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# On products in algebraic K-theory

Dominique Arlettaz, Grzegorz Banaszak and Wojciech Gajda

**Abstract.** This paper investigates the product structure in algebraic $K$-theory of rings. The first objective is to understand the relationships between products and the kernel of the Hurewicz homomorphism relating the algebraic $K$-theory of any ring to the integral homology of its linear groups. The second part of the paper is devoted to the ring of integers $\mathbb{Z}$. Using recent results of V. Voevodsky we completely determine the products in $K_*(\mathbb{Z})$ tensored with the ring of 2-adic integers.

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## 0. Introduction

The purpose of this paper is to study the Loday's product homomorphism

$$\star : K_i(R) \otimes K_k(\mathbb{Z}) \longrightarrow K_{i+k}(R)$$

in the algebraic $K$-theory of any ring $R$ with identity, for positive integers $i$ and $k$ (see [21]). Our first goal is to exhibit very strong connections between the image of that product and the kernel of the non-stable Hurewicz homomorphisms relating the $K$-groups of $R$ to the integral homology groups of its linear groups

$$h_i : K_i(R) = \pi_i BGL(R)^+ \longrightarrow H_i BGL(R)^+ \cong H_i GL(R) \quad \text{for} \quad i \geq 1\,,$$

respectively $h_i : K_i(R) \to H_i E(R)$ for $i \geq 2$ and $h_i : K_i(R) \longrightarrow H_i St(R)$ for $i \geq 3$, where $GL(R)$ is the infinite general linear group (considered as a discrete group) over $R$, $E(R)$ its subgroup generated by elementary matrices, and $St(R)$ the infinite Steinberg group over $R$. A universal approximation of the exponent of the kernel and some information on the cokernel of these Hurewicz homomorphisms

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have been obtained in [3], [4] and Section 5 of [5]. Our argument is based on the understanding, from various viewpoints, of the stable Hurewicz homomorphism between the algebraic $K$-theory and the homology of the $K$-theory spectrum. We establish in particular the following result (see Theorem 3.2):

*For any ring $R$ and any integer $i \geq 2$, the image of $\star : K_i(R) \otimes K_1(\mathbb{Z}) \to K_{i+1}(R)$ is contained in the kernel of the Hurewicz homomorphisms $h_{i+1} : K_{i+1}(R) \to H_{i+1}GL(R)$ and $h_{i+1} : K_{i+1}(R) \to H_{i+1}E(R)$; the same holds for $h_{i+1} : K_{i+1}(R) \to H_{i+1}St(R)$ if $i \geq 3$.*

In low dimensions, we actually prove exactness results for any ring $R$ (see Theorems 4.1 and 4.3).

(a) *There is an exact sequence*

$$K_4(R) \xrightarrow{h_4} H_4E(R) \longrightarrow \Gamma(K_2(R)) \longrightarrow K_3(R) \xrightarrow{h_3} H_3E(R) \longrightarrow 0\,,$$

*where $\Gamma(-)$ is the quadratic functor defined on abelian groups by J.H.C. Whitehead in Section 5 of* [37]*; moreover,* $\ker h_3$ *is isomorphic to $K_2(R) \star K_1(\mathbb{Z})$.*

(b) *There is an exact sequence*

$$K_5(R) \xrightarrow{h_5} H_5St(R) \longrightarrow K_3(R) \otimes K_1(\mathbb{Z}) \xrightarrow{\star} K_4(R) \xrightarrow{h_4} H_4St(R) \longrightarrow 0$$

*and the kernel of $h_5$ fits into a short exact sequence*

$$0 \longrightarrow K_4(R) \star K_1(\mathbb{Z}) \longrightarrow \ker h_5 \longrightarrow Q \longrightarrow 0\,,$$

*where $Q$ is a quotient of the subgroup of elements of order* 2 *in the group $K_3(R)$.*

The second objective of the paper is to compute explicitely products in the algebraic $K$-theory of the ring of integers $\mathbb{Z}$. First of all, we determine in low dimensions the products $K_i(\mathbb{Z}) \star K_k(\mathbb{Z})$, the homology groups of $SL(\mathbb{Z})$ and $St(\mathbb{Z})$, and the Hurewicz homomorphism (see Proposition 5.1). Secondly, we consider maps

$$K_i(\mathbb{Z}) \otimes K_k(\mathbb{Z}) \xrightarrow{\star} K_{i+k}(\mathbb{Z}) \longrightarrow K_{i+k}(\mathbb{Z}) \otimes \hat{\mathbb{Z}_2}$$

for all positive integers $i$ and $k$, where the second arrow is the tensor product of $K_{i+k}(\mathbb{Z})$ with the inclusion of $\mathbb{Z}$ into the ring of 2-adic integers $\hat{\mathbb{Z}_2}$. We call these maps 2-adic products for $K_*(\mathbb{Z})$ and continue to denote them by the symbol $\star$. We deduce from a topological argument based on results by M. Bökstedt [12], V. Voevodsky [33], J. Rognes and C. Weibel [35] and [28] the calculation of all such

2-adic products (see Theorems 5.6, 5.7, Corollary 5.8 and Theorem 5.9):

> *The 2-adic product* $\star : K_i(\mathbb{Z}) \otimes K_k(\mathbb{Z}) \longrightarrow K_{i+k}(\mathbb{Z}) \otimes \hat{\mathbb{Z}_2}$ *is trivial for all positive integers* $i$ *and* $k$*, except if* $i \equiv k \equiv 1$ *mod 8 or* $i \equiv 1$ *mod 8 and* $k \equiv 2$ *mod 8 (or* $i \equiv 2$ *mod 8 and* $k \equiv 1$ *mod 8) where its image is cyclic of order 2.*

We also mention in Proposition 5.11 an interesting relationship between products in algebraic $K$-theory and the kernel of the Dwyer-Friedlander map.

> *For any odd prime* $l$ *and any integer* $n \geq 2$*, the image of the product map*
>
> $$\star : K_{2n-1}(\mathbb{Z}) \otimes K_{2n-1}(\mathbb{Z}) \longrightarrow K_{4n-2}(\mathbb{Z})$$
>
> *is contained in the kernel of the Dwyer-Friedlander map* $K_{4n-2}(\mathbb{Z}) \rightarrow K_{4n-2}^{\text{ét}}(\mathbb{Z}[\frac{1}{l}])$.

Observe that the 2-adic products in the $K$-theory of $\mathbb{Z}$ have a very small image. On the other hand, we finally prove in Theorem 6.4 that the image of the product

$$\star : K_1(E) \otimes K_{2n-1}(E) \longrightarrow K_{2n}(E)$$

is huge when $E$ is a cyclotomic field and $n$ and odd integer.

The paper is organized as follows. In Section 1, we give a new construction of the Whitehead exact sequence for spectra. Section 2 presents another approach of the study of the Hurewicz homomorphism for spectra using the so-called Postnikov cofibrations. Section 3 is devoted to general results on the relations between products in algebraic $K$-theory and the kernel of the stable and of the non-stable Hurewicz homomorphism. Section 4 provides the above exact sequences involving the $K$-groups and the homology groups of the linear groups in dimensions $\leq 5$. In Section 5, we calculate the 2-adic products in the algebraic $K$-theory of the ring of integers $\mathbb{Z}$. We finally discuss in Section 6 products in the $K$-theory of cyclotomic fields.

Throughout the paper, all rings are supposed to have an identity. We consider all ordinary homology groups with (trivial) coefficients in $\mathbb{Z}$ except if explicitly mentioned. If $G$ is an abelian group, $G_l$ denotes the $l$-torsion subgroup of $G$ (for a prime $l$), $K(G, s)$ the Eilenberg-MacLane space having all homotopy groups trivial except for $G$ in dimension $s$ and $H(G)$ the Eilenberg-MacLane spectrum having all homotopy groups trivial except for $G$ in dimension 0. If $X$ is any CW-complex or any CW-spectrum and $i$ any integer, we write $\alpha_i : X \rightarrow X[i]$ for its $i$-th Postnikov section (i.e., $\pi_k X[i] = 0$ for $k > i$ and $(\alpha_i)_* : \pi_k X \xrightarrow{\cong} \pi_k X[i]$ for $k \leq i$) and $\gamma_i : X(i) \rightarrow X$ for the fiber of $\alpha_i$; in other words, $X(i)$ is the $i$-connected cover of $X$. For $j \geq i+1$, $X(i, j]$ denotes $X(i)[j]$, whose homotopy groups are $\pi_k X(i, j] = 0$ if $k \leq i$ or $k > j$ and $\pi_k X(i, j] \cong \pi_k X$ if $i + 1 \leq k \leq j$.

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## 1. The Whitehead exact sequence for spectra

Let $S$ be the sphere spectrum and $S \xrightarrow{\alpha_0} S[0] = H(\mathbb{Z})$ its 0-th Postnikov section. By taking the smash product of any spectrum $X$ with the cofibration $S(0) \xrightarrow{\gamma_0} S \xrightarrow{\alpha_0} S[0]$, where $S(0)$ is 0-connected, one obtains the cofibration of spectra

$$X \wedge S(0) \xrightarrow{\mathrm{id}\wedge\gamma_0} X \wedge S \simeq X \xrightarrow{\mathrm{id}\wedge\alpha_0} X \wedge H(\mathbb{Z}),$$

whose homotopy exact sequence is the long Whitehead exact sequence

$$\cdots \longrightarrow \pi_i(X \wedge S(0)) \xrightarrow{\bar{\chi}_i} \pi_i X \xrightarrow{\bar{h}_i} H_i X \xrightarrow{\bar{\nu}_i} \pi_{i-1}(X \wedge S(0)) \longrightarrow \cdots$$

of $X$; here $i$ is any integer, $\bar{\nu}_i$ is the connecting homomorphism, $\bar{\chi}_i$ is induced by $(\mathrm{id} \wedge \gamma_0)$ and $\bar{h}_i$ by $(\mathrm{id} \wedge \alpha_0)$, i.e., $\bar{h}_i$ is the stable Hurewicz homomorphism. The groups $\pi_i(X \wedge S(0))$ are usually denoted by $\Gamma_i(X)$: that definition coincides actually with the homotopy groups of the fiber of the Dold-Thom map (see [16]) and it was recently proved in [29] that they are isomorphic to the groups introduced in the original paper [37] by J.H.C. Whitehead.

Now, let us assume that the spectrum $X$ is $(r-1)$-connected for some integer $r$. The advantage of the above approach is that one can compute the groups $\Gamma_i(X)$ with the Atiyah-Hirzebruch spectral sequence for the $S(0)$-homology of $X$:

$$E^2_{s,t} \cong H_s(X; \pi_t S(0)) \Longrightarrow \Gamma_{s+t}(X).$$

Notice that $E^2_{s,t} = 0$ if $s \le r-1$ or $t \le 0$. This implies in particular that $\Gamma_i(X) = 0$ for $i \le r$ (Hurewicz theorem) and that $(\rho_1\rho_2\cdots\rho_{i-r})\Gamma_i(X) = 0$ for $i \ge r+1$, where $\rho_k$ denotes the exponent of the homotopy group $\pi_k S$ for $k \ge 1$ (see also [29] for another proof and [5] for corresponding results for the generalized Hurewicz homomorphisms). The first interesting Gamma group of an $(r-1)$-connected spectrum $X$ is

$$\Gamma_{r+1}(X) \cong E^2_{r,1} \cong \pi_r X \otimes \pi_1 S$$

(this was in fact established a long time ago by J.H.C. Whitehead, see for instance Section 14 of [37]). Our first goal is to understand the homomorphism $\bar{\chi}_{r+1} : \Gamma_{r+1}(X) \cong \pi_r X \otimes \pi_1 S \to \pi_{r+1} X$. Let us start with the following general result on the external product $\pi_i X \otimes \pi_k S \xrightarrow{\wedge} \pi_{i+k}(X \wedge S) \cong \pi_{i+k} X$ (see [32], p. 270 for the definition of the external product).

**Lemma 1.1.** *Let $X$ be any spectrum, $i$ and $k$ two integers with $k \ge 1$. Then the image of the external product $\wedge : \pi_i X \otimes \pi_k S \to \pi_{i+k} X$ is contained in the kernel* *of the stable Hurewicz homomorphism* $\bar{h}_{i+k} : \pi_{i+k}X \to H_{i+k}X$ *for all integers* $i$ *and all positive integers* $k$.

*Proof.* The commutative diagram

$$\begin{array}{ccc} \pi_i X \otimes \pi_k S(0) & \xrightarrow[\cong]{(\mathrm{id})_* \otimes (\gamma_0)_*} & \pi_i X \otimes \pi_k S \\ \downarrow \wedge & & \downarrow \wedge \\ \Gamma_{i+k}(X) & \xrightarrow{\bar{\chi}_{i+k}} & \pi_{i+k} X \end{array}$$

shows that the image of $\wedge : \pi_i X \otimes \pi_k S \to \pi_{i+k}X$ is contained in image $\bar{\chi}_{i+k} \cong \ker \bar{h}_{i+k}$. Another proof of this fact is given by Lemma 1 of [6]. □

In the case where $X$ is $(r-1)$-connected and $i = r$, $k = 1$, we have the following exactness result:

**Proposition 1.2.** *For an* $(r-1)$*-connected spectrum* $X$*, the homomorphism* $\bar{\chi}_{r+1} : \Gamma_{r+1}(X) \to \pi_{r+1}X$ *in the Whitehead exact sequence is exactly the external product* $\wedge : \pi_r X \otimes \pi_1 S \to \pi_{r+1}X$.

*Proof.* Consider the commutative diagram

$$\begin{array}{ccccc} H_r X \otimes H_1 S(0) & \xrightarrow[\cong]{\wedge} & H_{r+1}(X \wedge S(0)) & & \\ \uparrow \cong & & \uparrow \cong & & \\ \pi_r X \otimes \pi_1 S(0) & \xrightarrow{\wedge} & \pi_{r+1}(X \wedge S(0)) & = & \Gamma_{r+1}(X) \\ \cong \downarrow (\mathrm{id})_* \otimes (\gamma_0)_* & & \downarrow \bar{\chi}_{r+1} & & \\ \pi_r X \otimes \pi_1 S & \xrightarrow{\wedge} & \pi_{r+1}(X \wedge S) & \cong & \pi_{r+1} X\,. \end{array}$$

The top horizontal homomorphism is an isomorphism by Künneth formula and the two top vertical arrows, which are Hurewicz homomorphisms, are isomorphisms since $X$ is $(r-1)$-connected, $S(0)$ is 0-connected and $X \wedge S(0)$ is $r$-connected. Consequently, the external product in the middle of the diagram is an isomorphism. The homomorphism $(\mathrm{id})_* \otimes (\gamma_0)_*$ is an isomorphism because $(\gamma_0)_* : \pi_1 S(0) \xrightarrow{\cong} \pi_1 S$. Therefore, $\bar{\chi}_{r+1}$ is exactly the external product $\pi_r X \otimes \pi_1 S \xrightarrow{\wedge} \pi_{r+1}X$. □

**Corollary 1.3.** *For any $(r-1)$-connected spectrum $X$, the following sequence is exact:*

$$\cdots \longrightarrow \Gamma_{r+2}(X) \xrightarrow{\bar{\chi}_{r+2}} \pi_{r+2}X \xrightarrow{\bar{h}_{r+2}} H_{r+2}X \xrightarrow{\bar{\nu}_{r+2}}$$
$$\pi_r X \otimes \pi_1 S \xrightarrow{\wedge} \pi_{r+1}X \xrightarrow{\bar{h}_{r+1}} H_{r+1}X \longrightarrow 0\,.$$

## 2. Postnikov cofibrations

The purpose of this section is to present another approach of the study of the Hurewicz homomorphism. For an $(r-1)$-connected spectrum $X$, consider for all integers $i \geq r+1$ the cofibrations of spectra

$$\Sigma^i H(\pi_i X) \xrightarrow{\gamma_{i-1}} X[i] \xrightarrow{\alpha_{i-1}} X[i-1]\,,$$

where $\alpha_{i-1}$ is the $(i-1)$-st Postnikov section of $X[i]$: let us call them the Postnikov cofibrations of $X$. The associated homology exact sequences are

$$\cdots \longrightarrow H_{i+1}X[i] \xrightarrow{(\alpha_{i-1})_*} H_{i+1}X[i-1] \xrightarrow{\bar{\partial}} \underbrace{H_i(\Sigma^i H(\pi_i X))}_{\cong \pi_i X} \xrightarrow{(\gamma_{i-1})_*}$$
$$\underbrace{H_i X[i]}_{\cong H_i X} \xrightarrow{(\alpha_{i-1})_*} H_i X[i-1] \longrightarrow 0\,,$$

and it is easy to check that $(\gamma_{i-1})_*$ is the stable Hurewicz homomorphism $\bar{h}_i$. Thus, we obtain the following

**Proposition 2.1.** *Let $X$ be an $(r-1)$-connected spectrum and $i$ an integer $\geq r+1$. There is an exact sequence*

$$\cdots \longrightarrow H_{i+1}X[i] \xrightarrow{(\alpha_{i-1})_*} H_{i+1}X[i-1] \xrightarrow{\bar{\partial}} \pi_i X \xrightarrow{\bar{h}_i} H_i X \xrightarrow{(\alpha_{i-1})_*} H_i X[i-1] \longrightarrow 0\,.$$

Now let us try to understand the homomorphism $\bar{\partial}$ for the cases $i = r+1$ and $i = r+2$.

**Proposition 2.2.** *(a) For any $(r-1)$-connected spectrum $X$, there is an exact sequence*

$$0 \longrightarrow H_{r+2}X[r+1] \xrightarrow{\bar{\varphi}} \pi_r X \otimes \pi_1 S \xrightarrow{\bar{\partial}} \pi_{r+1}X \xrightarrow{\bar{h}_{r+1}} H_{r+1}X \longrightarrow 0\,.$$

*(b) The homomorphism* $\bar\partial$ *is again exactly the external product* $\wedge : \pi_r X \otimes \pi_1 S \to \pi_{r+1} X$.

*Proof.* Let us look at the Postnikov cofibration of $X$ for $i = r+1$,

$$\Sigma^{r+1} H(\pi_{r+1}X) \xrightarrow{\gamma_r} X[r+1] \xrightarrow{\alpha_r} X[r] \simeq \Sigma^r H(\pi_r X)\,,$$

and take its homology exact sequence

$$\underbrace{H_{r+2}(\Sigma^{r+1}H(\pi_{r+1}X))}_{=0} \longrightarrow H_{r+2}X[r+1] \xrightarrow{\bar\varphi} H_{r+2}(\Sigma^r H(\pi_r X)) \xrightarrow{\bar\partial}$$
$$\pi_{r+1}X \xrightarrow{\bar h_{r+1}} H_{r+1}X \longrightarrow \underbrace{H_{r+1}(\Sigma^r H(\pi_r X))}_{=0}\,,$$

where $\bar\varphi$ is written for $(\alpha_r)_*$. Since $\Sigma^r H(\pi_r X)$ is an Eilenberg-MacLane spectrum, it is clear that

$$H_{r+2}(\Sigma^r H(\pi_r X)) \cong \Gamma_{r+1}(\Sigma^r H(\pi_r X)) \cong \pi_r X \otimes \pi_1 S\,,$$

and we get assertion (a). Then, consider the map $\alpha_0 : S \to H(\mathbb{Z})$ and denote by $\zeta$ the composition

$$(\mathrm{id} \wedge \alpha_0)\,\gamma_r : \Sigma^{r+1}H(\pi_{r+1}X) \longrightarrow X[r+1] \simeq X[r+1] \wedge S \longrightarrow X[r+1] \wedge H(\mathbb{Z})\,,$$

and by $F$ its fiber. By smashing with $H(\mathbb{Z})$ the cofibration obtained by looping the base spectrum of the cofibration $\Sigma^{r+1}H(\pi_{r+1}X) \xrightarrow{\gamma_r} X[r+1] \xrightarrow{\alpha_r} \Sigma^r H(\pi_r X)$, we get the commutative diagram

$$\begin{array}{ccccc}
\Sigma^{r-1}H(\pi_r X) \wedge H(\mathbb{Z}) & \longrightarrow & \Sigma^{r+1}H(\pi_{r+1}X) \wedge H(\mathbb{Z}) & \xrightarrow{\gamma_r \wedge \mathrm{id}} & X[r+1] \wedge H(\mathbb{Z}) \\
\uparrow & & \uparrow{\scriptstyle \mathrm{id}\wedge\alpha_0} & & \uparrow{\scriptstyle \mathrm{id}} \\
F & \longrightarrow & \Sigma^{r+1}H(\pi_{r+1}X) & \xrightarrow{\zeta} & X[r+1] \wedge H(\mathbb{Z}) \\
\downarrow & & \downarrow{\scriptstyle \gamma_r} & & \downarrow{\scriptstyle \mathrm{id}} \\
X[r+1] \wedge S(0) & \xrightarrow{\mathrm{id}\wedge\gamma_0} & X[r+1] & \xrightarrow{\mathrm{id}\wedge\alpha_0} & X[r+1] \wedge H(\mathbb{Z})
\end{array}$$

in which all rows are cofibrations. Then look at their homotopy exact sequences

$$
\begin{array}{ccccccc}
\mathrm{H_{r+2}X[r+1]} & \longrightarrow & \overbrace{H_{r+1}(\Sigma^{r-1}H(\pi_r X))}^{\cong H_{r+2}(\Sigma^r H(\pi_r X))} & \xrightarrow{\bar\partial} & \overbrace{H_{r+1}(\Sigma^{r+1}H(\pi_{r+1}X))}^{\cong \pi_{r+1}X} & \xrightarrow[=\bar h_{r+1}]{(\gamma_r\wedge\mathrm{id})_*} & \overbrace{H_{r+1}X[r+1]}^{\cong H_{r+1}X} \\
\uparrow = & & \uparrow & & \text{Hurewicz}\uparrow\cong & & \uparrow = \\
H_{r+2}X[r+1] & \longrightarrow & \pi_{r+1}F & \longrightarrow & \pi_{r+1}X & \xrightarrow{\zeta_*} & H_{r+1}X[r+1] \\
\downarrow = & & \downarrow & & (\gamma_r)_*\downarrow\cong & & \downarrow = \\
H_{r+2}X[r+1] & \longrightarrow & \underbrace{\Gamma_{r+1}(X[r+1])}_{\cong\Gamma_{r+1}(X)} & \xrightarrow{\bar\chi_{r+1}} & \underbrace{\pi_{r+1}X[r+1]}_{\cong\pi_{r+1}X} & \xrightarrow{\bar h_{r+1}} & \underbrace{H_{r+1}X[r+1]}_{\cong H_{r+1}X}.
\end{array}
$$

Observe that the three horizontal arrows on the left of the diagram are injective and conclude by the five lemma that the two vertical arrows starting from $\pi_{r+1}F$ are isomorphisms: assertion (b) can then be deduced from Proposition 1.2. □

**Remark 2.3.** It follows from Corollary 1.3 and Proposition 2.2 that the cokernel of $\bar h_{r+2} : \pi_{r+2}X \to H_{r+2}X$ is isomorphic to image $\bar\nu_{r+2} = \ker(\wedge : \pi_r X \otimes \pi_1 S \to \pi_{r+1}X) \cong H_{r+2}X[r+1]$ for any $(r-1)$-connected spectrum $X$.

Similarly, we can investigate the stable Hurewicz homomorphism in dimension $r+2$. Consider the Postnikov cofibration of an $(r-1)$-connected spectrum $X$ for $i = r+2$ and its homology exact sequence

$$
\cdots \longrightarrow H_{r+3}X[r+1] \xrightarrow{\bar\psi} \underbrace{H_{r+2}(\Sigma^{r+2}H(\pi_{r+2}X))}_{\cong\pi_{r+2}X} \xrightarrow{\bar h_{r+2}}
$$
$$
\underbrace{H_{r+2}X[r+2]}_{\cong H_{r+2}X} \longrightarrow H_{r+2}X[r+1] \longrightarrow 0\,,
$$

where $\bar\psi$ is written for $\bar\partial$. The next two lemmas describe the group $H_{r+3}X[r+1]$ and the homomorphism $\bar\psi$.

**Lemma 2.4.** *There is an exact sequence*

$$
\cdots \longrightarrow \pi_{r+1}X \otimes \pi_1 S \xrightarrow{\bar\theta} H_{r+3}X[r+1] \xrightarrow{\bar\eta} {}_2(\pi_r X) \longrightarrow 0\,,
$$

*where* ${}_2(\pi_r X)$ *denotes the subgroup of elements of order dividing* 2 *in the group* $\pi_r X$.

*Proof.* Let us look again at the cofibration

$$\Sigma^{r+1}H(\pi_{r+1}X) \xrightarrow{\gamma_r} X[r+1] \xrightarrow{\alpha_r} \Sigma^r H(\pi_r X)$$

and at its homology exact sequence

$$\cdots \longrightarrow H_{r+3}(\Sigma^{r+1}H(\pi_{r+1}X)) \xrightarrow{\bar\theta} H_{r+3}X[r+1] \xrightarrow{\bar\eta} H_{r+3}(\Sigma^r H(\pi_r X)) \longrightarrow 0\,,$$

where $\bar\theta$ and $\bar\eta$ are the homomorphisms induced by $\gamma_r$ and $\alpha_r$ respectively. It turns out that
$H_{r+3}(\Sigma^{r+1}H(\pi_{r+1}X)) \cong \Gamma_{r+2}(\Sigma^{r+1}H(\pi_{r+1}X)) \cong \pi_{r+1}X \otimes \pi_1 S$ because of the results of Section 1 and that
$H_{r+3}(\Sigma^r H(\pi_r X)) \cong {}_2(\pi_r X)$, according to Théorème 2 of [14]. □

**Lemma 2.5.** *The composition $\bar\psi\bar\theta : \pi_{r+1}X \otimes \pi_1 S \to \pi_{r+2}X$ is the external product $\wedge$.*

*Proof.* The obvious map $X(r) \to X$ provides the commutative diagram of cofibrations

$$\begin{array}{ccccccc}
\Sigma^{r+2}H(\pi_{r+2}X) & \longrightarrow & X(r,r+2] & \longrightarrow & X(r,r+1] & \simeq & \Sigma^{r+1}H(\pi_{r+1}X) \\
\big\downarrow \mathrm{id} & & \big\downarrow & & \big\downarrow \gamma_r & & \\
\Sigma^{r+2}H(\pi_{r+2}X) & \longrightarrow & X[r+2] & \longrightarrow & X[r+1] & &
\end{array}$$

which induces the commutative square

$$\begin{array}{ccc}
\underbrace{H_{r+3}(\Sigma^{r+1}H(\pi_{r+1}X))}_{\cong \pi_{r+1}X \otimes \pi_1 S} & \longrightarrow & \underbrace{H_{r+2}(\Sigma^{r+2}H(\pi_{r+2}X))}_{\cong \pi_{r+2}X} \\
\big\downarrow \bar\theta & & \big\downarrow = \\
H_{r+3}X[r+1] & \xrightarrow{\bar\psi} & \underbrace{H_{r+2}(\Sigma^{r+2}H(\pi_{r+2}X))}_{\cong \pi_{r+2}X}\,.
\end{array}$$

Then, the statement of Proposition 2.2 for the $r$-connected spectrum $X(r)$ shows that the top horizontal arrow is the external product. □

We may summarize our results on the stable Hurewicz homomorphism $\bar{h}_{r+2}$ as follows.

**Proposition 2.6.** *Let $X$ be an $(r-1)$-connected spectrum.*
*(a) There is an exact sequence*

$$\cdots \longrightarrow H_{r+3}X[r+1] \xrightarrow{\bar{\psi}} \pi_{r+2}X \xrightarrow{\bar{h}_{r+2}} H_{r+2}X \longrightarrow H_{r+2}X[r+1] \longrightarrow 0\,.$$

*(b) The kernel of $\bar{h}_{r+2}$ fits into the short exact sequence*

$$0 \longrightarrow \wedge(\pi_{r+1}X \otimes \pi_1 S) \longrightarrow \ker \bar{h}_{r+2} \longrightarrow Q \longrightarrow 0\,,$$

*where $Q$ is a quotient of ${}_2(\pi_r X)$.*

**Remark 2.7.** Since $\pi_i X[r+1] = 0$ for $i \geq r+2$,

$$H_{r+3}X[r+1] \cong \Gamma_{r+2}(X[r+1]).$$

The Postnikov section $X \to X[r+1]$ induces a map $f$ between the Atiyah-Hirzebruch spectral sequences

$$H_s(X;\pi_t S(0)) \Longrightarrow \Gamma_{s+t}(X) \quad \text{and} \quad H_s(X[r+1];\pi_t S(0)) \Longrightarrow \Gamma_{s+t}(X[r+1])\,.$$

The lines $s+t = r+2$ in these spectral sequences give the following picture:

$$\begin{array}{ccccccccc}
H_{r+2}(X;\pi_1 S) & \xrightarrow{d^2} & \pi_r X \otimes \pi_2 S & \longrightarrow & \Gamma_{r+2}(X) & \longrightarrow & H_{r+1}(X;\pi_1 S) & \longrightarrow & 0 \\
\downarrow f_1 & & \cong\downarrow f_2 & & \downarrow f_3 & & \cong\downarrow f_4 & & \\
H_{r+2}(X[r+1];\pi_1 S) & \xrightarrow{d^2} & \pi_r X[r+1] \otimes \pi_2 S & \longrightarrow & \Gamma_{r+2}(X[r+1]) & \longrightarrow & H_{r+1}(X[r+1];\pi_1 S) & \longrightarrow & 0\,.
\end{array}$$

By the universal coefficient theorem, one has $H_{r+2}(X;\pi_1 S) \cong (H_{r+2}X \otimes \pi_1 S) \oplus \mathrm{Tor}(H_{r+1}X, \pi_1 S)$ and $H_{r+2}(X[r+1];\pi_1 S) \cong (H_{r+2}(X[r+1]) \otimes \pi_1 S) \oplus \mathrm{Tor}(H_{r+1}X, \pi_1 S)$; thus, one can check that $f_1$ is surjective because of Whitehead's theorem and deduce from the five lemma that

$$\Gamma_{r+2}(X) \cong \Gamma_{r+2}(X[r+1]) \cong H_{r+3}X[r+1]\,.$$

Moreover, one can show with the argument of the proof of Proposition 2.2 (b) that the homomorphism $\bar{\psi}$ of Proposition 2.6 (a) is actually $\bar{\chi}_{r+2} : \Gamma_{r+2}(X) \to \pi_{r+2}X$ of Corollary 1.3. Consequently, the part

$$H_{r+3}X[r+1] \xrightarrow{\bar{\psi}} \pi_{r+2}X \xrightarrow{\bar{h}_{r+2}} H_{r+2}X$$

of the sequence given by Proposition 2.6 (a) is a piece of the Whitehead exact sequence.

**Remark 2.8.** It follows from Lemma 2.4 and the previous remark that the group $\Gamma_{r+2}(X)$ is described by the exact sequence

$$\cdots \longrightarrow \pi_{r+1}X \otimes \pi_1 S \xrightarrow{\bar{\theta}} \Gamma_{r+2}(X) \xrightarrow{\bar{\eta}} {}_2(\pi_r X) \longrightarrow 0\,,$$

and in particular that its exponent divides 4 (this was already known by [11], Section 4).

**Remark 2.9.** All exact sequences introduced in Sections 1 and 2 are obviously natural in $X$.

## 3. Products and Hurewicz homomorphisms in algebraic $K$-theory

If $R$ is any ring, let us denote by $X_R$ the connective $K$-theory spectrum of $R$, i.e., a $(-1)$-connected $\Omega$-spectrum whose 0-th space is the infinite loop space $BGL(R)^+ \times K_0(R)$. We shall also consider the $(r-1)$-connected spectra $X_R(r-1)$ for $r \geq 0$, i.e., the fiber of the Postnikov section $X_R \to X_R[r-1]$, and call $\gamma_{r-1}$ the obvious map $X_R(r-1) \to X_R$. Observe that $K_i(R) \cong \pi_i X_R(r-1)$ for $i \geq r$. Remember that the infinite loop spaces corresponding to $X_R(0)$, $X_R(1)$ and $X_R(2)$ are $BGL(R)^+$, $BE(R)^+$ and $BSt(R)^+$ respectively. If $R$ and $R'$ are two rings, there is a pairing $\mu : X_R \wedge X_{R'} \to X_{R\otimes R'}$ and the product in algebraic $K$-theory is defined as follows

$$\begin{aligned}\star : K_i(R) \otimes K_k(R') \cong \pi_i X_R \otimes \pi_k X_{R'} &\xrightarrow{\wedge} \pi_{i+k}(X_R \wedge X_{R'}) \\ &\xrightarrow{\mu_*} \pi_{i+k} X_{R\otimes R'} \cong K_{i+k}(R \otimes R')\end{aligned}$$

for any two integers $i \geq 0$ and $k \geq 0$ (see for instance [21], Proposition 2.4.2). We shall actually concentrate our attention to the special case where $R'$ is the ring of integers $\mathbb{Z}$: the goal of Sections 3 and 4 is to investigate the relationships between the image of the product

$$\star : K_i(R) \otimes K_k(\mathbb{Z}) \longrightarrow K_{i+k}(R \otimes \mathbb{Z}) \cong K_{i+k}(R)$$

and the kernel of the stable and the non-stable Hurewicz homomorphism.

Remember that $X_{\mathbb{Z}}$ is a ring spectrum and let us call $j : S \to X_{\mathbb{Z}}$ its identity. Notice that $j$ corresponds to the map $B\Sigma_\infty^+ \to BGL(\mathbb{Z})^+$ given by the inclusion of the infinite symmetric group $\Sigma_\infty$ into $GL(\mathbb{Z})$. This map $j$ induces an isomorphism $j_* : \pi_1 S \xrightarrow{\cong} \pi_1 X_{\mathbb{Z}} \cong K_1(\mathbb{Z})$ and the image of $j_* : \pi_k S \to K_k(\mathbb{Z})$ for $k \geq 2$ is described in [22] and [26]. For any ring $R$, the above pairing $\mu$ provides then $X_R$

with an $X_{\mathbb{Z}}$-module structure. Let us first translate the results of Sections 1 and 2 in terms of algebraic $K$-theory.

**Proposition 3.1.** *Let $R$ be a ring, $i$ and $k$ two integers with $i \geq 0$, $k \geq 1$, and consider an element $x \in K_i(R)$ and an element $y \in K_k(\mathbb{Z})$ belonging to the image of $j_* : \pi_k S \to K_k(\mathbb{Z})$.*

*(a) For all $r \leq i$, $x \star y$ is an element of the kernel of the stable Hurewicz homomorphism $\bar{h}_{i+k} : K_{i+k}(R) \to H_{i+k} X_R(r-1)$.*

*(b) If $k \leq i-1$, then $x \star y$ is an element of the kernel of the non-stable Hurewicz homomorphisms $h_{i+k} : K_{i+k}(R) \to H_{i+k} E(R)$ and $h_{i+k} : K_{i+k}(R) \to H_{i+k} GL(R)$.*

*(c) If $i \geq 3$ and $k \leq i-1$, then $x \star y$ is an element of the kernel of $h_{i+k} : K_{i+k}(R) \to H_{i+k} St(R)$.*

*Proof.* The first assertion is a consequence of Lemma 1.1 and of the commutativity of the diagram

$$\begin{array}{ccccc}
\pi_i X_R(r-1) \otimes \pi_k S & \xrightarrow{\wedge} & \pi_{i+k} X_R(r-1) & \xrightarrow{\bar{h}_{i+k}} & H_{i+k} X_R(r-1) \\
\cong \downarrow (\gamma_{r-1})_* \otimes \mathrm{id} & & \cong \downarrow (\gamma_{r-1})_* & & \\
\pi_i X_R \otimes \pi_k S & \xrightarrow{\wedge} & \pi_{i+k} X_R & & \\
\downarrow \cong \otimes j_* & & \downarrow \cong & & \\
K_i(R) \otimes K_k(\mathbb{Z}) & \xrightarrow{\star} & K_{i+k}(R)\,, & &
\end{array}$$

where the bottom square commutes because $X_R$ is an $X_{\mathbb{Z}}$-module. In order to prove the last two assertions, consider the $(i-1)$-connected cover $BGL(R)^+(i-1)$ of the CW-complex $BGL(R)^+$, for $i \geq k+1 \geq 2$. The iterated homology suspension $\sigma : H_{i+k} BGL(R)^+(i-1) \to H_{i+k} X_R(i-1)$, which is an isomorphism since $k \leq i-1$, and the commutative diagram

$$\begin{array}{ccc}
K_{i+k}(R) & \xrightarrow{h_{i+k}} & H_{i+k} BGL(R)^+(i-1) \\
\downarrow = & & \sigma \downarrow \cong \\
K_{i+k}(R) & \xrightarrow{\bar{h}_{i+k}} & H_{i+k} X_R(i-1)
\end{array}$$

show that $h_{i+k} : K_{i+k}(R) \to H_{i+k} BGL(R)^+(i-1)$ fulfills $h_{i+k}(x \star y) = 0$ according to (a) for $r = i$. Since $i \geq 2$, assertion (b) then follows from the composition with the obvious homomorphism

$$H_{i+k}BGL(R)^+(i-1) \longrightarrow H_{i+k}BE(R)^+ \longrightarrow H_{i+k}BGL(R)^+\,.$$

If $i \geq 3$, this homomorphism factors even through $H_{i+k}BSt(R)^+$ and we get (c). □

Now, let us consider the case $k = 1$ and $i = r$: the fact that $j_* : \pi_1 S \to K_1(\mathbb{Z})$ is an isomorphism implies the following result, where $X_R(i-1, i+1]$ is written for $X_R(i-1)[i+1]$.

**Theorem 3.2.** *Let $R$ be any ring.*
*(a) For any integer $i \geq 0$, there is a natural exact sequence*

$$K_{i+2}(R) \xrightarrow{\bar{h}_{i+2}} H_{i+2}X_R(i-1) \xrightarrow{\bar{\nu}_{i+2}} K_i(R) \otimes K_1(\mathbb{Z}) \xrightarrow{\star} K_{i+1}(R) \xrightarrow{\bar{h}_{i+1}} H_{i+1}X_R(i-1) \longrightarrow 0\,.$$

*Moreover,* $\ker(\star) = \operatorname{image} \bar{\nu}_{i+2} \cong H_{i+2}X_R(i-1, i+1]$.
*(b) For any integer $i \geq 2$, the image of $\star : K_i(R) \otimes K_1(\mathbb{Z}) \to K_{i+1}(R)$ is contained in the kernel of the non-stable Hurewicz homomorphisms $h_{i+1} : K_{i+1}(R) \to H_{i+1}E(R)$ and $h_{i+1} : K_{i+1}(R) \to H_{i+1}GL(R)$.*
*(c) For any integer $i \geq 3$, the image of $\star : K_i(R) \otimes K_1(\mathbb{Z}) \to K_{i+1}(R)$ is contained in the kernel of $h_{i+1} : K_{i+1}(R) \to H_{i+1}St(R)$.*

*Proof.* Assertion (a) follows from Corollary 1.3 and Remark 2.3 for the spectrum $X_R(i-1)$ since the diagram

$$\begin{array}{ccc} K_i(R) \otimes \pi_1 S & \xrightarrow{\wedge} & \pi_{i+1} X_R \\ \cong \downarrow \mathrm{id}\otimes j_* & & \downarrow \cong \\ K_i(R) \otimes K_1(\mathbb{Z}) & \xrightarrow{\star} & K_{i+1}(R) \end{array}$$

commutes again because of the $X_{\mathbb{Z}}$-module structure of $X_R$. Assertions (b) and (c) are direct consequences of Proposition 3.1. (b) and (c). □

It is possible to obtain a similar information on the stable and the non-stable Hurewicz homomorphism in any dimension $i \geq r+1$. Proposition 2.1 provides the exact sequence

$$\cdots \longrightarrow H_{i+1}X_R(r-1, i] \xrightarrow{(\alpha_{i-1})_*} H_{i+1}X_R(r-1, i-1] \xrightarrow{\bar{\partial}} K_i(R) \xrightarrow{\bar{h}_i} H_iX_R(r-1) \xrightarrow{(\alpha_{i-1})_*} H_iX_R(r-1, i-1] \longrightarrow 0\,.$$

**Proposition 3.3.** *Let $R$ be any ring, $i$ and $r$ positive integers such that $r \leq i \leq 2r-1$, then the kernel of the non-stable Hurewicz homomorphism*

$$h_i : K_i(R) \to H_i BGL(R)^+(r-1)$$

*is exactly the image of $\bar{\partial}$.*

*Proof.* Let us consider the homology exact sequence of the Postnikov cofibration

$$\Sigma^i(H(K_i(R)) \xrightarrow{\gamma_{i-1}} X_R(r-1,i] \xrightarrow{\alpha_{i-1}} X_R(r-1,i-1]\,,$$

and the corresponding homology exact sequence obtained from the Serre spectral sequence of the fibration of CW-complexes

$$K(K_i(R),i) \longrightarrow BGL(R)^+(r-1,i] \longrightarrow BGL(R)^+(r-1,i-1]\,.$$

We obtain the commutative diagram

$$\begin{array}{ccccc}
H_{i+1}X_R(r-1,i-1] & \xrightarrow{\bar{\partial}} & K_i(R) & \xrightarrow{\bar{h}_i} & H_iX_R(r-1) \\
\uparrow \sigma & & \uparrow = & & \uparrow \\
H_{i+1}BGL(R)^+(r-1,i-1] & \xrightarrow{\partial} & K_i(R) & \xrightarrow{h_i} & H_iBGL(R)^+(r-1)\,,
\end{array}$$

where the horizontal sequences are exact and the three vertical arrows are iterated suspensions. The left iterated homology suspension $\sigma$ is surjective if $i+1 \leq 2r$ and even an isomorphism if $i+1 \leq 2r-1$ (see [36], p. 382); consequently we may conclude that image $\partial =$ image $\bar{\partial}$. □

## 4. Products and the non-stable Hurewicz homomorphism in low dimensions

The purpose of this section is to study the relationships between the algebraic $K$-theory of a ring $R$ and the integral homology of its linear groups in low dimensions. In dimension 2, the following isomorphisms are known (see [4]):

$$K_2(R) \cong H_2E(R) \quad \text{and} \quad H_2GL(R) \cong K_2(R) \oplus \Lambda^2(K_1(R))\,.$$

Let us start by looking at dimensions 3 and 4. Let $\Gamma(-)$ be the quadratic functor defined on abelian groups by J.H.C. Whitehead in Section 5 of [37]: if $Y$

is a simply connected CW-complex, then the group $\Gamma_3(Y)$ in the Whitehead exact sequence of the space $Y$ turns out to be isomorphic to $\Gamma(\pi_2 Y)$.

**Theorem 4.1.** *For any ring $R$, there is a natural exact sequence*

$$K_4(R) \xrightarrow{h_4} H_4E(R) \xrightarrow{\nu_4} \Gamma(K_2(R)) \xrightarrow{\chi_3} K_3(R) \xrightarrow{h_3} H_3E(R) \longrightarrow 0$$

*and* $\ker h_3$ *is isomorphic to the image of the product homomorphism* $\star : K_2(R) \otimes K_1(\mathbb{Z}) \to K_3(R)$. *In particular,* $H_3E(R) \cong K_3(R)/(K_2(R) \star K_1(\mathbb{Z}))$.

*Proof.* The exact sequence is just the Whitehead exact sequence (see [37]) of the space $BE(R)^+$ since $\Gamma_3(BE(R)^+) = \Gamma(K_2(R))$. In order to determine the image of $\chi_3$, consider the exact sequence given by Proposition 2.2 for $X = X_R(1)$ and $r = 2$, and also the corresponding exact sequence obtained from the Serre spectral sequence of the fibration of CW-complexes

$$K(K_3(R), 3) \longrightarrow BE(R)^+[3] \longrightarrow K(K_2(R), 2)\,.$$

We get the commutative diagram

$$\begin{array}{ccccccccccc}
0 & \longrightarrow & H_4X_R(1,3) & \xrightarrow{\bar{\varphi}} & \overbrace{H_4(\Sigma^2 H(K_2(R)))}^{\cong K_2(R)\otimes K_1(\mathbb{Z})} & \xrightarrow{\star} & K_3(R) & \xrightarrow{\bar{h}_3} & H_3X_R(1) & \longrightarrow & 0 \\
 & & \uparrow & & \uparrow \sigma & & \uparrow = & & \uparrow & & \\
0 & \longrightarrow & H_4BE(R)^+[3] & \xrightarrow{\varphi} & H_4(K(K_2(R),2)) & \xrightarrow{\partial} & K_3(R) & \xrightarrow{h_3} & H_3BE(R)^+ & \longrightarrow & 0\,,
\end{array}$$

where the vertical arrows are iterated suspensions. It turns out that

$$H_4(K(K_2(R), 2)) \cong \Gamma_3(K(K_2(R), 2)) \cong \Gamma(K_2(R))$$

and the argument of the proof of Proposition 2.2 shows again that the homomorphism $\chi_3$ in the Whitehed exact sequence is exactly $\partial$. According to the proof of Proposition 3.3, the iterated homology suspension $\sigma$ is surjective and one gets $\operatorname{image}\chi_3 = \operatorname{image}\partial = K_2(R) \star K_1(\mathbb{Z})$. Notice that this computation of the image of $\chi_3$ can also be deduced from Section 2.2.6 of [21]. □

**Remark 4.2.** This extends the result for fields given in [31], Corollary 5.2, to the case of any ring $R$.

In order to understand the 4-dimensional and 5-dimensional Hurewicz homomorphisms, let us use exactly the same idea (but now for $r = 3$) for the 2-connected CW-complex $BSt(R)^+$, respectively the 2-connected $K$-theory spectrum $X_R(2)$.

**Theorem 4.3.** *Let $R$ be any ring.*

*(a) There is a natural exact sequence*

$$H_6X_R(2,4] \xrightarrow{\bar\psi} K_5(R) \xrightarrow{h_5} H_5St(R) \xrightarrow{\nu_5} K_3(R)\otimes K_1(\mathbb{Z}) \xrightarrow{\star} K_4(R) \xrightarrow{h_4} H_4St(R) \longrightarrow 0\,.$$

*In particular,* $H_4St(R) \cong K_4(R)/(K_3(R)\star K_1(\mathbb{Z}))$.

*(b) There is a natural exact sequence*

$$K_4(R)\otimes K_1(\mathbb{Z}) \xrightarrow{\bar\theta} H_6X_R(2,4] \xrightarrow{\bar\eta} {}_2(K_3(R)) \longrightarrow 0\,.$$

*(c) The composition* $\bar\psi\,\bar\theta$ *is the product map* $\star : K_4(R)\otimes K_1(\mathbb{Z}) \to K_5(R)$. *Consequently, there is a natural short exact sequence*

$$0 \longrightarrow K_4(R)\star K_1(\mathbb{Z}) \longrightarrow \ker h_5 \longrightarrow Q \longrightarrow 0\,,$$

*where* $Q$ *is a quotient of* ${}_2(K_3(R))$.

*Proof.* As in the previous proof, we use Proposition 2.2, but consider in this case the following commutative diagram:

$$\begin{array}{ccccccccccc}
0 & \longrightarrow & H_5X_R(2,4] & \xrightarrow{\bar\varphi} & \overbrace{H_5(\Sigma^3H(K_3(R)))}^{\cong K_3(R)\otimes K_1(\mathbb{Z})} & \xrightarrow{\star} & K_4(R) & \xrightarrow{\bar h_4} & H_4X_R(2) & \longrightarrow & 0 \\
 & & \uparrow & & \uparrow\sigma & & \uparrow = & & \uparrow & & \\
0 & \longrightarrow & H_5BSt(R)^+[4] & \xrightarrow{\varphi} & H_5(K(K_3(R),3)) & \xrightarrow{\partial} & K_4(R) & \xrightarrow{h_4} & H_4BSt(R)^+ & \longrightarrow & 0\,.
\end{array}$$

However, this time, $\sigma$ is even an isomorphism. Observe that $H_5BSt(R)^+[4]$ is isomorphic to the kernel of $\star : K_3(R)\otimes K_1(\mathbb{Z}) \to K_4(R)$. The Whitehead exact sequence of $BSt(R)^+$ is

$$\cdots \longrightarrow \Gamma_5(BSt(R)^+) \xrightarrow{\chi_5} K_5(R) \xrightarrow{h_5} H_5St(R) \xrightarrow{\nu_5} \Gamma_4(BSt(R)^+) \xrightarrow{\chi_4} K_4(R) \xrightarrow{h_4} H_4St(R) \longrightarrow 0$$

and it is easy to check that $\Gamma_4(BSt(R)^+) \cong K_3(R)\otimes K_1(\mathbb{Z}) \cong H_5(K(K_3(R),3))$. In order to understand the kernel of $h_5$, let us use the exact sequence given by Proposition 2.6 for $r=3$, $i=5$, and the exact sequence coming from the homology Serre spectral sequence of the fibration of CW-complexes

$$K(K_5(R),5) \longrightarrow BSt(R)^+[5] \longrightarrow BSt(R)^+[4]\,.$$

We obtain the commutative diagram

$$\begin{array}{ccccccccc}
H_6X_R(2,4] & \xrightarrow{\bar{\psi}} & K_5(R) & \xrightarrow{\bar{h}_5} & H_5X_R(2) & \longrightarrow & H_5X_R(2,4] & \longrightarrow & 0 \\
\uparrow \sigma & & \uparrow = & & \uparrow & & \uparrow & & \\
H_6BSt(R)^+[4] & \xrightarrow{\psi} & K_5(R) & \xrightarrow{h_5} & H_5BSt(R)^+ & \longrightarrow & H_5BSt(R)^+[4] & \longrightarrow & 0\,,
\end{array}$$

where the vertical arrows are iterated suspensions. According to the proof of Proposition 3.3, the iterated homology suspension $\sigma$ is surjective and therefore $\operatorname{image}\psi = \operatorname{image}\bar{\psi}$. On the other hand, the group $H_6X_R(2,4]$ and the image of $\bar{\psi}$ may be described by Proposition 2.6 and Lemmas 2.4 and 2.5. □

The following corollary follows from the five lemma and the argument of the proofs of Theorems 4.1 and 4.3.

**Corollary 4.4.** *For any ring $R$, the iterated homology suspensions $H_3E(R) \cong H_3BE(R)^+ \to H_3X_R(1)$ and $H_4St(R) \cong H_4BSt(R)^+ \to H_4X_R(2)$ are isomorphisms.*

**Remark 4.5.** Observe that $h_4 : K_4(R) \to H_4St(R)$ is an isomorphism up to 2-torsion. This produces the following consequence of Proposition 9 of [9]. Let $l$ be an odd prime, $\xi_l$ a primitive $l$-root of unity of order $l$. Let $R = \mathbb{Z}[\xi_l + \xi_l^{-1}]$ be the ring of integers of the maximal real subfield of the cyclotomic field $\mathbb{Q}(\xi_l)$. The vanishing of the group $H_4St(R)$ in this case would imply the Kummer-Vandiver conjecture for the prime $l$.

## 5. Products in the algebraic $K$-theory of the ring of integers $\mathbb{Z}$

This section is devoted to the study of products in the algebraic $K$-theory of the ring of integers $\mathbb{Z}$:

$$\star : K_i(\mathbb{Z}) \otimes K_k(\mathbb{Z}) \longrightarrow K_{i+k}(\mathbb{Z})\,.$$

Let us start by describing the results on low-dimensional products given by Section 4 in the case where $R = \mathbb{Z}$.

**Proposition 5.1.**

*(a) The product homomorphism $\star : K_i(\mathbb{Z}) \otimes K_1(\mathbb{Z}) \to K_{i+1}(\mathbb{Z})$ is an isomorphism if $i = 1$, injective if $i = 2$, and trivial if $i \not\equiv 1$ or $2$ mod 8.*

*(b) The product homomorphism $\star : K_i(\mathbb{Z}) \otimes K_2(\mathbb{Z}) \to K_{i+2}(\mathbb{Z})$ is trivial if $i \not\equiv 1$ mod 8.*

*(c)* $H_4SL(\mathbb{Z}) \cong \mathbb{Z}/2$ *and* $H_4St(\mathbb{Z}) = 0$.
*(d) There is a short exact sequence*

$$0 \longrightarrow K_5(\mathbb{Z}) \xrightarrow{h_5} H_5St(\mathbb{Z}) \xrightarrow{\nu_5} \mathbb{Z}/2 \longrightarrow 0\,.$$

*Proof.* The assertion (a) is well known for $i = 1$. Theorem 4.1 produces the exact sequence

$$K_4(\mathbb{Z}) \xrightarrow{h_4} H_4SL(\mathbb{Z}) \xrightarrow{\nu_4} \underbrace{\Gamma(\mathbb{Z}/2)}_{\mathbb{Z}/4} \xrightarrow{\chi_3} \underbrace{K_3(\mathbb{Z})}_{\mathbb{Z}/48} \xrightarrow{h_3} \underbrace{H_3SL(\mathbb{Z})}_{\mathbb{Z}/24} \longrightarrow 0$$

(see [2], [19] and [37], Sections 5 and 13) and asserts that the product $\star : K_2(\mathbb{Z}) \otimes K_1(\mathbb{Z}) \to K_3(\mathbb{Z})$ is injective. Recently, J. Rognes and C. Weibel deduced from the work of V. Voevodsky [33] the complete calculation of the 2-torsion of the algebraic $K$-theory of $\mathbb{Z}$ (see Table 1 of [35] and Theorem 0.6 of [28]). This, together with another argument of J. Rognes, shows that $K_4(\mathbb{Z}) = 0$ and implies that $H_4SL(\mathbb{Z})$ is cyclic of order 2. Moreover, $K_i(\mathbb{Z})$ is a finite odd torsion group if $i$ is a positive integer $\equiv$ 0, 4, or 6 mod 8. Therefore, $K_i(\mathbb{Z}) \star K_1(\mathbb{Z}) = 0$ if $i \equiv$ 0, 4, or 6 mod 8 or if $i + 1 \equiv$ 0, 4, or 6 mod 8. This gives (a), and (b) follows from (a) since $K_2(\mathbb{Z}) = K_1(\mathbb{Z}) \star K_1(\mathbb{Z})$. Note that the first author proved the triviality of $\star : K_3(\mathbb{Z}) \otimes K_1(\mathbb{Z}) \to K_4(\mathbb{Z})$ in [6] before Rognes and Weibel's proof of the vanishing of $K_4(\mathbb{Z})$. The calculation of $K_i(\mathbb{Z}) \star K_1(\mathbb{Z})$ when $i \equiv 1$ or 2 mod 8 and of $K_i(\mathbb{Z}) \star K_2(\mathbb{Z})$ when $i \equiv 1$ mod 8 will be given by Theorems 5.7 and 5.9 below.

Now, let us apply Theorem 4.3. The map $\bar{\psi}$ is actually the connecting homomorphism of the homology exact sequence of the cofibration

$$\Sigma^5 H(K_5(\mathbb{Z})) \longrightarrow X_{\mathbb{Z}}(2,5] \longrightarrow X_{\mathbb{Z}}(2,4]\,.$$

It is of course possible to consider the analogous cofibration for the sphere spectrum $S$

$$\Sigma^5 H(\pi_5 S) \longrightarrow S(2,5] \longrightarrow S(2,4]\,.$$

The identity $j : S \to X_{\mathbb{Z}}$ of the ring spectrum $X_{\mathbb{Z}}$ induces the commutative diagram

$$\begin{array}{ccccc} H_6S(2,4] & \longrightarrow & \pi_5 S & = & 0 \\ \downarrow{\scriptstyle j_*} & & \downarrow & & \\ H_6X_{\mathbb{Z}}(2,4] & \xrightarrow{\bar{\psi}} & K_5(\mathbb{Z}) & & \end{array}$$

which shows that $\bar{\psi}\, j_* = 0$. Now, look at the following commutative diagram where the bottom homomorphism is given by the second assertion of Theorem 4.3:

$$\begin{array}{ccccc} H_6S(2,4] & \xrightarrow{\cong} & H_6(\Sigma^3 H(\pi_3 S)) & \cong & {}_2(\pi_3 S) \\ \downarrow{\scriptstyle j_*} & & & & \downarrow{\scriptstyle \cong} \\ H_6X_{\mathbb{Z}}(2,4] & \xrightarrow{\bar{\eta}} & H_6(\Sigma^3 H(K_3(\mathbb{Z}))) & \cong & {}_2(K_3(\mathbb{Z}))\,. \end{array}$$

The top horizontal arrow is an isomorphism because the vanishing of $\pi_4 S$ exhibits an equivalence $S(2,4] \simeq \Sigma^3 H(\pi_3 S)$. The right vertical arrow is an isomorphism since the homomorphism $\pi_3 S \to K_3(\mathbb{Z})$, induced by $j$, is injective (remember that $\pi_3 S \cong \mathbb{Z}/24$ and $K_3(\mathbb{Z}) \cong \mathbb{Z}/48$). Therefore, there exists a splitting $\tau : {}_2(K_3(\mathbb{Z})) \to H_6X_{\mathbb{Z}}(2,4]$ of $\bar{\eta}$ such that $\tau$ is the composition of an isomorphism ${}_2(K_3(\mathbb{Z})) \xrightarrow{\cong} H_6S(2,4]$ with $j_*$. It then follows from the vanishing of the composition $\bar{\psi}\, j_*$ that $\bar{\psi}\,\tau = 0$. Consequently, the group $Q$ of Theorem 4.3 (c) is trivial if $R = \mathbb{Z}$ since $Q = \mathrm{image}\,(\bar{\eta}\,\tau)$, and $\ker h_5 \cong \mathrm{image}\,(\bar{\psi}\,\bar{\theta})$ is the image of the product map $\star : K_4(\mathbb{Z}) \otimes K_1(\mathbb{Z}) \to K_5(\mathbb{Z})$. Consequently, there is an exact sequence

$$0 \longrightarrow K_4(\mathbb{Z}) \star K_1(\mathbb{Z}) \xrightarrow{\bar{\psi}\,\bar{\theta}} K_5(\mathbb{Z}) \xrightarrow{h_5} H_5St(\mathbb{Z}) \xrightarrow{\nu_5} K_3(\mathbb{Z}) \otimes K_1(\mathbb{Z}) \xrightarrow{\star}$$
$$K_4(\mathbb{Z}) \xrightarrow[\cong]{h_4} H_4St(\mathbb{Z}) \longrightarrow 0\,.$$

The fact that $K_4(\mathbb{Z}) = 0$ provides the vanishing of $H_4St(\mathbb{Z})$ and the short exact sequence (d) (which was already given in Section 6 of [7])

$$0 \longrightarrow K_5(\mathbb{Z}) \xrightarrow{h_5} H_5St(\mathbb{Z}) \xrightarrow{\nu_5} \mathbb{Z}/2 \longrightarrow 0\,.$$

According to Theorem 1 of [20], Theorem 1 of [30], Table 1 of [35] and Theorem 0.6 of [28], $K_5(\mathbb{Z}) \cong \mathbb{Z} \oplus T$, where $T$ is a finite abelian 3-group: thus, $H_5St(\mathbb{Z}) \cong \mathbb{Z} \oplus T$ and $h_5$ is multiplication by 2 on the infinite cyclic factor of $K_5(\mathbb{Z})$. □

**Corollary 5.2.** *Let $R$ be a ring such that the homomorphism $\ell_* : K_3(\mathbb{Z}) \to K_3(R)$ (induced by the obvious map $\ell : \mathbb{Z} \to R$) induces an isomorphism $\ell_* \otimes \mathrm{id} : K_3(\mathbb{Z}) \otimes \mathbb{Z}/2 \xrightarrow{\cong} K_3(R) \otimes \mathbb{Z}/2$ (for instance, if $R = \mathbb{Q}$ or any localization of $\mathbb{Z}$). Then the non-stable Hurewicz homomorphism $h_4 : K_4(R) \to H_4St(R)$ is an isomorphism.*

*Proof.* This follows from the commutativity of the diagram

$$\begin{array}{ccccccc}
K_3(\mathbb{Z}) \otimes K_1(\mathbb{Z}) & \xrightarrow{\star} & K_4(\mathbb{Z}) & = & 0 & & \\
\cong \big\downarrow \ell_* \otimes \mathrm{id} & & \big\downarrow \ell_* & & & & \\
K_3(R) \otimes K_1(\mathbb{Z}) & \xrightarrow{\star} & K_4(R) & \xrightarrow{h_4} & H_4 St(R) & \longrightarrow & 0\,,
\end{array}$$

where the rows are the exact sequences given by Theorem 4.3 (a). □

The argument of the proof of Proposition 5.1 (d) produces also the next two corollaries

**Corollary 5.3.** *Let $R$ be a ring such that the composition of $j : S \to X_{\mathbb{Z}}$ with the obvious map $X_{\mathbb{Z}} \to X_R$ induces an isomorphism ${}_2(\pi_3 S) \xrightarrow{\cong} {}_2(K_3(R))$. Then $K_4(R) \star K_1(\mathbb{Z}) \cong \ker(h_5 : K_5(R) \to H_5 St(R))$.*

**Corollary 5.4.** *If $R$ is a ring as in Corollary 5.3 (for instance, if $R = \mathbb{Z}$ or any localization of $\mathbb{Z}$), then the iterated homology suspension $H_5 St(R) \cong H_5 BSt(R)^+ \to H_5 X_R(2)$ is an isomorphism.*

*Proof.* By hypothesis, one has actually the exact sequences

$$\begin{array}{ccccccccc}
H_6(\Sigma^4 H(K_4(R))) & \xrightarrow{\bar\psi\,\bar\theta} & K_5(R) & \xrightarrow{\bar h_5} & H_5 X_R(2) & \longrightarrow & H_5 X_R(2,4] & \longrightarrow & 0 \\
\big\uparrow \sigma & & \big\uparrow = & & \big\uparrow & & \big\uparrow \sigma' & & \\
H_6 K(K_4(R),4) & \xrightarrow{\psi\theta} & K_5(R) & \xrightarrow{h_5} & H_5 BSt(R)^+ & \longrightarrow & H_5 BSt(R)^+[4] & \longrightarrow & 0\,,
\end{array}$$

and the iterated homology suspension $\sigma$ is clearly an isomorphism; the same is true for $\sigma'$ because of the commutativity of

$$\begin{array}{ccc}
H_5 X_R(2,4] & \xrightarrow{\cong} & H_5(\Sigma^3 H(K_3(R)) \\
\big\uparrow \sigma' & & \big\uparrow \cong \\
H_5 BSt(R)^+[4] & \xrightarrow{\cong} & H_5 K(K_3(R),3)\,.
\end{array}$$

The assertion then follows from the five lemma. □

Let us now consider maps

$$K_i(\mathbb{Z}) \otimes K_k(\mathbb{Z}) \xrightarrow{\star} K_{i+k}(\mathbb{Z}) \longrightarrow K_{i+k}(\mathbb{Z}) \otimes \hat{\mathbb{Z}_2}$$

for all positive integers $i$ and $k$, where the second arrow is the tensor product of $K_{i+k}(\mathbb{Z})$ with the inclusion of $\mathbb{Z}$ into the ring of 2-adic integers $\hat{\mathbb{Z}_2}$. We call these maps 2-adic products for $K_*(\mathbb{Z})$ and continue to denote them by the symbol $\star$. Again because of Table 1 of [35] (see also Theorem 0.6 of [28]), $K_i(\mathbb{Z})$ is a finite odd torsion group if $i$ is a positive integer $\equiv$ 0, 4, or 6 mod 8 and $\cong \mathbb{Z} \oplus$ (finite odd torsion group) if $i \equiv 5$ mod 8; thus, the only 2-adic products which can be non trivial are the following:

$$\begin{array}{rcl}
K_{8s+1}(\mathbb{Z}) \otimes K_{8t+1}(\mathbb{Z}) & \xrightarrow{\star} & K_{8(s+t)+2}(\mathbb{Z}) \otimes \hat{\mathbb{Z}_2} \\
K_{8s+1}(\mathbb{Z}) \otimes K_{8t+2}(\mathbb{Z}) & \xrightarrow{\star} & K_{8(s+t)+3}(\mathbb{Z}) \otimes \hat{\mathbb{Z}_2} \\
K_{8s+2}(\mathbb{Z}) \otimes K_{8t+5}(\mathbb{Z}) & \xrightarrow{\star} & K_{8(s+t)+7}(\mathbb{Z}) \otimes \hat{\mathbb{Z}_2} \\
K_{8s+2}(\mathbb{Z}) \otimes K_{8t+7}(\mathbb{Z}) & \xrightarrow{\star} & K_{8(s+t+1)+1}(\mathbb{Z}) \otimes \hat{\mathbb{Z}_2} \\
K_{8s+3}(\mathbb{Z}) \otimes K_{8t+7}(\mathbb{Z}) & \xrightarrow{\star} & K_{8(s+t+1)+2}(\mathbb{Z}) \otimes \hat{\mathbb{Z}_2} \\
K_{8s+5}(\mathbb{Z}) \otimes K_{8t+5}(\mathbb{Z}) & \xrightarrow{\star} & K_{8(s+t+1)+2}(\mathbb{Z}) \otimes \hat{\mathbb{Z}_2}
\end{array}$$

for $s$ and $t \geq 0$. We now want to determine these products.

The inclusion $\mathbb{Z} \hookrightarrow \mathbb{R}$ induces a map $\lambda : BGL(\mathbb{Z})^+ \to BO$ and the induced homomorphism

$$\lambda_* : K_*(\mathbb{Z}) \longrightarrow \pi_* BO$$

is a ring homomorphism since it can be written as the composition $\lambda_* : K_*(\mathbb{Z}) \to K_*(\mathbb{R}) \to \pi_* BO$, where both arrows are ring homomorphisms (see [13], p. 50 and Section 3). One can understand the kernel of $\lambda_*$ at the prime 2 by the following argument. If $p$ is a prime $\equiv$ 3 or 5 mod 8, M. Bökstedt introduced in [12] (see also [23] and Section 4 of [18]) a space $J(p)$ which is defined by the pull-back diagram

$$\begin{array}{ccc}
J(p) & \xrightarrow{\lambda'} & BO \\
\downarrow{\scriptstyle f_p} & & \downarrow{\scriptstyle c} \\
F\Psi^p & \xrightarrow{b} & BU\,,
\end{array}$$

where $F\Psi^p$ is the fiber of $(\Psi^p - 1) : BU \to BU$ (recall that $F\Psi^p \simeq BGL(\mathbb{F}_p)^+$ by Theorem 7 of [24]), $b$ the Brauer lifting and $c$ the complexification. The fibers of the horizontal maps are homotopy equivalent to the unitary group $U \simeq SU \times S^1$. More precisely, Bökstedt was interested in the covering space $JK(\mathbb{Z}, p)$ of $J(p)$ corresponding to the cyclic subgroup of order 2 of $\pi_1 J(p) \cong \mathbb{Z} \oplus \mathbb{Z}/2$. After completion at the prime 2, he constructed a map

$$\tilde{\varphi} : (BGL(\mathbb{Z})^+)\hat{_2} \longrightarrow JK(\mathbb{Z}, p)\hat{_2}$$

which induces a split surjection on all homotopy groups. Let us write $\widehat{\lambda}$ and $\widehat{\lambda'}$ for the 2-completion of the maps $\lambda$ and $\lambda'$ respectively: it turns out that the composition

$$(BGL(\mathbb{Z})^+)\hat{_2} \xrightarrow{\tilde{\varphi}} JK(\mathbb{Z}, p)\hat{_2} \longrightarrow J(p)\hat{_2} \xrightarrow{\widehat{\lambda'}} BO\hat{_2}$$

is exactly $\widehat{\lambda}$. Recall that the localization exact sequence in $K$-theory implies that

$$(BGL(\mathbb{Z}[\tfrac{1}{2}])^+)\hat{_2} \simeq (BGL(\mathbb{Z})^+)\hat{_2} \times (S^1)\hat{_2}\,.$$

Therefore, $\tilde{\varphi}$ provides a map

$$\varphi : (BGL(\mathbb{Z}[\tfrac{1}{2}])^+)\hat{_2} \longrightarrow J(p)\hat{_2}$$

which also induces a split surjection on all homotopy groups. Since the 2-torsion of $K_*(\mathbb{Z})$ is known by Table 1 of [35] and Theorem 0.6 of [28], it is easy to check that $\tilde{\varphi}$ and $\varphi$ are actually homotopy equivalences. Consequently, we obtain (see also Corollary 8 of [35]):

**Proposition 5.5.** *For all primes $p \equiv 3$ or $5$ mod 8, there is a pull-back diagram*

$$\begin{array}{ccc}
(BGL(\mathbb{Z})^+)\hat{_2} \times (S^1)\hat{_2} & \xrightarrow{\widehat{\lambda'}} & BO\hat{_2} \\
\Big\downarrow \widehat{f_p} & & \Big\downarrow \widehat{c} \\
(F\Psi^p)\hat{_2} & \xrightarrow{\widehat{b}} & BU\hat{_2}\,.
\end{array}$$

*Consequently, there is a fibration*

$$SU\hat{_2} \xrightarrow{\eta} (BGL(\mathbb{Z})^+)\hat{_2} \xrightarrow{\widehat{\lambda}} BO\hat{_2}\,.$$

This fibration induces the long exact sequence

$$\cdots \longrightarrow \pi_i SU \otimes \hat{\mathbb{Z}_2} \xrightarrow{\eta_*} K_i(\mathbb{Z}) \otimes \hat{\mathbb{Z}_2} \xrightarrow{\widehat{\lambda}_*} \pi_i BO \otimes \hat{\mathbb{Z}_2} \longrightarrow \pi_{i-1} SU \otimes \hat{\mathbb{Z}_2} \longrightarrow \cdots .$$

Remember that $\pi_i SU = 0$ if $i$ is even and $\pi_i SU \cong \mathbb{Z}$ if $i$ is odd $\geq 3$.

**Theorem 5.6.** *The 2-adic products*

$$K_{8s+3}(\mathbb{Z}) \otimes K_{8t+7}(\mathbb{Z}) \xrightarrow{\star} K_{8(s+t+1)+2}(\mathbb{Z}) \otimes \hat{\mathbb{Z}_2}$$

$$K_{8s+5}(\mathbb{Z}) \otimes K_{8t+5}(\mathbb{Z}) \xrightarrow{\star} K_{8(s+t+1)+2}(\mathbb{Z}) \otimes \hat{\mathbb{Z}_2}$$

*are trivial for all integers $s$ and $t \geq 0$.*

*Proof.* For any product mentioned in the statement of the theorem, let us consider the commutative diagram

$$\begin{array}{ccccc} K_i(\mathbb{Z}) \otimes K_k(\mathbb{Z}) & \xrightarrow{\star} & K_{i+k}(\mathbb{Z}) & \longrightarrow & K_{i+k}(\mathbb{Z}) \otimes \hat{\mathbb{Z}_2} \\ \downarrow \lambda_* \otimes \lambda_* & & \downarrow \lambda_* & & \downarrow \widehat{\lambda}_* \\ \pi_i BO \otimes \pi_k BO & \longrightarrow & \pi_{i+k} BO & \longrightarrow & \pi_{i+k} BO \otimes \hat{\mathbb{Z}_2}, \end{array}$$

where the bottom left horizontal arrow is the product map in $\pi_* BO$ and where the right horizontal arrows denote the tensor product with $\hat{\mathbb{Z}_2}$. Let $x \in K_i(\mathbb{Z})$ and $y \in K_k(\mathbb{Z})$. One has clearly $\lambda_*(y) = 0$ since $\pi_k BO = 0$ for $k = 8t+5$ or $k = 8t+7$. Thus, $\lambda_*(x \star y) = \lambda_*(x)\lambda_*(y) = 0$. This shows that the 2-adic product $x \star y \in K_{i+k}(\mathbb{Z}) \otimes \hat{\mathbb{Z}_2}$ belongs to the kernel of $\widehat{\lambda}_*$, and consequently to the image of $\eta_* : \pi_{i+k} SU \otimes \hat{\mathbb{Z}_2} \to K_{i+k}(\mathbb{Z}) \otimes \hat{\mathbb{Z}_2}$. Since $i+k$ is even, the group $\pi_{i+k} SU$ is trivial and $x \star y$ vanishes. □

In the next theorem, we look at the groups $K_{8s+1}(\mathbb{Z})$ for $s \geq 0$. Remember that $K_1(\mathbb{Z}) \cong \mathbb{Z}/2$ and that $K_{8s+1}(\mathbb{Z}) \cong \mathbb{Z} \oplus \mathbb{Z}/2 \oplus$ (finite odd torsion group) for $s \geq 1$ by Table 1 of [35] and Theorem 0.6 of [28]. The fibration given by Proposition 5.5 provides the exact sequence

$$0 \longrightarrow \pi_{8s+1} SU \otimes \hat{\mathbb{Z}_2} \cong \hat{\mathbb{Z}_2} \xrightarrow{\eta_*} K_{8s+1}(\mathbb{Z}) \otimes \hat{\mathbb{Z}_2} \xrightarrow{\widehat{\lambda}_*} \pi_{8s+1} BO \otimes \hat{\mathbb{Z}_2} \cong \mathbb{Z}/2 \longrightarrow 0$$

if $s \geq 1$ (if $s = 0$, $\pi_1 SU = 0$). Let us write $x_s$ for the element of order 2 in $K_{8s+1}(\mathbb{Z})$. We denote by $y_s$, for $s \geq 1$, the generator of the infinite cyclic summand of $K_{8s+1}(\mathbb{Z})$ whose image under the 2-adic completion $K_{8s+1}(\mathbb{Z}) \to K_{8s+1}(\mathbb{Z}) \otimes \hat{\mathbb{Z_2}}$ is exactly the image of a generator of $\pi_{8s+1}SU \otimes \hat{\mathbb{Z_2}}$ under the homomorphism $\eta_* : \pi_{8s+1}SU \otimes \hat{\mathbb{Z_2}} \cong \hat{\mathbb{Z_2}} \to K_{8s+1}(\mathbb{Z}) \otimes \hat{\mathbb{Z_2}}$.

**Theorem 5.7.** *Consider the 2-adic product*

$$K_{8s+1}(\mathbb{Z}) \otimes K_{8t+1}(\mathbb{Z}) \xrightarrow{\star} K_{8(s+t)+2}(\mathbb{Z}) \otimes \hat{\mathbb{Z_2}}$$

*for any integers $s$, $t \geq 0$.*
*(a) For all $s$ and $t \geq 1$, $y_s \star y_t = 0$.*
*(b) For all $s \geq 0$ and all $t \geq 1$, $x_s \star y_t = 0$.*
*(c) For all $s$ and $t \geq 0$, $x_s \star x_t$ is the generator of $K_{8(s+t)+2}(\mathbb{Z}) \otimes \hat{\mathbb{Z_2}} \cong \mathbb{Z}/2$.*

*Proof.* The commutativity of the square

$$\begin{array}{ccc} K_{8s+1}(\mathbb{Z}) & \longrightarrow & K_{8s+1}(\mathbb{Z}) \otimes \hat{\mathbb{Z_2}} \\ \downarrow \lambda_* & & \downarrow \widehat{\lambda}_* \\ \pi_{8s+1}BO & \xrightarrow{\cong} & \pi_{8s+1}BO \otimes \hat{\mathbb{Z_2}}\,, \end{array}$$

where the horizontal arrows denote the tensor product with $\hat{\mathbb{Z_2}}$, and the definition of $y_s$ show that $\lambda_*(y_s) = 0$. We deduce similarily that $\lambda_*(y_t) = 0$. This implies the vanishing of $\widehat{\lambda}_*(y_s \star y_t)$ and $\widehat{\lambda}_*(x_s \star y_t)$. The fact that $\pi_{8(s+t)+2}SU = 0$ then enables us to deduce the triviality of the products $y_s \star y_t$ and $x_s \star y_t$ in $K_{8(s+t)+2}(\mathbb{Z}) \otimes \hat{\mathbb{Z_2}}$. The image of $x_s$ under $\lambda_*$ is the generator $c_s$ of $\pi_{8s+1}BO \cong \mathbb{Z}/2$ and it is known that $c_s c_t$ is non trivial in $\pi_{8(s+t)+2}BO$ (see [32], p. 304). Therefore, it follows from the equality $\lambda_*(x_s \star x_t) = c_s c_t$ that the product $x_s \star x_t$ does not vanish. □

**Corollary 5.8.** *The 2-adic products*

$$K_{8s+2}(\mathbb{Z}) \otimes K_{8t+5}(\mathbb{Z}) \xrightarrow{\star} K_{8(s+t)+7}(\mathbb{Z}) \otimes \hat{\mathbb{Z_2}}$$

$$K_{8s+2}(\mathbb{Z}) \otimes K_{8t+7}(\mathbb{Z}) \xrightarrow{\star} K_{8(s+t+1)+1}(\mathbb{Z}) \otimes \hat{\mathbb{Z_2}}$$

*are trivial for all $s$ and $t \geq 0$.*

*Proof.* According to Theorem 5.7 (c),

$$K_{8s+2}(\mathbb{Z}) \otimes \hat{\mathbb{Z_2}} \cong (K_1(\mathbb{Z}) \star K_{8s+1}(\mathbb{Z})) \otimes \hat{\mathbb{Z_2}}\,.$$

This implies the assertion because the products

$$K_{8s+1}(\mathbb{Z}) \otimes K_{8t+5}(\mathbb{Z}) \xrightarrow{\star} K_{8(s+t)+6}(\mathbb{Z}) \otimes \hat{\mathbb{Z}_2} = 0$$

and

$$K_{8s+1}(\mathbb{Z}) \otimes K_{8t+7}(\mathbb{Z}) \xrightarrow{\star} K_{8(s+t+1)}(\mathbb{Z}) \otimes \hat{\mathbb{Z}_2} = 0$$

are obviously trivial (see Table 1 of [35] and Theorem 0.6 of [28]). □

For the next result, let us call $z_t$ the element of order 2 in $K_{8t+2}(\mathbb{Z}) \cong \mathbb{Z}/2 \oplus$ (finite odd torsion group).

**Theorem 5.9.** *Consider the 2-adic product*

$$K_{8s+1}(\mathbb{Z}) \otimes K_{8t+2}(\mathbb{Z}) \xrightarrow{\star} K_{8(s+t)+3}(\mathbb{Z}) \otimes \hat{\mathbb{Z}_2}$$

*for any integers $s$, $t \geq 0$.*
*(a) For all $s \geq 1$ and all $t \geq 0$, $y_s \star z_t = 0$.*
*(b) For all $s$ and $t \geq 0$, $x_s \star z_t$ is an element of order 2 in $K_{8(s+t)+3}(\mathbb{Z}) \otimes \hat{\mathbb{Z}_2}$*

*Proof.* Because of Theorem 5.7 (c),

$$(K_{8s+1}(\mathbb{Z}) \star K_{8t+2}(\mathbb{Z})) \otimes \hat{\mathbb{Z}_2} \cong (K_1(\mathbb{Z}) \star K_{8s+1}(\mathbb{Z}) \star K_{8t+1}(\mathbb{Z})) \otimes \hat{\mathbb{Z}_2}$$

and assertion (a) follows from $y_s \star z_t = x_0 \star y_s \star x_t = 0$ by Theorem 5.7 (b). Similarly, $x_s \star z_t = x_0 \star x_s \star x_t$ is non trivial according to Proposition 12.17 of [1] and Corollary 4.6 of [13]. □

We may summarize our results on the 2-adic products in the $K$-theory of $\mathbb{Z}$ as follows.

**Corollary 5.10.** *The 2-adic product*

$$\star : K_i(\mathbb{Z}) \otimes K_k(\mathbb{Z}) \longrightarrow K_{i+k}(\mathbb{Z}) \otimes \hat{\mathbb{Z}_2}$$

*is trivial for all positive integers $i$ and $k$, except if $i \equiv k \equiv 1$ mod 8 or $i \equiv 1$ mod 8 and $k \equiv 2$ mod 8 (or $i \equiv 2$ mod 8 and $k \equiv 1$ mod 8) where its image is cyclic of order 2.*

Let us conclude this section by the following observation about the relationships between products in algebraic $K$-theory of the ring of integers $\mathbb{Z}$ and the Dwyer-Friedlander map relating the algebraic $K$-theory of $\mathbb{Z}$ to its étale $K$-theory (see Section 4 of [17]).

**Proposition 5.11.** *For any odd prime $l$ and any integer $n \geq 2$, the image of the product map*

$$\star : K_{2n-1}(\mathbb{Z}) \otimes K_{2n-1}(\mathbb{Z}) \longrightarrow K_{4n-2}(\mathbb{Z})$$

*is contained in the kernel of the Dwyer-Friedlander map* $K_{4n-2}(\mathbb{Z}) \to K^{\text{ét}}_{4n-2}(\mathbb{Z}[\frac{1}{l}])$.

*Proof.* The products in algebraic $K$-theory and étale $K$-theory commute with the Dwyer-Friedlander map. Observe that

$$K^{\text{ét}}_{2n-1}(\mathbb{Z}[\tfrac{1}{l}]) = \begin{cases} \hat{\mathbb{Z}_l} & \text{if } n \text{ is odd,} \\ \mathbb{Z}/|w_n(\mathbb{Q})|_l^{-1} & \text{if } n \text{ is even.} \end{cases}$$

Hence $K^{\text{ét}}_{2n-1}(\mathbb{Z}[\frac{1}{l}]) \cong H^1_{\text{ét}}(\mathbb{Z}[\frac{1}{l}]; \mathbb{Z}_l(n))$ is cyclic. But the product in étale $K$-theory is just the cup product in étale cohomology. This shows that the product

$$K^{\text{ét}}_{2n-1}(\mathbb{Z}[\tfrac{1}{l}]) \otimes K^{\text{ét}}_{2n-1}(\mathbb{Z}[\tfrac{1}{l}]) \longrightarrow K^{\text{ét}}_{4n-2}(\mathbb{Z}[\tfrac{1}{l}])$$

is zero because the cup product $H^1_{\text{ét}} \otimes H^1_{\text{ét}} \to H^2_{\text{ét}}$ is anticommutative. □

## 6. Products in the algebraic $K$-theory of cyclotomic fields

The results of Section 5 indicate that the 2-adic products in $K_*(\mathbb{Z})$ are trivial or have a very small image. In this section, we show that in the case of products in the $K$-theory of number fields, the image of product maps can be quite big. In the proof, we use the methods of [9] and [10]. Let us consider an odd prime number $l$, a positive integer $m$, and the cyclotomic field $E = \mathbb{Q}(\xi_{l^m})$ obtained from $\mathbb{Q}$ by adding a primitive root of unity $\xi_{l^m}$ of order $l^m$. Our goal is to show that for $n$ odd, the product homomorphism

$$\star : K_1(E) \otimes K_{2n-1}(E)_l \longrightarrow K_{2n}(E)_l$$

has a big image.

If $R$ is a commutative ring, $X_R$ is a ring spectrum with respect to $\mu : X_R \wedge X_R \to X_{R\otimes R} \to X_R$, where the first map is the pairing which was also called $\mu$ at the beginning of Section 3 (see [21], Proposition 2.4.2) and the second is induced by the multiplication $R \otimes R \to R$. The product structure of $K_*(R)$, also denoted by $\star$, is given by the composition

$$\star : K_i(R) \otimes K_k(R) \cong \pi_i X_R \otimes \pi_k X_R \xrightarrow{\wedge} \pi_{i+k}(X_R \wedge X_R) \xrightarrow{\mu_*} \pi_{i+k} X_R \cong K_{i+k}(R)\,.$$

Recall that the $K$-theory with $\mathbb{Z}/l^m$-coefficients (for a prime $l$ and a positive integer m, with $m \geq 2$ if $l = 2$) may be defined by $K_k(R; \mathbb{Z}/l^m) = \pi_k(M \wedge X_R)$, where $M$ is the mod $l^m$ Moore spectrum (i.e., such that $H_0 M \cong \mathbb{Z}/l^m$, $H_k M = 0$ for $k \neq 0$). Notice that $M$ is a ring spectrum with identity $i_M$ and product $\mu_M$ (see [27], p. 22). We also consider the following products (see also [13]):

$$\star : K_i(R) \otimes K_k(R; \mathbb{Z}/l^m) \cong \pi_i X_R \otimes \pi_k(M \wedge X_R) \xrightarrow{\wedge} \pi_{i+k}(M \wedge X_R \wedge X_R) \xrightarrow{\mu_*}$$

$$\pi_{i+k}(M \wedge X_R) \cong K_{i+k}(R; \mathbb{Z}/l^m)$$

and

$$\star : K_i(R;\mathbb{Z}/l^m) \otimes K_k(R;\mathbb{Z}/l^m) \cong \pi_i(M \wedge X_R) \otimes \pi_k(M \wedge X_R) \xrightarrow{\wedge} \pi_{i+k}(M \wedge M \wedge X_R \wedge X_R) \xrightarrow{(\mu_M \wedge \mu)_*} \pi_{i+k}(M \wedge X_R) \cong K_{i+k}(R;\mathbb{Z}/l^m)\,.$$

**Remark 6.1.** Since $M$ is a ring spectrum, the diagram

$$\begin{array}{ccccc} S \wedge S & \xrightarrow{\mathrm{id}\wedge i_M} & S \wedge M & \xrightarrow{i_M \wedge \mathrm{id}} & M \wedge M \\ \downarrow \simeq & & \downarrow \simeq & & \downarrow \mu_M \\ S & \xrightarrow{i_M} & M & \xrightarrow{\mathrm{id}} & M \end{array}$$

commutes and implies the compatibility of the three products, i.e., the commutativity of the diagram

$$\begin{array}{ccccc} K_i(R) \otimes K_k(R) & \xrightarrow{\mathrm{id}\otimes\mathrm{red}} & K_i(R) \otimes K_k(R;\mathbb{Z}/l^m) & \xrightarrow{\mathrm{red}\otimes\mathrm{id}} & K_i(R;\mathbb{Z}/l^m) \otimes K_k(R;\mathbb{Z}/l^m) \\ \downarrow \star & & \downarrow \star & & \downarrow \star \\ K_{i+k}(R) & \xrightarrow{\mathrm{red}} & K_{i+k}(R;\mathbb{Z}/l^m) & \xrightarrow{\mathrm{id}} & K_{i+k}(R;\mathbb{Z}/l^m)\,, \end{array}$$

where red is the map which is induced on $K$-theory by the reduction of coefficients mod $l^m$.

For any ring $R$, there is the following Bockstein long exact sequence

$$\cdots \longrightarrow K_k(R) \xrightarrow{l^m} K_k(R) \longrightarrow K_k(R;\mathbb{Z}/l^m) \xrightarrow{\mathfrak{b}} K_{k-1}(R) \longrightarrow \cdots\,,$$

where $\mathfrak{b}$ is the Bockstein homomorphism.

**Lemma 6.2.** *For any $x \in K_i(R)$ and any $y \in K_k(R;\mathbb{Z}/l^m)$, one has $\mathfrak{b}(x \star y) = x \star \mathfrak{b}(y) \in K_{i+k-1}(R)$.*

*Proof.* The Bockstein homomorphism $\mathfrak{b}$ is induced by the obvious map $\varepsilon : M \to \Sigma^{-1}S$ which fits into the commutative diagram

$$\begin{array}{ccc} S \wedge M & \xrightarrow{\simeq} & M \\ \big\downarrow{\scriptstyle \mathrm{id}\wedge\varepsilon} & & \big\downarrow{\scriptstyle \varepsilon} \\ S \wedge \Sigma^{-1}S & \xrightarrow{\simeq} & \Sigma^{-1}S \end{array}$$

and provides the commutativity of

$$\begin{array}{ccc} K_i(R) \otimes K_k(R;\mathbb{Z}/l^m) & \xrightarrow{\star} & K_{i+k}(R;\mathbb{Z}/l^m) \\ \big\downarrow{\scriptstyle \mathrm{id}\otimes\mathfrak{b}} & & \big\downarrow{\scriptstyle \mathfrak{b}} \\ K_i(R) \otimes K_{k-1}(R) & \xrightarrow{\star} & K_{i+k-1}(R) \end{array}$$

and the statement of the lemma. □

This lemma implies the formula

$$\mathfrak{b}(Tr_{E/\mathbb{Q}}(u \star \beta_m^{\star n})) = Tr_{E/\mathbb{Q}}(u \star \mathfrak{b}(\beta_m^{\star n}))\,,$$

where $u$ is any element $\in K_1(E) = E^\times$, $\beta_m = \beta(\xi_{l^m}) \in K_2(E;\mathbb{Z}/l^m)$ is the Bott element (see Definition 2.7.2 of [34]) and $Tr_{E/\mathbb{Q}}$ is the transfer map (see [25], Section 4). Using this equality, we can rewrite the definition of the Stickelberger pseudosplitting homomorphism $\Lambda$ from [8], Section IV.1, or [10], Definition 3.2, as follows.

**Definition 6.3.** There is a homomorphism

$$\Lambda : \bigoplus_p K_{2n-1}(\mathbb{F}_p)_l \longrightarrow K_{2n}(\mathbb{Q})_l\,,$$

where $\Lambda = \prod_p \Lambda_p$ and $\Lambda_p : K_{2n-1}(\mathbb{F}_p)_l \to K_{2n}(\mathbb{Q})_l$ is given by the formula

$$\Lambda_p(\kappa_p) = \begin{cases} Tr_{E/\mathbb{Q}}(\lambda_b(\mathfrak{p}) \star \mathfrak{b}({\beta_k}^{\star n})^{b^n\gamma_l}) & \text{if } l \text{ does not divide } n, \\ Tr_{E/\mathbb{Q}}(\lambda_b(\mathfrak{p}) \star \mathfrak{b}({\beta_k}^{\star n})^{nb^n\gamma_l}) & \text{if } l \text{ divides } n, \end{cases}$$

where $b$ is a natural number such that $(b, w_{n+1}(\mathbb{Q})) = 1$ and $\kappa_p$ is a generator of the group $K_{2n-1}(\mathbb{F}_p)_l$. In addition, $\lambda_b(\mathfrak{p}) \in E^\times$ are the twisted Gauss sums (see [9], Definition 3) and $\gamma_l = 1 + l^n + l^{2n} + l^{3n} \cdots = \frac{1}{1-l^n} \in \hat{\mathbb{Z}_l}$.

**Theorem 6.4.** *Let $I$ be the image of the map*

$$K_1(E) \otimes K_{2n-1}(E)_l \xrightarrow{\star} K_{2n}(E)_l \xrightarrow{Tr_{E/\mathbb{Q}}} K_{2n}(\mathbb{Q})_l\,,$$

*where $n$ is an odd integer. Then the exponent of the group $K_{2n}(\mathbb{Q})_l/I$ divides the number $(\#K_{2n}(\mathbb{Z})_l)^2$.*

*Proof.* Consider the localization sequence

$$0 \longrightarrow K_{2n}(\mathbb{Z})_l \longrightarrow K_{2n}(\mathbb{Q})_l \xrightarrow{\partial} \bigoplus_p K_{2n-1}(\mathbb{F}_p)_l \longrightarrow 0\,.$$

By Definition 6.3, we see that image $\Lambda \subseteq I$. On the other hand, by Proposition 2 of [9], the composition $\partial\Lambda$ acts by raising into the power with exponent an integer $|(b^{n+1}-1)\zeta_{\mathbb{Q}}(-n)|_l^{-1}$ (recall that the Gauss sums used in the construction of $\Lambda$ depend on $b$), where $\zeta_{\mathbb{Q}}(s)$ is the Riemann zeta function. Consider now $x \in K_{2n}(\mathbb{Q})_l$ and $z = \partial(x) \in \bigoplus_p K_{2n-1}(\mathbb{F}_p)_l$. The computations above show that for every $b$,

$$\frac{\Lambda(z)}{x^{|(b^{n+1}-1)\zeta_{\mathbb{Q}}(-n)|_l^{-1}}} \in K_{2n}(\mathbb{Z})_l\,.$$

The greatest common divisor of all $|(b^{n+1}-1)|_l^{-1}$ over the integers $b$ which are relatively prime to $w_{n+1}(\mathbb{Q})$ equals the number $|w_{n+1}(\mathbb{Q})|_l^{-1}$, by Lemma 2.3 of [15]. Since the integer $|w_{n+1}(\mathbb{Q})\zeta_{\mathbb{Q}}(-n)|_l^{-1}$ divides the number $t$ of elements in the group $K_{2n}(\mathbb{Z})_l$, it follows that $x^{t^2} \in$ image $\Lambda \subseteq I$. □

**Remark 6.5.** Observe that in an abelian group the product of a torsion element and a nontorsion element is again nontorsion. Hence, every torsion element in $K_{2n-1}(E)$ can be written as a quotient of two nontorsion elements. Consequently, the subgroup of $K_1(E)\star K_{2n-1}(E)$ generated by the subset $\{x\star y \mid x \in K_1(E)\,,\ y \in K_{2n-1}(E)\,,\ y \text{ nontorsion}\}$ contains the subgroup $K_1(E) \star K_{2n-1}(E)_l$ of $K_{2n}(E)$.

**Remark 6.6.** It follows from Theorem 6.4 and Theorem 3.4 of [10] that the image of $K_1(E) \star K_{2n-1}(E)_l$ under the transfer $Tr_{E/\mathbb{Q}} : K_{2n}(E) \to K_{2n}(\mathbb{Q})$ contains the group of divisible elements $D(n)_l \cong K_{2n}^{\text{ét}}(\mathbb{Z}[\frac{1}{l}])$ from Section 5.2 of [9] (see also [8], Section IV.3).

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Dominique Arlettaz
Institut de mathématiques
Université de Lausanne
CH-1015 Lausanne
Switzerland
e-mail: dominique.arlettaz@ima.unil.ch

Wojciech Gajda
Department of Mathematics
Adam Mickiewicz University
60769 Poznań
Poland
e-mail: GAJDA@math.amu.edu.pl

Grzegorz Banaszak
Department of Mathematics
Adam Mickiewicz University
60769 Poznań
Poland
e-mail: BANASZAK@math.amu.edu.pl

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