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Boundary value problems with Atiyah–Patodi–Singer type conditions and spectral triples

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Abstract. We study realizations of pseudodifferential operators acting on sections of vector-bundles on a smooth, compact manifold with boundary, subject to conditions of Atiyah–Patodi–Singer type. Ellipticity and Fredholm property, compositions, adjoints and self-adjointness of such realizations are discussed. We construct regular spectral triples $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ for manifolds with boundary of arbitrary dimension, where $\mathcal{H}$ is the space of square integrable sections. Starting out from Dirac operators with APS-conditions, these triples are even in case of even dimensional manifolds; we show that the closure of $\mathcal{A}$ in $\mathcal{L}(\mathcal{H})$ coincides with the continuous functions on the manifold being constant on each connected component of the boundary.

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1. Introduction

Spectral triples play a fundamental role in non commutative geometry and provide a new approach to several fields in mathematics and physics. One of the most striking results involving spectral triples is Connes' famous reconstruction theorem [10], which shows that one can (re-)construct from a commutative spectral triple $(\mathcal{A}, \mathcal{H}, \mathcal{D})$, satisfying certain axioms, a compact oriented manifold *without boundary* M such that $\mathcal{A}$ is isomorphic to $\mathcal{C}^\infty(M)$. In the past years, the definition of spectral triple has been extended to different settings. For example by Lescure [21] to manifolds with conical singularities, by Lapidus [20], Cipriani et al. [6], and Christensen et al. [9] to fractals. Our paper provides a contribution to the analysis of spectral triples for manifolds with (smooth) boundary, mainly motivated by the recent work [17] of Iochum and Levy.

The central analytic tool our approach relies on is Boutet de Monvel's algebra of pseudodifferential boundary value problems [5], respectively a suitable extension of it going back to Schulze [24], cf. also Seiler [26]. This calculus provides an efficient framework for the application of microlocal methods in partial differential

equations, geometric analysis and index theory for manifolds with boundary. We shall use this calculus for a systematic study of *realizations* (i.e. closed extensions) of (pseudo)differential operators on compact manifolds subject to homogeneous boundary conditions. This study is inspired by and extends work of Grubb [16]. In comparison to her results, we allow a wider class of boundary conditions which we named *APS-type conditions*, since the classical spectral boundary conditions of Atiyah–Patodi–Singer [1] are a particular example of such conditions. Specifically, these boundary conditions are of the form

$$\mathcal{C}^\infty(\Omega, E) \longrightarrow \mathcal{C}^\infty(\partial\Omega, F), \quad u \mapsto Tu := P(S\rho + T')u,$$

where Ω is a smooth Riemannian manifold with boundary, E is a hermitian vector bundle over Ω, $F = F_0 \oplus \cdots \oplus F_{d-1}$ with F_j hermitian vector bundles over $\partial\Omega$ (possibly zero-dimensional), $T' = (T'_0, \ldots, T'_{d-1})$ with trace operators $T'_k : \mathcal{C}^\infty(\Omega, E) \to \mathcal{C}^\infty(\partial\Omega, F_j)$ of order $j + 1/2$, $\rho = (\gamma_0, \ldots, \gamma_{d-1})^t$ with γ_j denoting the operator of restriction to the boundary of the jth derivative in direction normal to the boundary, $S = (S_{jk})_{0\le j,k\le d-1}$ with $S_{jk} \in L^{j-k}_{\mathrm{cl}}(\partial\Omega; E|_{\partial\Omega}, F_j)$ being classical (i.e. step one poly-homogeneous) pseudodifferential operators of order $j - k$ on the boundary, and an *idempotent* $P = (P_{jk})_{0\le j,k\le d-1}$ with $P_{jk} \in L^{j-k}_{\mathrm{cl}}(\partial\Omega; F_k, F_j)$. We then consider operators with domain $\{u \in H^d(\Omega, E) \mid Tu = 0\}$ and with action given by a dth order operator from Boutet de Monvel's calculus acting between sections of E. In Section 3 we discuss ellipticity and Fredholm property, the adjoint (in particular, self-adjointness) and composition of such realizations.

In this context we prove and make use of a result on the invariance of the Fredholm index and the existence of inverses (parametrices) modulo projections onto the kernel for operators acting in families of Banach spaces, generalizing known, analogous results for pseudodifferential operator algebras to an abstract setting. This result is of independent interest as it applies to any operator algebra satisfying some very natural conditions, and is presented in the Appendix.

The framework developed in Section 3 allows us to introduce and analyze, in Section 4, spectral triples for manifolds with boundary. At a first glance, the approach is very similar to that of Iochum and Levy [17], however it provides a true extension of their results. The main example of [17] are spectral triples based on Dirac operators equipped with chiral boundary conditions; there are good physical and mathematical motivations to consider this kind of boundary conditions, as it has been already done in several other works, see [2] and [7] for example. Being local conditions, Iochum and Levy could rely on the results of Grubb [16] mentioned above. However, it is well known that chiral boundary condition cannot be defined in all settings. Indeed, it is always possible only in case the underlying manifold is of even dimension, in general a chirality operator is not naturally defined. In view of this lack of generality it seems natural to make use of *non-local* APS-type boundary conditions and, in fact, this is what our approach permits to do. We show how to define *regular*

spectral triples $\left(\mathcal{A}_{\mathcal{D}}^{\infty}, \mathcal{H}, \mathcal{D}\right)$ on every compact manifold with boundary, including, as particular example, those triples starting from Dirac operators equipped with APS boundary conditions. In the case of even-dimensional manifolds we show that the latter spectral triples respect the natural grading defined by the chirality, and therefore define so-called *even* spectral triples, see Remark 4.5.

Analogously to the case of chiral boundary conditions, the algebra $\mathcal{A}_{\mathcal{D}}^{\infty}$ is not the whole space $\mathcal{C}^{\infty}(\Omega)$, but a true subalgebra. In general, it is difficult to describe this algebra in explicit terms, but, in the case of a Dirac operator with APS conditions, we prove that the *closure* of $\mathcal{A}_{\mathcal{D}}^{\infty}$ with respect to the supremum norm is the C^*-algebra of continuous functions being constant on each connected component of the boundary. The knowledge of this closure is important, since it plays a key role in Connes' reconstruction Theorem. In this context, our result implies that the spectral triple does *not* fulfill the so-called Finiteness Axiom in [10], cf. Section 4.3. Roughly speaking, $\mathcal{A}_{\mathcal{D}}^{\infty}$ results to be *too small* to see the geometric properties of the boundary. This kind of negative result actually indicates that the correct notion of spectral triple able to reconstruct manifolds with boundary, taking properly into account the geometry of the boundary, still has to be found. For the time being we have to leave this as an open problem for future research.

Convention. Throughout the text, we denote by Ω a smooth, compact, Riemannian manifold with boundary. On a collar-neighborhood U of the boundary, identified with $\partial\Omega \times [0, \varepsilon)$ and using the splitting of variables $x = (x', x_n)$, we assume the metric to be of product-form $g_{\partial\Omega} + dx_n^2$. Vector bundles over Ω mean smooth, hermitian vector-bundles that respect the product structure near the boundary, i.e. if E denotes such a bundle, then $E|_U = \pi^* E|_{\partial\Omega}$ with $\pi(x', x_n) = x'$ the canonical projection of the collar-neighborhood onto the boundary. In writing $\mathcal{C}^{\infty}(\Omega)$ we mean functions smooth up to (i.e. including) the boundary.

2. Boutet de Monvel's calculus for Toeplitz type operators

Boutet de Monvel's algebra for boundary value problems on Ω consists of certain operators in block-matrix form,

$$\mathcal{A} = \begin{pmatrix} A_+ + G & K \\ T & Q \end{pmatrix} : \begin{array}{c} \mathcal{C}^{\infty}(\Omega, E_0) \\ \oplus \\ \mathcal{C}^{\infty}(\partial\Omega, F_0) \end{array} \longrightarrow \begin{array}{c} \mathcal{C}^{\infty}(\Omega, E_1) \\ \oplus \\ \mathcal{C}^{\infty}(\partial\Omega, F_1) \end{array}, \tag{2.1}$$

where E_j and F_j are vector bundles over Ω and $\partial\Omega$, respectively, which are allowed to be zero dimensional. Every such operator has an order, denoted by $\mu \in \mathbb{Z}$, and a

type, denoted by $d \in \mathbb{N}_0$.[1] To fix some terminology:

- A_+ is the "restriction" to the interior of Ω of a μth order, classical pseudodifferential operator A defined on the smooth double 2Ω, having the (two-sided) transmission property with respect to $\partial\Omega$,[2]
- G is a singular Green operator of order μ and type d,
- K is a μth order potential operator,
- T is a trace operator of order μ and type d,
- Q is a μth order, classical pseudodifferential operator on the boundary $\partial\Omega$.

The space of all such operators we shall denote by $\mathcal{B}^{\mu,d}(\Omega; (E_0, F_0), (E_1, F_1))$.

As a matter of fact, with $\mathcal{A}$ is associated a (homogeneous) principal symbol

$$\sigma^\mu(\mathcal{A}) = \big(\sigma^\mu_\psi(\mathcal{A}), \sigma^\mu_\partial(\mathcal{A})\big),$$

where

$$\sigma^\mu_\psi(\mathcal{A}) = \sigma^\mu_\psi(A) : \pi^*_\Omega E_0 \longrightarrow \pi^*_\Omega E_1$$

is the usual principal symbol of the pseudodifferential operator A (restricted to $T^*\Omega$), while

$$\sigma^\mu_\partial(\mathcal{A}) : \begin{array}{c} \pi^*_{\partial\Omega}(\mathcal{S}(\mathbb{R}_+) \otimes E_0) \\ \oplus \\ \pi^*_{\partial\Omega} F_0 \end{array} \longrightarrow \begin{array}{c} \pi^*_{\partial\Omega}(\mathcal{S}(\mathbb{R}_+) \otimes E_1) \\ \oplus \\ \pi^*_{\partial\Omega} F_1 \end{array}$$

is the so-called principal boundary symbol; here $\pi_M : T^*M \to M$ denotes the canonical projection of the tangent bundle to the manifold and π^*_M indicates pull-back of vector-bundles and $\mathcal{S}(\mathbb{R}_+) \otimes E$ denotes the bundle with fibre $\mathcal{S}(\mathbb{R}_+, E_y)$ in $y \in \partial\Omega$.

For convenience of the reader, in the following subsection we shall shortly describe the above mentioned structures in the model case of Ω being a half-space and the bundles involved being trivial one-dimensional. For more complete descriptions we refer the reader to the existing literature on Boutet de Monvel's calculus, for instance [5,16,22], and [23].

2.1. A few details on the structure of the operators. Let $\Omega = \mathbb{R}^{n-1} \times (0, +\infty)$ with variable $x = (x', x_n)$ and corresponding co-variable $\xi = (\xi', \xi_n)$. With $[\cdot]$ denote a smooth, positive function that coincides with the Euclidean norm outside a neighborhood of the origin. Let

$$k(x', \xi'; y_n) = k_0(x', \xi'; [\xi']y_n),$$

[1] It is possible to introcduce operators with negative type, cf. [16]. However, for our purpose it is sufficient to consider non-negative types only.

[2] $A_+ = r_+ A e_+$, where r_+ denotes the operator of restricting distributions from 2Ω to $\operatorname{int}\Omega$ and e_+ denotes the operator of extending (sufficiently regular) distributions by 0 from $\operatorname{int}\Omega$ to 2Ω. If A is differential, A_+ coincides with the standard action of A on distributions; occasionally we will therefore drop the subscript $+$ when dealing with differential operators.

where $k_0(x', \xi'; t)$ behaves like a classical pseudodifferential symbol of order $\mu + 1/2$ in the variables (x', ξ'), while in t like a rapidly decreasing function (smooth up to $t = 0$). Then

$$K\varphi(x', x_n) = (2\pi)^{-n+1} \int_{\mathbb{R}^{n-1}} e^{ix'\xi'} k(x', \xi'; x_n)\, \mathcal{F}\varphi(\xi')\, d\xi'$$

defines a Poisson operator of order μ, while

$$T_0 u(x') = (2\pi)^{-n+1} \int_{\mathbb{R}^{n-1}} \int_0^\infty e^{ix'\xi'} k(x', \xi'; y_n)\, \mathcal{F}_{y' \to \xi'} u(\xi', y_n)\, dy_n d\xi'$$

defines a trace operator of order μ and type 0 (note that taking formal adjoints with respect to the corresponding L_2-scalar products gives a one-to-one correspondence between these two types of operators). A trace operator of order μ and type d is of the form

$$Tu = \sum_{j=0}^{d-1} S_j \left(\frac{d^j u}{dx_n^j}\bigg|_{x_n=0} \right) + T_0 u$$

with classical pseudodifferential operators S_j of order $\mu - j - 1/2$ on the boundary $\mathbb{R}^{n-1}$. A singular Green operator of order μ and type 0 has the form

$$G_0 u(x', x_n) = (2\pi)^{-n+1} \int_{\mathbb{R}^{n-1}} \int_0^\infty e^{ix'\xi'} k(x', \xi'; x_n, y_n)\, \mathcal{F}_{y' \to \xi'} u(\xi', y_n)\, dy_n d\xi',$$

where

$$k(x', \xi'; x_n, y_n) = g_0(x', \xi'; [\xi']x_n, [\xi']y_n)$$

with a function $g_0(x', \xi'; s, t)$ that behaves like a classical pseudodifferential symbol of order $\mu + 1$ in (x', ξ'), while in (s, t) like a rapidly decreasing function (and smooth up to $s = 0$ and $t = 0$). A singular Green operator of order μ and type d is then of the form

$$Gu = \sum_{j=0}^{d-1} K_j \left(\frac{d^j u}{dx_n^j}\bigg|_{x_n=0} \right) + G_0 u$$

with Poisson operators K_j of order $\mu - j - 1/2$.

The corresponding principal boundary symbols are defined as

$$\sigma_\partial^\mu(K)(x', \xi') : \mathbb{C} \longrightarrow \mathcal{S}(\mathbb{R}_+), \quad c \mapsto c\, k_0^{(\mu+1/2)}(x', \xi'; |\xi'|\cdot)$$

$$\sigma_\partial^\mu(T_0)(x', \xi') : \mathcal{S}(\mathbb{R}_+) \longrightarrow \mathbb{C}, \quad u \mapsto \int_0^\infty k_0^{(\mu+1/2)}(x', \xi'; |\xi'|y_n) u(y_n)\, dy_n$$

for potential and trace operators of type 0, where $k_0^{(\mu+1/2)}$ denotes the homogeneous principal symbol of k_0 with respect to (x', ξ'). Moreover,

$$\sigma_\partial^\mu(T)(x', \xi')u = \sum_{j=0}^{d-1} \sigma_\psi^{\mu-j-1/2}(S_j)(x', \xi') \frac{d^j u}{dx_n^j}(0) + \sigma_\partial^\mu(T_0)(x', \xi')u.$$

Concerning the singular Green operators, we similarly have

$$\sigma_\partial^\mu(G_0)(x',\xi') : \mathcal{S}(\mathbb{R}_+) \longrightarrow \mathcal{S}(\mathbb{R}_+),$$
$$u \mapsto \int_0^\infty g_0^{(\mu+1)}(x',\xi';|\xi'|\cdot,|\xi'|y_n)u(y_n)\,dy_n,$$

and

$$\sigma_\partial^\mu(G)(x',\xi')u = \sum_{j=0}^{d-1} \sigma_\partial^{\mu-j-1/2}(K_j)(x',\xi')\frac{d^j u}{dx_n^j}(0) + \sigma_\partial^\mu(G_0)(x',\xi')u.$$

2.2. Basic properties of Boutet's calculus. The above described class of operators forms an "algebra" in the sense that composition of operators induces maps

$$\mathcal{B}^{\mu_1,d_1}(\Omega;(E_1,F_1),(E_2,F_2)) \times \mathcal{B}^{\mu_0,d_0}(\Omega;(E_0,F_0),(E_1,F_1)) \longrightarrow \mathcal{B}^{\mu,d}(\Omega;(E_0,F_0),(E_2,F_2)),$$

where the resulting order and type are

$$\mu = \mu_0 + \mu_1, \quad d = \max(d_0, d_1 + \mu_0).$$

The operators, initially acting on smooth sections, extend by density and continuity to Sobolev spaces, i.e. $\mathcal{A} \in \mathcal{B}^{\mu,d}(\Omega;(E_0,F_0),(E_1,F_1))$ induces maps

$$\begin{matrix} H_p^s(\Omega,E_0) \\ \oplus \\ B_{pp}^{s-(\frac{1}{p}-\frac{1}{2})}(\partial\Omega,F_0) \end{matrix} \longrightarrow \begin{matrix} H_p^{s-\mu}(\Omega,E_0) \\ \oplus \\ B_{pp}^{s-\mu-(\frac{1}{p}-\frac{1}{2})}(\partial\Omega,F_1) \end{matrix}, \quad s > d-1+\frac{1}{p}, \tag{2.2}$$

where $1 < p < \infty$ and H_p^s denotes the standard Sobolev (Bessel potential) spaces, while B_{pq}^s are the usual Besov spaces. Similarly, the boundary symbol extends to maps

$$\begin{matrix} \pi_{\partial\Omega}^*(H_p^s(\mathbb{R}_+)\otimes E_0) \\ \oplus \\ \pi_{\partial\Omega}^* F_0 \end{matrix} \longrightarrow \begin{matrix} \pi_{\partial\Omega}^*(H_p^{s-\mu}(\mathbb{R}_+)\otimes E_1) \\ \oplus \\ \pi_{\partial\Omega}^* F_1 \end{matrix}. \tag{2.3}$$

We shall employ these properties only in the Hilbert space case $p = 2$; in this case $B_{pp}^s = H_2^s$ and we eliminate the index $p = 2$ from the notation.

2.3. Toeplitz type operators and ellipticity. In this paper we shall need an extended version of Boutet de Monvel's calculus. As described here, this calculus was introduced in [24]; it can be also obtained as a special case from a general approach to operator-algebras of Toeplitz type developed in [26].

Let $P_j \in L^0_{\mathrm{cl}}(\partial\Omega; F_j, F_j)$, $j = 0, 1$, be two pseudodifferential projections on the boundary of Ω. We then denote by

$$\mathcal{B}^{\mu,d}(\Omega; (E_0, F_0; P_0), (E_1, F_1; P_1))$$

the space of all operators $\mathcal{A} \in \mathcal{B}^{\mu,d}(\Omega; (E_0, F_0), (E_1, F_1))$ such that

$$\mathcal{A}(1 - \mathcal{P}_0) = (1 - \mathcal{P}_1)\mathcal{A} = 0, \quad \mathcal{P}_j := \begin{pmatrix} 1 & 0 \\ 0 & P_j \end{pmatrix}.$$

Being projections, the range spaces $H^s(\partial\Omega, F_j, P_j) := P_j\big(H^s(\partial\Omega, F_j)\big)$ are closed subspaces of $H^s(\partial\Omega, F_j)$, and any such $\mathcal{A}$ induces continuous maps

$$\begin{matrix} H^s(\Omega, E_0) \\ \oplus \\ H^s(\partial\Omega, F_0, P_0) \end{matrix} \longrightarrow \begin{matrix} H^{s-\mu}(\Omega, E_0) \\ \oplus \\ H^{s-\mu}(\partial\Omega, F_1, P_1) \end{matrix}, \quad s > d - \frac{1}{2}, \tag{2.4}$$

according to (2.2). With P_j also the principal symbols $\sigma^0_\psi(P_j)$ are projections (as bundle morphisms) and thus define a subbundle $F_j(P_j)$ of $\pi^*_{\partial\Omega}F_j$. We then set

$$\sigma^\mu(\mathcal{A}; P_0, P_1) := \big(\sigma^\mu_\psi(\mathcal{A}), \sigma^\mu_\partial(\mathcal{A}; P_0, P_1)\big)$$

with $\sigma^\mu_\partial(\mathcal{A}; P_0, P_1)$ being the principal boundary symbol of $\mathcal{A}$ considered as a map

$$\begin{matrix} \pi^*_{\partial\Omega}(H^s(\mathbb{R}_+) \otimes E_0) \\ \oplus \\ F_0(P_0) \end{matrix} \longrightarrow \begin{matrix} \pi^*_{\partial\Omega}(H^{s-\mu}(\mathbb{R}_+) \otimes E_1) \\ \oplus \\ F_1(P_1) \end{matrix}, \quad s > d - \frac{1}{2}, \tag{2.5}$$

(or, alternatively, replacing the Sobolev spaces by $\mathcal{S}(\mathbb{R}_+)$).

Definition 2.1. $\mathcal{A} \in \mathcal{B}^{\mu,d}(\Omega; (E_0, F_0; P_0), (E_1, F_1; P_1))$ is called elliptic if both components of the principal symbol $\sigma^\mu(\mathcal{A}; P_0, P_1)$ are isomorphisms.[3]

The following result is the main theorem of elliptic theory of Toeplitz type operators. For details see Section 2.1 of [24] and Theorem 6.1 of [26].

Theorem 2.2. *For $\mathcal{A}_0 \in \mathcal{B}^{\mu,d}(\Omega; (E_0, F_0; P_0), (E_1, F_1; P_1))$ the following statements are equivalent*:

(1) *$\mathcal{A}_0$ is elliptic.*

(2) *There exists an $s > \max(\mu, d) - 1/2$ such that the map (2.4) associated with $\mathcal{A}_0$ is Fredholm.*

(3) *For every $s > \max(\mu, d) - 1/2$ the map (2.4) associated with $\mathcal{A}_0$ is Fredholm.*

(4) *There is an $\mathcal{A}_1 \in \mathcal{B}^{-\mu_0, \max(d-\mu, 0)}(\Omega; (E_1, F_1; P_1), (E_0, F_0; P_0))$ such that*

$$\mathcal{A}_1\mathcal{A}_0 - \mathcal{P}_0 \in \mathcal{B}^{-\infty, \max(\mu, d)}(\Omega; (E_0, F_0; P_0), (E_0, F_0; P_0)),$$
$$\mathcal{A}_0\mathcal{A}_1 - \mathcal{P}_1 \in \mathcal{B}^{-\infty, \max(d-\mu, 0)}(\Omega; (E_1, F_1; P_1), (E_1, F_1; P_1)).$$

Any such operator $\mathcal{A}_1$ is called a parametrix of $\mathcal{A}_0$.

[3]Invertibility of the principal boundary symbol as a map (2.5) is independent of the choice of s and, equivalently, one may replace the Sobolev spaces by $\mathcal{S}(\mathbb{R}_+)$.

3. Realizations subject to APS-type boundary conditions

In this section we shall study certain closed extensions of unbounded operators of the form

$$A_+ + G : \mathcal{C}^\infty(\Omega, E) \subset L_2(\Omega, E) \longrightarrow L_2(\Omega, E)$$

with $A_+ + G \in B^{d,d}(\Omega; E, E) := \mathcal{B}^{d,d}(\Omega; (E, 0; 1), (E, 0; 1))$, subject to (a vector of) boundary conditions of APS-type, which we shall describe in the following subsection. Our results extend those of Sections 1.4 and 1.6 of [16]; for convenience of the reader we shall employ similar notation.

3.1. APS-type boundary conditions. Let $d \in \mathbb{N}$ be a positive integer and let $\partial/\partial\nu$ denote the derivative in direction of the outer normal to $\partial\Omega$. We define, for $s > d + j - \frac{1}{2}$,

$$\gamma_j : H^s(\Omega, E) \to H^{s-j-\frac{1}{2}}(\partial\Omega, E|_{\partial\Omega}), \quad u \mapsto \frac{\partial^j u}{\partial \nu^j}\Big|_{\partial\Omega},$$

and $\rho = (\gamma_0, \dots, \gamma_{d-1})^t$, where $E|_{\partial\Omega}$ indicates the restriction of the bundle E to the boundary. Moreover,

$$T_j = \sum_{k=0}^{d-1} S_{jk}\gamma_k + T'_j : H^s(\Omega, E) \longrightarrow H^{s-j-\frac{1}{2}}(\partial\Omega, F_j), \tag{3.1}$$

with vector bundles F_j over $\partial\Omega$ (possibly zero-dimensional), pseudodifferential operators $S_{jk} \in L^{j-k}_{\mathrm{cl}}(\partial\Omega; E|_{\partial\Omega}, F_j)$ and trace operators T'_j of order $j + 1/2$ and type 0. We write $T' = (T'_0, \dots, T'_{d-1})^t$ and further introduce

$$\mathcal{H}^s(\partial\Omega, E) = \bigoplus_{j=0}^{d-1} H^{s+d-j-\frac{1}{2}}(\partial\Omega, E|_{\partial\Omega}),$$
$$\mathcal{H}^s(\partial\Omega, F) = \bigoplus_{j=0}^{d-1} H^{s+d-j-\frac{1}{2}}(\partial\Omega, F_k).$$

Definition 3.1. Using the previously introduced notation, an APS-type boundary condition T is an operator of the form

$$T = P(S\rho + T') : H^s(\Omega, E) \longrightarrow \mathcal{H}^{s-d}(\partial\Omega, F), \quad s > d - 1/2,$$

where $S = (S_{jk})_{0\le j,k\le d-1}$ and a projection (i.e. idempotent)

$$P = (P_{jk})_{0\le j,k\le d-1} \quad \text{with } P_{jk} \in L^{j-k}_{\mathrm{cl}}(\partial\Omega; F_k, F_j).$$

To give an example, let B_j be a pseudodifferential operator of integer order $0 \le j < d$ on the double of Ω satisfying the transmission condition with respect

to $\partial\Omega$ and $T_j := \gamma_0 \circ B_{j,+}$. Then T_j is as in (3.1), even with $S_{jk} = 0$ for $k > j$ and all S_{jj} are zero-order differential operators, i.e. induced by a bundle homomorphism $s_j : E|_{\partial\Omega} \to F_j$. Hence, $T = S\rho + T'$ with a left-lower triangular matrix S whose diagonal elements are zero-order differential.

The classical Atiyah–Patodi–Singer conditions are included in this setting by taking $d = 1$, $T' = 0$ and S equal to the identity in Definition 3.1.

Definition 3.2. Let $A_+ + G \in B^{d,d}(\Omega; E, E)$ and T be an APS-type boundary condition as described above. We write $(A_+ + G)_T$ for the operator acting like $A_+ + G$ on the domain

$$\operatorname{dom}((A_+ + G)_T) = \{u \in H^d(\Omega, E) \mid Tu = 0\}.$$

The operator $(A_+ + G)_T$ is often called the realization of $A_+ + G$ subject to the boundary condition T. We call two boundary conditions T_0 and T_1 equivalent, if they have the same kernel as maps on $H^d(\Omega, E)$; then, obviously, $(A_+ + G)_{T_0} = (A_+ + G)_{T_1}$.

3.2. Elliptic and normal realizations. Now let $\Lambda = \operatorname{diag}(\Lambda_0, \dots, \Lambda_{d-1})$ be a $(d \times d)$-diagonal matrix with invertible components $\Lambda_j \in L_{\mathrm{cl}}^{d-j-\frac{1}{2}}(\partial\Omega; F_j, F_j)$. Note that then

$$P_\Lambda := \Lambda P \Lambda^{-1} \in L_{\mathrm{cl}}^0(\partial\Omega; F_\partial, F_\partial), \quad F_\partial := F_0 \oplus \cdots \oplus F_{d-1},$$

is a zero order projection.

Definition 3.3. Consider the realization $(A_+ + G)_T$ with $T = P(S\rho + T')$.

(1) The realization is called elliptic if

$$\begin{pmatrix} A_+ + G \\ \Lambda T \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & P_\Lambda \end{pmatrix} \begin{pmatrix} A_+ + G \\ \Lambda(S\rho + T') \end{pmatrix}$$

is an elliptic element in $\mathcal{B}^{d,d}(\Omega; (E, 0; 1), (E, F_\partial; P_\Lambda))$.

(2) The boundary condition T is called normal if there exists a matrix

$$R = (R_{jk})_{0 \le j,k \le d-1}, \quad R_{jk} \in L_{\mathrm{cl}}^{j-k}(\partial\Omega; F_k, E|_{\partial\Omega}),$$

such that $PSR = P$. As way of speaking, we occasionally will call R the right-inverse of PS.

Note that ellipticity of $(A_+ + G)_T$ is equivalent to the Fredholm property of

$$\begin{pmatrix} A_+ + G \\ T \end{pmatrix} : H^s(\Omega, E) \longrightarrow \begin{matrix} H^{s-d}(\Omega, E) \\ \oplus \\ \mathcal{H}^{s-d}(\partial\Omega, F, P) \end{matrix}$$

for some (and then for all) $s > d - 1/2$, where, by definition,

$$\mathcal{H}^s(\partial\Omega, F, P) = P\big(\mathcal{H}^s(\partial\Omega, F)\big).$$

By abstract and well-known results on Fredholm operators (see, for example, Theorem 8.3 in [11]), this in turn is equivalent to the Fredholm property of

$$(A_+ + G)_T : \mathrm{dom}((A_+ + G)_T) \to L^2(\Omega, E)$$

together with the finiteness of the codimension of $T(H^d(\Omega, E))$ in $\mathcal{H}^0(\partial\Omega, F, P)$.

It is useful to observe that realizations with a normal boundary condition can be represented in a certain canonical form: If $T = P(S\rho + T')$ is normal as in Definition 3.3, then $\widetilde{T} := RT$ is a boundary condition equivalent to T in view of the injectivity of R on the range of P. Moreover,

$$\widetilde{T} = \widetilde{P}(\rho + \widetilde{T}'), \quad \widetilde{P} = RPS, \quad \widetilde{T}' = RT', \tag{3.2}$$

where $\widetilde{P}$ is a projection with components $\widetilde{P}_{jk} \in L^{j-k}_{\mathrm{cl}}(\partial\Omega; E|_{\partial\Omega}, E|_{\partial\Omega})$ and trace operators $\widetilde{T}'_j : H^s(\Omega, E) \to H^{s-j-\frac{1}{2}}(\partial\Omega, E|_{\partial\Omega})$ of order $j + 1/2$ and type 0.

Lemma 3.4. *A normal boundary condition $T = P(S\rho + T')$ induces surjective maps $H^s(\Omega, E) \to \mathcal{H}^{s-d}(\partial\Omega, F, P)$, $s > d - 1/2$.*

Proof. With the previously introduced notation, $T = PS(\rho + RT')$. By Proposition 1.6.5 of [16] we know that $\rho + RT' : H^s(\Omega, E) \to \mathcal{H}^{s-d}(\partial\Omega, E)$ is surjective. It remains to observe that $PS : \mathcal{H}^{s-d}(\partial\Omega, E) \to \mathcal{H}^{s-d}(\partial\Omega, F, P)$ surjectively, due to the existence of R with $PSR = P$. □

Lemma 3.5. *Let $T = P(S\rho + T')$ be a normal boundary condition and $\widetilde{T} = \widetilde{P}(\rho + \widetilde{T}')$ associated with T as in (3.2). Then*

$$\mathcal{H}^s(\partial\Omega, E, \widetilde{P}) = R\big(\mathcal{H}^s(\partial\Omega, F, P)\big).$$

In particular: The canonical form of a normal, elliptic realization is elliptic.

Proof. Applying the previous Lemma 3.4 with $T' = 0$, we obtain

$$\begin{aligned}\widetilde{P}\big(\mathcal{H}^s(\partial\Omega, E)\big) &= RPS\big(\mathcal{H}^s(\partial\Omega, E)\big)\\ &= RPS\rho\big(H^{s+d}(\Omega, E)\big) = R\big(\mathcal{H}^s(\partial\Omega, F, P)\big).\end{aligned}$$

This shows the first claim and that $\widetilde{T} = RT : H^s(\Omega, E) \to \mathcal{H}^{s-d}(\partial\Omega, F, \widetilde{P})$ surjectively. Thus the ellipticity follows from the relation with the Fredholm property of the realization, described after Definition 3.3. □

3.3. Key properties of realizations. In this section we are going to investigate compositions and adjoints of realizations. First, let us observe that normal realizations are always densily defined. In fact, writing $T = P(S\rho + T') = PS(\rho + \widetilde{T}')$, we see that the kernel of T on $H^d(\Omega, E)$ contains the kernel of $\rho + \widetilde{T}'$; this kernel, however, is known to be dense in $L_2(\Omega, E)$, cf. Lemma 1.6.8 of [16].

Theorem 3.6. *Let $B_j := (A_{j,+} + G_j)_{T_j}$, $j = 0, 1$, be two realizations of order d_j subject to APS-type boundary conditions $T_j = P_j(S_j\rho + T_j')$. Moreover, let*

$$A = A_1 A_0, \quad G = (A_{1,+} + G_1)(A_{0,+} + G_0) - A_+,$$

and define the boundary condition

$$T := \begin{pmatrix} T_1 \\ T_0(A_{1,+} + G_1) \end{pmatrix}.$$

Then the following statements are valid:

(1) *If B_0 is elliptic, then $B_1 B_0 = (A_+ + G)_T$.*

(2) *If both B_0 and B_1 are elliptic, then so is $B_1 B_0$.*

(3) *If both T_0 and T_1 are normal (and R_j denotes the right-inverse of $P_j S_j$), then the boundary condition $\widetilde{T} := \begin{pmatrix} R_1 T_1 \\ R_0 T_0 (A_{1,+} + G_1) \end{pmatrix}$ is normal and equivalent to T.*

Proof. The case of trivial projections, $P_0 = 1$ and $P_1 = 1$, is Theorem 1.4.6 of [16]. For (1) and (2) the same proof works also in the general case. Concerning (3), it is clear that $\widetilde{T}$ is equivalent to T, due to the injectiveness of $\operatorname{diag}(R_1, R_0)$. Moreover,

$$\widetilde{T} = \begin{pmatrix} \widetilde{P}_1 & 0 \\ 0 & \widetilde{P}_0 \end{pmatrix} \begin{pmatrix} \rho + \widetilde{T}_1' \\ (\rho + \widetilde{T}_0')(A_{1,+} + G_1) \end{pmatrix}$$

with $\widetilde{P}_j = R_j P_j S_j$ and $\widetilde{T}_j' = R_j T_j'$. According to Theorem 1.4.6 of [16], $(\rho + \widetilde{T}_0')(A_{1,+} + G_1)$ is a normal boundary condition of the form $S\rho + T'$. This yields the normality of $\widetilde{T}$. □

Let us now turn to the analysis of adjoints. First recall Green's formula (for details see Section 1.3 of [16], for example): If $A \in L^d(2\Omega, 2E)$ has the transmission property with respect to $\partial\Omega$, then there exists a matrix

$$\mathfrak{A} = (\mathfrak{A}_{jk})_{0 \le j,k \le d-1}, \quad \mathfrak{A}_{jk} \in L_{\mathrm{cl}}^{d-1-j-k}(\partial\Omega, E|_{\partial\Omega}),$$

whose components are differential operators (in particular, $\mathfrak{A}_{jk} = 0$ if $j + k \ge d$) such that

$$(A_+ u, v)_\Omega = (u, A_+^* v)_\Omega + (\mathfrak{A}\rho u, \rho v)_{\partial\Omega} \quad \forall\, u, v \in H^d(\Omega, E); \tag{3.3}$$

here $(\cdot,\cdot)_\Omega$ indicates the inner product of $L^2(\Omega, E)$, while $(\cdot,\cdot)_{\partial\Omega}$ is the inner product of $\oplus_{j=0}^{d-1} L^2(\partial\Omega, E|_{\partial\Omega})$. The skew-diagonal elements $\mathfrak{A}_{j(d-1-j)}$ are induced by endomorphisms in $E|_{\partial\Omega}$, acting like $i^d(-1)^{d-1-j}\sigma_\psi^d(A)(x,\nu(x))$ in the fibre over x. The boundary $\partial\Omega$ is called *non-characteristic for A* if all these endomorphisms are isomorphisms. In this case, $\mathfrak{A}$ is invertible.

Theorem 3.7. *Let $(A_+ + G)_T$ be a realization with $G = K\rho + G'$ and boundary condition $T = P(\rho + T')$ in canonical form* (*recall that any normal realization can be represented in this way*). *Assume that the boundary $\partial\Omega$ is non-characteristic for A and define*

$$G_{\mathrm{ad}} = -(\mathfrak{A}T')^*\rho + G'^* - (KT')^*, \quad T_{\mathrm{ad}} = P_{\mathrm{ad}}\big(\rho + (K\mathfrak{A}^{-1})^*\big),$$

with the so-called adjoint projection

$$P_{\mathrm{ad}} = \big(\mathfrak{A}(1-P)\mathfrak{A}^{-1}\big)^*.$$

The following is then true:

(1) $\mathrm{dom}((A_+ + G)_T^*) \cap H^d(\Omega, E) = \mathrm{dom}((A_+^* + G_{\mathrm{ad}})_{T_{\mathrm{ad}}})$.

(2) *If $(A_+ + G)_T$ is elliptic, its adjoint coincides with $(A_+^* + G_{\mathrm{ad}})_{T_{\mathrm{ad}}}$.*

Proof. For convenience set $B := (A_+ + G)_T$.

(1) Let $u, v \in H^d(\Omega, E)$. Using Green's formula and writing $\rho u = (\rho + T')u - T'u$ we obtain

$$\big((A_+ + G)u, v\big)_\Omega = \big(u, (A_+^* + G'^*)v\big)_\Omega - \big(u, T'^*(\mathfrak{A}^*\rho + K^*)v\big)_\Omega + \big((\rho + T')u, (\mathfrak{A}^*\rho + K^*)v\big)_{\partial\Omega}. \tag{3.4}$$

Now recall that $v \in \mathrm{dom}(B^*)$ if and only if $u \mapsto \big((A_+ + G)u, v\big)_\Omega$ is continuous on $\mathrm{dom}(B)$ with respect to the $L_2(\Omega, E)$-norm. Since the first two terms on the right-hand side of (3.4) are continuous in this sense, it follows that $v \in \mathrm{dom}(B^*)$ if and only if there exists a constant $C \geq 0$ such that

$$\big|\big((\rho + T')u, (\mathfrak{A}^*\rho + K^*)v\big)_{\partial\Omega}\big| \leq C\|u\|_{L_2(\Omega,E)} \quad \forall\, u \in \mathrm{dom}(B).$$

According to Proposition 1.6.5 of [16], for every $u \in H^d(\Omega, E)$ and $\varepsilon > 0$ there exists an $u_\varepsilon \in H^d(\Omega, E)$ with $\|u_\varepsilon\|_{L_2(\Omega,E)} < \varepsilon$ and $(\rho + T')u_\varepsilon = (\rho + T')u$. Hence $v \in \mathrm{dom}(B^*)$ if and only if

$$\big((\rho + T')u, (\mathfrak{A}^*\rho + K^*)v\big)_{\partial\Omega} = 0 \quad \forall\, u \in \mathrm{dom}(B).$$

The surjectivity of $\rho + T' : H^d(\Omega, E) \to \mathcal{H}^0(\partial\Omega, E)$ implies that

$$(\rho + T')(\mathrm{dom}(B)) = \ker P = \mathrm{im}\,(1 - P).$$

We conclude that $v \in \operatorname{dom}(B^*)$ if and only if

$$\big(\phi, (1-P^*)(\mathfrak{A}^*\rho + K^*)v\big)_{\partial\Omega} = 0 \quad \forall\, \phi \in \mathcal{H}^0(\partial\Omega, E).$$

Now the claim immediately follows, since $T_{\mathrm{ad}} = (\mathfrak{A}^{-1})^*(1-P^*)(\mathfrak{A}^*\rho + K^*)$.

(2) If the realization is elliptic, by Proposition 3.8, proved below, there exists an operator $R \in B^{-d,0}(\Omega; E, E)$ such that $R(L^2(\Omega, E)) \subset \operatorname{dom}(B)$ and $C := (A_+ + G)R - 1$ is smoothing, i.e. has range in $\mathcal{C}^\infty(\Omega, E)$. By general facts on the adjoint of compositions, $R^*B^* \subset (BR)^* = ((A_+ + G)R)^* = 1 + C^*$. Thus the result follows from (1). □

Proposition 3.8. *Let $(A_+ + G)_T$ be elliptic. Then there exists an operator $R \in B^{-d,0}(\Omega; E, E)$ such that*

(1) *$TR = 0$; in particular, R maps $H^d(\Omega, E)$ into the domain of $(A_+ + G)_T$.*

(2) *$C_0 := (A_+ + G)R - 1$ is a finite-rank smoothing Green operator of type 0.*

(3) *$C_1 =: R(A_+ + G) - 1$ coincides on every space $\{u \in H^s(\Omega, E) \mid Tu = 0\}$, $s > d - 1/2$, with a finite-rank smoothing Green operator of type d.*

Proof. It is a well-known fact that there exists a $\Lambda_\Omega \in B^{d,0}(\Omega; E, E)$ having inverse $\Lambda_\Omega^{-1} \in B^{-d,0}(\Omega; E, E)$. Employing the notation from Definition 3.3, let us define

$$\mathcal{A}_0 = \begin{pmatrix} A_0 \\ T_0 \end{pmatrix} := \begin{pmatrix} A_+ + G \\ \Lambda T \end{pmatrix} \Lambda_\Omega^{-1} \in \mathcal{B}^{0,0}(\Omega; (E, 0; 1), (E, F_\partial; P_\Lambda)).$$

By assumption, $\mathcal{A}_0$ is elliptic. We shall now define various projections; note that they all are smoothing Green operators of type 0, since they are integral operators with smooth kernels. Applying Theorem 2.2 and the results of the Appendix, or referring to Theorem 2.3 of [24], there exists a parametrix $\mathcal{A}_1 = (A_1 \quad K_1) \in \mathcal{B}^{0,0}(\Omega; (E, F_\partial; P_\Lambda), (E, 0; 1))$ of $\mathcal{A}_0$ such that

$$\mathcal{A}_1\mathcal{A}_0 = 1 - \pi_0, \quad \mathcal{A}_0\mathcal{A}_1 = 1 - \pi_1,$$

with projections of the form

$$\pi_0 = \sum_{j=1}^{n_0} \big(\cdot, v_j^0\big)_{L_2(\Omega,E)} v_j^0, \quad \pi_1 = \sum_{j=1}^{n_1} \big(\cdot, \mathcal{P}_\Lambda^* v_j^1\big)_{L_2(\Omega,E)\oplus L_2(\partial\Omega,F_{\partial\Omega})} v_j^1$$

with functions $\{v_1^0, \dots, v_{n_0}^0\} \subset \mathcal{C}^\infty(\Omega, E)$ being an L_2-orthogonal basis of $V_0 := \ker \mathcal{A}_0$, with $\{v_1^1, \dots, v_{n_1}^1\} \subset \mathcal{C}^\infty(\Omega, E) \oplus \mathcal{C}^\infty(\partial\Omega, F_{\partial\Omega}, P_\Lambda)$ being an L_2-orthogonal basis of a space V_1 that complements $\mathcal{A}_0(H^s(\Omega, E))$ in $H^s(\Omega, E) \oplus H^s(\partial\Omega, F_{\partial\Omega}, P_\Lambda)$ simultaneously for all s, and with $\mathcal{P}_\Lambda := \operatorname{diag}(1, P_\Lambda)$. Note that $(1 - \mathcal{P}_\Lambda)\pi_1 = \pi_1(1 - \mathcal{P}_\Lambda) = 0$. If we represent π_1 in block-matrix form,

$$\pi_1 = \begin{pmatrix} \pi_{11} & \pi_{12} \\ \pi_{21} & \pi_{22} \end{pmatrix} : \begin{matrix} H^s(\Omega, E) \\ \oplus \\ H^s(\partial\Omega, F_{\partial\Omega}, P_\Lambda) \end{matrix} \longrightarrow \begin{matrix} H^s(\Omega, E) \\ \oplus \\ H^s(\partial\Omega, F_{\partial\Omega}, P_\Lambda) \end{matrix},$$

then

$$\pi_{21}u = \sum_{j=1}^{n_1} (u, u_j)_{L_2(\Omega,E)} w_j$$

provided $v_j^1 = u_j \oplus w_j$ with suitable $u_j \in \mathcal{C}^\infty(\Omega, E)$ and $w_j \in \mathcal{C}^\infty(\partial\Omega, F_{\partial\Omega}, P_\Lambda)$. Now let $U = \operatorname{span}(u_1, \dots, u_{n_1})$ have the L_2-orthonormal basis $\{e_1, \dots, e_n\}$ and define

$$\pi_U = \sum_{j=1}^{n} (\cdot, e_j)_{L_2(\Omega,E)} e_j .$$

Then, by construction, $\pi_{21}(1-\pi_U) = 0$. We now claim that $R := \Lambda_\Omega^{-1} A_1 (1-\pi_U)$ is the desired operator. In fact,

$$\begin{aligned}\begin{pmatrix} A_+ + G \\ T \end{pmatrix} R &= \begin{pmatrix} 1 & 0 \\ 0 & \Lambda^{-1} \end{pmatrix} \mathcal{A}_0 A_1 (1-\pi_U) \\ &= \begin{pmatrix} 1 & 0 \\ 0 & \Lambda^{-1} \end{pmatrix} \begin{pmatrix} 1-\pi_{11} \\ -\pi_{21} \end{pmatrix} (1-\pi_U) = \begin{pmatrix} (1-\pi_{11})(1-\pi_U) \\ 0 \end{pmatrix}\end{aligned}$$

shows that $TR = 0$ and that C_0 is a finite-rank smoothing Green operator of type 0. This shows (1) and (2). Finally, on $H^s(\Omega, E) \cap \ker T$,

$$\begin{aligned} C_1 &= \Lambda_\Omega^{-1} A_1 (1-\pi_U)(A_+ + G) - 1 = \Lambda_\Omega^{-1} A_1 (1-\pi_U) A_0 \Lambda_\Omega - 1 \\ &= \Lambda_\Omega^{-1} (A_1 A_0 - K_1 T_0) \Lambda_\Omega - 1 - \Lambda_\Omega^{-1} A_1 \pi_U A_0 \Lambda_\Omega \\ &= \Lambda_\Omega^{-1} (\mathcal{A}_1 \mathcal{A}_0 - 1) \Lambda_\Omega - \Lambda_\Omega^{-1} A_1 \pi_U A_0 \Lambda_\Omega \\ &= -\Lambda_\Omega^{-1} (\pi_0 + A_1 \pi_U A_0) \Lambda_\Omega , \end{aligned}$$

proving claim (3). □

Corollary 3.9. *If $(A_+ + G)_T$ is elliptic, it is a closed operator in $L_2(\Omega, E)$.*

Proof. Let (u_n) be a sequence in $\operatorname{dom}((A_+ + G)_T)$ such that both $u := \lim_{n\to+\infty} u_n$ and $v := \lim_{n\to+\infty} A u_n$ exist in $L_2(\Omega, E)$.

Let $R \in B^{-d,0}(\Omega; E, E)$ be the parametrix constructed in Proposition 3.8 and C_1 the respective smoothing Green operator. Then $C_1 u_n = R(A_+ + G)u_n - u_n$ is convergent in $L_2(\Omega, E)$. Since C_1 maps the domain of $(A_+ + G)_T$ into a finite-dimensional subspace of $\mathcal{C}^\infty(\Omega, E)$, the sequence $(C_1 u_n)$ is also convergent in $H^d(\Omega, E)$. Thus $u_n = R(A_+ + G)u_n - C_1 u_n$ converges in $H^d(\Omega, E)$. We conclude that $u \in H^d(\Omega, E)$, $v = (A_+ + G)u$, and $Tu = \lim_{n\to+\infty} T u_n = 0$. □

3.4. Self-adjoint realizations. A realization may be represented in many different ways. In the present section we analize this fact systematically and then characterize the self-adjoint realizations.

Proposition 3.10. *Let $T_j = P_j(\rho + T'_j)$, $j = 0, 1$, be two boundary conditions in normal form. Then T_0 and T_1 are equivalent if, and only if, $P_j(1 - P_{1-j}) = 0$ and $P_j T'_j = P_j T'_{1-j}$ for $j = 0, 1$.*

Note that the property $P_j(1 - P_{1-j}) = 0$ for $j = 0, 1$ is equivalent to $\ker P_0 = \ker P_1$ for P_0 and P_1 considered as maps on $\mathcal{H}^s(\partial\Omega, E)$ for some (and then every) choice of s. Then $P_0 T'_0 = P_0 T'_1$ is equivalent to $P_1 T'_1 = P_1 T'_0$.

Proof of Proposition 3.10. Recall that the boundary conditions are called equivalent if their kernels on $H^d(\Omega, E)$ coincide.

First let us show that the stated conditions imply the equivalence. Clearly $T_0 u = 0$ means $(\rho + T'_0)u \in \ker P_0$. Thus, by assumption, also $0 = P_1(\rho + T'_0)u = P_1(\rho + T'_1)u = T_1 u$. Interchanging roles of T_0 and T_1 thus shows $\ker T_0 = \ker T_1$.

Now let us assume that the kernels coincide. According to Lemma 1.6.8 of [16] there exists a right-inverse K to ρ such that $\Lambda := 1 + K T'_0$ is an isomorphism in $H^s(\Omega, E)$ simultaneously for all $s \geq 0$. Note that $P_0 \rho \Lambda = T_0$. Thus, for $u \in H^d(\Omega, E)$,

$$\begin{aligned} P_0 \rho \Lambda u = 0 &\iff T_1 u = 0 \\ &\iff P_1 \rho \Lambda u + P_1\big(\rho(1 - \Lambda) + T'_1\big)u = 0 \\ &\iff P_1 \rho \Lambda u + P_1(T'_1 - T'_0)u = 0. \end{aligned} \tag{3.5}$$

This equivalence implies, in particular, that

$$P_1(T'_1 - T'_0)u = 0 \quad \forall\, u \in U := \Lambda^{-1}\big(\mathcal{C}_0^\infty(\operatorname{int}\Omega, E)\big).$$

Since U is dense in $L_2(\Omega, E)$ and T'_j is of type 0, it follows that $P_1(T'_1 - T'_0) = 0$, i.e. $P_1 T'_0 = P_1 T'_1$. Then (3.5) and the surjectivity of $\rho\Lambda : H^d(\Omega, E) \to \mathcal{H}^0(\partial\Omega, E)$ show that P_0 and P_1 have the same kernel on $\mathcal{H}^0(\partial\Omega, E)$. Interchanging roles of T_0 and T_1 yields also $P_0 T'_1 = P_0 T'_0$. □

Let $B = (A_+ + G)_T$ with $T = P(\rho + T')$. One can always choose $G = K\rho + G'$ in a certain *reduced form*, namely with K satisfying $KP = 0$. In fact, if initially $G = K_0\rho + G'_0$ and $Tu = 0$ (i.e. $P\rho u = -T'u$), we can write

$$Gu = K_0(P\rho u + (1 - P)\rho u) + G'_0 u = K_0(1 - P_0)\rho u + (G'_0 - K_0 P T')u$$

and then set $K := K_0(1 - P)$ and $G' := G'_0 - K_0 P T'$.

Proposition 3.11. *With $j = 0, 1$ let $B_j = (A_+ + G_j)_T$ be two realizations with $T = P(\rho + T')$ and $G_j = K_j\rho + G'_j$ in reduced form, i.e. $K_j P = 0$. Then $B_0 = B_1$ if, and only if, $K_0 = K_1$ and $G'_0 = G'_1$.*

Proof. Clearly $B_0 = B_1$ if, and only if,

$$(A_+ + G_0)u = (A_+ + G_1)u \quad \forall\, u \in H^d(\Omega, E) \cap \ker T.$$

If Λ is an isomorphism associated with T as in the proof of Proposition 3.10, this is equivalent to

$$(G_0 - G_1)\Lambda^{-1}v = 0 \quad \forall\, v \in H^d(\Omega, E) \cap \ker P\rho.$$

For such v we can write

$$(G_0 - G_1)\Lambda^{-1}v = (K_0 - K_1)\rho v + Gv$$

with $G := (K_0 - K_1)\rho(\Lambda^{-1} - 1) + (G_0' - G_1')\Lambda^{-1}$ having type 0, according to Lemma 1.6.8 of [16]. Choosing $v \in \mathcal{C}_0^\infty(\operatorname{int}\Omega, E)$ we derive that $G = 0$ and that

$$(K_0 - K_1)\rho v = 0 \quad \forall\, v \in H^d(\Omega, E) \cap \ker P\rho.$$

since $\rho : H^d(\Omega, E) \to \mathcal{H}^0(\partial\Omega, E)$ surjectively, this means

$$(K_0 - K_1)\phi = 0 \quad \forall\, \phi \in \mathcal{H}^0(\partial\Omega, E) \cap \ker P.$$

Since $\ker P = \operatorname{im}(1-P)$ we derive that $(K_0-K_1)(1-P) = 0$ and thus $K_0-K_1 = 0$, since $(K_0 - K_1)P = 0$ by assumption. From $G = 0$ we then obtain $G_0' = G_1'$. $\square$

As a consequence we obtain the following description of self-adjointness for realizations:

Theorem 3.12. *Consider an elliptic realization $B = (A_+ + G)_T$ with A being symmetric, $T = P(\rho + T')$ and $G = K\rho + G'$ in reduced form. Assume that $\partial\Omega$ is non-characteristic for A. Then*:

(1) $\operatorname{dom}(B^*) = \operatorname{dom}(B)$ *if, and only if,* $\mathfrak{A} : \ker P \to (\ker P)^\perp$ *isomorphically and* $P(T' + \mathfrak{A}^{-1}K^*) = 0$.

(2) *If* $\operatorname{dom}(B^*) = \operatorname{dom}(B)$ *then* $B = B^*$ *if, and only if,*

$$G' = G'^* - (KT')^* - T'^*\mathfrak{A}T'.$$

Let us note that one always may assume that $T' = PT'$ in the representation of T. In this case, the term $(KT')^*$ in (2) *vanishes, since $KP = 0$ by assumption.*

Proof. We have $B^* = (A_+ + G_{\mathrm{ad}})_{T_{\mathrm{ad}}}$ as described Theorem 3.7. The symmetry of A implies that $\mathfrak{A}^* = -\mathfrak{A}$ and therefore

$$\begin{aligned} G_{\mathrm{ad}} &= T'^*\mathfrak{A}\rho + G'^* - (KT')^*, \\ T_{\mathrm{ad}} &= P_{\mathrm{ad}}\big(\rho - \mathfrak{A}^{-1}K^*\big), \\ P_{\mathrm{ad}} &= \mathfrak{A}^{-1}(1 - P^*)\mathfrak{A}. \end{aligned}$$

Hence, by Proposition 3.10 and the comment given thereafter, the domains of B and B^* coincide if, and only if, $\ker P_{\mathrm{ad}} = \ker P$ and $PT' = -P\mathfrak{A}^{-1}K^*$. Now (1) follows, since

$$u \in \ker P_{\mathrm{ad}} \iff \mathfrak{A}u \in \ker(1 - P^*) = \operatorname{im} P^* = (\ker P)^{\perp}.$$

Let us now show (2). We have $B^* = (A_+ + G_{\mathrm{ad}})_T$, since T and T_{ad} are equivalent by assumption. Writing $\rho = (1 - P)\rho + P\rho$ and using the fact that $P\rho u = -T'u$ provided $Tu = 0$, the reduced form of G_{ad} is

$$G_{\mathrm{ad}} = T'^*\mathfrak{A}(1 - P)\rho + G'^* - (KT')^* - T'^*\mathfrak{A}T'.$$

According to Proposition 3.11, $B = B^*$ is equivalent to

$$K = T'^*\mathfrak{A}(1 - P) \quad \text{and} \quad G' = G'^* - (KT')^* - T'^*\mathfrak{A}T'.$$

Since $KP = 0$ by assumption,

$$\begin{aligned} K = T'^*\mathfrak{A}(1 - P) &\iff (K - T'^*\mathfrak{A})(1 - P) = 0 \\ &\iff (1 - P^*)(K^* + \mathfrak{A}T') = 0 \\ &\iff P_{\mathrm{ad}}(T' + \mathfrak{A}^{-1}K^*) = 0. \end{aligned}$$

However, this is true by (1) (P_{ad} can be equivalently replaced by P, since P_{ad} and P have the same kernel). □

Theorem 3.12 in case of $B = (A_+)_T$ with symmetric A and $T = P\rho$, states that the self-adjointness of B is equivalent to $\mathfrak{A} : \ker P \to (\ker P)^{\perp}$ being an isomorphism.

4. Spectral triples for manifolds with boundary

A triple $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ is called a spectral triple of dimension $n \in \mathbb{N}$ if

(a) $\mathcal{H}$ is a Hilbert space and $\mathcal{A}$ is a unital, involutive algebra, faithfully represented in $\mathcal{L}(H)$,

(b) $\mathcal{D}$ is a closed, self-adjoint operator in $\mathcal{H}$ with compact resolvent and such that the sequence of eigenvalues $\mu_1 \le \mu_2 \le \cdots$ of $|\mathcal{D}|$ satisfies $\mu_j \sim j^{1/n}$.

(c) for every $a \in \mathcal{A}$, application of a preserves $\operatorname{dom}(\mathcal{D})$ and the commutator $[\mathcal{D}, a]$, initially defined on $\operatorname{dom}(\mathcal{D})$, extends by continuity to an operator in $\mathcal{L}(H)$, denoted by da (thus $da = \overline{[\mathcal{D}, a]}$).

To define the notion of regular spectral triple, we shall need the operator

$$\delta : \operatorname{dom}(\delta) \longrightarrow \mathcal{L}(\mathcal{H}), \quad L \mapsto \delta(L) := \overline{[|\mathcal{D}|, L]},$$

whose domain consists of those operators $L \in \mathcal{L}(\mathcal{H})$ that map $\operatorname{dom}(\mathcal{D})$ into itself and whose commutator $[|\mathcal{D}|, L]$ extends by continuity to a bounded operator in $\mathcal{H}$.

Definition 4.1. A spectral triple $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ is called regular if, for every $a \in \mathcal{A}$,

$$a, da \in \operatorname{dom}(\delta^k) \quad \forall\, k \in \mathbb{N}.$$

In the sequel we shall focus on the case that $\mathcal{H} := L^2(\Omega, E)$ with $n = \dim \Omega$ and that $\mathcal{A} \subseteq \mathcal{C}^\infty(\Omega)$, represented in $\mathcal{L}(\mathcal{H})$ as operators of multiplication with functions. We now shall analyze when a first order, elliptic, self-adjoint realization $\mathcal{D} = (A_+ + G)_T$ subject to APS-type conditions leads to a spectral triple of dimension n.

First of all we note that if $\mathcal{A}^0_{\mathcal{D}}$ is defined as

$$\mathcal{A}^0_{\mathcal{D}} := \{a \in \mathcal{C}^\infty(\Omega) \mid \text{both } a \text{ and } a^* \text{ map } \operatorname{dom}(\mathcal{D}) \text{ into itself}\}, \tag{4.1}$$

then $(\mathcal{A}^0_{\mathcal{D}}, \mathcal{H}, \mathcal{D})$ is a spectral triple provided G is a Green operator of order and type 0. In fact, for any $a \in \mathcal{A}^0_{\mathcal{D}}$,

$$[A_+ + G, a] = [A, a]_+ + [G, a] \in \mathcal{L}(\mathcal{H}),$$

since $[A, a]$ is a pseudodifferential operator of order 0 and $[G, a]$ is Green operator of order and type 0. Moreover, by self-adjointness, $(\mathcal{D} - i)^n$ induces a bijection $\operatorname{dom}(\mathcal{D}^n) \to \mathcal{H}$. The fact that $\operatorname{dom}(\mathcal{D}^n) \subset H^n(\Omega, E)$ together with a general, functional-analytic result (see e.g. Lemma A.4 in [16]) now implies that $\mu_j(|\mathcal{D}|) \sim j^{1/n}$.

The situation for regular spectral triples is more complicated:

Theorem 4.2. *Let $\mathcal{H} = L^2(\Omega, E)$ and $\mathcal{D} := (A_+)_T$ be an elliptic, self-adjoint*[4] *realization of first order with boundary condition of the form $Tu = P(Su|_{\partial\Omega} + T'u)$ (cf. Definition 3.1 with $d = 1$). Assume that A^2 has scalar principal symbol and that*

$$A_+ P_+ = (AP)_+ \quad \forall \text{ non-negative order pseudodifferential operators } P.^{5} \tag{4.2}$$

Let $\mathcal{A}^\infty_{\mathcal{D}}$ be defined as

$$\mathcal{A}^\infty_{\mathcal{D}} := \{a \in \mathcal{A}^0_{\mathcal{D}} \mid \text{both } a \text{ and } a^* \text{ map } \mathcal{H}^\infty_{\mathcal{D}} \text{ into itself}\},$$

where $\mathcal{A}^0_{\mathcal{D}}$ is as in (4.1) and

$$\mathcal{H}^\infty_{\mathcal{D}} = \bigcap_{k \in \mathbb{N}} \operatorname{dom}(\mathcal{D}^k)$$

(note that in the definition of $\mathcal{H}^\infty_{\mathcal{D}}$ one may also use the operator $|\mathcal{D}|$ in place of $\mathcal{D}$). Then $(\mathcal{A}^\infty_{\mathcal{D}}, \mathcal{H}, \mathcal{D})$ is a regular spectral triple. Moreover, $\mathcal{A}^\infty_{\mathcal{D}}$ is the largest subalgebra of $\mathcal{A}^0_{\mathcal{D}}$ that, together with $\mathcal{D}$, leads to a regular spectral triple.

[4]Recall that $\mathcal{D}$ is self-adjoint if, and only if, A is symmetric and $\ker T = \ker T_{\mathrm{ad}}$.

[5]For example this is the case if $A = A_0 + A_1$ where A_0 is a differential operator and A_1 is pseudodifferential with $A_1 = A_1\varphi$ with a smooth function φ supported in the interior of Ω. Also more general A_1 are possible, but we shall not enter details here.

Proof. The proof is along the lines of that of Theorem 4.5 in [17]. Clearly, $(\mathcal{A}_{\mathcal{D}}^{\infty}, \mathcal{H}, \mathcal{D})$ is a spectral triple of dimension n, since $\mathcal{A}_{\mathcal{D}}^{\infty}$ is a $*$-subalgebra of $\mathcal{A}_{\mathcal{D}}^{0}$. Now observe that, by construction, if $b = a$ or $b = [\mathcal{D}, a]$ with $a \in \mathcal{A}_{\mathcal{D}}^{\infty}$, then b maps $\mathcal{H}_{\mathcal{D}}^{\infty}$ into itself. Thus we can define the iterated commutators

$$b^{(k)} := [\mathcal{D}^2, \cdot]^k(b) : \mathcal{H}_{\mathcal{D}}^{\infty} \longrightarrow \mathcal{H}_{\mathcal{D}}^{\infty}, \quad k \in \mathbb{N}.$$

By Lemma 2.6 of [17], to prove regularity of the spectral triple, it suffices to show that

$$\|b^{(k)}u\|_{L^2(\Omega,E)} \leq C_k \|u\|_{H^k(\Omega,E)} \quad \forall\, u \in \mathcal{H}_{\mathcal{D}}^{\infty}, \quad \forall\, k \in \mathbb{N}, \tag{4.3}$$

with constants C_k not depending on u.

Due to assumption (4.2), $(A_+)^\ell = (A^\ell)_+$ for every $\ell \in \mathbb{N}$. Therefore, in case $b = a$, property (4.3) immediately follows, since

$$b^{(k)} = ([A^2, \cdot]^{(k)}(a))_+$$

is (the restriction to Ω of) a pseudodifferential operator of order k, since A^2 has scalar principal symbol.

To verify (4.3) in case of $b = [\mathcal{D}, a] = [A_+, a] = [A, a]_+$, first observe that with A also b satisfies condition (4.2), since

$$\begin{aligned}
[A_+, a]P_+ &= A_+ a P_+ - a A_+ P_+ \\
&= A_+(aP)_+ - a(AP)_+ \\
&= (AaP)_+ - (aAP)_+ = ([A, a]P)_+;
\end{aligned}$$

here we have used that a is zero order differential. Then $b^{(k)} = ([A_+^2, \cdot]^k(b)$ is a linear combination of terms $A_+^{2k_1} b A_+^{2k_2}$ with $k_1 + k_2 = k$. Applying repeatedly Property (4.2), any such term equals $(A^{2k_1}[A, a]A^{2k_2})_+$. We conclude that

$$([A_+^2, \cdot]^k(b) = \big([A^2, \cdot]^k([A, a])\big)_+,$$

and then can argue again as before to obtain the mapping property (4.3).

For the last affirmation of the theorem, assume that $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ is a regular spectral triple with $\mathcal{A}$ being a $*$-subalgebra of $\mathcal{A}_{\mathcal{D}}^{0}$. Then, for $b = a$ or $b = a^*$ with $a \in \mathcal{A}$, the following identity holds (see Lemma 2.1 in [10]):

$$|\mathcal{D}|^n(bu) = \sum_{j=0}^{n} \binom{n}{j} \delta^j(b)|\mathcal{D}|^{n-j}u \quad \forall\, u \in \mathrm{dom}(|\mathcal{D}|^n).$$

This shows at once that $bu \in \mathcal{H}_{\mathcal{D}}^{\infty}$ provided $u \in \mathcal{H}_{\mathcal{D}}^{\infty}$. Thus $\mathcal{A} \subseteq \mathcal{A}_{\mathcal{D}}^{\infty}$. □

The precise description of $\mathcal{A}_{\mathcal{D}}^{\infty}$ is in general very cumbersome and even in specific examples it appears very difficult to provide an explicit expression of this algebra. However, the following is valid:

Lemma 4.3. *Let $\mathcal{D} = (A_+)_T$ be as described in Theorem 4.2 with A differential and assume that the boundary condition is of the form $Tu = PSu|_{\partial\Omega}$ with a projection P and a bundle homomorphsm S on the boundary. Let $a \in \mathcal{C}^\infty(\Omega)$ and assume that there exists a smooth function φ which is constant near every connected coponent of the boundary of Ω such that $a - \varphi$ vanishes to infinite order at $\partial\Omega$. Then $a \in \mathcal{A}^\infty_{\mathcal{D}}$.*

Proof. Since φ belongs to $\mathcal{A}^\infty_{\mathcal{D}}$, we may assume that $\varphi \equiv 0$. If $u \in \mathcal{H}^\infty_{\mathcal{D}}$ then au also vanishes to infinite order at the boundary, hence so does $A_+^N(au)$ for arbitrary $N \in \mathbb{N}$. Therefore

$$T(A_+^N(au)) = PS(A_+^N(au)|_{\partial\Omega}) = 0,$$

showing that $au \in \mathrm{dom}(\mathcal{D}^{N+1})$. For the same reason, $a^*u \in \mathrm{dom}(\mathcal{D}^{N+1})$. Since N is arbitrary it follows that both a and a^* preserve $\mathcal{H}^\infty_{\mathcal{D}}$. □

As we shall see in the sequel, cf. Theorem 4.9 below, it seems easier to describe the closure in $\mathcal{L}(\mathcal{H})$ of $\mathcal{A}^\infty_{\mathcal{D}}$ (or, if we consider $\mathcal{A}^\infty_{\mathcal{D}}$ as a subspace of the continuous functions on Ω, its closure with respect to the supremum-norm). As already said, this is of significance in view of Connes' reconstruction Theorem; in fact, in case a spectral triple $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ satisfies Connes' axioms, the reconstructed manifold is homeomorphic, as a topological space, to $\mathrm{Spec}(\overline{\mathcal{A}})$. We finish this subsection with a technical lemma which we shall employ below in this context.

Lemma 4.4. *Let $(A_+)_T$ be as described in Theorem 4.2 with A differential and assume that the boundary condition is of the form $Tu = PSu|_{\partial\Omega}$ with an orthogonal projection P and a bundle isomorphism S on the boundary. Let $a \in \mathcal{C}^\infty(\Omega)$ and assume that both a and a^* preserve* $\mathrm{dom}\,((A_+)_T)$. *Then $a|_{\partial\Omega}$ commutes with P.*

Proof. Let $\varphi \in \mathcal{C}^\infty(\partial\Omega, E|_{\partial\Omega})$ arbitrary and u be some function in $\mathcal{C}^\infty(\Omega, E)$ such that $u|_{\partial\Omega} = S^{-1}(1-P)\varphi$. Then $u \in \mathrm{Dom}(\mathcal{D})$ and hence, by assumption,

$$0 = T(au) = P(\widetilde{a}(1-P)\varphi), \quad \widetilde{a} := a|_{\partial\Omega}.$$

Thus $P\widetilde{a}(1-P) = 0$, i.e. $P\widetilde{a} = P\widetilde{a}P$. Replacing a by a^* shows that $P\widetilde{a}^* = P\widetilde{a}^*P$. Passing to adjoints yields $\widetilde{a}P = P\widetilde{a}P = P\widetilde{a}$. □

4.1. Self-adjoint realizations of Dirac operators. In order to define a Dirac operator we suppose that (Ω, E) is a Clifford module and that the bundle E has an Hermitian structure $\langle\cdot,\cdot\rangle$ and a connection ∇ compatible with the Clifford module structure. We call D the associated Dirac operator; it is symmetric and locally has the form

$$(Du)(x) = \sum_{j=1}^{n} c\left(e_j\right)\left(\nabla_{e_j} u\right)(x), \quad u \in \mathcal{C}^\infty(\Omega, E)$$

where $\{e_1, \ldots, e_n\}$ is an orthonormal frame of $T\Omega$ at $x \in \Omega$ and $c(\cdot)$ is the Clifford multiplication.

Depending on the parity of the dimension of Ω, we can complete D with APS-type boundary conditions to an elliptic, self-adjoint realization.

The case of even dimension. In this case the bundle E canonically splits in two subbundles E_+ and E_- via the chirality operator, i.e. $E = E_+ \oplus E_-$, and $D : \mathcal{C}^\infty(\Omega, E) \to \mathcal{C}^\infty(\Omega, E)$ can be identified with

$$D = \begin{pmatrix} 0 & D^- \\ D^+ & 0 \end{pmatrix}, \quad D^\pm : \mathcal{C}^\infty(\Omega, E_\pm) \to \mathcal{C}^\infty(\Omega, E_\mp), \tag{4.4}$$

where $\left(D^+\right)^* = D^-$. Recall that, in a collar neighborhood of the boundary $\partial\Omega$, the metric is assumed to be of product type (in case of Dirac operators this is not a restrictive assumption as it can be always achieved up to conjugation by unitary isomorphism, see the appendix of [3], for instance). Then one can write, near the boundary,

$$D = \Gamma(x')\,(\partial_{x_n} + B) \tag{4.5}$$

where (x', x_n) are the normal and tangential coordinates, respectively, and an endomorphism $\Gamma : E_+ \oplus E_- \to E_- \oplus E_+$ which inverts the chirality and does not depend on the normal direction. In fact, it corresponds to the Clifford multiplication with the inward normal vector; in particular, $\Gamma^2 = -\mathrm{Id}_E$. The so-called tangential operator

$$B : \mathcal{C}^\infty(\partial\Omega, E_+|_{\partial\Omega} \oplus E_-|_{\partial\Omega}) \to \mathcal{C}^\infty(\partial\Omega, E_+|_{\partial\Omega} \oplus E_-|_{\partial\Omega})$$

is a self-adjoint elliptic differential operator of first order preserving the splitting $E = E_+ \oplus E_-$. Since B is elliptic and self-adjoint, there are well defined eigenvalues $\{\lambda_k\}_{k\in\mathbb{Z}}$ and eigenfunctions $\{f_k\}_{k\in\mathbb{Z}}$ which form an orthonormal base of $L^2(\partial\Omega, E)$. We consider

$$P_\geq : L^2(\partial\Omega, E) \longrightarrow L^2(\partial\Omega, E), \quad u \mapsto \sum_{\lambda_k \geq 0} \langle u, f_k\rangle f_k,$$

that is the orthogonal projection onto the span of the eigenfunctions corresponding to non-negative eigenvalues. We set $P_< = 1 - P_\geq$. The APS boundary condition is then defined as

$$T_{\mathrm{APS}} = \begin{pmatrix} P_\geq \gamma_0 & 0 \\ 0 & P_< \Gamma^* \gamma_0 \end{pmatrix} : H^s(\Omega, E_+ \oplus E_-) \longrightarrow H^{s-1/2}(\partial\Omega, E_+ \oplus E_-).$$

Then we let $\mathcal{D} = \mathcal{D}_{\mathrm{APS}}$ denote the realization of D with subject to the condition T_{APS}. The splitting $E = E_+ \oplus E_-$ induces an identification $\mathcal{D}_{\mathrm{APS}} = \begin{pmatrix} 0 & D^-_{\mathrm{APS}} \\ D^+_{\mathrm{APS}} & 0 \end{pmatrix}$ with the operators $D^\pm_{\mathrm{APS}}$ given by the action of $D^\pm$ on the domains

$$\begin{aligned} \mathrm{dom}(D^+_{\mathrm{APS}}) &= \left\{u \in H^1(\Omega, E_+) \mid P_\geq u|_{\partial\Omega} = 0\right\}, \\ \mathrm{dom}(D^-_{\mathrm{APS}}) &= \left\{u \in H^1(\Omega, E_-) \mid P_<(\Gamma^* u|_{\partial\Omega}) = 0\right\}. \end{aligned} \tag{4.6}$$

Theorem 3.12 implies that $\mathcal{D}_{\mathrm{APS}}$ is self-adjoint. Moreover, by Theorem 4.2, both $(\mathcal{A}^0_{\mathcal{D}}, \mathcal{H}, \mathcal{D})$ and $(\mathcal{A}^\infty_{\mathcal{D}}, \mathcal{H}, \mathcal{D})$ with $\mathcal{H} = L^2(\Omega, E)$ and $\mathcal{D} = \mathcal{D}_{\mathrm{APS}}$ are spectral triples of dimension n, the second one being regular.

Remark 4.5. If Ω were without boundary, it is known that $(\mathcal{C}^\infty(\Omega), L^2(\Omega, E), \mathcal{D})$ defines a regular spectral triple. Moreover, in the even dimensional case, it is possible to define a grading γ of $L^2(\Omega, E)$ such that the spectral triple is even, i.e. $\gamma^2 = \mathrm{Id}$, $\gamma^* = \gamma$, $\gamma a = a\gamma$ for all $a \in \mathcal{C}^\infty(\Omega)$ and $\gamma\mathcal{D} + \mathcal{D}\gamma = 0$. The grading is related to the splitting induced by the chirality.

The spectral triples $(\mathcal{A}^0_{\mathcal{D}}, \mathcal{H}, \mathcal{D})$ and $(\mathcal{A}^\infty_{\mathcal{D}}, \mathcal{H}, \mathcal{D})$ introduced above are also even, since the grading operator preserves the domain of $\mathcal{D}_{\mathrm{APS}}$. This is not true for the spectral triples introduced in [18], based on chiral boundary conditions. Dealing with local boundary conditions, as chiral boundary conditions, the grading does not preserve the domain of the Dirac operator, see e.g. the example on the disk in [8].

The case of odd dimension. We suppose again that the metric is of product type near the boundary and thus D has the form (4.5). Also in this case it is well known that the APS boundary conditions define elliptic realizations of the Dirac operator. Since D is essentially self-adjoint,

$$\Gamma B = B\Gamma^* = -B\Gamma, \tag{4.7}$$

i.e. Γ inverts the splitting induced by the spectrum of B. For Green's formula (3.3) we find $\mathfrak{A} = \Gamma^* = -\Gamma$. By (4.7), we obtain

$$\mathfrak{A} = -\Gamma : \ker P_{\geq} \longrightarrow \ker P_{\leq}.$$

In case B is invertible, $\ker P_{\leq} = \ker P_{<} = (\ker P_{\geq})^{\perp}$. Hence, by Theorem 3.12, the realization $\mathcal{D} = \mathcal{D}_{\mathrm{APS}}$ of D subject to the boundary condition $T = P_{\geq 0}\gamma_0$ is self-adjoint.

In case B is not invertible, the usual APS-boundary condition does not give a self-adjoint realization and one has to proceed differently, as is discussed in detail in [12]. Let $a^2 \in \mathbb{R} \setminus \sigma(B^2)$ and let $E(\lambda) \subset L^2(\partial\Omega, E|_{\partial\Omega})$, denote the eigenspace associated to the eigenvalue $\lambda \in \sigma(B)$. Since the boundary is even dimensional, there is a splitting $E|_{\partial\Omega} = E|^+_{\partial\Omega} \oplus E|^-_{\partial\Omega}$, inducing a splitting of each eigenspace. We set

$$K^{\pm}_{\partial\Omega}(a) = \bigoplus_{-a<\lambda<a} E^{\pm}(\lambda)$$

and let $P_g \in \mathcal{L}(L^2(\partial\Omega, E|_{\partial\Omega}))$ be the orthogonal projection onto the graph of an L_2-unitary map $g : K^+_{\partial\Omega}(a) \to K^-_{\partial\Omega}(a)$. Then the trace operator

$$T = \left(P_{>a} + P_g\right)\gamma_0$$

is an APS-type boundary condition as defined in Section 3 (note that $P_{>a}P_g = P_g P_{>a} = 0$, hence $P_{>a} + P_g$ indeed is a projection) and induces an elliptic, self-adjoint realization of the Dirac operator, again denoted by $\mathcal{D}$ (of course this operator depends on the choice of a and g).

In any case, by Theorem 4.2, we can conclude that both $(\mathcal{A}^0_{\mathcal{D}}, L^2(\Omega, E), \mathcal{D})$ and $(\mathcal{A}^\infty_{\mathcal{D}}, L^2(\Omega, E), \mathcal{D})$ are spectral triples of dimension n, the second one being regular.

Remark 4.6. In [25], the realizations of the Dirac operator subject to APS-type conditions $T = P\gamma_0$ are analyzed, where P is a zero order pseudodifferential projection on the boundary. In case of odd dimension and invertible tangential operator, it is proven that such a realization is elliptic and self-adjoint if, and only if, $P = P_g$ is the orthogonal projection onto the graph of an L^2-unitary isomorphism $g : E^+_{\partial\Omega} \to E^-_{\partial\Omega}$. The case of a non-invertible tangential operator is studied in [12]. It is proven that $T = P\gamma_0$ leads to a elliptic and self-adjoint extension of $D_{P_{>-a}}\gamma_0$ if and only if $P = P_{>a} + P_g$ with P_g described above.

We want to stress once more that all these boundary conditions are of APS-type and fit in our general framework.

4.2. Spectral triples based on Dirac operators. We shall now study $\mathcal{A}^0_{\mathcal{D}}$ and $\mathcal{A}^\infty_{\mathcal{D}}$ in case of $\mathcal{D}$ being an above described realization of the Dirac operator.

Proposition 4.7. *Let $(\mathcal{A}^0_{\mathcal{D}}, \mathcal{H}, \mathcal{D})$ be the spectral triple associated with a Dirac operator D and* APS*-type boundary conditions as described in the previous Subsection 4.1. Let $a \in \mathcal{A}^0_{\mathcal{D}}$ such that both a and a^* preserve* Dom$(\mathcal{D})$. *Then $a|_{\partial\Omega}$ is locally constant.*

Proof. In case Ω is of even dimension, both a and a^* preserve the domain of D^+_{APS}, cf. (4.6). In the case of odd dimension let us first assume that the tangential operator is invertible. In both cases, the boundary condition is $T = P_{\geq}\gamma_0$ and Lemma 4.4 implies that both a and a^* commute with $P_{\geq}$.

It is well known, see [4, §14], that $P_{\geq}$ is a classical pseudodifferential operator of order zero, having principal symbol

$$p_0(x', \xi') = \frac{b'(x', \xi') + 1}{2}, \tag{4.8}$$

where $b'(x', \xi')$ is the principal symbol of the operator $(1 + B^2)^{-1/2}B$. In the even dimensional case, B can be identified with a Dirac operator on the boundary, therefore

$$b'(x', \xi') = \frac{i\xi'\cdot}{|\xi'|}, \quad \xi'\cdot \text{ being the Clifford multiplication on } \partial\Omega.$$

In particular, $b'(x', \xi')$ is not constant as a function of ξ'. In the odd dimensional case it is also possible to give the explicit expression of the principal symbol of B

and state that it is not constant, see e.g. [13].[6] For notational convenience, let us now simply write a instead of $a|_{\partial\Omega}$. Since $aP_{\geq} = P_{\geq}a$, in particular, the local symbols of $aP_{\geq}$ and $P_{\geq}a$ are equal. Let us suppose that the symbol of $P_{\geq}$ has the asymptotic expansion into homogeneous components $\sum_{j=0}^{+\infty} p_{-j}(x', \xi')$. Then we obtain

$$a(x') \sum_{j=0}^{+\infty} p_{-j}(x', \xi') \sim \sum_{|\alpha|=0}^{+\infty} \frac{1}{\alpha!} \partial_{\xi'}^{\alpha} \Big(\sum_{j=0}^{+\infty} p_{-j}(x', \xi') \Big) D_{x'}^{\alpha} a(x').$$

The components of zero order coincide, since a is scalar-valued. Equality of the components of order -1 means

$$\sum_{|\alpha|=1} \partial_{\xi'}^{\alpha} p_0(x', \xi') D_{x'}^{\alpha} a(x') = (\nabla_{\xi'} p_0(x', \xi'), \nabla a(x')) = 0. \tag{4.9}$$

Since both a and a^* preserve the domain also $a + a^*$ and $(a - a^*)/i$ preserves the domain. Therefore, it is not a restriction to suppose that a is real valued. Since p_0 is not constant, the following Lemma 4.8 implies $\nabla a = 0$, i.e. a is locally constant on the boundary.

In the odd case, if B is not invertible, the involved projection is $P = P_{>0} + P_g$. Since it differs from $P_{\geq}$ by a finite-dimensional, smoothing operator, the homogeneous components of P coincide with those of $P_{\geq}$ and we can argue as above. □

Notice that Proposition 4.7 holds infact true for all boundary conditions described in Remark 4.6.

Lemma 4.8. *Let $q \in \mathcal{C}^1(\mathbb{R}^m \setminus \{0\})$ be positively homogeneous of degree 0, $v \in \mathbb{R}^m$, and*

$$\big(\nabla q(\eta), v\big) = 0 \quad \forall\, \eta \neq 0.$$

Then either $v = 0$ or $q \equiv$ const.

Proof. Assume $v \neq 0$. Let V denote the span of v. Then, for arbirary $\eta \in \mathbb{R}^m \setminus V$,

$$\frac{d}{dt} q(\eta + tv) = \big(\nabla q(\eta + tv), v\big) = 0 \quad \forall\, t \in \mathbb{R}.$$

Thus $t \mapsto q(\eta + tv)$ is constant in t. Hence, using the homogeneity,

$$q(\eta) = q(\eta + tv) = q(\eta/t + v) \xrightarrow{t \to +\infty} q(v) \quad \forall\, \eta \in \mathbb{R}^m \setminus V.$$

By continuity of q on $\mathbb{R}^m \setminus \{0\}$, it follows that $q \equiv q(v)$. □

[6]Here, we are supposing the dimension to be at least two. The one dimensional case is trivial: clearly the restriction to the boundary is constant since it is the evaluation at one point.

Theorem 4.9. *Let $(\mathcal{A}_{\mathcal{D}}^{\infty}, \mathcal{H}, \mathcal{D})$ be the spectral triple associated with a Dirac operator D and* APS*-type boundary conditions as described in the previous Subsection 4.1. Then the closure of $\mathcal{A}_{\mathcal{D}}^{\infty}$ in $\mathcal{L}(\mathcal{H})$ is isomorphic to*

$$\mathcal{C}_{\partial}(\Omega) := \left\{a \in \mathcal{C}(\Omega) \mid a|_{\partial\Omega} \text{ is locally constant}\right\}.^{7}$$

Proof. Let us introduce the space V consisting of those functions $a \in \mathcal{C}^{\infty}(\Omega) \cap \mathcal{C}_{\partial}(\Omega)$ such that $a - a|_{\partial\Omega}$ vanishes to infinite order at the boundary. Then, by Lemmas 4.3 and 4.7, $V \subset \mathcal{A}_{\mathcal{D}}^{\infty} \subset \mathcal{C}_{\partial}(\Omega)$. However, it is an elementary fact that the closure of V with repect to the supremum norm coincides with $\mathcal{C}_{\partial}(\Omega)$. □

Denoting by $\widehat{\Omega}$ the topological space obtained from Ω by collapsing each connected component of $\partial\Omega$ to a separate, single point, $\mathcal{C}_{\partial}(\Omega)$ is isomorphic to $\mathcal{C}(\widehat{\Omega})$. In this sense, the spectral triple constructed above does not see the boundary of the manifold.

4.3. Example: A spectral triple on the disk. It is natural to ask which of the hypotheses of Connes' reconstruction theorem are not met when considering the regular spectral triple of a manifold with boundary $(\mathcal{A}_{\mathcal{D}}^{\infty}, \mathcal{H}, \mathcal{D})$ introduced above. The answer is that the so called finiteness axiom is violated. Indeed, it is not true that $\mathcal{H}_{\mathcal{D}}^{\infty}$ is a finitely generated projective $\mathcal{A}_{\mathcal{D}}^{\infty}$-module. Indeed, if this were true, then

$$\mathcal{H}_{\mathcal{D}}^{\infty} \cong p(\mathcal{A}_{\mathcal{D}}^{\infty} \oplus \cdots \oplus \mathcal{A}_{\mathcal{D}}^{\infty}) \quad (N \text{ summands})$$

for a suitable N and a suitable projection p given by an $N \times N$-matrix with entries from $\mathcal{A}_{\mathcal{D}}^{\infty}$. Therefore, in view of Theorem 4.9, restricting $\mathcal{H}_{\mathcal{D}}^{\infty}$ to the boundary would give a finite-dimensional space. However, as we shall verify in an explicit example, this is not true in general.

Following the exposition in [14], we consider on $\mathbb{B} := \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 \leq 1\}$ the Dirac operator

$$D = i \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \partial_x + i \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \partial_y,$$

acting on $\mathbb{C}^2$-valued functions. Passing to polar coordinates (r, θ),[8]

$$D = i \begin{pmatrix} 0 & e^{-i\theta} \\ e^{i\theta} & 0 \end{pmatrix} \partial_r + \frac{i}{r} \begin{pmatrix} 0 & -i e^{-i\theta} \\ i e^{i\theta} & 0 \end{pmatrix} \partial_\theta,$$

respectively

$$D = \begin{pmatrix} 0 & i e^{-i\theta} (\partial_r + B(r)) \\ -i e^{i\theta} (-\partial_r + B(r)) & 0 \end{pmatrix}, \quad B(r) = -\frac{i}{r} \partial_\theta.$$

[7]Here, we identify functions from $\mathcal{C}(\Omega)$ with their operators of multiplication; the operator norm as an element in $\mathcal{L}(L^2(\Omega, E))$ then coincides with the supremum norm of the function. Hence taking the closure refers to uniform convergence on Ω.

[8]In the literature sometimes a different sign convention is used, cf. [8] for example. This is due to the change of coordinates $\theta \to \frac{\pi}{2} - \tilde{\theta}$.

The operator $B := B(1)$, acting as unbounded operator in $L^2(\partial\mathbb{B})$, has spectrum consisting of the eigenvalues $n \in \mathbb{Z}$, with corresponding eigenfunctions $e^{in\theta}$. If $P_{\geq}$ and $P_{\leq}$ denote the orthonormal projections in $L^2(\partial\mathbb{B})$ onto the span of the $e^{in\theta}$ with $n \geq 0$ and $n \leq 0$, respectively, then $\mathcal{D} := D_{\mathrm{APS}}$ has domain

$$\mathrm{dom}\,(\mathcal{D}) = \big\{\psi = (\psi_1, \psi_2) \in H^1(\mathbb{B}, \mathbb{C}^2) \mid P_{\geq}\gamma_0\psi_1 = P_{\leq}\gamma_0\psi_2 = 0\big\},$$

where $\gamma_0 u = u|_{\partial\mathbb{B}}$ is the restriction to the boundary of $\mathbb{B}$.

Proposition 4.10. *The restriction of $\mathcal{H}^{\infty}_{\mathcal{D}}$ to the boundary of $\mathbb{B}$ is an infinite-dimensional space.*

Proof. With an integer $k > 0$ let us consider $\psi_k = (\psi_{k,1}, 0) \in \mathcal{C}^{\infty}(\mathbb{B}, \mathbb{C}^2)$ with

$$\psi_{k,1}(r,\theta) = \chi(r)\,(\cosh(-k\log r) + \sinh(-k\log r))\,e^{-ik\theta}, \tag{4.10}$$

where χ is a smooth function identically equal to 1 near $r = 1$ and identically equal to zero near $r = 0$. Obviously, $\psi_k|_{\partial\mathbb{B}} = (e^{-ik\theta}, 0)$ and ψ_k belongs to the domain of $\mathcal{D}$. Moreover, $D\psi_k = (0, \varphi_{k,2})$ with

$$\varphi_{k,2}(r,\theta) = -i\,e^{i\theta}\Big(-\partial_r - \frac{i}{r}\partial_\theta\Big)\Big[\chi(r)\,(\cosh(-k\log r) + \sinh(-k\log r))\,e^{-ik\theta}\Big].$$

A straightforward calculation now reveals that

$$\varphi_{k,2}(r,\theta) = i\,e^{i\theta}\,(\partial_r\chi)\,(r)\Big(\big(\cosh(-k\log r) + \sinh(-k\log r)\big)e^{-ik\theta}\Big).$$

Therefore $D\psi_k$ is supported in the interior of $\mathbb{B}$, since $\partial_r\chi$ vanishes near $r = 1$. Since D is a differential operator, this is then also true for $D^n\psi_k$ for every $n \geq 1$, hence the APS-boundary conditions are trivially fulfilled. □

A. Ellipticity and Fredholm property

Let $\Sigma \subset \mathbb{R}^n$ and assume we have two families of Banach spaces

$$\{H^s_j\}_{s\in\Sigma}, \quad j = 0, 1,$$

having the following properties, for every s:

(1) $H^{\infty}_j := \cap_{s\in\Sigma} H^s_j$ is a dense subspace of H^s_j.

(2) Any continuous operator $T : H^s_j \to H^s_j$ with $\mathrm{im}\,T \subset H^{\infty}_j$ is compact.

A.1. Invariance of the index. Let us consider two operators

$$A_j : H_j^\infty \longrightarrow H_{1-j}^\infty, \quad j = 0, 1,$$

that extend by continuity to operators

$$A_j^s : H_j^s \longrightarrow H_{1-j}^s, \quad s \in \Sigma,$$

and such that

$$C_j := 1 - A_{1-j} A_j : H_j^\infty \longrightarrow H_j^\infty$$

are regularizing operators, in the sense that the extensions C_j^s satisfy

$$\operatorname{im} C_j^s \subset H_j^\infty, \quad s \in \Sigma.$$

Due to assumption (2) on the compactness, each A_j^s is a Fredholm operator. We shall see, in particular, that the corresponding index does not depend on $s \in \Sigma$. This has already been observed in Lemma 1.2.94 of [19] in case of one-parameter scales of Hilbert spaces (requiring, in particular, continuous embeddings $H^s \hookrightarrow H^t$ for $s \geq t$, which are compact in case $s > t$); the proof we give here extends to multi-parameter families of Banach spaces.

Example A.1. In connection with boundary value problems, typical families arising are of the form $\Sigma = \{s = (r, p) \mid p > 1,\ r > 1/p\} \subset \mathbb{R}^2$ and

$$H_j^s = H_p^r(\Omega, E_j) \oplus B_{pp}^{r-1/p}(\partial\Omega, F_j), \quad s = (r, p) \in \Sigma,$$

with a smooth compact manifold with boundary Ω and vector-bundles E_j and F_j (direct sum of Sobolev (Bessel potential) and Besov spaces). Then

$$H_j^\infty = \mathcal{C}^\infty(\Omega, E_j) \oplus \mathcal{C}^\infty(\partial\Omega, F_j)$$

and (1), (2) hold due to well-known embedding theorems.

Let us first observe the following consequence, refered to as *elliptic regularity* in the sequel: Let $f \in H_1^\infty$ and $A_0^s u = f$ with $u \in H_0^s$ for some $s \in \Sigma$. Then $u \in H_0^\infty$ and $f = A_0 u$. In fact,

$$A_1 f = A_1^s f = A_1^s A_0^s u = (1 - C_0^s)u = u - C_0^s u$$

shows that $u = A_1 f + C_0^s u$ belongs to H_0^∞.

Lemma A.2. *Let V_1 be a finite-dimensional subspace of H_1^∞ such that*

(1) $V_1 \cap \operatorname{im} A_0^s = \{0\}$, (2) $V_1 + \operatorname{im} A_0^s = H_1^s$

for some fixed value $s = s_0$. Then both (1) *and* (2) *hold for arbitrary $s \in \Sigma$.*

Proof. (1) is a direct consequence of elliptic regularity: If $f = A_0^s u \in V_1$ then $u \in H_0^\infty$ and, in particular, $f = A_0^{s_0} u \in V_1 \cap \operatorname{im} A_0^{s_0} = \{0\}$.

For (2) let $f \in H_1^s$ with some $s \in \Sigma$ be given. Choose a sequence $(f_n) \subset H_1^\infty$ converging to f in H_1^s. By (2) for $s = s_0$ and elliptic regularity, we can write

$$f_n = v_n + A_0 u_n, \qquad v_n \in V_1, \quad u_n \in H_0^\infty.$$

Since A_0^s is a Fredholm operator, $\operatorname{im} A_0^s$ is a closed subspace of H_1^s. By (1) it has a complement of the form $V_1 \oplus V$ with some finite-dimensional V; this is then a topological complement. It follows that (f_n) converges in H_1^s if, and only if, both $(A_0^s u_n)$ and (v_n) converge in H_1^s. Thus there exists a $u \in H_0^s$ and a $v \in V_1$ such that, in H_1^s,

$$f_n \xrightarrow{n\to+\infty} f, \quad f_n = v_n + A_0^s u_n \xrightarrow{n\to+\infty} v + A_0^s u.$$

Therefore $f = v + A_0^s u \in V_1 + \operatorname{im} A_0^s$. □

Proposition A.3. *Under the above assumptions there exist finite-dimensional subspaces $V_j \subset H_j^\infty$ such that, for every $s \in \Sigma$,*

$$\ker A_0^s = V_0, \quad H_1^s = V_1 \oplus \operatorname{im} A^s.$$

In particular, $\operatorname{ind} A_0^s = \dim V_0 - \dim V_1$ *does not depend on s. Moreover, if A_0^s is invertible for some $s \in \Sigma$ then it is for all s.*

Proof. By elliptic regularity, $V_0 := \ker A_0^s$ is independent of s. Due to the Fredholm property it is finite-dimensional.

By the previous Lemma A.2 it suffices to find a subspace $V_1 \subset H_1^\infty$ that is a complement to $\operatorname{im} A_0^s$ for some fixed choice of s. Let $W = \operatorname{span}\{w_1, \dots, w_n\}$ be a complement to $\operatorname{im} A_0^s$ in H_1^s. Write

$$w_k = (C_1^s + A_0^s A_1^s) w_k = v_k + A_0^s u_k, \quad k = 1, \dots, n,$$

with $v_k := C_1^s w_k \in H_1^\infty$ and $u_k := A_1^s w_k \in H_0^s$. Then $V_1 = \operatorname{span}\{v_1, \dots, v_n\}$ is the desired complement. □

A.2. Inverses modulo projections. Now let L_{jk}^0, $j, k \in \{0, 1\}$, denote certain vector spaces of operators $H_j^\infty \to H_k^\infty$, whose elements extend by continuity to operators in $\mathcal{L}(H_j^s, H_k^s)$ for every $s \in \Sigma$. Let $L_{jk}^{-\infty}$ be subspaces of regularizing (in the sense described above) operators. Assume that, for every choice of $j, k, \ell \in \{0, 1\}$, compostion of operators induces maps

$$(A, B) \mapsto AB, \quad L_{k\ell}^s \times L_{jk}^t \longrightarrow L_{j\ell}^{s+t}, \quad s, t \in \{-\infty, 0\}.$$

By definition, a parametrix to $A_0 \in L_{01}^0$ is any operator $A_1 \in L_{10}^0$ such that

$$1 - A_{10} A_{01} \in L_{00}^{-\infty} \quad \text{and} \quad 1 - A_{01} A_{10} \in L_{11}^{-\infty}.$$

An operator possessing a parametrix is called elliptic.

Proposition A.4. *Let $A_0 \in L^0_{01}$ be elliptic and $V_j \subset H^\infty_j$ be two spaces as described in Proposition A.3. Assume that $\pi_j \in L^{-\infty}_{jj}$ are projections with $\operatorname{im}\pi_j = V_j$. Furthermore assume that, for every $s \in \Sigma$, compostion of operators induces maps*

$$(A, B, C) \mapsto ABC, \quad L^{-\infty}_{00} \times \mathcal{L}(H^s_1, H^s_0) \times L^{-\infty}_{11} \longrightarrow L^{-\infty}_{10}, \tag{A.1}$$

i.e. sandwiching a continuous operator between two smoothing operators results in a smoothing operator. Then there exists a parametrix $A_1 \in L^0_{10}$ such that

$$A_1 A_0 = 1 - \pi_0, \quad A_0 A_1 = 1 - \pi_1.$$

Proof. Fix some $s \in \Sigma$. Then π^s_j are projections in H^s_j with image V_j. Note that $A^s_0 : \ker \pi^s_0 \to \operatorname{im} A^s_0$ is bijective, hence has an inverse. Let $A^s_1 \in \mathcal{L}(H^s_1, H^s_0)$ be the operator acting like this inverse on $\operatorname{im} A^s_0$ and vanishing on V_1. By construction we thus have

$$A^s_1 A^s_0 = 1 - \pi^s_0, \quad A^s_0 A^s_1 = 1 - \pi^s_1.$$

Now let $\widetilde{A}_1$ be a parametrix to A_0 and $C_0 = 1 - \widetilde{A}_1 A_0$ and $C_1 = 1 - A_0 \widetilde{A}_1$. Then

$$\begin{aligned} A^s_1 - \widetilde{A}^s_1 &= \big(\widetilde{A}^s_1 A^s_0 + C^s_0\big)\big(A^s_1 - \widetilde{A}^s_1\big) = \widetilde{A}^s_1\big(C^s_1 - \pi^s_1\big) + C^s_0\big(A^s_1 - \widetilde{A}^s_1\big) \\ A^s_1 - \widetilde{A}^s_1 &= \big(A^s_1 - \widetilde{A}^s_1\big)\big(A^s_0 \widetilde{A}^s_1 + C^s_1\big) = \big(C^s_0 - \pi^s_0\big)\widetilde{A}^s_1 + \big(A^s_1 - \widetilde{A}^s_1\big)C^s_1. \end{aligned}$$

Substituting the second equation into the first yields

$$A^s_1 - \widetilde{A}^s_1 = \widetilde{A}^s_1\big(C^s_1 - \pi^s_1\big) + C^s_0\big(C^s_0 - \pi^s_0\big)\widetilde{A}^s_1 + C^s_0\big(A^s_1 - \widetilde{A}^s_1\big)C^s_1.$$

Hence the desired parametrix is

$$A_1 = \widetilde{A}_1 + \widetilde{A}_1\big(C_1 - \pi_1\big) + C_0\big(C_0 - \pi_0\big)\widetilde{A}_1 + C_0\big(A^s_1 - \widetilde{A}^s_1\big)C_1,$$

since the last term belongs to $L^{-\infty}_{10}$ due to assumption (A.1). □

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