ (u_1, u_2) &\longmapsto u_1 \underset{\pi_2}{\times} u_2. \end{aligned} \tag{2.26}$$

If $u_j \in \mathcal{D}'_{\pi_j}(M_j)$, $j = 1, 2$ then the equality

$$u_1 \underset{\pi_1}{\times} u_2 = u_1 \underset{\pi_2}{\times} u_2 \tag{2.27}$$

holds and both previous maps restrict to a separately continuous bilinear map

$$\begin{aligned} \mathcal{D}'_{\sigma_1, \pi_1}(M_1) \times \mathcal{D}'_{\pi_2, \sigma_2}(M_2) &\longrightarrow \mathcal{D}'_{\sigma_1 \circ \mathrm{pr}_1, \pi, \sigma_2 \circ \mathrm{pr}_2}(M_1 \underset{B}{\times} M_2) \\ (u_1, u_2) &\longmapsto u_1 \underset{\pi_1}{\times} u_2. \end{aligned} \tag{2.28}$$

Proof. We just need to combine Propositions 2.14 and 2.15. By Proposition 2.15, the maps ${\mathrm{pr}_1}^* : \mathcal{D}'_{\pi_1}(M_1) \to \mathcal{D}'_{\mathrm{pr}_2}(M_1 \underset{B}{\times} M_2)$ and ${\mathrm{pr}_2}^* : \mathcal{D}'_{\pi_2}(M_1) \to \mathcal{D}'_{\mathrm{pr}_1}(M_1 \underset{B}{\times} M_2)$ are continuous. Then, using Proposition 2.14, denoting

$$u_1 \underset{\pi_1}{\times} u_2 = u_2 \bullet ({\mathrm{pr}_1}^*(u_1)) \quad \text{and} \quad u_1 \underset{\pi_2}{\times} u_2 = u_1 \bullet ({\mathrm{pr}_2}^*(u_2)),$$

we get separately continuous maps:

$$\mathcal{D}'_{\pi_1}(M_1) \times \mathcal{D}'_{\sigma_2}(M_2) \longrightarrow \mathcal{D}'_{\sigma_2 \circ \mathrm{pr}_2}(M_1 \underset{B}{\times} M_2)$$
$$(u_1, u_2) \longmapsto u_1 \underset{\pi_1}{\times} u_2$$

and

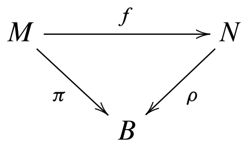
$$\mathcal{D}'_{\sigma_1}(M_1) \times \mathcal{D}'_{\pi_2}(M_2) \longrightarrow \mathcal{D}'_{\sigma_1 \circ \mathrm{pr}_1}(M_1 \underset{B}{\times} M_2)$$
$$(u_1, u_2) \longmapsto u_1 \underset{\pi_2}{\times} u_2. \tag{2.29}$$

Now, observe that in the case where u_1 and u_2 are smooth, then we have

$$\begin{aligned} u_1 \underset{\pi_1}{\times} u_2 &= u_2 \bullet (\mathrm{pr}_1{}^*(u_1)) \\ &= (u_2 \circ \mathrm{pr}_2) \times (u_1 \circ \mathrm{pr}_1) \\ &= u_1 \bullet (\mathrm{pr}_2{}^*(u_2)) = u_1 \underset{\pi_2}{\times} u_2. \end{aligned}$$

Hence the equality extends by continuity when $u_j \in \mathcal{D}'_{\pi_j}(M_j)$, $j = 1, 2$ and this also allows to take into account the extra transversality assumptions (2.28) in order to conclude, by the previous method, that $u_1 \underset{\pi_2}{\times} u_2$ is transversal with respect to $\sigma_1 \circ \mathrm{pr}_1, \pi$ and $\sigma_2 \circ \mathrm{pr}_2$ and depends continuously on u_1 and u_2. □

Consider a commutative diagram



(2.30)

where f is a C^∞ map and π, ρ are submersions. If $u \in \mathcal{E}'(M, \Omega_M)$, the push-forward of u by f is given by $\langle f_* u, g \rangle = \langle u, g \circ f \rangle$ and if moreover u is transversal with respect to π, then $f_* u$ is given by the C^∞ family $((f|_{M_b})_* u_b)$, $b \in B$. We obtain a map

$$f_* : \mathcal{E}'_\pi(M, \Omega_M) \longrightarrow \mathcal{E}'_\rho(N, \Omega_N). \tag{2.31}$$

Since f is not necessarily proper, we can not extend f_* to $\mathcal{D}'_\pi$, nevertheless:

Proposition 2.18. *Let $\varphi \in C^\infty(M)$ such that $f : \mathrm{supp}(\varphi) \longrightarrow N$ is proper. Then the map*

$$\mathcal{D}'_\pi(M, \Omega_M) \longrightarrow \mathcal{D}'_\rho(N, \Omega_N) \tag{2.32}$$
$$u \longmapsto f_*(\varphi u) \tag{2.33}$$

is well defined and continuous.

Proof. Under the assumption on the support of φ, we easily get that $g \longmapsto \varphi.g \circ f$ maps continuously $C_c^\infty(N)$ into $C_c^\infty(M)$. The result follows. □

3. Convolution of transversal distributions on groupoids

We apply these observations in the context of Lie groupoids. A Lie groupoid is a manifold G endowed with the additional following structures:

- two surjective submersions $r, s : G \rightrightarrows G^{(0)}$ onto a manifold $G^{(0)}$ called the space of units.
- An embedding $u : G^{(0)} \longrightarrow G$, which allows to consider $G^{(0)}$ as a submanifold of G and then such that
$$r(x) = x,\ s(x) = x, \quad \text{for all } x \in G^{(0)}. \tag{3.1}$$
- A C^∞ map
$$i : G \longrightarrow G, \quad \gamma \longmapsto \gamma^{-1} \tag{3.2}$$
called inversion and satisfying $s(\gamma^{-1}) = r(\gamma)$ and $r(\gamma^{-1}) = s(\gamma)$ for any γ.
- a C^∞ map
$$m : G^{(2)} = \big\{(\gamma_1, \gamma_2) \in G^2\ ;\ s(\gamma_1) = r(\gamma_2)\big\} \longrightarrow G, \quad (\gamma_1, \gamma_2) \longmapsto \gamma_1\gamma_2 \tag{3.3}$$
called the multiplication, satisfying the relations, whenever they make sense
$$\gamma\gamma^{-1} = r(\gamma) \quad \gamma^{-1}\gamma = s(\gamma) \quad r(\gamma_1\gamma_2) = r(\gamma_1), \quad s(\gamma_1\gamma_2) = s(\gamma_2) \tag{3.4}$$
$$(\gamma_1\gamma_2)\gamma_3 = \gamma_1(\gamma_2\gamma_3) \quad r(\gamma)\gamma = \gamma \quad \gamma s(\gamma) = \gamma. \tag{3.5}$$

It follows from these axioms that i is a diffeomorphism equal to its inverse, m is a surjective submersion and γ^{-1} is the unique inverse of γ, for any γ, that is the only element of G satisfying $\gamma\gamma^{-1} = r(\gamma)$, $\gamma^{-1}\gamma = s(\gamma)$. These assertions need a proof, and the unfamiliar reader is invited to consult for instance [16] and references therein.

It is customary to write

$$G_x = s^{-1}(x), \quad G^x = r^{-1}(x), \quad G_x^y = G_x \cap G^y,$$
$$m_x = m|_{G^x \times G_x} : G^x \times G_x \longrightarrow G.$$

G_x, G^x are submanifolds and G_x^x is a Lie group. The submersion $d : (\gamma_1, \gamma_2) \mapsto \gamma_1\gamma_2^{-1}$ defined on $G \underset{s}{\times} G$ is called the division of G.

Obviously, Lie groups, C^∞ vector bundles, principal bundles, are Lie groupoids. Also, for any manifold X, the manifold $X \times X$ inherits a canonical structure of Lie groupoid with unit space X and multiplication given by $(x, y).(y, z) = (x, z)$. The reader can find in [3,5,7–9,18–20,22,25,26,28] more concrete examples.

The Lie algebroid AG of a Lie groupoid G is the fiber bundle $TG|_{G^{(0)}}/TG^{(0)}$ over $G^{(0)}$. It can be identified with $\operatorname{Ker} ds|_{G^{(0)}}$ or $\operatorname{Ker} dr|_{G^{(0)}}$. Its dual A^*G is the conormal bundle of $G^{(0)}$.

We recall the construction of the canonical convolution algebra $C_c^\infty(G, \Omega^{1/2})$ [3,6] associated with any Lie groupoid G. The product of convolution

$$C_c^\infty(G, \Omega^{1/2}) \times C_c^\infty(G, \Omega^{1/2}) \xrightarrow{*} C_c^\infty(G, \Omega^{1/2}) \tag{3.6}$$

is given by the integral

$$f * g(\gamma) = \int_{\gamma_1\gamma_2=\gamma} f(\gamma_1)g(\gamma_2), \quad \gamma \in G \tag{3.7}$$

which is well defined and gives an internal operation as soon as we take

$$\Omega^{1/2} = \Omega^{1/2}(\ker dr) \otimes \Omega^{1/2}(\ker ds) = \Omega^{1/2}(\ker dr \oplus \ker ds). \tag{3.8}$$

To understand this point, we recall:

Lemma 3.1 ([3,6]). *Denoting by m the multiplication map of G and by* $\mathrm{pr}_1, \mathrm{pr}_2 : G^{(2)} \to G$ *the natural projection maps, we have a canonical isomorphism*

$$\mathrm{pr}_1^*(\Omega^{1/2}) \otimes \mathrm{pr}_2^*(\Omega^{1/2}) \simeq \Omega(\ker dm) \otimes m^*(\Omega^{1/2}). \tag{3.9}$$

Proof. Let

$$\begin{array}{ccc} M & \xrightarrow{p_2} & M_2 \\ \downarrow{\scriptstyle p_1} & & \downarrow{\scriptstyle \pi_2} \\ M_1 & \xrightarrow{\pi_1} & B \end{array}$$

be a fibered product where π_2 is a submersion. Then dp_2 induces an isomorphism between $\ker(dp_1)$ and $p_2^*(\ker(d\pi_2))$. The three following diagrams being fibered products (we use the natural diffeomorphism $G^{(2)} \to G \underset{s}{\times} G$ given by $(\gamma_1, \gamma_2) \mapsto (\gamma_1\gamma_2, \gamma_2)$ for the first two)

$$\begin{array}{ccc} G^{(2)} & \xrightarrow{\mathrm{pr}_2} & G \\ \downarrow{\scriptstyle m} & & \downarrow{\scriptstyle s} \\ G & \xrightarrow{s} & G^{(0)} \end{array} \qquad \begin{array}{ccc} G^{(2)} & \xrightarrow{m} & G \\ \downarrow{\scriptstyle \mathrm{pr}_2} & & \downarrow{\scriptstyle s} \\ G & \xrightarrow{s} & G^{(0)} \end{array} \qquad \begin{array}{ccc} G^{(2)} & \xrightarrow{\mathrm{pr}_1} & G \\ \downarrow{\scriptstyle \mathrm{pr}_2} & & \downarrow{\scriptstyle s} \\ G & \xrightarrow{r} & G^{(0)}, \end{array}$$

we get $\ker dm \simeq {\mathrm{pr}_2}^*(\ker ds)$ and $m^*(\ker ds) \simeq \ker d\,\mathrm{pr}_2 = {\mathrm{pr}_1}^*(\ker ds)$, and similarly $\ker dm \simeq {\mathrm{pr}_1}^*(\ker dr)$ and $m^*(\ker dr) \simeq \ker d\,\mathrm{pr}_1 = {\mathrm{pr}_2}^*(\ker dr)$. We then have the following isomorphism of vector bundles:

$$\begin{aligned} &{\mathrm{pr}_1}^*(\ker ds \oplus \ker dr) \oplus {\mathrm{pr}_2}^*(\ker ds \oplus \ker dr) \\ &\qquad \simeq m^*(\ker ds) \oplus \ker dm \oplus \ker dm \oplus m^*(\ker dr), \end{aligned}$$

and taking half-densities on these bundles gives exactly (3.9). □

Since in the basic formula (3.7) the function under sign of integration

$$G_\gamma^{(2)} \ni (\gamma_1, \gamma_2) \mapsto f(\gamma_1)g(\gamma_2) \in \big(\mathrm{pr}_1^*(\Omega^{1/2}) \otimes \mathrm{pr}_2^*(\Omega^{1/2})\big)_{(\gamma_1,\gamma_2)}$$

is a C^∞ section of the bundle $\big(\mathrm{pr}_1^*(\Omega^{1/2}) \otimes \mathrm{pr}_2^*(\Omega^{1/2})\big)|_{G_\gamma^{(2)}}$, Lemma 3.1 shows that (3.7) is the integral of a one density, canonically associated with f, g over the submanifold $m^{-1}(\gamma)$ and that the result is a C^∞ section of $\Omega^{1/2}$. Further computations on densities show that the statement

$$f * g(\gamma) = \int_{G^{r(\gamma)}} f(\gamma_1)g(\gamma_1^{-1}\gamma) = \int_{G_{s(\gamma)}} f(\gamma\gamma_2^{-1})g(\gamma_2) \tag{3.10}$$

makes sense and is true. The involution on $C_c^\infty(G, \Omega^{1/2})$ is also natural in terms of densities

$$f^\star = \overline{i^*(f)}, \quad f \in C_c^\infty(G, \Omega^{1/2})$$

where i is the induced vector bundle isomorphism over the inversion map of G

$$\ker dr \oplus \ker ds \longrightarrow \ker dr \oplus \ker ds, \quad (\gamma, X_1, X_2) \longmapsto (\gamma^{-1}, di(X_2), di(X_1)).$$

By the usual convention, the spaces $\mathcal{E}'(G, \Omega^{1/2})$ and $\mathcal{D}'(G, \Omega^{1/2})$ are the topological duals of, respectively, the spaces $C^\infty(G, \Omega^{-1/2} \otimes \Omega_G)$ and $C_c^\infty(G, \Omega^{-1/2} \otimes \Omega_G)$ endowed with their usual Fréchet and LF topological vector space structures. The choice of densities is made so that we have canonical embeddings

$$C^\infty(G, \Omega^{1/2}) \hookrightarrow \mathcal{D}'(G, \Omega^{1/2}) \quad \text{and} \quad C_c^\infty(G, \Omega^{1/2}) \hookrightarrow \mathcal{E}'(G, \Omega^{1/2}).$$

Actually, the bundle of densities used in the spaces of test functions above can be replaced by a rather simpler one. Indeed, using the exact sequence $0 \to \ker dr \to TG \to r^*(TG^{(0)}) \to 0$, one gets $\Omega_G^{1/2} = \Omega^{1/2}(\ker dr) \otimes r^*(\Omega_{G^{(0)}}^{1/2})$. Doing the same with s instead of r and combining the two gives

$$\Omega_G = \Omega^{1/2}(\ker dr) \otimes \Omega^{1/2}(\ker ds) \otimes (r^* \otimes s^*)(\Omega_{G^{(0)}}^{1/2}) = \Omega^{1/2} \otimes (r^* \otimes s^*)(\Omega_{G^{(0)}}^{1/2}),$$

hence, we have a canonical isomorphism

$$\mathcal{D}'(G, \Omega^{1/2}) \simeq \big(C_c^\infty(G, (r^* \otimes s^*)(\Omega_{G^{(0)}}^{1/2}))\big)'.$$

For simplicity, we assume in the sequel that $G^{(0)}$ is compact.

Theorem 3.2. *The bilinear map*

$$\begin{aligned} \mathcal{E}_s'(G, \Omega^{1/2}) \times \mathcal{E}'(G, \Omega^{1/2}) &\xrightarrow{\ *\ } \mathcal{E}'(G, \Omega^{1/2}) \\ (u, v) &\longmapsto u * v = m_*(u \underset{s}{\times} v) \end{aligned} \tag{3.11}$$

is well defined and separately continuous and extends the convolution product defined on $C_c^\infty(G,\Omega^{1/2})$. Also, the maps

$$\begin{aligned}\mathcal{D}'(G,\Omega^{1/2}) &\xrightarrow{\;*\;} \mathcal{D}'(G,\Omega^{1/2})\\ v &\longmapsto u_0 * v = m_*(u_0 \underset{s}{\times} v)\end{aligned}$$

and

$$\begin{aligned}\mathcal{D}'_s(G,\Omega^{1/2}) &\xrightarrow{\;*\;} \mathcal{D}'_s(G,\Omega^{1/2})\\ u &\longmapsto u * v_0 = m_*(u \underset{s}{\times} v_0)\end{aligned} \tag{3.12}$$

are well defined and continuous for any $u_0 \in \mathcal{E}'_s(G,\Omega^{1/2})$ and $v_0 \in \mathcal{E}'(G,\Omega^{1/2})$. Similar statements are available for r-transversal distributions used as right variables. We get by restriction separately continuous bilinear maps

$$\mathcal{E}'_\pi(G,\Omega^{1/2}) \times \mathcal{E}'_\pi(G,\Omega^{1/2}) \xrightarrow{\;*\;} \mathcal{E}'_\pi(G,\Omega^{1/2}) \tag{3.13}$$

for $\pi = r$ and $\pi = s$. The space $(\mathcal{E}'_\pi(G,\Omega^{1/2}),*)$ is an associative algebra with unit given by

$$\langle \delta, f\rangle = \int_{G^{(0)}} f, \quad f \in C^\infty(G,\Omega^{-1/2}\otimes\Omega_G). \tag{3.14}$$

In particular $(\mathcal{E}'_{r,s}(G,\Omega^{1/2}),*)$ is an associative unital algebra with involution given by

$$u^\star = \overline{i^*(u)}. \tag{3.15}$$

Proof. Applying Proposition 2.17 to the case $M_1 = M_2 = G$, $B = G^{(0)}$, $\pi_1 = s$, $\pi_2 = r$ and $\sigma_2 : G \to \{\mathrm{pt}\}$, one gets a distribution $u \underset{s}{\times} v \in \mathcal{D}'(G^{(2)},\Omega^{1/2})$ which depends continuously on u and v. Since $u \in \mathcal{E}'$ one can choose $\phi \in C_c^\infty(G)$ such that $u = \phi u$. Then

$$u \underset{s}{\times} v = \varphi u \underset{s}{\times} v$$

where $\varphi = \phi \circ \mathrm{pr}_1|_{G^{(2)}}$ and Proposition 2.18 can be applied to the case $f = m$ with $B = \{\mathrm{pt}\}$. This gives that $u * v$ is well defined for $v \in \mathcal{D}'$ and the continuity of $v \mapsto u * v$ on $\mathcal{E}', \mathcal{D}'$ for fixed $u \in \mathcal{E}'_s$ as well. For fixed $v \in \mathcal{E}'$, one gets the continuity of $u \mapsto u * v$ on $\mathcal{E}'_s, \mathcal{D}'_s$ in the same way.

To prove the statement involving (3.13) for $\pi = s$ we apply Proposition 2.17 to $M_1 = M_2 = G$, $B = G^{(0)}$, $\pi_1 = s$, $\pi_2 = r$ and $\sigma_2 = s$, and Proposition 2.18 to $M = G^{(2)}$, $N = G$, $B = G^{(0)}$ with $\rho = s$ and $\pi = s \circ \mathrm{pr}_2$.

The associativity of $*$ on distributions follows by continuity and density of $C_c^\infty(G,\Omega^{1/2})$.

To check that the integral defining δ has an intrinsic meaning, we identify $\Omega^{-1/2} \otimes \Omega_G \simeq (r^* \otimes s^*)(\Omega^{1/2}_{G^{(0)}})$ and observe that the restriction to $G^{(0)}$ of any $f \in C_c^\infty(G,(r^*\otimes s^*)(\Omega^{1/2}_{G^{(0)}}))$ gives an element in $C^\infty(G^{(0)},\Omega_{G^{(0)}})$, so that $\delta(f) = \int_{G^{(0)}} f_{|G^{(0)}}$ is well defined.

Moreover, we have

$$r_*(f\delta) = s_*(f\delta) = f|_{G^{(0)}} \in C^\infty(G^{(0)}) \subset \mathcal{D}'(G^{(0)}), \quad \text{for any } f \in C^\infty(G),$$

in particular $\delta \in \mathcal{E}'_{r,s}(G,\Omega^{1/2})$. If $\delta^x \in \mathcal{D}'(G^x)$, $x \in G^{(0)}$ is the associated C^∞ family, we then get by Remark 2.8

$$\langle \delta^x, \phi\rangle = r_*(\delta\widetilde{\phi})(x) = \phi(x),$$
$$\text{for any } \phi \in C_c^\infty(G^x) \text{ and } \widetilde{\phi} \in C_c^\infty(G) \text{ such that } \widetilde{\phi}|_{G^x} = \phi.$$

It follows that for any $f \in C^\infty(G,\Omega^{-1/2}\otimes\Omega_G)$,

$$\langle u*\delta, f\rangle = \int_{x\in G^{(0)}} \langle u_x\otimes\delta^x, (f\circ m)|_{G_x\times G^x}\rangle = \int_{x\in G^{(0)}} \langle u_x, f|_{G_x}\rangle = \langle u, f\rangle.$$

Therefore, δ is a left unit and that it is a right unit is proved similarly. The assertion about the involution is obvious. □

In particular, when one of the two factors is in C_c^∞, the convolution product is defined without any restriction on the other factor. We give a sufficient condition for the result to be C^∞.

Proposition 3.3. *The convolution product gives by restriction a bilinear separetely continuous map*

$$\mathcal{D}'_r(G,\Omega^{1/2}) \times C_c^\infty(G,\Omega^{1/2}) \xrightarrow{*} C^\infty(G,\Omega^{1/2}).$$

The analogous statement with C^∞ functions on the left and s-transversal distributions on the right also holds. The map $u \mapsto u*\cdot$ mapping $\mathcal{D}'_r(G,\Omega^{1/2})$ to $\mathcal{L}(C_c^\infty(G,\Omega^{1/2}), C^\infty(G,\Omega^{1/2}))$ is injective.

Proof. If $u = (u^y)_y \in \mathcal{D}'_r$, the map

$$\gamma \mapsto \langle u^{r(\gamma)}(\cdot), f((\cdot)^{-1}\gamma)\rangle \tag{3.16}$$

is C^∞ and by definition of the convolution product we get

$$\langle u*f, \phi\rangle = \int_{\gamma_2\in G} \langle u^{r(\gamma_2)}(\cdot), f((\cdot)^{-1}\gamma_2)\rangle\phi(\gamma_2).$$

Thus $u*f$ coincides with the C^∞ function (3.16). The continuity of $u \mapsto u*f$ is given by Theorem 3.2 and repeating the argument given in its proof, one gets the continuity of $f \mapsto u*f$ on $C_0^\infty(K,\Omega^{1/2}) = \{f \in C^\infty\ ;\ \mathrm{supp}(f)\subset K\}$ for any compact $K\subset G$. The results follows by inductive limit.

Finally, the vanishing of $u*f$ for any f and the previous expression for $u*f$ shows that $u^x = 0$, for any x, and thus $u = 0$. □

Remark 3.4. (1) Note that if in the previous proposition we suppose that u has compact support $K \subset G$, then $u * f$ can be defined for any map $f \in C^\infty(G, \Omega^{1/2})$. Moreover for any $f \in C_c^\infty(G, \Omega^{1/2})$, then $u * f$ is also compactly supported and $\mathrm{supp}(u * f) \subset K.\mathrm{supp}(f)$.

(2) If $G^{(0)}$ is non compact, to get an involutive unital algebra, one should rather consider the subalgebra $\mathcal{P}'_{r,s}(G, \Omega^{1/2})$ of $\mathcal{D}'_{r,s}(G, \Omega^{1/2})$ of distributions whose support has the property that the restrictions of r and s to it are proper maps.

4. G-operators

We recall the notion of G-operators given in [19] and we add a notion of adjoint for them.

Definition 4.1. A (left) G-operator is a continuous linear map $P : C_c^\infty(G, \Omega^{1/2}) \to C^\infty(G, \Omega^{1/2})$ such that there exists a family $P_x : C_c^\infty(G_x, \Omega^{1/2}_{G_x}) \longrightarrow C^\infty(G_x, \Omega^{1/2}_{G_x})$, $x \in G^{(0)}$ of operators such that

$$P(f)|_{G_x} = P_x(f|_{G_x}), \quad \forall f \in C_c^\infty(G, \Omega^{1/2}),\ \forall x \in G^{(0)} \tag{4.1}$$

$$P_{r(\gamma)} \circ R_\gamma = R_\gamma \circ P_{s(\gamma)}, \quad \forall \gamma \in G. \tag{4.2}$$

A G-operator P is said to be adjointable if there exists a G-operator Q such that

$$(P(f)|g) = (f|Q(g))\ ; \quad f, g \in C_c^\infty(G, \Omega^{1/2}). \tag{4.3}$$

Here $(f|g) = f^\star * g$ is the $C_c^\infty(G, \Omega^{1/2})$-valued pre-hilbertian product of $C_c^\infty(G, \Omega^{1/2})$.

We note Op_G and $\mathrm{Op}_G^\star$ respectively the linear spaces of G-operators and adjointable ones.

We say that a G-operator P is supported in K if $\mathrm{supp}(P(f)) \subset K.\mathrm{supp}(f)$ for all f. The subspaces of compactly supported G-operators are denoted $\mathrm{Op}_{G,c}$, $\mathrm{Op}^\star_{G,c}$.

Looking at $C_c^\infty(G, \Omega^{1/2})$ and $C^\infty(G, \Omega^{1/2})$ as right $C_c^\infty(G, \Omega^{1/2})$-modules for the convolution product, G-operators can be characterized in a simple way.

Proposition 4.2. *A linear operator $P : C_c^\infty(G, \Omega^{1/2}) \to C^\infty(G, \Omega^{1/2})$ is a G-operator if and only if it is continuous and*

$$P(f * g) = P(f) * g, \quad \forall f, g \in C_c^\infty(G, \Omega^{1/2}).$$

In other words, $\mathrm{Op}_G = \mathcal{L}_{C_c^\infty(G,\Omega^{1/2})}(C_c^\infty(G, \Omega^{1/2}), C^\infty(G, \Omega^{1/2}))$.

Proof. Let $P \in \mathrm{Op}_G$. Let us write p_x for the Schwartz kernel of P_x. For any f, g compactly supported and $\gamma \in G$

$$\begin{aligned} P(f*g)(\gamma) &= \int_{\gamma_2 \in G_{s(\gamma)}} \int_{\gamma_1 \in G_{s(\gamma)}} p_{s(\gamma)}(\gamma,\gamma_2) f(\gamma_2\gamma_1^{-1}) g(\gamma_1) \\ &= \int_{\gamma_1 \in G_{s(\gamma)}} \Big(\int_{\gamma_2 \in G_{s(\gamma)}} p_{s(\gamma)}(\gamma,\gamma_2)(R_{\gamma_1^{-1}} f)(\gamma_2) \Big) g(\gamma_1) \\ &= \int_{\gamma_1 \in G_{s(\gamma)}} \Big(\int_{\gamma_2 \in G_{s(\gamma)}} p_{r(\gamma_1)}(\gamma\gamma_1^{-1},\gamma_2) f(\gamma_2) \Big) g(\gamma_1) \\ &= \int_{\gamma_1 \in G_{s(\gamma)}} P(f)(\gamma\gamma_1^{-1}) g(\gamma_1) = P(f)*g(\gamma). \end{aligned}$$

Conversely, let $f \in C_c^\infty(G, \Omega^{1/2})$ and $x \in G^{(0)}$ such that $f|_{G_x} = 0$. Observe that $(g*f)|_{G_x} = 0$ for any $g \in C_c^\infty(G, \Omega^{1/2})$. It follows that $P(g*f)|_{G_x} = P(g)*f|_{G_x} = 0$. Choose a sequence $\phi_n \in C_c^\infty(G, \Omega^{1/2})$ converging to δ in $\mathcal{E}'_r$. Then $\phi_n * f$ converges to f in $C_c^\infty(G, \Omega^{1/2})$ and therefore

$$P(f)(\gamma) = \lim P(\phi_n * f)(\gamma) = 0, \quad \forall \gamma \in G_x.$$

In other words, $P(f)|_{G_x}$ only depends on $f|_{G_x}$ and we can define P_x for any x by

$$\begin{aligned} &P_x(f) = P(\widetilde{f})|_{G_x}, \\ &\qquad \forall f \in C_c^\infty(G_x, \Omega_{G_x}^{1/2}) \text{ and } \widetilde{f} \in C_c^\infty(G, \Omega^{1/2}) \text{ such that } \widetilde{f}|_{G_x} = f. \end{aligned}$$

Let $\gamma \in G_x^y$. Then for any $\gamma' \in G^y$ and $f \in C_c^\infty(G, \Omega^{1/2})$, we have

$$\begin{aligned} R_\gamma(P_x(\phi_n * f))(\gamma') &= P(\phi_n * f)(\gamma'\gamma) \\ &= P(\phi_n) * f(\gamma'\gamma) \\ &= P(\phi_n) * (R_\gamma f)(\gamma') = P(\phi_n * (R_\gamma f))(\gamma'). \end{aligned}$$

Taking the limit in this equality gives (4.2).

The set Op_G being obviously closed in $\mathcal{L}(C_c^\infty(G, \Omega^{1/2}), C^\infty(G, \Omega^{1/2}))$ this proof shows in particular that Op_G is the closure of operators of the form $f \mapsto u * f$ where $u \in C^\infty(G, \Omega^{1/2})$. □

Let $u \in \mathcal{D}'_r(G, \Omega^{1/2})$. Using Propositions 3.3 and 4.2, we can define $P \in \mathrm{Op}_G$ by setting $P(f) = u * f$ for any $f \in C_c^\infty(G, \Omega^{1/2})$.

Conversely, let $P \in \mathrm{Op}_G$ and $p_x \in \mathcal{D}'(G_x \times G_x)$ the Schwartz kernel of P_x, $x \in G^{(0)}$. Since

$$\gamma \longmapsto P(f)(\gamma) = \int p_{s(\gamma)}(\gamma, \gamma_1) f(\gamma_1)$$

is C^∞ for any f, we get that $\gamma \mapsto p_{s(\gamma)}(\gamma,\cdot)$ belongs to $\mathcal{D}'_{\mathrm{pr}_1}(G \underset{s}{\times} G)$ and then using the second part of Proposition 2.15, it restricts to the map $G^{(0)} \ni x \mapsto p_x(x,\cdot)$ belonging to $\mathcal{D}'_s(G)$. Defining $k_P \in \mathcal{D}'_r(G)$ by $k_P(\gamma) = p_{r(\gamma)}(r(\gamma),\gamma^{-1})$, we get for any $f \in C_c^\infty(G,\Omega^{1/2})$, $x, y \in G^{(0)}$ and $\gamma \in G_x^y$

$$\begin{aligned} P(f)(\gamma) &= \int_{G_x} p_x(\gamma,\gamma_1) f(\gamma_1) = \int_{G_y} p_y(y,\gamma_1) f(\gamma_1\gamma) \\ &= \int_{G^y} p_y(y,\gamma_1^{-1}) f(\gamma_1^{-1}\gamma) = \langle (k_P)_y, f((\cdot)^{-1}\gamma)\rangle_{G^y} = k_P * f(\gamma). \end{aligned} \tag{4.4}$$

Thus P the operator given by left convolution with k_P. We call k_P *the convolution distributional kernel* of P. Note that $\mathrm{supp}(P) = \mathrm{supp}(k_P)$. We have proved

Theorem 4.3. *The map $P \mapsto k_P$ gives the isomorphisms*

$$\mathrm{Op}_G \simeq \mathcal{D}'_r(G,\Omega^{1/2}) \quad \textit{and} \quad \mathrm{Op}_{G,c} \simeq \mathcal{E}'_r(G,\Omega^{1/2}). \tag{4.5}$$

If $k_P \in \mathcal{D}'_{r,s}(G,\Omega^{1/2})$ then P is obviously adjointable and $k_{P^\star} = (k_P)^\star$. Conversely, if P as an adjoint Q then

$$(k_P * f)^\star * g = (f^\star * k_P^\star) * g = f^\star * (k_Q * g)\,; \quad f, g \in C_c^\infty(G,\Omega^{1/2}), \tag{4.6}$$

hence $k_P^\star = k_Q \in \mathcal{D}'_s(G,\Omega^{1/2}) \cap \mathcal{D}'_r(G,\Omega^{1/2})$. Thus Theorem 4.3 yields:

Corollary 4.4. *The map $P \to k_P$ induces an isomorphism*

$$\mathrm{Op}_G^\star \simeq \mathcal{D}'_{r,s}(G,\Omega^{1/2}). \tag{4.7}$$

Remark 4.5. Rephrazing the previous results, we have, for instance

$$\mathrm{Op}_G \simeq \mathcal{L}_s(C_c^\infty(G,\Omega^{1/2}), C^\infty(G^{(0)},\Omega^{1/2}(AG))).$$

where we have replaced $\mathcal{L}_{C^\infty(G^{(0)})}$ by $\mathcal{L}_s$ to emphasize that the $C^\infty(G^{(0)})$-module structure on $C_c^\infty(G,\Omega^{1/2})$ is given by s. Also

$$\mathrm{Op}_G^\star \simeq \mathcal{L}_{r,s}(C_c^\infty(G,\Omega^{1/2}), C^\infty(G^{(0)},\Omega^{1/2}(AG))).$$

where $\mathcal{L}_{r,s} = \mathcal{L}_s \cap \mathcal{L}_r$. In terms of Schwartz kernel theorems for submersions, G-operators thus appear as "semi-regular" distributions (see Treves [24, p. 532]) since, for $\pi = s$ or $\pi = r$

$$\mathcal{D}'(G,\Omega^{1/2}) \simeq \mathcal{L}_\pi(C_c^\infty(G,\Omega^{1/2}), \mathcal{D}'(G^{(0)},\Omega^{1/2}(AG))).$$

Now observe that if $k_P \in \mathcal{E}'_{r,s}(G,\Omega^{1/2})$, Theorem 3.2 implies that P extends continuously to a map $\mathcal{D}'(G,\Omega^{1/2}) \longrightarrow \mathcal{D}'(G,\Omega^{1/2})$ sending the subspace $\mathcal{E}'_{r,s}$ to $\mathcal{E}'_{r,s}$. This leads to another characterization of adjointness.

Proposition 4.6. *A compactly supported G-operator P is adjointable if and only if it extends continuously to a map*

$$\widetilde{P} : \mathcal{D}'(G, \Omega^{1/2}) \longrightarrow \mathcal{D}'(G, \Omega^{1/2})$$

such that $\widetilde{P}(\delta) \in \mathcal{D}'_{r,s}(G, \Omega^{1/2})$. *In that case,* $\widetilde{P} = k_P * \cdot$.

Proof. Let $u \in \mathcal{D}'(G, \Omega^{1/2})$ and $(u_n) \subset C_c^\infty(G, \Omega^{1/2})$ a sequence converging to u in $\mathcal{D}'$. We have

$$\widetilde{P}(u * f) = \lim P(u_n * f) = \lim P(u_n) * f = \widetilde{P}(u) * f, \quad \forall f \in C_c^\infty(G, \Omega^{1/2}).$$

Thus $\widetilde{P}$ is automatically $C_c^\infty(G, \Omega^{1/2})$-right linear. It follows that

$$k_P * f = P(f) = P(\delta * f) = \widetilde{P}(\delta) * f, \quad \forall f \in C_c^\infty(G, \Omega^{1/2})$$

which proves that $k_P = \widetilde{P}(\delta) \in \mathcal{D}'_{r,s}(G, \Omega^{1/2})$ and that $\widetilde{P}$ is given by left convolution with k_P. □

5. Convolution on groupoids and wave front sets

We now turn to some microlocal aspects of the convolution of distributions on groupoids. In view of Proposition 2.9, it is natural to call r-transversal any (conic) subset $W \subset T^*G \setminus 0$ such that $W \cap \ker dr^\perp = \emptyset$, indeed in that case

$$\mathcal{D}'_W(G, \Omega^{1/2}) \subset \mathcal{D}'_r(G, \Omega^{1/2}). \tag{5.1}$$

Similarly, W is called s-transversal if $W \cap \ker ds^\perp = \emptyset$ and we call bi-transversal any set which is both r and s-transversal. We then introduce

$$\mathcal{D}'_a(G, \Omega^{1/2}) = \{u \in \mathcal{D}'(G, \Omega^{1/2})\ ;\ \mathrm{WF}(u) \text{ is bi-transversal}\} \tag{5.2}$$

and $\mathcal{E}'_a = \mathcal{D}'_a \cap \mathcal{E}'$. We call them *admissible* distributions. From Proposition 2.9, we get

$$\mathcal{D}'_a(G, \Omega^{1/2}) \subset \mathcal{D}'_{r,s}(G, \Omega^{1/2}). \tag{5.3}$$

Example 5.1. Observe that $A^*G \setminus 0$ is bi-transversal. Since $\Psi(G) = I(G, G^{(0)}) \subset \mathcal{D}'_{A^*G}(G)$ (see [17]) we get

$$\Psi(G) \subset \mathcal{D}'_a(G, \Omega^{1/2}). \tag{5.4}$$

Theorem 3.2 and Proposition 3.3 can be reused in various ways for subspaces of distributions with transversal wave front sets. We only record the main one: the convolution product restricts to a bilinear map

$$\mathcal{E}'_a(G, \Omega^{1/2}) \times \mathcal{E}'_a(G, \Omega^{1/2}) \stackrel{*}{\longrightarrow} \mathcal{E}'_{r,s}(G, \Omega^{1/2}), \tag{5.5}$$

and we strenghthen this result as follows, by using the cotangent groupoid structure of Coste–Dazord–Weinstein (see Section A).

Theorem 5.2. *For any $u_1, u_2 \in \mathcal{E}'_a(G, \Omega^{1/2})$, we have $u_1 * u_2 \in \mathcal{E}'_a(G, \Omega^{1/2})$ and*

$$\mathrm{WF}(u_1 * u_2) \subset \mathrm{WF}(u_1) * \mathrm{WF}(u_2) \tag{5.6}$$

where on the right, $*$ denotes the product of the symplectic groupoid $T^*G \rightrightarrows A^*G$. In particular $(\mathcal{E}'_a(G, \Omega^{1/2}), *)$ is a unital involutive subalgebra of $(\mathcal{E}'_{r,s}(G, \Omega^{1/2}), *)$.

Proof. Let $u_j \in \mathcal{E}'_a(G, \Omega^{1/2})$ and set $W_j = \mathrm{WF}(u_j)$, $j = 1, 2$. We first show that the fibered product $u_1 \underset{\pi}{\times} u_2$ (where $\pi = r, s$ indifferently) given by Proposition 2.17, coincides with the distribution obtained by the functorial operations in [13, Theorems 8.2.9, 8.2.4]:

$$u_1 \underset{\pi}{\times} u_2 = \rho^*(u_1 \otimes u_2) \in \mathcal{D}'(G^{(2)}, \Omega(\ker dm) \otimes m^*(\Omega^{1/2})), \tag{5.7}$$

where $\rho : G^{(2)} \hookrightarrow G^2$. By [13, Theorem 8.2.9], we know that

$$\mathrm{WF}(u_1 \otimes u_2) \subset W_1 \times W_2 \cup W_1 \times (G \times \{0\}) \cup (G \times \{0\}) \times W_2, \tag{5.8}$$

and to apply [13, Theorem 8.2.4], we just need to check that

$$\mathrm{WF}(u_1 \otimes u_2) \cap N^*G^{(2)} = \emptyset. \tag{5.9}$$

Observe that $N^*G^{(2)} = \ker m_\Gamma \subset \Gamma^{(2)}$ and $\ker ds^\perp = \ker r_\Gamma$. Thus, if

$$\delta_j = (\gamma_j, \xi_j) \in T^*_{\gamma_j} G \quad \text{and} \quad (\delta_1, \delta_2) \in \mathrm{WF}(u_1 \otimes u_2) \cap N^*G^{(2)}$$

then $(\delta_1, \delta_2) \in \Gamma^{(2)}$ and

$$r_\Gamma(\delta_1) = r_\Gamma(\delta_1 \delta_2) = (r(\gamma_1), 0). \tag{5.10}$$

By the s-transversality assumption on W_1 and the relation (5.8), this implies $\delta_1 = (\gamma_1, 0)$ and $\delta_2 \in W_2$. On the other hand

$$s_\Gamma(\delta_2) = s_\Gamma(\delta_1 \delta_2) = (s(\gamma_2), 0), \tag{5.11}$$

which contradicts the r-transversality of W_2, and this proves (5.9). Therefore, the right hand side in (5.7) is well defined by [13, Theorem 8.2.4] and it coincides with the left hand side, which is obvious after pairing with test functions. Now

$$u_1 * u_2 = m_*(u_1 \underset{\pi}{\times} u_2) = m_* \rho^*(u_1 \otimes u_2) \tag{5.12}$$

and thus, using [13, Theorem 8.2.4] and [11, (3.6), p. 328],

$$\mathrm{WF}(u_1 * u_2) \subset m_* \rho^* \, \mathrm{WF}(u_1 \otimes u_2)\,. \tag{5.13}$$

Here $\rho^*\colon T^*G^2 \longrightarrow T^*G^{(2)}$ is the restriction of linear forms and, for any $\widetilde{W} \subset T^*G^{(2)}$,

$$m_*(\widetilde{W}) = \big\{(\gamma,\xi) \in T^*G \;;\; \exists(\gamma_1,\gamma_2) \in m^{-1}(\gamma), \\ (\gamma_1,\gamma_2,{}^t dm_{\gamma_1,\gamma_2}(\xi)) \in \widetilde{W} \cup G^{(2)} \times 0\big\}.$$

Since m is submersive, ${}^t dm_{\gamma_1,\gamma_2}$ is injective and the term $G^{(2)} \times 0$ can be removed. By definition of the multiplication of $\Gamma = T^*G$, we get, for any $W \subset T^*G^2$, the equivalence

$$\gamma_1\gamma_2 = \gamma \quad \text{and} \quad (\gamma_1,\gamma_2,{}^t dm_{\gamma_1,\gamma_2}(\xi)) \in \rho^*(W) \\ \Leftrightarrow \exists(\delta_1,\delta_2) \in \Gamma^{(2)} \cap W, \quad \delta_1\delta_2 = (\gamma,\xi). \tag{5.14}$$

Thus,

$$m_*\rho^* W = m_\Gamma(W \cap \Gamma^{(2)}). \tag{5.15}$$

By r-transversality of $\mathrm{WF}(u_1)$, we have $s_\Gamma(\mathrm{WF}(u_1)) \subset A^*G \setminus 0$, so

$$\mathrm{WF}(u_1) \times (G \times \{0\}) \cap \Gamma^{(2)} = \emptyset.$$

Similarly, s-transversality of $\mathrm{WF}(u_2)$ gives $(G \times \{0\}) \times \mathrm{WF}(u_2) \cap \Gamma^{(2)} = \emptyset$. It follows that

$$\mathrm{WF}(u_1 \otimes u_2) \cap \Gamma^{(2)} = (\mathrm{WF}(u_1) \times \mathrm{WF}(u_2)) \cap \Gamma^{(2)},$$

and therefore

$$m_\Gamma(\mathrm{WF}(u_1 \otimes u_2) \cap \Gamma^{(2)}) = m_\Gamma((\mathrm{WF}(u_1) \times \mathrm{WF}(u_2)) \cap \Gamma^{(2)}) \\ = \mathrm{WF}(u_1) * \mathrm{WF}(u_2),$$

which proves (5.6). Clearly, $W_1 * W_2$ is s or r-transversal if the same holds respectively for W_1 and W_2, so (5.6) implies $u_1 * u_2 \in \mathcal{E}'_a$, therefore $\mathcal{E}'_a$ is a subalgebra of $\mathcal{E}'_{r,s}$.

Finally, since $\mathrm{WF}(\delta) = A^*G \setminus 0$, we have $\delta \in \mathcal{E}'_a$ and since $\mathrm{WF}(u^\star) = i_\Gamma(\mathrm{WF}(u))$, we conclude that $\mathcal{E}'_a$ is unital and involutive. □

Looking at the proof of this theorem, we see that the assumptions on $\mathrm{WF}(u_j)$ can be significanlty relaxed in order to conserve the property (5.9) and then to be able to define the convolution product $u_1 * u_2$ by the right hand side of (5.12).

Firstly, if $W \subset T^*G \setminus 0$, then $W \times (G \times 0) \cap \ker m_\Gamma = \emptyset$. Indeed, if $(\gamma_1,\xi_1,\gamma_2,0) \in W \times (G \times \{0\}) \cap \Gamma^{(2)}$, we can choose $t_1 \in T_{\gamma_1}G$ such that $\xi_1(t_1) \neq 0$ since $\xi_1 \neq 0$ by assumption. Using a local section β of r such that $\beta(s(\gamma_1)) = \gamma_2$ and setting $t_2 = d\beta ds(t_1) \in T_{\gamma_2}G$, we get $(t_1,t_2) \in T_{(\gamma_1,\gamma_2)}G^{(2)}$ and $\xi_1(t_1) + 0(t_2) \neq 0$, that is $\xi_1 \oplus 0 \neq 0$ which proves that $(\gamma_1,\xi_1,\gamma_2,0) \notin \ker m_\Gamma$.

Arguing identically on $(G \times 0) \times W$ we get the equivalence, for any distributions u_1, u_2

$$\mathrm{WF}(u_1 \otimes u_2) \cap \ker m_\Gamma = \emptyset \Leftrightarrow \mathrm{WF}(u_1) \times \mathrm{WF}(u_2) \cap \ker m_\Gamma = \emptyset. \tag{5.16}$$

This is again the condition (5.9) which is sufficient to define $\rho^*(u_1 \otimes u_2) = u_1 \otimes u_2|_{G^{(2)}}$ and there the convolution product under additional suitable supports conditions.

Theorem 5.3. *Let $W_j \subset T^*G \setminus 0$ be closed cones such that*

$$W_1 \times W_2 \cap \ker m_\Gamma = \emptyset \tag{5.17}$$

and set $W_1 \bar{*} W_2 = m_\Gamma((W_1 \times W_2 \cup W_1 \times 0 \cup 0 \times W_2) \cap \Gamma^{(2)})$. Then the map

$$\mathcal{E}'_{W_1}(G, \Omega^{1/2}) \times \mathcal{E}'_{W_2}(G, \Omega^{1/2}) \xrightarrow{*} \mathcal{E}'_{W_1 \bar{*} W_2}(G, \Omega^{1/2}) \tag{5.18}$$

$$(u_1, u_2) \longmapsto m_*(u_1 \otimes u_2|_{G^{(2)}}) \tag{5.19}$$

is separately sequentially continuous and coincides with the convolution product on $C_c^\infty(G, \Omega^{1/2})$.

Proof. Under the assumption made on W_1, W_2, we can apply [13, Theorems 8.2.4, 8.2.9] to find that the bilinear map

$$\begin{aligned} \mathcal{D}'_{W_1}(G, \Omega^{1/2}) \times \mathcal{D}'_{W_2}(G, \Omega^{1/2}) &\longrightarrow \mathcal{D}'_{\rho^*(W_1 \bar{\times} W_2)}(G^{(2)}, \Omega^{1/2}) \\ (u_1, u_2) &\longmapsto u_1 \otimes u_2|_{G^{(2)}} \end{aligned} \tag{5.20}$$

is well defined, sequentially separately continuous for the natural notion of convergence of sequences in the spaces $\mathcal{D}'_W$ [11,13], and also separately continuous for the normal topology of these spaces [1]. Above, we have set for convenience $W_1 \bar{\times} W_2 = W_1 \times W_2 \cup W_1 \times 0 \cup 0 \times W_2$.

To apply m_* and get a continuous map for the same topologies, we restrict ourselves to compactly supported distributions and we get

$$\begin{aligned} &\mathcal{E}'_{W_1}(G, \Omega^{1/2}) \times \mathcal{E}'_{W_2}(G, \Omega^{1/2}) \\ &\qquad \xrightarrow{(\cdot \otimes \cdot)|_{G^{(2)}}} \mathcal{E}'_{\rho^*(W_1 \bar{\times} W_2)}(G^{(2)}, \Omega^{1/2}) \xrightarrow{m_*} \mathcal{E}'_{W_1 \bar{*} W_2}(G, \Omega^{1/2}). \end{aligned} \tag{5.21}$$

Indeed, the formulas (5.13) and (5.15) are still valid here and give the last distribution space above. □

If u_1 or u_2 is smooth then $\mathrm{WF}(u_1) \times \mathrm{WF}(u_2)$ is empty and (5.17) is trivially satisfied, thus

Corollary 5.4. *The convolution product of Theorem 5.3 gives by restriction the maps*

$$\mathcal{E}'(G, \Omega^{1/2}) \times C_c^\infty(G, \Omega^{1/2}) \xrightarrow{*} \mathcal{E}'_{s_\Gamma^{-1}(0)}(G, \Omega^{1/2}), \tag{5.22}$$

$$C_c^\infty(G, \Omega^{1/2}) \times \mathcal{E}'(G, \Omega^{1/2}) \xrightarrow{*} \mathcal{E}'_{r_\Gamma^{-1}(0)}(G, \Omega^{1/2}). \tag{5.23}$$

As we said, bi-transversal subsets of $T^*G \setminus 0$ satisfy (5.17). Actually,

Corollary 5.5. *Let W_1, W_2 be any subsets of $T^*G\setminus 0$. If W_1 is s-transversal (resp. W_2 is r-transversal) then the assumption (5.17) is satisfied and $W_1 * W_2$ is s-transversal (resp. W_2 r-transversal) .*

Proof. Use the equalities $s_\Gamma \circ m_\Gamma = s_\Gamma \circ \mathrm{pr}_2$ and $r_\Gamma \circ m_\Gamma = r_\Gamma \circ \mathrm{pr}_1$. □

Remark 5.6. Theorems 3.2 and 5.3 do not apply exactly to the same situations. For instance, consider the pair groupoid $G = \mathbb{R} \times \mathbb{R}$. On one hand, using the relation $\ker m_\Gamma = ((\ker ds)^\perp \times (\ker dr)^\perp) \cap (T^*G)^{(2)}$ and Remark 2.11, it is easy to obtain pairs of distributions $(u_1, u_2) \in \mathcal{E}'_s(\mathbb{R}^2) \times \mathcal{E}'(\mathbb{R}^2)$ for which only Theorem 3.2 can be applied to define $u_1 * u_2$. On the other hand, consider the distributions $u_1 = \delta_{(0,0)}$ and $u_2 = \delta_{(1,1)}$, whose wave fronts are respectively $W_1 = \{(0, 0, \xi, \eta)\ ;\ (\xi, \eta) \neq (0, 0)\}$ and $W_2 = \{(1, 1, \xi, \eta)\ ;\ (\xi, \eta) \neq (0, 0)\}$. These distributions are neither s nor r transversal, but $W_1 \times W_2 \cap \Gamma^{(2)} = \emptyset$, hence the convolution $u_1 * u_2$ on G can only be defined by Theorem 5.3 (note that $u_1 * u_2 = 0$; less peculiar examples can be easily constructed).

Of course, both convolution products coincide when both make sense, since the equality (5.7) is valid as soon as $(\mathrm{WF}(u_1) \times \mathrm{WF}(u_2)) \cap \ker m_\Gamma = \emptyset$.

A. The cotangent groupoid of Coste–Dazord–Weinstein

We recall the definition of the cotangent groupoid of Coste–Dazord–Weinstein. We explain the construction of the source and target map given in [4] and we enlighten the role played by the differential of the multiplication map of G. This is a pedestrian approach based on concrete differential geometry while more conceptual developments can be found in [16,23].

Let G be a Lie groupoid whose multiplication is denoted by m, source and target by s, r and inversion by i. Differentiating all the structure maps of G, we get that $TG \rightrightarrows TG^{(0)}$ is a Lie groupoid whose multiplication is given by dm, source and target by ds, dr and inversion by di. Hence, it is natural to try to transpose everything to get a groupoid structure on $\Gamma = T^*G$. Following this idea, it is natural to decide that the product $(\gamma_1, \xi_1).(\gamma_2, \xi_2) \in T^*G$ of two elements $(\gamma_j, \xi_j) \in T^*G$ is defined by $(\gamma_1\gamma_2, \xi)$ where ξ is the solution of the equation

$$ {}^t dm_{(\gamma_1,\gamma_2)}(\xi) = (\xi_1, \xi_2)|_{T_{(\gamma_1,\gamma_2)}G^{(2)}}. \tag{A.1}$$

Indeed, $m : G^{(2)} \longrightarrow G$ being a submersion, ${}^t dm_{(\gamma_1,\gamma_2)}$ is injective for all $(\gamma_1, \gamma_2) \in G^{(2)}$ and ξ, when it exists, is therefore unique. In that case, we have

$$ \xi = {}^t dm^{-1}_{(\gamma_1,\gamma_2)} \rho(\xi_1, \xi_2) \tag{A.2}$$

where $\rho : T^*_{G^{(2)}}G^2 \longrightarrow T^*G^{(2)}$ is the restriction of linear forms and we introduce the notations

$$\xi = \xi_1 \oplus \xi_2 \quad \text{and} \quad m_\Gamma(\gamma_1, \xi_1, \gamma_2, \xi_2) = (\gamma_1\gamma_2, \xi_1 \oplus \xi_2). \tag{A.3}$$

The equation (A.1) has a solution ξ if and only if

$$(\xi_1, \xi_2) \in \operatorname{Im} {}^t dm_{(\gamma_1,\gamma_2)}. \tag{A.4}$$

Since $\operatorname{Im} {}^t dm_{(\gamma_1,\gamma_2)} = (\ker dm_{(\gamma_1,\gamma_2)})^\perp$, this is equivalent to

$$\xi_1(t_1) + \xi_2(t_2) = 0, \quad \forall (t_1, t_2) \in \ker dm_{(\gamma_1,\gamma_2)}. \tag{A.5}$$

Let us explicit $\ker dm \subset TG^{(2)}$. Let

$$L_\gamma : G^{s(\gamma)} \longrightarrow G^{r(\gamma)},\ \gamma' \mapsto \gamma\gamma' \quad \text{and} \quad R_\gamma : G_{r(\gamma)} \longrightarrow G_{s(\gamma)},\ \gamma' \mapsto \gamma'\gamma$$

be the left and right multiplication maps of G. Let $(\gamma_1, \gamma_2) \in G^{(2)}$ and set $\gamma = \gamma_1\gamma_2$, $x = s(\gamma_1)$. Parametrizing $G^{(2)}_\gamma = m^{-1}(\gamma)$ by $G^{r(\gamma)} \ni \eta \mapsto (\eta, \eta^{-1}\gamma)$, we find, after a routine computation:

$$(t_1, t_2) \in \ker dm_{(\gamma_1,\gamma_2)} \Leftrightarrow t_1 = dL_{\gamma_1} di(t),\ t_2 = dR_{\gamma_2}(t), \quad \text{for some } t \in T_x G_x. \tag{A.6}$$

It follows that (A.4) is equivalent to the equality

$$^t dR_{\gamma_2}(\xi_2) = -{}^t d(L_{\gamma_1} \circ i)(\xi_1) \in (T_x G_x)^*, \tag{A.7}$$

where it is understood that R_{γ_2} and $L_{\gamma_1} \circ i$ are differentiated at $\gamma = x$ and that the linear forms ξ_1, ξ_2 are restricted to the ranges of the corresponding differential maps. The same abuse of notations is used below without further notice. We then define elements $\overline{s}(\xi_1), \overline{r}(\xi_2)$ belonging to $A^*_x G = (T_x G / T_x G^{(0)})^*$ by

$$\overline{s}(\xi_1)(t + u) = {}^t dL_{\gamma_1}(\xi_1)(t), \quad \text{for all } t + u \in T_x G^x \oplus T_x G^{(0)} = T_x G, \tag{A.8}$$

$$\overline{r}(\xi_2)(t + u) = {}^t dR_{\gamma_2}(\xi_2)(t), \quad \text{for all } t + u \in T_x G_x \oplus T_x G^{(0)} = T_x G. \tag{A.9}$$

Differentiating the relation $\gamma^{-1}\gamma = s(\gamma)$ at $\gamma = x$ we get the relation

$$di + \mathrm{id} = ds + dr \tag{A.10}$$

which yields $-di(t) \equiv t \bmod T_x G^{(0)}$, $\forall t \in T_x G$. Thus, (A.7), and then (A.4), is equivalent to

$$\overline{r}(\xi_2) = \overline{s}(\xi_1) \in A^*_x G. \tag{A.11}$$

This leads to the definitions

$$s_\Gamma(\gamma, \xi) = (s(\gamma), \overline{s}(\xi)) \in A^*G \quad \text{and} \quad r_\Gamma(\gamma, \xi) = (r(\gamma), \overline{r}(\xi)) \in A^*G, \quad \forall (\gamma, \xi) \in T^*G. \tag{A.12}$$

Finally, we denote $u_\Gamma : A^*G \hookrightarrow T^*G$ the canonical inclusion and we set

$$i_\Gamma(\gamma, \xi) = (\gamma^{-1}, -({}^t di_\gamma)^{-1}(\xi)), \quad \forall (\gamma, \xi) \in T^*G. \tag{A.13}$$

Theorem A.1 ([4]). *Let G be a Lie groupoid. The space $\Gamma = T^*G$ is a Lie groupoid with unit space A^*G and structural maps given by $s_\Gamma, r_\Gamma, m_\Gamma, i_\Gamma$ and u_Γ (respectively, source, target, multiplication, inversion and inclusion of unit maps).*

Remark A.2. (1) The Lie algebroid of G is sometimes defined by $AG = \ker ds|_{G^{(0)}}$. In that picture, we deduce from (A.7) that s_Γ and r_Γ have to be defined by replacing $\overline{s}, \overline{r}$ by

$$\widetilde{s}(\xi) = -{}^t d(L_\gamma \circ i)(\xi) \quad \text{and} \quad \widetilde{r}(\xi) = {}^t dR_\gamma(\xi). \tag{A.14}$$

(2) The submanifold $\Gamma^{(2)}$ of composable pairs in Γ is given by

$$\Gamma^{(2)} = \left\{(\delta_1, \delta_2) \in T^*_{G^{(2)}} G^2 \ ; \ \rho(\delta_1, \delta_2) \in (\ker dm)^\perp\right\} \tag{A.15}$$

and $m_\Gamma = {}^t dm^{-1} \circ \rho$.

(3) The graph of m_Γ is canonically isomorphic to the conormal space of the graph of m:

$$\mathrm{Gr}(m_\Gamma) \ni (\gamma, \xi, \gamma_1, \xi_1, \gamma_2, \xi_2) \longrightarrow (\gamma, -\xi, \gamma_1, \xi_1, \gamma_2, \xi_2) \in N^*\mathrm{Gr}(m). \tag{A.16}$$

Since $N^*\mathrm{Gr}(m)$ is Lagrangian in $T^*G \times T^*G \times T^*G$, we get that $\mathrm{Gr}(m_\Gamma)$ is Lagrangian in $(-T^*G) \times T^*G \times T^*G$, that is, Γ is a symplectic groupoid.

Finally, we recall that T^*G is also a vector bundle over G, and we note $p : T^*G \to G$ the projection map. The following result is useful and obvious from the construction detailed above.

Proposition A.3. (1) *The subspace of composable pairs $\Gamma^{(2)}$ is a vector bundle over $G^{(2)}$ and $m_\Gamma : \Gamma^{(2)} \to \Gamma$ is a vector bundle homomorphism:*

$$\begin{array}{ccc} \Gamma^{(2)} & \xrightarrow{m_\Gamma} & \Gamma \\ \downarrow{\scriptstyle (p,p)} & & \downarrow{\scriptstyle p} \\ G^{(2)} & \xrightarrow{m} & G \end{array} \tag{A.17}$$

whose kernel is the conormal space of $G^{(2)}$ into G^2: $\ker m_\Gamma = N^*G^{(2)}$.

(2) *The maps $r_\Gamma, s_\Gamma : \Gamma \to A^*G$ are also vector bundle homomorphisms:*

$$\begin{array}{ccc} \Gamma & \xrightarrow{s_\Gamma} & A^*G \\ \downarrow{\scriptstyle p} & & \downarrow{\scriptstyle p} \\ G & \xrightarrow{s} & G^{(0)} \end{array} \qquad \begin{array}{ccc} \Gamma & \xrightarrow{r_\Gamma} & A^*G \\ \downarrow{\scriptstyle p} & & \downarrow{\scriptstyle p} \\ G & \xrightarrow{r} & G^{(0)} \end{array} \tag{A.18}$$

and $\ker r_\Gamma = (\ker ds)^\perp$*,* $\ker s_\Gamma = (\ker dr)^\perp$.

We finish this review with two basic examples, the first one being the historical one [4].

Example A.4. Let G be a Lie group with Lie algebra $\mathfrak{g}$. We have immediately

$$s_\Gamma(g,\xi) = L_g^*\xi \in \mathfrak{g}^* \quad \text{and} \quad r_\Gamma(g,\xi) = R_g^*\xi \in \mathfrak{g}^*. \tag{A.19}$$

When $s_\Gamma(g_1,\xi_1) = r_\Gamma(g_2,\xi_2)$, we get $(g_1,\xi_1)(g_2,\xi_2) = (g_1g_2,\xi)$ with ξ characterized by:

$$\xi(dm_{(g_1,g_2)}(t_1,t_2)) = \xi_1(t_1) + \xi_2(t_2). \tag{A.20}$$

Since $dm_{(g_1,g_2)}(t_1,t_2) = dR_{g_2}(t_1) + dL_{g_1}(t_2)$, we obtain $\xi = R^*_{g_2^{-1}}\xi_1 = L^*_{g_1^{-1}}\xi_2$. Thus

$$(g_1,\xi_1)(g_2,\xi_2) = (g_1g_2, R^*_{g_2^{-1}}\xi_1) \quad \text{when } L^*_{g_1}\xi_1 = R^*_{g_2}\xi_2. \tag{A.21}$$

On the other hand, we recall that G acts on $\mathfrak{g}^*$ by

$$\mathrm{Ad}^*_g.\xi = L^*_g R^*_{g^{-1}}\xi. \tag{A.22}$$

This gives rise to the transformation groupoid $G \ltimes \mathfrak{g}^* \rightrightarrows \mathfrak{g}^*$ whose source, target, multiplication and inversion are thus given by

$$\begin{gathered} s(g,\xi) = \mathrm{Ad}^*_g.\xi, \quad r(g,\xi) = \xi, \\ (g_1,\xi_1)(g_2,\mathrm{Ad}^*_{g_1}.\xi_1) = (g_1g_2,\xi_1), \quad (g,\xi)^{-1} = (g^{-1},\mathrm{Ad}^*_g.\xi). \end{gathered} \tag{A.23}$$

Now, the vector bundle trivialization $\Phi : T^*G \longrightarrow G \times \mathfrak{g}^*$, $(g,\xi) \longmapsto (g, R^*_g\xi)$, gives a Lie groupoid isomorphism $\Phi : T^*G \longrightarrow G \ltimes \mathfrak{g}^*$. For instance, we check

$$\begin{aligned} \Phi((g_1,\xi_1)(g_2,\xi_2)) &= \Phi(g_1g_2, R^*_{g_2^{-1}}\xi_1) \\ &= (g_1g_2, R^*_{g_1g_2}R^*_{g_2^{-1}}\xi_1) \\ &= (g_1g_2, R^*_{g_1}\xi_1) \\ &= (g_1, R^*_{g_1}\xi_1).(g_2, R^*_{g_2}\xi_2) \text{ [since } \mathrm{Ad}^*_{g_1}.R^*_{g_1}\xi_1 = L^*_{g_1}\xi_1 = R^*_{g_2}\xi_2] \\ &= \Phi(g_1,\xi_1).\Phi(g_2,\xi_2). \end{aligned}$$

Example A.5. We take $G = X \times X \times Z \rightrightarrows X \times Z$ (cartesian product of the pair groupoid $X \times X$ with the space Z). Here we have

$$\Gamma^{(0)} = A^*G = \{(x,x,z,\xi,-\xi,0) \,;\, (x,\xi) \in T^*X, z \in Z\}.$$

Let $\gamma = (x,y,z)$ and $\xi = (\zeta,\eta,\sigma) \in T^*_\gamma G$. Then $\overline{s}(\xi) \in T^*_{(y,y,z)}X \times X \times Z$ is given by $\eta \in T^*_y X \simeq 0 \times T^*_y X \times 0$ after extension by 0 onto the subspace of vectors of the form (u,u,w). This is similar for $\overline{r}(\xi) \in T^*_{(x,x,z)}X \times X \times Z$, starting with $\zeta \in T^*_x X \simeq T^*_x X \times 0 \times 0$. Using

$$(u,v,w) = (u-v,0,0) + (v,v,w) = (0,v-u,0) + (u,u,w),$$

we get

$$s_\Gamma(x,y,z,\xi,\eta,\sigma) = (y,y,z,-\eta,\eta,0), \quad r_\Gamma(x,y,z,\xi,\eta,\sigma) = (x,x,z,\xi,-\xi,0)$$

and

$$(x,y,z,\xi,\eta,\sigma).(y,x',z,-\eta,\xi',\sigma') = (x,x',z,\xi,\xi',\sigma+\sigma'). \tag{A.24}$$

Note that if $Z = \{\mathrm{pt}\}$, $\Gamma = T^*(X \times X)$ is isomorphic to the pair groupoid $T^*X \times T^*X$, with isomorphism given by

$$T^*(X\times X) \longrightarrow T^*X \times T^*X \ ; \ (x,y,\zeta,\eta) \mapsto (x,\zeta,y,-\eta).$$

References

[1] C. Brouder, N. V. Dang, and F. Hélein, Continuity of the fundamental operations on distributions having a specified wave front set (with a counter example by Semyon Alesker), *Studia Math.*, **232** (2016), no. 3, 201–226. Zbl 06586860 MR 3499105

[2] A. Connes, Sur la théorie non commutative de l'intégration, in *Algèbres d'opérateurs (Sém., Les Plans-sur-Bex, 1978)*, 19–143, Lecture Notes in Math., 725, Springer, Berlin, 1979. Zbl 0412.46053 MR 548112

[3] A. Connes, *Noncommutative Geometry*, Academic Press, San Diego, CA, 1994. Zbl 0818.46076 MR 1303779

[4] A. Coste, P. Dazord, and A. Weinstein, Groupoïdes symplectiques (French), in *Publications du Dep. de Maths. Nouvelle Série. A, Vol. 2, i–ii*, 1–62, Publ. Dép. Math. Nouvelle Sér. A, 87-2, Univ. Claude-Bernard, Lyon, 1987. Zbl 0668.58017 MR 996653

[5] C. Debord, J.-M. Lescure, and F. Rochon, Pseudodifferential operators on manifolds with fibred corners, *Ann. Inst. Fourier*, **65** (2015), no. 3, 1799–1880. Zbl 06541668 MR 3449197

[6] C. Debord and G. Skandalis, Adiabatic groupoid, crossed product by $\mathbb{R}_+^*$ and pseudodifferential calculus, *Adv. Math.*, **257** (2014), 66–91. Zbl 1300.58007 MR 3187645

[7] C. Debord, Holonomy groupoids of singular foliations, *J. Differential Geom.*, **58** (2001), no. 3, 467–500. Zbl 1034.58017 MR 1906783

[8] C. Debord and J.-M. Lescure, K-duality for stratified pseudomanifolds, *Geom. Topol.*, **13** (2009), no. 1, 49–86. Zbl 1159.19303 MR 2469513

[9] C. Debord, J.-M. Lescure, and V. Nistor, Groupoids and an index theorem for conical pseudo-manifolds, *J. Reine Angew. Math.*, **628** (2009), 1–35. Zbl 1169.58005 MR 2503234

[10] E. Van Erp and R. Yuncken, A groupoid approach to pseudodifferential operators, 2015. arXiv:1511.01041

[11] V. Guillemin and S. Sternberg, *Geometric asymptotics*, Mathematical Surveys, 14, American Mathematical Society, 1977. Zbl 0364.53011 MR 516965

[12] L. Hörmander, Fourier integral operators. I, *Acta Math.*, **127** (1971), no. 1-2, 79–183. Zbl 0212.46601 MR 0388463

[13] L. Hörmander, *The analysis of linear partial differential operators. I. Distribution theory and Fourier analysis*, second edition, Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], 256, Springer-Verlag, Berlin, 1990. Zbl 0712.35001 MR 1065136

[14] L. Hörmander, *The analysis of linear partial differential operators. IV. Fourier integral operators*, corrected reprint of the 1985 original, Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], 275, Springer-Verlag, Berlin, 1994. Zbl 0612.35001 MR 1481433

[15] G. Skandalis and I. Androulidakis, Pseudodifferential calculus on a singular foliation, *Journal of Noncommutative Geometry*, **5** (2011), 125–152. Zbl 1216.53029 MR 2746653

[16] K. Mackenzie, *General theory of Lie groupoids and Lie algebroids*, London Mathematical Society Lecture Note Series, 213, Cambridge University Press, Cambridge, 2005. Zbl 1078.58011 MR 2157566

[17] B. Monthubert, *Groupoïdes et calcul pseudo-différentiel sur les variétés à coins*, Ph. D. thesis, Université Paris VII-Denis Diderot, 1998.

[18] B. Monthubert, Pseudodifferential calculus on manifolds with corners and groupoids, *Proc. Amer. Math. Soc.*, **127** (1999), no. 10, 2871–2881. Zbl 0939.35202 MR 1600121

[19] B. Monthubert and F. Pierrot, Indice analytique et groupoïde de Lie, *C.R.A.S Série 1*, **325** (1997), 193–198. Zbl 0955.22004 MR 1467076

[20] V. Nistor, A. Weinstein, and P. Xu, Pseudodifferential operators on differential groupoids, *Pacific J. of Math.*, **181** (1999), no. 1, 117–152. Zbl 0940.58014 MR 1687747

[21] P. Pérez Carreras and J. Bonet, *Barrelled locally convex spaces*, North-Holland Mathematics Studies, 131, North-Holland Publishing Co., Amsterdam, 1987. Zbl 0614.46001 MR 880207

[22] J. Pradines, How to define the differentiable graph of a singular foliation, *C. de Top. et Geom. Diff. Cat.*, **XXVI** (1986), no. 4, 339–381. Zbl 0576.57023 MR 816646

[23] J. Pradines, Remarque sur le groupoïde cotangent de Weinstein–Dazord, *C.R. Acad Sc. Paris Série I*, **306** (1988), 557–560. Zbl 0659.18009 MR 941624

[24] F. Trèves, *Topological vector spaces, distributions and kernels*, Academic Press, New York-London, 1967. Zbl 0171.10402 MR 225131

[25] E. van Erp, The Atiyah–Singer index formula for subelliptic operators on contact manifolds. Part I, *Ann. of Math. (2)*, **171** (2010), no. 3, 1647–1681. Zbl 1206.19004 MR 2680395

[26] E. van Erp, The Atiyah–Singer index formula for subelliptic operators on contact manifolds. Part II, *Ann. of Math. (2)*, **171** (2010), no. 3, 1683–1706. Zbl 1206.19005 MR 2680396

[27] S. Vassout, Unbounded pseudodifferential calculus on Lie groupoids, *J. Funct. Anal.*, **236** (2006), no. 1, 161–200. Zbl 1105.58014 MR 2227132

[28] H. E. Winkelnkemper, The graph of a foliation, *Ann. Global Anal. Geom.*, **1** (1983), no. 3, 51–75. Zbl 0526.53039 MR 0739904

Received 04 November, 2015

J.-M. Lescure, Laboratoire de Mathématiques, UMR6620, Université Blaise Pascal, Clermont-Ferrand, 24, Avenue des Landais, 63177 Aubière cedex, France
E-mail: lescure@math.univ-bpclermont.fr

D. Manchon, Laboratoire de Mathématiques, UMR6620, Université Blaise Pascal, Clermont-Ferrand, 24, Avenue des Landais, 63177 Aubière cedex, France
E-mail: manchon@math.univ-bpclermont.fr

S. Vassout, Institut de Mathématiques de Jussieu - Paris Rive Gauche (IMJ-PRG), UP7D - Campus des Grands Moulins, Bâtiment Sophie Germain,
Boite Courrier 7012 - 75205 Paris cedex 13, France
E-mail: stephane.vassout@imj-prg.fr