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Cohomology of $\mathcal{A}_\theta^{\mathrm{alg}} \rtimes \mathbb{Z}_2$ and its Chern–Connes pairing

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Abstract. We calculate the Hochschild and cyclic cohomology of the noncommutative $\mathbb{Z}_2$ toroidal algebraic orbifold $\mathcal{A}_\theta^{\mathrm{alg}} \rtimes \mathbb{Z}_2$. We also calculate the Chern–Connes pairing of the even periodic cyclic cocycles with the known elements of $K_0(\mathcal{A}_\theta^{\mathrm{alg}} \rtimes \mathbb{Z}_2)$.

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1. Introduction and statement

In the classic paper [4], Connes constructed a noncommutative analogue of the Chern map of differential geometry. He considered the map from the K_0 group of a noncommutative algebra to its even cyclic homology and paired a projection with the cyclic cocycle to give a numerical invariant.

Let $\mathcal{S}(\mathbb{Z}^2)$ be the Schwartz space on $\mathbb{Z}^2$, consisting of all complex sequences $a_{n,m}$ satisfying:

$$\sup_{(n,m)\in\mathbb{Z}^2} \big(|n| + |m|\big)^q |a_{n,m}| < \infty, \quad \text{for all } q \in \mathbb{N}.$$

For given $\theta \in \mathbb{R}$, we associate the algebra $\mathcal{A}_\theta$ defined below.

$$\mathcal{A}_\theta = \Big\{ a = \sum_{(n,m)\in\mathbb{Z}^2} a_{n,m} U_1^n U_2^m \mid a_{n,m} \in \mathcal{S}(\mathbb{Z}^2) \Big\},$$

where U_1 and U_2 are unitary generators satisfying $U_2 U_1 = \lambda U_1 U_2$, $\lambda = e^{2\pi i\theta}$. Connes [4] computed the cyclic cohomology and Chern–Connes index for the smooth algebra $\mathcal{A}_\theta$. The group $SL(2,\mathbb{Z})$ has the following action on $\mathcal{A}_\theta$. An element

$$g = \begin{bmatrix} g_{1,1} & g_{1,2} \\ g_{2,1} & g_{2,2} \end{bmatrix} \in SL(2,\mathbb{Z})$$

acts on the generators U_1 and U_2 as described below:

$$g \cdot U_1 = e^{(\pi i g_{1,1} g_{2,1})\theta} U_1^{g_{1,1}} U_2^{g_{2,1}} \quad \text{and} \quad g \cdot U_2 = e^{(\pi i g_{1,2} g_{2,2})\theta} U_1^{g_{1,2}} U_2^{g_{2,2}}.$$

Let $\mathcal{A}_\theta^{\text{alg}}$ consists of all finitely supported elements of $\mathcal{A}_\theta$. We shall study the crossed product of the subalgebra $\mathcal{A}_\theta^{\text{alg}}$ with the group $\mathbb{Z}_2$ identified as a subgroup of $SL(2, \mathbb{Z})$.

Berest et al. [3] calculated the Picard group of $\mathcal{A}_\theta^{\text{alg}}$. Some of the Hochschild homology groups of $\mathcal{A}_\theta^{\text{alg}} \rtimes \mathbb{Z}_k$ for $k = 2, 3, 4$ and 6 were known for many years [2, 10]. All the Hochschild and cylic homology groups of the orbifolds $\mathcal{A}_\theta^{\text{alg}} \rtimes \mathbb{Z}_k$, for $k = 2, 3, 4$ and 6, were recently calculated [12]. Further the Hochschild homology of the Weyl algebra was studied by Alev and Lambre in [1]. In this article we shall compute the Hochschild and cyclic cohomology of $\mathcal{A}_\theta^{\text{alg}} \rtimes \mathbb{Z}_2$, the $\mathbb{Z}_2$ noncommutative algebraic toroidal orbifold. We also compute the Chern–Connes index for this orbifold by pairing these cocycles with the algebraic projections of the group $K_0(\mathcal{A}_\theta \rtimes \mathbb{Z}_2)$, which was calculated in [5]. In this article we adopt the notation from [4] and [12] and prove the following results.

Theorem 1.1. *If $\theta \notin \mathbb{Q}$, then the Hochschild cohomology groups of $\mathcal{A}_\theta^{\text{alg}} \rtimes \mathbb{Z}_2$ are:*

$$H^0(\mathcal{A}_\theta^{\text{alg}} \rtimes \mathbb{Z}_2, (\mathcal{A}_\theta^{\text{alg}} \rtimes \mathbb{Z}_2)^*) \cong \mathbb{C}^5,$$
$$H^1(\mathcal{A}_\theta^{\text{alg}} \rtimes \mathbb{Z}_2, (\mathcal{A}_\theta^{\text{alg}} \rtimes \mathbb{Z}_2)^*) = 0,$$
and
$$H^2(\mathcal{A}_\theta^{\text{alg}} \rtimes \mathbb{Z}_2, (\mathcal{A}_\theta^{\text{alg}} \rtimes \mathbb{Z}_2)^*) \cong \mathbb{C}.$$

Theorem 1.2. $HP^{\text{even}}(\mathcal{A}_\theta^{\text{alg}} \rtimes \mathbb{Z}_2) \cong \mathbb{C}^6$ *and* $HP^{\text{odd}}(\mathcal{A}_\theta^{\text{alg}} \rtimes \mathbb{Z}_2) = 0.$

Theorem 1.3. *The following is the description of the Chern–Connes pairing of the six dimensional group $HP^{\text{even}}(\mathcal{A}_\theta^{\text{alg}} \rtimes \mathbb{Z}_2)$ generated by cocycles $S\tau$, $S\mathfrak{D}_{1,1}$, $S\mathfrak{D}_{0,0}$, $S\mathfrak{D}_{0,1}$, $S\mathfrak{D}_{1,0}$ and φ, with the five known independent projections of $\mathcal{A}_\theta^{\text{alg}} \rtimes \mathbb{Z}_2$ namely 1, p^θ, q_1^θ, q_2^θ and r^θ [5].*

	$S\tau$	$S\mathfrak{D}_{1,1}$	$S\mathfrak{D}_{0,0}$	$S\mathfrak{D}_{0,1}$	$S\mathfrak{D}_{1,0}$	$S\varphi$
1	1	0	0	0	0	0
p^θ	$\frac{1}{2}$	$\frac{1}{2}$	0	0	0	0
q_1^θ	$\frac{1}{2}$	0	0	$-\frac{1}{2}$	0	0
q_2^θ	$\frac{1}{2}$	0	$-\frac{1}{2}$	0	0	0
r^θ	$\frac{1}{2}$	0	0	0	$-\frac{\lambda}{2}$	0

We end the article with a conjecture over the dimension of the unknown group $K_0(\mathcal{A}_\theta^{\text{alg}} \rtimes \mathbb{Z}_2)$.

2. Hochschild cohomology of $\mathcal{A}_\theta^{\mathrm{alg}} \rtimes \mathbb{Z}_2$

We note that the dual of the algebraic noncommutative torus $\mathcal{A}_\theta^{\mathrm{alg}}$ is

$$\mathcal{A}_\theta^{\mathrm{alg}\,*} = \Big\{ a \mid a = \sum_{(n,m)\in\mathbb{Z}^2} a_{n,m} U_1^n U_2^m \Big\},$$

where U_1 and U_2 are unitaries satisfying $U_2U_1 = \lambda U_1 U_2$. For $a \in \mathcal{A}_\theta^{\mathrm{alg}}$, let the trace τ on the algebra $\mathcal{A}_\theta^{\mathrm{alg}}$ be defined as

$$\tau(a) = a_{0,0}.$$

Then an element $a \in \mathcal{A}_\theta^{\mathrm{alg}\,*}$ acts on $b \in \mathcal{A}_\theta^{\mathrm{alg}}$ as $a(b) = \tau(ab)$. Using the results of Getzler and John [7], the cohomology group $H^\bullet(\mathcal{A}_\theta^{\mathrm{alg}} \rtimes \mathbb{Z}_2, (\mathcal{A}_\theta^{\mathrm{alg}} \rtimes \mathbb{Z}_2)^*)$ has the following decomposition:

$$\begin{aligned} H^\bullet\big(\mathcal{A}_\theta^{\mathrm{alg}} \rtimes \mathbb{Z}_2, \big(\mathcal{A}_\theta^{\mathrm{alg}} \rtimes \mathbb{Z}_2\big)^*\big) &= \bigoplus_{g\in\mathbb{Z}_2} H^\bullet\big(\mathcal{A}_\theta^{\mathrm{alg}}, {}_g\mathcal{A}_\theta^{\mathrm{alg}\,*}\big)^{\mathbb{Z}_2} \\ &= H^\bullet\big(\mathcal{A}_\theta^{\mathrm{alg}}, \mathcal{A}_\theta^{\mathrm{alg}\,*}\big)^{\mathbb{Z}_2} \bigoplus H^\bullet\big(\mathcal{A}_\theta^{\mathrm{alg}}, {}_{-1}\mathcal{A}_\theta^{\mathrm{alg}\,*}\big)^{\mathbb{Z}_2}. \end{aligned}$$

In the above equation, ${}_{-1}\mathcal{A}_\theta^{\mathrm{alg}\,*}$ consists of elements of $\mathcal{A}_\theta^{\mathrm{alg}\,*}$ with the following twisted $\mathcal{A}_\theta^{\mathrm{alg}}$ bimodule structure. For $a \in {}_{-1}\mathcal{A}_\theta^{\mathrm{alg}\,*}$ and $\alpha \in \mathcal{A}_\theta^{\mathrm{alg}}$,

$$\alpha \cdot a = (-1\cdot\alpha)a \quad \text{and} \quad a \cdot \alpha = a\alpha,$$

where $a\alpha$ is the product of a and α in $\mathcal{A}_\theta^{\mathrm{alg}\,*}$. We recall the modified Connes projective resolution:

$$\mathcal{A}_\theta^{\mathrm{alg}} \xleftarrow{\epsilon} \mathcal{B}_\theta^{\mathrm{alg}} \xleftarrow{b_1} \mathcal{B}_\theta^{\mathrm{alg}} \bigoplus \mathcal{B}_\theta^{\mathrm{alg}} \xleftarrow{b_2} \mathcal{B}_\theta^{\mathrm{alg}},$$

where

$$\begin{gathered} \mathcal{B}_\theta^{\mathrm{alg}} = \mathcal{A}_\theta^{\mathrm{alg}} \otimes (\mathcal{A}_\theta^{\mathrm{alg}})^{\mathrm{op}}, \\ \epsilon(a\otimes b) = ab, \\ b_1(1\otimes e_j) = 1\otimes U_j - U_j \otimes 1, \\ b_2(1\otimes(e_1\wedge e_2)) = (U_2\otimes 1 - \lambda\otimes U_2)\otimes e_1 - (\lambda U_1 \otimes 1 - 1\otimes U_1)\otimes e_2. \end{gathered}$$

The above resolution was used in [12] to calculate the Hochschild and cyclic homology groups of the algebra $\mathcal{A}_\theta^{\mathrm{alg}} \rtimes \mathbb{Z}_k$ for $k = 2, 3, 4$ and 6. We use it to construct the twisted cochain complex corresponding to each of the two elelments of the group $\mathbb{Z}_2$. Thereafter we compute the cohomology groups of $\mathcal{A}_\theta^{\mathrm{alg}} \rtimes \mathbb{Z}_2$ by locating the $\mathbb{Z}_2$ invariant cocycles.

While the bar resolution is not computationally convenient, modified Connes resolution for the algebraic noncommutative torus does make it easier to compute the cohomology groups. In order to locate the $\mathbb{Z}_2$ invariant cocycles of $H^\bullet(\mathcal{A}_\theta^{\text{alg}}, {}_g\mathcal{A}_\theta^{\text{alg}\,*})$, we need to use the resolution homotopy maps

$$h_* : C_*(\mathcal{A}_\theta^{\text{alg}}) \to J_*(\mathcal{A}_\theta^{\text{alg}})$$

and

$$k_* : J_*(\mathcal{A}_\theta^{\text{alg}}) \to C_*(\mathcal{A}_\theta^{\text{alg}}),$$

where $J_*(\mathcal{A}_\theta^{\text{alg}})$ is the standard bar resolution ($J_k(\mathcal{A}_\theta^{\text{alg}}) = \mathcal{B}_\theta^{\text{alg}} \otimes (\mathcal{A}_\theta^{\text{alg}})^{\otimes k}$) and $C_*(\mathcal{A}_\theta^{\text{alg}})$ is the Connes resolution. We push a cocylcle $\mathcal{D}$ into the bar complex and let $\mathbb{Z}_2$ act on it. Then, in the Connes complex, we compare the pullback of this $\mathbb{Z}_2$-acted cocycle with $\mathcal{D}$ to check the $\mathbb{Z}_2$ invariance. These maps were explicitly calculated in [4] and [12].

It is worthwhile to note that $\operatorname{Hom}_{\mathcal{B}_\theta^{\text{alg}}}(\mathcal{B}_\theta^{\text{alg}}, {}_{-1}\mathcal{A}_\theta^{\text{alg}\,*})$ and $\operatorname{Hom}_{\mathcal{B}_\theta^{\text{alg}}}(\mathcal{B}_\theta^{\text{alg}}, \mathcal{A}_\theta^{\text{alg}\,*})$ can be identified with ${}_{-1}\mathcal{A}_\theta^{\text{alg}\,*}$ and $\mathcal{A}_\theta^{\text{alg}\,*}$, respectively. Hence for $g = -1$ we have the following Hochschild cohomology complex:

$${}_{-1}\mathcal{A}_\theta^{\text{alg}\,*} \xrightarrow{{}_{-1}\alpha_1} {}_{-1}\mathcal{A}_\theta^{\text{alg}\,*} \oplus {}_{-1}\mathcal{A}_\theta^{\text{alg}\,*} \xrightarrow{{}_{-1}\alpha_2} {}_{-1}\mathcal{A}_\theta^{\text{alg}\,*} \to 0,$$

where for φ, φ_1 and φ_2 in $\mathcal{A}_\theta^{\text{alg}\,*}$, the maps ${}_{-1}\alpha_1$ and ${}_{-1}\alpha_2$ are as follows:

$$\begin{aligned} {}_{-1}\alpha_1(\varphi) &= (U_1^{-1}\varphi - \varphi U_1, U_2^{-1}\varphi - \varphi U_2), \\ {}_{-1}\alpha_2(\varphi_1, \varphi_2) &= U_2^{-1}\varphi_1 - \lambda\varphi_1 U_2 - \lambda U_1^{-1}\varphi_2 + \varphi_2 U_1. \end{aligned}$$

Lemma 2.1. $H^0(\mathcal{A}_\theta^{\text{alg}}, {}_{-1}\mathcal{A}_\theta^{\text{alg}\,*})^{\mathbb{Z}_2} \cong \mathbb{C}^4$.

Proof. Let $\varphi = \sum \varphi_{n,m} U_1^n U_2^m$ be an element of ${}_{-1}\mathcal{A}_\theta^{\text{alg}\,*}$. Then φ is a 0-cocycle if and only if ${}_{-1}\alpha_1(\varphi) = 0$, which implies that $U_1^{-1}\varphi - \varphi U_1 = U_2^{-1}\varphi - \varphi U_2 = 0$. This further gives the relation $\varphi_{n+1,m} = \lambda^m \varphi_{n-1,m} = \lambda^{m+n-1}\varphi_{n-1,m-2}$ on its coefficients. Hence we see that

$$H^0(\mathcal{A}_\theta^{\text{alg}}, {}_{-1}\mathcal{A}_\theta^{\text{alg}\,*}) \cong \mathbb{C}^4.$$

The generators of this group are the cocycles generated by $\varphi_{0,0}, \varphi_{0,1}, \varphi_{1,0}$ and $\varphi_{1,1}$.

Let us denote by $\mathcal{D}_{i,j}$ the cocycle generated by $\varphi_{i,j}$, for $0 \le i, j \le 1$. First consider the cocycle $\mathcal{D}_{0,0}$. The above relation on the coefficients of $\mathcal{D}_{0,0}$ gives $\varphi_{2n,2m} = \lambda^{2mn}\varphi_{0,0}$ for all $(2n, 2m) \in \mathbb{Z}^2$. The maps k_0 and h_0 are idenity and hence the action of $\mathbb{Z}_2$ on $H^0(\mathcal{A}_\theta^{\text{alg}}, {}_{-1}\mathcal{A}_\theta^{\text{alg}\,*})$ is given by $U_j \to U_j^{-1}$ for $j = 1, 2$. Thus we conclude that the $\mathbb{Z}_2$ action leaves $\mathcal{D}_{0,0}$ invariant.

For the cocycle $\mathcal{D}_{0,1}$, we infer that $\varphi_{2k,2l+1} = \lambda^{2kl+k}\varphi_{0,1}$. Now from

$$\varphi_{-2k,-2l-1} = \varphi_{2(-k),2(-l-1)+1} = \lambda^{2(-k)(-l-1)+(-k)}\varphi_{0,1} = \lambda^{2kl+k}\varphi_{0,1}$$

it follows that $\mathcal{D}_{0,1}$ is a $\mathbb{Z}_2$ invariant element of $H^0(\mathcal{A}_\theta^{\text{alg}}, {}_{-1}\mathcal{A}_\theta^{\text{alg}\,*})$. In the case of $\mathcal{D}_{1,0}$, its coeffcients satisfy the relation $\varphi_{2k+1,2l} = \lambda^{2kl+l}\varphi_{1,0}$. Since

$$\varphi_{-2k-1,-2l} = \varphi_{2(-k-1)+1,2(-l)} = \lambda^{2(-k-1)(-l)+(-l)}\varphi_{1,0} = \lambda^{2kl+l}\varphi_{1,0},$$

the cocycle $\mathcal{D}_{1,0} \in H^0(\mathcal{A}_\theta^{\text{alg}}, {}_{-1}\mathcal{A}_\theta^{\text{alg}\,*})^{\mathbb{Z}_2}$. Finally for $\mathcal{D}_{1,1}$, we have

$$\varphi_{2k+1,2l+1} = \lambda^{2kl+k+l}\varphi_{1,1} \quad \text{and} \quad \varphi_{-2k-1,-2l-1} = \lambda^{2kl+k+l}\varphi_{1,1}.$$

Hence $\mathcal{D}_{1,1} \in H^0(\mathcal{A}_\theta^{\text{alg}}, {}_{-1}\mathcal{A}_\theta^{\text{alg}\,*})^{\mathbb{Z}_2}$. □

Lemma 2.2. $H^2(\mathcal{A}_\theta^{\text{alg}}, {}_{-1}\mathcal{A}_\theta^{\text{alg}\,*})^{\mathbb{Z}_2} = 0$.

Proof. Let $\varphi \in \mathcal{A}_\theta^{\text{alg}\,*}$ and let $\widetilde{\varphi}$ be the corresponding element of $\text{Hom}\mathcal{B}_\theta^{\text{alg}}(J_2, \mathcal{A}_\theta^{\text{alg}\,*})$. Then

$$\widetilde{\varphi}(a \otimes b \otimes e_1 \wedge e_2)(x) = \varphi((-1 \cdot b)xa),$$

for all $a, b, x \in \mathcal{A}_\theta^{\text{alg}}$. Let $\psi = k_2^*\widetilde{\varphi} = \widetilde{\varphi} \circ k_2$. We have

$$\psi(x_0, x_1, x_2) = \widetilde{\varphi}(k_2(I \otimes x_1 \otimes x_2))(x_0),$$

for all $x_0, x_1, x_2 \in \mathcal{A}_\theta^{\text{alg}}$. The group $\mathbb{Z}_2$ acts on $\mathcal{A}_\theta^{\text{alg}}$ in the bar complex as

$$-1 \cdot \chi(x_0, x_1, x_2) = \chi(-1 \cdot x_0, -1 \cdot x_1, -1 \cdot x_2).$$

Further we pull the map ${}_{-1}\psi = -1 \cdot \psi$ back on to the Connes complex via the map h_2^*. Let $w = h_2^*({}_{-1}\psi)$ denote the pullback of ${}_{-1}\psi$ on the Connes complex. We have

$$\begin{aligned} w(x_0) &= {}_{-1}\psi(x_0, U_2, U_1) - \lambda_{-1}\psi(x_0, U_1, U_2) \\ &= \psi(-1 \cdot x_0, U_2^{-1}, U_1^{-1}) - \lambda\psi(-1 \cdot x_0, U_1^{-1}, U_2^{-1}) \\ &= \widetilde{\varphi}\big(k_2(I \otimes U_2^{-1} \otimes U_1^{-1})\big)(-1 \cdot x_0) - \lambda\widetilde{\varphi}\big(k_2(I \otimes U_1^{-1} \otimes U_2^{-1})\big)(-1 \cdot x_0). \end{aligned}$$

Following the calculations from [12, Section 6], we have

$$k_2(I \otimes U_2^{-1} \otimes U_1^{-1}) - \lambda k_2(I \otimes U_1^{-1} \otimes U_2^{-1}) = (U_1^{-1}U_2^{-1} \otimes U_1^{-1}U_2^{-1}).$$

Applying this we conclude that

$$\begin{aligned} &\widetilde{\varphi}(k_2(I \otimes U_2^{-1} \otimes U_1^{-1}))(-1 \cdot x_0) - \lambda\widetilde{\varphi}(k_2(I \otimes U_1^{-1} \otimes U_2^{-1}))(-1 \cdot x_0) \\ &\qquad = \widetilde{\varphi}(U_1^{-1}U_2^{-1} \otimes U_1^{-1}U_2^{-1})(-1 \cdot x_0) = \varphi(U_1U_2 \cdot (-1 \cdot x_0) \cdot U_1^{-1}U_2^{-1}). \end{aligned}$$

Hence we need to compare $\varphi(x)$ with $\varphi(U_1U_2 \cdot (-1 \cdot x) \cdot U_1^{-1}U_2^{-1})$. Using the Connes complex, we see that $H^2(\mathcal{A}_\theta^{\text{alg}}, {}_{-1}\mathcal{A}_\theta^{\text{alg}\,*})^{\mathbb{Z}_2} = {}_{-1}\mathcal{A}_\theta^{\text{alg}\,*}/\operatorname{Im}({}_{-1}\alpha_2)$. Since ${}_{-1}\alpha_2(U_2, 0) = 1 - \lambda U_2^2$ and ${}_{-1}\alpha_2(0, U_1) = U_1^2 - \lambda$, we have $H^2(\mathcal{A}_\theta^{\text{alg}}, {}_{-1}\mathcal{A}_\theta^{\text{alg}\,*}) \cong \mathbb{C}^4$ generated by the cocycles supported at $\varphi_{0,0}$, $\varphi_{1,0}$, $\varphi_{0.1}$ and $\varphi_{1,1}$.

Case 1. We check the invariance of $\varphi_{0,0}$. From

$$\varphi_{0,0}(U_1U_2 \cdot (-1 \cdot x) \cdot U_1^{-1}U_2^{-1}) = \lambda^{-1}x_{0,0} \quad \text{and} \quad \varphi_{0,0}(x) = x_{0,0},$$

we see that $\varphi_{0,0}$ is *not* invariant under the $\mathbb{Z}_2$ action.

Case 2. Observe that for $\varphi_{1,0}$, we have

$$\varphi_{1,0}(U_1U_2 \cdot (-1 \cdot x) \cdot U_1^{-1}U_2^{-1}) = x_{-1,0} \quad \text{and} \quad \varphi_{1,0}(x) = x_{1,0}.$$

Since the cocycle class $\varphi_{1,0}$ is equivalent to the class $\lambda\varphi_{-1,0}$, it is *not* invariant under the $\mathbb{Z}_2$ action.

Case 3. We check the invariance of $\varphi_{0,1}$. We have

$$\varphi_{0,1}(U_1U_2 \cdot (-1 \cdot x) \cdot U_1^{-1}U_2^{-1}) = x_{0,-1} \quad \text{and} \quad \varphi_{0,1}(x) = x_{0,1}.$$

Since the cocycle class $\varphi_{0,1}$ is equivalent to the class $\lambda^{-1}\varphi_{0,-1}$, it is *not* invariant under the $\mathbb{Z}_2$ action.

Case 4. Finally, we check the invariance of $\varphi_{1,1}$. We have

$$\varphi_{1,1}(U_1U_2 \cdot (-1 \cdot x) \cdot U_1^{-1}U_2^{-1}) = \lambda^{-1}x_{-1,-1} \quad \text{and} \quad \varphi_{1,1}(x) = x_{1,1}.$$

Since the cocycle class $\varphi_{1,1}$ is equivalent to the cocycle class $\varphi_{-1,-1}$, the cocycle is *not* invariant under the $\mathbb{Z}_2$ action.

For $\psi = a\varphi_{0,0} + b\varphi_{1,0} + c\varphi_{0,1} + d\varphi_{1,1}$, if Ψ is the pullback of the corresponding cocycle in the bar complex after the $\mathbb{Z}_2$ action, then:

$$\Psi = a\lambda^{-1}\varphi_{0,0} + b\lambda^{-1}\varphi_{1,0} + c\lambda^{-1}\varphi_{0,1} + d\lambda^{-1}\varphi_{1,1}.$$

We see that the coefficients of this pullback are different from those of the original cocycle. Therefore we conclude that

$$H^2(\mathcal{A}_\theta^{\text{alg}}, {}_{-1}\mathcal{A}_\theta^{\text{alg}\,*})^{\mathbb{Z}_2} = 0.$$

We remark in this computation that although $H^2(\mathcal{A}_\theta^{\text{alg}}, {}_{-1}\mathcal{A}_\theta^{\text{alg}\,*})$ is of 4 dimension, there is no nontrivial $\mathbb{Z}_2$ invariant cocycle. □

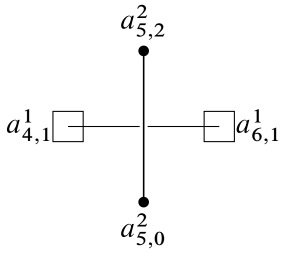
For $\varphi \in \mathcal{A}_\theta^{\text{alg}\,*} \oplus \mathcal{A}_\theta^{\text{alg}\,*}$, we define the diagram $\text{Dgm}(\varphi)$ $(\subset \mathbb{Z}^2 \oplus \mathbb{Z}^2 \oplus \mathbb{Z}^2)$ associated to it [12, Section 7]. Two elements $a, b \in \mathbb{C}$ indexed by the lattice $\mathbb{Z}^2$ are said to be f-connected and drawn on the lattice plane as

$$a \bullet\!\!-\!\!\!-\!\!\!-\!\!\!-\!\!\bullet\, b$$

if there exists $f \in \mathbb{C}[x, y, w, z]$ whose roots are a and b. For example, consider the following equation

$$a_{6,1}^1 - a_{5,0}^2 = \lambda a_{4,1}^1 - \lambda^{-5}a_{5,2}^2.$$

The corresponding diagram is as below, where the boxes represent elements of $a^1_{\bullet,\bullet}$ and the thick dots that of $a^2_{\bullet,\bullet}$.



In the above example we see that $f(x, y, w, z) = x - y + \lambda^{-5} z - \lambda w$ has its roots as $a^1_{4,1}, a^1_{6,1}, a^2_{5,0}$ and $a^2_{5,2}$. For $\varphi = (\varphi^1, \varphi^2) \in \mathcal{A}_\theta^{\mathrm{alg}\,*} \oplus \mathcal{A}_\theta^{\mathrm{alg}\,*}$, we use all the ${}_{-1}\alpha_1$ equations to ${}_{-1}\alpha_1$-connect the non-zero elements $(\varphi^1_{n,m}, \varphi^2_{r,s})$. We call this lattice graph as $\mathrm{Dgm}(\varphi)$.

We notice that, for a given lattice point (n, m), there are three possible values at that point. They are:

(1) $\varphi^1_{n,m}$,
(2) $\varphi^2_{n,m}$,
(3) 0.

Hence we conclude that the kernel diagram $\mathrm{Dgm}(\varphi)$ of φ is a subset of $\mathbb{Z}^2 \oplus \mathbb{Z}^2 \oplus \mathbb{Z}^2$. It can be easily figured out [12] that there are no edges to the graph $\mathrm{Dgm}(\varphi)$, and the graph is a disjoint union of closed graphs with no open edges. These graphs can be infinitely supported as $\mathcal{A}_\theta^{\mathrm{alg}\,*}$ consists of elements which are infinitely supported. For $1 \le i \le 3$, let the maps $\pi_i : \mathbb{Z}^2 \oplus \mathbb{Z}^2 \oplus \mathbb{Z}^2 \to \mathbb{Z}^2$ be the ith projection, projecting the diagram $\mathrm{Dgm}(\varphi)$ to the ith $\mathbb{Z}^2$. From now onwards we shall deal with the map π_1 and similar arguments will hold for π_2 and π_3.

Definition 2.3 (Lines). For $s_0 \in \mathbb{Z}$ and $\varphi(= (\varphi_1, \varphi_2)) \in \ker({}_{-1}\alpha_2)$, we define a $\mathbb{Z}^2$ lattice H_{s_0} such that

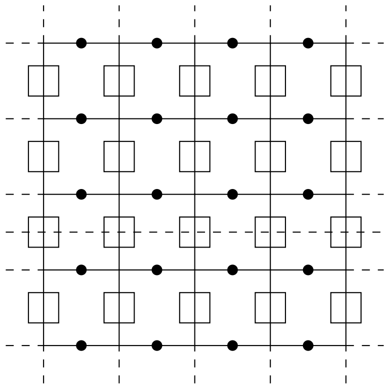
$$(H_{s_0})_{w,s} := \begin{cases} (\pi_1(\mathrm{Dgm}(\varphi)))_{w,s}, & \text{for } s = s_0, \\ 0, & \text{else.} \end{cases}$$

Lemma 2.4. *Given* $s_0 \in \mathbb{Z}$ *and* $\varphi \in \ker({}_{-1}\alpha_2)$*, there exists* $\gamma_{s_0} \in \mathcal{A}_\theta^{\mathrm{alg}\,*}$ *such that* $\pi_1(\mathrm{Dgm}({}_{-1}\alpha_1(\gamma_{s_0}))))_{w,s_0} = (H_{s_0})_{w,s_0}$ *for all* $w \in \mathbb{Z}$.

Proof. We know that ${}_{-1}\alpha_1(\varphi) = (U_1^{-1}\varphi - \varphi U_1, U_2^{-1}\varphi - \varphi U_2)$. If ${}_{-1}\alpha_1(\varphi) = (\varphi_1, \varphi_2)$, then

$$\varphi^1_{n,m} = \varphi_{n+1,m} - \lambda^m \varphi_{n-1,m} \quad \text{and} \quad \varphi^2_{n,m} = \lambda^{-n} \varphi_{n,m+1} - \varphi_{n,m-1}.$$

The diagram $\pi_1(\mathrm{Dgm}({}_{-1}\alpha_1(\varphi)))$ can be infinitely supported, a connected component of it resembles the one below:



Assume that $\varphi^1_{0,s_0} \neq 0$. It is clear from the diagram that in the row $y = s_0$ of the lattice $\pi_1(\mathrm{Dgm}(\varphi))$, $\varphi^2_{w,s_0} = 0$ for all $w \in \mathbb{Z}$. Define

$$(\gamma^{(1)}_{s_0})_{w,s} = \begin{cases} -\lambda^{-s_0}\varphi^1_{0,s}, & \text{for } (w,s) = (-1, s_0), \\ 0, & \text{else.} \end{cases}$$

We have $\pi_1(\mathrm{Dgm}({}_{-1}\alpha_1(\gamma^{(1)}_{s_0})))_{0,s_0} - (H_{s_0})_{0,s_0} = 0$. We define

$$(\gamma^{(2)}_{s_0})_{w,s} = \begin{cases} -\lambda^{-s_0}(\varphi^1_{-2,s} - \lambda^{-s_0}\varphi^1_{0,s}), & \text{for } (w,s) = (-3, s_0), \\ (\gamma^{(1)}_{s_0})_{w,s}, & \text{else.} \end{cases}$$

Then we have

$$\begin{aligned} \pi_1(\mathrm{Dgm}({}_{-1}\alpha_1(\gamma^{(2)}_{s_0})))_{-2,s_0} - (H_{s_0})_{-2,s_0} &= \pi_1(\mathrm{Dgm}({}_{-1}\alpha_1(\gamma^{(2)}_{s_0})))_{0,s_0} - (H_{s_0})_{0,s_0} \\ &= 0. \end{aligned}$$

Similarly we can construct a sequence $\gamma^{(n)}_{s_0}$ which satisfies the required condition for finitely many lattice points. Define $\gamma^{\leq}_{s_0} := \lim_{n\to\infty} \gamma^{(n)}_{s_0}$. Since $\gamma^{\leq}_{s_0} \in \mathcal{A}^{\mathrm{alg}\,*}_{\theta}$, we have

$$\pi_1\big(\mathrm{Dgm}({}_{-1}\alpha_1(\gamma^{\leq}_{s_0}))\big)_{\bullet,s_0} - (H_{s_0})_{\bullet,s_0} = 0 \qquad \text{for } \bullet \leq 0.$$

We can similarly define $\gamma^{>}_{s_0}$ such that

$$\pi_1\big(\mathrm{Dgm}(({}_{-1}\alpha_1(\gamma^{>}_{s_0})))\big)_{\bullet,s_0} - (H_{s_0})_{\bullet,s_0} = 0 \qquad \text{for } \bullet > 0.$$

Then $\gamma_{s_0} := \gamma^{\leq}_{s_0} + \gamma^{>}_{s_0}$ satisfies the following equation:

$$\pi_1\big(\mathrm{Dgm}(({}_{-1}\alpha_1(\gamma_{s_0})))\big)_{\bullet,s_0} - (H_{s_0})_{\bullet,s_0} = 0 \qquad \text{for } \bullet \in \mathbb{Z}.$$

This completes the proof. □

It is interesting to note the degree of freedom that we had while constructing the above γ_{s_0}. This can be traced back to the fact that the kernel of ${}_{-1}\alpha_1$ is a 4 dimensional vector space. As we shall prove that an arbitrary cocycle is a coboundary, it is worthwhile to note the various possibilities we have in doing so; hence revealing the nature of the map ${}_{-1}\alpha_1$.

Lemma 2.5. $H^1(\mathcal{A}_\theta^{\text{alg}}, {}_{-1}\mathcal{A}_\theta^{\text{alg}\,*}) = 0$.

Proof. Let φ belonging to $\ker({}_{-1}\alpha_2)$ be a 1-cochain in the Connes complex. Let us understand the construction of $\pi_1(\text{Dgm}(\varphi))$. It consists of alternate non-zero entries, meaning one considering a row/column will find zeros at alternate positions. It has rows/columns of φ^2's and φ^1's alternately placed.

For $s_0 \in \mathbb{Z}$, we get a $\gamma_{s_0} \in \mathcal{A}_\theta^{\text{alg}\,*}$ as in Lemma 2.4. Define $\gamma = \gamma_0 + \gamma_2 + \gamma_{-2} + \cdots \in \mathcal{A}_\theta^{\text{alg}\,*}$. We observe that the lattice

$$\pi_1\big(\text{Dgm}({}_{-1}\alpha_1(\gamma) - (\varphi))\big)$$

has zero rows placed alternately. These rows are precisely the rows of φ^1 in $\pi_1(\text{Dgm}(\varphi))$. The other alternate set of rows is the rows of the φ'^2's, where $\varphi'^2 \in \mathcal{A}_\theta^{\text{alg}\,*}$ are the bulleted points ($\bullet$) in the lattice diagram

$$\pi_1\big(\text{Dgm}({}_{-1}\alpha_1(\gamma) - (\varphi))\big).$$

We state that $\pi_1(\text{Dgm}({}_{-1}\alpha_1(\gamma) - (\varphi)))$ is the diagram of an image element, that is, there exists $\rho \in \mathcal{A}_\theta^{\text{alg}\,*}$ such that

$$\pi_1\big(\text{Dgm}({}_{-1}\alpha_1(\rho))\big) = \pi_1\big(\text{Dgm}({}_{-1}\alpha_1(\gamma) - (\varphi))\big).$$

It is easy to see as there is no kernel equation that relates $(\varphi_2')_{p,q}$ with $(\varphi_2')_{l,w}$ for $q \neq w$. Also note that if there is even a single zero entry in any of these rows, then the whole row is ought to be a *zero row*. This can be seen by the repetitive application of the kernel equation to the row starting with the kernel equation containing the zero entry.

Lemma 2.6. *For $w_0 \in \mathbb{Z}$. There exist $\rho_{w_0} \in \mathcal{A}_\theta^{\text{alg}\,*}$ such that:*

$$\pi_1\big(\text{Dgm}({}_{-1}\alpha_1(\rho))\big)_{w,s} := \begin{cases} \pi_1\big(\text{Dgm}({}_{-1}\alpha_1(\gamma) - (\varphi))\big)_{w,s}, & \textit{for } (w,s) = (w_0, s), \\ 0, & \textit{else}. \end{cases}$$

Proof. We define ρ_{w_0} such that

$$(\rho_{w_0})_{n,m} = \begin{cases} \varphi^1_{0,0}, & \text{for } (n,m) = (w_0, -1), \\ 0, & \text{for } (n,m) = (w_0, 1) \text{ and for } (n,m) \text{ such that } n \neq w_0. \end{cases}$$

Thereafter we define $(\rho_{w_0})_{n,m}$ for $m < -1$ and $n = w_0$ in the following iterated way.

$$(\rho_0)_{n,m} = -\varphi^{2\prime}{}_{n+1,m}.$$

Where $\varphi^{2\prime}{}_{n+1,m}$ is second entry of $\pi_1(\mathrm{Dgm}({}_{-1}\alpha_1(\gamma + \rho_{w_0}) - (\varphi)))_{n+1,m}$. Clearly, $\pi_1(\mathrm{Dgm}({}_{-1}\alpha_1(\gamma + \rho_{w_0}) - (\varphi)))_{w_0,s} = 0$ for all $s < 0$. Similarly, we define ρ_{w_0} for $m > 0$ and hence we have ρ_{w_0} satisfying

$$\pi_1\big(\mathrm{Dgm}({}_{-1}\alpha_1(\gamma + \rho_{w_0}) - (\varphi))\big)_{w_0,s} = 0 \quad \text{for all } s \in \mathbb{Z}. \qquad \square$$

Now we prove Lemma 2.5. The element $\rho = \sum_{s\in\mathbb{N}}(\rho_{w_0})$ has the following property:

$$\pi_1\big(\mathrm{Dgm}({}_{-1}\alpha_1(\rho))\big) = \pi_1\big(\mathrm{Dgm}({}_{-1}\alpha_1(\gamma) - (\varphi))\big).$$

Hence, $H^1(\mathcal{A}_\theta^{\mathrm{alg}}, {}_{-1}\mathcal{A}_\theta^{\mathrm{alg}\,*}) = 0$. $\square$

3. The $\mathbb{Z}_2$ invariant Hochschild cohomology $H^\bullet(\mathcal{A}_\theta^{\mathrm{alg}}, \mathcal{A}_\theta^{\mathrm{alg}\,*})^{\mathbb{Z}_2}$

For $g = 1$, we have the following cohomology complex

$$\mathcal{A}_\theta^{\mathrm{alg}\,*} \xrightarrow{\alpha_1} \mathcal{A}_\theta^{\mathrm{alg}\,*} \oplus \mathcal{A}_\theta^{\mathrm{alg}\,*} \xrightarrow{\alpha_2} \mathcal{A}_\theta^{\mathrm{alg}\,*} \to 0,$$

where the maps α_1 and α_2 are as follows:

$$\alpha_1(\varphi) = (U_1\varphi - \varphi U_1, U_2\varphi - \varphi U_2),$$
$$\alpha_2(\varphi_1, \varphi_2) = U_2\varphi_1 - \lambda\varphi_1 U_2 - \lambda U_1\varphi_2 + \varphi_2 U_1.$$

Lemma 3.1. $H^0(\mathcal{A}_\theta^{\mathrm{alg}}, \mathcal{A}_\theta^{\mathrm{alg}\,*})^{\mathbb{Z}_2} \cong \mathbb{C}$.

Proof. Let $\varphi = \sum \varphi_{n,m} U_1^n U_2^m$ be an element of $\mathcal{A}_\theta^{\mathrm{alg}\,*}$. If $\alpha_1(\varphi) = 0$, then we have $U_1\varphi - \varphi U_1 = U_2\varphi - \varphi U_2 = 0$. This imples that we have the following relations on the coefficients:

$$\varphi_{n-1,m} = \lambda^m \varphi_{n-1,m} = \lambda^{m+n-1}\varphi_{n-1,m}.$$

We see that these relations are satisfied only for $m = n - 1 = 0$. Hence, we have

$$H^0(\mathcal{A}_\theta^{\mathrm{alg}}, \mathcal{A}_\theta^{\mathrm{alg}\,*}) \cong \mathbb{C}$$

and is generated by $\varphi_{0,0}$. Since the action of $\mathbb{Z}_2$ on the bar complex is the same as on the Connes complex, we deduce that $H^0(\mathcal{A}_\theta^{\mathrm{alg}}, \mathcal{A}_\theta^{\mathrm{alg}\,*})^{\mathbb{Z}_2} \cong \mathbb{C}$. $\square$

Lemma 3.2. $H^2(\mathcal{A}_\theta^{\text{alg}}, \mathcal{A}_\theta^{\text{alg}\,*})^{\mathbb{Z}_2} \cong \mathbb{C}$.

Proof. We see from the calculations as in [4] that $H^2(\mathcal{A}_\theta^{\text{alg}}, \mathcal{A}_\theta^{\text{alg}\,*})^{\mathbb{Z}_2} = \mathcal{A}_\theta^{\text{alg}\,*}/\text{Im}(\alpha_2)$. Since $\alpha_2(U_2, 0) = (1-\lambda)(U_2^2)$ and $\alpha_2(0, U_1) = (1-\lambda)(U_1^2)$, we have $H^2(\mathcal{A}_\theta^{\text{alg}}, \mathcal{A}_\theta^{\text{alg}\,*}) \cong \mathbb{C}$ and is generated by the cocycle equivalent to $\varphi_{-1,-1}$. Let $\widetilde{\varphi}_{-1,-1}$ be the corresponding element in the Connes complex. For $\varphi \in \mathcal{A}_\theta^{\text{alg}\,*}$ to be $\mathbb{Z}_2$ invariant, we need to check that

$$\widetilde{\varphi}(x_0) = \widetilde{\varphi}(k_2(I \otimes U_2^{-1} \otimes U_1^{-1}))(-1 \cdot x_0) - \lambda\widetilde{\varphi}(k_2(I \otimes U_1^{-1} \otimes U_2^{-1}))(-1 \cdot x_0) = \widetilde{\varphi}(U_1^{-1}U_2^{-1} \otimes U_2^{-1}U_1^{-1})(-1 \cdot x_0) = \varphi(U_1^{-1}U_2^{-1} \cdot (-1 \cdot x_0) \cdot U_2^{-1}U_1^{-1}).$$

In the above, $\widetilde{\varphi}$ is the element corresponding to φ in the Connes complex. Considering the cocycle $\widetilde{\varphi}_{-1,-1} \in H^2(\mathcal{A}_\theta^{\text{alg}}, \mathcal{A}_\theta^{\text{alg}\,*})$, we see that $\widetilde{\varphi}_{-1,-1}(x) = x_{-1,-1}$ and $\widetilde{\varphi}(U_1^{-1}U_2^{-1} \cdot (-1 \cdot x) \cdot U_2^{-1}U_1^{-1}) = x_{-1,-1}$. Hence we conclude that $\widetilde{\varphi}_{-1,-1}$ is invariant under the $\mathbb{Z}_2$ action. □

Lemma 3.3. $H^1(\mathcal{A}_\theta^{\text{alg}}, \mathcal{A}_\theta^{\text{alg}\,*})^{\mathbb{Z}_2} = 0$.

Proof. We recall from [4] that:

$$H^1(\mathcal{A}_\theta^{\text{alg}}, \mathcal{A}_\theta^{\text{alg}\,*}) \cong \mathbb{C}^2$$

and is generated by $\varphi^1_{-1,0}$ and $\varphi^2_{0,-1}$. In order to locate the $\mathbb{Z}_2$ invariant subgroup of $H^1(\mathcal{A}_\theta^{\text{alg}}, \mathcal{A}_\theta^{\text{alg}\,*})$, we use the chain homotopy maps h_1 and k_1. For $a, b \in \mathbb{C}$, we consider the cocycle $\chi := (a\varphi^1_{-1,0}, b\varphi^2_{0,-1}) \in \mathcal{A}_\theta^{\text{alg}\,*} \oplus \mathcal{A}_\theta^{\text{alg}\,*}$ in the Connes complex, and let $\widetilde{\chi} := (a\widetilde{\varphi^1}_{-1,0}, b\widetilde{\varphi^2}_{0,-1}) \in \text{Hom}_{\mathcal{B}_\theta^{\text{alg}}}(J_1, \mathcal{A}_\theta^{\text{alg}})$ be the corresponding cocycle in the bar complex. It satisfyies the following relation:

$$\widetilde{\varphi^1}_{-1,0}(a \otimes b \otimes e_1)(x) = \varphi^1_{-1,0}(bxa), \quad \text{for } a, b, x \in \mathcal{A}_\theta^{\text{alg}\,*}.$$

Let $\psi = k_1^*(\widetilde{\chi}) = k_1^*(a\widetilde{\varphi^1}_{-1,0}, b\widetilde{\varphi^2}_{0,-1}) = (a\widetilde{\varphi^1}_{-1,0}, b\widetilde{\varphi^2}_{0,-1}) \circ k_1$ be the pushforward of $\widetilde{\chi}$. We have the following explicit description of ψ:

$$\psi(x_0, x_1) = (a\widetilde{\varphi^1}_{-1,0}, b\widetilde{\varphi^2}_{0,-1})(k_1(I \otimes x_1))(x_0), \quad \text{for } x_0, x_1 \in \mathcal{A}_\theta^{\text{alg}\,*}.$$

After the $\mathbb{Z}_2$ action ψ is transformed to ${}_{-1}\psi(x_0, x_1) := \psi(-1 \cdot x_0, -1 \cdot x_1)$. We now pullback ${}_{-1}\psi$ on to the Connes complex to compare with the cocycle χ. The pullback $w := (w_1, w_2)$ can be described as follows:

$$(w_1, w_2) = h_1^*({}_{-1}\psi), \quad \text{where } w_i(x) := {}_{-1}\psi(x, U_i).$$

We observe that

$$w_1(x) = {}_{-1}\psi(x, U_1) = \psi(-1 \cdot x, U_1^{-1}) = \widetilde{a\varphi^1_{-1,0}}(k_1(I \otimes U_1^{-1}))(-1 \cdot x).$$

We know from our computations [12, Proof of Theorem 4.1] that

$$k_1(I \otimes U_1^{-1}) = -(U_1^{-1} \otimes U_1^{-1}),$$

using this we have:

$$\begin{aligned}\widetilde{a\varphi^1_{-1,0}}\big(k_1(I \otimes U_1^{-1})\big)(-1 \cdot x) &= -\widetilde{\varphi^1}_{-1,0}(U_1^{-1} \otimes U_1^{-1})(-1 \cdot x)\\ &= -\varphi^1_{-1,0}(U_1^{-1} \otimes U_1^{-1})(-1 \cdot x)\\ &= -\varphi^1_{-1,0}\big(U_1^{-1} \cdot (-1 \cdot x) \cdot U_1^{-1}\big) = -x_{-1,0}.\end{aligned}$$

Similarly, we can calculate w_2 and hence we finally conclude that

$$h_1^*\big(-1 \cdot \big(k_1^*\big(a\widetilde{\varphi^1}_{-1,0}, b\widetilde{\varphi^2}_{0,-1}\big)\big)\big) = -\big(a\widetilde{\varphi^1}_{-1,0}, b\widetilde{\varphi^2}_{0,-1}\big).$$

Hence $\chi \notin H^1(\mathcal{A}_\theta^{\text{alg}}, \mathcal{A}_\theta^{\text{alg}\,*})^{\mathbb{Z}_2}$. □

Proof of Theorem 1.1. We know that the cohomology group $H^0(\mathcal{A}_\theta^{\text{alg}}, {}_{-1}\mathcal{A}_\theta^{\text{alg}\,*})^{\mathbb{Z}_2} \cong \mathbb{C}^4$ and the group $H^0(\mathcal{A}_\theta^{\text{alg}}, \mathcal{A}_\theta^{\text{alg}\,*})^{\mathbb{Z}_2} \cong \mathbb{C}$. Hence, we conclude that

$$H^0\big(\mathcal{A}_\theta^{\text{alg}} \rtimes \mathbb{Z}_2, \big(\mathcal{A}_\theta^{\text{alg}} \rtimes \mathbb{Z}_2\big)^*\big) \cong \mathbb{C}^5.$$

We also notice that,

$$H^1\big(\mathcal{A}_\theta^{\text{alg}} \rtimes \mathbb{Z}_2, \big(\mathcal{A}_\theta^{\text{alg}} \rtimes \mathbb{Z}_2\big)^*\big) = H^1\big(\mathcal{A}_\theta^{\text{alg}}, \mathcal{A}_\theta^{\text{alg}\,*}\big)^{\mathbb{Z}_2} \oplus H^1\big(\mathcal{A}_\theta^{\text{alg}}, {}_{-1}\mathcal{A}_\theta^{\text{alg}\,*}\big)^{\mathbb{Z}_2} \cong 0$$

is clear as each of these summands is zero. As for the second Hochschild cohomology group $H^2(\mathcal{A}_\theta^{\text{alg}} \rtimes \mathbb{Z}_2, (\mathcal{A}_\theta^{\text{alg}} \rtimes \mathbb{Z}_2)^*)$, we observe that $H^2(\mathcal{A}_\theta^{\text{alg}}, \mathcal{A}_\theta^{\text{alg}\,*})^{\mathbb{Z}_2} \cong \mathbb{C}$ and $H^2(\mathcal{A}_\theta^{\text{alg}}, {}_{-1}\mathcal{A}_\theta^{\text{alg}\,*})^{\mathbb{Z}_2} = 0$. Hence, we finally conclude that

$$H^2\big(\mathcal{A}_\theta^{\text{alg}} \rtimes \mathbb{Z}_2, \big(\mathcal{A}_\theta^{\text{alg}} \rtimes \mathbb{Z}_2\big)^*\big)^{\mathbb{Z}_2} \cong \mathbb{C}. \qquad \square$$

4. Cyclic cohomology of $\mathcal{A}_\theta^{\text{alg}} \rtimes \mathbb{Z}_2$

Theorem 4.1. *For the algebraic noncommutative toroidal orbifold $\mathcal{A}_\theta^{\text{alg}} \rtimes \mathbb{Z}_2$, we have,*

$$HC^0\big(\mathcal{A}_\theta^{\text{alg}} \rtimes \mathbb{Z}_2\big) \cong \mathbb{C}^5, \quad HC^1\big(\mathcal{A}_\theta^{\text{alg}} \rtimes \mathbb{Z}_2\big) \cong 0,$$
$$HC^2\big(\mathcal{A}_\theta^{\text{alg}} \rtimes \mathbb{Z}_2\big) \cong \mathbb{C}^6.$$

Proof. We consider the S, B, I sequence for cohomology exact sequence.

$$\cdots \to H^1\big(\mathcal{A}_\theta^{\text{alg}}, {}_{-1}\mathcal{A}_\theta^{\text{alg}\,*}\big)^{\mathbb{Z}_2} \xrightarrow{B} HC^0\big(\mathcal{A}_\theta^{\text{alg}}, {}_{-1}\mathcal{A}_\theta^{\text{alg}\,*}\big)^{\mathbb{Z}_2} \xrightarrow{I} HC^2\big(\mathcal{A}_\theta^{\text{alg}}, {}_{-1}\mathcal{A}_\theta^{\text{alg}\,*}\big)^{\mathbb{Z}_2} \xrightarrow{S} H^2\big(\mathcal{A}_\theta^{\text{alg}}, {}_{-1}\mathcal{A}_\theta^{\text{alg}\,*}\big)^{\mathbb{Z}_2} \xrightarrow{B} HC^1\big(\mathcal{A}_\theta^{\text{alg}}, {}_{-1}\mathcal{A}_\theta^{\text{alg}\,*}\big)^{\mathbb{Z}_2} \xrightarrow{I} \cdots$$

Since, $HC^1(\mathcal{A}_\theta^{\text{alg}}, {}_{-1}\mathcal{A}_\theta^{\text{alg}\,*}) = H^1(\mathcal{A}_\theta^{\text{alg}}, {}_{-1}\mathcal{A}_\theta^{\text{alg}\,*}) = 0$. We get

$$HC^2\big(\mathcal{A}_\theta^{\text{alg}}, {}_{-1}\mathcal{A}_\theta^{\text{alg}\,*}\big) \cong \mathbb{C}^4.$$

Since $H^0(\mathcal{A}_\theta^{\text{alg}}, {}_{-1}\mathcal{A}_\theta^{\text{alg}\,*})^{\mathbb{Z}_2} \cong \mathbb{C}^4$, while, $H^0(\mathcal{A}_\theta^{\text{alg}}, \mathcal{A}_\theta^{\text{alg}\,*})^{\mathbb{Z}_2} \cong \mathbb{C}$, we have,

$$HC^0\big(\mathcal{A}_\theta^{\text{alg}} \rtimes \mathbb{Z}_2\big) \cong \mathbb{C}^5.$$

We see that $HC^1(\mathcal{A}_\theta^{\text{alg}}, {}_{\pm 1}\mathcal{A}_\theta^{\text{alg}\,*})^{\mathbb{Z}_2} = 0$, and hence we have

$$HC^1\big(\mathcal{A}_\theta^{\text{alg}} \rtimes \mathbb{Z}_2\big) \cong 0.$$

Also since $HC^2(\mathcal{A}_\theta^{\text{alg}}, \mathcal{A}_\theta^{\text{alg}\,*})^{\mathbb{Z}_2} \cong \mathbb{C}^2$ and $HC^2(\mathcal{A}_\theta^{\text{alg}}, {}_{-1}\mathcal{A}_\theta^{\text{alg}\,*})^{\mathbb{Z}_2} \cong \mathbb{C}^4$, we have

$$HC^2\big(\mathcal{A}_\theta^{\text{alg}} \rtimes \mathbb{Z}_2\big) \cong \mathbb{C}^6. \qquad \square$$

Now we can easily compute the periodic cyclic homology of the $\mathcal{A}_\theta^{\text{alg}} \rtimes \mathbb{Z}_2$.

Proof of Theorem 1.2. From the modified Connes complex we have

$$H^\bullet\big(\mathcal{A}_\theta^{\text{alg}} \rtimes \mathbb{Z}_2, (\mathcal{A}_\theta^{\text{alg}} \rtimes \mathbb{Z}_2)^*\big) = 0$$

for $\bullet \geq 3$, and we have the isomorphism

$$HC^\bullet\big(\mathcal{A}_\theta^{\text{alg}} \rtimes \mathbb{Z}_2, (\mathcal{A}_\theta^{\text{alg}} \rtimes \mathbb{Z}_2)^*\big) \cong HC^{\bullet+2}\big(\mathcal{A}_\theta^{\text{alg}} \rtimes \mathbb{Z}_2, (\mathcal{A}_\theta^{\text{alg}} \rtimes \mathbb{Z}_2)^*\big)$$

for $\bullet > 1$. Now, using the results of Theorem 4.1 we arrive at the desired results:

$$HP^{\text{even}}\big(\mathcal{A}_\theta^{\text{alg}} \rtimes \mathbb{Z}_2\big) \cong \mathbb{C}^6 \quad \text{and} \quad HP^{\text{odd}}\big(\mathcal{A}_\theta^{\text{alg}} \rtimes \mathbb{Z}_2\big) = 0. \qquad \square$$

5. Chern–Connes pairing for $\mathcal{A}_\theta^{\text{alg}} \rtimes \mathbb{Z}_2$

In this section we calculate the Chern–Connes pairing associated with the toroidal orbifold $\mathcal{A}_\theta^{\text{alg}} \rtimes \mathbb{Z}_2$. There are six projections generating $K_0(\mathcal{A}_\theta \rtimes \mathbb{Z}_2)$ [5]. Five of them belong to the algebra $\mathcal{A}_\theta^{\text{alg}} \rtimes \mathbb{Z}_2$ and they are the following:

(i) [1]

(ii) $[p^\theta]$, where $p^\theta = \frac{1}{2}(1+t)$.

(iii) $[q_0^\theta]$, where $q_0^\theta = \frac{1}{2}(1-U_1 t)$.

(iv) $[q_1^\theta]$, where $q_1^\theta = \frac{1}{2}(1-U_2 t)$.

(v) $[r^\theta]$, where $r^\theta = \frac{1}{2}(1-\sqrt{\lambda}U_1U_2 t)$,

where t is an unitary satisfying the relations $t^2 = 1$ and $tU_it^{-1} = U_i^{-1}$ for $1 \leq i \leq 2$. A complete description of the group $K_0(\mathcal{A}_\theta^{\text{alg}} \rtimes \mathbb{Z}_2)$ is unknown, with the Chern–Connes pairing of these five generators with the cyclic cocycles we will have some understanding of its noncommutative index theory. We describe pairing table for these projections with the cyclic cocyles of $HP^{\text{even}}(\mathcal{A}_\theta^{\text{alg}} \rtimes \mathbb{Z}_2)$ computed in Theorem 1.2. Using the fact that $\langle [e], [S\phi] \rangle = \langle [e], [\phi] \rangle$, we have the following computations.

Proof of Theorem 1.3.
Pairing of $[S\tau]$. The following are the pairings with the element

$$[S\tau] \in HP^{\text{even}}(\mathcal{A}_\theta^{\text{alg}} \rtimes \mathbb{Z}_2).$$

1. $\langle [1], [\tau] \rangle = 1$
2. $\langle [p^\theta], [\tau] \rangle = \frac{1}{2}$
3. $\langle [q_0^\theta], [\tau] \rangle = \frac{1}{2}$
4. $\langle [q_1^\theta], [\tau] \rangle = \frac{1}{2}$
5. $\langle [r^\theta], [\tau] \rangle = \frac{1}{2}$.

Pairing of $[S\mathcal{D}_{0,0}]$. The following are the pairings with the element

$$[S\mathcal{D}_{0,0}] \in HP^{\text{even}}(\mathcal{A}_\theta^{\text{alg}} \rtimes \mathbb{Z}_2).$$

1. $\langle [1], [\mathcal{D}_{0,0}] \rangle = 0$
2. $\langle [p^\theta], [\mathcal{D}_{0,0}] \rangle = \frac{1}{2}$
3. $\langle [q_0^\theta], [\mathcal{D}_{0,0}] \rangle = 0$
4. $\langle [q_1^\theta], [\mathcal{D}_{0,0}] \rangle = 0$
5. $\langle [r^\theta], [\mathcal{D}_{0,0}] \rangle = 0$.

Pairing of $[S\,\mathcal{D}_{1,0}]$. The following are the pairings with the element

$$[S\,\mathcal{D}_{1,0}] \in HP^{\text{even}}(\mathcal{A}_\theta^{\text{alg}} \rtimes \mathbb{Z}_2).$$

1. $\langle [1], [\mathcal{D}_{1,0}] \rangle = 0$
2. $\langle [p^\theta], [\mathcal{D}_{1,0}] \rangle = 0$
3. $\langle [q_0^\theta], [\mathcal{D}_{1,0}] \rangle = -\dfrac{1}{2}$
4. $\langle [q_1^\theta], [\mathcal{D}_{1,0}] \rangle = 0$
5. $\langle [r^\theta], [\mathcal{D}_{1,0}] \rangle = 0.$

Pairing of $[S\,\mathcal{D}_{0,1}]$. The following are the pairings with the element

$$[S\,\mathcal{D}_{0,1}] \in HP^{\text{even}}(\mathcal{A}_\theta^{\text{alg}} \rtimes \mathbb{Z}_2).$$

1. $\langle [1], [\mathcal{D}_{0,1}] \rangle = 0$
2. $\langle [p^\theta], [\mathcal{D}_{0,1}] \rangle = 0$
3. $\langle [q_0^\theta], [\mathcal{D}_{0,1}] \rangle = 0$
4. $\langle [q_1^\theta], [\mathcal{D}_{0,1}] \rangle = -\dfrac{1}{2}$
5. $\langle [r^\theta], [\mathcal{D}_{0,1}] \rangle = 0.$

Pairing of $[S\,\mathcal{D}_{1,1}]$. The following are the pairings with the element

$$[S\,\mathcal{D}_{1,1}] \in HP^{\text{even}}(\mathcal{A}_\theta^{\text{alg}} \rtimes \mathbb{Z}_2).$$

1. $\langle [1], [\mathcal{D}_{1,1}] \rangle = 0$
2. $\langle [p^\theta], [\mathcal{D}_{1,1}] \rangle = 0$
3. $\langle [q_0^\theta], [\mathcal{D}_{1,1}] \rangle = 0$
4. $\langle [q_1^\theta], [\mathcal{D}_{1,1}] \rangle = 0$
5. $\langle [r^\theta], [\mathcal{D}_{1,1}] \rangle = -\dfrac{\sqrt{\lambda}}{2}.$

Pairing of $[\varphi]$. The following are the pairings with the element

$$[\varphi] \in HP^{\text{even}}(\mathcal{A}_\theta^{\text{alg}} \rtimes \mathbb{Z}_2),$$

where φ is the even cocycle computed in the paper of A. Connes [4].

1. $\langle [1], [\varphi] \rangle = 0$
2. $\langle [p^\theta], [\varphi] \rangle = 0$
3. $\langle [q_0^\theta], [\varphi] \rangle = 0$
4. $\langle [q_1^\theta], [\varphi] \rangle = 0$
5. $\langle [r^\theta], [\varphi] \rangle = 0.$ □

We observe that since these five projections of the algebraic noncommutative toroidal orbifold $\mathcal{A}_\theta^{\text{alg}} \rtimes \mathbb{Z}_2$ are projections of the smooth orbifold, $\mathcal{A}_\theta \rtimes \mathbb{Z}_2$; their linear independence in $K_0(\mathcal{A}_\theta \rtimes \mathbb{Z}_2)$ implies that they are linearly independent in $K_0(\mathcal{A}_\theta^{\text{alg}} \rtimes \mathbb{Z}_2)$. We conjecture that these five projections span the group $K_0(\mathcal{A}_\theta^{\text{alg}} \rtimes \mathbb{Z}_2)$.

Conjecture 5.1. $K_0(\mathcal{A}_\theta^{\text{alg}} \rtimes \mathbb{Z}_2) \cong \mathbb{Z}^5$.

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