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Representability of cohomological functors over extension fields

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Abstract. We generalize a result of Orlov and Van den Bergh on the representability of a cohomological functor $H : D^b_{\mathrm{Coh}}(X) \to \underline{\mathrm{mod}}_L$ to the case where L is a field extension of the base field k of the variety X, with $\mathrm{trdeg}_k L \le 1$ or L purely transcendental of degree 2.

This result can be applied to investigate the behavior of an exact functor $F : D^b_{\mathrm{Coh}}(X) \to D^b_{\mathrm{Coh}}(Y)$ with X and Y smooth projective varieties and $\dim Y \le 1$ or Y a rational surface. We show that for any such F there exists a "generic kernel" A in $D^b_{\mathrm{Coh}}(X \times Y)$, such that F is isomorphic to the Fourier–Mukai transform with kernel A after composing both with the pullback to the generic point of Y.

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1. Introduction

Let X be a smooth projective variety over an algebraically closed field k. In this paper we will generalize a result of Orlov and Van den Bergh [4, Lemma 2.14] on the representability of a functor $H : D^b_{\mathrm{Coh}}(X) \to \underline{\mathrm{mod}}_k$ to the case of an extension field $k \subset L$:

Theorem 1.1. *Let X be a smooth projective variety over a field k. Let L be a finitely generated separable field extension of k with* $\mathrm{trdeg}_k L \le 1$*, or a purely transcendental field extension of transcendence degree 2 over k. Consider a contravariant, cohomological, finite type functor*

$$H : D^b_{\mathrm{Coh}}(X) \to \underline{\mathrm{mod}}_L$$

Then H is representable by an object $E \in D^b_{\mathrm{Coh}}(X_L)$, i.e. there exists E such that for every $C \in D^b_{\mathrm{Coh}}(X)$ we have

$$H(C) = \mathrm{Mor}_{D^b_{\mathrm{Coh}}(X_L)}(j^*C, E)$$

where $j^* : X_L \to X$ is the base change morphism.

An interesting example of a functor as in Theorem 1.1 can be obtained from an exact functor $F : D^b_{\mathrm{Coh}}(X) \to D^b_{\mathrm{Coh}}(Y)$ between the bounded derived categories of two smooth projective varieties X and Y, where $\dim Y \leq 1$ or Y is a rational surface. To produce a functor as in the above theorem, we compose F with the pullback to the generic point of Y, take cohomology, then dualize to get a contravariant functor:

$$D^b_{\mathrm{Coh}}(X) \xrightarrow{F} D^b_{\mathrm{Coh}}(Y) \xrightarrow{i^*} D^b_{\mathrm{Coh}}(\eta) \xrightarrow{H^0} \underline{\mathrm{mod}}_{K(Y)} \xrightarrow{D} \underline{\mathrm{mod}}_{K(Y)}, \qquad H = D \circ H^0 \circ i^* \circ F$$

Theorem 1.1 will thus allow us to tackle the question of whether a functor between the bounded derived categories of two smooth projective varieties is representable by a Fourier–Mukai transform. When $\dim Y \leq 1$ or Y is a rational surface we can answer positively to the question above after restricting to the generic point of Y:

Theorem 1.2. *Let X, Y be smooth projective varieties over a field k, where $\dim Y \leq 1$ or Y is a rational surface. Consider a covariant exact functor*

$$F : D^b_{\mathrm{Coh}}(X) \to D^b_{\mathrm{Coh}}(Y)$$

let $i : \eta \to Y$ the inclusion of the generic point of Y. Then there exists an object $A \in D^b_{\mathrm{Coh}}(X \times Y)$ such that

$$i^* \circ F = i^* \circ \Phi_A,$$

where $\Phi_A(\cdot) := Rp_{2*}(A \overset{L}{\otimes} Lp_1^*(\cdot))$ is the Fourier–Mukai trasform with kernel A and $p_1 : X \times Y \to X$, $p_2 : X \times Y \to Y$ are the projection morphisms.

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2. The base change category

In what follows, an abelian category $\mathcal{A}$ does not automatically have any limits or colimits apart from the finite ones.

Given a field K, we will denote with $\underline{\mathrm{mod}}_K$ the category of finite dimensional K-vector spaces, whereas $\underline{\mathrm{Mod}}_K$ will denote the category of possibly infinite-dimensional K-vector spaces. $D(\mathcal{A})$ will denote the derived category of an abelian category $\mathcal{A}$.

Given an R-linear abelian category $\mathcal{A}$ and an inclusion of rings $R \hookrightarrow S$, we can define the base change category $\mathcal{A}_S$ as in [6, §4]:

Definition 2.1. The category $\mathcal{A}_S$ is given by pairs (C, ρ_C) where $C \in \mathrm{Ob}(\mathcal{A})$ and $\rho_C : S \to \mathrm{Hom}_{\mathcal{A}}(C, C)$ is an R-algebra map such that the composition $R \to S \to \mathrm{Hom}_{\mathcal{A}}(C, C)$ gives back the R-algebra structure on $\mathcal{A}$. The morphisms in $\mathcal{A}_S$ are the morphisms in $\mathcal{A}$ compatible with the S-structure.

Definition 2.2. For each element $C \in \mathcal{A}$, the functor

$$C \otimes_R - : \underline{\mathrm{mod}}(R) \to \mathcal{A}$$

is the unique finite colimit preserving functor with $C \otimes R = C$.

This gives for each finitely presented R-algebra S a functor

$$- \otimes S : \mathcal{A} \to \mathcal{A}_S$$

to the base change category $\mathcal{A}_S$.

Proposition 2.3 ([6, Proposition 4.3]). *The functor $- \otimes S$ is left adjoint to the forgetful functor*

$$\begin{aligned} \mathrm{Forget} : \mathcal{A}_S &\to \mathcal{A} \\ (C, \rho_C) &\mapsto C \end{aligned}$$

Whenever the context is clear, given an object $B \in \mathcal{A}_S$, we will still denote by B the corresponding object of $\mathcal{A}$ obtained via the forgetful functor.

For the purposes of this discussion we will need a more general setting for base change — specifically, we need to be able to talk about base change for a bigger category of rings and not just the ones that are finitely presented over the base. Let us extend Definition 2.2 as follows:

Definition 2.4. Let $\mathcal{A}$ be an R-linear abelian category satisfying AB5. Using the fact that any R-module is the filtered colimit of finitely presented R-modules, we can extend definition 2.2 to the general case of

$$- \otimes S : \mathcal{A} \to \mathcal{A}_S$$

for any R-algebra S.

The notion of base change category can be extended to the case of the derived category $D(\mathcal{A})$ of an abelian R-linear category $\mathcal{A}$ in the obvious way:

Definition 2.5. Given an inclusion of rings $R \hookrightarrow S$, the category $D(\mathcal{A})_S$ is given by pairs (C, ρ_C) where $C \in \mathrm{Ob}(D(\mathcal{A}))$ and $\rho_C : S \to \mathrm{Hom}_{D(\mathcal{A})}(C, C)$ is an R-algebra map such that the composition $R \to S \to \mathrm{Hom}_{D(\mathcal{A})}(C, C)$ gives back the R-algebra structure on $D(\mathcal{A})$. The morphisms in $D(\mathcal{A})_S$ are the morphisms in $D(\mathcal{A})$ compatible with the S-structure.

Again, we have a notion of tensor product:

Definition 2.6. Let R be a ring, let $\mathcal{A}$ be an R-linear abelian category satisfying AB5, and let $C^\bullet$ be a complex of objects in $\mathcal{A}$:

$$C^\bullet = \cdots \to C^{i-1} \xrightarrow{d^{i-1}} C^i \xrightarrow{d^i} C^{i+1} \to \cdots$$

Let S be a ring, with a map $R \hookrightarrow S$. Then we can define $C^\bullet \otimes S$, as an object of $D(\mathcal{A}_S)$, as

$$C^\bullet \otimes S = \cdots \to C^{i-1} \otimes S \xrightarrow{d^{i-1}\otimes 1} C^i \otimes S \xrightarrow{d^i \otimes 1} C^{i+1} \otimes S \to \cdots$$

The complex $C^\bullet \otimes S$ can also be considered as an object of $D(\mathcal{A})_S$.

Remark 2.7. Suppose that $\mathcal{A}$ is a k-linear abelian category satisfying AB5 and $k \subset K$ is an extension of fields. In the situation of Definitions 2.4 and 2.6, similarly to the case of 2.3, it is easy to show that again tensoring with K is left adjoint to the forgetful functor

- as a functor $\mathcal{A} \to \mathcal{A}_K$;
- as a functor $D(\mathcal{A}) \to D(\mathcal{A}_K)$;
- as a functor $D(\mathcal{A}) \to D(\mathcal{A})_K$.

Remark 2.8. Let R be a ring, let $\mathcal{A}$ be an R-linear abelian category satisfying AB5, and let $C^\bullet$ be a complex of objects in $\mathcal{A}$,

$$C^\bullet = \cdots \to C^{i-1} \xrightarrow{d^{i-1}} C^i \xrightarrow{d^i} C^{i+1} \to \cdots$$

Let $S \subset R$ a multiplicative system. In this case $C^\bullet \otimes_R S^{-1}R$, as an object of $D(\mathcal{A})$, is the same as

$$\cdots \to \operatorname*{colim}_{f\in S} f^{-1}C^{i-1} \xrightarrow{d^{i-1}} \operatorname*{colim}_{f\in S} f^{-1}C^i \xrightarrow{d^i} \operatorname*{colim}_{f\in S} f^{-1}C^{i+1} \to \cdots$$

where $\operatorname{colim}_{f\in S} f^{-1}C^i$ is obtained by taking for every $f \in S$ a copy of C^i and as morphisms only the maps

$$f^{-1}C^i \longrightarrow (fg)^{-1}C^i$$

given by multiplication by $g : C^i \to C^i$.

Lemma 2.9. *In the situation of the remark above, if for every element $f \in S$ the multiplication by f is a quasi-isomorphism of $C^\bullet$, then the map*

$$C^\bullet \to C^\bullet \otimes_R S^{-1}R$$

is a quasi-isomorphism in $D(\mathcal{A})$.

Proof. Since taking cohomology commutes with directed colimits we have

$$H^i(C^\bullet \otimes_R S^{-1}R) = \operatorname*{colim}_{f \in S} f^{-1} H^i(C^\bullet)$$

but since multiplication by any $g \in S$ is a quasi-isomorphism we get

$$f^{-1} H^i(C^\bullet) \xrightarrow[g]{\cong} (fg)^{-1} H^i(C^\bullet)$$

hence the cohomology of $C^\bullet \otimes_R S^{-1}R$ consists of only one copy of $H^i(C^\bullet)$, and the map $C^\bullet \to C^\bullet \otimes_R S^{-1}R$ is a quasi-isomorphism. □

3. A result on base change for derived categories

The purpose of this section is to analyze the functor $D(\mathcal{A}_K) \to D(\mathcal{A})_K$ that sends an object in $D(\mathcal{A}_K)$ to the same object considered as an object of $D(\mathcal{A})$, together with its K-action. Specifically, we will prove the following:

Theorem 3.1. *Let $\mathcal{A}$ be a k-linear abelian category satisfying AB5, where k is a field. Let $K = k(T)$ or $K = k(T, T')$. Then the functor*

$$\begin{aligned} D(\mathcal{A}_K) &\to D(\mathcal{A})_K \\ C^\bullet &\mapsto (C^\bullet, \rho_C) \end{aligned}$$

is essentially surjective, where $\rho_C : K \to \operatorname{Aut}(C^\bullet)$ is the obvious map.

Moreover, if L is a finite separable extension of $K = k(T)$ with $L = K(\alpha) = K[T]/P(T)$ then we can lift an object $(C^\bullet, \rho_C) \in D(\mathcal{A})_L$ to an object $N^\bullet$ of $D(\mathcal{A}_K)$ endowed with a map $\psi_\alpha \in \operatorname{End}(N^\bullet)$ such that $P(\psi_\alpha)$ is zero on all cohomology groups, and the action of ψ_α on $N^\bullet$ corresponds to the action of α on $C^\bullet$.

A stronger results for the case of a finite extension K/k was obtained in [8]. In this case, there is actually an equivalence $D(\mathcal{A}_K) \to D(\mathcal{A})_K$.

The proof of this theorem will be carried out in several steps. First we will notice that, in the purely transcendental case $K = k(T, T')$, this comes down to lifting the actions of the two variables T and T' on a complex $C^\bullet \in D(\mathcal{A})_K$, given by $\rho_C(T)$ and $\rho_C(T')$, to actions coming from morphisms in $\mathcal{A}$ that commute with each other.

Then in Lemma 3.3 we will tackle the case of one variable and obtain a complex $M^\bullet \in D(\mathcal{A}_{k[T]})$ with a quasi-isomorphism to $C^\bullet$ as objects of $D(\mathcal{A})$, and such that the T-actions on $M^\bullet$ and $C^\bullet$ cohincide. At this point, since $\rho_C(T)$ is an automorphism of $C^\bullet$, tensoring with $k(T)$ will give us a complex in $D(\mathcal{A}_{k(T)})$ which is still quasi-isomorphic to $C^\bullet$.

A similar process can be repeated twice, as we will show in Lemma 3.4.

Lemma 3.2. *Let $e^n D(\mathcal{A})$ be the category whose*

(1) *Objects are pairs*

$$(C, \varphi_1, \dots, \varphi_n)$$

where $E \in \mathrm{Ob}(D(\mathcal{A}))$, $\varphi_i \in \mathrm{End}_{D(\mathcal{A})}(C)$ for all i, and φ_i commutes with φ_j for all i, j;

(2) *Morphisms*

$$a : (C, \varphi_1, \dots, \varphi_n) \to (C', \varphi'_1, \dots, \varphi'_n)$$

are elements $a \in \mathrm{Hom}_{D(\mathcal{A})}(C, C')$ such that $a \circ \varphi_i = \varphi'_i \circ a$.

Consider the full subcategory $e^n D'(\mathcal{A}) \subset e^n D(\mathcal{A})$ whose objects consist of those pairs $(C, \varphi_1, \dots, \varphi_n)$ such that for every nonzero $f \in k[T_1, \dots, T_n]$ the map

$$f(\varphi_1, \dots, \varphi_n) : C \to C$$

is an isomorphism in $D(\mathcal{A})$.

The category $D(\mathcal{A})_{k(T_1,\dots,T_n)}$ is equivalent to the category $e^n D'(\mathcal{A})$. The equivalence is given by the functor

$$D(\mathcal{A})_{k(T_1,\dots,T_n)} \longrightarrow e^n D'(\mathcal{A}), \quad (C, \rho_C) \longmapsto (C, \rho_C(T_1), \dots, \rho_C(T_n)).$$

Proof. The equivalence is given by the inverse functor

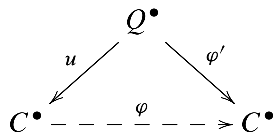
$$\begin{aligned} e^n D'(\mathcal{A}) &\longrightarrow D(\mathcal{A})_{k(T_1,\dots,T_n)} \\ (C, \varphi_1, \dots, \varphi_n) &\mapsto \left(C, \begin{array}{rcl} \rho_C : k(T_1, \dots, T_n) & \to & \mathrm{Aut} \\ T_i & \mapsto & \varphi_i \end{array} \right). \end{aligned}$$

□

Lemma 3.3. *Let $\mathcal{A}$ be a k-linear abelian category satisfying AB5, where k is a field. Let $C^\bullet$ be a complex in $D(\mathcal{A})$. Let $\varphi \in \mathrm{Hom}_{D(\mathcal{A})}(C^\bullet, C^\bullet)$. Then there exists a complex $M^\bullet \in D(\mathcal{A}_{k[T]})$ and a quasi-isomorphism $C^\bullet \to M^\bullet$ as objects of $D(\mathcal{A})$ such that the action of multiplication by T on $M^\bullet$ corresponds to the action of multiplication by φ on $C^\bullet$.*

Note that when $\mathcal{A}$ is a Grothendieck category, the same result can be achieved by considering the morphism corresponding to φ on a K-injective replacement of $C^\bullet$ (which exists by [5, Prop. 3.2]) and defining the action of T accordingly. However, in what follows we will need this specific form for the complex $M^\bullet$.

Proof. The map $\varphi : C^\bullet \to C^\bullet$ in $D(\mathcal{A})$ corresponds to a diagram of complexes in $D(\mathcal{A})$



where u is a quasi-isomorphism.

Let $C^\bullet[T] = C^\bullet \otimes_k k[T]$ as a complex in $D(\mathcal{A}_{k[T]})$. Consider the morphism

$$\varphi \otimes 1 - 1 \otimes T : C^\bullet[T] \to C^\bullet[T]$$

in $D(\mathcal{A}_{k[T]})$. This can be represented by actual maps of complexes

$$\begin{array}{ccc} & Q^\bullet[T] & \\ {}^{u\otimes 1}\swarrow & & \searrow^{\varphi'\otimes 1 - u\otimes T} \\ C^\bullet[T] & \overset{\varphi\otimes 1 - 1\otimes T}{- - - - - - - - \to} & C^\bullet[T] \end{array}$$

The map $\varphi' \otimes 1 - u \otimes T$ is injective on all cohomology objects: to prove this we need to show that

$$\varphi' \otimes 1 - u \otimes T : H^r(Q^\bullet[T]) \to H^r(C^\bullet[T])$$

is injective for every r.

Let $\alpha \in H^r(Q^\bullet[T])$, $\alpha \neq 0$, then

$$\alpha = \sum_{i=0}^{n} \alpha_i T^i$$

where all of the α_i are different from zero in $H^r(Q^\bullet)$. If

$$0 = (\varphi' \otimes 1 - u \otimes T)\alpha = \sum_{i=0}^{n} \varphi'(\alpha_i) T^i - \sum_{i=0}^{n} u(\alpha_i) T^{i+1}$$

then the only term of degree $n+1$ in T, $u(\alpha_n)T^{n+1}$, must be zero in $H^r(C^\bullet)$, hence $u(\alpha_n) = 0$, hence $\alpha_n = 0$ since u is a quasi-isomorphism. This contradicts our assumption that $\alpha_i \neq 0\ \forall i$, and so this proves injectivity.

Now set

$$M^\bullet = \mathrm{Cone}(Q^\bullet[T] \xrightarrow{\varphi'\otimes 1 - u\otimes T} C^\bullet[T])$$

Then we have a distinguished triangle

$$Q^\bullet[T] \xrightarrow{\varphi'\otimes 1 - u\otimes T} C^\bullet[T] \longrightarrow M^\bullet \longrightarrow (Q^\bullet[T])[1] \tag{3.1}$$

and by injectivity of the map $\varphi' \otimes 1 - u \otimes T$ on the cohomology objects we get a short exact sequence in cohomology

$$0 \to H^r(Q^\bullet[T]) \xrightarrow{\varphi'\otimes 1 - u\otimes T} H^r(C^\bullet[T]) \longrightarrow H^r(M^\bullet) \to 0$$

hence we get

$$H^r(M^\bullet) = \mathrm{Coker}(H^r(Q^\bullet[T]) \xrightarrow{\varphi'\otimes 1 - u\otimes T} H^r(C^\bullet[T]))$$

for any r.

Now consider the composition

$$C^\bullet \longrightarrow C^\bullet[T] \xrightarrow{c} M^\bullet.$$

This map is a quasi-isomorphism; to prove this we just need to show that under the map above,

$$H^r(C^\bullet) \cong \operatorname{Coker}(H^r(Q^\bullet[T]) \xrightarrow{\varphi' \otimes 1 - u \otimes T} H^r(C^\bullet[T]))$$

for every r.

Proceed as follows: first of all, considered as a sub-object of $H^r(C^\bullet[T])$ via the obvious map $C^\bullet \to C^\bullet[T]$, $H^r(C^\bullet)$ is not in the image of $\varphi' \otimes 1 - u \otimes T$, since, for any element $\alpha = \sum_{i=1}^n \alpha_i T^i$ of $H^r(Q^\bullet[T])$, its image $\sum_{i=1}^n \varphi(\alpha_i)T^i - \sum_{i=0}^n u(\alpha_i)T^{i+1}$ is either zero or has a nonzero term of positive degree. To prove that any term of positive degree $\beta = \sum_{i=1}^n \beta_i T^i$ is in the image up to an element of degree zero, notice that it can be written as an element of lower degree plus an element of the image as follows:

$$\begin{aligned}
\sum_{i=0}^n \beta_i T^i &= \sum_{i=0}^n \beta_i T^i - (\varphi' \otimes 1 - u \otimes T)(u^{-1}(\beta_n)T^{n-1}) \\
&\qquad\qquad + (\varphi' \otimes 1 - u \otimes T)(u^{-1}(\beta_n)T^{n-1}) \\
&= \sum_{i=0}^n \beta_i T^i - \varphi'(u^{-1}(\beta_n))T^{n-1} \\
&\qquad\qquad + \beta_n T^n + (\varphi' \otimes 1 - u \otimes T)(u^{-1}(\beta_n)T^{n-1}) \\
&= \sum_{i=0}^{n-1} \beta_i T^i - \varphi'(u^{-1}(\beta_n))T^{n-1} + (\varphi' \otimes 1 - u \otimes T)(u^{-1}(\beta_n)T^{n-1})
\end{aligned}$$

Hence we found a complex $M^\bullet \in D(\mathcal{A}_{k[T]})$ which is quasi-isomorphic to $C^\bullet$ as an object of $D(\mathcal{A})$; moreover the action of multiplication by φ on $C^\bullet$ corresponds to the action by multiplication by T on $M^\bullet$, because the following diagram is commutative in $D(\mathcal{A})$:

$$\begin{array}{ccccc}
C^\bullet & \longrightarrow & C^\bullet[T] & \xrightarrow{c} & M^\bullet \\
\downarrow{\scriptstyle \varphi} & & \downarrow{\scriptstyle \varphi\otimes 1} & & \downarrow{\scriptstyle T} \\
C^\bullet & \longrightarrow & C^\bullet[T] & \xrightarrow{c} & M^\bullet
\end{array}$$

this follows from the fact that

$$\begin{aligned}
c \circ (1 \otimes T) - (\varphi \otimes 1) \circ c &= (1 \otimes T) \circ c - (\varphi \otimes 1) \circ c \\
&= (1 \otimes T - \varphi \otimes 1) \circ c = 0
\end{aligned}$$

since those are two consecutive maps in a triangle. □

Lemma 3.4. *Let $\mathcal{A}$ be a k-linear abelian category satisfying AB5, where k is a field. Let $C^\bullet$ be a complex in $D(\mathcal{A})$.*

Let $\varphi \in \mathrm{Hom}_{D(\mathcal{A})}(C^\bullet, C^\bullet)$ such that $f(\varphi)$ is an isomorphism for all $f \in k[T]$ monic. Then there exists a complex $N^\bullet \in D(\mathcal{A}_{k(T)})$ and a quasi-isomorphism $C^\bullet \to N^\bullet$ as objects of $D(\mathcal{A})$ such that the action of multiplication by T on $N^\bullet$ corresponds to the action by multiplication by φ on $C^\bullet$.

Likewise, let $\varphi, \psi \in \mathrm{Hom}_{D(\mathcal{A})}(C^\bullet, C^\bullet)$ such that φ and ψ commute with each other and such that $f(\varphi, \psi)$ is a quasi-isomorphisms for all $f \in k[T, T']$ nonzero. Then there exists a complex $N^\bullet \in D(\mathcal{A}_{k(T,T')})$ and a quasi-isomorphism $j : C^\bullet \to N^\bullet$ as objects of $D(\mathcal{A})$ such that the action of multiplication by T (resp. T') on $N^\bullet$ corresponds to the action by multiplication by φ (resp. ψ) on $C^\bullet$.

Proof. By Lemma 3.3 we can find a complex $M^\bullet \in \mathcal{A}_{k[T]}$ and a quasi-isomorphism $j : C^\bullet \to M^\bullet$ as objects of $D(\mathcal{A})$ such that the action of multiplication by T on $M^\bullet$ corresponds to the action by multiplication by φ on $C^\bullet$. This implies that multiplication by $f(T)$ gives a quasi-isomorphism of $M^\bullet$ for all f monic.

Now let $N^\bullet := M^\bullet \otimes_{k[T]} k(T)$ as in Definition 2.6 above. This is a complex in $D(\mathcal{A}_{k(T)})$ and it is quasi-isomorphic to $C^\bullet$ as objects of $D(\mathcal{A})$, by Lemma 2.9. The action of φ on $C^\bullet$ corresponds to the action of T on $N^\bullet$.

For the second case, again by Lemma 3.3 we can find a complex $M^\bullet \in \mathcal{A}_{k[T]}$ and a quasi-isomorphism $j : C^\bullet \to M^\bullet$ as objects of $D(\mathcal{A})$ such that the action of multiplication by T on $M^\bullet$ corresponds to the action by multiplication by φ on $C^\bullet$.

Moreover, we have an exact triangle

$$C^\bullet[T] \xrightarrow{\varphi\otimes 1 - 1\otimes T} C^\bullet[T] \longrightarrow M^\bullet$$

in $D(\mathcal{A}_{k[T]})$, see (3.1).

Then, since φ and ψ commute with each other, we get a diagram in $D(\mathcal{A}_{k[T]})$:

$$\begin{array}{ccc} C^\bullet[T] & \xrightarrow{\varphi\otimes 1 - 1\otimes T} & C^\bullet[T] \\ \downarrow{\scriptstyle \psi\otimes 1} & & \downarrow{\scriptstyle \psi\otimes 1} \\ C^\bullet[T] & \xrightarrow{\varphi\otimes 1 - 1\otimes T} & C^\bullet[T] \end{array}$$

This diagram is commutative: this follows from the fact that $\varphi \circ \psi = \psi \circ \varphi$ in $D(\mathcal{A})$, hence $\varphi\psi \otimes 1 = \psi\varphi \otimes 1$ in $D(\mathcal{A}_{k[T]})$. Therefore we can find a map $\tilde{\psi}$ on $M^\bullet$ so that the following diagram commutes:

$$\begin{array}{ccccccc} C^\bullet[T] & \xrightarrow{\varphi\otimes 1 - 1\otimes T} & C^\bullet[T] & \longrightarrow & M^\bullet & \longrightarrow & (C^\bullet[T])[1] \\ \downarrow{\scriptstyle \psi\otimes 1} & & \downarrow{\scriptstyle \psi\otimes 1} & & \downarrow{\scriptstyle \tilde{\psi}} & & \downarrow{\scriptstyle \psi\otimes 1} \\ C^\bullet[T] & \xrightarrow{\varphi\otimes 1 - 1\otimes T} & C^\bullet[T] & \longrightarrow & M^\bullet & \longrightarrow & (C^\bullet[T])[1] \end{array}$$

it follows that the action of $\tilde{\psi}$ on $M^\bullet$ is the same as the action of ψ on $C^\bullet$, thanks to the commutativity of

$$\begin{array}{ccccc} C^\bullet & \longrightarrow & C^\bullet[T] & \longrightarrow & M^\bullet \\ \downarrow \psi & & \downarrow \psi\otimes 1 & & \downarrow \tilde{\psi} \\ C^\bullet & \longrightarrow & C^\bullet[T] & \longrightarrow & M^\bullet \end{array}$$

taking into account the fact that, as we mentioned already, the composition of the two horizontal maps is a quasi-isomorphism. As before we can then construct $P^\bullet = M^\bullet \otimes_{k[T]} k(T)$, which is quasi-isomorphic to $M^\bullet$ and hence we get a corresponding map $\tilde{\psi} : P^\bullet \to P^\bullet$.

So we are in the following situation: we have a complex $P^\bullet \in D(\mathcal{A}_{k(T)})$ and a map $\tilde{\psi} : P^\bullet \to P^\bullet$ so that $f(\tilde{\psi})$ is a quasi-isomorphism for all $f \in k(T)[T']$ monic. By Lemma 3.3 again, we get a complex $Q^\bullet \in D((\mathcal{A}_{k(T)})_{k(T)[T']}) = D(\mathcal{A}_{k(T)[T']})$ which is quasi-isomorphic to $P^\bullet$.

Then define

$$N^\bullet := Q^\bullet \otimes_{k(T)[T']} k(T, T').$$

By Lemma 2.9, since $f(T, \psi)$ is a quasi-isomorphisms for all nonzero $f \in k(T)[T']$, the complex $N^\bullet \in D(\mathcal{A}_{k(T,T')})$ is quasi isomorphic to $Q^\bullet$ as objects of $D(\mathcal{A}_{k(T)[T']})$ hence it is quasi-isomorphic to $C^\bullet$ as objects of $D(\mathcal{A})$. The action of φ and ψ correspond to the action of T and T' respectively. □

The last thing we need to do is to tackle the case of a general separable field extension of transcendence degree one, corresponding to the last statement of Theorem 3.1:

Lemma 3.5. *Let $\mathcal{A}$ be a k-linear abelian category satisfying AB5, where k is a field. Let $C^\bullet$ be a complex in $D(\mathcal{A})$. Let $\varphi, \psi \in \mathrm{Hom}_{D(\mathcal{A})}(C^\bullet, C^\bullet)$ such that φ and ψ commute with each other, and such that $f(\varphi)$ is a quasi-isomorphisms for all $f \in k[T]$ monic and there exists an irreducible $P \in k[T, T']$ with $P(\varphi, \psi) = 0$.*

Then there exists a complex $N^\bullet \in D(\mathcal{A}_{k(T)})$ and a quasi-isomorphism $j : C^\bullet \to N^\bullet$ as objects of $D(\mathcal{A})$ such that the action of multiplication by T on $N^\bullet$ corresponds to the action by multiplication by φ on $C^\bullet$. Moreover there is a morphism $\tilde{\psi} \in \mathrm{End}(N^\bullet)$ such that the action of ψ on $C^\bullet$ corresponds to the action of $\tilde{\psi}$ on $N^\bullet$ and $P(T, \tilde{\psi})$ induces the zero map on all cohomology groups of $N^\bullet$.

Proof. By Lemma 3.3 we can find a complex $M^\bullet \in \mathcal{A}_{k[T]}$ and a quasi-isomorphism $j : C^\bullet \to M^\bullet$ as objects of $D(\mathcal{A})$ such that the action of multiplication by T on $M^\bullet$ corresponds to the action by multiplication by φ on $C^\bullet$.

Moreover, we have an exact triangle

$$C^\bullet[T] \xrightarrow{\varphi\otimes 1 - 1\otimes T} C^\bullet[T] \longrightarrow M^\bullet$$

in $D(\mathcal{A}_{k[T]})$, see (3.1).

Then, since φ and ψ commute with each other, we get a commutative diagram in $D(\mathcal{A}_{k[T]})$

$$\begin{array}{ccc} C^\bullet[T] & \xrightarrow{\varphi\otimes 1 - 1\otimes T} & C^\bullet[T] \\ \downarrow{\scriptstyle \psi\otimes 1} & & \downarrow{\scriptstyle \psi\otimes 1} \\ C^\bullet[T] & \xrightarrow{\varphi\otimes 1 - 1\otimes T} & C^\bullet[T] \end{array}$$

Therefore we can find a map $\tilde{\psi}$ on $M^\bullet$ so that the following diagram commutes:

$$\begin{array}{ccccccc} C^\bullet[T] & \xrightarrow{\varphi\otimes 1 - 1\otimes T} & C^\bullet[T] & \longrightarrow & M^\bullet & \longrightarrow & (C^\bullet[T])[1] \\ \downarrow{\scriptstyle \psi\otimes 1} & & \downarrow{\scriptstyle \psi\otimes 1} & & \downarrow{\scriptstyle \tilde{\psi}} & & \downarrow{\scriptstyle \psi\otimes 1} \\ C^\bullet[T] & \xrightarrow{\varphi\otimes 1 - 1\otimes T} & C^\bullet[T] & \longrightarrow & M^\bullet & \longrightarrow & (C^\bullet[T])[1] \end{array}$$

Since $P(T, \psi) = 0$, we obtain that $P(\varphi \otimes 1, \psi \otimes 1) = 0$ in $D(\mathcal{A}_{k[T]})$, hence $P(T, \psi \otimes 1)$ is zero on $C[T]$.

As before we can construct $N^\bullet = M^\bullet \otimes_{k[T]} k(T)$, which is quasi-isomorphic to $M^\bullet$ and hence we get a corresponding map $\tilde{\psi} : N^\bullet \to N^\bullet$ and the action of ψ on $C^\bullet$ corresponds to the action of $\tilde{\psi}$ on $N^\bullet$.

Finally, since $P(T, \psi \otimes 1)$ is zero on $C[T]$, it follows that $P(T, \tilde{\psi}) = 0$ induces the zero map on all cohomology of $M^\bullet$ and hence of $N^\bullet$. □

We are now ready to prove Theorem 3.1:

Proof of Theorem 3.1. By Lemma 3.2, we just need to show that the functors

$$\begin{aligned} D(\mathcal{A}_K) &\to e^1 D'(\mathcal{A}) \\ C^\bullet &\mapsto (C^\bullet, \cdot T) \end{aligned}$$

and

$$\begin{aligned} D(\mathcal{A}_K) &\to e^2 D'(\mathcal{A}) \\ C^\bullet &\mapsto (C^\bullet, \cdot T, \cdot T') \end{aligned}$$

are essentially surjective.

Let $(E, \varphi) \in e^1 D'(\mathcal{A})$. Then by Lemma 3.4 there exists $N^\bullet \in \mathcal{A}_{k(T)}$ such that N is quasi isomorphic to E and the action of φ on $E^\bullet$ corresponds to the action of T on $N^\bullet$. This proves the case $i = 1$.

Similarly, let $(E, \varphi, \varphi') \in e^2 D'(\mathcal{A})$. Then by Lemma 3.4 there exists $N^\bullet \in \mathcal{A}_{k(T,T')}$ such that N is quasi isomorphic to E and the action of φ and φ' on $E^\bullet$ correspond to the action of T and T' respectively on $N^\bullet$. This proves the case $i = 2$.

The last part follows from Lemma 3.5 by setting $\psi_\alpha := \tilde{\psi}$. □

Let us now apply this theorem to the case $\mathcal{A} = \mathrm{QCoh}(X)$, where X is a quasi-compact, separated scheme over a field k. This is possible since $\mathrm{QCoh}(X)$ satisfies AB5. Moreover, note that in this case we have an equivalence $D_{\mathrm{Qcoh}}(X) \cong D(\mathrm{Qcoh}(X))$. As a preliminary step, we will prove the following technical lemma:

Lemma 3.6. *Let $k \subset K$ be a field extension, X a quasi-compact and separated scheme. Let $X_K \xrightarrow{j} X$ the base change morphism. Then there is an equivalence of categories*

$$D_{\mathrm{QCoh}}(X_K) \xrightarrow{\Psi} D(\mathrm{QCoh}(X)_K)$$

under this equivalence, the functors

$$Lj^*, \cdot \otimes K : D_{\mathrm{QCoh}}(X) \to D(\mathrm{QCoh}(X_K))$$

and

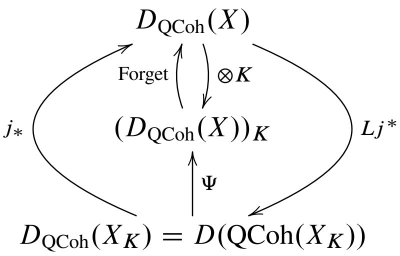
$$Rj_*, \mathrm{Forget} : D_{\mathrm{QCoh}}(X_K) \to D(\mathrm{QCoh}(X))$$

coincide.

In other words,

$$Rj_* = \mathrm{Forget} \circ \Psi : D(\mathrm{QCoh}(X)_K) \to D_{\mathrm{QCoh}}(X)$$
$$\Psi \circ Lj^* = - \otimes K : D_{\mathrm{QCoh}}(X) \to (D_{\mathrm{QCoh}}(X))_K.$$

This is summarized in the following diagram:



Proof. There is an equivalence of categories induced by j_* between quasi-coherent $\mathcal{O}_{X_K}$-modules and quasi-coherent $j_*\mathcal{O}_{X_K}$-modules on X. But $j_*\mathcal{O}_{X_K} = \mathcal{O}_X \otimes K$ and an $(\mathcal{O}_X \otimes K)$-module is the same thing as an $\mathcal{O}_X$-module with a K-structure which is compatible with its k-structure.

Hence we get an equivalence

$$\Psi : \mathrm{QCoh}(X_K) \to \mathrm{QCoh}(X)_K$$
$$C \mapsto (j_*C, \rho_C)$$

where ρ_C is the composition $K \to \mathcal{O}_X \otimes K \to \mathrm{End}(j_*C)$.

Under this equivalence, the two functors j_* and "Forget" coincide; moreover, always under the same equivalence, both j^* and $- \otimes K$ are left adjoint to j_*, hence they also coincide.

Thus all of this also holds for the corresponding derived categories; hence the statement follows since $D_{\mathrm{QCoh}}(X) = D(\mathrm{QCoh}(X))$ for X quasi compact and separated. □

Corollary 3.7. *Let X be a quasi compact, separated scheme over a field k. Let $K = k(T)$ or $K = k(T, T')$. The map*

$$\begin{aligned} D_{\mathrm{QCoh}}(X_K) &\longrightarrow (D_{\mathrm{QCoh}}(X))_K \\ C^\bullet &\mapsto (\mathrm{Forget}(C^\bullet), \rho_C) \end{aligned}$$

is essentially surjective, where ρ_C is the obvious K-structure on C.

Moreover, if L is a finite separable extension of $K = k(T)$ with $L = K(\alpha) = K[T]/P(T)$ then we can lift an object $(C^\bullet, \rho_C) \in (D_{\mathrm{QCoh}}(X))_L$ to an object $N^\bullet$ of $D_{\mathrm{QCoh}}(X_K)$ endowed with a map $\tilde{\psi} \in \mathrm{End}(N^\bullet)$ such that $P(\tilde{\psi})$ induces the zero map on all cohomology groups of $N^\bullet$.

Proof. By Lemma 3.6, there is an equivalence between $D_{\mathrm{QCoh}}(X_K)$ and $D(\mathrm{QCoh}(X)_K)$, hence it is sufficient to show that the map

$$\begin{aligned} D(\mathrm{QCoh}(X)_K) &\to (D(\mathrm{QCoh}(X)))_K \\ C^\bullet &\mapsto (\mathrm{Forget}(C^\bullet), \rho_C) \end{aligned}$$

is essentially surjective.

Let $\mathcal{A} = \mathrm{QCoh}(X)$. This category satisfies AB5, hence Theorem 3.1 applies in this case. □

4. A representability theorem for derived categories

The results of the previous section will become handy to study functors from $D^b_{\mathrm{Coh}}(X)$, where X is defined over a field k, to a vector space over a bigger field in light of the following theorem:

Theorem 4.1. *Let k be a field, $\mathcal{A}$ be a k-linear abelian category satisfying AB5, and let $k \hookrightarrow K$ an inclusion of fields. Let $D(\mathcal{A})^c$ denote the full subcategory of compact objects in $D(\mathcal{A})$.*

Given an exact, contravariant functor

$$F : D(\mathcal{A})^c \to \underline{\mathrm{mod}}_K$$

there exists a $T \in D(\mathcal{A})_K$ such that

$$F(C) = \mathrm{Mor}_{D(\mathcal{A})_K}(C \otimes K, T)$$

for all $C \in D(\mathcal{A})^c$.

To prove this we will use the ideas from [4, Lemma 2.14] where the version of this theorem with $k = K$ has been proved for a general triangulated category.

Proof of Theorem 4.1. Let D be the functor taking a K-vector space to its dual. Then $G = D \circ F$ is exact and covariant. Let $\mathcal{T}$ be the cocomplete triangulated subcategory generated by $D(\mathcal{A})^c$, i.e. the smallest full triangulated subcategory of $D(\mathcal{A})$ containing $D(\mathcal{A})^c$ which is closed under colimits.

Let $\tilde{G} : \mathcal{T} \to \underline{\mathrm{Mod}}_K$ be the Kan extension of G to $\mathcal{T}$: this is defined as

$$\tilde{G}(C) = \operatorname*{colim}_{\substack{B \to C \\ B \in D(\mathcal{A})^c}} G(B).$$

Since $\tilde{G}$ is exact and commutes with coproducts, it follows that $D \circ \tilde{G}$ is exact and takes coproducts to products. Hence by the Brown representability theorem [7, Theorem 8.3.3] the functor $D \circ \tilde{G}$ is representable, as a functor to $\underline{\mathrm{Mod}}_k$, by an object $U \in \mathcal{T} \subset D(\mathcal{A})$.

The K-action on $\underline{\mathrm{Mod}}_K$ induces a K-action $\tilde{\rho}$ on $D \circ \tilde{G} = h_U$, hence by Yoneda we get a K-action ρ on U, given by $K \xrightarrow{\rho} \mathrm{Nat}(h_U, h_U) = \mathrm{Aut}(U)$. Therefore we obtain an object $(U, \rho) \in D(\mathcal{A})_K$. We need to show that

$$D \circ \tilde{G}(C) = \mathrm{Mor}_{D(\mathcal{A})_K}(C \otimes K, (U, \rho))$$

for all $C \in D(\mathcal{A})^c$.

To do so, first of all notice that as k-vector spaces

$$D \circ \tilde{G}(C) = \mathrm{Mor}_{D(\mathcal{A})}(C, U) = \mathrm{Mor}_{D(\mathcal{A})_K}(C \otimes K, (U, \rho))$$

because $K \otimes_k -$ is left adjoint to the functor forgetting the K-structure. By our definition of the K-action on $\mathrm{Mor}_{D(\mathcal{A})}(C, U)$, this is the same as the K-action on $D \circ \tilde{G}(C)$; moreover the k-vector space map

$$\begin{aligned} \mathrm{Mor}_{D(\mathcal{A})}(C, U) &\xrightarrow{\gamma} \mathrm{Mor}_{D(\mathcal{A})_K}(C \otimes K, (U, \rho)) \\ f &\mapsto f \otimes \rho \end{aligned}$$

is compatible with the K-action since, for any $\alpha \in K$,

$$\gamma(\alpha \cdot f) = \gamma(\tilde{\rho}(\alpha) f) = \tilde{\rho}(\alpha) f \otimes \rho(\cdot) = f \otimes \rho(\alpha)\rho(\cdot) = \alpha \cdot (f \otimes \rho(\cdot))$$

hence we found that the two actions coincide and so

$$D \circ \tilde{G}(C) = \mathrm{Mor}_{D(\mathcal{A})_K}(C \otimes K, (U, \rho)).$$

Let $T = (U, \rho)$. Now since F is of finite type, we get

$$\begin{aligned} F(C) = (D \circ D \circ F)(C) &= (D \circ G)(C) \\ &= (D \circ \tilde{G})(C) = \mathrm{Mor}_{D(\mathcal{A})_K}(C \otimes K, T). \end{aligned}$$ □

Lemma 4.2. *Let k and K be two fields, $k \hookrightarrow K$. Consider the equivalence of categories*

$$D^b(\underline{\text{mod}}(\Lambda)) \xrightarrow{\theta} D^b(\text{Coh}(\mathbb{P}^n_k))$$

as described in [1].

Then there is also an equivalence of categories

$$D^b(\underline{\text{mod}}(\Lambda \otimes K)) \xrightarrow{\theta_K} D^b(\text{Coh}(\mathbb{P}^n_K))$$

and the diagram

$$\begin{array}{ccc} D^b(\underline{\text{mod}}(\Lambda)) & \xrightarrow{\theta} & D^b(\text{Coh}(\mathbb{P}^n_k)) \\ \downarrow & & \downarrow \\ D^b(\underline{\text{mod}}(\Lambda \otimes K)) & \xrightarrow{\theta_K} & D^b(\text{Coh}(\mathbb{P}^n_K)) \end{array}$$

is commutative.

Proof. By [1], we have $\Lambda = \text{End}(\mathcal{M})$ where $\mathcal{M} = \bigoplus_{i=0}^n \mathcal{O}_{\mathbb{P}^n_k}(i)$. Set $\mathcal{M}_K = \bigoplus_{i=0}^n \mathcal{O}_{\mathbb{P}^n_K}(i)$, then

$$\begin{aligned} \text{End}_{\mathbb{P}^n_K}(\mathcal{M}_K) &= \text{End}_{\mathbb{P}^n_K}\left(\bigoplus_{i=0}^n \mathcal{O}_{\mathbb{P}^n_K}(i)\right) = \bigoplus_{i,j=0}^n \text{End}_{\mathbb{P}^n_K}\big(\mathcal{O}_{\mathbb{P}^n_K}(i), \mathcal{O}_{\mathbb{P}^n_K}(j)\big) \\ &= \bigoplus_{i,j=0}^n K[x_0, \ldots, x_n]_{j-i} = \bigoplus_{i,j=0}^n k[x_0, \ldots, x_n]_{j-i} \otimes K \\ &= \left(\bigoplus_{i,j=0}^n k[x_0, \ldots, x_n]_{j-i}\right) \otimes K = \Lambda \otimes K \end{aligned}$$

Moreover, the equivalence θ is induced by the map

$$\underline{\text{mod}}(\Lambda) \xrightarrow{-\otimes_\Lambda \mathcal{M}} \text{Coh}(\mathbb{P}^n_k)$$

and if we let $h : \mathbb{P}^n_K \to \mathbb{P}^n_k$ be the base change morphism, we obtain the following commutative diagram:

$$\begin{array}{ccc} \underline{\text{mod}}(\Lambda) & \xrightarrow{-\otimes_\Lambda \mathcal{M}} & \text{Coh}(\mathbb{P}^n_k) \\ \otimes K \downarrow & & \downarrow h^* \\ \underline{\text{mod}}(\Lambda \otimes K) & \xrightarrow{-\otimes_{\Lambda \otimes K} \mathcal{M}_{\mathcal{K}}} & \text{Coh}(\mathbb{P}^n_K) \end{array}$$

this proves the last assertion. □

We are now almost ready to prove Theorem 1.1, but first we will prove the version of the theorem for the purely transcendental case. The following proof uses ideas from [3, Theorem A.1].

Theorem 4.3. *Let X be a smooth projective variety over a field k. Let $K = k(T)$ or $K = k(T, T')$. Consider a contravariant, cohomological, finite type functor*

$$H : D^b_{\mathrm{Coh}}(X) \to \underline{\mathrm{mod}}_K$$

Then the complex T of Theorem 4.1 lifts to a complex $S \in D^b_{\mathrm{Coh}}(X_K)$ such that H is representable by S, i.e. for every $C \in D^b_{\mathrm{Coh}}(X)$ we have

$$H(C) = \mathrm{Mor}_{D^b_{\mathrm{Coh}}(X_K)}(Lj^*C, S)$$

where $j : X_K \to X$ is the base change morphism.

Proof. By Lemma 4.1, the functor H is representable by an element $T \in (D_{\mathrm{QCoh}}(X))_K$, i.e.

$$H(C) = \mathrm{Mor}_{(D_{\mathrm{QCoh}}(X))_K}(C \otimes K, T)$$

Let S be a lift of T to $D_{\mathrm{QCoh}}(X_K)$ (this is possible by Corollary 3.7). Let C be an element of $D^b_{\mathrm{Coh}}(X)$. By applying the functors in Lemmas 3.1 and 3.6 we get a K-linear map

$$\mathrm{Mor}_{D_{\mathrm{QCoh}}(X_K)}(Lj^*C, S) \xrightarrow{\Psi(\cdot)} \mathrm{Mor}_{(D_{\mathrm{QCoh}}(X))_K}(\Psi \circ Lj^*C, T)$$

and, since by Lemma 3.6, $\Psi \circ Lj^*C = C \otimes K$, we have

$$\mathrm{Mor}_{(D_{\mathrm{QCoh}}(X))_K}(\Psi \circ Lj^*C, T) = \mathrm{Mor}_{(D_{\mathrm{QCoh}}(X))_K}(C \otimes K, T) = H(C)$$

Hence to show that H is represented by S we just need to show that $\Psi(\cdot)$ is an isomorphism. It suffices to show that it is an isomorphism of k-vector spaces, which follows from the following diagram of k-vector spaces:

$$\begin{array}{ccc}
\mathrm{Mor}_{D_{\mathrm{QCoh}}(X_K)}(Lj^*C, S) & \xrightarrow{\Psi(\cdot)} & \mathrm{Mor}_{(D_{\mathrm{QCoh}}(X))_K}(\Psi \circ Lj^*C, T) \\
\| & & \| \\
\mathrm{Mor}_{D_{\mathrm{QCoh}}(X)}(C, Rj_*S) & & \mathrm{Mor}_{(D_{\mathrm{QCoh}}(X))_K}(C \otimes K, T) \\
\| & & \| \\
\mathrm{Mor}_{D_{\mathrm{QCoh}}(X)}(C, \mathrm{Forget}(\Psi(S))) & = & \mathrm{Mor}_{(D_{\mathrm{QCoh}}(X))}(C, \mathrm{Forget}(T))
\end{array}$$

here we used the fact that $Rj_* = \mathrm{Forget} \circ \Psi$, again from Lemma 3.6.

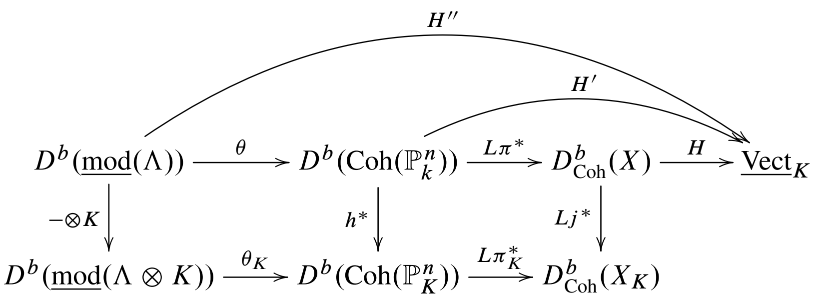
So $\Psi(\cdot)$ is an isomorphism, and hence H is represented by $S \in D_{\mathrm{QCoh}}(X_K)$. We still have to show that S is actually in $D^b_{\mathrm{Coh}}(X_K)$.

Choose an embedding $\pi : X \to \mathbb{P}^n_k$. Let $H' = H \circ L\pi^*$. Let

$$\theta\colon D^b(\underline{\mathrm{mod}}(\Lambda)) \to D^b(\mathrm{Coh}(\mathbb{P}^n_k)) \quad \text{and} \quad \theta_K\colon D^b(\underline{\mathrm{mod}}(\Lambda \otimes K)) \to D^b(\mathrm{Coh}(\mathbb{P}^n_K))$$

as defined in Lemma 4.2 above. Let $H'' = H' \circ \theta$. Let $h : \mathbb{P}^n_K \to \mathbb{P}^n_k$ be the base change morphism.

Consider the following diagram:



and let $C \in D^b(\mathrm{Coh}(\mathbb{P}^n_k))$.

$$\begin{aligned} H'(C) &= H(L\pi^*(C)) = \mathrm{Mor}_{D_{\mathrm{QCoh}}(X_K)}(Lj^*L\pi^*C, S) \\ &= \mathrm{Mor}_{D_{\mathrm{QCoh}}(X_K)}(L\pi^*_K h^*C, S) = \mathrm{Mor}_{D_{\mathrm{QCoh}}(\mathbb{P}^n_K)}(h^*C, R\pi_{K*}S) \end{aligned}$$

so H' is represented by $R\pi_{K*}S \in D_{\mathrm{QCoh}}(\mathbb{P}^n_K)$.

Let $V = \theta_K^{-1}(R\pi_{K*}(S))$ so that H'' is represented by V. Then

$$H''(\Lambda) = \mathrm{Mor}_{\Lambda\otimes K}(\Lambda \otimes K, V)$$

and

$$\begin{aligned} \sum_n \dim H''(\Lambda[n]) &= \sum_n \dim \mathrm{Mor}(\Lambda[n] \otimes K, V) \\ &= \sum_n \dim \mathrm{Mor}((\Lambda \otimes K)[n], V) < \infty \end{aligned}$$

since H'' is of finite type. Therefore $V \in D^b(\underline{\mathrm{mod}}(\Lambda \otimes K))$.

This implies that $R\pi_* S \in D^b(\mathrm{Coh}(\mathbb{P}^n_K))$ hence $S \in D^b(\mathrm{Coh}(X_K))$. □

Proof of Theorem 1.1. The case where L is purely transcendental of degree 2 over k was treated in Theorem 4.3. Let L be a finitely generated separable field extension of k with $\mathrm{trdeg}_k L \leq 1$. There exists a field K such that K is a purely transcendental extension of k of degree less than or equal 1, and $K \subset L$ is a finite extension.

Set $L = K(\alpha) = K[T]/P(T)$, where $P(T)$ is a separable polynomial. Consider the composition

$$D^b_{\mathrm{Coh}}(X) \xrightarrow{H} \underline{\mathrm{mod}}_L \xrightarrow{\mathrm{Forget}} \underline{\mathrm{mod}}_K, \qquad H' = \mathrm{Forget} \circ H$$

By Theorem 4.3, H' is representable by an object $S \in D^b_{\mathrm{Coh}}(X_K)$. Moreover, by Corollary 3.7, S is endowed with a map ψ_α such that $P(\psi_\alpha) = 0$ is zero on all the cohomology groups of S.

First of all, this implies that there exists an n such that $P(\psi_\alpha)^n = 0$. In fact, considering the good truncations $\tau_{\leq i} S$,

$$\cdots \to S^{i-1} \to Z^i \to 0,$$

the claim follows inductively considering the distinguished triangles

$$\tau_{\leq i-1} S \to \tau_{\leq i} S \to H^i(S)[-i]$$

and recalling the fact that if (f_1, f_2, f_3) is a morphism between two triangles and two of the morphisms are nilpotent, then so is the third.

Now let $h : X_L \to X_K$ be the base change morphism, and consider the pullback $Lh^*S \in D^b_{\mathrm{Coh}}(X_L)$. It has an $L[T]$ action induced by the morphism $Lh^*\psi_\alpha$, and $P(Lh^*\psi_\alpha)^n = 0$ so Lh^*S has in fact an $L[T]/P^n(T)$-action. But since P is a separable polynomial, the map $L[T]/P^n(T) \to L[T]/P(T)$ splits as L-algebra map hence $L[T]/P(T)$ also acts on Lh^*S.

Since, over L, $P(T)$ factors as $(T - \alpha)Q(T)$, we can find two elements e_1, e_2 of $L[T]/P(T)$ such that $e_1^2 = e_1$, $e_2^2 = e_2$, $e_1 e_2 = 0$, $e_1 + e_2 = 1$. But since $L[T]/P(T)$ acts on Lh^*S, this gives two idempotent operators e_1, e_2 in $\mathrm{Aut}_{D^b_{\mathrm{Coh}}(X_L)}(Lh^*S)$ such that $e_1 e_2 = 0$, $e_1 + e_2 = \mathrm{id}_{Lh^*S}$.

Now since $D^b_{\mathrm{Coh}}(X_L)$ is Karoubian by [2, Proposition 3.2] we have obtained that $Lh^*S = E \oplus S_2$ and $Lh^*\psi_\alpha$ acts as multiplication by α on E.

We claim that $Rh_*E = S$. Consider the map

$$S \to Rh_*Lh^*S \xrightarrow{pr_1} Rh_*E$$

Under the identification $D_{\mathrm{QCoh}}(X_L) \xrightarrow{\Psi} D(\mathrm{QCoh}(X)_L)$ this corresponds to

$$S \to S \otimes L \to \mathrm{Forget}(E),$$

so this is actually the identity map on S.

Then for every $C \in D^b_{\mathrm{Coh}}(X)$ we have a map of L-vector spaces

$$\mathrm{Mor}_{D_{\mathrm{QCoh}}(X_L)}(Lj^*C, E) \to \mathrm{Mor}_{(D_{\mathrm{QCoh}}(X))_L}(C \otimes L, T) = H(C)$$

where $j : X_L \to X$ is the base change morphism, since E is a lift of S to $D^b_{\mathrm{Coh}}(X_L)$ with the correct L-action. This map is an isomorphism because it is an isomorphism of K-vector spaces:

$$\begin{aligned}\mathrm{Mor}_{D_{\mathrm{QCoh}}(X_L)}(Lj^*C, E) &= \mathrm{Mor}_{(D_{\mathrm{QCoh}}(X_K))}(Li^*C, Rh_*E)\\ &= \mathrm{Mor}_{(D_{\mathrm{QCoh}}(X_K))}(Li^*C, S) = H'(C)\end{aligned}$$

where $i : X_K \to X$ is the base change morphism. □

Proof of Theorem 1.2. Consider the composition

$$D^b_{\mathrm{Coh}}(X) \xrightarrow{F} D^b_{\mathrm{Coh}}(Y) \xrightarrow{i^*} D^b_{\mathrm{Coh}}(\eta) \xrightarrow{H^0} \underline{\mathrm{mod}}_{K(Y)} \xrightarrow{D} \underline{\mathrm{mod}}_{K(Y)}, \qquad H : D^b_{\mathrm{Coh}}(X) \to \underline{\mathrm{mod}}_{K(Y)}$$

where $H^0(-) = H^0(\eta, -)$ and D is the dual as $K(Y)$-vector space. H is an exact contravariant finite type functor, hence by Theorem 1.1 it is representable by $E \in D^b_{\mathrm{Coh}}(X_{K(Y)})$.

Now consider the following diagram:

$$\begin{array}{ccc} X_{K(Y)} & \xrightarrow{p} & \eta \\ \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle i} \\ X \times Y & \xrightarrow{\pi_2} & Y \\ \downarrow{\scriptstyle \pi_1} & & \downarrow \\ X & \longrightarrow & \mathrm{Spec}\, k \end{array}$$

$$j : X_{K(Y)} \to X$$

Both squares are cartesian because $X_{K(Y)} = X \times_{\mathrm{Spec}\, k} K(Y) = X \times Y \times_Y \eta$. Note that g is a flat map, so the derived pullback is just regular pullback in every degree. Also, g is an affine map so that pushforward is also exact.

Let $E^\vee = \underline{\mathrm{RHom}}_{X_{K(Y)}}(E, \mathcal{O}_{X_{K(Y)}})$. Let us construct a complex

$$A \in D^b_{\mathrm{Coh}}(X \times Y)$$

such that

$$Lg^*A = E^\vee \otimes \omega_{X_{K(Y)}}[\dim X_{K(Y)}].$$

Let $\mathcal{L} \in \mathrm{Coh}(X \times Y)$ be a line bundle such that

$$E^\vee \otimes \omega_{X_{K(Y)}}[\dim X_{K(Y)}] \otimes g^*\mathcal{L}^{\otimes n} \in D^b_{\mathrm{Coh}}(X_{K(Y)})$$

is generated by its global sections in each degree. Let $\{s_{i,\ell}\}$ be a set of generators in degree ℓ. Consider the complex

$$g_*(E^\vee \otimes \omega_{X_{K(Y)}}[\dim X_{K(Y)}] \otimes g^*\mathcal{L}^{\otimes n})$$

on $D^b_{\mathrm{Coh}}(X \times Y)$. Then take the subcomplex generated in each degree by

$$\{g_* s_{i,\ell}\} \cup \{g_* ds_{i,\ell-1}\},$$

and twist it down by $\mathcal{L}^{-n}$. This gives the desired complex $A \in D^b_{\mathrm{Coh}}(X \times Y)$.

Then we get the following:

$$\begin{aligned}
H^0 \circ i^* \circ \Phi_A(C) &= H^0 i^* R\pi_{2*}(A \otimes \pi_1^* C) \\
&= H^0 Rp_*(g^* A \otimes g^* \pi_1^* C) \qquad \text{(by flat base change)} \\
&= H^0 Rp_*(E^\vee \otimes \omega_{X_{K(Y)}}[\dim X_{K(Y)}] \otimes j^* C) \\
&= \mathrm{Mor}(\mathcal{O}_\eta, Rp_*(E^\vee \otimes \omega_{X_{K(Y)}}[\dim X_{K(Y)}] \otimes j^* C)) \\
&= \mathrm{Mor}(p^*\mathcal{O}_\eta, E^\vee \otimes \omega_{X_{K(Y)}}[\dim X_{K(Y)}] \otimes j^* C) \\
&= \mathrm{Mor}(\mathcal{O}_{X_{K(Y)}}, E^\vee \otimes \omega_{X_{K(Y)}}[\dim X_{K(Y)}] \otimes j^* C) \\
&= \mathrm{Mor}(E, \omega_{X_{K(Y)}}[\dim X_{K(Y)}] \otimes j^* C) \\
&= D \circ \mathrm{Mor}(j^* C, E) \\
&= D \circ H(C) \\
&= H^0 \circ i^* \circ F(C)
\end{aligned}$$

for every $C \in D^b_{\mathrm{Coh}}(X)$.

Now since F is an exact functor,

$$\begin{aligned}
H^i \circ i^* \circ F &= H^0(i^* \circ F(C)[i]) = H^0(i^* \circ F(C[i])) \\
&= H^0(i^* \circ \Phi_A(C[i])) = H^0(i^* \circ \Phi_A(C)[i]) \\
&= H^i \circ i^* \circ \Phi_A(C)
\end{aligned}$$

Hence, since all cohomology groups agree and $D^b_{\mathrm{Coh}}(K(Y))$ is equivalent to the category of graded vector spaces over $K(Y)$, F and Φ_A agree after restricting to the generic point of Y. □

References

[1] A. Beĭlinson, Coherent sheaves on $\mathbb{P}^n$ and problems of linear algebra, *Funktsional. Anal. i Prilozhen.*, **12** (1978), no. 3, 68–69. Zbl 0402.14006 MR 509388

[2] M. Bökstedt and A. Neeman, Homotopy limits in triangulated categories, *Compositio Math.*, **86** (1993), no. 2, 209–234. Zbl 0802.18008 MR 1214458

[3] A. Bondal and M. Van den Bergh, Generators and representability of functors in commutative and noncommutative geometry, *Mosc. Math. J.*, **3** (2003), no. 1, 1–36. Zbl 1135.18302 MR 1996800

[4] D. Christensen, B. Keller, and A. Neeman, Failure of Brown representability in derived categories, *Topology*, **40** (2001), no. 6, 1339–1361. Zbl 0997.18007 MR 1867248

[5] J. Franke, On the Brown representability theorem for triangulated categories, *Topology*, **40** (2001), no. 4, 667–680. Zbl 1006.18012 MR 1851557

[6] W. Lowen and M. Van den Bergh, Deformation theory of abelian categories, *Trans. Amer. Math. Soc.*, **358** (2006), no. 12, 5441–5483. Zbl 1113.13009 MR 2238922

[7] A. Neeman, *Triangulated categories*, Annals of Mathematics Studies, 148, Princeton University Press, 2001. Zbl 0974.18008 MR 1812507

[8] P. Sosna, Scalar extensions of triangulated categories, *Appl. Categ. Structures*, **22** (2014), no. 1, 211–227. Zbl 1327.18023 MR 3163514

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