J. Noncommut. Geom. 11 (2017), 1001–1036
DOI 10.4171/JNCG/11-3-7

Journal of Noncommutative Geometry

Langlands functorality in K-theory for C^*-algebras. I. Base change

Kuok Fai Chao* and Hang Wang**

Abstract. We compare representations of the real and complex general linear groups and special linear groups in the framework of K-theory, using base change on L-parameters. We introduce a notion of base change on K-theory involving the fixed point set of the reduced dual of a complex group. For general linear groups, we prove that the base change map is compatible with the Connes–Kasparov isomorphism.

Mathematics Subject Classification (2010). 22D25; 46L80, 20G05.

Keywords. K-theory, local Langlands correspondence, base change, reduced group C^*-algebra, tempered representation.

1. Introduction

Local Langlands correspondence (LLC), proposed by Langlands [20,21] and then verified for many algebraic groups (see for example [8,9,11,22,23]), roughly speaking, is a deep philosophy connecting arithmetics and representation theory. The philosophy claims that the irreducible admissible representations of a reductive algebraic group G over a local field $\mathbb{F}$ are parametrised by L-parameters

$$\phi : W_{\mathbb{F}} \to {}^L G$$

where $W_{\mathbb{F}}$ is the Weil group (for $\mathbb{F}$ being an archimedean field) or the Weil–Deligne group (for $\mathbb{F}$ being non-archimedean) and ${}^L G$ is the so-called L-group. In this paper, we will only investigate the case of archimedean local fields. When the group G is a general linear group, LLC gives rise to a bijective map

$$f : \Pi(G) \to \Phi(G), \tag{1.1}$$

*K. F. Chao is supported by National Natural Science Foundation of China, Grant no: 11401105.
**H. Wang is supported by the Australian Research Council Discovery Early Career Researcher Award DE160100525.

where

$$\Pi(G) := \{\text{equivalence classes of irreducible admissible representations of } G\};$$
$$\Phi(G) := \{G^\vee\text{-conjugacy classes of } L\text{-parameters: } \phi : W_{\mathbb{F}} \to {}^L G\}.$$

Here, $G^\vee$ is the Langlands dual group. For a general algebraic group, the map f fails to be one-to-one. This means that two distinct representations ρ, τ in $\Pi(G)$ may share the same L-parameter, i.e. ρ and τ are in the same L-packet. LLC states that there is a bijection between $\Phi(G)$ and equivalence classes of $\Pi(G)$ determined by L-packets.

Thanks to the philosophy of Langlands, comparisons of irreducible admissible representations of algebraic groups $G(\mathbb{C})$ and $G(\mathbb{R})$ can be made by comparing the corresponding L-parameters. The inclusion of Weil groups $W_{\mathbb{C}} \to W_{\mathbb{R}}$ gives rise to a well defined base change map on L-parameters over different fields:

$$bc : \{L\text{-parameters of } G(\mathbb{R})\} \to \{L\text{-parameters of } G(\mathbb{C})\}.$$

Because of LLC (1.1), there is a base change map on admissible representations between $G(\mathbb{C})$ and $G(\mathbb{R})$ (see Section 5.1):

$$bc : \Pi(G(\mathbb{R})) \to \Pi(G(\mathbb{C})).$$

The induced base change map on representations gives rise to the commutative diagram:

$$\begin{array}{ccc} f : \Pi(G(\mathbb{R})) & \xrightarrow{LLC} & \Phi(G(\mathbb{R})) \\ \downarrow{\scriptstyle bc} & & \downarrow{\scriptstyle bc} \\ f : \Pi(G(\mathbb{C})) & \xrightarrow{LLC} & \Phi(G(\mathbb{C})) \end{array}$$

In this paper, we study the above diagram using $K_*(C_r^*(G))$, the K-theory for group C^*-algebras. Because $C_r^*(G)$ encodes information of the tempered dual of G, we consider admissible representations that are *tempered representations*.

The first main result of our paper is Theorem 6.9. When G is a general linear group or a special linear group, there is a nontrivial base change map on operator algebra K-theory,

$$bc : K_i(C_r^*(G(\mathbb{C}))) \to K_j(C_r^*(G(\mathbb{R}))), \tag{1.2}$$

where i (resp. j) has the same parity as $\dim[G(\mathbb{R})/K(\mathbb{R})]$ (resp. $\dim[G(\mathbb{C})/K(\mathbb{C})]$).

A featured treatment of our K-theory characterisation of base change is the possible degree shift in K-theory of (1.2). Namely, i and j may not be the same. This is a finer definition than that of Mendes and Plymen [25]. In fact, in their formulation in our archimedean examples, their base change map is trivial when $n > 1$. In our modified definition of base change on K-theory, we have taken into account the subset of $\Pi(G(\mathbb{C}))$ fixed by the Galois group $\mathrm{Gal}(\mathbb{C}/\mathbb{R})$, and obtain a *nontrivial* base change map on K-theory. The Galois fixed point set has lower dimension than the connected component where it locates, causing the degree shift in K-theory. Another feature of our formulations is that for the case $SL(n)$, the L-packet information is encoded in the base change map (1.2) on K-theory (see for example Corollary 6.11).

Our second main result is Theorem 7.7. Let G be a general linear group and K a maximal compact subgroup, then through the Connes–Kasparov isomorphism

$$R(K) \cong K_*(C_r^*(G)), \tag{1.3}$$

there is a base change map on the representation ring $R(K)$ of K:

$$bc_* : R(K_{G(\mathbb{C})}) \to R(K_{G(\mathbb{R})}).$$

We show that the base change maps (1.2) on $GL(n)$ and on its maximal compact subgroup K are compatible with the Connes–Kasparov isomorphism (1.3).

This paper is the first series of papers about functorality of K-theory for C^*-algebras. Our second paper in the series will be focused on the case of general classical groups.

We would like to mention a recent preprint by Mendes–Plymen [26]. The paper has some common terms as ours in the exposition part. However, our results are parallel to theirs and represents an independent work.

Acknowledgements. We are grateful to Prof. King Fai Lai for his encouragements and suggestions. We would like to thank Shanghai Center of Mathematical Sciences at Fudan University and University of Adelaide for the hospitality and support, during our visits to each other. A large part of this work is completed during these visits. We appreciate the very helpful comments and suggestions from the referee of our article.

2. L-parameters

In this section, we review L-parameters and some examples. Details can be found in notes such as [16,17].

Let $\mathbb{F}$ be an archimedean local field and $\overline{\mathbb{F}}$ be the algebraic closure. Recall that a *Weil group* $W_{\mathbb{F}}$ is an extension of the multiplicative group of $\overline{\mathbb{F}}^\times$ by the Galois group $\mathrm{Gal}(\overline{\mathbb{F}}/\mathbb{F})$, i.e.

$$1 \to \overline{\mathbb{F}}^\times \to W_{\mathbb{F}} \to \mathrm{Gal}(\overline{\mathbb{F}}/\mathbb{F}) \to 1.$$

Example 2.1. (1) If $\mathbb{F} = \mathbb{C}$, the Galois group $\mathrm{Gal}(\mathbb{C}/\mathbb{C})$ is trivial, so

$$W_{\mathbb{C}} = \mathbb{C}^\times.$$

(2) If $\mathbb{F} = \mathbb{R}$, then $\mathrm{Gal}(\mathbb{C}/\mathbb{R}) = \{1, \sigma\} \cong \mathbb{Z}_2$ and $W_{\mathbb{R}}$ is $\mathbb{C}^\times \cup k\mathbb{C}^\times$ subject to relations

$$k^2 = -1 \quad \text{and} \quad kz = \bar{z}k.$$

The action on $\mathbb{C}$ by conjugacy of z (resp. kz) corresponds the trivial element 1 (resp. nontrivial element σ) of $\mathrm{Gal}(\mathbb{C}/\mathbb{R})$. This determines a map $j : W_{\mathbb{R}} \to \mathrm{Gal}(\mathbb{C}/\mathbb{R})$ with kernel $\mathbb{C}^\times$ and defines a nontrivial extension.

Denote by $G^\vee$ the Langlands dual group of G. It is a simply connected complex reductive group whose root datum is dual to that of the group G. For the field $\mathbb{F}$, the *L-group* associated to $G(\mathbb{F})$ is defined by the crossed product,

$${}^L G_{\mathbb{F}} = G^\vee \rtimes \mathrm{Gal}(\overline{\mathbb{F}}/\mathbb{F}).$$

Example 2.2. (1) For $G = GL(2)$, the Langlands dual group is $G^\vee = GL(2, \mathbb{C})$. When $\mathbb{F} = \mathbb{C}$ or $\mathbb{R}$, we have

$${}^L G_{\mathbb{C}} = GL(2, \mathbb{C}), \quad {}^L G_{\mathbb{R}} = GL(2, \mathbb{C}) \rtimes \mathrm{Gal}(\mathbb{C}/\mathbb{R}).$$

Here, the nontrivial element σ of the Galois group $\mathrm{Gal}(\mathbb{C}/\mathbb{R})$ acts on $G^\vee$ by the following outer involution

$$\sigma : g \mapsto J(g^{-1})^T J^{-1}$$

where $g \in GL(2, \mathbb{C})$ and $J = \left[\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right]$.

(2) For $G = SL(2)$, the Langlands dual group is $G^\vee = PGL(n, \mathbb{C})$ and the L-groups for $G(\mathbb{C})$ and $G(\mathbb{R})$ are

$${}^L G_{\mathbb{C}} = PGL(2, \mathbb{C}), \quad {}^L G_{\mathbb{R}} = PGL(2, \mathbb{C}) \rtimes \mathrm{Gal}(\mathbb{C}/\mathbb{R}).$$

Definition 2.3. An *L-parameter* is a $G^\vee$-conjugacy class of homomorphisms

$$\phi : W_{\mathbb{F}} \to^L G_{\mathbb{F}}$$

where the image contains only semisimple elements and is compatible with the canonical projections from ${}^L G_{\mathbb{F}}$ to $\mathrm{Gal}(\overline{\mathbb{F}}/\mathbb{F})$, i.e. the diagram commutes:

$$\begin{array}{ccc} W_{\mathbb{F}} & \longrightarrow & {}^L G_{\mathbb{F}} \\ {\scriptstyle j}\downarrow & & \downarrow \\ \mathrm{Gal}(\overline{\mathbb{F}}/\mathbb{F}) & \xrightarrow{=} & \mathrm{Gal}(\overline{\mathbb{F}}/\mathbb{F}). \end{array}$$

Example 2.4. When $\mathbb{F} = \mathbb{R}$, there is a canonical map

$$j : W_{\mathbb{R}} \to \mathrm{Gal}(\mathbb{C}/\mathbb{R}).$$

Thus, an L-parameter can also be expressed in the form $\phi(w) = \phi_0(w) \times w$ for $w \in W_{\mathbb{R}}$ where ϕ_0 is a continuous 1-cocycle of $W_{\mathbb{R}}$ into semisimple elements of $G^\vee$.

2.1. L-parameters of $GL(2)$. For $\mathbb{F} = \mathbb{C}$, L-parameters for $GL(2,\mathbb{C})$ are given by the $G^\vee$-conjugacy classes of homomorphisms $W_\mathbb{C} \to GL(2,\mathbb{C})$ with semisimple images. As $W_\mathbb{C} = \mathbb{C}^\times$ is abelian, all its irreducible representations are 1-dimensional. Also, denoting a complex number by $z = \rho e^{i\theta}$, a group homomorphism $\mathbb{C}^\times \to \mathbb{C}^\times$ has the form

$$\chi_{\lambda,n} : \mathbb{C}^\times \to \mathbb{C}^\times, \quad z \mapsto \rho^\lambda e^{in\theta}, \qquad \text{for some } n \in \mathbb{Z},\ \lambda \in \mathbb{C}.$$

Hence, Langlands parameters for $GL(2,\mathbb{C})$ up to conjugacy have the following form

$$\phi_{\lambda_1,n_1,\lambda_2,n_2}(z) = \begin{bmatrix} \chi_{\lambda_1,n_1}(z) & 0 \\ 0 & \chi_{\lambda_2,n_2}(z) \end{bmatrix}$$

for some $\lambda_1, \lambda_2 \in \mathbb{C}$ and $n_1, n_2 \in \mathbb{Z}$. Note also that up to conjugacy

$$\phi_{\lambda_1,n_1,\lambda_2,n_2} \simeq \phi_{\lambda_2,n_2,\lambda_1,n_1}.$$

Remark 2.5. In this note, we are interested in tempered representations (hence unitary). With these requirements the corresponding L-parameters are bounded and unitary, i.e. $\lambda_1, \lambda_2 \in i\mathbb{R}$. We shall assume this condition for L-parameters for $GL(2,\mathbb{C})$ and for later cases: $GL(2,\mathbb{R})$, $SL(2,\mathbb{C})$ and $SL(2,\mathbb{R})$.

The case for $\mathbb{F} = \mathbb{R}$ is slightly complicated. A bounded and unitary L-parameter $W_\mathbb{R} \to GL(2,\mathbb{C})$ for $GL(2,\mathbb{R})$ up to conjugacy has the following two forms:

(1) For some $\lambda_1, \lambda_2 \in i\mathbb{R}$ and $m_1, m_2 \in \mathbb{Z}_2$,

$$\phi_{\lambda_1,m_1,\lambda_2,m_2}(z) = \begin{bmatrix} \chi_{\lambda_1,0}(z) & 0 \\ 0 & \chi_{\lambda_2,0} \end{bmatrix}, \tag{2.1}$$

$$\phi_{\lambda_1,m_1,\lambda_2,m_2}(\sigma) = \begin{bmatrix} (-1)^{m_1} & 0 \\ 0 & (-1)^{m_2} \end{bmatrix}. \tag{2.2}$$

Note that $\phi_{\lambda_1,m_1,\lambda_2,m_2} \simeq \phi_{\lambda_2,m_2,\lambda_1,m_1}$.

(2) For some $\lambda \in i\mathbb{R}, n \in \mathbb{Z}$,

$$\phi_{\lambda,n}(z) = \begin{bmatrix} \chi_{\lambda,n}(z) & 0 \\ 0 & \chi_{\lambda,-n}(z) \end{bmatrix},$$

$$\phi_{\lambda,n}(\sigma) = \begin{bmatrix} 0 & (-1)^n \\ 1 & 0 \end{bmatrix}.$$

Note that $\phi_{\lambda,n} \simeq \phi_{\lambda,-n}$ and on the intersection of (1) and (2)

$$\phi_{\lambda,0} \simeq \phi_{\lambda,1,\lambda,0} \simeq \phi_{\lambda,0,\lambda,1}.$$

2.2. L-parameters of $SL(2)$. L-parameters for $SL(2)$ are homomorphisms from the Weil group $W_{\mathbb{F}}$ to the L-group $PGL(2,\mathbb{C}) \rtimes \mathrm{Gal}(\overline{\mathbb{F}}/\mathbb{F})$. Notice that an L-parameter for $GL(2)$ gives rise to an L-parameter for $SL(2)$ by the natural projection $GL(2,\mathbb{C}) \to PGL(2,\mathbb{C})$:

$$\begin{array}{ccc} W_{\mathbb{F}} & \longrightarrow & GL(2,\mathbb{C}) \rtimes \mathrm{Gal}(\overline{\mathbb{F}}/\mathbb{F}) \\ & \searrow & \downarrow \\ & & PGL(2,\mathbb{C}) \rtimes \mathrm{Gal}(\overline{\mathbb{F}}/\mathbb{F}) \end{array}$$

The converse is also true due to the following theorem:

Theorem 2.6 ([22], [18, Theorem 7.1]). *Let $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$. Every projective representation of $W_{\mathbb{F}}$ can be lifted to a representation of $W_{\mathbb{F}}$. Moreover, there is a surjective map of L-parameters for $GL(n,\mathbb{F})$ and $SL(n,\mathbb{F})$:*

$$\Phi(GL(n,\mathbb{F})) \to \Phi(SL(n,\mathbb{F})). \tag{2.3}$$

Because of this theorem, all L-parameters for $SL(2)$ are obtained from L-parameters of $GL(2)$ composed with the projection $GL(2,\mathbb{C}) \to PGL(2,\mathbb{C})$. As above, let $z = \rho e^{i\theta}$ and $\chi_{\lambda,n}(z) = \rho^{\lambda} e^{in\theta}$.

When $\mathbb{F} = \mathbb{C}$, a bounded unitary L-parameter for $SL(2,\mathbb{C})$ has the following form:

(1) For some $\lambda \in i\mathbb{R}$ and $m \in \mathbb{Z}$,

$$\phi_{\lambda,m}(z) = \begin{bmatrix} \chi_{\lambda,m}(z) & 0 \\ 0 & 1 \end{bmatrix}.$$

Note that $\phi_{\lambda,m} \simeq \phi_{-\lambda,-m}$.

When $\mathbb{F} = \mathbb{R}$, a bounded L-parameter for $SL(2,\mathbb{R})$ has either of the following forms up to conjugation.

(1) For some $\lambda \in i\mathbb{R}$ and $m \in \mathbb{Z}_2$,

$$\phi_{\lambda,m}(z) = \begin{bmatrix} \chi_{\lambda,0}(z) & 0 \\ 0 & 1 \end{bmatrix}, \quad \phi_{\lambda,m}(\sigma) = \begin{bmatrix} (-1)^m & 0 \\ 0 & 1 \end{bmatrix}.$$

(2) For some $n \in \mathbb{Z}$,

$$\phi_n(z) = \begin{bmatrix} \chi_{0,n}(z) & 0 \\ 0 & \chi_{0,-n}(z) \end{bmatrix}, \quad \phi_n(\sigma) = \begin{bmatrix} 0 & (-1)^n \\ 1 & 0 \end{bmatrix}.$$

Note that $\phi_n \simeq \phi_{-n}$.

3. Representations

In this section, we recall the classification of irreducible tempered representations of general and special linear groups. Without loss of generality, we shall focus on the example when $G = GL(2,\mathbb{F})$ or $SL(2,\mathbb{F})$ for $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$. By Harish-Chandra's subquotient theorem, all such representations (admissible representations in general) arise as subquotients of some principal series representations. A *principal series representation* $p(\mu_1, \mu_2)$ is a representation induced from the Borel subgroup $P = MAN$ of G (upper triangular matrices) determined by unitary characters μ_1, μ_2 of $\mathbb{F}^\times \cong MA$. Here MA is the Levi subgroup with a maximal torus M and an abelian group A. The induced representation $p(\mu_1, \mu_2) \in \mathrm{Ind}_P^G(\mu_1, \mu_2)$ consists of functions on G where

$$f\left(\begin{bmatrix} a & x \\ 0 & b \end{bmatrix} g\right) = \mu_1(a)\mu_2(b)\left|\frac{a}{b}\right|^{\frac{1}{2}} f(g) \quad \forall g \in K.$$

Denote the character $\mu_1\mu_2^{-1}$ of $\mathbb{F}^\times$ by μ.

When $\mathbb{F} = \mathbb{R}$, any character $\mu : \mathbb{R}^\times \to \mathbb{C}^\times$ has the form

$$\mu(x) = [\mathrm{sign}(x)]^m |x|^\lambda \quad \text{for some } m \in \mathbb{Z}_2,\ \lambda \in \mathbb{C}.$$

Here, sign is the sign character where $\mathrm{sign}(x) = 1$ when $x > 0$ and $\mathrm{sign}(x) = -1$ when $x < 0$.

When $\mathbb{F} = \mathbb{C}$, any character $\mu : \mathbb{C}^\times \to \mathbb{C}^\times$ has the form

$$\mu(\rho e^{i\theta}) = e^{in\theta}\rho^\lambda \quad \text{for some } n \in \mathbb{Z},\ \lambda \in \mathbb{C}.$$

In general, all irreducible representations of G can be constructed using characters μ. In a particular case when G is a complex semisimple Lie group, e.g. $SL(n,\mathbb{C})$, every principal series representation

$$p(\mu_1, \mu_2, \ldots, \mu_n)$$

is irreducible. Thus, all irreducible tempered representations of G can be identified as the irreducible principal series up to the action by the Weyl group W of G:

$$\widehat{G} \cong (\widehat{A} \times \widehat{M})/W.$$

In the case when a principal series is reducible, it corresponds more than one irreducible representations.

3.1. Representations of $GL(2,\mathbb{F})$.

When $\mathbb{F} = \mathbb{R}$, consider the character

$$\mu = \mu_1\mu_2^{-1} : \mathbb{R}^\times \to \mathbb{C}^\times$$

where $\mu(x) = [\text{sign}(x)]^m |x|^\lambda$ for some $m \in \mathbb{Z}_2, \lambda \in \mathbb{C}$. It can be verified that

$$\phi_n\left(\begin{bmatrix} a & x \\ 0 & b \end{bmatrix}\begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}\right) = \mu_1(a)\mu_2(b)\left|\frac{a}{b}\right|^{\frac{1}{2}} e^{in\theta}$$

form a basis for the principal series $p(\mu_1, \mu_2)$. According to [12], if $\lambda - m$ is not an odd integer or if $\lambda = 0$, $p(\mu_1, \mu_2)$ are irreducible. If $\lambda - m$ is an odd integer and $\lambda \neq 0$ the principal series $p(\mu_1, \mu_2)$ is reducible and we have two cases:

(1) If $\lambda > 0$, then the submodule

$$\sigma(\mu_1, \mu_2) := \bigoplus_{\substack{n \geq \lambda+1, \\ n \leq -\lambda-1}} \mathbb{C}\phi_n$$

is irreducible and the quotient representation $\pi(\mu_1, \mu_2)$ is a finite dimensional irreducible representation.

(2) If $\lambda < 0$, then

$$\pi(\mu_1, \mu_2) := \bigoplus_{\substack{1+\lambda \leq n \leq -1-\lambda, \\ n \leq -\lambda-1}} \mathbb{C}\phi_n$$

is a finite dimensional irreducible representation and the quotient representation $\sigma(\mu_1, \mu_2)$ is also irreducible.

As we shall focus on irreducible tempered representations, we disregard finite dimensional representations, the trivial representation and complementary series, because they do not contribute to generators of K-theory.

Theorem 3.1. *The irreducible tempered representations of $GL(2, \mathbb{R})$ up to unitary equivalence are:*

(1) *Discrete series given by the subquotient $\sigma(\mu_1, \mu_2)$ of the principal series $p(\mu_1, \mu_2)$ where*

$$\mu(x) = |x|^n [\text{sign}(x)]^{n+1} \qquad n \neq 0.$$

Note: $[\text{sign}(x)]^{n+1+2k} = [\text{sign}(x)]^{n+1}$.

(2) *Irreducible principal series defined by $p(\mu_1, \mu_2)$ where*

$$\mu(x) = |x|^{i\nu} [\text{sign}(x)]^m \quad (\textit{if } \nu = 0, \textit{ m is not odd}).$$

The representations are also equivalent under permutation of μ_1, μ_2.

When $\mathbb{F} = \mathbb{C}$, consider the character

$$\mu = \mu_1 \mu_2^{-1} : \mathbb{C}^\times \to \mathbb{C}^\times$$

where $\mu(z) = z^p\bar{z}^q$, $p, q \in \mathbb{C}, 2p - 2q \in \mathbb{Z}$. According to [12], the principal series $p(\mu_1, \mu_2)$ is irreducible whenever p, q are not both positive or both negative. When p, q are both positive or both negative, $p(\mu_1, \mu_2)$ splits into two irreducible subquotients, one finitely dimensional, the other is isomorphic to $p(\mu_3, \mu_4)$ for some μ_3, μ_4 where $\mu_3\mu_4^{-1} = \mu$. From this we see that in this case all irreducible tempered representations come from irreducible principal series.

Theorem 3.2. *The irreducible tempered representations of $GL(2, \mathbb{C})$ up to unitary equivalence are: Irreducible principal series, obtained from $p(\mu_1, \mu_2)$ where $\mu(z) = \rho^{i\nu}e^{in\theta}$ with $\nu \in \mathbb{R}$ and $n \in \mathbb{N}$.*

3.2. Representations of $SL(2, \mathbb{F})$. For $SL(2, \mathbb{F})$, we only need one character $\mu = \mu_1\mu_2^{-1}$. A principal series $\pi(\mu)$ is a representation induced from the Borel subgroup $P = MAN$ determined by a unitary character μ of $\mathbb{R}^\times \cong MA$ and the principal series $p(\mu) \in \mathrm{Ind}_P^G(\mu)$ consists of functions f on $SL(2, \mathbb{R})$ where

$$f\left(\begin{bmatrix} a & x \\ 0 & a^{-1} \end{bmatrix} g\right) = \mu(a)|a|f(g), \quad g \in K.$$

The unitary character μ is parametrised by $(m, \lambda) \in \mathbb{Z}_2 \times \mathbb{C}$, i.e.

$$\mu(x) = [\mathrm{sign}(x)]^m|x|^\lambda \quad x \in \mathbb{R}^\times.$$

Note that μ is determined by $m = \mathbb{Z}_2 = \{\pm\}$ and $\lambda \in \mathbb{C}$. Denote

$$I_{\pm,\lambda} = p(\mu).$$

Note that for $\pi(\mu)$ to be a unitary representation, we need $\lambda \in i\mathbb{R}$. The representation $I_{-,0}$ splits into direct summands $D_0^- \oplus D_0^+$ underlie two limit of discrete series representations. When n is a positive integer and $\pm$ is the parity of $n + 1$, $I_{\pm,n}$ has two irreducible submodules D_n^-, D_n^+ underlie discrete series. All other $I_{\pm,\lambda}$ are irreducible. Thus, the space of irreducible unitarisable representations of $SL(2, \mathbb{R})$ are classified as follows:

Theorem 3.3 ([15, Theorem 16.3]). *The irreducible tempered representations of $SL(2, \mathbb{R})$ up to unitary equivalence are:*

(1) *Discrete series $D_n^\pm (n > 0)$ obtained from $I_{\pm,n}$ or $\mu(x) = |x|^n[\mathrm{sign}(x)]^{n+1}$.*

(2) *Limit of discrete series $D_0^\pm$ obtained from $I_{-,0}$ or $\mu(x) = \mathrm{sign}(x)$.*

(3) *Irreducible principal series $I_{+,i\nu} (\nu \in \mathbb{R})$ and $I_{-,i\nu} (\nu \in \mathbb{R}^\times)$ (or $\mu(x) = |x|^{i\nu}[\mathrm{sign}(x)]^m$). Denote them by $PS_{i\nu,1}$, $PS_{i\nu,-1}$, $\nu \in \mathbb{R}$.*

Admissible representations of $SL(2, \mathbb{C})$ are much easier. The reason is that the principal series $p(\mu)$ depending on a unitary character μ of $\mathbb{C}^\times \cong AM$ is always irreducible. Let $z = \rho e^{i\theta}$ and denote by $I_{n,\lambda}$, $n \in \mathbb{N}$, $\lambda \in i\mathbb{R}$, the principal series $p(\mu)$ is induced by

$$\mu(z) = \rho^\lambda e^{in\theta}.$$

The space of irreducible tempered unitary representations of $SL(2,\mathbb{C})$ is classified as follows.

Theorem 3.4 ([15, Theorem 16.2]). *The irreducible tempered representations of $SL(2,\mathbb{C})$ up to unitary equivalence are: Irreducible principal series are obtained from $I_{i\nu,n} = \pi(\mu)$ where $\mu(z) = \rho^{i\nu}e^{in\theta}$ with ν real and $n \in \mathbb{N}$. Denote them by $PS_{i\nu,n}$, $\nu \in \mathbb{R}$, $n \in \mathbb{N}$. Note that $PS_{i\nu,n} \simeq PS_{-i\nu,-n}$.*

4. Local Langlands correspondence

In this section, we review the local Langlands correspondence. The correspondence was conjectured by Langlands [20,21]. The conjecture is proved for $GL(n)$ over archimedean fields [22] and over p-adic fields [8,9], and proved for $SL(n)$ in [11, 23]. Following from the Langlands program, the study of admissible irreducible representations of an algebraic group can be parametrised by L-parameters.

Recall that all admissible irreducible representations of a group G over an archimedean field can be characterised by irreducible $(\mathfrak{g}, K)$-modules. In the philosophy of Langlands, L-parameters produce a way to classify admissible representations of G in the sense that each L-parameter corresponds a finite set of irreducible $(\mathfrak{g}, K)$-modules, called an L-packet. This bijection between L-packets of admissible irreducible representations of $G_{\mathbb{F}}$ and conjugacy classes of admissible homomorphisms of the Weil group $W_{\mathbb{F}}$ in the L-group ${}^{L}G_{\mathbb{F}}$, satisfying some other suitable conditions, is called a local Langland correspondence.

For example, due to the local Langlands correspondence, the surjective map (2.3) from L-parameters of $GL(2,\mathbb{F})$ to that of $SL(2,\mathbb{F})$ implies that every class of irreducible admissible representations of $SL(2,\mathbb{F})$ can be constructed from an element in $\Pi(GL(2,\mathbb{F}))$. For $GL(2,\mathbb{F})$, each L-packet contains only one element from $\Pi(GL(2,\mathbb{F}))$. But for $SL(2,\mathbb{F})$, an L-packet may contain more than one elements of $\Pi(SL(2,\mathbb{F}))$. Therefore, in view of Theorem 2.6, different classes of representations from a same L-packet for $SL(2,\mathbb{F})$ in fact come from a same representation of $GL(2,\mathbb{F})$. These facts are related deeply to the reason why the unitary dual of $GL(2,\mathbb{F})$ is Hausdorff but the topology for the unitary dual of $SL(2,\mathbb{F})$ is slightly more complicated (see Section 6).

We shall review local Langlands correspondence in the context of the following examples.

4.1. Local Langlands correspondence for $GL(2,\mathbb{F})$.

Let us first consider $\mathbb{F} = \mathbb{R}$. The central object constructing local Langlands correspondence for a group over $\mathbb{R}$ is the characters of $\mathbb{R}^{\times}$. For $GL(2,\mathbb{R})$, we know that an irreducible tempered representation is either a principal series $p(\mu_1,\mu_2)$ or its subquotient (as discrete series), depending on the parameter $\lambda \in i\mathbb{R}, m \in \mathbb{Z}_2$ for the character

$$\mu : \mathbb{R}^\times \to \mathbb{C}^\times \quad x \mapsto |x|^\lambda [\operatorname{sign} x]^m.$$

Here, $\mu = \mu_1 \mu_2^{-1}$ and μ_1, μ_2 are characters of $\mathbb{R}^\times$ given by

$$\mu_i(x) = |x|^{\lambda_i} [\operatorname{sign} x]^{m_i}$$

for some $\lambda_i \in \mathbb{C}, m_i \in \mathbb{Z}_2$. We will only need to construct a unique L-parameter

$$\phi : W_\mathbb{R} \to GL(2, \mathbb{R}) \rtimes \operatorname{Gal}(\mathbb{C}/\mathbb{R})$$

associated to the characters μ_1, μ_2 of $\mathbb{R}^\times$. Note there is a natural map $W_\mathbb{R} \to \operatorname{Gal}(\mathbb{C}/\mathbb{R})$ coming from the extension

$$0 \to \mathbb{R}^\times \to W_\mathbb{R} \to \operatorname{Gal}(\mathbb{C}/\mathbb{R}) \to 0.$$

Thus, we only need to construct a homomorphism $W_\mathbb{R} \to GL(2, \mathbb{C})$. There are two facts to be used in the construction:

(1) The norm map given by the homomorphism

$$N : W_\mathbb{R} \to \mathbb{R}^\times \qquad z \mapsto |z|, \sigma \mapsto -1.$$

gives rise to a bijection between the characters of $W_\mathbb{R}$ and characters of $\mathbb{R}^\times$:

$$\begin{array}{ccc} W_\mathbb{R} & \xrightarrow{\hat{\chi}} & \mathbb{C}^\times \\ {\scriptstyle N}\downarrow & \nearrow_{\chi} & \\ \mathbb{R}^\times & & \end{array}$$

(2) The character $\mu_1\mu_2 : \mathbb{R}^\times \to \mathbb{C}^\times$ lifted to the character of $W_\mathbb{R}$ determines a homomorphism $W_\mathbb{R} \to GL(2, \mathbb{C})$ through the determinant map:

$$\begin{array}{ccc} W_\mathbb{R} & \xrightarrow{\phi} & GL(2, \mathbb{C}) \\ & \searrow^{\widehat{\mu_1\mu_2}} & \downarrow{\scriptstyle \det} \\ & & \mathbb{C}^\times \end{array}$$

Bearing these two facts in mind, there are two cases for constructing the homomorphism

$$\phi : W_\mathbb{R} \to GL(2, \mathbb{C}).$$

(1) ϕ is made up of characters of $W_{\mathbb{R}}$. In other words, it is given by 1-dimensional representations:

$$\phi : W_{\mathbb{R}} \to GL(2,\mathbb{C}) \quad w \mapsto \begin{bmatrix} \mu_1(N(w)) & 0 \\ 0 & \mu_2(N(w)) \end{bmatrix}.$$

Thus, an irreducible principal series $p(\mu_1,\mu_2)$, where $\mu_i(x) = |x|^{\lambda_i}[\operatorname{sign} x]^{m_i}$ and $\mu = |x|^{i\nu}[\operatorname{sign} x]^m$, corresponds uniquely to the L-parameter given by $\phi_{\lambda_1,m_1,\lambda_2,m_2}$ in the notation of (2.1).

(2) ϕ is not made of characters. In other words, it is a 2-dimensional irreducible representation. In this case, ϕ is always induced from a character χ of $\mathbb{C}^\times$ where $\chi \neq \mathrm{Id}$, i.e.

$$\phi = \operatorname{Ind}_{\mathbb{C}^\times}^{W_{\mathbb{R}}} \chi.$$

The induced representation is given by

$$\phi(z) = \begin{bmatrix} \chi(z) & 0 \\ 0 & \overline{\chi}(z) \end{bmatrix} \quad \phi(\sigma) = \begin{bmatrix} 0 & \chi(-1) \\ 1 & 0 \end{bmatrix}.$$

Let $\chi(z) = |z|^\lambda e^{in\theta}$. Then the character of $W_{\mathbb{R}}$ obtained by taking the determinant of the above induced representation

$$z \mapsto |z|^{2\lambda} \quad \sigma \mapsto (-1)^{n+1}$$

corresponds to the character $x \mapsto |x|^{2\lambda}[\operatorname{sign} x]^{n+1}$ which we requested to be $\mu_1\mu_2$. Therefore, a discrete series $\sigma(\mu_1,\mu_2)$ where $\mu_i(x) = |x|^{\lambda_i}[\operatorname{sign} x]^{m_i}$ and $\mu = |x|^n[\operatorname{sign} x]^{n+1}$ corresponds uniquely to $\phi_{\lambda,n}$ where λ, n are determined by $\mu_1\mu_2 = |x|^{2\lambda}[\operatorname{sign} x]^{n+1}$.

We summarise this construction in the following lemma:

Lemma 4.1. *The local Langlands correspondence between irreducible tempered representations of $GL(2,\mathbb{R})$ and L-parameters is the following:*

$\Pi(GL(2,\mathbb{R}))$	$\mu_1,\mu_2,\mu = \mu_1\mu_2^{-1}$	$\Phi(GL(2,\mathbb{R}))$
$\sigma(\mu_1,\mu_2)$	$\mu_1\mu_2(x) = \|x\|^{2\lambda}[\operatorname{sign} x]^{n+1}$ $\mu = \|x\|^n[\operatorname{sign} x]^{n+1}$	$\phi_{\lambda,n}$
$p(\mu_1,\mu_2)$ $\mu_i(x) = \|x\|^{\lambda_i}[\operatorname{sign} x]^{m_i}$	$\mu(x) = \|x\|^{i\nu}[\operatorname{sign} x]^m$ $(\nu = 0 \Rightarrow m$ even)	$\phi_{\lambda_1,m_1,\lambda_2,m_2}$

Table 1.

This is a one-to-one correspondence.

When $\mathbb{F} = \mathbb{C}$, life is much simpler. An L-parameter $W_{\mathbb{C}} \to GL(2,\mathbb{C})$ can be diagonalised, which corresponds to two characters μ_1,μ_2 of $\mathbb{C}^\times$. This L-parameter corresponds to the irreducible principal series $p(\mu_1,\mu_2)$ of $GL(2,\mathbb{C})$ uniquely.

Lemma 4.2. *The local Langlands correspondence between irreducible tempered representations of $GL(2,\mathbb{C})$ and L-parameters is the following:*

$\Pi(GL(2,\mathbb{C}))$	$\mu_1,\mu_2,\mu=\mu_1\mu_2^{-1}$	$\Phi(GL(2,\mathbb{C}))$
$p(\mu_1,\mu_2)$ $\mu_i(\rho e^{i\theta})=\rho^{\lambda_i}e^{in_i\theta}$	$\mu(\rho e^{i\theta})=\rho^{i\nu}e^{in\theta}$	$\phi_{\lambda_1,n_1,\lambda_2,n_2}$

Table 2.

This is a one-to-one correspondence.

4.2. Local Langlands correspondence for $SL(2,\mathbb{F})$. From the previous subsection, we find that the local Langlands correspondence is constructed using characters μ_1,μ_2 of $\mathbb{F}^\times$. The same idea can be applied to $SL(2,\mathbb{F})$, but by definition we restrict only to the case $\mu_1=\mu_2^{-1}$. Thus, an irreducible representation of $SL(2,\mathbb{F})$ depends only on μ. Because the composition of $GL(2,\mathbb{F})\to PGL(2,\mathbb{F})$ with L-parameters of $GL(2,\mathbb{F})$ gives rise to L-parameters of $SL(2,\mathbb{F})$, so we have

$$\phi_{\lambda_1,n_1,\lambda_2,n_2}=\phi_{\lambda_1-\lambda_2,n_1-n_2,0,0},$$

i.e. an L-parameter for $SL(2,\mathbb{F})$ depends only on μ as well. Hence, we only need the character $\mu=\mu_1\mu_2^{-1}$ to parametrise the representations and L-parameters.

Relative to the example of $SL(2,\mathbb{R})$, the parameter for the principal series $\pi(\mu)$ where $\mu(x)=|x|^\lambda[\text{sign}(x)]^m$ is defined by $\phi_{\lambda,m},\lambda\in\mathbb{C},m\in\mathbb{Z}_2$.

Lemma 4.3. *The local Langlands correspondence between irreducible tempered representations of $SL(2,\mathbb{R})$ and the space of L-parameters is the following:*

$\Pi(SL(2,\mathbb{R}))$	$\mu(x)$	$\Phi(SL(2,\mathbb{R}))$
$D_n^\pm, n>0$	$\lvert x\rvert^n[\text{sign}(x)]^{n+1}$	ϕ_n
$D_0^\pm$	$\text{sign}(x)$	ϕ_0
$PS_{i\nu,\pm1}$	$[\text{sign}(x)]^m\lvert x\rvert^{i\nu}$	$\phi_{i\nu,m}$

Table 3.

The correspondence is 2 *to* 1 *on (limit of) discrete series.*

Remark 4.4. Therefore, the principal series $PS_{s,\pm1}$ corresponds one-to-one with L-parameters, while the (limit of) discrete series corresponds two-to-one with L-parameters, i.e. two (limit of) discrete series $D_n^\pm$ have the same L-parameter. This situation is more complicated than the case of $GL(2,\mathbb{R})$. The reason follows from the fact that principal representations for some character μ corresponds to more than one irreducible representations of $SL(2,\mathbb{R})$, while only one irreducible representations for $GL(2,\mathbb{R})$.

For $\mathbb{F} = \mathbb{C}$, all principal series are irreducible. Irreducible tempered representations of $SL(2, \mathbb{C})$ are in one-to-one correspondence with their L-parameters.

Lemma 4.5. *The local Langlands correspondence of $SL(2, \mathbb{C})$ is a bijection between irreducible tempered representations of $SL(2, \mathbb{C})$ and the space of L-parameters, given by the following table:*

$\Pi(SL(2, \mathbb{C}))$	$\mu(z)$	$\Phi(SL(2, \mathbb{C}))$
$PS_{i\nu,m}$	$e^{im\theta}\rho^{i\nu}$	$\phi_{i\nu,m}$

Table 4.

5. Base change on L-parameters and representations

In this section, we first recall base change on L-parameters. Then in view of the local Langlands correspondence, base change is obtained as a map from admissible representations of $G(\mathbb{R})$ to the fixed point set of $G(\mathbb{C})$ under the Galois group action of $\mathrm{Gal}(\mathbb{C}/\mathbb{R})$. We work out explicitly the examples of $GL(2)$ and $SL(2)$ over $\mathbb{R}$ and over $\mathbb{C}$.

5.1. Base change on L-parameters. Let F be a local field and E be a field extension. Then the Weil group W_E is a subgroup of W_F. Let G_E, G_F be reductive groups over E, F and ${}^LG_E$ and ${}^LG_F$ be their respective L-groups. Recall that a map $u :{}^L G_E \to {}^L G_F$ is an L-homomorphism if it is a continuous homomorphism over $\mathrm{Gal}(\overline{F}/F)$, i.e. the following diagram commutes:

$$\begin{array}{ccc} {}^LG_E & \xrightarrow{\;u\;} & {}^LG_F \\ \downarrow & & \downarrow \\ \mathrm{Gal}(\overline{E}/E) & \longrightarrow & \mathrm{Gal}(\overline{F}/F). \end{array}$$

and the restriction $u|_{{}^LG_E} :{}^L G_E \to {}^L G_F$ is analytic. An L-parameter ϕ_E is a lift of an L-parameter ϕ_F if there is an L-homomorphism from ${}^LG_E$ to ${}^LG_F$ such that the following diagram commutes:

$$\begin{array}{ccc} W_F & \xrightarrow{\;\phi_F\;} & {}^LG_F \\ \uparrow & & \downarrow \\ W_E & \xrightarrow{\;\phi_E\;} & {}^LG_E \end{array} \tag{5.1}$$

In our situation we are mainly interested in $F = \mathbb{R}$, $E = \mathbb{C}$ and $G = GL(2)$ or $SL(2)$, the projection map from ${}^LG_{\mathbb{R}} ={}^L G_{\mathbb{C}} \rtimes \mathrm{Gal}(\mathbb{C}/\mathbb{R})$ to the first factor is an L-homomorphism making the above diagram commute. Therefore, every L-parameter $W_{\mathbb{R}} \to^L G_{\mathbb{R}}$ gives rise to a unique L-parameter $W_{\mathbb{C}} \to^L G_{\mathbb{C}}$. This is defined as the base change map on L-parameters.

Definition 5.1. A base change is a map

$$bc : \Phi(G(\mathbb{R})) \to \Phi(G(\mathbb{C}))$$

from an L-parameter $\varphi = \phi_{\mathbb{R}} : W_{\mathbb{R}} \to^L G_{\mathbb{R}}$ to an L-parameter $\Phi = \phi_{\mathbb{C}} : W_{\mathbb{C}} \to^L G_{\mathbb{C}}$ where the restriction of $\phi_{\mathbb{R}}$ to $W_{\mathbb{C}}$ has images in ${}^LG_{\mathbb{C}}$, i.e. the diagram (5.1) commutes.

Remark 5.2. The base change map is not onto, i.e. not every L-parameter Φ : $W_{\mathbb{C}} \to^L G_{\mathbb{C}}$ extends to an L-parameter $\varphi : W_{\mathbb{R}} \to^L G_{\mathbb{R}}$ such that $\varphi|_{W_{\mathbb{C}}} = \Phi$. If Φ is in the image of ϕ under the base change map, it is called a *lift* of φ.

5.2. Galois fixed points and base change on representations. An element being in the image of a base change map or not is closely related to the L-parameters Φ fixed by the action of the Galois group $\mathrm{Gal}(\mathbb{C}/\mathbb{R})$. Given a representation $\Pi \in \Pi(G(\mathbb{C}))$ and the Galois group $\mathrm{Gal}(\mathbb{C}/\mathbb{R}) = \{1, \sigma\}$, the action of σ on Π is defined by

$$\Pi^{\sigma}(g) = \Pi(\overline{g}) \quad g \in G(\mathbb{C}).$$

Let $\Phi(z) = (a(z), j(z)) \in G^{\vee} \rtimes \mathrm{Gal}(\mathbb{C}/\mathbb{C})$ be the L-parameter corresponding to the representation Π. Then according to [30, Proposition 1], the corresponding L-parameter Φ^{σ} is given by

$$\Phi^{\sigma}(z) = (\sigma \cdot a(\overline{z}), j(z)) \quad z \in W_{\mathbb{C}} = \mathbb{C}^{\times}.$$

Suppose Φ is in the image of the base change map, i.e. a lifting of ϕ. Let $\varphi(1, \sigma) = (h, j(1, \sigma))$ and $g = \sigma \cdot h$. Then

$$\Phi^{\sigma} = Ad(g)\Phi \quad g \in G^{\vee},$$

i.e. Φ^{σ} and Φ represent the same L-parameter. So Φ is fixed by σ. If $G = GL(n)$, the converse is true as well, i.e. If $\Phi^{\sigma} = \Phi$, then Φ is a lift of some φ. This is not true in particular for $G = SL(2)$. Repka has a criteria of Φ being a lift. We summarise it into the following proposition.

Proposition 5.3 ([30]). *Suppose Φ is an L-parameter of $GL(n, \mathbb{C})$ being the fixed point under the action of the Galois group* $\mathrm{Gal}(\mathbb{C}/\mathbb{R})$. *Then a lift φ of Φ always exists, i.e. there exists φ such that*

$$bc(\varphi) = \Phi.$$

Suppose Φ *is an* L*-parameter of* $SL(n,\mathbb{C})$ *being the fixed point under the action of the Galois group* $\mathrm{Gal}(\mathbb{C}/\mathbb{R})$*. Then* Φ *is a lift of* φ *if*

(1) $\varphi(1,\sigma)\Phi(z)\varphi(1,\sigma)^{-1} = \Phi(\overline{z})$*;*

(2) $\varphi(1,\sigma)^2 = \Phi(-1)$.

Through the local Langlands correspondence, the base change map between L-parameters gives rise to the base change correspondence on representations

$$bc : \Pi(G(\mathbb{R})) \to \Pi(G(\mathbb{C})).$$

Remark 5.4. We use "correspondence" instead of "map" here because of the existence of L-packets. But for all the examples discussed in this paper, the base change on representations is in fact a map.

As the images of the base change map on $\Phi(G(\mathbb{R}))$ is a subset of the Galois fixed point sets $\Phi(G(\mathbb{C}))^{\mathrm{Gal}(\mathbb{C}/\mathbb{R})}$, in the definition of base change map the range is contained in the Galois fixed point set of admissible representations of the complex group $G(\mathbb{C})$:

Definition 5.5. The base change between representations is a correspondence from representations of $G(\mathbb{R})$ to the Galois fixed point of representations of $G(\mathbb{C})$

$$bc : \Pi(G(\mathbb{R})) \to \Pi(G(\mathbb{C}))^{\mathrm{Gal}(\mathbb{C}/\mathbb{R})} \tag{5.2}$$

which is compatible with the base change between L-parameters.

5.3. Base change for $\Pi(GL(2,\mathbb{F}))$.

In view of Lemma 4.1 and Lemma 4.2, we obtain the following base change map on the L-parameters and irreducible tempered representations of $G = GL(2)$.

Theorem 5.6. *Under the local Langlands correspondence, the base change map from* $\Phi(GL(2,\mathbb{R}))$ *to* $\Phi(GL(2,\mathbb{C}))$ *gives rise to a base change map on the irreducible tempered unitary representations of* $GL(2)$*:*

$$bc : \Pi(GL(2,\mathbb{R})) \to \Pi(GL(2,\mathbb{C}))^{\mathrm{Gal}(\mathbb{C}/\mathbb{R})}.$$

The base change map is explicitly described in the following table:

$\pi \in \Pi(GL(2,\mathbb{R}))$	$\varphi \in \Phi(GL(2,\mathbb{R}))$	$\Phi \in \Phi(GL(2,\mathbb{C}))$	$\Pi \in \Pi(GL(2,\mathbb{C}))$
$p(\mu_1,\mu_2)$ $\mu_i(x) = \lvert x\rvert^{\lambda_i}[\mathrm{sign}\, x]^{m_i}$	$\phi^{\mathbb{R}}_{\lambda_1,m_1,\lambda_2,m_2}$	$\phi^{\mathbb{C}}_{\lambda_1,0,\lambda_2,0}$	$p(\mu_1,\mu_2)$ $\mu_i(\rho e^{i\theta}) = \rho^{\lambda_i}$
$\sigma(\mu_1,\mu_2)$ $\mu_1\mu_2 = \lvert x\rvert^{2\lambda}[\mathrm{sign}\, x]^{n+1}$ $\mu_1\mu_2^{-1} = \lvert x\rvert^{n}[\mathrm{sign}\, x]^{n+1}$	$\phi^{\mathbb{R}}_{\lambda,n}$	$\phi^{\mathbb{C}}_{\lambda,n,\lambda,-n}$	$p(\mu_1,\mu_2)$ $\mu_1(\rho e^{i\theta}) = \rho^{\lambda}e^{in\theta}$ $\mu_2(\rho e^{i\theta}) = \rho^{\lambda}e^{-in\theta}$

Table 5.

This map is onto.

Proof. We only need to restrict an L-parameter $\phi^{\mathbb{R}}$ to $W_{\mathbb{C}} = \mathbb{C}^{\times}$. The discrete series $\sigma(\mu_1, \mu_2)$ for $GL(2,\mathbb{R})$ corresponds the L-parameter $\phi^{\mathbb{R}}_{\lambda,n} : W_{\mathbb{R}} \to^L G_{\mathbb{R}}$ given by

$$\phi^{\mathbb{R}}_{\lambda,n}(z) = \left(\begin{bmatrix} \rho^{\lambda} e^{in\theta} & 0 \\ 0 & \rho^{\lambda} e^{-in\theta} \end{bmatrix}, j(z) \right) \quad \phi^{\mathbb{R}}_{\lambda,n}(\sigma) = \left(\begin{bmatrix} 0 & (-1)^n \\ 1 & 0 \end{bmatrix}, j(z) \right)$$

where $z = \rho e^{i\theta} \in \mathbb{C}^{\times}$. The restriction of $\phi^{\mathbb{R}}_{\lambda,n}$ to $\mathbb{C}^{\times}$ is the same as the complex L-parameter $\phi^{\mathbb{C}}_{\lambda,n,\lambda,-n}$ evaluated at $z \in \mathbb{C}^{\times}$. The irreducible principal series $p(\mu_1, \mu_2)$ of $GL(2,\mathbb{R})$ corresponds the L-parameter $\phi^{\mathbb{R}}_{\lambda_1,m_1,\lambda_2,m_2}$ whose restriction to $\mathbb{C}^{\times}$ is $\phi^{\mathbb{C}}_{\lambda_1,0,\lambda_2,0}$. □

5.4. Base change for $\Pi(SL(2,\mathbb{F}))$. With the preparation of Lemma 4.3 and Lemma 4.5, we obtain the following base change map on the L-parameters and irreducible tempered representations of $G = SL(2)$.

Theorem 5.7. *Under the local Langlands correspondence, the base change map from $\Phi(SL(2,\mathbb{R}))$ to $\Phi(SL(2,\mathbb{C}))$ gives rise to a base change map on the irreducible tempered unitary representations of $SL(2)$:*

$$bc : \Pi(SL(2,\mathbb{R})) \to \Pi(SL(2,\mathbb{C}))^{\mathrm{Gal}(\mathbb{C}/\mathbb{R})}.$$

The base change map is explicitly described in the following table:

$\pi \in \Pi(SL(2,\mathbb{R}))$	$\varphi \in \Phi(SL(2,\mathbb{R}))$	$\Phi \in \Phi(SL(2,\mathbb{C}))$	$\Pi \in \Pi(SL(2,\mathbb{C}))$
$D^{\pm}_n, n > 0$	$\phi^{\mathbb{R}}_n$	$\phi^{\mathbb{C}}_{0,2n}$	$PS_{0,2n}$
$D^{\pm}_0$	$\phi^{\mathbb{R}}_0$	$\phi^{\mathbb{C}}_{0,0}$	$PS_{0,0}$
$PS_{i\nu,\pm 1}$	$\phi^{\mathbb{R}}_{i\nu,m}$	$\phi^{\mathbb{C}}_{i\nu,0}$	$PS_{i\nu,0}$

Table 6.

This map is not onto.

Proof. We only need to restrict an L-parameter $\phi^{\mathbb{R}}$ to $W_{\mathbb{C}} = \mathbb{C}^{\times}$. For the (limit of) discrete series D^+_n and D^-_n for $SL(2,\mathbb{R})$, they are in the same L-packet and correspond to the L-parameter $\phi^{\mathbb{R}}_n : W_{\mathbb{R}} \to^L G_{\mathbb{R}}$:

$$\phi_n(z) = \left(\begin{bmatrix} e^{in\theta} & 0 \\ 0 & e^{-in\theta} \end{bmatrix}, j(z) \right) \quad \phi_n(\sigma) = \left(\begin{bmatrix} 0 & (-1)^n \\ 1 & 0 \end{bmatrix}, j(z) \right)$$

where $z = \rho e^{i\theta} \in \mathbb{C}^{\times}$. Because we consider the matrix to be an element of $PGL(2,\mathbb{C})$, the restriction of $\phi^{\mathbb{R}}_n$ to $\mathbb{C}^{\times}$ is the same as

$$\begin{bmatrix} e^{i2n\theta} & 0 \\ 0 & 1 \end{bmatrix} \quad z \in \mathbb{C}^{\times},$$

which is equal to the complex L-parameter $\phi^{\mathbb{C}}_{0,2n}$ evaluated at $z \in \mathbb{C}^{\times}$. Other cases are obvious. □

Remark 5.8. (1) The base change map is *not* onto for $G = SL(2)$. In particular, the $(2n-1)$-th lines corresponding the principal series $PS_{i\nu,2n-1}$ are never images of the base change map (cf. [30]).

(2) The base change maps on $\Phi(SL(2,\mathbb{R}))$ and on $\Pi(SL(2,\mathbb{R}))$ are not one-to-one. In particular, the base change map cannot distinguish representations from the same L-packet.

6. Base change on K-theory

This section is the central part of the paper. Tempered representations of an almost connected reductive group over local field can be classified by the K-theory of group C^*-algebra. Together with our study on the Galois group action on the reduced dual of $G(\mathbb{C})$, we define a base change map on the level of K-theory.

6.1. K-theory and unitary representations. Let G be an almost connected reductive Lie group, i.e. the quotient of G by the connected component of the group identity is finite. Let $\widehat{G}$ be equivalence classes of irreducible unitary representations of G. The set $\widehat{G}$ equipped with Fell topology is a T_0 space and is called the unitary dual of G. The space $\widehat{G}$ carries a Plancherel measure μ, supported on the subset $\widehat{G}_t$, the tempered dual of G, of irreducible tempered representations of G. The Fell topology on $\widehat{G}_t$ is not Hausdorff in general. Group C^*-algebras are introduced and extensively studied (cf. [7]) as an important tool to cope with the poor topology of $\widehat{G}$ and $\widehat{G}_t$.

A unitary representation π of G in a Hilbert space H gives rise to a representation of the convolution algebra $L^1(G)$ by

$$\pi(f) = \int_G f(g)\pi(g)dg \quad f \in L^1(G).$$

The maximal group C^*-algebra $C^*(G)$ is the enveloping algebra of $L^1(G)$, i.e. completion of $L^1(G)$ under the C^*-norm given by

$$\|f\| := \sup_{(\pi,H)} \|\pi(f)\|_{\mathcal{L}(H)}.$$

If we choose the left regular representation λ represented in $L^2(G)$, then the reduced group C^*-algebra $C_r^*(G)$ is the completion of the image of

$$\lambda : L^1(G) \to \mathcal{L}(L^2(G)) \quad f \mapsto (g \mapsto f * g)$$

under the operator norm. $C^*(G)$ is universal in the sense that every unitary representation of G gives rise to a representation of $C^*(G)$. In particular, there is a surjective homomorphism $C^*(G) \to C_r^*(G)$.

When G is a reductive group over an archimedean field, $\widehat{G}$ (resp. $\widehat{G}_t$) corresponds bijectively to primitive ideals of $C^*(G)$ (resp. $C_r^*(G)$). (Reduced) group C^*-algebras are algebraic analogue of (tempered) unitary dual. *Philosophically* speaking, we have

$$C_r^*(G) \text{ is Morita equivalent to } C_0(\widehat{G}_t). \tag{6.1}$$

Then because K-theory is stable under Morita equivalence, we have the isomorphism

$$K_*(C_r^*(G)) \cong K^*\left(\widehat{G}_t\right). \tag{6.2}$$

The right hand side of (6.2) is the topological K-theory of $\widehat{G}_t$. Unfortunately, (6.1) and (6.2) are false in general. (See [3, Section 9] for examples of this isomorphisms and a counterexample.) However, for the examples considered in our paper, $\widehat{G}_t$ is *almost* Hausdorff and $C_r^*(G)$ is Morita equivalent to a C^*-algebra which is *almost* $C_0(\widehat{G}_t)$ (quote from [10, Section 6]). Being more explicit, when G is a complex semisimple group, (6.1) and (6.2) are proved by Plymen [28]. This covers the case when $G = SL(n,\mathbb{C})$. When G is $GL(n)$, the Morita equivalence (6.1) is proved by Plymen [29] (see also [24]). A case we are interested but not included in the above examples is a connected real reductive group (for example, $G = SL(n,\mathbb{R})$), where the right hand side in (6.1) does not make sense because $\widehat{G}_t$ is not Hausdorff. However, $\widehat{G}_t$ has an orbifold structure whose K-theory can be locally calculated by equivariant topological K-theory [5,31]. Therefore, it is possible to compute K-theory for $GL(n)$ and $SL(n)$ over an archimedean field explicitly.

K-theory for complex semisimple Lie groups was studied by Penington and Plymen [28]. Let G be a complex semisimple group with the Borel (minimal parabolic) subgroup P and Langlands decomposition $P = MAN$. In this case, all principal series are irreducible. Thus, the tempered dual $\widehat{G}_t$ is parametrised by $\widehat{A}\times\widehat{M}$. Note also that the principle series are equivalent for each orbit of parameters in $\widehat{A}\times\widehat{M}$ under the action of the Weyl group $W = N_K(A)/Z_K(A)$. Therefore,

$$\widehat{G}_t \cong \left(\widehat{M}\times\widehat{A}\right)/W \cong \bigoplus_{\sigma\in\widehat{M}/W} \widehat{A}/W_\sigma \tag{6.3}$$

Here, W_σ is the stabiliser of $\sigma \in \widehat{M}$. It is a subgroup of W. It can be observed that $\widehat{G}_t$ is Hausdorff. This leads to the calculation of K-theory (cf. [28]):

$$K_*(C_r^*(G)) \cong \bigoplus_{\sigma\in\widehat{M}/W} K^*\left(\widehat{A}/W_\sigma\right) \tag{6.4}$$

where

$$K^j\left(\widehat{A}/W_\sigma\right) = \begin{cases}\mathbb{Z} & W_\sigma = \{1\} \text{ and } j \equiv \dim(G/K) \bmod 2,\\ 0 & \text{otherwise}.\end{cases} \tag{6.5}$$

When G is in general a connected reductive group, the situation is more complicated in the sense that $\widehat{G}_t$ consists of not only irreducible principal series. Moreover, principal series are not always irreducible ($G = SL(2,\mathbb{R})$, for example). This more general case was studied by Wassermann in his short note [31]. We also refer to the recent paper by Clare–Crisp–Higson [5, Theorem 6.8 and Remark 6.9] for a proof with full details.

Theorem 6.1 ([5,31]). *Let G be a connected reductive group. Then*

$$K_i(C_r^*(G)) \cong \bigoplus_P \bigoplus_{\sigma \in \widehat{M}_d} K^i_{R_\sigma}\left(\widehat{A}/W'_\sigma\right). \tag{6.6}$$

The first summand is over all cuspidal parabolic subgroup P of G, i.e. when $\widehat{M}$ in the Langlands decomposition of P has discrete series. The second summand is over all discrete series of M, denoted $\widehat{M}_d$. The R-group R_σ and W'_σ are subgroups of W_σ where $W_\sigma = W'_\sigma \rtimes R_\sigma$ (cf. [15]). Moreover,

$$K^i_{R_\sigma}\left(\widehat{A}/W'_\sigma\right) = \mathbb{Z} \quad i \equiv \dim(G/K) \bmod 2.$$

Remark 6.2. General linear groups over $\mathbb{R}$ are not connected, but almost connected. From investigating the topological structure of the reduced dual, the same formula (6.6) still applies to $G = GL(n,\mathbb{R})$. For $GL(n,\mathbb{R})$, the R-group is always trivial. Hence, $W_\sigma = W'_\sigma$.

Example 6.3 ($GL(2)$). For $GL(2,\mathbb{C})$, its only cuspidal parabolic subgroup is the upper triangular invertible matrices. In this case,

$$A = \left\{\begin{bmatrix} r_1 & 0 \\ 0 & r_2 \end{bmatrix} : r, s > 0\right\}, \quad M = \left\{\begin{bmatrix} e^{in_1\theta} & 0 \\ 0 & e^{in_2\theta} \end{bmatrix}\right\}, \quad N = \left\{\begin{bmatrix} 1 & * \\ 0 & 1 \end{bmatrix}\right\}$$

and $\widehat{A} \cong \mathbb{R}^2$, $\widehat{M} \cong \mathbb{Z}_2$. The Weyl group is S_2 where the nontrivial element identifies $(\lambda_1, n_1, \lambda_2, n_2)$ and $(\lambda_2, n_2, \lambda_1, n_1)$ in $\widehat{A} \times \widehat{M}$. Thus,

$$\widehat{M}/W = \left\{(n_1, n_2) \in \mathbb{Z}^2 : n_1 \geq n_2\right\}.$$

When $n_1 = n_2 \in \mathbb{Z}$, the corresponding character $\sigma \in \widehat{M}$ has the stabiliser subgroup of $W_\sigma \cong \mathbb{Z}_2$ and so $\widehat{A}/W_\sigma$ is a closed half plane. When $n_1 > n_2$, W_σ is trivial and then $\widehat{A}/W_\sigma \cong \mathbb{R}^2$. Note also that every R-group R_σ is trivial because all principal series of $GL(2,\mathbb{C})$ are irreducible. Therefore according to Theorem 6.1,

$$\widehat{GL(2,\mathbb{C})}_t \cong (\oplus_{n_1=n_2}\mathbb{R}^2/\mathbb{Z}_2) \bigoplus (\oplus_{n_1>n_2}\mathbb{R}^2)$$

and the K-group is

$$K_i(C_r^*(GL(2,\mathbb{C}))) = \begin{cases} \oplus_{n_1>n_2}\mathbb{Z}, & i = 0, \\ 0, & i = 1. \end{cases}$$

For $GL(2,\mathbb{R})$, its cuspidal parabolic subgroups are the Borel subgroup (upper triangular matrices) P_1 and $P_2 = GL(2,\mathbb{R})$ itself. For P_1, similar as above $\widehat{A} \cong \mathbb{R}^2$, $\widehat{M} \cong \mathbb{Z}_2^2$. For $m_1 = m_2 \in \mathbb{Z}_2$, the corresponding principal series are irreducible and parametrised by points on a closed half plane $\mathbb{R}^2/\mathbb{Z}_2$. For $m_1 > m_2$ in $\mathbb{Z}_2$ the corresponding principal series is parametrised by $\mathbb{R}^2$, providing a generator for K_0. (Even though the principal series is reducible when $\lambda_1 = \lambda_2$, there are no double points here). For P_2, A consists of positive scalar matrices, M consists of matrices with ± 1 determinant and N is trivial. The discrete series in $\widehat{M}$ are indexed by $\mathbb{Z}_{>0}$ The Weyl group and the R-group are trivial. Thus, group C^*-algebra of $GL(2,\mathbb{R})$ is morita equivalence to

$$(C_0(\mathbb{R}^2/\mathbb{Z}_2) \oplus C_0(\mathbb{R}^2)) \bigoplus (\oplus_{\mathbb{Z}_{>0}} C_0(\mathbb{R}))$$

and the K-group is

$$K_i(C_r^*(GL(2,\mathbb{R}))) = \begin{cases} \oplus_{\mathbb{Z}_{>0}} \mathbb{Z}, & i = 1, \\ \mathbb{Z}, & i = 0. \end{cases}$$

Example 6.4 ($SL(2)$, see also [2]). The space $\widehat{SL(2,\mathbb{R})}_t$ consists of the following components: principal series $PS_{i\nu,+}$, $\nu \geq 0$ (parametrised by $\mathbb{R}/\mathbb{Z}_2$), principal series $PS_{i\nu,-}$, $\nu > 0$ and limit of discrete series $D_0^{\pm}$ (parametrised by an open ray $\mathbb{R}_{>0}$ with double points attached at 0, or equivalently, by $\mathbb{R} \rtimes \mathbb{Z}_2$), discrete series $D_n^{\pm}$, $n > 0$ (pairs of points indexed by $\pm n$). Hence, the topological K-theory of $\widehat{SL(2,\mathbb{R})}_t$ is given by

$$K_i(C_r^*(SL(2,\mathbb{R}))) = K^i\left(\widehat{SL(2,\mathbb{R})}_t\right) = \begin{cases} \oplus_{n\in\mathbb{Z}} \mathbb{Z}, & i = 0, \\ 0, & i = 1. \end{cases}$$

The space $\widehat{SL(2,\mathbb{C})}_t$ consists of the following components: principal series $PS_{i\nu,n}$, $n > 0$ (parametrised by $\mathbb{R}$) and principal series $PS_{i\nu,0}$, $\nu \geq 0$ (parametrised by $\mathbb{R}/\mathbb{Z}_2$). Hence, the topological K-theory of $\widehat{SL(2,\mathbb{C})}_t$ is given by

$$K_i(C_r^*(SL(2,\mathbb{C}))) = K^i\left(\widehat{SL(2,\mathbb{C})}_t\right) = \begin{cases} 0, & i = 0, \\ \oplus_{n\in\mathbb{Z}_{>0}} \mathbb{Z}, & i = 1. \end{cases}$$

6.2. Base change on K-theory. When G is a general linear group, K-theory of reduced group C^*-algebras is related to the topological K-theory of the space of irreducible tempered representations. So we can use base change on representations, i.e. topological space of tempered dual of a group to define a base change map on K-theory:

$$K_i(C_r^*(G(\mathbb{C}))) \cong K^i\left(\widehat{G(\mathbb{C})}_t\right) \longrightarrow K^i\left(\widehat{G(\mathbb{R})}_t\right) \cong K_i(C_r^*(G(\mathbb{R}))). \tag{6.7}$$

Such a base change map on K-theory was defined by Mendes and Plymen [25] for $GL(n)$ for Galois extensions of *p-adic* fields. This gives an important arithmetic method for comparing operator K-theory for group C^*-algebras over different number fields. However, for *archimedean* fields the K-theory base change map is 0 for $GL(n), n > 1$, thus the base change map (6.7) does not provide useful information for this arithmetic invariant. To cope with this problem, we use Galois fixed point set of $\widehat{G(\mathbb{C})}_t$ to include the image of base change map on representations (5.2). This gives rise to a *new* map on topological K-theory:

$$K^i\left(\widehat{G(\mathbb{C})}_t^{\mathrm{Gal}(\mathbb{C}/\mathbb{R})}\right) \to K^i\left(\widehat{G(\mathbb{R})}_t\right). \tag{6.8}$$

Then, we extend the definition of base change to K-theory of operator algebras. In particular, we look for a C^*-algebra analogue for the Galois fixed points of $\widehat{G(\mathbb{C})}_t$. Using (6.3) we can find the Galois fixed point set of $\widehat{G(\mathbb{C})}_t$ by finding fixed point set in each summand:

$$\widehat{G(\mathbb{C})}_t^{\mathrm{Gal}(\mathbb{C}/\mathbb{R})} = \oplus_{\sigma\in\widehat{M}/W}\left(\widehat{A}/W_\sigma\right)^{\mathrm{Gal}(\mathbb{C}/\mathbb{R})}. \tag{6.9}$$

In view of (6.5), when $W_\sigma = \{1\}$, the Galois group $\mathrm{Gal}(\mathbb{C}/\mathbb{R})$ acts on $\widehat{A}$ by reflection, and then the fixed point set $\widehat{A}^{\mathrm{Gal}(\mathbb{C}/\mathbb{R})}$, if exists, is a subspace of $\widehat{A}$.

Remark 6.5. When $G = SL(n)$, the space $\widehat{G(\mathbb{R})}_t$ is not Hausdorff. We can replace the last isomorphism in (6.7) by the isomorphism (6.6). Note that on the level of characters the base change map (5.2) gives rise to

$$K^i\left(\widehat{G(\mathbb{C})}_t^{\mathrm{Gal}(\mathbb{C}/\mathbb{R})}\right) \to \bigoplus_P \bigoplus_{\sigma\in\widehat{M}_d} K^i_{R_\sigma}\left(\widehat{A}/W'_\sigma\right).$$

Thus, base change map on K-theory can also be developed for special linear groups. This is not the focus of the paper but we will do some calculations. (See Example 6.8 and Corollary 6.11.)

Due to (6.5), we know that when $\{W_\sigma\} \neq \{1\}$, the corresponding component does not contribute to a nontrivial element in K-theory. So when we compare K-theory of $\widehat{A}/W_\sigma$ and that of its Galois fixed point set, we only consider the case when $W_\sigma = \{1\}$.

The following two lemmas are the key tool to determine the Galois fixed points. Here, the group G is assumed to be $GL(n,\mathbb{C})$ or $SL(n,\mathbb{C})$.

Lemma 6.6. *Let $P = MAN$ be* the *Borel subgroup of a* complex *connected reductive group G. Let $\widehat{A}$ be the component indexed by the character $\sigma = (n_1,\ldots,n_k) \in \widehat{M}$ where $W_\sigma = \{1\}$. Then*

(1) *$n_i \neq n_j$ for all $i \neq j$.*

(2) *If we fix a discrete series* $(n_1, \dots, n_k)$ *in* $\widehat{M}$, *the Galois group action on* $\widehat{A}$ *has a nonempty fixed point set only when* $\{n_1, \dots, n_k\}$ *up to permutation satisfies* $n_i = -n_{k+1-i}$ *for all* $i = 1, \dots, k$ *and all* n_i*s are distinct.*

Proof. (1) If $n_i = n_j$ for some $i \neq j$, then $(i\ \ j) \in W$ fixes the character $\sigma = (n_1, \dots, n_k) \in \widehat{M}$, i.e. the stabiliser $W_\sigma \neq \{1\}$. This is a contradiction to our assumption. So (1) is proved.

(2) For a character $\mu = (\lambda_1, \dots, \lambda_k) \in \widehat{A}$, the action on $(\sigma, \mu) \in \widehat{M} \times \widehat{A}$ by the nontrivial element of the Galois group $\mathrm{Gal}(\mathbb{C}/\mathbb{R})$ is

$$(n_1, \dots, n_k, \lambda_1, \dots, \lambda_k) \mapsto (-n_1, \dots, -n_k, \lambda_1, \dots, \lambda_k) \tag{6.10}$$

because the action is given by complex conjugation. $\widehat{A}$ over σ having a Galois fixed point means that the characters $\sigma = (n_1, \dots, n_k)$ and $-\sigma = (-n_1, \dots, -n_k)$ are related by a Weyl group element, i.e. $\{-n_1, \dots, -n_k\}$ is a permutation of $\{n_1, \dots, n_k\}$. This together with (1) prove (2). □

Lemma 6.7. *Assume all conditions in Lemma 6.6. If* $\dim_{\mathbb{R}} \widehat{A} = k$ *and* $\widehat{A}^{\mathrm{Gal}(\mathbb{C}/\mathbb{R})} \neq \varnothing$, *then*

$$\dim_{\mathbb{R}} \widehat{A}^{\mathrm{Gal}(\mathbb{C}/\mathbb{R})} = \lceil k/2 \rceil.$$

Denote $j = \dim_{\mathbb{R}} \widehat{A} - \lceil (\dim_{\mathbb{R}} \widehat{A})/2 \rceil$, *then*

$$K^*\left(\widehat{A}\right) = K^{*+j}\left(\widehat{A}^{\mathrm{Gal}(\mathbb{C}/\mathbb{R})}\right).$$

Proof. Using the notation from the proof of Lemma 6.6(2), the fixed point set of $\widehat{A}$ over $\sigma \in \widehat{M}$ is the subspace determined by the subspace

$$\left\{\lambda_i = \lambda_{k+1-i}, i = 1, 2, \dots, \lfloor k/2 \rfloor\right\},$$

which has dimension $\lceil k/2 \rceil$. Noting that $K^*(X \times \mathbb{R}^j) = K^{*+j}(X)$ for any Hausdorff space X, the lemma is then proved. □

Example 6.8. When $G = SL(2)$, each component $\widehat{A} \cong \mathbb{R}$ (where $W_\sigma = \{1\}$) possess a Galois fixed point, i.e. $\dim \widehat{A}^{\mathrm{Gal}(\mathbb{C}/\mathbb{R})} = 0$. Then

$$K^{i+1}\left(\widehat{G(\mathbb{C})}_t^{\mathrm{Gal}(\mathbb{C}/\mathbb{R})}\right) = K^i\left(\widehat{G(\mathbb{C})}_t\right).$$

In general, when $K^*(\widehat{A}/W_\sigma) \neq 0$, i.e. $* \equiv \dim(G/K)$ and $W_\sigma = \{1\}$, we have

$$K^*(\widehat{A}/W_\sigma) = \begin{cases} K^{*+j}\left(\left(\widehat{A}/W_\sigma\right)^{\mathrm{Gal}(\mathbb{C}/\mathbb{R})}\right), & \widehat{A}^{\mathrm{Gal}(\mathbb{C}/\mathbb{R})} \neq \varnothing, \\ 0, & \text{otherwise.} \end{cases}$$

Here, $j = \dim_{\mathbb{R}} \widehat{A} - \lceil \dim_{\mathbb{R}} \widehat{A}/2 \rceil$. This together with (6.3), (6.9) and (6.4) or (6.6) give rise to a map

$$K_*(C_r^*(G(\mathbb{C}))) \to K^{*+j}\left(\widehat{G(\mathbb{C})}_t^{\mathrm{Gal}(\mathbb{C}/\mathbb{R})}\right).$$

This composed with (6.8) gives rise to the desired base change map

$$bc : K_*(C_r^*(G(\mathbb{C}))) \to K_{*+j}(C_r^*(G(\mathbb{R}))) \tag{6.11}$$

as is concluded in the following theorem.

Theorem 6.9. *Let G be a general linear group or a special linear group. There exists a base change map on K-theory of operator algebras*

$$bc : K_i(C_r^*(G(\mathbb{C}))) \to K_j(C_r^*(G(\mathbb{R}))) \tag{6.12}$$

where $i = \dim(G(\mathbb{C})/K(\mathbb{C}))$ mod 2 and $j = \dim(G(\mathbb{R})/K(\mathbb{R}))$ mod 2. This map is compatible with the base change map (6.8) on topological K-theory.

Proof. We need only to verify that the degrees of K-theory in (6.11) and in (6.12) are compatible. As we know from the Connes–Kasparov isomorphism, for a general linear group or a special linear group G and its maximal compact subgroup K, we have

$$K_j(C_r^*(G)) \cong R(K) \quad \dim(G/K) \equiv j \bmod 2.$$

Let P be the maximal cuspidal parabolic subgroup of G and $P = MAN$. Then

$$\dim G/K \equiv \dim \widehat{A} \bmod 2.$$

For a group $G(\mathbb{C})$ over the complex field, the maximal cuspidal parabolic subgroup is the Borel subgroup and $A_{\mathbb{C}}$ is the diagonal matrix subgroup whose entries are positive real numbers. For a group $G(\mathbb{R})$ over the real field, the maximal parabolic subgroup is the upper triangular matrices where we have the maximal possible number of 2×2 blocks on the diagonal. In this case, $A_{\mathbb{R}}$ has the form $\mathrm{diag}\{c_1 I_2, c_2 I_2, \ldots, c_n I_2\}$ or $\mathrm{diag}\{c_1 I_2, c_2 I_2, \ldots, c_n I_2, c_{n+1}\}$ where $c_i \in \mathbb{R}_+$. Hence $\dim_{\mathbb{R}} \widehat{A}_{\mathbb{R}} = \lceil \dim_{\mathbb{R}} \widehat{A}_{\mathbb{C}}/2 \rceil$ and

$$\dim G(\mathbb{C})/K(\mathbb{C}) \equiv \dim_{\mathbb{R}} \widehat{A}_{\mathbb{C}} \bmod 2,$$
$$\dim G(\mathbb{R})/K(\mathbb{R}) \equiv \lceil \dim_{\mathbb{R}} \widehat{A}_{\mathbb{C}}/2 \rceil \bmod 2.$$

Therefore $j - i$ in (6.12) is equal to $\dim_{\mathbb{R}} \widehat{A}_{\mathbb{C}} - \lceil \dim_{\mathbb{R}} \widehat{A}_{\mathbb{C}}/2 \rceil$ mod 2, the same parity as j in (6.11). The theorem is then proved. ☐

We examine the base change map on K-theory on two typical examples.

Corollary 6.10. *The base change map $K_0(C_r^*(GL(2,\mathbb{C}))) \to K_1(C_r^*(GL(2,\mathbb{R})))$ for $G = GL(2)$ is explicitly calculated as follows: In $\oplus_{n_1>n_2}\mathbb{Z} \to \oplus_{n>0}\mathbb{Z}$, the generator corresponding to (n_1,n_2) is mapped to 0 unless $n_1+n_2 = 0$ and the generator parametrised by $(n,-n)$ is mapped to the generator parametrised by $|n|$.*

Proof. From Lemma 6.6, when $W_\sigma = \{1\}$ a component of $\widehat{GL(2,\mathbb{C})_t}$ has a Galois fixed point set only when $n_1 = -n_2$. In this case the Galois fixed point set has co-dimension one in $\widehat{A}$. Thus, we have a map

$$K_0(C_r^*(GL(2,\mathbb{C}))) \to K^1\left(\widehat{GL(2,\mathbb{C})}_t^{\mathrm{Gal}(\mathbb{C}/\mathbb{R})}\right)$$

which is non-vanishing and isomorphic on components with Galois fixed points. Then applying Theorem 5.6 to the base change for topological K-theory leads to the result. □

Corollary 6.11. *The base change map $K_1(C_r^*(SL(2,\mathbb{C}))) \to K_0(C_r^*(SL(2,\mathbb{R})))$ on for $G = SL(2)$ is explicitly calculated as follows:*

$$\oplus_{n\in\mathbb{Z}_+}\mathbb{Z} \to \oplus_{n\in\mathbb{Z}}\mathbb{Z} \quad (x_1,x_2,\ldots) \mapsto (\ldots,x_4,x_2,0,x_2,x_4,\ldots).$$

Proof. This follows from Example 6.8 and adapting Theorem 5.7 to topological K-theory. □

7. Base change on representations of maximal compact subgroups

In this section, we investigate base change maps in relation to the Connes–Kasparov isomorphism. The groups under consideration are general linear or special linear groups.

7.1. Connes–Kasparov isomorphism. Elements of $K_*(C_r^*(G))$ for a reductive group G can be realised as higher indices of Dirac operators via Dirac induction based on representation of a fixed maximal compact subgroup K in G. Let $R(K)$ be the ring of irreducible representations of K. To each $(\pi, V_\pi) \in R(K)$, there is an associated Dirac operator,

$$D_\pi = \sum_{i=1}^{\dim G/K} X_i \otimes c(X_i) \otimes 1 : (L^2(G)\otimes S\otimes V_\pi)^K \to (L^2(G)\otimes S\otimes V_\pi)^K,$$

where $\{X_i\}$ is an orthonormal basis for $\mathfrak{p} = \mathfrak{g}/\mathfrak{k}$ and S is the spin representation of K in $\mathfrak{p}$. The Dirac operator D_π is essentially selfadjoint. The structure of S depends on $\dim\mathfrak{p}$. The higher index $\mu(D_\pi)$ is an element of $K_*(C_r^*(G))$, where $*$ is equal to the parity of $\dim \mathfrak{p}$. This is known as the Connes–Kasparov conjecture [6,13,14] and proved for almost connected groups and all reductive groups respectively by Charbert–Eschterhoff–Nest [4] and Lafforgue [19].

Theorem 7.1 (Connes–Kasparov isomorphism). *The higher index map gives rise to an isomorphism*

$$R(K) \to K_*(C_r^*(G)) \quad [\pi] \mapsto \mu(D_\pi) \quad * = \dim \mathfrak{p} \bmod 2. \tag{7.1}$$

The theorem reveals further a striking correspondence between the representations of G and the representations of its maximal compact subgroup K. This is observed using representation theory by Parthasarathy [27], then Atiyah–Schmid developed a geometric approach using Dirac operator [1] for semisimple groups. We shall use Atiyah–Schmid's construction to describe the Connes–Kasparov isomorphism (7.1) for *semisimple* groups. The Dirac operator D_π gives rise to elements of $K_*(C_r^*(G)) \cong K^*(\widehat{G}_t)$ via Fourier transformation. Applying the Plancherel theorem to the Hilbert space on which D_π acts gives rise to the direct integral decomposition (see also [28])

$$(L^2(G) \otimes S \otimes V_\pi)^K \cong \int_{\widehat{G}} H_\sigma \otimes W_\sigma d\mu(\sigma) \quad W_\sigma = (H_\sigma^* \otimes S \otimes V_\pi)^K.$$

The G-invariant differential operator D_π acting on the left thus decomposes as

$$\widehat{D}_\pi = \int_{\widehat{G}} 1 \otimes \phi(\sigma) d\mu(\sigma)$$

where $\phi(\sigma) \in \mathrm{End}(W_\sigma)$ if $\dim \mathfrak{p}$ is odd and $\phi(\sigma) \in \mathrm{Hom}(W_\sigma^\pm, W_\sigma^\mp)$ if $\mathfrak{p}$ is even. The Fourier transform $\widehat{D}_\pi$ is a family of endomorphisms indexed by $\sigma \in \widehat{G}_t$. It determines naturally an unbounded KK-cycle

$$[C_0(\widehat{G}_t, H \otimes W), \widehat{D}_\pi] \in KK^*(\mathbb{C}, C_0(\widehat{G}_t)) \quad * = \dim \mathfrak{p} \bmod 2.$$

Here, $H \otimes W$ stands for the family of Hilbert spaces $\{H_\sigma \otimes W_\sigma\}_{\sigma \in \widehat{G}_r}$. We only care the tempered representation because the the points outside $\widehat{G}_t$ have 0 Plancherel measure.

Note that if the dimension of W_σ vanishes or if $\widehat{D}_\pi$ is invertible on a "connected component" in $\widehat{G}_t$ containing σ (when it makes sense), then $[\widehat{D}_\pi]$ does not generate the K-element corresponding to the component containing σ.

Theorem 7.2 ([1]). *Let G be a connected noncompact semisimple Lie group with finite center and a maximal compact subgroup K. Denote by ρ_c the half sum of compact positive roots of G. Suppose G has discrete series, i.e,* $\mathrm{rank}(G) = \mathrm{rank}(K)$. *Then for each discrete series representation π_σ of G labeled by a regular character σ of a maximal torus T of G, it can be realised by the the kernel of Dirac operator twisted by the irreducible representation of K with highest weight $\sigma - \rho_c$, i.e.*

$$\mathrm{Ker}\, D^+_{\sigma-\rho_c} \cong \pi_\sigma \quad \mathrm{Ker}\, D^-_{\sigma-\rho_c} = 0.$$

Remark 7.3. This theorem is stated only for semisimple groups having discrete series. But this follows from a general observation for *any* reductive group G, the map $R(K) \to K_*(C_r^*(G))$ reveals a connection of the highest weights of characters of a maximal torus of K and characters of the torus M in the Langlands decomposition of the maximal cuspidal parabolic subgroup of P (see [31] for example). This philosophy is used in the following example (see also [2]) and in the proof of Theorem 7.7.

Example 7.4. The K_0-group of $SL(2,\mathbb{R})$ is isomorphic to the representation ring $R(SO(2))$ of the maximal compact subgroup $SO(2)$. Denote by π_k the representation of $SO(2)$ with weight k. Then the isomorphism is given by the following:

$$\begin{aligned} R(SO(2)) &\to K_0(C_r^*(SL(2,\mathbb{R}))), \\ [\pi_1] &\mapsto [PS_{i\nu,-}D_0^{\pm}], \\ [\pi_k] &\mapsto [D_{k-1}^{-}], \qquad && k > 1 \\ [\pi_k] &\mapsto [D_{1-k}^{+}], \qquad && k < 1. \end{aligned}$$

The K_1-group of $SL(2,\mathbb{C})$ is isomorphic to the representation ring $R(SU(2))$ of the maximal compact subgroup $SU(2)$. Denote by σ_k the representation of $SU(2)$ with highest weight $k, k \geq 0$. Then the isomorphism is given by the following:

$$\begin{aligned} R(SU(2)) &\to K_0(C_r^*(SL(2,\mathbb{C}))), \\ [\sigma_k] &\mapsto [PS_{i\nu,k+1}]. \end{aligned}$$

7.2. Base change on maximal compact subgroup. The base change map on K-theory given by Theorem 6.9 leads to a base change on the representation of maximal compact subgroups via the Connes–Kasparov isomorphism:

$$\begin{array}{ccc} K_i(C_r^*(G(\mathbb{C}))) & \xrightarrow{\cong} & R(K_{\mathbb{C}}) \\ \big\downarrow bc & & \big\downarrow bc \\ K_j(C_r^*(G(\mathbb{R}))) & \xrightarrow{\cong} & R(K_{\mathbb{R}}). \end{array}$$

Example 7.5. The base change map on the representation of the maximal compact spaces is defined by the diagram:

$$\begin{array}{ccc} K_1(C_r^*(SL(2,\mathbb{C}))) & \xrightarrow{\cong} & R(SU(2)) \\ \big\downarrow bc & & \big\downarrow bc \\ K_0(C_r^*(SL(2,\mathbb{R}))) & \xrightarrow{\cong} & R(SO(2)). \end{array}$$

Using Corollary 6.11, the map between generators of $R(SU(2))$ and $R(SO(2))$ is given by

$$\begin{aligned}&R(SU(2)) \to R(SO(2)),\\&(x_k[\sigma_k])_{k>0} \mapsto (\ldots, x_5[\pi_{-2}], x_3[\pi_{-1}], x_1[\pi_0], 0[\pi_1], x_1[\pi_2], x_3[\pi_3], x_5[\pi_4], \ldots).\end{aligned}$$

Remark 7.6. Note that $SU(2)$ is not a complex Lie group and is not the same type as $SO(2)$. Thus, for $SL(2)$, the induced base change on its maximal compact subgroups is different from the base change discussed in previous sections.

Nevertheless, the case when $G = GL(n)$ is particular interesting as the respective maximal compact subgroups $U(n)$ and $O(n)$ have the same type and defined over $\mathbb{C}$ and $\mathbb{R}$, respectively. For this situation, we can directly define base change map on representation

$$bc : \widehat{O(n)} \to \widehat{U(n)}^{\mathrm{Gal}(\mathbb{C}/\mathbb{R})} \subset \widehat{U(n)}$$

and the corresponding base change map on K-theory

$$bc : R(U(n)) \to R(O(n)).$$

Note that in this case, our definition and the one used in [25] to such case of compact groups coincide, because the reduced dual of a compact group are 0 dimensional (and discrete), causing no degree shift when comparing K-theory of $\widehat{U(n)}^{\mathrm{Gal}(\mathbb{C}/\mathbb{R})}$ and $\widehat{U(n)}$.

We conclude the paper with the following compatibility theorem. Namely, base change maps on $GL(n)$ and on their maximal compact subgroups are homomorphisms compatible with the Connes–Kasparov isomorphism.

Recall that a character of the maximal torus T of a compact Lie group K is regular if its stabiliser in the Weyl group W is trivial. An irreducible representation of K is called a regular element of $\widehat{K}$ if it corresponds a regular character of T in the identification:

$$\widehat{K} = \widehat{T}/W.$$

Theorem 7.7. *Over regular points of $\widehat{U(n)}$, the following diagram commutes:*

$$\begin{array}{ccc} R(U(n)) & \xrightarrow{CK} & K_i(C_r^*(GL(n,\mathbb{C}))) \\ \downarrow{\scriptstyle bc} & & \downarrow{\scriptstyle bc} \\ R(O(n)) & \xrightarrow{CK} & K_j(C_r^*(GL(n,\mathbb{R}))) \end{array} \tag{7.2}$$

where $i \equiv n^2$, $j \equiv \frac{n^2+n}{2}$ mod 2. The vertical maps are base change maps defined in Theorem 6.9 and the horizontal maps are the Connes–Kasparov isomorphisms.

Proof. To prove the theorem we study explicit maps involved in the diagram over regular points.

The representation of a compact group is characterised by the characters of its maximal torus up to a Weyl group action. Recall that the unitary group $U(n)$ has maximal torus

$$T = \{\operatorname{diag}(z_1, \dots, z_n) : z_i \in \mathbb{C}, |z_i| = 1\}.$$

and Weyl group $W = S_n$, the permutation group. The dual of $U(n)$ is labeled by the highest weight $\sigma = (m_1, \dots, m_n) \in \mathbb{Z}^n$ of an element in $\widehat{U(n)}$:

$$\widehat{U(n)} = \widehat{T}/W \cong \{(m_1, m_2, \dots, m_n) \in \mathbb{Z}^n : m_1 \geq m_2 \geq \cdots \geq m_n\}.$$

An element $\sigma \in \widehat{T}$ is regular if the stabiliser W_σ of W at σ is trivial, i.e. $m_i \neq m_j$ for all $i \neq j$. Then the regular elements of $\widehat{U(n)}$ is indexed by

$$\{(m_1, \dots, m_n) \in \mathbb{Z}^n : m_1 > m_2 \cdots > m_n\}.$$

Recall also that the Galois group action on T is given by conjugation on $\{z_i\}$. Then over $\widehat{T}$, the nontrivial element of $\operatorname{Gal}(\mathbb{C}/\mathbb{R})$ acts by

$$(m_1, \dots, m_n) \mapsto (-m_1, \dots, -m_n).$$

Similar to the proof of Lemma 6.6, $\sigma \in \widehat{U(n)}$ is fixed by the Galois action if $m_j = -m_{n+1-j}$ for $j = 0, 1, \dots, n-1$.

The dual of $O(n)$ is similar to that of $\widehat{U(n)}$. It has maximal torus

$$T = \begin{cases} \operatorname{diag}(O(2), \dots, O(2), \pm 1), & n \text{ odd}, \\ \operatorname{diag}(O(2), \dots, O(2)), & n \text{ even}, \end{cases}$$

and Weyl group $S_{\lfloor \frac{n}{2} \rfloor} \times \mathbb{Z}_2$ (odd case) or $S_{\lfloor \frac{n}{2} \rfloor}$ (even case). Recall that all irreducible representations of $O(2)$ is labeled by $\mathbb{Z}_{\geq -1}$ where $n > 0$ corresponds the induced representation $\operatorname{Ind}_{SO(2)}^{O(2)} \chi_n$ and $-1, 0$ accommodate two irreducible subrepresentations of $\operatorname{Ind}_{SO(2)}^{O(2)} \chi_0$. Thus,

$$\widehat{O(n)} = \widehat{T}/W = \{(m_1, \dots, m_{\lfloor \frac{n}{2} \rfloor}; s) \in \mathbb{Z}_{\geq -1}^{\lfloor \frac{n}{2} \rfloor} \times \mathbb{Z}_2 : m_1 \geq \cdots \geq m_{\lfloor \frac{n}{2} \rfloor} \geq -1\}$$

when n is odd. In the even case, remove label $s \in \mathbb{Z}_2$.

The base change map $\widehat{O(n)} \to \widehat{U(n)}^{\operatorname{Gal}(\mathbb{C}/\mathbb{R})}$ is then given by

$$(m_1, \dots, m_{\lfloor \frac{n}{2} \rfloor}) \mapsto (l_1, \dots, l_{\lfloor \frac{n}{2} \rfloor}, -l_{\lfloor \frac{n}{2} \rfloor}, \dots, -l_1)$$

when n is even and by

$$(m_1, \dots, m_{\lfloor \frac{n}{2} \rfloor}; s) \mapsto (l_1, \dots, l_{\lfloor \frac{n}{2} \rfloor}, 0, -l_{\lfloor \frac{n}{2} \rfloor}, \dots, -l_1)$$

when n is odd. Here $l_k = m_k$ if $m_k \geq 0$ and $l_k = 0$ if $m_k = -1$. Note that preimage(s) of a regular element in $\widehat{U(n)}^{\mathrm{Gal}(\mathbb{C}/\mathbb{R})}$ is(are) regular in $\widehat{O(n)}$, i.e. $m_1 > m_2 > \cdots > m_{\lfloor \frac{n}{2} \rfloor} > 0$.

Because $K_0(C_r^*(G)) \cong R(G)$ for a compact group G, we can describe the base change map

$$bc : R(U(n)) \to R(O(n))$$

over regular elements $(m_1, \ldots, m_n) \in \mathbb{Z}^n$ of $\widehat{U(n)}$. The base change vanishes unless $m_i = -m_{n+1-i}$ for all i. Otherwise, it is the projection map to the first $m_{\lfloor \frac{n}{2} \rfloor}$ entries:

$$\bigoplus_{\substack{m_1 > \cdots > m_n, \\ m_i = -m_{n+1-i}}} \mathbb{Z} \to \bigoplus_{m_1 > \cdots > m_{\lfloor \frac{n}{2} \rfloor} > 0, s} \mathbb{Z}.$$

Note, $s \in \mathbb{Z}_2$ only appears in the odd case and if $m_1, \ldots, m_{\lfloor \frac{n}{2} \rfloor}$ are fixed, the labels $(m_1, \ldots, m_{\lfloor \frac{n}{2} \rfloor}; 1)$ and $(m_1, \ldots, m_{\lfloor \frac{n}{2} \rfloor}; -1)$ have the same value in the image.

Now we will describe the Connes–Kasparov isomorphism in the complex case. Similar to Example 6.3, we have

$$\widehat{GL(n, \mathbb{C})}_t \cong \bigoplus_{\widehat{M}/W} \widehat{A}/W_\sigma$$

where $\widehat{A} \cong \mathbb{R}^n$ and $\widehat{M}/W$ is labeled by

$$\{\sigma = (m_1, \ldots, m_n) \in \mathbb{Z}^n : m_1 \geq m_2 \geq \cdots \geq m_n\}.$$

and $W_\sigma = \{1\}$ if and only if $m_1 > \cdots > m_n$. The K-theory of the component of σ vanishes if W_σ is not trivial. So

$$K_0(C_r^*(GL(n, \mathbb{C}))) = \bigoplus_{m_1 > \cdots > m_n} \mathbb{Z}. \tag{7.3}$$

The isomorphism is given by translating the labels by $\rho_c = (\frac{n-1}{2}, \frac{n-3}{2}, \ldots, -\frac{n-1}{2})$, half sum of the compact positive roots of $GL(n, \mathbb{C})$:

$$\begin{array}{ccc} R(U(n)) & \longrightarrow & K_*(C_r^*(GL(n, \mathbb{C}))) \\ \bigoplus_{m_1 \geq \cdots \geq m_n} \mathbb{Z} & \mapsto & \bigoplus_{(m_1, \ldots, m_n) + \rho_c} \mathbb{Z}. \end{array}$$

It is straight forward to verify that $(k_1, \ldots, k_n) = (m_1, \ldots, m_n) + \rho_c$ satisfies $k_1 > \cdots > k_n$ and the map is an isomorphism. But note that when n is even, the labels $(m_1, \ldots, m_n) + \rho_c$ are not exactly the same as labels in (7.3) but differ by a half integer lattice, but this will not affect K-theory calculation.

The Connes–Kasparov isomorphism for the real case is more complicated to describe (not all principal series are irreducible). But as we only need to consider regular elements in $\widehat{O(n)}$, we only need to understand labels coming from a maximal cuspidal parabolic subgroup P_{max} of $GL(n,\mathbb{R})$. Denote by M_0 the group of real 2×2 matrices whose determinant is ± 1, i.e. a double cover of $SL(2,\mathbb{R})$. Then P_{max} consists of upper block diagonal matrixes where the dimension of each block is at most 2 and there is at most one 1-dimensional block. Thus, in a similar fashion as Example 6.3, we have

$$M = \begin{cases} \mathrm{diag}(M_0,\dots,M_0,\pm 1), & n \text{ odd}, \\ \mathrm{diag}(M_0,\dots,M_0), & n \text{ even}, \end{cases}$$

and $\widehat{A} \cong \mathbb{R}^{\lfloor \frac{n}{2} \rfloor}$. The discrete series of M_0 is induced by discrete series $DS_{\pm n}(n>0)$ of $SL(2,\mathbb{R})$ but the induced representation is the same for n and $-n$. So $\widehat{M}_d = \mathbb{Z}_{>0}^{\lfloor \frac{n}{2} \rfloor} \otimes \mathbb{Z}_2$ if n is odd and $\mathbb{Z}_{>0}^{\lfloor \frac{n}{2} \rfloor}$ if n is even. So when n is odd

$$K_*(C_r^*(GL(n,\mathbb{R}))) = \Big[\bigoplus_{\mathbb{Z}_{>0}^{\lfloor \frac{n}{2} \rfloor} \otimes \mathbb{Z}_2} \mathbb{Z} \Big] \bigoplus \cdots . \tag{7.4}$$

where $\cdots$ stands for generators contributed by *nonmaximal* cuspidal parabolic subgroups. In the even case we just drop $\mathbb{Z}_2$ in (7.4). In the odd case and over *regular* points of $\widehat{O(n)}$, the Connes–Kasparov isomorphism is given by translating labels by $\rho_c = (\frac{n-1}{2}, \frac{n-3}{2}, \dots, 1)$:

$$\widehat{O(n)} \longrightarrow K_*(C_r^*(GL(n,\mathbb{R}))) \tag{7.5}$$

$$\bigoplus_{m_1 > \cdots > m_{\lfloor \frac{n}{2} \rfloor} > 0, s} \mathbb{Z} \mapsto \bigoplus_{(m_1,\dots,m_{\lfloor \frac{n}{2} \rfloor}) + \rho_c; s} \mathbb{Z}. \tag{7.6}$$

In the even case, the isomorphism is also given by (7.5)–(7.6) but $s \in \mathbb{Z}_2$ in the parameter is dropped.

The last step is to describe the base change map on the right hand side of (7.2). The case $n=2$ is described in Corollary 6.10. For the general case, we do not need to reexamine base changes on L-parameters. This is because images of Weil groups $W_{\mathbb{R}}$ or $W_{\mathbb{C}}$ in $GL(n,\mathbb{C})$ are isomorphic to direct sum of irreducible representations of at most dimension 2. Thus, from Corollary 6.10, the base change is defined by (we only list those corresponding to regular points of $\widehat{U(n)}$):

$$K_i(C_r^*(GL(n,\mathbb{C}))) \longrightarrow K_j(C_r^*(GL(n,\mathbb{R}))) \tag{7.7}$$

$$\bigoplus_{\substack{m_1 > m_2 > \cdots > m_n, \\ m_i = -m_{n+1-i}}} \mathbb{Z} \mapsto \bigoplus_{(m_1,\dots,m_{\lfloor \frac{n}{2} \rfloor}; s)} \mathbb{Z}; \tag{7.8}$$

$$\bigoplus_{m_1 > m_2 > \cdots > m_n, \text{others}} \mapsto 0. \tag{7.9}$$

Again, s is removed from (7.7)–(7.9) in the even case.

In summary, over regular point $(m_1,\dots,m_n)\in\mathbb{Z}^n$ where $m_1>m_2>\cdots>m_n$, if $m_i=-m_{n+1-i}$ for all i we have a diagram

$$\begin{array}{ccc}\bigoplus_{m_1>m_2>\cdots m_n, m_i=-m_{n+1-i}}\mathbb{Z} & \xrightarrow{CK} & \bigoplus_{(m_1,\dots,m_n)+\rho_c}\mathbb{Z}\\ \downarrow bc & & \downarrow bc\\ \bigoplus_{m_1>\cdots m_{\lfloor\frac{n}{2}\rfloor};s}\mathbb{Z} & \xrightarrow{CK} & \bigoplus_{(m_1,\dots,m_{\lfloor\frac{n}{2}\rfloor})+\rho'_c;s}\mathbb{Z}\end{array}$$

where ρ_c,ρ'_c are half sums of compact positive roots of $GL(n,\mathbb{C})$, $GL(n,\mathbb{R})$ respectively. (Drop $s\in\mathbb{Z}_2$ when n is even.) The diagram commutes because the first $\lfloor\frac{n}{2}\rfloor$ entries of ρ_c and ρ'_c are identical. If $m_i=-m_{n+1-i}$ for all i is not satisfied, the diagram commutes because both base change maps vanish. The theorem is then proved. □

Example 7.8. For $G=GL(3)$, in the complex case we have

$$R(U(3))\cong\bigoplus_{m_1\geq m_2\geq m_3}\mathbb{Z},$$
$$K_1(C_r^*(GL(3,\mathbb{C})))\cong\bigoplus_{m_1>m_2>m_3}\mathbb{Z}$$

and the Connes–Kasparov isomorphism is to translate the label by $(1,0,-1)$. In the real case we have

$$R(O(3))\cong[\oplus_{m\geq-1}\mathbb{Z}]\bigoplus\mathbb{Z}_2,$$
$$K_0(C_r^*(GL(3,\mathbb{R})))\cong[\oplus_{m>0}\mathbb{Z}]\bigoplus\mathbb{Z}_2$$

where the isomorphism is to translate the first label by 1. Vertically, the base change maps vanishes if $m_1=-m_3\geq 0, m_2=0$ are not satisfied. Otherwise, the commutative diagram (7.2) for $GL(3)$ is described in Figure 1. In the figure, "·" stands for K-theory generator and $\circ$ in the up-left corner stands for the regular points in $\widehat{U(3)}$ fixed by the Galois group action, and $\circ$ elsewhere is to label the image of the regular points in $\widehat{U(3)}^{\mathrm{Gal}(\mathbb{C}/\mathbb{R})}$.

This example also shows that over singular points, Theorem 7.7 is not true. For $GL(3)$, $(0,0,0)$ is the only singular point of $\widehat{U(3)}$ fixed by $\mathrm{Gal}(\mathbb{C}/\mathbb{R})$ (labeled by $\star$ in Figure 1). Through the down-right arrows, the value in $\mathbb{Z}$ at $(0,0,0)$ should be transferred to the value at $0,1$ in $K_0(C_r^*(GL(3,\mathbb{R})))$. But through the right-down arrows , it is transferred to the value at 1 in $K_0(C_r^*(GL(3,\mathbb{R})))$, while the value at 0 in $K_0(C_r^*(GL(3,\mathbb{R})))$ is always 0. This problem could be solved if in our definition of base change on K-theory (6.12), instead of considering Galois fixed point of $\widehat{G}_{\mathbb{C}}$, we consider the crossed product of $\widehat{G}_{\mathbb{C}}$ by $\mathbb{Z}_2$. But we shall treat the issue here in a future paper.

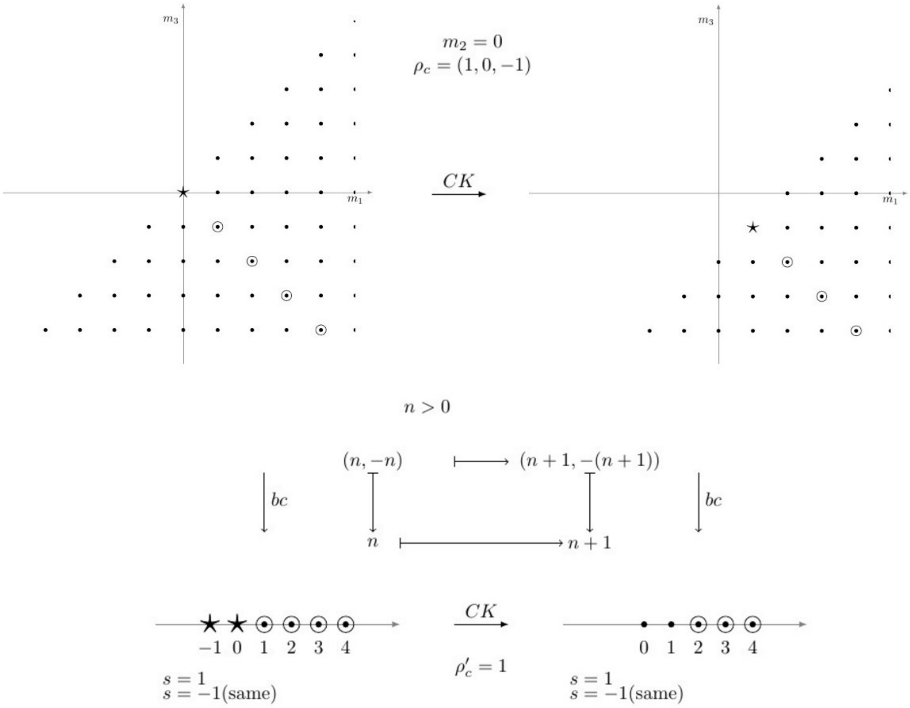

Figure 1. Commutative diagram for $GL(3)$.

7.3. On $SL(n)$: inductive method. As in the above section, we can combine the Langlands classification and the criterion of discrete series. We have:

Lemma 7.9. *The tempered representations of G are parametrised by $\sigma \in \widehat{M}$ and $\chi \in \widehat{A}$, where $\sigma = (\sigma_{11}, \dots, \sigma_{1r}; \sigma_{21}, \dots, \sigma_{2q})$, σ_{1i} is a character of $\mathbb{F}$, and σ_{2j} is a discrete series of M. The pattern is the combination of several pieces of rank 2 and rank 1.*

The above lemma serves as the building blocks of tempered representations. Using this we have the following:

Lemma 7.10. *Let $G = SL(n, \mathbb{C})$, $n \geq 3$, and P be a parabolic subgroup of G. A tempered representation of G is built up by representations of $SL(2)$ and $SL(1)$.*

Remark 7.11. We could also give the corresponding relation in terms of L-parameters due to the local Langlands correspondence of special linear groups. This is just a game of combination. The key point is to determine the corresponding R-groups R_σ. Referring the construction of Galois fixed point of $\widehat{G}_t$, we can work it out in the same way.

References

[1] M. Atiyah and W. Schmid, A geometric construction of the discrete series for semisimple Lie groups, *Invent. Math.*, **42** (1977), 1–62. Zbl 0373.22001 MR 463358

[2] P. Baum, A. Connes, and N. Higson, Classifying space for proper actions and K-theory of group C^*-algebras, in *C^*-algebras: 1943?1993 (San Antonio, TX, 1993)*, 240–291, Contemp. Math., 167, Amer. Math. Soc., Providence, RI, 1994. Zbl 0830.46061 MR 1292018

[3] P. F. Baum, N. Higson, and R. J. Plymen, Representation theory of p-adic groups: a view from operator algebras, in *The mathematical legacy of Harish-Chandra (Baltimore, MD, 1998)*, 111–149, Proc. Sympos. Pure Math., 68, Amer. Math. Soc., Providence, RI, 2000. Zbl 0982.19006 MR 1767895

[4] J. Chabert, S. Echterhoff, and R. Nest, The Connes–Kasparov conjecture for almost connected groups and for linear p-adic groups, *Publ. Math. Inst. Hautes Études Sci.*, (2003), no. 97, 239–278. Zbl 1048.46057 MR 2010742

[5] P. Clare, T. Crisp, and N. Higson, Parabolic induction and restriction via C^*-algebras and Hilbert C^*-modules, *Compos. Math.*, **152** (2016), no. 6, 1286–1318. Zbl 1346.22005 MR 3518312

[6] A. Connes, An analogue of the Thom isomorphism for crossed products of a C^*-algebra by an action of $\mathbb{R}$, *Adv. Math.*, **39** (1981), 31–55. Zbl 0461.46043 MR 605351

[7] J. Dixmier, *Les C^*-algebres et Leurs Representations*, Gauthier-Villars, Paris, 1964. Zbl 0152.32902 MR 171173

[8] M. Harris and R. Taylor, *The geometry and cohomology of certain simple Shimura varieties*, with an appendix by Vladimir G. Berkovich, Annals of Maths Studies 151, Princeton University Press, 2001. Zbl 1036.11027

[9] G. Henniart, Une preuve preuve simple des correspondance de Langlands pour $GL(n)$ use un corps p-adic, *Invent. math*, **139** (2000), 439–455. Zbl 1048.11092 MR 1738446

[10] N. Higson, The Baum-Connes conjecture, in *Proceedings of the International Congress of Mathematicians, Vol. II (Berlin, 1998)*, Extra Vol. II, 637–646, Doc. Math., 1998. Zbl 0911.46041 MR 1648112

[11] K. Hiraga and H. Saito, On L-packets of inner form of SL_n, *Mem. Amer. Math. Soc.*, **215** (2012), no. 1013, vi+97pp. Zbl 1242.22023 MR 2918491

[12] H. Jacquet and R. P. Langlands, *Automorphic Forms on $GL(2)$. Part 1*, Lecture Notes in Mathematics, 114, Springer-Verlag, Berlin-New York, 1970. Zbl 0236.12010 MR 401654

[13] G. G. Kasparov, K-theory, Group C^*-algebras, and higher signatures (conspectus), in *Novikov conjectures, index theorems and rigidity, Vol. 1 (Oberwolfach, 1993)*, 101–146, London Math. Soc. Lecture Note Ser., 226, Cambridge Univ. Press, Cambridge, 1995. Zbl 0957.58020 MR 1388299

[14] G. G. Kasparov, Equivariant KK-theory and the Novikov conjecture, *Invent. Math.*, **91** (1988), no. 1, 147–201. Zbl 0647.46053 MR 918241

[15] A. W. Knapp, *Representation theory of semisimple groups. An overview based on examples*, Princeton Mathematical Series, 36. Princeton University Press, Princeton, NJ, 1986. Zbl 0604.22001 MR 855239

[16] A. W. Knapp, Introduction to the Langlands program, in *Representation theory and automorphic forms. Proceedings of an instructional conference (Edinburgh, 1996)*, 245–302, Proc. Symp. Pure Math., 61, American Mathematical Society, 1997. Zbl 0903.11015 MR 1476501

[17] J.-P. Labesse, Introduction to endoscopy, in *Representation Theory of real reductive Lie groups*, 175–213, Contemporary Mathematics, 472, AMS, 2008. Zbl 1161.22010 MR 2454335

[18] J.-P. Labesse, Cohomologie, L-groupes et fonctorialité (French), *Compositio Math.*, **55** (1985), no. 2, 163–184. Zbl 0595.12004 MR 795713

[19] V. Lafforgue, KK-théorie bivariante pour les algèbres de Banach et conjecture de Baum–Connes, *Invent. Math.*, **149** (2002), no. 1, 1–95. Zbl 1084.19003 MR 1914617

[20] R. P. Langlands, Letter to Prof. Weil, 1967.

[21] R. P. Langlands, Problems in the theory of automorphic forms, in *Lectures in modern analysis and applications*, III, 18–61, Lecture Notes in Math., 170, Springer-Verlag, 1970. Zbl 0225.14022 MR 302614

[22] R. Langlands, On the classification of irreducible representations of real algebraic groups, in *Representation Theory and Harmonic Analysis on Semisimple Lie Groups*, 101–170, Mathematical Surveys and Monographs, 13, AMS, 1989. Zbl 0741.22009 MR 1011897

[23] J.-P. Labesse and R. Langlands, L-indistinguisabilty for $SL(2)$, *Canad. J. Math.*, **31** (1979), no. 4, 726–785. Zbl 0421.12014 MR 540902

[24] S. Mendes, Galois-fixed points and K-theory for $GL(n)$, in *Operator algebras, operator theory and applications*, 309–320, Oper. Theory Adv. Appl., 181, Birkhäuser Verlag, Basel, 2008. Zbl 1153.22019 MR 2681894

[25] S. Mendes and R. Plymen, Base change and K-theory for $GL(n)$, *J. Noncommut. Geom.*, **1** (2007), 311–331. Zbl 1149.22015 MR 2314099

[26] S. Mendes and R. Plymen, Functoriality and K-theory for $GL_n(\mathbb{R})$, *Münster J. Math.*, **10** (2017), no. 1, 39–58. Zbl 06728991 MR 3624100

[27] R. Parthasarathy, Dirac operator and the discrete series, *Ann. of Math. (2)*, **96** (1972), 1–30. Zbl 0249.22003 MR 318398

[28] M. G. Penington and R. J. Plymen, The Dirac operator and the principal series for complex semisimple Lie groups, *Journal of Functional Analysis*, **53** (1983), 269–286. Zbl 0542.22013 MR 724030

[29] R. J. Plymen, Reduced C^*-algebra for the p-adic group GL(n), *J. Functional Analysis*, **72** (1987), 1–12. Zbl 0625.46064 MR 883498

[30] J. Repka, Base change lifting and Galois invariance, *Pacific J. Math.*, **95** (1981), no. 1, 205–212. Zbl 0431.12015 MR 631670

[31] A. Wassermann, Une démonstration de la conjecture de Connes–Kasparov pour les groupers de Lie linéaries connexes réductifs, *C. R. Acad. Sci. Paris Sér. I Math.*, **304** (1987), no. 18, 559–562. Zbl 0615.22011 MR 894996

Received 14 June, 2015; revised 14 November, 2015

K. F. Chao, Department of Mathematics, College of Sciences, Shanghai University,
Shanghai 200436, China

E-mail: kchao@shu.edu.cn

H. Wang, Research Center for Operator Algebras, East China Normal University,
Shanghai 200062, China; and
School of Mathematical Sciences, University of Adelaide,
Adelaide, SA 5005, Australia

E-mail: wanghang@math.ecnu.edu.cn; hang.wang01@adelaide.edu.au