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Koszul pairs and applications

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Abstract. Let R be a semisimple ring. A pair (A, C) is called almost-Koszul if A is a connected graded R-ring and C is a compatible connected graded R-coring. To an almost-Koszul pair one associates three chain complexes and three cochain complexes such that one of them is exact if and only if the others are so. In this situation (A, C) is said to be Koszul. One proves that a connected R-ring A is Koszul if and only if there is a connected R-coring C such that (A, C) is Koszul. This result allows us to investigate the Hochschild (co)homology of Koszul rings. We apply our method to show that the twisted tensor product of two Koszul rings is Koszul. More examples and applications of Koszul pairs, including a generalization of Fröberg Theorem [12], are discussed in the last part of the paper.

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1. Introduction

Koszul algebras were introduced by Priddy in [18]. By definition, the $\mathbb{N}$-graded algebra $A := \oplus_{n\in\mathbb{N}} A^n$ over a field $\Bbbk$ is said to be (left) Koszul if A is connected, that is $A^0 = \Bbbk$, and there is a resolution P_* of A^0 by projective graded left A-modules such that each P_n is generated by homogeneous elements of degree n. This class of algebras has outstanding applications in numerous fields of Mathematics, including Representation Theory, Algebraic Geometry, Algebraic Topology, Quantum Groups and Combinatorics; see [17] and the references therein.

Koszul algebras have been generalized by Beilinson, Ginzburg and Soergel. Following [5], we say that a graded ring A is Koszul if A^0 is a semisimple ring and it has a resolution P_* with the same properties as above. Many fundamental properties of Koszul algebras still hold in this more general setting. For instance, such a ring is always quadratic. The Koszulity of a ring is equivalent to the exactness of the Koszul complex. Moreover, for any Koszul ring A such that A^n is a finitely generated left A^0-module, the graded ring $A^\# := \mathrm{Ext}^*_A(A^0, A^0)$ is Koszul as well and $(A^\#)^\# = A$. Here the functors $\mathrm{Ext}^*_A(-,-)$ are defined on the category of all left A-modules. The opposite ring A^{op} of a left Koszul ring A is left Koszul over $(A^0)^{\mathrm{op}}$. Thus, a left Koszul ring is right Koszul, and conversely.

The Hochschild cohomology of an algebra A with coefficients in an A-bimodule M is defined by the relation $\mathrm{HH}^*(A,M) := \mathrm{Ext}_{A\otimes A^{\mathrm{op}}}(A,M)$. Hochschild cohomology may also be defined using the standard Hochschild complex [25, Chapter 9.1]. Rather recently, a lot of effort has been paid to the explicit calculation of these cohomology groups, see for example [8–10,14,20,21,24]. Although the standard complex is an important tool for the study of Hochschild cohomology, it is not helpful for computational purposes. In general, in order to compute Hochschild cohomology, ad hoc complexes are constructed, depending on the algebra that one works with. One of our main goals is to show that in the case of Koszul rings such complexes can be obtained using Coring Theory.

In order to explain our approach we need some terminology and notation. Let R be a given ring and let $({}_R\mathrm{Mod}_R, \otimes, R)$ denote the tensor category of R-bimodules with respect to the tensor product $\otimes$ of R-bimodules. Note that a graded ring $A := \oplus_{n\in\mathbb{N}} A^n$ with $A^0 = R$ may be seen as a connected graded algebra in this tensor category, and conversely. For short, we shall say that A is a connected R-ring. A connected R-ring is said to be strongly graded if, in addition, $A^nA^m = A^{n+m}$, for any $n,m \in \mathbb{N}$. Connected and strongly graded R-corings can be defined by duality, as coalgebras in the tensor category ${}_R\mathrm{Mod}_R$. Since we work with graded structures, the multiplication of a ring A and the comultiplication of a coring C are uniquely determined by some R-bimodule morphisms $m^{p,q}\colon A^p\otimes A^q \to A^{p+q}$ and $\Delta_{p,q}\colon C_{p+q} \to C_p \otimes C_q$.

We can now introduce *almost-Koszul pairs*, one of the the main tools that we use for studying Koszul rings. By definition, such a pair consists of a connected R-ring A and a connected R-coring C, together with an isomorphism of R-bimodules $\theta_{C,A}\colon C_1 \to A^1$ which satisfies the relation

$$m^{1,1}\circ(\theta_{C,A}\otimes\theta_{C,A})\circ\Delta_{1,1} = 0. \tag{1.1}$$

To every strongly graded R-ring A corresponds a canonical almost-Koszul pair $(A, T(A))$. By construction, the homogeneous component of degree n of $T(A)$ is the R-bimodule $T_n(A) := \mathrm{Tor}_n^A(R,R)$. Note that $\mathrm{Tor}_*^A(R,R)$ is the homology of the chain R-coring $R\otimes_A \beta_*^l(A)$, where $\beta_*^l(A)$ denotes the normalized bar resolution of R regarded as a left A-module. Thus $T(A)$ has a natural connected R-coring structure. In this example, the fact that A is strongly graded guarantees the existence of the R-bimodule isomorphism $\theta_{T(A),A}\colon T_1(A) \to A^1$. By duality, to every strongly graded R-coring C corresponds an almost-Koszul pair $(E(C), C)$, where $E(C) := \mathrm{Ext}_C^*(R,R)$. Here, the functors $\mathrm{Ext}_C^*(-,-)$ are defined on the category of right C-comodules.

For each almost-Koszul pair (A,C) we associate three chain complexes: $\mathrm{K}_*^l(A,C)$, $\mathrm{K}_*^r(A,C)$ and $\mathrm{K}_*(A,C)$. The first and the second are complexes of graded left and right A-modules, respectively. The third one lives in the category of graded A-bimodules. By duality, we also define three cochain complexes $\mathrm{K}_l^*(A,C)$, $\mathrm{K}_r^*(A,C)$ and $\mathrm{K}^*(A,C)$ in the categories of left, right and two-sided comodules over C.

By Theorem 3.3, all six complexes associated to an almost-Koszul pair (A, C) are exact, provided that one of them is so. In this case we shall say that (A, C) is a *Koszul pair*. Note that, for any Koszul pair (A, C), the complexes $\mathrm{K}^l_*(A, C)$ and $\mathrm{K}^r_*(A, C)$ provide projective resolutions of R in the categories of left and right graded A-modules, respectively. Similarly, $\mathrm{K}^*_l(A, C)$ and $\mathrm{K}^*_r(A, C)$ are injective resolutions of R in suitable categories of C-comodules. Supposing in addition that R is a separable algebra over a field $\Bbbk$, then $\mathrm{K}_*(A, C)$ is a resolution of A by projective graded A-bimodules and $\mathrm{K}^*(A, C)$ is a resolution of C by injective graded C-bicomodules.

Some useful properties of Koszul pairs are investigated in the second section of the paper. In Theorem 3.9 one shows that, for such a pair (A, C), both components are strongly graded and, moreover, $(A, T(A))$ and $(E(C), C)$ are Koszul as well. Moreover, in this situation, it follows that C and $T(A)$ are isomorphic as graded corings. The relationship between Koszul pairs and Koszul rings is explained in Theorem 3.13: A is such a ring if and only if there exists a Koszul pair (A, C). Taking into account that the components of a Koszul pair uniquely determine each other, it is easy to see that

$$E(T(A)) \cong A \quad \text{and} \quad T(E(C)) \cong C,$$

without any finiteness condition imposed on A or C. These isomorphisms suggest that the coring $T(A)$ and the ring $E(C)$ may be thought of as (Koszul) duals of A and C, respectively. For example, the Koszul dual of a tensor R-ring $T^a_R(V)$ is the unique connected R-coring $C := R \oplus V$, which is concentrated in degree 0 and 1. In the last part of the paper we compute the dual coring for other Koszul R-rings, such as: trivial extension, multiparametric quantum spaces and quotients of quiver algebras by ideals generated by 2-paths.

On the other hand, in view of Theorem 3.13, we say that a strongly graded R-coring C is Koszul if and only if $(E(C), C)$ is a Koszul pair. With this definition in hand, it follows that the functors T and E preserve Koszulity. Multiparametric quantum spaces are Koszul both as a ring and a coring, cf. Theorem 7.2 and §7.7. Since Koszul corings might be useful for the study of other quantum groups, their properties will be investigated in a subsequent article.

We have already noticed that $\mathrm{K}_*(A, C)$ is a resolution of A as a projective (graded) bimodule over itself, for any separable algebra R over a field $\Bbbk$ and any Koszul pair (A, C). In the third section we use this resolution to get a new (co)chain complex that computes the Hochschild (co)homology of A with coefficients in an arbitrary bimodule. As an immediate corollary we show that, for any Koszul R-ring A, the projective dimension of A in the category of A-bimodules (i.e. the Hochschild dimension of A) can be computed using the formula

$$\mathrm{Hdim}(A) = \sup\{n \mid T_n(A) \neq 0\}.$$

It is well known that the class of Koszul algebras is closed under twisted tensor products. As another application of Koszul pairs we prove a similar result for Koszul rings. Let A and B be two strongly graded R-rings. To every invertible graded twisting map $\sigma: B \otimes A \to A \otimes B$ we associate an invertible graded twisting map of R-corings $\tau: T(A) \otimes T(B) \to T(B) \otimes T(A)$. Thus the twisted tensor products $A \otimes_\sigma B$ and $T(A) \otimes_\tau T(B)$ make sense and, in fact, they always define an almost-Koszul pair $(A \otimes_\sigma B, T(A) \otimes_\tau T(B))$. By Theorem 5.18 this pair is Koszul, provided that A and B are Koszul R-rings. In particular, $A \otimes_\sigma B$ is Koszul too. We have already mentioned that, for any Koszul pair (A, C), the coring structure of $T(A)$ is captured on the fly by the isomorphism $T(A) \cong C$. In particular, for a twisted tensor product of two Koszul rings we get $T(A \otimes_\sigma B) \cong T(A) \otimes_\tau T(B)$.

We next use the above Koszul pair to identify the Hochschild (co)homology of a twisted tensor product $A \otimes_\sigma B$ with the total (co)homology of a certain double complex. In homology, the column-wise and row-wise filtrations of the double complex lead us to two spectral sequences converging to the Hochschild homology of $A \otimes_\sigma B$, see Theorem 6.2. Under some additional conditions, similar spectral sequences are obtained in cohomology. By definition generalized Ore extensions are examples of twisted tensor products. We specialize our results on Hochschild homology to this more particular setting in the last part of the fifth section.

Our method based on coring techniques is also useful for the investigation of the Gerstenhaber structure of $\mathrm{HH}^*(A, A)$, in the case when A is a Koszul ring (for example the smashed product between a Koszul ring and a finite dimensional group algebra over a field of characteristic zero). Details about these results will be given in a sequel of this paper.

Some more examples of Koszul pairs, related to braided bialgebras in the tensor category ${}_R\mathrm{Mod}_R$, are discussed in the last section. First of all, in Theorem 7.2 we prove that any couple of strongly graded braided commutative bialgebras in ${}_R\mathrm{Mod}_R$, under some conditions on their braidings, defines a Koszul pair. In particular we prove that any symmetric braided bialgebra in ${}_R\mathrm{Mod}_R$ is Koszul, provided that the braiding is an involution. The incidence algebra of the power set of a finite set is a nontrivial example of such bialgebras, cf. Theorem 7.9. For a different approach to Koszulity of (reduced) incidence algebras the reader is referred to [19]. As a last application, in Theorem 7.11, we extend a result of Fröberg [12].

2. Almost-Koszul pairs

In this section we introduce almost-Koszul pairs and we investigate their basic properties. We start by fixing the terminology and the notation that we use. Throughout, R will denote a semisimple ring.

2.1. *R*-rings. The main objects that we work with are (co)unital and (co)associative (co)algebras in the tensor category of R-bimodules $({}_R\mathcal{M}_R, \otimes, R)$. For the tensor product of two R-bimodules we shall always use the unadorned tensor product symbol $\otimes$. The unit object in ${}_R\mathcal{M}_R$ is R, regarded as a bimodule with respect to the left and right actions induced by the multiplication of R.

By definition, an *R-ring* is an associative and unital algebra in ${}_R\mathcal{M}_R$. Therefore an R-ring consists of an associative and unital ring A together with a morphism of unital rings $u\colon R \to A$. An R-ring A is *graded* if it is equipped with a decomposition $A = \oplus_{n\in\mathbb{N}} A^n$ in ${}_R\mathcal{M}_R$, such that the multiplication $m\colon A \otimes A \to A$ maps $A^p \otimes A^q$ to A^{p+q}. If $A^0 = R$, then we shall say that A is *connected*. The multiplication m induces an R-bilinear map $m^{p,q}\colon A^p \otimes A^q \to A^{p+q}$, for all non-negative integers p and q. The R-ring A is said to be *strongly graded* if and only if it is connected and all maps $m^{p,q}$ are surjective. Obviously, A is strongly graded if and only if $m^{p,1}$ is surjective, for all p. The projection of A onto A^n will be denoted by π_A^n.

We denote the ideal $\oplus_{n>0} A^n$ by $\bar{A}$. The multiplication of A induces a bimodule map $\bar{m}\colon \bar{A} \otimes \bar{A} \to \bar{A}$. Let R^{op} be the opposite ring of R. If V and W are R-bimodules, then they become R^{op}-bimodules by interchanging their left and right module structures. The group morphism $\Lambda_{V,W}\colon V \otimes_{R^{\mathrm{op}}} W \to W \otimes_R V$ that maps $v \otimes_{R^{\mathrm{op}}} w$ to $w \otimes_R v$ is an isomorphism. If (A, m, u) is an R-ring, then the multiplication and the unit of the *opposite R^{op}-ring* A^{op} are the maps $m^{\mathrm{op}} := m \circ \Lambda_{A,A}$ and u, respectively.

2.2. *R*-corings. An *R-coring* is a coassociative and counital coalgebra in ${}_R\mathcal{M}_R$. Thus, an R-coring is an R-bimodule C together with a coassociative *comultiplication* $\Delta\colon C \to C \otimes C$ and a *counit* $\varepsilon\colon C \to R$, which are morphisms in ${}_R\mathcal{M}_R$. An R-coring (C, Δ, ε) is said to be *graded* if, in addition, C is the direct sum of a family $\{C_n\}_{n\in\mathbb{N}}$ of sub-bimodules, such that the counit vanishes on C_m, for any $m > 0$, and

$$\Delta(C_n) \subseteq \bigoplus_{p=0}^{n} C_p \otimes C_{n-p}.$$

The comultiplication of a graded coring is defined by a family of R-bilinear maps $\Delta_{p,q}\colon C_{p+q} \to C_p \otimes C_q$. In the graded case we shall use a special form of Sweedler notation, namely $\Delta_{p,q}(c) = \sum c_{(1,p)} \otimes c_{(2,q)}$. Therefore, in a graded coring, coassociativity is equivalent to the following relations

$$\sum c_{(1,p+q)(1,p)} \otimes c_{(1,p+q)(2,q)} \otimes c_{(2,r)} = \sum c_{(1,p)} \otimes c_{(2,q+r)(1,q)} \otimes c_{(2,q+r)(2,r)}, \quad (2.1)$$

where p, q, r are arbitrary non-negative integers and $c \in C_{p+q+r}$. For short, we shall write the sums from equation (2.1) as $\sum c_{(1,p)} \otimes c_{(2,q)} \otimes c_{(3,r)}$. The counit satisfies the relations

$$\sum \varepsilon(c_{(1,0)}) c_{(2,n)} = c = \sum c_{(1,n)} \varepsilon(c_{(2,0)}), \quad (2.2)$$

for every $c \in C_n$. By definition, a graded R-coring C is *connected* if C_0 is isomorphic as an R-bimodule to R. Thus, $\Delta_{0,0}$ and the restriction of ε to C_0 are uniquely determined by

$$\Delta_{0,0}(c_0) = r_0 c_0 \otimes c_0 \quad \text{and} \quad \varepsilon(c_0) = r_0^{-1},$$

where c_0 is a certain element in C_0 that commutes with all $r \in R$, and r_0 is an invertible element in the center of R. In this paper we shall always assume that c_0 is a group-like element, that is $r_0 = 1$. For such a connected coring we shall identify C_0 with R via the counit of C. In conclusion, without loss in generality, we may assume that $C_0 = R$ and that the restriction of the counit to the zero degree component is the identity map of R. Of course, the counit vanishes on all other homogeneous components of C. It is easy to see that for any $c \in C_n$ we have

$$\Delta_{0,n}(c) = 1 \otimes c \quad \text{and} \quad \Delta_{n,0}(c) = c \otimes 1.$$

For any graded R-coring (C, Δ, ε) the comultiplication Δ factors through a map $\bar{\Delta}: \bar{C} \to \bar{C} \otimes \bar{C}$, where $\bar{C} := C/C_0$. Note that $\bar{\Delta}$ is coassociative. Let $p_C: C \to \bar{C}$ and $\pi_n^C: C \to C_n$ denote the canonical projections. The comultiplication and the counit of the *opposite* R^{op}*-coring* C^{op} are the bimodule maps $\Delta^{\mathrm{op}} := \Lambda_{C,C}^{-1} \circ \Delta$ and ε, respectively.

For a connected R-coring C one defines the maps $\Delta(n): C_n \to C_1^{\otimes n}$ by setting $\Delta(1) := \mathrm{I}_{C_1}$ and then using the recursive relation

$$\Delta(n) = \big(\mathrm{I}_{C_1} \otimes \Delta(n-1)\big) \circ \Delta_{1,n-1}. \tag{2.3}$$

A graded coring C is said to be *strongly graded* if and only if it is connected and $\Delta(n)$ is injective for all $n > 0$. By induction, it follows that $\Delta(p+q) = (\Delta(p) \otimes \Delta(q)) \circ \Delta_{p,q}$, hence C is strongly graded if and only if $\Delta_{p,q}$ is injective for all p and q. To check that a coring is strongly graded it is enough to prove that $\Delta_{1,n}$ is injective for all n or, equivalently, that $\Delta_{n,1}$ is injective for all n.

2.3. Almost-Koszul pairs. We can now introduce one of the main tools that we shall use to investigate Koszul rings. An *almost-Koszul pair* (A, C) consists of a connected R-ring A and a connected R-coring C, together with an isomorphism of R-bimodules $\theta_{C,A}: C_1 \to A^1$ that satisfies the relation (1.1). Using Sweedler notation this is equivalent to the fact that, for any $c \in C_2$,

$$\sum \theta_{C,A}(c_{(1,1)})\theta_{C,A}(c_{(2,1)}) = 0. \tag{2.4}$$

Remark 2.4. Let (A, C) be an almost-Koszul pair. We have noticed that A^{op} and C^{op} are an R^{op}-ring and an R^{op}-coring, respectively. Obviously, $(A^{\mathrm{op}}, C^{\mathrm{op}})$ is an almost-Koszul pair over the ring R^{op}, with respect to $\theta_{C^{\mathrm{op}},A^{\mathrm{op}}} := \theta_{C,A}$, regarded as an R^{op}-bimodule map in the canonical way.

2.5. The normalized bar resolution of R (as a right A-module). We now want to show that, for every strongly graded R-ring A, there is a canonical graded coring C such that (A, C) is almost-Koszul. By [25, p. 283], the groups $\mathrm{Tor}^A_*(R, R)$ may be computed using the *normalized right bar resolution* $\beta^r_*(A)$, that is the exact sequence

$$0 \longleftarrow R \xleftarrow{\delta_0} A \xleftarrow{\delta_1} \bar{A} \otimes A \longleftarrow \cdots \longleftarrow \bar{A}^{\otimes n-1} \otimes A \xleftarrow{\delta_n} \bar{A}^{\otimes n} \otimes A \longleftarrow \cdots, \quad (2.5)$$

where $\delta_0 = \pi^0_A$ and, for $n > 0$, the arrows are given by

$$\delta_n(a_1 \otimes \cdots \otimes a_n \otimes a_{n+1}) := \sum_{i=1}^{n} (-1)^i a_1 \otimes \cdots \otimes a_i a_{i+1} \otimes \cdots \otimes a_{n+1}.$$

Note that, since R is semisimple, $\bar{A}^{\otimes n} \otimes A$ is a projective right A-module. Hence $\mathrm{Tor}^A_*(R, R)$ is the homology of the *normalized bar complex* $(\Omega_*(A), \partial_*)$

$$0 \longleftarrow \Omega_0(A) \xleftarrow{\partial_1} \Omega_1(A) \longleftarrow \cdots \longleftarrow \Omega_{n-1}(A) \xleftarrow{\partial_n} \Omega_n(A) \longleftarrow \cdots, \quad (2.6)$$

where $\Omega_0(A) = R$ and $\Omega_n(A) := \bar{A}^{\otimes n}$, for $n > 0$. The differential map ∂_1 is zero, and ∂_n is the restriction of δ_{n-1} to $\Omega_n(A) \subseteq \bar{A}^{\otimes n-1} \otimes A$. We shall use the notation

$$T_n(A) := \mathrm{H}_n(\Omega_*(A)) \quad \text{and} \quad T(A) := \bigoplus_{n \in \mathbb{N}} T_n(A).$$

The homology class of $x \in \bar{A}$ in $T_1(A)$ will be denoted by $[x]$.

The *normalized left bar resolution* is defined by $\beta^l_*(A) := \beta^r_*(A^{\mathrm{op}})$. Note that $\beta^l_n(A) \cong A \otimes \bar{A}^{\otimes n}$.

2.6. The R-corings $T^c_R(V)$ and $\Omega(A)$. Let V denote an R-bimodule. On $T^c_R(V) := \oplus_{n \in \mathbb{N}} V^{\otimes n}$ one defines a graded R-coring structure such that $\Delta_{p,q}$ is the isomorphism $V^{\otimes p+q} \cong V^{\otimes p} \otimes V^{\otimes q}$. The counit of $T^c_R(V)$ is the projection onto $V^{\otimes 0} := R$, the zero degree homogeneous component of $T^c_R(V)$.

The coring $\Omega(A) := T^c_R(\bar{A})$ will play an important role in our work. Let us first show that $\Omega(A)$ is a chain coring, i.e. it is a coalgebra in the tensor category of chain complexes of R-bimodules.

Lemma 2.7. *The comultiplication and the counit of $\Omega(A)$ are chain maps. In particular, $T(A)$ is a connected R-coring, and $T_1(A)$ is the cokernel of*

$$\bar{m}\colon \Omega_2(A) \to \Omega_1(A).$$

Proof. By definition, the n-chains set of $\Omega_*(A) \otimes \Omega_*(A)$ is the R-bimodule $\bigoplus_{p=0}^n \Omega_p(A) \otimes \Omega_{n-p}(A)$, while the restriction of the differential d_n to $\Omega_p(A) \otimes \Omega_{n-p}(A)$ is given by

$$d_n = \partial_p \otimes \mathrm{I}_{\Omega_{n-p}(A)} + (-1)^p \mathrm{I}_{\Omega_p(A)} \otimes \partial_{n-p}.$$

Proceeding as in [17, Chapter 1.1] one shows that the comultiplication of $\Omega(A)$ is a chain map. Clearly the counit is a chain map, regarding R as a complex concentrated in degree zero. In conclusion, $\Omega(A)$ is a chain R-coring and $T(A)$ is a graded R-coring, as the homology of a chain coring always inherits a graded coring structure. Since $\Omega_0(A) = R$, the map ∂_1 is trivial and $\partial_2 = \overline{m}$, it follows that $T(A)$ is connected and $T_1(A) = \operatorname{Coker} \overline{m}$. □

Proposition 2.8. *If A is a connected strongly graded R-ring, then $(A, T(A))$ is almost-Koszul.*

Proof. We have proved that $T(A)$ is connected. Since A is strongly graded, $T_1(A) = \overline{A}/\overline{A}^2 = A^1$ and the projection π_A^1 induces an R-bimodule isomorphism $\theta_{T(A),A}$: $T_1(A) \to A^1$. Every $\omega \in T_2(A)$ is the homology class of a certain $\zeta \in \operatorname{Ker} \overline{m}$. Hence $\zeta = \sum_{i=1}^n x_i \otimes y_i$, for some $x_1, \ldots, x_n$ and $y_1, \ldots, y_n$ in $\overline{A}$ that satisfy the equation $\sum_{i=1}^n x_i y_i = 0$. Note that $x_i y_i - \pi_A^1(x_i)\pi_A^1(y_i)$ belongs to $\sum_{n>2} A^n$ and

$$\Delta_{1,1}(\omega) = \sum_{i=1}^n [x_i] \otimes [y_i].$$

Thus, the relation (2.4) follows by the following computation

$$\sum_{i=1}^n \theta_{T(A),A}\big([x_i]\big)\theta_{T(A),A}\big([y_i]\big) = \sum_{i=1}^n \pi_A^1(x_i)\pi_A^1(y_i) = \pi_A^2\Big(\sum_{i=1}^n x_i y_i\Big) = 0. \quad \square$$

Our goal now is to associate to an almost-Koszul pair (A, C) three cochain complexes: one in the category of graded left C-comodules and, symmetrically, one in the category of graded right C-comodules. By combining these constructions, we shall get the third cochain complex, that lives in the category of graded C-bicomodules.

2.9. The categories $\mathfrak{M}^C$, ${}^C\mathfrak{M}$ and ${}^C\mathfrak{M}^C$. Let C be an R-coring. The pair (M, ρ^M) is a right C-*comodule* if M is a right R-module and $\rho^M \colon M \to M \otimes C$ is a morphisms of right R-modules such that, using the Sweedler notation $\rho^M(m) = \sum m_{\langle 0 \rangle} \otimes m_{\langle 1 \rangle}$, the relations below hold true for any $m \in M$

$$\sum m_{\langle 0 \rangle \langle 0 \rangle} \otimes m_{\langle 0 \rangle \langle 1 \rangle} \otimes m_{\langle 1 \rangle} = \sum m_{\langle 0 \rangle} \otimes m_{\langle 1 \rangle (1)} \otimes m_{\langle 1 \rangle (2)}$$
$$\sum m_{\langle 0 \rangle} \varepsilon\big(m_{\langle 1 \rangle}\big) = m.$$

A morphism of comodules is a right R-linear map that commutes with the comodule structure maps. The category $\mathfrak{M}^C$ of right C-comodules is Grothendieck, as C is flat as a left R-module [6, p. 264].

A right comodule M is *graded* if $M := \oplus_{n\in\mathbb{N}} M_n$, and $\rho^M(M_n) \subseteq \oplus_{p=0}^n M_p \otimes C_{n-p}$, for all n. Then ρ^M is uniquely defined by the induced R-linear maps

$$\rho^M_{p,q}: M_{p+q} \to M_p \otimes C_q.$$

The category ${}^C\mathfrak{M}$ of left C-comodules is constructed in a similar way. For a left C-comodule (N, ρ^N) we use the Sweedler notation $\rho^N(n) = \sum n_{\langle -1\rangle} \otimes n_{\langle 0\rangle}$.

A *C-bicomodule* is a triple (M, ρ_l^M, ρ_r^M) such that M is an R-bimodule, (M, ρ_l^M) is a left comodule, (M, ρ_r^M) is a right comodule and, for every $m \in M$,

$$\sum m_{\langle -1\rangle} \otimes m_{\langle 0\rangle\langle 0\rangle} \otimes m_{\langle 0\rangle\langle 1\rangle} = \sum m_{\langle 0\rangle\langle -1\rangle} \otimes m_{\langle 0\rangle\langle 0\rangle} \otimes m_{\langle 1\rangle}.$$

Note that, by definition, the structure maps ρ_l^M and ρ_r^M must be morphisms of R-bimodules, otherwise the above compatibility relation does not make sense. A morphism of C-bicomodules is a map which is left and right C-colinear. For the category of C-bicomodules we shall use the notation ${}^C\mathfrak{M}^C$.

Lemma 2.10. *If V is a right R-module, then $V \otimes C$ is an injective right C-comodule. A similar result holds for left C-comodules and C-bicomodules.*

Proof. For any C-comodule (M, ρ^M) the natural transformation

$$\Theta_{M,V}: \mathrm{Hom}_R(M, V) \to \mathrm{Hom}^C(M, V \otimes C), \quad \Theta_{M,V}(f) := (f \otimes \mathrm{I}_C) \circ \rho^M \quad (2.7)$$

is an isomorphism. The inverse of $\Theta_{M,V}$ maps $g \in \mathrm{Hom}^C(M, V \otimes C)$ to $(\mathrm{I}_V \otimes \varepsilon) \circ g$. Let U denote the functor that associates to a right C-comodule the underlying R-module structure, forgetting the coaction. Therefore the functors $\mathrm{Hom}^C(-, V \otimes C)$ and $\mathrm{Hom}_R(-, V) \circ U$ are isomorphic. As R is semisimple, $\mathrm{Hom}_R(-, V) \circ U$ is exact. Thus $\mathrm{Hom}^C(-, V \otimes C)$ is exact as well meaning that $V \otimes C$ is injective.

For any injective R-bimodule W the functorial isomorphism

$$\Theta'_{M,W}: \mathrm{Hom}_{R,R}(M, W) \to \mathrm{Hom}^{C-C}(M, C \otimes W \otimes C),$$
$$\Theta'_{M,W}(f) := (\mathrm{I}_C \otimes f \otimes \mathrm{I}_C) \circ (\rho_l^M \otimes \mathrm{I}_C) \circ \rho_r^M$$

can be used as above to show that $C \otimes W \otimes C$ is injective as a bicomodule. □

2.11. The complexes $\mathrm{K}_l^*(A, C)$ and $\mathrm{K}_r^*(A, C)$. Let (A, C) be an almost-Koszul pair. We set

$$\mathrm{K}_l^{-1}(A, C) := R \quad \text{and} \quad \mathrm{K}_l^n(A, C) := C \otimes A^n,$$

for any $n \geq 0$. We regard R as a graded left C-comodule with respect to the trivial coaction. On the other hand, it is easy to see that $\Delta \otimes \mathrm{I}_{A^n}$ defines a graded C-comodule structure on $\mathrm{K}_l^n(A, C)$, whose homogeneous component of degree p is $C_{p-n} \otimes A^n$ (by convention, $C_k = 0$ for any $k < 0$). The differential maps $d_l^n: \mathrm{K}_l^n(A, C) \to \mathrm{K}_l^{n+1}(A, C)$ are defined as follows. If $n = -1$, then we take d_l^n to

be the canonical bimodule morphisms $R \to C \otimes A^0$ that maps 1 to $1 \otimes 1 \in C_0 \otimes A^0$. For $n \geq 0$ the map d_l^n vanishes on $C_0 \otimes A^n$ and, if $p > 0$ and $c \otimes a \in C_p \otimes A^n$, then

$$d_l^n(c \otimes a) := \sum c_{(1,p-1)} \otimes \theta_{C,A}\big(c_{(2,1)}\big)a.$$

Obviously, d_l^n respects the gradings on $\mathrm{K}_l^n(A, C)$ and $\mathrm{K}_l^{n+1}(A, C)$.

Recall that $(A^{\mathrm{op}}, C^{\mathrm{op}})$ is an almost-Koszul pair (over R^{op}), see Remark 2.4. Hence we may consider

$$\mathrm{K}_r^*(A, C) := \mathrm{K}_l^*(A^{\mathrm{op}}, C^{\mathrm{op}}).$$

For any $n \geq 0$, in view of the isomorphism $C^{\mathrm{op}} \otimes_{R^{\mathrm{op}}} (A^{\mathrm{op}})^n \cong A^n \otimes_R C$, we identify $\mathrm{K}_r^n(A, C)$ and $A^n \otimes_R C$. Through this identification the differential of $\mathrm{K}_l^*(A^{\mathrm{op}}, C^{\mathrm{op}})$ in degree n corresponds to the map $d_r^n \colon A^n \otimes C \to A^{n+1} \otimes C$ which is zero on $A^n \otimes C_0$ and, for $p > 0$ and $a \otimes c \in A^n \otimes C_p$, is given by

$$d_r^n(a \otimes c) = \sum a\theta_{C,A}\big(c_{(1,1)}\big) \otimes c_{(2,p-1)}.$$

Note that $d_r^{-1} \colon R \to A^0 \otimes C$ is uniquely defined by the relation $d_r^{-1}(1) = 1 \otimes 1$.

Proposition 2.12. *If (A, C) is an almost-Koszul pair, then $(\mathrm{K}_l^*(A, C), d_l^*)$ and $(\mathrm{K}_r^*(A, C), d_r^*)$ are cochain complexes of graded C-comodules (left and right, respectively).*

Proof. Clearly $d_l^0 \circ d_l^{-1} = 0$, as d_l^0 vanishes on $C_0 \otimes A^0$. Let us show that $d_l^{n+1} \circ d_l^n(c \otimes a) = 0$, for any $n \geq 0$ and $c \otimes a \in C_p \otimes A^n$. We may assume that $p \geq 2$, otherwise the relation is trivially satisfied, as $d_l^k(1 \otimes a) = 0$, for any $a \in A$ and $k > 0$. Let $\theta := \theta_{C,A}$. Coassociativity and the relation (2.4) imply

$$\begin{aligned}\big(d_l^{n+1} \circ d_l^n\big)(c \otimes a) &= \sum c_{(1,p-1)(1,p-2)} \otimes \theta\big(c_{(1,p-1)(2,1)}\big)\theta\big(c_{(2,1)}\big)a \\ &= \sum c_{(1,p-2)} \otimes \theta\big(c_{(2,2)(1,1)}\big)\theta\big(c_{(2,2)(2,1)}\big)a = 0.\end{aligned}$$

Hence $\mathrm{K}_l^*(A, C)$ is a complex. Let us prove that the maps d_l^n are morphisms of C-comodules. For $n = -1$ we have nothing to show. Let $n \geq 0$ and $p > 0$. For $c \otimes a \in C_p \otimes A^n$, we have

$$\begin{aligned}\big(\mathrm{I}_C \otimes d_l^n\big)\big(\rho(c \otimes a)\big) &= \sum_{r=0}^{p} c_{(1,r)} \otimes d_l^n\big(c_{(2,p-r)} \otimes a\big) \\ &= \sum_{r=0}^{p-1} \sum c_{(1,r)} \otimes c_{(2,p-r)(1,p-r-1)} \otimes \theta\big(c_{(2,p-r)(2,1)}\big)a.\end{aligned}$$

For the second identity we used that $\Delta_{p,0}(c) = c \otimes 1$ and the fact that d_l^n vanishes on $C_0 \otimes A^n$. Since the comultiplication of C is coassociative we get

$$\rho\big(d_l^n(c \otimes a)\big) = \rho\Big(\sum c_{(1,p-1)} \otimes \theta\big(c_{(2,1)}\big)a\Big)$$
$$= \sum_{u=0}^{p-1} \sum c_{(1,p-1)(1,u)} \otimes c_{(1,p-1)(2,p-1-u)} \otimes \theta\big(c_{(2,1)}\big)a.$$

Thus $(\mathrm{I}_C \otimes d_l^n)(\rho(c \otimes a)) = \rho(d_l^n(c \otimes a))$. We conclude the proof that d_l^n is a morphism of C-comodules, by remarking that the above relation trivially holds for $c \otimes a \in C_0 \otimes A^n$, as $d_l^n(c \otimes a) = 0$. □

2.13. The complex $\mathbf{K^*(A, C)}$. We are going to construct a cochain complex $(\mathrm{K}^*(A, C), d^*)$ in the category of C-bicomodules. By definition, $\mathrm{K}^{-1}(A, C) := C$ and $\mathrm{K}^n(A, C) = C \otimes A^n \otimes C$. The differential maps are defined by the relations $d^{-1} := \Delta$ and

$$d^n = d_l^n \otimes \mathrm{I}_C + (-1)^{n+1}\mathrm{I}_C \otimes d_r^n, \tag{2.8}$$

for $n \geq 0$. Since d_l^n is left C-colinear and d_r^n is right C-colinear it follows that d^n is a morphism of bicomodules. It is not difficult to show that $(d^0 \circ d^{-1})(c) = 0$, for any $c \in C_p$. Indeed, for $p = 0$ the relation is obvious. Let us assume that $p > 0$. Thus, $d^{-1}(c) = \sum_{i=0}^{p} c_{(1,i)} \otimes c_{(2,p-i)}$. Henceforth,

$$\big(d^0 \circ d^{-1}\big)(c) = \sum_{i=1}^{p} c_{(1,i-1)} \otimes \theta_{C,A}\big(c_{(2,1)}\big) \otimes c_{(1,p-i)}$$
$$- \sum_{i=0}^{p-1} c_{(1,i)} \otimes \theta_{C,A}\big(c_{(2,1)}\big) \otimes c_{(1,p-i-1)} = 0.$$

Let $n \geq 0$. Since $\mathrm{K}_l^*(A, C)$ and $\mathrm{K}_r^*(A, C)$ are complexes, that is $d_l^{n+1} \circ d_l^n = d_r^{n+1} \circ d_r^n = 0$, we get

$$d^{n+1} \circ d^n = (-1)^{n+1}\big[\big(d_l^{n+1} \otimes \mathrm{I}_C\big) \circ \big(\mathrm{I}_C \otimes d_r^n\big) - \big(\mathrm{I}_C \otimes d_r^{n+1}\big) \circ \big(d_l^n \otimes \mathrm{I}_C\big)\big].$$

Using the formulae that define d_l^* and d_r^* and the fact that the multiplication in A is associative, it follows that $d^{n+1} \circ d^n = 0$. Hence $(\mathrm{K}^*(A, C), d^*)$ is a complex.

2.14. The subcomplexes $\mathbf{K_l^*(A, C, m)}$ and $\mathbf{K_r^*(A, C, m)}$. The complex $\mathrm{K}_l^*(A, C)$ decomposes as a direct sum of subcomplexes in the category of R-bimodules $\oplus_{m \in \mathbb{N}} \mathrm{K}_l^*(A, C, m)$. By definition, $\mathrm{K}_l^*(A, C, 0)$ is the complex

$$0 \longrightarrow R \xrightarrow{\cong} C_0 \otimes A^0 \longrightarrow 0.$$

By construction, the elements of R are cochains of degree -1. Note that $\mathrm{K}_l^*(A, C, 0)$ is always exact. If $m > 0$, then d_l^n maps $C_{m-n} \otimes A^n$ to $C_{m-n-1} \otimes A^{n+1}$. Therefore,

$\mathrm{K}_l^n(A,C,m) := C_{m-n} \otimes A^n$ define a subcomplex $\mathrm{K}_l^*(A,C,m)$ of $\mathrm{K}_l^*(A,C)$, which can be displayed as follows

$$0 \longrightarrow C_m \otimes A^0 \longrightarrow \cdots \longrightarrow C_{m-n} \otimes A^n \xrightarrow{d_l^n} C_{m-n-1} \otimes A^{n+1} \longrightarrow \cdots$$
$$\cdots \longrightarrow C_0 \otimes A^m \longrightarrow 0.$$

Obviously, $\mathrm{K}_l^*(A,C) = \oplus_{m\geq 0} \mathrm{K}_l^*(A,C,m)$. The complex $\mathrm{K}_r^*(A,C)$ admits a similar decomposition as a direct sum of subcomplexes $\oplus_{m\geq 0} \mathrm{K}_r^*(A,C,m)$.

2.15. The normalized bar resolution of R (as a right C-comodule). We are going to sketch how the preceding constructions and results can be dualized. Let C be an R-coring. We assume that C is connected. Then R is a right C-comodule with respect to the trivial coaction. Recall that $\overline{C} := C/C_0$ and that $\overline{\Delta}\colon \overline{C} \to \overline{C} \otimes \overline{C}$ is the unique map such that $\overline{\Delta} \circ p_C = (p_C \otimes p_C) \circ \Delta$, where p_C is the canonical projection. We also use the notation $\widetilde{\Delta} := (p_C \otimes \mathrm{I}_C) \circ \Delta$.

The *right normalized bar resolution* $\beta_r^*(C)$ of R is the exact sequence of right C-comodules

$$0 \longrightarrow R \xrightarrow{\delta^{-1}} C \xrightarrow{\delta^0} \overline{C} \otimes C \longrightarrow \cdots \longrightarrow \overline{C}^{\otimes n} \otimes C \xrightarrow{\delta^n} \overline{C}^{\otimes n+1} \otimes C \longrightarrow \cdots, \tag{2.9}$$

where δ^{-1} is the canonical inclusion and, for $n \geq 0$,

$$\delta^n = \sum_{i=1}^{n} (-1)^{i-1} \mathrm{I}_{\overline{C}^{\otimes i-1}} \otimes \overline{\Delta} \otimes \mathrm{I}_{\overline{C}^{\otimes n-i} \otimes C} + (-1)^n \mathrm{I}_{\overline{C}^{\otimes n}} \otimes \widetilde{\Delta}.$$

The normalized resolution $\beta_l^*(C)$ of R in the category of left C-comodules is defined in a similar way.

We use $\beta_r^*(C)$ to compute $\mathrm{Ext}_C^*(R,R)$. Applying the functor $\mathrm{Hom}^C(R,-)$ and using the isomorphisms $\Theta_{R,-}$, we obtain the *normalized bar complex* $(\Omega^*(C), \partial^*)$

$$0 \longrightarrow \Omega^0(C) \xrightarrow{\partial^0} \Omega^1(C) \xrightarrow{\partial^1} \cdots \longrightarrow \Omega^n(C) \xrightarrow{\partial^n} \Omega^{n+1}(C) \longrightarrow \cdots, \tag{2.10}$$

where $\Omega^0(C) = R$ and $\Omega^n(C) := \overline{C}^{\otimes n}$, for each positive n. The differential morphisms are defined by the formulae $\partial^0 = 0$ and, for $n > 0$,

$$\partial^n = \sum_{i=1}^{n} (-1)^{i-1} \mathrm{I}_{\overline{C}^{\otimes i-1}} \otimes \overline{\Delta} \otimes \mathrm{I}_{\overline{C}^{\otimes n-i}}.$$

We shall use the notation $E^n(C) := \mathrm{H}^n(\Omega^*(C))$ and $E(C) := \oplus_{n\in\mathbb{N}} E^n(C)$. Thus $E^n(C) \cong \mathrm{Ext}_C^n(R,R)$.

2.16. The R-rings $T_R^a(V)$ and $\Omega(C)$. Let V be an R-bimodule and let $T_R^a(V)$ denote the free R-ring generated by V. Therefore, $T_R^a(V) := \oplus_{n\in\mathbb{N}} V^{\otimes n}$ and the graded ring

structure is defined by the bimodule isomorphisms $m^{p,q}\colon V^{\otimes p}\otimes V^{\otimes q}\to V^{\otimes p+q}$. The unit of R is a unit for $T_R^a(V)$ too. In particular, to every connected coring C we associate a connected R-ring $\Omega(C):=T_R^a(\bar{C})$.

Lemma 2.17. *The multiplication and the unit of $\Omega(C)$ are cochain maps. Thus $E(C):=\operatorname{Ext}_C^*(R,R)$ is a connected R-ring and $E^1(C)=\operatorname{Ker}\bar{\Delta}$.*

Proof. As in the ordinary case of algebras over a commutative ring, one shows that the multiplication is a morphism of cochain complexes from $\Omega^*(C)\otimes\Omega^*(C)$ to $\Omega^*(C)$, where the set of n-cochains of the former complex is $\oplus_{p=0}^n\Omega^p(C)\otimes\Omega^{n-p}(C)$ and the differential d^n on $\Omega^p(C)\otimes\Omega^{n-p}(C)$ is given by

$$d^n=\partial^p\otimes \mathrm{I}_{\Omega^{n-p}(C)}+(-1)^p\mathrm{I}_{\Omega^p(C)}\otimes\partial^{n-p}.$$

Clearly, the unit of $\Omega(C)$ is a morphism of cochain complexes, where R is regarded as a complex concentrated in degree zero. Since the cohomology of a cochain R-ring inherits a canonical graded R-ring structure, $\partial^0=0$ and $\partial^1=\bar{\Delta}$, it follows that $E(C)$ is a connected graded R-ring and $E^1(C)=\operatorname{Ker}\bar{\Delta}$. □

Proposition 2.18. *Let C be a strongly graded R-coring. Then $(E(C),C)$ is an almost-Koszul pair.*

Proof. We already know that $E(C)$ is a connected R-ring. Let $\bar{C}_n:=p_C(C_n)$. Hence, $\bar{C}=\oplus_{n>0}\bar{C}_n$ and p_C is injective on each component of positive degree. In particular, $C_n\cong\bar{C}_n$.

Let $\theta\colon C_1\to\bar{C}_1$ denote the restriction of p_C to C_1. Since $\Delta_{0,1}(c)=1\otimes c$ and $\Delta_{1,0}(c)=c\otimes 1$ it follows that the image of θ is included into $\operatorname{Ker}\bar{\Delta}=E^1(C)$, so we may regard θ as a map from C_1 to $E^1(C)$. We claim that the pair $(E(C),C)$ satisfies the identity (2.4) with respect to $\theta_{C,E(C)}:=\theta$. If $B^2(C)$ is the group of 2-coboundaries in the normalized bar complex $\Omega^*(C)$ and $c\in C_2$, then

$$\begin{aligned}\sum\theta\big(c_{(1,1)}\big)\cdot\theta\big(c_{(2,1)}\big)&=\sum p_C\big(c_{(1,1)}\big)\otimes p_C\big(c_{(2,1)}\big)+B^2(C)\\&=\bar{\Delta}\big(p_C(c)\big)+B^2(C)=\partial^1\big(p_C(c)\big)+B^2(C)=0.\end{aligned}$$

Note that the first equality is a consequence of the definition of multiplication in $E(C)$, while for the second identity we used the definition of $\bar{\Delta}$ and the relations $\Delta_{0,2}(c)=1\otimes c$ and $\Delta_{2,0}(c)=c\otimes 1$. It remains to prove that θ is bijective. Let $p_C(c)$ be an element in $\operatorname{Ker}\bar{\Delta}$ and let $c=\sum_{n=1}^d c_n$, where $c_n\in C_n$. We claim that $c_n=0$ for $n\geq 2$. We have

$$\begin{aligned}\sum_{n=2}^{d}\sum_{r=1}^{n-1}(p_C\otimes p_C)\big(\Delta_{r,n-r}(c_n)\big)&=\sum_{n=0}^{d}\sum_{r=0}^{n}(p_C\otimes p_C)\big(\Delta_{r,n-r}(c_n)\big)\\&=(p_C\otimes p_C)\big(\Delta(c)\big)=\bar{\Delta}\big(p_C(c)\big)=0.\end{aligned}$$

We fix $n \geq 2$. Since $(p_C \otimes p_C)(\Delta_{r,n-r}(c_n))$ belongs to $\bar{C}_r \otimes \bar{C}_{n-r}$, in view of the foregoing remarks, we deduce that this element vanishes for every $0 < r < n$. Since $p_C \otimes p_C$ is an R-bimodule isomorphism between $C_r \otimes C_{n-r}$ and $\bar{C}_r \otimes \bar{C}_{n-r}$ it follows that $\Delta_{r,n-r}(c_n) = 0$. As C is strongly graded, we get that the kernel of $\bar{\Delta}$ is included into $\bar{C}_1$. The other inclusion is obvious, so θ is an isomorphism. □

2.19. The cotensor product. Recall that the cotensor product $N \Box_C M$ between a right C-comodule (N, ρ^N) and a left C-comodule (M, ρ^M) is defined as the kernel of $\rho^N \otimes \mathrm{I}_M - \mathrm{I}_N \otimes \rho^M$.

For any right R-module V the tensor product $V \otimes C$ is a right C-comodule with respect to the coaction given by $\mathrm{I}_V \otimes \Delta$. If M is a left C-comodule, then

$$\zeta\colon (V \otimes C)\Box_C M \to V \otimes M, \quad \zeta\Big(\sum_{i=1}^{n} v_i \otimes c_i \otimes m_i\Big) = \sum_{i=1}^{n} v_i \otimes \varepsilon(c_i) m_i \tag{2.11}$$

is an isomorphism. The inverse of ζ maps $v \otimes m$ to $\sum v \otimes m_{\langle -1 \rangle} \otimes m_{\langle 0 \rangle}$. In particular, for $V = R$, we get that $C \Box_C M$ and M are isomorphic as R-bimodules.

In the case when C is connected, we know that R is a left and a right C-comodule with respect to the trivial coactions. For such a coring C, there is a canonical isomorphism $R \Box_C M \cong M^{\mathrm{co}\, C}$, where the set of coinvariant elements $M^{\mathrm{co}\, C}$ contains all $m \in M$ such that $\rho_M(m) = 1 \otimes m$. Of course, the above discussion applies for right C-comodules as well.

2.20. The complexes $\mathrm{K}^l_*(A, C)$, $\mathrm{K}^r_*(A, C)$ and $\mathrm{K}_*(A, C)$. For any almost-Koszul pair (A, C), by duality, we can also construct three chain complexes. We first define a complex $\mathrm{K}^r_*(A, C)$ of graded right A-modules. Let $\mathrm{K}^r_{-1}(A, C) := R$ and $\mathrm{K}^r_n(A, C) := C_n \otimes A$. The differential $d^r_0\colon C_0 \otimes A \to R$ maps $x \otimes a$ to $x\pi^0_A(a)$. For $n > 0$ one defines $d^r_n\colon C_n \otimes A \to C_{n-1} \otimes A$ by

$$d^r_n(c \otimes a) = \sum c_{(1,n-1)} \otimes \theta_{C,A}\big(c_{(2,1)}\big)a.$$

To show that $\mathrm{K}^r_*(A, C)$ is a complex one proceeds as in §2.11, so we omit the proof. Applying the previous construction to the opposite almost-Koszul pair $(A^{\mathrm{op}}, C^{\mathrm{op}})$ we obtain a complex $(\mathrm{K}^l_*(A, C), d^l_*)$ of graded left A-modules. Explicitly, $\mathrm{K}^l_{-1}(A, C) := R$ and $\mathrm{K}^l_n(A, C) := A \otimes C_n$. The differential d^l_0 maps $a \otimes x$ to $\pi^0_A(a)x$. For $n > 0$ the maps d^l_n are given by

$$d^l_n(a \otimes c) = \sum a\theta_{C,A}\big(c_{(1,1)}\big) \otimes c_{(2,n-1)}.$$

By combining the above two constructions we obtain a new complex $(\mathrm{K}_*(A, C), d_*)$ in the category of A-bimodules. By definition, $\mathrm{K}_{-1}(A, C) := A$ and $\mathrm{K}_n(A, C) := A \otimes C_n \otimes A$. The map d_0 is induced by the multiplication of A and, for $n > 0$, we have

$$d_n = d^l_n \otimes \mathrm{I}_A + (-1)^n \mathrm{I}_A \otimes d^r_n.$$

2.21. The subcomplexes $\mathrm{K}^r_*(A, C, m)$ and $\mathrm{K}^l_*(A, C, m)$. The complex $\mathrm{K}^r_*(A, C)$ can be written as a direct sum of subcomplexes $\oplus_{m\in\mathbb{N}} \mathrm{K}^r_*(A, C, m)$. In the case when $m = 0$ we take $\mathrm{K}^r_*(A, C, 0)$ to be the exact cochain complex

$$0 \longrightarrow R \xrightarrow{\cong} C_0 \otimes A^0 \longrightarrow 0.$$

If $m > 0$, then $\mathrm{K}^r_n(A, C, m) := C_{m-n} \otimes A^n$. Therefore in this case $\mathrm{K}^r_*(A, C, m)$ is the complex

$$0 \longleftarrow C_0 \otimes A^m \longleftarrow \cdots \longleftarrow C_{n-1} \otimes A^{m-n+1} \xleftarrow{d^r_n} C_n \otimes A^{m-n} \longleftarrow \cdots$$
$$\cdots \longleftarrow C_m \otimes A^0 \longleftarrow 0.$$

Analogously, $\mathrm{K}^l_*(A, C)$ decomposes as a direct sum of subcomplexes $\mathrm{K}^l_*(A, C) \cong \oplus_{m\in\mathbb{N}} \mathrm{K}^l_*(A, C, m)$.

Proposition 2.22. *Let (A, C) be an almost-Koszul pair. Then the following statements are true.*

(1) *There are canonical isomorphisms*

$$\mathrm{K}_*(A, C) \otimes_A R \cong \mathrm{K}^l_*(A, C) \quad \textit{and} \quad R \otimes_A \mathrm{K}_*(A, C) \cong \mathrm{K}^r_*(A, C).$$

(2) *There are canonical isomorphisms*

$$\mathrm{K}^*(A, C) \,\square_C R \cong \mathrm{K}^*_l(A, C) \quad \textit{and} \quad R \,\square_C \mathrm{K}^*(A, C) \cong \mathrm{K}^*_r(A, C).$$

Proof. We shall only prove the isomorphisms for $\mathrm{K}^l_*(A, C)$ and $\mathrm{K}^*_l(A, C)$. The other two isomorphisms can be proved in a similar way.

The relation $\xi_{-1}(r) = 1 \otimes_A r$ defines a left A-module bijective map $\xi_{-1}\colon R \to A \otimes_A R$. For $n \geq 0$, let $\xi_n\colon \mathrm{K}^l_n(A, C) \to \mathrm{K}_n(A, C) \otimes_A R$ be the left A-module isomorphism $\xi_n(a\otimes c) = (a\otimes c\otimes 1)\otimes_A 1$. One can prove easily that $(d_0\otimes_A \mathrm{I}_R)\circ\xi_0 = \xi_{-1} \circ d^l_0$. Furthermore, for $n > 0$,

$$\big[\big(d_n \otimes_A \mathrm{I}_R\big) \circ \xi_n\big](a \otimes c) = \sum \big(a\theta_{C,A}\big(c_{(1,1)}\big) \otimes c_{(2,n-1)} \otimes 1\big) \otimes_A 1$$
$$+ (-1)^n \sum \big(a \otimes c_{(1,n-1)} \otimes \theta_{C,A}\big(c_{(2,1)}\big)\big) \otimes_A 1.$$

Since the left A-module structure of R is induced by the algebra morphism $\pi^0_A\colon A \to R$ and $\theta_{C,A}(c_{(2,1)})$ is an element in A^1, it follows that the second sum in the right-hand side of the above relation is zero. Hence ξ_* is a morphism of chain complexes between $\mathrm{K}^l_*(A, C)$ and $\mathrm{K}_*(A, C) \otimes_A R$, as

$$\big[\big(d_n \otimes_A \mathrm{I}_R\big) \circ \xi_n\big](a \otimes c) = \sum \big(a\theta_{C,A}\big(c_{(1,1)}\big) \otimes c_{(2,n-1)} \otimes 1\big) \otimes_A 1$$
$$= \big(\xi_{n-1} \circ d^l_n\big)(a \otimes c).$$

In order to prove the second part of the proposition, we define $\xi^{-1}\colon R \to C \,\square_C R$ by $\xi^{-1}(r) = 1 \otimes r$. If $n \geq 0$, then $\xi^n\colon \mathrm{K}^n_l(A,C) \to \mathrm{K}^n(A,C)\square_C R$ is given by $\xi^n(c \otimes a) = (c \otimes a \otimes 1) \otimes 1$. Clearly, ξ^n is a morphism of left C-comodules and it is bijective by §2.19. To check that ξ_* is a morphism of complexes we first notice that

$$\big(\xi^0 \circ d_l^{-1}\big)(1) = (1 \otimes 1 \otimes 1) \otimes 1 = \big[\big(d^{-1} \,\square_C \mathrm{I}_R\big) \circ \xi^0\big](1).$$

Let $n \geq 0$. Since $d^n_r(a \otimes 1) = 0$, for any $a \in A^n$, we conclude the proof by the computation below

$$\begin{aligned}\big[\big(d^n \square_C \mathrm{I}_R\big) \circ \xi^n\big](c \otimes a) &= \big[d^n_l(c \otimes a) \otimes 1\big] \otimes 1 + (-1)^{n+1}\big[c \otimes d^n_r(a \otimes 1)\big] \otimes 1 \\ &= \big[d^n_l(c \otimes a) \otimes 1\big] \otimes 1 = \big(\xi^{n+1} \circ d^n_l\big)(c \otimes a). \qquad \square\end{aligned}$$

Let us regard the graded R-bimodule A^* as a cochain complex with trivial differential maps. Let $R \longrightarrow A^*$ denote the augmented complex with respect to the identity of $R = A^0$ (we regard R as the component of degree -1). The coaugmented complex $R \longleftarrow C_*$ is defined similarly.

Proposition 2.23. *Let (A, C) be an almost-Koszul pair.*

(1) *The complexes $R \otimes_A \mathrm{K}^l_*(A,C)$ and $\mathrm{K}^r_*(A,C) \otimes_A R$ are isomorphic to $R \longleftarrow C_*$.*

(2) *The complexes $\mathrm{Hom}^C(R, \mathrm{K}^*_l(A,C))$ and $\mathrm{Hom}^C(R, \mathrm{K}^*_r(A,C))$ are isomorphic to $R \longrightarrow A^*$.*

Proof. Let $\psi_{-1}\colon R \to R \otimes_A R$ be the canonical isomorphism. For $n \geq 0$ we define the isomorphism $\psi_n\colon C_n \to R \otimes_A \mathrm{K}^l_n(A,C)$ by $\psi_n(c) = 1 \otimes_A (1 \otimes c)$. Since $\psi_{-1} = (\mathrm{I}_R \otimes_A d^l_0) \circ \psi_0$ and for $n > 0$ we have

$$\begin{aligned}\big(\big(\mathrm{I}_A \otimes d^l_n\big) \circ \psi_n\big)(c) &= 1 \otimes_A d^l_n(1 \otimes c) \\ &= \sum \pi^0_A\big(\theta_{C,A}\big(c_{(1,1)}\big)\big) \otimes_A \big(1 \otimes c_{(2,n-1)}\big) = 0,\end{aligned}$$

it follows that ψ_* is a morphism of complexes.

Let $\psi^{-1}\colon R \to \mathrm{Hom}^C(R,R)$ be the bijective map $\psi^{-1}(r) = f_r$, where $f_r(x) = xr$. For $n \geq 0$ we define $\psi^*\colon A^n \to \mathrm{Hom}^C(R, C \otimes A^n))$ by $\psi^n(a) = f_a$, where f_a is the unique C-comodule map such that $f_a(1) = 1 \otimes a$. Obviously ψ_n is bijective. It is easy to see that $\psi^0 = \mathrm{Hom}^C(R, d_l^{-1}) \circ \psi^{-1}$. By the definition of d^n_l we have $d^n_l(1 \otimes a) = 0$. Thus $\mathrm{Hom}^C(R, d^n_l) \circ \psi^n = 0$, for any $n \geq 0$. In conclusion, ψ^* is an isomorphisms of complexes.

The other two isomorphisms can be proved in a similar way. $\square$

Let (A, C) be an almost-Koszul pair. Our next goal is to compare the complexes $\mathrm{K}^l_*(A,C)$ and $\mathrm{K}^*_r(A,C)$ with the bar resolutions $\beta^l_*(A)$ and $\beta^*_r(C)$, respectively. We start by introducing some notation. For $n > 0$, let $\overline{\pi}^C_n\colon \overline{C} \to C_n$ be the map induced by the projection $\pi^C_n\colon C \to C_n$ and let $\overline{\theta} := \theta_{C,A} \circ \overline{\pi}^C_1$. If x is a class

in $\overline{C}$ then we introduce the Sweedler notation $\overline{\Delta}(x) = \sum x_{(1)} \otimes x_{(2)}$. Note that, if $x = p_C(c)$ then $\sum x_{(1)} \otimes x_{(2)} = \sum p_C(c_{(1)}) \otimes p_C(c_{(2)})$.

To relate $\beta_r^*(C)$ and $(\mathrm{K}_r^*(A,C), (-1)^* d^*)$ we construct the maps $\phi^n \colon \beta_r^n(C) \to \mathrm{K}_r^n(A,C)$, for any $n \geq -1$. By definition, ϕ_{-1} and ϕ_0 are I_R and the canonical isomorphism $C \cong A^0 \otimes C$, respectively. Furthermore, for $n > 0$, we define $\phi^n \colon \overline{C}^{\otimes n} \otimes C \to A^n \otimes C$ by

$$\phi^n(x^1 \otimes \cdots \otimes x^n \otimes c) = \overline{\theta}(x^1)\overline{\theta}(x^2)\cdots\overline{\theta}(x^n) \otimes c.$$

For every $n \geq -1$ we now construct a map $\phi_n \colon \mathrm{K}_n^l(A,C) \to \beta_n^l(A)$, as follows. First, we set $\phi_{-1} := \mathrm{I}_R$ and we take ϕ_0 to be the canonical isomorphism $A \otimes C_0 \cong A$. Then, for $n > 0$, we define ϕ_n by

$$\phi_n(a \otimes c) := \sum a \otimes \theta_{C,A}\big(c_{(1,1)}\big) \otimes \theta_{C,A}\big(c_{(2,1)}\big) \otimes \cdots \otimes \theta_{C,A}\big(c_{(n,1)}\big),$$

where in the above equation we used the Sweedler notation

$$\Delta(n)(c) = \sum c_{(1,1)} \otimes c_{(2,1)} \otimes \cdots \otimes c_{(n,1)}.$$

Proposition 2.24. *If (A,C) is an almost-Koszul pair, then*

$$\phi^* \colon \beta_r^*(C) \to \big(\mathrm{K}_r^*(A,C), (-1)^* d_r^*\big) \quad \textit{and} \quad \phi_* \colon \mathrm{K}_*^l(A,C) \to \beta_*^l(A)$$

are morphisms of complexes.

Proof. We need the relation $\phi^{n+1} \circ \delta^n = (-1)^n d_r^n \circ \phi^n$, for all $n \geq -1$. Let $x = p_C(c)$ be some class in $\overline{C}$. Since $\overline{\theta} \circ p_C = \theta \circ \pi_1^C$ and $\Delta(c) = \sum_{u,v \geq 0} c_{(1,u)} \otimes c_{(2,v)}$, we get

$$\begin{aligned} \sum \overline{\theta}(x_{(1)})\overline{\theta}(x_{(2)}) &= \sum_{u,v \geq 0} \theta\big(\pi_1^C\big(c_{(1,u)}\big)\big)\theta\big(\pi_1^C\big(c_{(2,v)}\big)\big) \\ &= \sum \theta\big(c_{(1,1)}\big)\theta\big(c_{(2,1)}\big) = 0. \end{aligned} \tag{2.12}$$

We can now prove the claimed relation. Let $x^1, \ldots, x^n$ be elements in $\overline{C}$. If $p > 0$ and $c \in C_p$, then

$$\begin{aligned} &(\phi^{n+1} \circ \delta^n)(x^1 \otimes \cdots \otimes x^n \otimes c) \\ &\quad = \sum_{i=1}^{n} \sum (-1)^{i-1} \overline{\theta}(x^1)\cdots\overline{\theta}(x^{i-1})\overline{\theta}(x_{(1)}^i)\overline{\theta}(x_{(2)}^i)\overline{\theta}(x^{i+1})\cdots\overline{\theta}(x^n) \otimes c \\ &\qquad\qquad + (-1)^n \sum \overline{\theta}(x^1)\cdots\overline{\theta}(x^n)\overline{\theta}\big(p_C(c_{(1)})\big) \otimes c_{(2)}. \end{aligned}$$

The double sum is zero, by (2.12). As $\Delta(c) = \sum_{u=0}^{p} \sum c_{(1,u)} \otimes c_{(2,p-u)}$, it follows that

$$\sum \overline{\theta}\big(p_C(c_{(1)})\big) \otimes c_{(2)} = \sum \theta(c_{(1,1)}) \otimes c_{(2,p-1)}.$$

Using this relation we get

$$(-1)^n(d_r^n\circ\phi^n)(x^1\otimes\cdots\otimes x^n\otimes c)=(-1)^n\sum\bar{\theta}(x^1)\cdots\bar{\theta}(x^n)\bar{\theta}\big(p_C(c_{(1)})\big)\otimes c_{(2)}.$$

To complete the proof of the fact that ϕ^* is a morphism we remark that both $(-1)^n d_r^n\circ\phi^n$ and $\phi^{n+1}\circ\delta^n$ vanish on $\bar{C}^{\otimes n}\otimes C_0$.

Let us now show that ϕ_* is a morphism of complexes. We have to prove that $\phi_{n-1}\circ d_n^l=\delta_n\circ\phi_n$. For $n=0$ this relation is obvious. Let $a\otimes c$ be a tensor monomial in $A\otimes C_n$. In the case when $n=1$, both sides of the equation map $a\otimes c$ to $a\theta(c)$. Let us assume that $n>1$. By (2.3), we get

$$\big(\phi_{n-1}\circ d_n^l\big)(a\otimes c)=\sum a\theta\big(c_{(1,1)}\big)\otimes\theta\big(c_{(2,1)}\big)\otimes\cdots\otimes\theta\big(c_{(n,1)}\big).$$

On the other hand, by the definition of ϕ_n and δ_n, we have

$$\begin{aligned}(\delta_n\circ\phi_n)(a\otimes c)&=\sum a\theta\big(c_{(1,1)}\big)\otimes\theta\big(c_{(2,1)}\big)\otimes\cdots\otimes\theta\big(c_{(n,1)}\big)\\&\quad+\sum_{i=1}^{n-1}\sum(-1)^i a\otimes\theta\big(c_{(1,1)}\big)\otimes\cdots\otimes\theta\big(c_{(i,1)}\big)\theta\big(c_{(i+1,1)}\big)\otimes\cdots\otimes\theta\big(c_{(n,1)}\big).\end{aligned}$$

By coassociativity and using the condition (2.4), it follows that the double sum vanishes. □

3. Koszul pairs

In this section we shall investigate the exactness of the (co)chain complexes that we have associated to an almost-Koszul pair (A,C). Roughly speaking, we shall show that one of these complexes is exact if and only if the other five are so. Furthermore, these complexes are exact precisely when A is a Koszul R-ring.

Lemma 3.1. *Let (A,C) be an almost-Koszul pair.*

(1) *If A is strongly graded, then $\mathrm{H}_0(\mathrm{K}_*(A,C))=0$.*

(2) *If C is strongly graded, then $\mathrm{H}^0(\mathrm{K}^*(A,C))=0$.*

Proof. Let $\theta:=\theta_{C,A}$. Since C is connected we have $A\otimes C_0\otimes A\cong A\otimes A$. Via this isomorphism, one identifies d_0 with the multiplication m of the R-ring A. Moreover,

$$d_1(a\otimes c\otimes b)=a\theta(c)\otimes b-a\otimes\theta(c)b,$$

for any $a,b\in A$ and $c\in C_1$. Let us assume that $z=\sum_{i=1}^d a_i\otimes b_i$ is an element in the kernel of m. We have to show that z is in the image of d_1. Since A is

strongly graded and θ is bijective we may assume that $b_i = \theta(c_1^i)\cdots\theta(c_{n_i}^i)$, for some $c_1^i, \ldots, c_{n_i}^i \in C_1$. It is easy to see that $z = d_1(x)$, where

$$x := -\sum_{i=1}^{d}\sum_{j=1}^{n_i} a_i\theta(c_1^i)\cdots\theta(c_{j-1}^i)\otimes c_j^i\otimes\theta(c_{j+1}^i)\cdots\theta(c_{n_i}^i).$$

Let us now prove the second part of the lemma. Let $\nu\colon C\otimes C\to C\otimes C_1\otimes C$ be the unique bimodule morphism such that

$$\nu|_{C_p\otimes C_q} = \Delta_{p-1,1}\otimes \mathrm{I}_{C_q} - \mathrm{I}_{C_p}\otimes\Delta_{1,q-1},$$

where $\Delta_{u,v} = 0$ whenever $u < 0$ or $v < 0$. Let us consider the following diagram

$$\begin{array}{ccccccc}
0 & \longrightarrow & C & \xrightarrow{d^{-1}} & C\otimes A^0\otimes C & \xrightarrow{d^0} & C\otimes A^1\otimes C \\
 & & \| & & \uparrow\cong & & \uparrow{\scriptstyle \mathrm{I}_C\otimes\theta_{C,A}\otimes\mathrm{I}_C} \\
0 & \longrightarrow & C & \xrightarrow[\Delta]{} & C\otimes C & \xrightarrow[\nu]{} & C\otimes C_1\otimes C
\end{array}$$

Since the squares are commutative and the vertical arrows are isomorphisms it is enough to prove that $\operatorname{Ker}\nu = \operatorname{Im}\Delta$. We claim that the sequence

$$C \xrightarrow{\Delta} C\otimes C \xrightarrow{\mu} C\otimes C\otimes C$$

is exact, where $\mu = \Delta\otimes\mathrm{I}_C - \mathrm{I}_C\otimes\Delta$. Indeed, the inclusion $\operatorname{Im}\Delta\subseteq\operatorname{Ker}\mu$ is trivial as the comultiplication is coassociative. Let $s_{-1} := \mathrm{I}_C\otimes\varepsilon$ and $s_0 := \mathrm{I}_C\otimes\mathrm{I}_C\otimes\varepsilon$. Since

$$-s_0\mu + \Delta s_{-1} = \mathrm{I}_{C\otimes C},$$

any element in the kernel of μ belongs to the image of Δ, so our claim has been proved. Summarizing, we must show that $\operatorname{Ker}\mu = \operatorname{Ker}\nu$. Let x be an element in $C\otimes C$. There are finitely many non-zero elements $x_{p,q}\in C_p\otimes C_q$ such that $x = \sum_{p,q\geq 0} x_{p,q}$. Note that $C\otimes C\otimes C = \bigoplus_{u,v,w\geq 0} C_u\otimes C_v\otimes C_w$. Computing the component of $\mu(x)$ in $C_u\otimes C_v\otimes C_w$, we deduce that x is in the kernel of μ if and only if

$$(\Delta_{u,v}\otimes\mathrm{I}_{C_w})(x_{u+v,w}) = (\mathrm{I}_{C_u}\otimes\Delta_{v,w})(x_{u,v+w}), \tag{3.1}$$

for any $u, v, w\geq 0$. Proceeding similarly, we deduce that x is in the kernel of ν if and only if (3.1) holds for $v = 1$ and any $u, w\geq 0$. Thus the inclusion $\operatorname{Ker}\mu\subseteq\operatorname{Ker}\nu$ is trivial.

To prove the other inclusion we pick $x\in\operatorname{Ker}\nu$. Since C is connected the equation (3.1) holds for $v = 0$ and any $u, w\geq 0$. As x is in the kernel of ν,

it remains to prove (3.1) for any $v \geq 2$. By induction, let us assume that (3.1) holds for some $v \geq 1$. By coassociativity and the induction hypothesis we have

$$\begin{aligned}\big[(\mathrm{I}_{C_u} \otimes \Delta_{1,v} \otimes \mathrm{I}_{C_w}) \circ (\Delta_{u,v+1} \otimes \mathrm{I}_{C_w})\big](x_{u+v+1,w}) \\ &= \big[(\Delta_{u,1} \otimes \mathrm{I}_{C_v} \otimes \mathrm{I}_{C_w}) \circ (\Delta_{u+1,v} \otimes \mathrm{I}_{C_w})\big](x_{u+v+1,w}) \\ &= \big[(\Delta_{u,1} \otimes \mathrm{I}_{C_v} \otimes \mathrm{I}_{C_w}) \circ (\mathrm{I}_{C_{u+1}} \otimes \Delta_{v,w})\big](x_{u+1,v+w}) \\ &= (\Delta_{u,1} \otimes \Delta_{v,w})(x_{u+1,v+w}).\end{aligned}$$

On the other hand, using coassociativity once again and then the fact that $x \in \operatorname{Ker} \nu$, we get

$$\begin{aligned}\big[(\mathrm{I}_{C_u} \otimes \Delta_{1,v} \otimes \mathrm{I}_{C_w}) \circ (\mathrm{I}_{C_u} \otimes \Delta_{v+1,w})\big](x_{u,v+1+w}) \\ &= \big[(\mathrm{I}_{C_u} \otimes \mathrm{I}_{C_1} \otimes \Delta_{v,w}) \circ (\mathrm{I}_{C_u} \otimes \Delta_{1,v+w})\big](x_{u,v+w+1}) \\ &= (\Delta_{u,1} \otimes \Delta_{v,w})(x_{u+1,v+w}).\end{aligned}$$

We conclude the proof of the lemma by remarking that $\mathrm{I}_{C_u} \otimes \Delta_{1,v} \otimes \mathrm{I}_{C_w}$ is injective, since C is strongly graded and any R-bimodule is flat as a left and a right module. □

Proposition 3.2. *Let (A, C) be an almost-Koszul pair.*

(1) *The complexes $\mathrm{K}^l_*(A, C)$, $\mathrm{K}^r_*(A, C)$ and $\mathrm{K}_*(A, C)$ are exact, provided that one of them is so.*

(2) *The complexes $\mathrm{K}^*_l(A, C)$, $\mathrm{K}^*_r(A, C)$ and $\mathrm{K}^*(A, C)$ are exact, provided that one of them is so.*

Proof. Assuming that $\mathrm{K}_*(A, C)$ is exact we deduce that it splits in the category of right A-modules, as $\mathrm{K}_n(A, C)$ is projective in this category for any $n \geq -1$. By Proposition 2.22 (1) we have

$$\mathrm{K}^l_*(A, C) \cong \mathrm{K}_*(A, C) \otimes_A R,$$

so this complex is exact too. Let us now suppose that $\mathrm{K}^l_*(A, C)$ is exact. We first remark that, by construction, d^l_0 is zero on $A^n \otimes C_0$ for any $n > 0$. Moreover, for $c \in C_1$ and $a \in A$ we have

$$d^l_1(a \otimes c) = a\theta_{C,A}(c) \otimes 1.$$

Therefore the exactness of $\mathrm{K}^l_*(A, C)$ in degree 0 and the fact that $\theta_{C,A}$ is bijective imply the relation $A^n = A^{n-1}A^1$, for all $n > 0$. Hence A is strongly graded. In view of the preceding lemma we get that $\mathrm{K}_*(A, C)$ is exact in degree 0.

Since d_0 is surjective it remains to show that $\mathrm{H}_n(\mathrm{K}_*(A, C)) = 0$, for all $n > 0$. For short, let $\mathrm{K}_* := \mathrm{K}_*(A, C)$. For $n, p \geq 0$ we define $X_{n,p} := A^{p-n} \otimes C_n$, where $A^{p-n} = 0$ in the case when $p < n$. Thus $\mathrm{K}_n := \oplus_{p \geq 0} X_{n,p} \otimes A$. Obviously, the following relations hold

$$\begin{aligned}(d^l_n \otimes \mathrm{I}_A)(X_{n,p} \otimes A) &\subseteq X_{n-1,p} \otimes A \\ (\mathrm{I}_A \otimes d^r_n)(X_{n,p} \otimes A) &\subseteq X_{n-1,p-1} \otimes A.\end{aligned} \tag{3.2}$$

Taking into account the above inclusions we get that $\mathrm{K}_*^i := \oplus_{p=0}^i X_{*,p} \otimes A$ is a subcomplex of K_*. Let $L_*^i := \mathrm{K}_*^i / \mathrm{K}_*^{i-1}$. For any $k \geq 0$ we can identify L_k^i and $A^{i-k} \otimes C_k \otimes A$ as R-bimodules. By (3.2) it follows that L_*^i is isomorphic to $\mathrm{K}_*^l(A, C, i) \otimes A$, see §2.21 for the definition of the latter complex. We deduce that L_*^i is exact, being isomorphic to a direct summand of the exact complex $\mathrm{K}_*^l(A, C) \otimes A$. We now fix $n > 0$. Note that $\mathrm{K}_n^0 = 0$. In particular, K_*^0 is exact in degree n. Writing the long exact sequence in homology for

$$0 \longrightarrow \mathrm{K}_*^{i-1} \longrightarrow \mathrm{K}_*^i \longrightarrow L_*^i \longrightarrow 0$$

we deduce by induction that K_*^i is exact for any $i \geq 0$. We can now prove that K_* is exact in degree n. Indeed, if ω is an n-cycle in K_*, then there is i such that $\omega \in \mathrm{K}_n^i$. Since ω is a cycle in K_*^i it follows that ω is a boundary in this complex. Thus, a fortiori, ω is a boundary in K_*, so $\mathrm{K}_*(A, C)$ is exact.

To show that $\mathrm{K}_*(A, C)$ is exact if and only if $\mathrm{K}_*^r(A, C)$ is exact we proceed as follows. By definition, the latter sequence is equal to $\mathrm{K}_*^l(A^{\mathrm{op}}, C^{\mathrm{op}})$ which, in turn, is exact if and only if $\mathrm{K}_*(A^{\mathrm{op}}, C^{\mathrm{op}})$ is so. Let d_*^{op} denote the differential of $\mathrm{K}_*(A^{\mathrm{op}}, C^{\mathrm{op}})$. For every $n \geq 0$, the map

$$\eta_n \colon A^{\mathrm{op}} \otimes_{R^{\mathrm{op}}} (C^{\mathrm{op}})_n \otimes_{R^{\mathrm{op}}} A^{\mathrm{op}} \to A \otimes_R C_n \otimes_R A,$$
$$\eta_n(a \otimes_{R^{\mathrm{op}}} c \otimes_{R^{\mathrm{op}}} b) = b \otimes_R c \otimes_R a$$

is bijective and $\eta_{n-1} \circ d_n^{\mathrm{op}} = (-1)^n d_n \circ \eta_n$. In conclusion, the homology of $\mathrm{K}_*(A^{\mathrm{op}}, C^{\mathrm{op}})$ is trivial if and only if the homology groups of $\mathrm{K}_*(A, C)$ vanish as well.

We begin the proof of the second part of the proposition by showing that $\mathrm{K}_l^*(A, C)$ is exact, provided that $\mathrm{K}^*(A, C)$ satisfies this property. The latter complex splits in the category of right C-comodules, as it is exact and $\mathrm{K}^n(A, C)$ is injective as a right comodule for any n. In view of Proposition 2.22 (2) we conclude that $\mathrm{K}_l^*(A, C)$ is exact.

Let us now assume that $\mathrm{K}_l^*(A, C)$ is exact. Since the comultiplication of a coring is always injective, this property holds in degree -1.

Since A is connected we identify $\mathrm{K}_l^0(A, C)$ with C. Via this identification, d_l^0 is a map from C to $C \otimes A^0$ which vanishes on C_0. On the other hand, for $p > 0$ and $c \in C_p$, we have $d_l^0(c) = \sum c_{(1,p-1)} \otimes \theta_{C,A}(c_{(2,1)})$. Since $\theta_{C,A}$ is an isomorphism it follows

$$\mathrm{H}^0(\mathrm{K}_l^*(A, C)) \cong \bigoplus_{p \geq 1} \mathrm{Ker}\, \Delta_{p-1,1}.$$

By assumption we get that C is strongly graded, so $\mathrm{K}^*(A, C)$ is exact in degree zero, cf. Lemma 3.1 (2).

It remains to show that $\mathrm{K}^*(A,C)$ is exact in degree $n > 0$. For the sake of simplicity, let us denote this complex by (K^*, d^*). If $X_{n,p} := C_{p-n} \otimes A^n$, then $\mathrm{K}^n = \oplus_{p\geq 0} X_{n,p} \otimes C$. Moreover,

$$(d_l^n \otimes \mathrm{I}_C)(X_{n,p} \otimes C) \subseteq X_{n+1,p} \otimes C$$
$$(\mathrm{I}_C \otimes d_r^n)(X_{n,p} \otimes C) \subseteq X_{n+1,p+1} \otimes C.$$

In particular, K_i^* is a subcomplex of K^*, where $\mathrm{K}_i^n := \oplus_{p\geq i} X_{n,p} \otimes C$. Let $L_i^* := \mathrm{K}_i^* / \mathrm{K}_{i+1}^*$.

Note that $X_{n+1,i+1} \otimes C$ is a subset of K_{i+1}^{n+1} and the differential of L_i^* maps an n-chain $x \otimes c + \mathrm{K}_{i+1}^n$ to $d_l^n(x) \otimes c + \mathrm{K}_{i+1}^{n+1}$. Thus the R-bilinear isomorphism

$$L_i^n \cong \mathrm{K}_l^n(A, C, i) \otimes C$$

allows us to identify the complexes L_i^* and $\mathrm{K}_l^*(A,C,i) \otimes C$. Since $\mathrm{K}_l^*(A,C,i)$ is a direct summand of $\mathrm{K}_l^*(A,C)$ and C is flat as a left R-module we conclude that L_i^* is exact. Our goal now is to show that all quotients $\mathrm{K}^* / \mathrm{K}_i^*$ are exact. We proceed by induction. Clearly $\mathrm{K}^* / \mathrm{K}_0^* = 0$. Let us assume that $\mathrm{K}^* / \mathrm{K}_i^*$ is exact. Since in the short exact sequence

$$0 \longrightarrow L_i^* \longrightarrow \mathrm{K}^* / \mathrm{K}_{i+1}^* \longrightarrow \mathrm{K}^* / \mathrm{K}_i^* \longrightarrow 0$$

L_i^* is also exact we deduce that the middle term has the same property. We can now prove that K^* is exact. Let ω be an n-cocycle in K^*. We choose a positive integer i such that ω belongs to $M_i^n := \bigoplus_{p\leq i} X_{n,p} \otimes C$. The projection induces an R-bimodule isomorphism $\nu^n : \mathrm{K}^n / \mathrm{K}_{i+1}^n \to M_i^n$. Using ν^* one transports the differential maps of $\mathrm{K}^* / \mathrm{K}_{i+1}^*$ to get a cochain complex (M_i^*, ∂^*) which is isomorphic to $\mathrm{K}^* / \mathrm{K}_{i+1}^*$. Clearly, M_i^* is exact and ω is a cocycle in M_i^*. Hence, there is $\zeta \in M_i^{n-1}$ such that $\omega = \partial^n(\zeta)$. On the other hand $\partial^n(\zeta) = d^n(\zeta)$ as, by construction, ∂^n and d^n are equal on M_i^n. Thus the proposition is proved. □

Theorem 3.3. *If one of the six complexes from the preceding proposition is exact, then all are so.*

Proof. Recall that for the complexes $\mathrm{K}_l^*(A,C)$ and $\mathrm{K}_*^r(A,C)$ we have the decompositions

$$\mathrm{K}_l^*(A,C) = \bigoplus_{m\in\mathbb{N}} \mathrm{K}_l^*(A,C,m) \quad \text{and} \quad \mathrm{K}_*^r(A,C) = \bigoplus_{m\in\mathbb{N}} \mathrm{K}_*^r(A,C,m).$$

Let $m > 0$. By construction, $\mathrm{K}_l^p(A,C,m) = \mathrm{K}_{m-p}^r(A,C,m)$ and $d_l^p = d_{m-p}^r$, for any integer p. Thus

$$\mathrm{H}^p(\mathrm{K}_l^*(A,C,m)) = \mathrm{H}_{m-p}(\mathrm{K}_*^r(A,C,m)).$$

Since, by definition, both $\mathrm{K}_l^*(A,C,0)$ and $\mathrm{K}_*^r(A,C,0)$ are always exact, it follows that the cohomology groups of $\mathrm{K}_l^*(A,C)$ vanish if and only if the homology groups of $\mathrm{K}_*^r(A,C)$ vanish as well. We conclude the proof of the theorem by applying Proposition 3.2. □

Definition 3.4. An almost-Koszul pair (A,C) is said to be *Koszul* if and only if the complexes from Proposition 3.2 are exact.

Corollary 3.5. *Let (A,C) be a Koszul pair. Then the following statements hold:*

(1) *The complex $\mathrm{K}_l^*(A,C)$ is a resolution of R by injective graded left C-comodules.*

(2) *The complex $\mathrm{K}_r^*(A,C)$ is a resolution of R by injective graded right C-comodules.*

(3) *The complex $\mathrm{K}_*^l(A,C)$ is a resolution of R by projective graded left A-modules.*

(4) *The complex $\mathrm{K}_*^r(A,C)$ is a resolution of R by projective graded right A-modules.*

(5) *If A is an injective R-bimodule, then $\mathrm{K}^*(A,C)$ is a resolution of C by injective graded C-bicomodules.*

(6) *If C is a projective R-bimodule, then $\mathrm{K}_*(A,C)$ is a resolution of A by projective graded A-bimodules.*

Proof. The first four statements are immediate consequences of the preceding theorem. Let us assume that C is projective as an R-bimodule. Thus the subbimodule C_n is also projective. It follows that $A \otimes C_n \otimes A$ is projective as an A-bimodule, so $\mathrm{K}_*(A,C)$ is a resolution of A by projective A-bimodules. Analogously, by Lemma 2.10, if A is an injective R-bimodule then $C \otimes A^n \otimes C$ is an injective bicomodule. Therefore, the last statement holds as well. □

Remark 3.6. Note that, although R is semisimple, there might exist R-bimodules which are neither projective nor injective. However, if R is a separable algebra over a field $\Bbbk$ (i.e. R is projective as a bimodule over itself), then every R-bimodule is both projective and injective. Henceforth, for a Koszul pair (A,C), under this additional assumption on R, the complex $\mathrm{K}^*(A,C)$ is a resolution of C by injective C-bicomodules and $\mathrm{K}_*(A,C)$ is a resolution by projective A-bimodules.

Corollary 3.7. *Let (A,C) be a Koszul pair. Then the R-ring A and the R-coring C are strongly graded. In particular, $(A,T(A))$ and $(E(C),C)$ are almost-Koszul pairs.*

Proof. The complex $\mathrm{K}_*^l(A,C)$ is exact, as (A,C) is Koszul. Hence, by the proof of Proposition 3.2, it follows that A is strongly graded. By Proposition 2.8, the pair $(A,T(A))$ is almost-Koszul. The statements concerning the coring C can be proved in a similar way. □

Corollary 3.8. *The pair (A, C) is Koszul if and only if $(A^{\mathrm{op}}, C^{\mathrm{op}})$ is Koszul too.*

Proof. The homology of $\mathrm{K}_*(A^{\mathrm{op}}, C^{\mathrm{op}})$ is trivial if and only if the homology groups of $\mathrm{K}_*(A, C)$ vanishes, by the proof of Proposition 3.2. □

Theorem 3.9. *Let (A, C) be a Koszul pair. The graded R-coring $T(A)$ is isomorphic to C and $(A, T(A))$ is a Koszul pair. Dually, $E(C)$ and A are isomorphic as graded R-rings and $(E(C), C)$ is Koszul.*

Proof. We know that A is strongly graded and that $(A, T(A))$ is almost-Koszul. Let ϕ_* be the morphism of complexes from $\mathrm{K}^l_*(A, C)$ to $\beta^l_*(A)$, that we constructed in Proposition 2.24. Since $\mathrm{K}^l_*(A, C)$ is a projective resolution of R and $\phi_{-1} = \mathrm{I}_R$ is invertible, there exists a morphism $\phi'_*\colon \beta^l_*(A) \to \mathrm{K}^l_*(A, C)$ that lifts the identity of R. By the Comparison Theorem [25, Theorem 2.2.6] it follows that $\phi_* \circ \phi'_*$ and $\phi'_* \circ \phi_*$ are homotopic to $\mathrm{I}_{\beta^l_*(A)}$ and $\mathrm{I}_{\mathrm{K}^l_*(A,C)}$, respectively. Obviously, $\mathrm{I}_R \otimes_A \phi_*$ and $\mathrm{I}_R \otimes_A \phi'_*$ are inverses of each other up to a homotopy. We deduce that $\mathrm{H}_n(\mathrm{I}_R \otimes_A \phi_*)$ is an isomorphism between $\mathrm{H}_n(R \otimes_A \mathrm{K}^l_*(A, C))$ and $\mathrm{H}_n(\Omega_*(A))$, for any $n \geq 0$. On the other hand, by Proposition 2.23, the former homology group is isomorphic to C_n. In conclusion, the map $\gamma_n\colon C_n \to T_n(A)$ defined by

$$\gamma_n(c) \coloneqq \Delta(n)(c) + B_n(A)$$

is an R-bimodule isomorphism. Here, $\Delta(0) \coloneqq \mathrm{I}_R$ and $B_n(A)$ denotes the group of n-boundaries in $\Omega_n(A)$. Now it is not difficult to see that $\{\gamma_n\}_{n\in\mathbb{N}}$ is an isomorphism of graded R-corings between C and $T(A)$. In conclusion $\mathrm{I}_A \otimes \gamma_*$ is an isomorphism of chain complexes from $\mathrm{K}^l_*(A, C)$ to $\mathrm{K}^l_*(A, T(A))$. Thus the complex $\mathrm{K}^l_*(A, T(A))$ is exact, meaning that $(A, T(A))$ is Koszul.

We have seen that C is strongly graded and that $(E(C), C)$ is an almost-Koszul pair. The morphism ϕ^* from Proposition 2.24 lifts the identity of R. By hypothesis $\beta^*_r(C)$ and $(\mathrm{K}^*_r(A, C), (-1)^* d^*_r)$ are resolutions of R in the category of right C-comodules, so $\mathrm{Hom}^C(R, \phi^*)$ is a quasi-isomorphism of complexes. By applying the functor $\mathrm{Hom}^C(R, -)$ to $\beta^*_r(C)$ and $\mathrm{K}^*_r(A, C)$ we get an isomorphism γ^* of cochain complexes from $(\Omega^*(C), \partial^*)$ to $(A^*, 0)$.

Of course, $\gamma^0 \coloneqq \mathrm{I}_R$. Let us recall that $\overline{\pi}^C_1\colon \overline{C} \to C_1$ denotes the map induced by the projection π^C_1 and $\overline{\theta} \coloneqq \theta_{C,A} \circ \overline{\pi}^C_1$. For $n > 0$ the map $\gamma^n\colon \overline{C}^{\otimes n} \to A^n$ is explicitely given by the formula

$$\gamma^n(x^1 \otimes \cdots \otimes x^n) = \overline{\theta}(x^1) \cdots \overline{\theta}(x^n).$$

As $\Omega^*(C)$ is the free R-ring generated by $\overline{C}$, it follows that γ^* is a morphism of cochain R-rings. Thus, it induces an isomorphism of graded R-rings $E(C) \cong A$, which in turn can be used to identify $\mathrm{K}^*_r(E(C), C)$ and $\mathrm{K}^*_r(A, C)$. □

Corollary 3.10. *Let (A, C') and (A, C'') be Koszul pairs. Then $C' \cong C''$ as graded R-corings. Dually, if (A', C) and (A'', C) are Koszul, then $A' \cong A''$ as graded R-rings.*

Proof. By the preceding theorem, there are canonical coring isomorphisms $C' \cong T(A)$ and $C'' \cong T(A)$. Analogously, we have $A' \cong E(C)$ and $A'' \cong E(C)$. □

Corollary 3.11. *If (A, C) is a Koszul pair then $E(T(A)) \cong A$ and $T(E(C)) \cong C$.*

Proof. By Theorem 3.9, both $(A, T(A))$ and $(E(T(A)), T(A))$ are Koszul. Using the preceding corollary we conclude that $A \cong E(T(A))$. The proof of the second isomorphism is similar. □

3.12. Koszul rings. Let A be a connected R-ring. Following [5] we shall say that A is a left Koszul ring if R has a resolution $P_* \to R$ by projective graded left A-modules such that every P_n is generated by its homogeneous elements of degree n.

Theorem 3.13. *Let A be a connected R-ring. Then the following assertions are equivalent:*

(1) *A is Koszul.*

(2) *A is strongly graded and $(A, T(A))$ is Koszul.*

(3) *There exists a graded R-coring C such that (A, C) is Koszul.*

Proof. Let us assume that A is strongly graded and that $(A, T(A))$ is Koszul. Then $\mathrm{K}^l_*(A, T(A))$ is a resolution of R by graded projective A-modules. Clearly, by definition, every $\mathrm{K}^l_n(A, T(A))$ is generated as an A-module by $A^0 \otimes T_n(A)$, its homogeneous component of degree n. Hence A is Koszul.

Let A be a Koszul ring and let $V := A^1$. By [5], any Koszul ring is quadratic and generated by V, so there is $W \subseteq V \otimes V$ such that A is isomorphic as a graded R-ring with the quotient of $T^a_R(V)$ by the two-sided ideal generated by W. Furthermore, we define $C_0 := R$, $C_1 := V$ and

$$C_n = \bigcap_{p=0}^{n-2} V^{\otimes p} \otimes W \otimes V^{\otimes n-p-2},$$

for every $n \geq 2$. We claim that the R-bimodule $C := \bigoplus_{n\in\mathbb{N}} C_n$ is a graded subcoring of $T^c_R(V)$. It is enough to prove that $\Delta_{p,q}(c) \in C_p \otimes C_q$, for any $p, q \geq 0$ and $c \in C_{p+q}$. In the case when $p = 0$ or $q = 0$, we have nothing to prove. Let us assume that $p > 0$ and $q > 0$. Obviously, $\Delta_{p,q}(c) \in C_p \otimes T^q$, if $p = 1$. In the case

when $p \geq 2$ we have $\Delta_{p,q}(c) = c$, where in the right-hand side of this identity c is regarded as an element in $V^{\otimes p} \otimes V^{\otimes q}$. Thus

$$\Delta_{p,q}(c) \in \bigcap_{i=0}^{p+q-2} V^{\otimes i} \otimes W \otimes V^{\otimes p+q-i-2} \subseteq \bigcap_{i=0}^{p-2} V^{\otimes i} \otimes W \otimes V^{\otimes p+q-i-2} = C_p \otimes V^{\otimes q}.$$

Similarly, $\Delta_{p,q}(c) \in V^{\otimes p} \otimes C_q$. Thus $\Delta_{p,q}(c) \in C_p \otimes C_q$, that is our claim is proved.

By construction C is connected and $A^1 = C_1 = V$. Let $c \in C_2 = W$. Since $\Delta_{1,1}(c) = c$ and the multiplication of $T_R^a(V)$ is defined by the canonical isomorphisms $V^{\otimes p} \otimes V^{\otimes q} \cong V^{\otimes p+q}$, we have $m_A^{1,1}(\Delta_{1,1}(c)) = c + W = 0$. Thus (A, C) is almost-Koszul with respect to $\theta_{C,A} = \mathrm{I}_V$. Taking into account the definition of C it follows that $\mathrm{K}_*^l(A, C)$ coincides up to a degree shifting with the Koszul complex [5, p. 483], which is exact by assumption. In conclusion if A is Koszul, then there is a Koszul pair (A, C). Finally, by Theorem 3.9 we deduce that A is strongly graded and $(A, T(A))$ is Koszul, provided that there exists a Koszul pair (A, C). □

Corollary 3.14. *If A is left Koszul, then A is right Koszul, and conversely.*

Proof. Let A be a left Koszul ring. Then, by the preceding theorem, A is strongly graded and $(A, T(A))$ is Koszul. Since A^{op} is strongly graded and $(A^{\mathrm{op}}, T(A)^{\mathrm{op}})$ is Koszul, it follows that A^{op} is a left Koszul ring. Of course, this is equivalent to the fact that A is right Koszul. □

Remark 3.15. The results of this section suggest the following definition. A connected coring C is called Koszul if and only if C is strongly graded and $(E(C), C)$ is a Koszul pair. Koszul corings and some of their applications will be investigated in a sequel of this paper.

4. Hochschild (co)homology of Koszul rings

In this section R denotes a separable algebra over a field $\Bbbk$. Therefore, for any Koszul pair (A, C) the complex $\mathrm{K}_*(A, C)$ is a projective resolution of A in the category of right A-modules, cf. Corollary 3.5. We shall use this resolution to investigate the Hochschild (co)homology of A with coefficients in an A-bimodule M.

4.1. The cyclic tensor product. The tensor product algebra $R^e := R \otimes_{\Bbbk} R^{\mathrm{op}}$, between R and its opposite algebra R^{op}, is called the *enveloping algebra* of R.

Since R is an algebra over the field $\Bbbk$, we have to adapt the definition of R-bimodules to reflect this extra structure. By an R-bimodule we mean a left (or a right) R^e-module V. Equivalently, V is a left and a right R module which satisfies the usual associativity relation $(rv)s = r(vs)$ and, in addition, the condition $xv = vx$ holds for all $x \in \Bbbk$ and $v \in V$.

For an R-bimodule V let $[R, V]$ be the linear space spanned by all commutators $[r, v] := rv - vr$, with $r \in R$ and $v \in V$. Let $V_R := V/[R, V]$.

Recall that $\otimes = \otimes_R$. For any R-bimodules $V_1, \dots, V_n$ the tensor product $V_1 \otimes \cdots \otimes V_n$ is a bimodule, so we may define *the cyclic tensor product* of $V_1, \dots, V_n$ by the relation

$$V_1 \mathbin{\widehat{\otimes}} \cdots \mathbin{\widehat{\otimes}} V_n := (V_1 \otimes \cdots \otimes V_n)_R.$$

For the equivalence class of $v_1 \otimes \cdots \otimes v_n$ in the cyclic tensor product we use the notation $v_1 \mathbin{\widehat{\otimes}} \cdots \mathbin{\widehat{\otimes}} v_n$. If V and W are R-bimodules, then the map $v \mathbin{\widehat{\otimes}} w \mapsto v \otimes_{R^e} w$ is an isomorphism $V \mathbin{\widehat{\otimes}} W \cong V \otimes_{R^e} W$, so

$$V_1 \mathbin{\widehat{\otimes}} \cdots \mathbin{\widehat{\otimes}} V_n \cong (V_1 \otimes \cdots \otimes V_i) \otimes_{R^e} (V_{i+1} \otimes \cdots \otimes V_n).$$

The cyclic tensor products $V \mathbin{\widehat{\otimes}} W$ and $W \mathbin{\widehat{\otimes}} V$ are isomorphic via the linear map $v \mathbin{\widehat{\otimes}} w \mapsto w \mathbin{\widehat{\otimes}} v$. Thus

$$V_1 \mathbin{\widehat{\otimes}} \cdots \mathbin{\widehat{\otimes}} V_n \cong V_2 \mathbin{\widehat{\otimes}} V_3 \mathbin{\widehat{\otimes}} \cdots \mathbin{\widehat{\otimes}} V_n \mathbin{\widehat{\otimes}} V_1 \cong \cdots \cong V_n \mathbin{\widehat{\otimes}} V_1 \mathbin{\widehat{\otimes}} \cdots \mathbin{\widehat{\otimes}} V_{n-1}.$$

4.2. The complex $\mathbf{K_*(A, M)}$. Let A be a Koszul ring over a separable $\Bbbk$-algebra R. The R-ring structure of A induces a canonical ring morphism from $\Bbbk$ to A. Since we are interested in the Hochschild (co)homology of A, we assume that the image of this map is central in A, that is A is a $\Bbbk$-algebra. In this setting we also define an A-bimodule as a left module over the enveloping algebra $A^e := A \otimes_{\Bbbk} A^{\mathrm{op}}$ of A. We fix a connected R-coring C such that (A, C) is Koszul. Recall that $C \cong T(A)$ as R-corings. By the above assumption, $T(A)$ is an R^e-module, so we may assume that C has the same property.

The Hochschild homology of A with coefficients in an A-bimodule M is defined by the relation $\mathrm{HH}_*(A, M) := \mathrm{Tor}_*^{A^e}(A, M)$. Let $\mathrm{K}'_*(A, C)$ denote the complex obtained from $\mathrm{K}_*(A, C)$ by dropping the part of degree -1. Thus the Hochschild homology of A with coefficients in M may be computed as the homology of the complex $\mathrm{K}'_*(A, C) \otimes_{A^e} M$. We identify the vector spaces $\mathrm{K}_n(A, C) \otimes_{A^e} M$ and $M \mathbin{\widehat{\otimes}} C_n \cong C_n \mathbin{\widehat{\otimes}} M$ via the map φ_n defined by

$$\varphi_n\big((x \otimes c \otimes y) \otimes_{A^e} m\big) = (ymx) \mathbin{\widehat{\otimes}} c.$$

Its inverse is given by $\varphi_n^{-1}(m \mathbin{\widehat{\otimes}} c) = (1 \otimes c \otimes 1) \otimes_{A^e} m$. Let $\partial_n \colon M \mathbin{\widehat{\otimes}} C_n \to M \mathbin{\widehat{\otimes}} C_{n-1}$ be the map

$$\partial_n := \varphi_{n-1} \circ (d_n \otimes_{A^e} \mathrm{I}_M) \circ \varphi_n^{-1},$$

where d_n denotes the differential of $\mathrm{K}_*(A, C)$. It is easy to see that ∂_n satisfies the following equation

$$\partial_n(m \mathbin{\widehat{\otimes}} c) = \sum m\theta_{C,A}\big(c_{(1,1)}\big) \mathbin{\widehat{\otimes}} c_{(2,n-1)} + (-1)^n \sum \theta_{C,A}\big(c_{(2,1)}\big)m \mathbin{\widehat{\otimes}} c_{(1,n-1)}. \quad (4.1)$$

Clearly, $\varphi_*\colon M \otimes_{A^e} \mathrm{K}'_*(A, C) \to (M \mathbin{\widehat{\otimes}} C_*, \partial_*)$ is an isomorphism. Hence we have the following result.

Theorem 4.3. *Let (A, C) be a Koszul pair over a separable $\Bbbk$-algebra R. The Hochschild homology of A with coefficients in M is the homology of the chain complex $\mathrm{K}_*(A, M) = M \mathbin{\widehat{\otimes}} C_*$. For $m \in M$ and $c \in C_n$, the differential ∂_n of this complex is given by equation (4.1).*

Recall that the Hochschild cohomology of A with coefficients in an A-bimodule M is defined by $\mathrm{HH}^*(A, M) := \mathrm{Ext}^*_{A^e}(A, M)$. For a Koszul pair (A, C), proceeding as in the proof of the preceding theorem, we obtain a complex to compute Hochschild cohomology of A.

Theorem 4.4. *Let (A, C) be a Koszul pair over a separable $\Bbbk$-algebra R. The Hochschild cohomology of A with coefficients in M is the cohomology of the cochain complex $\mathrm{K}^*(A, M) = \mathrm{Hom}_{R^e}(C_*, M)$. For $c \in C_{n+1}$ and $f \in \mathrm{Hom}_{R^e}(C_n, M)$ the differential ∂^n of this complex is given by*

$$\partial^n(f)(c) = \sum \theta_{C,A}\big(c_{(1,1)}\big) f\big(c_{(2,n)}\big) + (-1)^{n+1} \sum f\big(c_{(1,n)}\big)\theta_{C,A}\big(c_{(2,1)}\big). \quad (4.2)$$

As an application of our previous results, we compute $\mathrm{Hdim}\, A$, the Hochschild dimension of a Koszul ring A. By definition, $\mathrm{Hdim}\, A$ is the projective dimension of A as a left (or right) A^e-module. Hence $\mathrm{Hdim}\, A = n$ if and only if $\mathrm{HH}^{n+1}(A, M) = 0$ for any bimodule M, but there is at least one bimodule M_0 such that $\mathrm{HH}^n(A, M_0) \neq 0$. Of course, if such an n does not exist, then we say that the Hochschild dimension of A is infinite. The projective dimension of R as a left A-module will be denoted by $\mathrm{pd}({}_A R)$. For the projective dimension of the right A-module R we shall use the notation $\mathrm{pd}(R_A)$.

Theorem 4.5. *If (A, C) is a Koszul pair over a separable $\Bbbk$-algebra R, then*

$$\mathrm{Hdim}\, A = \mathrm{pd}({}_A R) = \mathrm{pd}(R_A) = \sup\{n \mid C_n \neq 0\}.$$

Proof. Obviously, $\mathrm{pd}({}_A R) = \mathrm{pd}(R_A)$. Let us assume that $C_{d+1} = 0$, for some d. Then $\mathrm{K}^l_*(A, C)$ provides a projective resolution of R in the category of left A-modules of length at most d . Thus, for $n > d$, we have $C_n \cong \mathrm{Tor}^A_n(R, R)$=0. It follows that $S := \sup\{n \mid C_n \neq 0\} \leq d$. In particular, if $\mathrm{pd}(R_A) \leq d$, then $S \leq d$. On the other hand, assuming that the latter inequality holds, we get $C_{d+1} = 0$. Thus, in view of the foregoing remarks, $\mathrm{pd}({}_R A) \leq d$. In conclusion, $S = \mathrm{pd}({}_R A)$.

It remains to prove that $\mathrm{Hdim}\, A \leq d$ if and only if $\mathrm{pd}({}_R A) \leq d$. Let us suppose that the Hochschild dimension of A is less than or equal to d. Therefore, the d-syzygy M of the resolution $\mathrm{K}_*(A, C)$ is a projective A-bimodule such that the sequence

$$0 \longleftarrow A \longleftarrow \mathrm{K}_0(A, C) \longleftarrow \cdots \longleftarrow \mathrm{K}_{d-2}(A, C) \longleftarrow \mathrm{K}_{d-1}(A, C) \longleftarrow M \longleftarrow 0.$$

is exact. Proceeding as in the proof of Proposition 3.2, one shows that this sequence splits in the category of right A-modules. Hence, by applying the functor $(-)\otimes_A R$ we get a resolution of R by projective left A-modules. Since the length of this resolution is at most d we conclude that $\mathrm{pd}({}_R A) \leq d$. Conversely, if the projective dimension of R as a left A-module is less than or equal to d, then $C_{d+1} \cong T_d(A) = 0$. Thus $\mathrm{K}_*(A, C)$ is a resolution of A of length less than or equal to d, that is $\mathrm{Hdim}\, A \leq d$. □

We conclude this section by giving a first example of Koszul pair, which we shall use later for the computation of Hochschild (co)homology of generalized Ore extensions. For every R-bimodule V let $T := T_R^a(V)$. Let $C = R \oplus V$ denote the connected coring with the comultiplication Δ uniquely defined such that $\Delta(v) = v \otimes 1 + 1 \otimes v$, for any $v \in V$. We take $\theta_{T,C}$ to be the identity map.

Proposition 4.6. *The pair (T, C) is Koszul.*

Proof. The identity from the definition of almost-Koszul pairs is automatically verified, as $C_2 = 0$. Thus (T, C) is such a pair. Furthermore, $\mathrm{K}_*^l(T, C)$ is the complex

$$0 \longleftarrow R \longleftarrow T \longleftarrow T \otimes V \longleftarrow 0,$$

whose non-zero arrows are the projection π_T^0 of T on T^0 and d_1^l. Since the multiplication in T is given by concatenation of tensor monomials, and $\Delta_{1,0}(v) = v \otimes 1$ we deduce that $d_1^l(x \otimes v) = x \otimes v$. Therefore, d_1^l is the identity map of $T \otimes V = \overline{T}$. Hence $\mathrm{K}_*^l(T, C)$ is exact. □

Remark 4.7. By Theorem 4.5 it follows that $\mathrm{Hdim}\, T_R^a(V) = 1$, for any separable algebra R and any R-bimodule V. This property of tensor algebras was proved in [11], where the algebras of Hochschild dimension one are called quasi-free and they represent the key ingredient in the definition of nonsingularity in Noncommutative Geometry. The Koszulity of tensor algebras is also proved in the last section of the paper, as a consequence of the fact that they are braided bialgebras.

5. Almost-Koszul pairs associated to twisted tensor products

We keep the notation and the assumptions from the first section. In this section we consider two connected strongly graded R-rings A and B together with an invertible

graded twisting map $\sigma\colon B \otimes A \to A \otimes B$. Our first aim is to show that σ induces a canonical graded twisting map of R-corings

$$\tau\colon T(A) \otimes T(B) \to T(B) \otimes T(A).$$

This construction is performed such that $(A \otimes_\sigma B, T(A) \otimes_\tau T(B))$ is an almost-Koszul pair, where $A \otimes_\sigma B$ and $T(A) \otimes_\tau T(B)$ denote the twisted tensor product R-ring and the twisted tensor product R-coring with respect σ and τ, respectively. Furthermore, if A and B are Koszul and σ is invertible, then we shall show that $A \otimes_\sigma B$ is a Koszul R-ring.

5.1. Twisting maps of R-rings. Let A and B be R-rings. A *twisting map* between A and B is given by an R-bilinear map $\sigma\colon B \otimes A \to A \otimes B$ which is compatible with the multiplication of A and B, i.e.

$$\sigma \circ (\mathrm{I}_B \otimes m_A) = (m_A \otimes \mathrm{I}_B) \circ (\mathrm{I}_A \otimes \sigma) \circ (\sigma \otimes \mathrm{I}_A), \tag{5.1}$$

$$\sigma \circ (m_B \otimes \mathrm{I}_A) = (\mathrm{I}_A \otimes m_B) \circ (\sigma \otimes \mathrm{I}_B) \circ (\mathrm{I}_B \otimes \sigma). \tag{5.2}$$

By definition, σ must be compatible with the units of A and B as well. Therefore, $\sigma(1_B \otimes a) = a \otimes 1_B$ and $\sigma(b \otimes 1_A) = 1_A \otimes b$, for all $a \in A$ and $b \in B$. In computations we shall write $\sigma(b \otimes a) \in A \otimes B$ as a formal sum $\sum a_\sigma \otimes b_\sigma$. Thus, for instance

$$\big[(m_A \otimes \mathrm{I}_B) \circ (\mathrm{I}_A \otimes \sigma) \circ (\sigma \otimes \mathrm{I}_A)\big](b \otimes a' \otimes a'') = \sum a'_\sigma a''_{\sigma'} \otimes (b_\sigma)_{\sigma'}.$$

The occurrence of σ and σ' in the above identity indicates that the twisting map is applied twice.

If σ is a twisting map then $A \otimes B$ has a canonical R-ring structure with respect to the multiplication

$$(a' \otimes b')(a'' \otimes b'') = \sum a' a''_\sigma \otimes b'_\sigma b''$$

and the unit $1_A \otimes 1_B$. The twisted tensor product will be denoted by $A \otimes_\sigma B$. In the case when R is commutative and A and B are R-algebras, the twisted tensor product $A \otimes_\sigma B$ may be seen as a deformation of the usual tensor product algebra.

Let us now assume that the R-rings A and B are both graded. A twisting map σ between A and B is called *graded* if $\sigma(B^p \otimes A^q) \subseteq A^q \otimes B^p$. The restrictions of σ to $B^p \otimes A^q$ will be denoted by $\sigma^{p,q}$. For such a σ the R-ring $A \otimes_\sigma B$ is graded and its homogeneous component of degree n is the direct sum of all R-bimodules $A^p \otimes B^q$ with $p + q = n$.

5.2. Twisting maps of cochain R-rings. We now assume that (Ω^*, d^*_Ω) and (Γ^*, d^*_Γ) are cochain R-rings. A graded twisting map $\sigma^{*,*}\colon \Gamma^* \otimes \Omega^* \to \Omega^* \otimes \Gamma^*$ is called a *twisting map of cochain R-rings* if $\sigma^{*,*}$ is compatible with the differential maps of Ω^* and Γ^*, in the sense that $\sigma^{*,q}$ is a map of complexes from $(\Gamma^* \otimes \Omega^q, d^*_\Gamma \otimes \mathrm{I}_{\Omega^q})$

to $(\Omega^q \otimes \Gamma^*, \mathrm{I}_{\Omega^q} \otimes d_\Gamma^*)$, for all q. Symmetrically, $\sigma^{p,*}$ must be a morphism of complexes from $(\Gamma^p \otimes \Omega^*, \mathrm{I}_{\Gamma^p} \otimes d_\Omega^*)$ to $(\Omega^* \otimes \Gamma^p, d_\Omega^* \otimes \mathrm{I}_{\Gamma^p})$, for all p.

Proposition 5.3. *Let (Ω^*, d_Ω^*) and (Γ^*, d_Γ^*) be cochain R-rings. Suppose that V is an R-bimodule.*

(1) *If $\varphi^*: (V \otimes \Omega^*, \mathrm{I}_V \otimes d_\Omega^*) \to (\Omega^* \otimes V, d_\Omega^* \otimes \mathrm{I}_V)$ is a morphism of complexes which is compatible with the multiplication and the unit of Ω^*, then $\overline{\varphi}^* := \mathrm{H}^*(\varphi^*)$ is compatible with the multiplication and the unit of the graded R-ring $\mathrm{H}^*(\Omega^*)$.*

(2) *If $\chi^*: (\Gamma^* \otimes V, d_\Gamma^* \otimes \mathrm{I}_V) \to (V \otimes \Gamma^*, \mathrm{I}_V \otimes d_\Gamma^*)$ is a morphism of complexes which is compatible with the multiplication and the unit of Γ^*, then $\overline{\chi}^* := \mathrm{H}^*(\chi^*)$ is compatible with the multiplication and the unit of the graded R-ring $\mathrm{H}^*(\Gamma^*)$.*

(3) *Every twisting map of cochain R-rings $\sigma^{*,*}: \Gamma^* \otimes \Omega^* \to \Omega^* \otimes \Gamma^*$ induces a twisting map of graded R-rings $\overline{\sigma}^{*,*}: \mathrm{H}^*(\Gamma^*) \otimes \mathrm{H}^*(\Omega^*) \to \mathrm{H}^*(\Omega^*) \otimes \mathrm{H}^*(\Gamma^*)$.*

Proof. For every $p > 0$ the morphism φ^* induces a map

$$\overline{\varphi}^p: \mathrm{H}^p(V \otimes \Omega^*) \to \mathrm{H}^p(\Omega^* \otimes V).$$

By assumption every left or right R-module is flat. Hence, $\overline{\varphi}^p$ can be seen as a map from $V \otimes \mathrm{H}^p(\Omega^*)$ to $\mathrm{H}^p(\Omega^*) \otimes V$. For $x \in \mathrm{Z}^p(\Omega^*)$ let $[x]$ denote its cohomology class. Note that $\varphi^p(v \otimes x)$ is an element in $\mathrm{Z}^p(\Omega^*) \otimes V$, as φ^* is a morphism of complexes. So $\varphi^p(v \otimes x) = \sum x_\varphi \otimes v_\varphi$, for some $x_\varphi \in \mathrm{Z}^p(\Omega^*)$. Hence

$$\overline{\varphi}^p\big(v \otimes [x]\big) = \sum [x_\varphi] \otimes v_\varphi. \tag{5.3}$$

Since φ^* is compatible with the multiplication of Ω^*, we get

$$\sum \big[(xy)_\varphi\big] \otimes v_\varphi = \sum [x_\varphi y_{\varphi'}] \otimes (v_\varphi)_{\varphi'}.$$

Thus, by the definition of the multiplication in $\mathrm{H}^*(\Omega^*)$, it follows

$$\sum \big([x][y]\big)_{\overline{\varphi}} \otimes v_{\overline{\varphi}} = \sum [x]_{\overline{\varphi}}[y]_{\overline{\varphi}'} \otimes (v_{\overline{\varphi}})_{\overline{\varphi}'}.$$

In conclusion, $\overline{\varphi}^*$ is also compatible with the multiplication of $\mathrm{H}^*(\Omega^*)$. By the definition of $\overline{\varphi}^*$ one can easily see that this family of R-bilinear maps is compatible with the unit of $\mathrm{H}^*(\Omega^*)$.

We omit the proof of the second statement, being similar to the above one.

Let σ be a twisting map of cochain R-rings. If $p \geq 0$ then $\sigma^{p,*}$ is a morphism of complexes from $\Gamma^p \otimes \Omega^*$ to $\Omega^* \otimes \Gamma^p$ which is compatible with the multiplication and the unit of Ω^*. By the first part of the proposition it follows that

$$\mathrm{H}^*(\sigma^{p,*}): \Gamma^p \otimes \mathrm{H}^*(\Omega) \to \mathrm{H}^*(\Omega^*) \otimes \Gamma^p$$

is compatible with the multiplication and the unit of $\mathrm{H}^*(\Omega^*)$. For a given $q \geq 0$, the family of R-bilinear maps $\{\mathrm{H}^q(\sigma^{p,*})\}_{p \geq 0}$ is a morphism of complexes from

$\Gamma^* \otimes \mathrm{H}^q(\Omega)$ to $\mathrm{H}^q(\Omega^*) \otimes \Gamma^*$ which is compatible with the multiplication and the unit of Γ^*. Applying the second part of the proposition, for every p and q, we get a map

$$\overline{\sigma}^{\,p,q}\colon \mathrm{H}^p(\Gamma^*) \otimes \mathrm{H}^q(\Omega^*) \to \mathrm{H}^q(\Omega^*) \otimes \mathrm{H}^p\,\Gamma^*, \quad \overline{\sigma}^{\,p,q}\big([x] \otimes [y]\big) = \sum [y_\sigma] \otimes [x_\sigma],$$

such that $\overline{\sigma}^{\,*,*}$ is a graded twisting map of graded R-rings. □

5.4. Twisting maps of R-corings. Let C and D be R-corings. A *twisting map between C and D* is an R-bilinear map $\tau\colon C \otimes D \to D \otimes C$ compatible with the comultiplication of C and D, that is

$$(\Delta_D \otimes \mathrm{I}_C) \circ \tau = (\mathrm{I}_D \otimes \tau) \circ (\tau \otimes \mathrm{I}_D) \circ (\mathrm{I}_C \otimes \Delta_D), \tag{5.4}$$

$$(\mathrm{I}_D \otimes \Delta_C) \circ \tau = (\tau \otimes \mathrm{I}_C) \circ (\mathrm{I}_C \otimes \tau) \circ (\Delta_C \otimes \mathrm{I}_D). \tag{5.5}$$

By definition, τ must be compatible with the counits of C and D as well. Thus, $(\mathrm{I}_D \otimes \varepsilon_C) \circ \tau = \varepsilon_C \otimes \mathrm{I}_D$ and $(\varepsilon_D \otimes \mathrm{I}_C) \circ \tau = \mathrm{I}_C \otimes \varepsilon_D$.

For a twisting map of corings we use the notation $\tau(c \otimes d) = \sum d_\tau \otimes c_\tau$, for all $c \in C$ and $d \in D$. The tensor product $C \otimes D$ has a canonical R-coring structure, that will be denoted by $C \otimes_\tau D$. The counit of this R-coring is $\varepsilon_C \otimes \varepsilon_D$, and its comultiplication Δ is defined by the formula

$$\Delta := (\mathrm{I}_C \otimes \tau \otimes \mathrm{I}_D) \circ (\Delta_C \otimes \Delta_D).$$

Let us now assume that C and D are graded corings. A twisting map of corings $\tau\colon C \otimes D \to D \otimes C$ is called *graded* if $\tau(C_p \otimes D_q) \subseteq D_q \otimes C_p$. The restriction of τ to $C_p \otimes D_q$ will be denoted by $\tau_{p,q}$. Clearly, in this case $C \otimes_\tau D$ is a graded R-coring, whose homogeneous component of degree n is the direct sum of all bimodules $C_p \otimes D_q$ with $p + q = n$.

5.5. Twisting maps of chain corings. We now assume that (Ω_*, d^Ω_*) and (Γ_*, d^Γ_*) are chain R-corings. A graded twisting map $\tau_{*,*}\colon \Omega_* \otimes \Gamma_* \to \Gamma_* \otimes \Omega_*$ is called a *twisting map of chain R-corings* if $\tau_{*,q}$ is a map of complexes from $(\Omega_* \otimes \Gamma_q, d^\Omega_* \otimes \mathrm{I}_{\Gamma_q})$ to $(\Gamma_q \otimes \Omega_*, \mathrm{I}_{\Gamma_q} \otimes d^\Omega_*)$, for any q. In addition, $\tau_{p,*}$ is a morphism of complexes from $(\Omega_p \otimes \Gamma_*, \mathrm{I}_{\Omega_p} \otimes d^\Gamma_*)$ to $(\Gamma_* \otimes \Omega_p, d^\Gamma_* \otimes \mathrm{I}_{\Omega_p})$, for any p.

Reasoning as in the proof of Proposition 5.3, one can show that the following result holds true.

Proposition 5.6. *Let (Ω_*, d^Ω_*) and (Γ_*, d^Γ_*) be chain corings. Suppose that V is an R-bimodule.*

(1) *If $\varphi_*\colon (V \otimes \Gamma_*, \mathrm{I}_V \otimes d^\Gamma_*) \to (\Gamma_* \otimes V, d^\Gamma_* \otimes \mathrm{I}_V)$ is a morphism of complexes that is compatible with the comultiplication and the counit of Γ_* then $\overline{\varphi}_* := \mathrm{H}_*(\varphi_*)$ is also compatible with the comultiplication and the counit of the graded R-coring $\mathrm{H}_*(\Gamma_*)$.*

(2) *If* $\chi_*: (\Omega_* \otimes V, d_*^{\Omega} \otimes \mathrm{I}_V) \to (V \otimes \Omega_*, \mathrm{I}_V \otimes d_*^{\Omega})$ *is a morphism of complexes that is compatible with the comultiplication and the counit of* Ω_* *then* $\overline{\chi}_* := \mathrm{H}_*(\chi_*)$ *is also compatible with the comultiplication and the counit of the graded* R*-coring* $\mathrm{H}_*(\Omega_*)$.

(3) *If* $\tau_{*,*}: \Omega_* \otimes \Gamma_* \to \Gamma_* \otimes \Omega_*$ *is a twisting map of chain corings, then* $\tau_{*,*}$ *induces a twisting map of graded* R*-corings*

$$\overline{\tau}_{*,*}: \mathrm{H}_*(\Omega_*) \otimes \mathrm{H}_*(\Gamma_*) \to \mathrm{H}_*(\Gamma_*) \otimes \mathrm{H}_*(\Omega_*).$$

5.7. Entwining maps. Let A be an R-ring, and let C be an R-coring. We say that a bimodule morphism $\lambda: C \otimes A \to A \otimes C$ is an *entwining map* if $\lambda(c \otimes 1_A) = 1_A \otimes c$ and $(\mathrm{I}_A \otimes \varepsilon_C) \circ \lambda = \varepsilon_C \otimes \mathrm{I}_A$, and the following relations hold

$$\lambda \circ (\mathrm{I}_C \otimes m_A) = (m_A \otimes \mathrm{I}_C) \circ (\mathrm{I}_A \otimes \lambda) \circ (\lambda \otimes \mathrm{I}_A), \tag{5.6}$$

$$(\mathrm{I}_A \otimes \Delta_C) \circ \lambda = (\lambda \otimes \mathrm{I}_C) \circ (\mathrm{I}_C \otimes \lambda) \circ (\Delta_C \otimes \mathrm{I}_A). \tag{5.7}$$

Similarly one can define an entwining structure $\nu: A \otimes C \to C \otimes A$.

Let us now assume that A and C are both graded. An entwining map $\lambda: C \otimes A \to A \otimes C$ is called *graded* if $\lambda(C_p \otimes A^q) \subseteq A^q \otimes C_p$. The restriction of λ to $C_p \otimes A^q$ will be denoted by λ_p^q.

Let (Ω^*, d_{Ω}^*) and (Γ_*, d_*^{Γ}) be a cochain R-ring and a chain R-coring, respectively. A graded entwining map $\lambda_*^*: \Gamma_* \otimes \Omega^* \to \Omega^* \otimes \Gamma_*$ is called a *differential entwining map* if $\lambda_p^*: \Gamma_p \otimes \Omega^* \to \Omega^* \otimes \Gamma_p$ and $\lambda_*^q: \Gamma_* \otimes \Omega^q \to \Omega^q \otimes \Gamma_*$ are morphisms of complexes, for any p and q. We state for future reference, without proof, the following proposition.

Proposition 5.8. *Let* (Ω^*, d_{Ω}^*) *and* (Γ_*, d_*^{Γ}) *be a cochain* R*-ring and a chain* R*-coring, respectively. Any differential entwining map* $\lambda_*^*: \Gamma_* \otimes \Omega^* \to \Omega^* \otimes \Gamma_*$ *induces a graded entwining map*

$$\overline{\lambda}_*^*: \mathrm{H}_*(\Gamma_*) \otimes \mathrm{H}^*(\Omega^*) \to \mathrm{H}^*(\Omega^*) \otimes \mathrm{H}_*(\Gamma_*).$$

It is well known that any $\Bbbk$-linear map $\sigma^{1,1}: N \otimes_{\Bbbk} M \to M \otimes_{\Bbbk} N$ can be extended in a unique way to a graded twisting map $\sigma: T(N) \otimes_{\Bbbk} T(M) \to T(M) \otimes_{\Bbbk} T(N)$ between the free algebras generated by M and N. We shall adapt the method from [4] in order to produce examples of twisting maps of chain coalgebras and differential entwining maps. Some of them will be used later on in the paper to show that the twisting tensor product of two Koszul R-rings is Koszul.

Recall that, for every R-bimodule V, the graded R-ring $T_R^a(V)$ and the graded R-coring $T_R^c(V)$ have the same homogeneous component of degree n, namely $V^{\otimes n}$. Their multiplication and comultiplication are defined by the canonical isomorphism $V^{\otimes p} \otimes V^{\otimes q} \xrightarrow{\cong} V^{\otimes p+q}$ and its inverse, respectively.

Proposition 5.9. *Let $\varphi^{V,W}: W \otimes V \to V \otimes W$ be an R-bimodule map.*

(1) *There exists a unique bimodule map $\varphi_*^{V,W}: T_R^c(W) \otimes V \to V \otimes T_R^c(W)$ verifying the relation $\varphi_1^{V,W} = \varphi^{V,W}$ and which is compatible with the graded coring structure of $T_R^c(W)$.*

(2) *If A is an R-ring and $\varphi^{A,W}$ is compatible with the multiplication and the unit of A, then $\varphi_*^{A,W}$ is an entwining map. Moreover, $\varphi_*^{A,W}$ is graded, provided that A is graded and $\varphi^{A,W}$ maps $W \otimes A^q$ to $A^q \otimes W$ for all q.*

(3) *If, in addition, B is a connected R-ring and $\varphi^{A,\overline{B}}$ is compatible with the multiplication and the unit of $\overline{B}$, then $\varphi_*^{A,\overline{B}}$ is an entwining map from $(\Omega_*(B) \otimes A, \partial_* \otimes \mathrm{I}_A)$ to $(A \otimes \Omega_*(B), \mathrm{I}_A \otimes \partial_*)$ which commutes with the differentials.*

(4) *If (A, d^*) is a cochain R-ring and $\varphi^{A,\overline{B}}$ is a morphism of complexes as in (3), then $\varphi_*^{A,\overline{B}}$ is a differential entwining map.*

Proof. In the case when R is a field, the first part of the lemma is proved in [4]. The same proof works in our setting as well. Let us assume that we have already constructed $\varphi_*^{V,W}$. By the definition of the coring $T_R^c(W)$, the compatibility of $\varphi_*^{V,W}$ with $\Delta_{p,q}$ is equivalent to the relation

$$\varphi_{p+q}^{V,W} = \left(\varphi_p^{V,W} \otimes \mathrm{I}_{W^{\otimes q}}\right) \circ \left(\mathrm{I}_{W^{\otimes p}} \otimes \varphi_q^{V,W}\right), \tag{5.8}$$

where p and q are arbitrary nonnegative integers. In particular, if the map $\varphi_*^{V,W}$ exists it is uniquely defined by the condition $\varphi_1^{V,W} = \varphi^{V,W}$. On the other hand, to prove the existence of $\varphi_*^{V,W}$ we can proceed as follows. The map $\varphi_0^{V,W}$ must be the canonical identification $R \otimes V \cong V \otimes R$, as $\varphi_*^{V,W}$ is compatible with the counit of $T_R^c(W)$. We set $\varphi_1^{V,W} = \varphi^{V,W}$ and, for $p > 1$, we define $\varphi_p^{V,W}$ by

$$\begin{aligned}\varphi_p^{V,W} := \left(\varphi^{V,W} \otimes \mathrm{I}_{W^{\otimes p-1}}\right) \circ \left(\mathrm{I}_W \otimes \varphi^{V,W} \otimes \mathrm{I}_{W^{\otimes p-2}}\right) \circ \cdots \\ \cdots \circ \left(\mathrm{I}_{W^{\otimes p-2}} \otimes \varphi^{V,W} \otimes \mathrm{I}_W\right) \circ \left(\mathrm{I}_{W^{\otimes p-1}} \otimes \varphi^{V,W}\right).\end{aligned}$$

It follows easily by induction on q that the relation (5.8) is true for any p and q.

For the second part of the proposition we have to prove that $\varphi_*^{A,W}$ is compatible with the unit and the multiplication of A. Both compatibility conditions follow by induction on p, using on the one hand the relation (5.8) written for $q = 1$ and the fact that $\varphi_1^{A,W} = \varphi^{A,W}$ is compatible with the ring structure of A, on the other hand. Clearly, if A is graded, then $\varphi_*^{A,W}$ maps $W^{\otimes p} \otimes A^q$ to $A^q \otimes W^{\otimes p}$. In the graded case we shall denote the restriction of $\varphi_p^{A,W}$ to $W^{\otimes p} \otimes A^q$ by $\varphi_{p,q}^{A,W}$.

Let us prove the third part of the proposition. Recall that $\Omega_*(B) = T^c_R(\overline{B})$ as R-corings. Using the relation (5.8) we get

$$\varphi^{A,\overline{B}}_p = \left(\varphi^{A,\overline{B}}_i \otimes \mathrm{I}_{\overline{B}^{\otimes p-i}}\right) \circ \left(\mathrm{I}_{\overline{B}^{\otimes i}} \otimes \varphi^{A,\overline{B}}_1 \otimes \mathrm{I}_{\overline{B}^{\otimes p-i-1}}\right)$$
$$\circ \left(\mathrm{I}_{\overline{B}^{\otimes i+1}} \otimes \varphi^{A,\overline{B}}_1 \otimes \mathrm{I}_{\overline{B}^{\otimes p-i-2}}\right) \circ \left(\mathrm{I}_{\overline{B}^{\otimes i+2}} \otimes \varphi^{A,\overline{B}}_{p-i-2}\right),$$
$$\varphi^{A,\overline{B}}_{p-1} = \left(\varphi^{A,\overline{B}}_i \otimes \mathrm{I}_{\overline{B}^{\otimes p-i-1}}\right) \circ \left(\mathrm{I}_{\overline{B}^{\otimes i}} \otimes \varphi^{A,\overline{B}}_1 \otimes \mathrm{I}_{\overline{B}^{\otimes p-i-2}}\right) \circ \left(\mathrm{I}_{\overline{B}^{\otimes i+1}} \otimes \varphi^{A,\overline{B}}_{p-i-2}\right).$$

Since, by assumption, $\varphi^{A,\overline{B}}_1 = \varphi^{A,\overline{B}}$ is compatible with the multiplication of B, we deduce that

$$\varphi^{A,\overline{B}}_{p-1} \circ \left(\mathrm{I}_{\overline{B}^{\otimes i}} \otimes m_B \otimes \mathrm{I}_{\overline{B}^{\otimes p-i-2}} \otimes \mathrm{I}_A\right) = \left(\mathrm{I}_A \otimes \mathrm{I}_{\overline{B}^{\otimes i}} \otimes m_B \otimes \mathrm{I}_{\overline{B}^{\otimes p-i-2}}\right) \circ \varphi^{A,\overline{B}}_p,$$

for any $i \in \{0, \dots, p-2\}$. Taking into account the definition of ∂_* (see §2.5), the above identity implies that $\varphi^{A,\overline{B}}_*$ is a morphism of complexes.

Let us show that $\varphi^{A,\overline{B}}_*$ is a differential entwining map, provided that (A, d^*) is a cochain R-ring and $\varphi^{A,\overline{B}}$ is a morphism of complexes. Hence, it remains to prove that

$$\varphi^{A,\overline{B}}_{p,q+1} \circ \left(\mathrm{I}_{\Omega_p(B)} \otimes d^q\right) = \left(d^q \otimes \mathrm{I}_{\Omega_p(B)}\right) \circ \varphi^{A,\overline{B}}_{p,q}, \tag{5.9}$$

for any $p, q \geq 0$. If $p = 0$ then this relation is trivially true for any q, as $\varphi^{A,\overline{B}}_{0,q}$ is the canonical identification $R \otimes A^q \cong A^q \otimes R$. For $p = 1$ the equation holds as well, since $\varphi^{A,\overline{B}}_{1,*} : \overline{B} \otimes A^* \to A^* \otimes \overline{B}$ is a morphism of complexes by assumption. Let us suppose that (5.9) is true for a given p and any $q \geq 0$. By using the recurrence relation that defines $\varphi^{A,\overline{B}}_*$ and the fact that $\varphi^{A,\overline{B}}_{p,*}$ and $\varphi^{A,\overline{B}}_{1,*}$ are morphisms of complexes we deduce that (5.9) holds for $p+1$ and any q. □

Proceeding in a similar way one proves the proposition below. Starting with a bimodule map, we now produce examples of twisting maps of (graded or chain) R-corings.

Proposition 5.10. *Let $\psi^{V,W} : W \otimes V \to V \otimes W$ be an R-bimodule map.*

(1) *There exists a unique graded R-bimodule map $\psi^{V,W}_* : W \otimes T^c_R(V) \to T^c_R(V) \otimes W$ which verifies the relation $\psi^{V,W}_1 = \psi^{V,W}$ and is compatible with the coring structure of $T^c_R(V)$.*

(2) *If C is a R-coring and $\psi^{V,C}$ is compatible with the comultiplication and the counit of C, then $\psi^{V,C}_*$ is a twisting map of corings. Moreover, $\psi^{V,C}_*$ is graded, provided that C is graded and $\psi^{V,C}$ maps $C_p \otimes V$ to $V \otimes C_p$ for all p. In this case we shall use the notation $\psi^{V,C}_{p,q} := \psi^{V,C}_p|_{C_p \otimes V^{\otimes q}}$.*

(3) *If, in addition A is a connected R-ring and $\psi^{\overline{A},C}_*$ is compatible with the comultiplication and the counit of C, then the twisting map $\psi^{\overline{A},C}_*$ is a morphism of chain complexes from $(C \otimes \Omega_*(A), \mathrm{I}_C \otimes \partial_*)$ to $(\Omega_*(A) \otimes C, \partial_* \otimes \mathrm{I}_C)$.*

(4) *If (C, d_*) is a chain coring and $\psi^{\bar{A},C}$ is a morphism of complexes as in (3), then $\psi_*^{\bar{A},C}$ is a twisting map of chain corings.*

Remark 5.11. If we apply Proposition 5.9 and Proposition 5.10 to the same map $\varphi^{V,W} = \psi^{V,W}$, then we get two different morphisms $\varphi_*^{V,W}$ and $\psi_*^{V,W}$, that can be seen as left-right symmetric version of each other. By analogy, we can also define the morphisms

$$\widetilde{\varphi}_*^{V,B}\colon B \otimes T_R^c(V) \to T_R^c(V) \otimes B \quad \text{and} \quad \widetilde{\varphi}_*^{\bar{A},B}\colon B \otimes \Omega_*(A) \to \Omega_*(A) \otimes B,$$

the symmetric versions of $\varphi^{A,W}$ and $\varphi^{\bar{A},B}$, respectively. Again by symmetry, there are morphisms

$$\widetilde{\psi}_*^{D,W}\colon T_R^c(W) \otimes D \to D \otimes T_R^c(W) \quad \text{and} \quad \widetilde{\psi}_*^{D,\bar{B}}\colon \Omega_*(B) \otimes D \to D \otimes \Omega_*(B),$$

for any R-coring D. For the sake of completeness, let us mention that the results from Proposition 5.9 and Proposition 5.10 can be easily dualized. In this way we obtain twisted tensor products of (graded or chain) R-rings in which one of the factors is either $T_R^a(V)$ or $\Omega^*(C)$. We do not state the dual results in detail, as we shall not use them in this paper.

5.12. The entwining maps λ and ν. Let A and B be two connected strongly graded R-rings. Then it makes sense to consider the almost-Koszul pairs $(A, T(A))$ and $(B, T(B))$. For ease of notation, we shall write C and D instead of $T(A)$ and $T(B)$, respectively. Recall that, by definition, (A, C) and (B, D) are endowed with two R-bimodule isomorphisms $\theta_{C,A}\colon C_1 \to A^1$ and $\theta_{D,B}\colon D_1 \to B^1$.

We now assume, in addition, that $\sigma\colon B \otimes A \to A \otimes B$ is a given invertible graded twisting map. Obviously, the inverse σ^{-1} of σ is also a twisting map or rings. Note that $(\sigma^{-1})^{p,q} = (\sigma^{q,p})^{-1}$. We claim that, under these assumptions, there is a graded entwining map $\lambda\colon C \otimes B \to B \otimes C$ that extends in a certain sense the inverse of σ. Indeed, by taking $\varphi^{B,\bar{A}} := \sigma^{-1}|_{\bar{A}\otimes B}$ in Proposition 5.9, we get an entwining map $\varphi_*^{B,\bar{A}}$ between $\Omega_*(A)$ and B, which is a morphism of complexes. Since any left and right R-module is flat we get the following relations

$$\begin{aligned} \mathrm{H}_p(\Omega_*(A) \otimes B) &\cong \mathrm{H}_p(\Omega_*(A)) \otimes B = C_p \otimes B, \\ \mathrm{H}_p(B \otimes \Omega_*(A)) &\cong B \otimes \mathrm{H}_p(\Omega_*(A)) = B \otimes C_p. \end{aligned}$$

Therefore, by Proposition 5.6 (2), the induced morphism $\lambda\colon C \otimes B \to B \otimes C$ is a graded entwining map.

By symmetry (see Remark 5.11), if we take $\varphi^{\bar{B},A}$ to be the restriction of σ^{-1} to $A \otimes \bar{B}$, then $\widetilde{\varphi}_*^{\bar{B},A}$ induces another graded entwining map $\nu\colon A \otimes D \to D \otimes A$.

5.13. The twisting map τ'. Under the same assumptions as above, we can now construct a twisting map τ between C and D. We apply Proposition 5.10 for $\psi^{\bar{B},C} := \lambda|_{C\otimes\bar{B}}$ to get a twisting map $\psi_*^{\bar{B},C}: C \otimes \Omega_*(B) \to \Omega_*(B) \otimes C$ of graded corings. In fact, if we regard C as a chain coring with trivial differential maps, then $\psi_*^{\bar{B},C}$ is a twisting map of chain corings, so it induces a graded twisting map $\tau': C \otimes D \to D \otimes C$, cf. Proposition 5.6 (3).

Some useful properties of λ, τ' and ν are collected in the theorem below.

Theorem 5.14. *Let $\sigma: B \otimes A \to A \otimes B$ be an invertible graded twisting map. The twisting map τ' and the entwining maps λ and ν constructed in §5.13 and §5.12 verify the following relations:*

$$\big(\mathrm{I}_{B^p} \otimes \theta_{C,A}\big) \circ \lambda_1^p = (\sigma^{p,1})^{-1} \circ \big(\theta_{C,A} \otimes \mathrm{I}_{B^p}\big), \tag{5.10}$$

$$\lambda_p^1 \circ \big(\mathrm{I}_{C_p} \otimes \theta_{D,B}\big) = \big(\theta_{D,B} \otimes \mathrm{I}_{C_p}\big) \circ \tau'_{p,1}, \tag{5.11}$$

$$\nu_p^1 \circ \big(\theta_{C,A} \otimes \mathrm{I}_{D_p}\big) = \big(\mathrm{I}_{D_p} \otimes \theta_{C,A}\big) \circ \tau'_{1,p}, \tag{5.12}$$

$$\big(\theta_{D,B} \otimes \mathrm{I}_{A^p}\big) \circ \nu_1^p = (\sigma^{1,p})^{-1} \circ \big(\mathrm{I}_{A^p} \otimes \theta_{D,B}\big). \tag{5.13}$$

Proof. We know that $C = T(A)$ is the homology of $(\Omega_*(A), \partial_*)$, so $C_1 = \bar{A}/\bar{A}^2$. Let us denote the class of $a \in \bar{A}$ by $[a]$. By definition, $\varphi_1^{B,\bar{A}} = \sigma^{-1}|_{\bar{A}\otimes B}$. Therefore, for any $a \in \bar{A}$ and $b \in B^q$, we have

$$\lambda_1\big([a] \otimes b\big) = \sum b_{\sigma^{-1}} \otimes [a_{\sigma^{-1}}].$$

On the other hand, $\theta_{C,A}$ maps $[a]$ to the homogeneous component of degree 1 of a. The equation (5.10) now follows by a simple computation.

To prove the second identity we first note that

$$D_1 = \bar{B}/\bar{B}^2 \quad \text{and} \quad \psi_1^{\bar{B},C} = \psi^{\bar{B},C} = \lambda|_{C\otimes\bar{B}}.$$

Since τ' is the morphism induced by $\psi_*^{\bar{B},C}$ and $\theta_{D,B}$ maps the class of $b \in \bar{B}$ modulo $\bar{B}^2$ to its homogeneous component of degree 1, we conclude the proof as in the case of the previous relation. To show that (5.12) holds one proceeds in a similar way. □

5.15. Notation and assumptions. Our goal is to show that the pair

$$\big(A \otimes_\sigma B, T(A) \otimes_\tau T(B)\big)$$

is almost-Koszul, see Remark 5.16 for the definition of τ. Then we shall show that this pair is Koszul, provided that A and B are Koszul R-rings. In fact we are able to

prove these results for any almost-Koszul pairs (A, C) and (B, D) which are equipped with the following extra data:

(1) An invertible graded twisting map $\sigma\colon B \otimes A \to A \otimes B$.

(2) An invertible graded twisting map $\tau'\colon C \otimes D \to D \otimes C$.

(3) An invertible entwining map $\lambda\colon C \otimes B \to B \otimes C$.

(4) An invertible entwining map $\nu\colon A \otimes D \to D \otimes A$.

We assume that the conditions (5.10)–(5.13) are satisfied, where $\theta_{C,A}\colon C_1 \to A^1$ and $\theta_{D,B}\colon D_1 \to B^1$ are the isomorphisms corresponding to (A, C) and (B, D), respectively. We have already seen that, for any invertible twisting map σ, the pairs $(A, T(A))$ and $(B, T(B))$ fulfill the conditions (5.10)–(5.13), where τ', λ and ν are as in Theorem 5.14.

In the case when $p = q = 1$ the above identities imply the following equation:

$$\sigma^{1,1} \circ \big(\theta_{D,B} \otimes \theta_{C,A}\big) \circ \tau'_{1,1} = \theta_{C,A} \otimes \theta_{D,B}.$$

Equivalently, for $c \in C_1$ and $d \in D_1$, we have

$$\sum \theta_{C,A}(c_{\tau'})_\sigma \otimes \theta_{D,B}(d_{\tau'})_\sigma = \theta_{C,A}(c) \otimes \theta_{D,B}(d). \tag{5.14}$$

Remark 5.16. If $\tau'\colon C \otimes D \to D \otimes C$ is a graded twisting map of graded R-corings, then the map τ defined by $\tau_{p,q} := (-1)^{pq}\tau'_{p,q}$ is also a graded twisting map between C and D.

Proposition 5.17. *With the notation and assumptions from §5.15 and the preceding remark, the pair $(A \otimes_\sigma B, C \otimes_\tau D)$ is almost-Koszul.*

Proof. It is obvious that $A \otimes_\sigma B$ and $C \otimes_\tau D$ are connected. By definition we have

$$(A \otimes_\sigma B)^1 = (R \otimes B^1) \oplus (A^1 \otimes R) \quad \text{and} \quad (C \otimes_\tau D)_1 = (R \otimes D_1) \oplus (C_1 \otimes R).$$

We define $\theta\colon (C \otimes_\tau D)_1 \to (A \otimes_\sigma B)^1$ such that it coincides with $(\theta_{C,A} \otimes \mathrm{I}_R)$ and $(\mathrm{I}_R \otimes \theta_{D,B})$ on $C_1 \otimes R$ and $R \otimes D_1$, respectively. We claim that θ satisfies the relation (2.4). Indeed, if $\cdot_\sigma$ denotes the multiplication on $A \otimes_\sigma B$, then we have to show that

$$\sum \theta\big(x_{(1,1)}\big) \cdot_\sigma \theta\big(x_{(2,1)}\big) = 0, \tag{5.15}$$

for any x in $(C \otimes_\tau D)_2 = (C_2 \otimes R) \oplus (C_1 \otimes D_1) \oplus (R \otimes D_2)$. Hence for proving (5.15) we may assume that x belongs to one of the three direct summands. Let us consider the case $x \in C_2 \otimes R$, so $x = c \otimes 1$ for some $c \in C_2$. By definition of the comultiplication on $C \otimes_\tau D$, we have

$$\Delta_{1,1}(c \otimes 1) = \sum \big(c_{(1,1)} \otimes 1\big) \otimes \big(c_{(2,1)} \otimes 1\big).$$

Henceforth, in this case we have

$$\sum \theta\big(x_{(1,1)}\big) \cdot_\sigma \theta\big(x_{(2,1)}\big) = \sum \big(\theta_{C,A}\big(c_{(1,1)}\big) \otimes 1\big) \cdot_\sigma \big(\theta_{C,A}\big(c_{(2,1)}\big) \otimes 1\big)$$
$$= \sum \theta_{C,A}\big(c_{(1,1)}\big)\theta_{C,A}\big(c_{(2,1)}\big) \otimes 1 = 0,$$

since (A, C) is an almost-Koszul pair. If $x \in C_0 \otimes D_2$, the computations are done in a similar way.

Let us finally assume that $x = c \otimes d$ with $c \in C_1$, $d \in D_1$. Since $c \in C_1$ we have $\Delta c = 1 \otimes c + c \otimes 1$, and $\tau_{1,1} = -\tau'_{1,1}$. Thus

$$\Delta(c \otimes d) = (c \otimes d) \otimes (1 \otimes 1) + (c \otimes 1) \otimes (1 \otimes d)$$
$$+ (1 \otimes 1) \otimes (c \otimes d) - 1 \otimes \tau'_{1,1}(c \otimes d) \otimes 1.$$

The component of the latest expression belonging to $(C \otimes_\tau D)_1 \otimes (C \otimes_\tau D)_1$ is precisely

$$\Delta_{1,1}(c \otimes d) = (c \otimes 1) \otimes (1 \otimes d) - \sum (1 \otimes d_{\tau'}) \otimes (c_{\tau'} \otimes 1).$$

Henceforth, applying first $\theta \otimes \theta$ and then the product in $A \otimes_\sigma B$, we get

$$\sum \theta(x_{(1,1)}) \cdot_\sigma \theta(x_{(2,1)}) = \theta_{C,A}(c) \otimes \theta_{D,B}(d) - \sum \theta_{C,A}(c_{\tau'})_\sigma \otimes \theta_{D,B}(d_{\tau'})_\sigma.$$

In view of the relation (5.14) it follows that the equation (2.4) holds in this case as well. □

Theorem 5.18. *We keep the notation and the assumptions from §5.15. If (A, C) and (B, D) are Koszul, then $(A \otimes_\sigma B, C \otimes_\tau D)$ is Koszul too.*

Proof. We have already proved that $(A \otimes_\sigma B, C \otimes_\tau D)$ is an almost-Koszul pair. Let K_* be the complex that is obtained from $\mathrm{K}^l_*(A \otimes_\sigma B, C \otimes_\tau D)$ dropping the part in degree -1. We define (K'_*, d'_*) and (K''_*, d''_*) in a similar way from $\mathrm{K}^l_*(A, C)$ and $\mathrm{K}^l_*(B, D)$, respectively. We claim that

$$\mathrm{I}_A \otimes \lambda \otimes \mathrm{I}_D \colon \mathrm{K}'_* \otimes \mathrm{K}''_* \to \mathrm{K}_*$$

is an isomorphism of complexes. Let

$$\partial_n \colon (A \otimes_\sigma B) \otimes (C \otimes_\tau D)_n \to (A \otimes_\sigma B) \otimes (C \otimes_\tau D)_{n-1}$$

denote the differential map in K_*. We fix p and q such that $p + q = n$. For $c \in C_p$ and $d \in D_q$ we have

$$\Delta(c \otimes d) = \sum_{u=0}^{p} \sum_{v=0}^{q} (-1)^{(p-u)v} c_{(1,u)} \otimes d_{(1,v)_{\tau'}} \otimes c_{(2,p-u)_{\tau'}} \otimes d_{(2,q-v)}.$$

Thus the component of $\Delta(c \otimes d)$ in $(C \otimes_\tau D)_1 \otimes (C \otimes_\tau D)_{p+q-1}$ is obtained from the above equality by dropping all summands but the ones with either $u = 1$ and $v = 0$, or $u = 0$ and $v = 1$. Therefore,

$$\Delta_{1,p+q-1}(c \otimes d) = \sum c_{(1,1)} \otimes 1 \otimes c_{(2,p-1)} \otimes d + (-1)^p \sum 1 \otimes d_{(1,1)_{\tau'}} \otimes c_{\tau'} \otimes d_{(2,q-1)}.$$

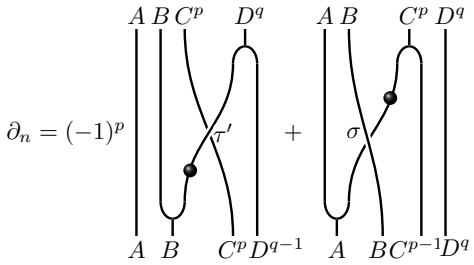
Hence, for any $\zeta = a \otimes b \otimes c \otimes d$ in $A \otimes B \otimes C_p \otimes D_q$, with $p + q = n$, we get

$$\partial_n(\zeta) = \sum a\theta_{C,A}(c_{(1,1)})_\sigma \otimes b_\sigma \otimes c_{(2,p-1)} \otimes d + \sum (-1)^p a \otimes b\theta_{D,B}(d_{(1,1)_{\tau'}}) \otimes c_{\tau'} \otimes d_{(2,q-1)}.$$

To make computations with morphisms in the category of R-bimodules we use string representation of morphisms in a tensor category, which is explained for example in [16, Chapter XIV.1]. Each morphism will be represented downwards, as a black bead. Sometimes, to avoid confusion, we shall write the name of the morphism near the corresponding bead. For the identity of a bimodule we shall draw only the string. The tensor product and the composition of two morphisms will be represented by horizontal and vertical juxtaposition, respectively. In conclusion, every string diagram may be interpreted as the representation of a composition $f_1 \circ \cdots \circ f_n$, where each f_i is a tensor product $f_i = \mathrm{I}_{X_i} \otimes g_i \otimes \mathrm{I}_{Y_i}$. The corresponding diagrams will be drawn one under the other, starting with f_n on the top.

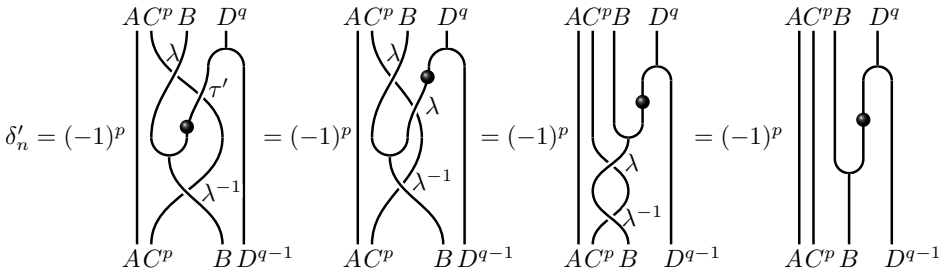
As usual, the multiplication of an R-ring is drawn by joining two strings. For the components $\Delta_{p,q}$ of the comultiplication of a coring C we shall use the 'dual' representation, in which the string representing C_{p+q} is split in two strings that corresponds to C_p and C_q, respectively.

As an example, let us have a look at the picture below, which represents ∂_n. Here the beads symbolize the morphism $\theta_{D,B}$ and $\theta_{C,A}$, respectively. Note the notation of σ as a crossing. For τ' and λ we shall use the inverse crossing representation, to put stress on the fact that they were obtained using σ^{-1}.

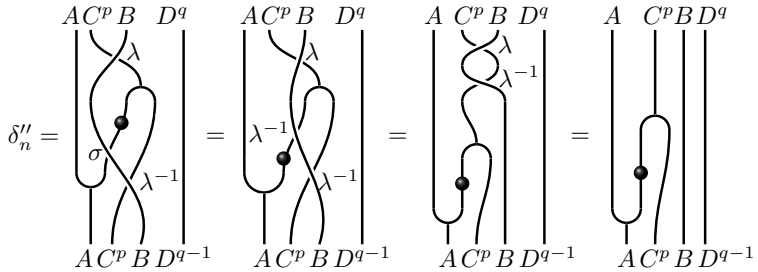


Let $\delta_n := (\mathrm{I}_A \otimes \lambda^{-1} \otimes \mathrm{I}_D) \circ \partial_n \circ (\mathrm{I}_A \otimes \lambda \otimes \mathrm{I}_D)$. Then $\delta_n = \delta'_n + \delta''_n$, where δ'_n is the first term of ∂_n composed to the left by $\mathrm{I}_A \otimes \lambda^{-1} \otimes \mathrm{I}_D$ and to the right by $\mathrm{I}_A \otimes \lambda \otimes \mathrm{I}_D$. The map δ''_n is obtained in a similar way from the second term of ∂_n.

The computation of δ'_n is performed in the diagram below. For the first equality we are using (5.11). The second one means that λ is compatible with the multiplication of B, while the third one is obvious, as λ and λ^{-1} are inverses each other.



The morphism δ''_n is computed in the next diagram using the same method. To deduce the first identity we use (5.10). The second equality follows by the fact that the coring structure of C and the entwining map λ are compatible. The third one is obvious, as λ^{-1} is the inverse of λ.



The above computations show that

$$\delta'_n = (-1)^p \mathrm{I}_{\mathrm{K}'_p} \otimes d''_q \quad \text{and} \quad \delta''_n = d'_p \otimes \mathrm{I}_{\mathrm{K}''_q}$$

for any p and q such that $p + q = n$. Therefore, K_* is the tensor product of the complexes K'_* and K''_*.

Let us now assume that (A, C) and (B, D) are Koszul. By definition, then the complexes K'_* and K''_* are acyclic and their homology groups in degree zero are isomorphic to R. If K_* has the same properties, then $\mathrm{K}^l_*(A \otimes_\sigma B, C \otimes_\tau D)$ will be exact. Using Künneth formula [25, Theorem 3.6.3] and the fact that R is semisimple, we get

$$\mathrm{H}_n\big(\mathrm{K}^l_*(A \otimes_\sigma B, C \otimes_\tau D)\big) \cong \bigoplus_{p+q=n} \mathrm{H}_p\big(\mathrm{K}^l_*(A, C)\big) \otimes \mathrm{H}_q\big(\mathrm{K}^l_*(B, D)\big).$$

Thus $\mathrm{K}^l_n(A \otimes_\sigma B, C \otimes_\tau D)$ clearly is acyclic. On the other hand,

$$\begin{aligned}\mathrm{H}_0\big(\mathrm{K}^l_*(A \otimes_\sigma B, C \otimes_\tau D)\big) &\cong \mathrm{H}_0\big(\mathrm{K}^l_*(A, C)\big) \otimes \mathrm{H}_0\big(\mathrm{K}^l_*(B, D)\big)\\ &\cong R \otimes R \cong R.\end{aligned}$$

□

Corollary 5.19. *Let $\sigma\colon B\otimes A\to A\otimes B$ be an invertible graded twisting map between two Koszul R-rings. Then $A\otimes_\sigma B$ is Koszul and $T(A\otimes_\sigma B)\cong T(A)\otimes_\tau T(B)$.*

Proof. Let τ', λ and ν be the maps that we constructed in §5.12 and §5.13. By the preceding theorem and Theorem 3.13, the pairs $(A\otimes_\sigma B, T(A)\otimes_\tau T(B))$ and $(A\otimes_\sigma B, T(A\otimes_\sigma B))$ are Koszul. To end the proof we apply Corollary 3.10. □

Corollary 5.20. *Let A and B be Koszul rings over a separable $\Bbbk$-algebra R. Let σ be an invertible graded twisting map between A and B. If $\operatorname{Hdim} A = n$ and $\operatorname{Hdim} B = m$, then $\operatorname{Hdim} A\otimes_\sigma B\le n+m$. Moreover, $\operatorname{Hdim}(A\otimes_\sigma B) = m+n$ if and only if $T_n(A)\otimes T_m(B)\neq 0$.*

Proof. By Theorem 4.5, we have $T_n(A)\neq 0$, $T_m(B)\neq 0$ and $T_p(A)=0=T_q(B)$, for all $p>n$ and $q>m$. It follows that the homogeneous component of degree $m+n$ of $T(A)\otimes_\tau T(B)$ is $T_n(A)\otimes T_m(B)$. Since the homogeneous component of degree $r>m+n$ of the twisted tensor product coring is zero and $A\otimes_\sigma B$ is a Koszul ring, we conclude the proof using Theorem 4.5 once again. □

6. The Hochschild (co)homology of a twisted tensor product

As usual, when we speak about the Hochschild (co)homology of an R-ring we assume that R is a separable algebra over a field $\Bbbk$ and that any R-bimodule is symmetric as a $\Bbbk$-bimodule (with respect to the bimodule structure induced by restriction of scalars). Let $\sigma\colon B\otimes A\to A\otimes B$ be an invertible graded twisting map between two Koszul R-rings. By Theorem 5.14 and Theorem 5.18 there is a twisting map τ between the corings $T(A)$ and $T(B))$ such that $(A\otimes_\sigma B, T(A)\otimes_\tau T(B))$ is a Koszul pair. In the first part of this section, for more flexibility, instead of using this pair as a tool we prefer to place ourselves in the setting of §5.15, adding the assumption that (A, C) and (B, D) are Koszul.

Our aim is to show that the Hochschild homology of $A\otimes_\sigma B$ can be computed as the homology of the total complex associated to a suitable double complex. A similar result will be obtained for Hochschild cohomology. Then, as an application, we introduce generalized Ore extensions of an R-ring A and we show that such an extension is Koszul, provided that A is so. Furthermore, using the results from this section, we investigate the Hochschild cohomology of generalized Ore extensions.

6.1. The Hochschild homology of twisted tensor products of Koszul rings. By Theorem 5.18 the pair $(A\otimes_\sigma B, C\otimes_\tau D)$ is Koszul, so we can use Theorem 4.3 to compute the Hochschild homology of $A\otimes_\sigma B$. The decomposition

$$(C\otimes_\tau D)_n=\bigoplus_{p+q=n} C_p\otimes D_q$$

suggests that $\mathrm{K}_*(A \otimes_\sigma B, M)$ is the total complex of a double complex

$$\big(\mathrm{K}_{**}(A \otimes_\sigma B, M), \partial^h_{**}, \partial^v_{**}\big).$$

Before proving this, let us write the latter chain complex in an equivalent way. Recall that

$$\begin{aligned}\mathrm{K}_n(A \otimes_\sigma B, M) = M \mathbin{\widehat{\otimes}} (C \otimes_\tau D)_n &\cong \bigoplus_{p+q=n} (M \mathbin{\widehat{\otimes}} C_p \mathbin{\widehat{\otimes}} D_q) \\ &\cong \bigoplus_{p+q=n} D_q \mathbin{\widehat{\otimes}} M \mathbin{\widehat{\otimes}} C_p = \mathrm{K}'_n(A \otimes_\sigma B, M).\end{aligned}$$

Through these identifications, to the differential map ∂_n of $\mathrm{K}_*(A \otimes_\sigma B, M)$ corresponds a morphism

$$\partial'_n \colon \mathrm{K}'_n(A \otimes_\sigma B, M) \to \mathrm{K}'_{n-1}(A \otimes_\sigma B, M).$$

In view of Theorem 4.3, to compute ∂'_n we need $\Delta^{C \otimes_\tau D}_{1,n-1}(c \otimes d)$ and $\Delta^{C \otimes_\tau D}_{n-1,1}(c \otimes d)$ for any $c \otimes d \in C_p \otimes D_q$ with $p + q = n$. The first element is given by the formula

$$\begin{aligned}\Delta^{C \otimes_\tau D}_{1,n-1}(c \otimes d) = \sum c_{(1,1)} \otimes 1 \otimes c_{(2,p-1)} \otimes d \\ + (-1)^p \sum 1 \otimes d_{(1,1)_{\tau'}} \otimes c_{\tau'} \otimes d_{(2,q-1)},\end{aligned}$$

cf. the proof of Theorem 5.18. We can compute the second element in a similar way, obtaining

$$\begin{aligned}\Delta^{C \otimes_\tau D}_{n-1,1}(c \otimes d) = \sum c \otimes d_{(1,q-1)} \otimes 1 \otimes d_{(2,1)} \\ + (-1)^q \sum c_{(1,p-1)} \otimes d_{\tau'} \otimes c_{(2,1)_{\tau'}} \otimes 1.\end{aligned}$$

Let $m \in M$. Taking into account the relation (4.1) it follows that $\partial'_n(d \mathbin{\widehat{\otimes}} m \mathbin{\widehat{\otimes}} c)$ can be written as a sum of two elements. The first one, belonging to $D_q \mathbin{\widehat{\otimes}} M \mathbin{\widehat{\otimes}} C_{p-1}$, has the following form

$$\begin{aligned}\partial^h_{p,q}(d \mathbin{\widehat{\otimes}} m \mathbin{\widehat{\otimes}} c) = \sum d \mathbin{\widehat{\otimes}} m\big(\theta_{C,A}\big(c_{(1,1)}\big) \otimes 1\big) \mathbin{\widehat{\otimes}} c_{(2,p-1)} \\ + (-1)^p \sum d_{\tau'} \mathbin{\widehat{\otimes}} \big(\theta_{C,A}\big(c_{(2,1)_{\tau'}}\big) \otimes 1\big) m \mathbin{\widehat{\otimes}} c_{(1,p-1)}. \quad (6.1)\end{aligned}$$

The other one, which is an element of $D_{q-1} \mathbin{\widehat{\otimes}} M \mathbin{\widehat{\otimes}} C_p$, can be written as $(-1)^p \partial^v_{p,q}(d \mathbin{\widehat{\otimes}} m \mathbin{\widehat{\otimes}} c)$, where

$$\begin{aligned}\partial^v_{p,q}(d \mathbin{\widehat{\otimes}} m \mathbin{\widehat{\otimes}} c) = \sum d_{(2,q-1)} \mathbin{\widehat{\otimes}} m\big(1 \otimes \theta_{D,B}\big(d_{(1,1)_{\tau'}}\big)\big) \mathbin{\widehat{\otimes}} c_{\tau'} \\ + (-1)^q \sum d_{(1,q-1)} \mathbin{\widehat{\otimes}} \big(1 \otimes \theta_{D,B}\big(d_{(2,1)}\big)\big) m \mathbin{\widehat{\otimes}} c. \quad (6.2)\end{aligned}$$

Thus the complexes $(\mathrm{K}_*(A\otimes_\sigma B,M),\partial_*)$ and $(\mathrm{K}'_*(A\otimes_\sigma B,M),\partial'_*)$ are isomorphic. We have also proved that the restriction of the differential map ∂'_n to $D_q\,\widehat{\otimes}\,M\,\widehat{\otimes}\,C_p$ satisfies the relation $\partial'_n=\partial^h_{p,q}+(-1)^p\partial^v_{p,q}$, where $\partial^h_{p,q}$ and $\partial^v_{p,q}$ are defined as above. Let

$$\mathrm{K}_{p,q}(A\otimes_\sigma B,M):=D_q\widehat{\otimes}\,M\,\widehat{\otimes}C_p.$$

By a straightforward but tedious computation, based on the relations (5.10)–(5.13), one shows that $(\mathrm{K}_{**}(A\otimes_\sigma B,M),\partial^h_{**},\partial^v_{**})$ is a double complex, that is the diagram

$$\begin{array}{ccc}\mathrm{K}_{p,q}(A\otimes_\sigma B,M) & \xrightarrow{\partial^h_{p,q}} & \mathrm{K}_{p-1,q}(A\otimes_\sigma B,M)\\ \downarrow{\scriptstyle\partial^v_{p,q}} & & \downarrow{\scriptstyle\partial^v_{p-1,q}}\\ \mathrm{K}_{p,q-1}(A\otimes_\sigma B,M) & \xrightarrow[\partial^h_{p,q-1}]{} & \mathrm{K}_{p-1,q-1}(A\otimes_\sigma B,M)\end{array}\tag{6.3}$$

is commutative for all nonzero p and q. Obviously, its total complex is

$$\big(\mathrm{K}'_*(A\otimes_\sigma B,M),\partial'_*\big).$$

We reinterpret the double complex (6.3) to relate the homology groups of the rows and columns with the Hochschild homology groups of A and B, respectively. First, let us notice that $D_q\otimes M$ is an A-bimodule with respect to the actions

$$a(d\otimes m)a':=\sum d_\nu\otimes(a_\nu\otimes 1)m(a'\otimes 1),$$

where for the left module structure we used the entwining map $\nu\colon A\otimes D\to D\otimes A$. Similarly, we can endow $M\otimes C_p$ with a B-bimodule structure by

$$b(m\otimes c)b':=\sum(1\otimes b)m(1\otimes b'_\lambda)\otimes c_\lambda.$$

Using the relations (5.11) and (5.12), it follows that the complexes $\mathrm{K}_*(B,M\otimes C_p)$ and $\mathrm{K}_*(A,D_q\otimes M)$ are isomorphic to $\mathrm{K}_{p*}(A\otimes_\sigma B,M)$ and $\mathrm{K}_{*q}(A\otimes_\sigma B,M)$, respectively.

There are two filtrations on the total complex of $\mathrm{K}_{**}(A\otimes_\sigma B,M)$, cf. [25, Section 5.6]. They give rise to two spectral sequences, both converging to the Hochschild homology of $A\otimes_\sigma B$ with coefficients in M. For future reference, we summarize the above results in the theorem below.

Theorem 6.2. *We keep the notation and the assumptions of §5.15. If (A,C) and (B,D) are Koszul pairs over a separable R-algebra and M is an $A\otimes_\sigma B$-bimodule, then the Hochschild homology of $A\otimes_\sigma B$ with coefficients in M is the homology of the total complex of $(\mathrm{K}_{**}(A\otimes_\sigma B,M),\partial^h_{**},\partial^v_{**})$. The pages ${}^{I}E^1_{**}$ and ${}^{II}E^1_{**}$ of the spectral sequences that correspond to the column-wise and row-wise filtrations are*

$${}^{I}E^1_{pq}=\mathrm{HH}_q(B,M\otimes C_p)\quad\text{and}\quad{}^{II}E^1_{pq}=\mathrm{HH}_q(A,D_p\otimes M).\tag{6.4}$$

Both spectral sequences converge to the Hochschild homology of $A \otimes_\sigma B$ with coefficients in M.

6.3. The Hochschild cohomology of twisted tensor products of Koszul rings. For the computation of Hochschild cohomology of $A \otimes_\sigma B$ with coefficients in a bimodule M one may also use a similar double complex. With the notation from §6.1, let us remark that

$$\mathrm{K}^n(A \otimes_\sigma B, M) = \mathrm{Hom}_{R^e}((C \otimes_\tau D)_n, M) \cong \bigoplus_{p+q=n} \mathrm{Hom}_{R^e}(C_p \otimes D_q, M). \tag{6.5}$$

For every R-bilinear morphism $f: C_p \otimes D_q \to M$ we define

$$\partial_h^{p,q}(f)(c \otimes d) = \sum \big(\theta_{C,A}\big(c_{(1,1)}\big) \otimes 1\big) f\big(c_{(2,p)} \otimes d\big) + (-1)^{p+1} \sum f\big(c_{(1,p)} \otimes d_{\tau'}\big)\big(\theta_{C,A}\big(c_{(2,1)_{\tau'}}\big) \otimes 1\big).$$

Note that $\partial_h^{p,q}(f)$ is a morphism of R-bimodules from $C_{p+1} \otimes D_q$ to M. Similarly, for any f as above we define the map $\partial_v^{p,q}(f): C_p \otimes D_{q+1} \to M$ by

$$\partial_v^{p,q}(f)(c \otimes d) = \sum \big(1 \otimes \theta_{D,B}\big(d_{(1,1)_\tau}\big)\big) f\big(c_\tau \otimes d_{(2,q)}\big) + (-1)^{q+1} \sum f\big(c \otimes d_{(1,q)}\big)\big(1 \otimes \theta_{D,B}\big(d_{(2,1)}\big)\big).$$

Taking into account the identification (6.5), by direct computation, we see that the differential maps of the complex $\mathrm{K}^*(A \otimes_\sigma B, M)$ satisfy the relations $\partial^n = \partial_h^{p,q} + (-1)^p \partial_v^{p,q}$, and that the diagram

$$\begin{array}{ccc} \mathrm{Hom}_{R^e}(C_p \otimes D_{q+1}, M) & \xrightarrow{\partial_h^{p,q+1}} & \mathrm{Hom}_{R^e}(C_{p+1} \otimes D_{q+1}, M) \\ \uparrow{\scriptstyle \partial_v^{p,q}} & & \uparrow{\scriptstyle \partial_v^{p+1,q}} \\ \mathrm{Hom}_{R^e}(C_p \otimes D_q, M) & \xrightarrow[\partial_h^{p,q}]{} & \mathrm{Hom}_{R^e}(C_{p+1} \otimes D_q, M) \end{array} \tag{6.6}$$

is commutative. The resulting double complex will be denoted by

$$\big(\mathrm{K}^{**}(A \otimes_\sigma B, M), \partial_h^{**}, \partial_v^{**}\big).$$

We have seen that the homology groups of the rows and columns of the double complex in Theorem 3.3 computes the Hochschild of A and B with respect to appropriate A and B bimodules, respectively. For the Hochschild cohomology of $A \otimes_\sigma B$ a similar result does not hold in general. Nevertheless, supposing that R is a separable commutative $\Bbbk$-algebra and that all R-bimodules that we work with are symmetric, in view of the adjunction formula, we can rewrite $\mathrm{K}^{pq}(A \otimes_\sigma B, M)$ as follows

$$\mathrm{Hom}_{R^e}(C_p \otimes D_q, M) = \mathrm{Hom}_R(C_p \otimes D_q, M) \cong \mathrm{Hom}_R\big(C_p, \mathrm{Hom}_R(D_q, M)\big).$$

Through this isomorphism, for a given q, the row $\mathrm{K}^{*q}(A\otimes_\sigma B, M)$ can be identified with the complex $\mathrm{K}^*(A, \mathrm{Hom}_R(D_q, M))$, where the linear space of R-module morphisms from D_q to M is regarded as an A-bimodule via the actions

$$(afa')(d) = \sum (a\otimes 1) f(d_\nu)(a'_\nu \otimes 1).$$

Therefore, the page ${}_{II}E_1^{pq}$ of the spectral sequence associated to the row-wise filtration of $\mathrm{K}^{**}(A\otimes_\sigma B, M)$ has in bidegree (p,q) the Hochschild cohomology group $\mathrm{HH}^p(A, \mathrm{Hom}_R(D_q, M))$.

To give an analogous description of the columns of $\mathrm{K}^{**}(A\otimes_\sigma B, M)$ we first endow $\mathrm{Hom}_R(C_p, M)$ with a B-bimodule structure, using the following left and right actions

$$(bgb')(c) = \sum (1\otimes b_\lambda) g(c_\lambda)(b\otimes 1).$$

Then, by the adjunction formula,

$$\mathrm{Hom}_{R^e}(C_p\otimes D_q, M) \cong \mathrm{Hom}_R\left(D_q, \mathrm{Hom}_R(C_p, M)\right).$$

Thus, the column $\mathrm{K}^{p*}(A\otimes_\sigma B, M)$ is isomorphic to the complex

$$\mathrm{K}^*\left(B, \mathrm{Hom}_R(C_p, M)\right),$$

for any p. In particular, the page ${}_{I}E_1^{pq}$ of the spectral sequence associated to the column-wise filtration has in the spot (p,q) the Hochschild cohomology group of $\mathrm{HH}^q(B, \mathrm{Hom}_R(C_p, M))$. Summarizing, we sketched the proof of the following.

Theorem 6.4. *We keep the notation and assumptions of §5.15. If (A, C) and (B, D) are Koszul pairs over a separable R-algebra and M is an $A\otimes_\sigma B$-bimodule, then the Hochschild cohomology of $A\otimes_\sigma B$ with coefficients in M is the cohomology of the total complex of $(\mathrm{K}^{**}(A\otimes_\sigma B, M), \partial_h^{**}, \partial_v^{**})$. Assuming that R is commutative and that the R-bimodules A, B and M are symmetric, then the pages ${}_{I}E_1^{**}$ and ${}_{II}E_1^{**}$ of the spectral sequences that correspond to the column-wise and row-wise filtrations are given by*

$${}_{I}E_1^{pq} = \mathrm{HH}^q\left(B, \mathrm{Hom}_R(C_p, M)\right) \quad \textit{and} \quad {}_{II}E_1^{pq} = \mathrm{HH}^q\left(A, \mathrm{Hom}_R(D_p, M)\right).$$

Both spectral sequences converge to the Hochschild cohomology of $A\otimes_\sigma B$ with coefficients in M.

6.5. Generalized Ore extensions of R-rings. Let A be an R-ring, where R is a semisimple ring. If $\sigma\colon A\to M_n(A)$ is a morphism of R-rings then, for every couple (i, j) of positive integers which are less than or equal to n, there exists an R-bimodule endomorphism σ_{ij} of A such that $\sigma_{ij}(a)$ is the (i, j)-element of the matrix $\sigma(a)$, and

$$\sigma_{ij}(ab) = \sum_{p=1}^{n} \sigma_{ip}(a)\sigma_{pj}(b) \quad \text{and} \quad \sigma_{ij}(1) = \delta_{i,j}1. \tag{6.7}$$

Let $\{e_1, \ldots, e_n\}$ denote the canonical basis (both as a left and as a right R-module) on $V := R^n$. For any morphism σ as above there exists a unique twisting map

$$\widetilde{\sigma}: T_R^a(V) \otimes A \to A \otimes T_R^a(V)$$

such that $\widetilde{\sigma}(1 \otimes a) = a \otimes 1$ and

$$\widetilde{\sigma}(e_{i_1} \otimes \cdots \otimes e_{i_p} \otimes a) = \sum_{j_1, \ldots, j_p = 1}^{n} (\sigma_{i_1 j_1} \circ \cdots \circ \sigma_{i_p j_p})(a) \otimes e_{j_1} \otimes \cdots \otimes e_{j_p}. \quad (6.8)$$

Since $\widetilde{\sigma}$ is a twisting map of R-rings, the twisted tensor product $A \otimes_{\widetilde{\sigma}} T_R^a(V)$ makes sense. The set

$$\mathcal{B} := \{e_{i_1} \otimes \cdots \otimes e_{i_m} \mid m \geq 0 \text{ and } 1 \leq i_1, \ldots, i_m \leq n\}$$

is a basis of the left A-module $A \otimes_{\widetilde{\sigma}} T_R^a(V)$. Therefore, by identifying $e_{i_1} \otimes \cdots \otimes e_{i_m}$ with $X_{i_1} \cdots X_{i_m}$, any element in $A \otimes_{\widetilde{\sigma}} T_R^a(V)$ can be written in a unique way as a left linear combination of noncommutative monomials in the indeterminates $X_1, \ldots, X_n$ with coefficients in A. Via this identification, the multiplication in $A \otimes_{\widetilde{\sigma}} T_R^a(V)$ is determined by the relation

$$X_i a = \sum_{j=1}^{n} \sigma_{ij}(a) X_j.$$

If $n = 1$ then σ identifies with an algebra automorphism σ_{11} of A, and $A \otimes_{\widetilde{\sigma}} T_R^a(V)$ is the usual Ore extension $A_{\sigma_{11}}[X]$. For this reason, in the case when σ is an arbitrary R-ring morphism from A to $M_n(A)$, the corresponding twisted tensor product $A \otimes_{\widetilde{\sigma}} T_R^a(V)$ will be called the *generalized Ore extension* of A (with respect to σ) and it will be denoted by $A_\sigma\langle X_1, \ldots, X_n\rangle$.

The twisting map $\widetilde{\sigma}$, associated to an algebra morphism $\sigma: A \to M_n(A)$, is invertible if and only if there is a matrix $(\sigma'_{ij})_{i,j}$ whose elements are R-bimodule endomorphisms of A satisfying the equations

$$\sum_{p=1}^{n} \sigma_{pi} \circ \sigma'_{jp} = \sum_{p=1}^{n} \sigma'_{pi} \circ \sigma_{jp} = \delta_{i,j} \mathrm{I}_A.$$

The matrix $(\sigma'_{ij})_{i,j}$ determines the inverse of $\widetilde{\sigma}$ by the formula

$$\widetilde{\sigma}^{-1}(a \otimes e_{i_1} \otimes \cdots \otimes e_{i_p}) = \sum_{j_1, \ldots, j_p = 1}^{n} e_{j_1} \otimes \cdots \otimes e_{j_p} \otimes \sigma'_{i_p j_p} \cdots \sigma'_{i_1 j_1}(a). \quad (6.9)$$

Let us now assume that $A := \bigoplus_{p \in \mathbb{N}} A^p$ is a connected graded R-ring. Obviously, $\widetilde{\sigma}$ is a graded twisting map if and only if every σ_{ij} is a morphism of graded R-bimodules. From now on we assume that A is a connected R-ring and that $\sigma: A \to M_n(A)$ is a morphism of R-rings such that the corresponding twisting map $\widetilde{\sigma}$ is graded and invertible.

Theorem 6.6. *Let R and A denote a semisimple ring and a Koszul R-ring, respectively. If σ is a morphism of R-rings from A to $M_n(A)$ such that the corresponding twisting map $\widetilde{\sigma}$ is graded and invertible, then the generalized Ore extension $A_\sigma\langle X_1,\dots,X_n\rangle$ is a Koszul R-ring.*

Proof. We know that A^1 generates A and $(A, T(A))$ is a Koszul pair, see Theorem 3.10. Let V denote the R-bimodule R^n. By Proposition 4.6, the pair $(T_R^a(V), C)$ is Koszul, where $C := C_0 \oplus C_1$ and $C_1 = V$. Applying Proposition 5.14 for

$$\widetilde{\sigma}\colon T_R^a(V) \otimes A \to A \otimes T_R^a(V),$$

we get a twisting map of corings

$$\tau'\colon T(A) \otimes C \to C \otimes T(A)$$

and the entwining maps

$$\lambda\colon T(A) \otimes T_R^a(V) \to T_R^a(V) \otimes T(A) \quad \text{and} \quad \nu\colon A \otimes C \to C \otimes A,$$

such that the relations (5.10)–(5.13) hold with σ replaced by $\widetilde{\sigma}$. Hence by Theorem 5.18 the pair $(A \otimes_{\widetilde{\sigma}} T_R^a(V), T(A) \otimes_\tau C)$ is Koszul. In particular, $A_\sigma\langle X_1,\dots,X_n\rangle$ is a Koszul R-ring. □

Remark 6.7. The matrix $(\sigma'_{ij})_{i,j}$ that gives the inverse of $\widetilde{\sigma}$ from relation (6.9) can be used to construct explicitly the maps τ', λ and ν from the proof of the preceding theorem.

To construct τ' we define the R-bimodule map $\tau'_p\colon \Omega_p(A) \otimes V \to V \otimes \Omega_p(A)$ as follows. If $p = 0$ then we take τ'_p to be the isomorphism $R \otimes V \cong V \otimes R$. On the other hand, if p is positive, then we set

$$\tau'_p(a_1 \otimes \cdots \otimes a_p \otimes e_i) := \sum_{j_1,\dots,j_p=1}^{n} e_{j_1} \otimes \sigma'_{j_2 j_1}(a_1) \otimes \sigma'_{j_3 j_2}(a_2) \otimes \cdots \otimes \sigma'_{j_p j_{p-1}}(a_{p-1}) \otimes \sigma'_{i j_p}(a_p),$$

for any $a_1,\dots,a_p \in A$. Since

$$\sum_{j_k=1}^{n} \sigma'_{j_k j_{k-1}}(a_{k-1})\sigma'_{j_{k+1} j_k}(a_k) = \sigma'_{j_{k+1} j_{k-1}}(a_{k-1}a_k)$$

it follows easily that τ'_* is a morphism of chain complexes from $(\Omega_*(A) \otimes V, \partial_* \otimes \mathrm{I}_V)$ to $(V \otimes \Omega_*(A), \mathrm{I}_V \otimes \partial_*)$. To show that τ'_* is compatible with the comultiplication of $\Omega_*(A)$ one uses the fact that this coring structure is given by the canonical

identifications $\Omega_{p+q}(A) \cong \Omega_p(A) \otimes \Omega_q(A)$. Thus τ'_* satisfies the conditions from Proposition 5.3, so for every p there is a bimodule map

$$\tau'_{p,1}: T_p(A) \otimes V \to V \otimes T_p(A)$$

such that $\{\tau'_{p,1}\}_{p\in\mathbb{N}}$ is compatible with the coring structure of $T(A)$. Let

$$\tau'_{p,0}: T_p(A) \otimes R \to R \otimes T_p(A)$$

be the canonical R-bilinear isomorphisms. The family $\{\tau'_{p,0}\}_{p\in\mathbb{N}}$ together with $\{\tau'_{p,1}\}_{p\in\mathbb{N}}$ define the required twisting map τ' between $T(A)$ and C.

We can now define the entwining structure λ. For any p we take λ^0_p to be the canonical isomorphism $T_p(A) \otimes R \cong R \otimes T_p(A)$, and for $q > 0$ we define

$$\lambda^q_p: T_p(A) \otimes V^{\otimes q} \to V^{\otimes q} \otimes T_p(A)$$

by

$$\lambda^q_p := \left(\mathrm{I}_{V^{\otimes q-1}} \otimes \tau'_{p,1}\right) \circ \left(\mathrm{I}_{V^{\otimes q-2}} \otimes \tau'_{p,1} \otimes V\right) \circ \cdots \circ \left(\tau'_{p,1} \otimes \mathrm{I}_{V^{\otimes q-1}}\right).$$

Finally, the entwining map $\nu: C \otimes A \to A \otimes C$ coincides with the isomorphism $R \otimes A \cong A \otimes R$ on $C_0 \otimes A$ and with $\widetilde{\sigma}^{-1}$ on $C_1 \otimes A$. Plainly, by construction, $\widetilde{\sigma}$, τ', λ, and ν satisfies the conditions from §5.15.

6.8. Hochschild (co)homology of generalized Ore extensions. We keep the notation and the assumptions from Theorem 6.6, and we suppose that M is an $A_\sigma\langle X_1, \ldots, X_n\rangle$-bimodule. By the proof of the aforementioned result

$$\left(A_\sigma\langle X_1, \ldots, X_n\rangle, T(A) \otimes_\tau C\right)$$

is Koszul. Thus to compute the Hochschild homology of $A_\sigma\langle X_1, \ldots, X_n\rangle$ we may use the double complex constructed in Theorem 4.3. In this particular case, the double complex has only two non-trivial rows as $C_q = 0$ for any $q > 1$. Therefore, for the generalized Ore extension $A_\sigma\langle X_1, \ldots, X_n\rangle$ the double complex becomes

$$\begin{array}{ccccccccc}
0 & \longleftarrow & V \mathbin{\widehat{\otimes}} M \mathbin{\widehat{\otimes}} T_0(A) & \xleftarrow{\partial^h_{1,1}} & V \mathbin{\widehat{\otimes}} M \mathbin{\widehat{\otimes}} T_1(A) & \xleftarrow{d^h_{2,1}} & V \mathbin{\widehat{\otimes}} M \mathbin{\widehat{\otimes}} T_2(A) & \xleftarrow{\partial^h_{3,1}} & \cdots \\
 & & \downarrow{\scriptstyle \partial^v_{0,1}} & & \downarrow{\scriptstyle \partial^v_{1,1}} & & \downarrow{\scriptstyle \partial^v_{2,1}} & & \\
0 & \longleftarrow & M \mathbin{\widehat{\otimes}} T_0(A) & \xleftarrow[\partial^h_{1,0}]{} & M \mathbin{\widehat{\otimes}} T_1(A) & \xleftarrow[\partial^h_{2,0}]{} & M \mathbin{\widehat{\otimes}} T_2(A) & \xleftarrow[\partial^h_{3,0}]{} & \cdots
\end{array}$$

The arrows of this double complex are described in the next proposition. Recall that the comultiplication on C is taken such that for all elements in V we have $\Delta_{1,0}(v) = v \otimes 1$ and $\Delta_{0,1}(v) = 1 \otimes v$.

Proposition 6.9. *The Hochschild homology of $A_\sigma\langle X_1,\dots,X_n\rangle$ is isomorphic to the homology of the total complex of the above double complex. For $c \in T_p(A)$, $m \in M$ and $v \in V$ the arrows are given by*

$$\begin{aligned}
\partial^h_{p,0}(m \,\widehat{\otimes}\, c) &= \sum m\big(\theta\big(c_{(1,1)}\big)\otimes 1\big)\,\widehat{\otimes}\, c_{(2,p-1)} \\
&\quad + (-1)^p \sum \big(\theta\big(c_{(2,1)}\big)\otimes 1\big) m \,\widehat{\otimes}\, c_{(1,p-1)}, \\
\partial^h_{p,1}(v \,\widehat{\otimes}\, m \,\widehat{\otimes}\, c) &= \sum v \,\widehat{\otimes} m\big(\theta\big(c_{(1,1)}\big)\otimes 1\big)\,\widehat{\otimes}\, c_{(2,p-1)} \\
&\quad + (-1)^p \sum v_{\tau'} \,\widehat{\otimes}\, \big(\theta\big(c_{(2,1)_{\tau'}}\big)\otimes 1\big) m \,\widehat{\otimes}\, c_{(1,p-1)}, \\
\partial^v_{p,1}(v \widehat{\otimes} m \widehat{\otimes} c) &= \sum m(1\otimes v_{\tau'})\,\widehat{\otimes}\, c_{\tau'} - (1\otimes v) m \,\widehat{\otimes}\, c.
\end{aligned}$$

Similarly, the double complex from Theorem 4.4 computing the Hochschild cohomology of the generalized Ore extension $A_\sigma\langle X_1,\dots,X_n\rangle$ has only two nonzero rows

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathrm{Hom}_{R^e}(V,M) & \xrightarrow{\partial_h^{0,1}} & \mathrm{Hom}_{R^e}(T_1(A)\otimes V, M) & \xrightarrow{\partial_h^{1,1}} & \mathrm{Hom}_{R^e}(T_2(A)\otimes V, M) & \longrightarrow & \cdots \\
 & & \uparrow{\scriptstyle \partial_v^{0,0}} & & \uparrow{\scriptstyle \partial_v^{1,0}} & & \uparrow{\scriptstyle \partial_v^{2,0}} & & \\
0 & \longrightarrow & \mathrm{Hom}_{R^e}(R,M) & \xrightarrow[\partial_h^{0,0}]{} & \mathrm{Hom}_{R^e}(T_1(A),M) & \xrightarrow[\partial_h^{1,0}]{} & \mathrm{Hom}_{R^e}(T_2(A),M) & \longrightarrow & \cdots
\end{array}$$

The arrows of this double complex are described in the next proposition.

Proposition 6.10. *The Hochschild cohomology of the generalized Ore extension $A_\sigma\langle X_1,\dots,X_n\rangle$ is isomorphic to the cohomology of the total complex of the above double complex. For $c \in T_p(A)$, $m \in M$ and $v \in V$ the arrows are given by*

$$\begin{aligned}
\partial_v^{p,0}(f)(c\otimes v) &= \sum (1\otimes v_{\tau'}) f(c_{\tau'}\otimes 1) - f(c\otimes 1)(1\otimes v), \\
\partial_h^{p,0}(f)(c) &= \sum (c_{(1,1)}\otimes 1) f(c_{(2,p)}) + (-1)^p \sum f(c_{(1,p)})(c_{(2,1)}\otimes 1), \\
\partial_h^{p,1}(f)(c\otimes v) &= \sum (c_{(1,1)}\otimes 1) f(c_{(2,p)}\otimes v) \\
&\quad + (-1)^p \sum f(c_{(1,p)}\otimes v_{\tau'})(c_{(2,1)_{\tau'}}\otimes 1).
\end{aligned}$$

Remark 6.11. The p-degree component of $T(A)\otimes_\tau C$ is

$$\big(T_p(A)\otimes R\big)\oplus\big(T_{p-1}(A)\otimes V\big).$$

We deduce that

$$\mathrm{Hdim}\, A \le \mathrm{Hdim}\, A_\sigma\langle X_1,\dots,X_n\rangle \le \mathrm{Hdim}\, A + 1,$$

see Theorem 4.5. Therefore, if the Hochschild dimension of A is n, then the Hochschild dimension of the generalized Ore extensions is $n+1$. Moreover, the Hochschild dimension of A is infinite if and only if the Hochschild dimension of the ore extension $A_\sigma\langle X_1,\dots,X_n\rangle$ is so.

7. Koszul braided R-bialgebras

In this section we are going to give some examples of Koszul rings and indicate some applications of our previous results. We start by showing that, under certain mild assumptions, any almost-Koszul pair that consists of connected braided commutative bialgebras is always Koszul. Some particular instances of such pairs, including trivial extensions, quantum affine spaces and the incidence algebra of the power poset $(\mathcal{P}(X), \subseteq)$ of a finite set X, will be discussed.

7.1. Braided R-bialgebras. A pair $(V, \mathfrak{c})$ is called *braided R-bimodule* if V is an R-bimodule and $\mathfrak{c}\colon V \otimes V \to V \otimes V$ is an R-bimodule map which satisfies the braid equation

$$\mathfrak{c}_1 \circ \mathfrak{c}_2 \circ \mathfrak{c}_1 = \mathfrak{c}_2 \circ \mathfrak{c}_1 \circ \mathfrak{c}_2,$$

where $\mathfrak{c}_1 := \mathfrak{c} \otimes \mathrm{I}_V$ and $\mathfrak{c}_2 := \mathrm{I}_V \otimes \mathfrak{c}$. A morphism of braided bimodules from $(V, \mathfrak{c}_V)$ to $(W, \mathfrak{c}_W)$ is a bimodule map $f\colon V \to W$ such that $\mathfrak{c}_W \circ (f \otimes f) = (f \otimes f) \circ \mathfrak{c}_V$.

The definition of braided algebras from [2] can be adapted for R-rings without difficulty. See also [1] for more details about braided algebras, coalgebras and bialgebras. The quadruple $(A, m, u, \mathfrak{c})$ is called a *braided R-ring* if $(A, \mathfrak{c})$ is a braided R-bimodule and (A, m, u) is an R-ring such that $\mathfrak{c}$ is a twisting map of R-rings. A *morphism* of braided R-rings is, by definition, a morphism of ordinary R-rings which, in addition, is a morphism of braided bimodules. The braided ring $(A, m, u, \mathfrak{c})$ is called *braided commutative*, or $\mathfrak{c}$-*commutative* if $m \circ \mathfrak{c} = m$.

We shall say that $(A, m, u, \mathfrak{c})$ is a *graded* braided R-ring if and only if A is graded and $\mathfrak{c}$ is a graded twisting map of graded R-rings. In this case we shall denote the restriction of $\mathfrak{c}$ to $A^p \otimes A^q$ by $\mathfrak{c}^{p,q}$. Note that, for a braided R-ring $(A, \mathfrak{c})$, we can consider the twisted tensor product $A \otimes_{\mathfrak{c}} A$ with respect to the twisting map $\mathfrak{c}$.

Braided R-corings, and their graded version, are defined in a dual manner. Braided R-bialgebras generalize braided bialgebras, introduced by Takeuchi in [23]. By definition, a sextuple $(A, m, u, \Delta, \varepsilon, \mathfrak{c})$ is a *braided R-bialgebra* if $(A, m, u, \mathfrak{c})$ is a braided R-ring and $(A, \Delta, \varepsilon, \mathfrak{c})$ is a braided R-coring such that Δ and ε are morphisms of R-rings (on the R-bimodule $A \otimes A$ we take the ring structure $A \otimes_{\mathfrak{c}} A$). A braided R-bialgebra A is *graded* if the underlying ring, coring and braiding structures are so (with respect to the same decomposition $A = \oplus_{n \in \mathbb{N}} A^n$).

The free R-ring $T_R^a(V)$ and the symmetric R-ring $S_R(V, \mathfrak{c})$ of a braided R-bimodule $(V, \mathfrak{c})$ are the main examples of braided R-bialgebras. To define them one follows the same steps as in [1], where the case of braided bialgebras over a base field is considered. First, one shows that there exists a unique R-bimodule map

$$\mathfrak{c}_T\colon T_R^a(V) \otimes T_R^a(V) \to T_R^a(V) \otimes T_R^a(V),$$

which extends $\mathfrak{c}$ and is a solution of the braid equation that respects the grading on $T_R^a(V) \otimes T_R^a(V)$. Since $\mathfrak{c}$ is a solution of the braid equation, $\mathfrak{c}_T$ is a twisting map of R-rings. Thus $(T_R^a(V), \mathfrak{c}_T)$ is a graded braided R-ring. Using the universal property

of $T_R^a(V)$, one constructs a unique comultiplication $\Delta\colon T_R^a(V) \to T_R^a(V) \otimes_{\mathfrak{c}_T} T_R^a(V)$ such that $\Delta(v) = v \otimes 1 + 1 \otimes v$, for any $v \in V$. We also define $\varepsilon\colon T_R^a(V) \to R$ to be the unique R-ring morphism that coincides with I_R on the 0-degree component of $T_R^a(V)$ and vanishes on V. As in [1] one shows that $(T_R^a(V), \Delta, \varepsilon, \mathfrak{c}_T)$ is a braided R-bialgebra.

Let $(V, \mathfrak{c})$ be a symmetric braided bimodule, that is $\mathfrak{c}$ satisfies the additional condition $\mathfrak{c}^2 = \mathrm{I}_{V\otimes V}$. Since $W := \mathrm{Im}(\mathrm{I}_{V\otimes V} - \mathfrak{c})$ contains only primitive elements, the two-sided ideal I generated by W is a coideal in $T_R^a(V)$. Let $S_R(V, \mathfrak{c}) := T_R^a(V)/I$. Since $\mathfrak{c}_T$ maps $I \otimes T_R^a(V)$ and $T_R^a(V) \otimes I$ to $T_R^a(V) \otimes I$ and $I \otimes T_R^a(V)$, respectively, we conclude that $S_R(V, \mathfrak{c})$ inherits a canonical structure of braided R-bialgebra. We denote the braiding of $S_R(V, \mathfrak{c})$ by $\mathfrak{c}_S$. We shall say that $S_R(V, \mathfrak{c})$ is the braided symmetric R-ring of $(V, \mathfrak{c})$. This R-ring is $\mathfrak{c}_S$-commutative by construction.

Theorem 7.2. *Let R be a separable algebra over a field $\Bbbk$ of characteristic zero. Let $(A, \mathfrak{c}_A)$ and $(H, \mathfrak{c}_H)$ denote two connected braided R-bialgebras such that $A^1 = H^1$ and $\mathfrak{c}_A^{1,1} = -\mathfrak{c}_H^{1,1}$. If A and H are strongly graded and braided commutative (as R-rings), then (A, H) and (H, A) are Koszul pairs. In particular, $(S_R(V, \mathfrak{c}), S_R(V, -\mathfrak{c}))$ is a Koszul pair.*

Proof. Let $\{m_A^{p,q}\}_{p,q\in\mathbb{N}}$ and $\{\Delta_{p,q}^A\}_{p,q\in\mathbb{N}}$ denote the components of the multiplication and of the comultiplication of A. We claim that

$$m_A^{n,1} \circ \Delta_{n,1}^A = (n+1)\mathrm{I}_{A^{n+1}} \quad \text{and} \quad m_A^{1,n}\Delta_{1,n}^A = (n+1)\mathrm{I}_{A^{n+1}}, \tag{7.1}$$

for any $n \geq 0$. We shall prove only the first identity, the other one can be obtained in a similar way. Let us note that the proof of [1, eq. (21)] works for arbitrary graded braided R-bialgebras, not only for usual bialgebras. Hence for any $x \in A^n$ and any $a \in A^1$ we have

$$\Delta_{n,1}^A(xa) = x \otimes a + \sum x_{(1,n-1)}a_{\mathfrak{c}_A} \otimes x_{(2,1)_{\mathfrak{c}_A}}, \tag{7.2}$$

where in the above relation we used the notation $\mathfrak{c}_A(a' \otimes a'') = \sum a''_{\mathfrak{c}_A} \otimes a'_{\mathfrak{c}_A}$. On the other hand, since A is braided commutative by hypothesis, we get $\sum a''_{\mathfrak{c}_A} a'_{\mathfrak{c}_A} = a'a''$. This relation together with (7.2) yield

$$\begin{aligned}\left(m_A^{n,1} \circ \Delta_{n,1}^A\right)(xa) &= xa + \sum x_{(1,n-1)}a_{\mathfrak{c}_A}x_{(2,1)_{\mathfrak{c}_A}} \\ &= xa + \sum x_{(1,n-1)}x_{(2,1)}a = xa + \left(m_A^{n-1,1} \circ \Delta_{n-1,1}^A\right)(x)a.\end{aligned}$$

In conclusion, the first equality in (7.1) follows by induction, using the fact that the products xa with $x \in A^n$ and $a \in A^1$ generates A^{n+1} as a linear space, since A is a strongly graded R-ring.

As H is braided commutative, we also have

$$m_H^{n,1} \circ \Delta_{n,1}^H = (n+1)\mathrm{I}_{H^{n+1}} \quad \text{and} \quad m_H^{1,n} \circ \Delta_{1,n}^H = (n+1)\mathrm{I}_{H^{n+1}}. \tag{7.3}$$

Note that the relation (7.2) holds for the braided bialgebra H as well. In particular, taking into account that $\mathfrak{c}_H^{1,1} = -\mathfrak{c}_A^{1,1}$, we get

$$\Delta_{1,1}^H(xy) = x \otimes y + \mathfrak{c}_H^{1,1}(x \otimes y) = x \otimes y - \mathfrak{c}_A^{1,1}(x \otimes y),$$

for all $x, y \in H^1$. Since H is strongly graded and A is braided commutative we deduce that $m_A^{1,1}\Delta_{1,1}^H = 0$, that is (A, H) is almost-Koszul.

Now we can prove that (A, H) is Koszul, showing that $\mathrm{K}_l^*(A, H)$ is exact. We know that this complex is the direct sum of its subcomplexes $\mathrm{K}_l^*(A, H, p)$, where p is an arbitrary nonnegative integer. Since $\mathrm{K}_l^*(A, H, 0)$ is always exact, it is enough to show that $\mathrm{I}_{\mathrm{K}_l^*(A,H,p)}$ is null homotopic, for any $p > 0$. We need a sequence of maps $s^*: \mathrm{K}_l^{*+1}(A, H, p) \to \mathrm{K}_l^*(A, H, p)$ such that

$$d_l^{n-1} \circ s^{n-1} + s^n \circ d_l^n = \mathrm{I}_{\mathrm{K}^n(A,H,p)}. \tag{7.4}$$

We take $s^{-1} = 0 = s^p$, and for $0 \le n \le p-1$ we set

$$s^n(h \otimes a) = p^{-1} \sum ha_{(1,1)} \otimes a_{(2,n)}.$$

Since $\Delta_{1,0}^A(a) = a \otimes 1$ and $\Delta_{0,1}^H(h) = 1 \otimes h$, it is easy to prove (7.4) in the case when either $n = 0$ or $n = p-1$. We now suppose that $0 < n < p-1$ and we pick $h \in H^{p-n}$ and $a \in A^n$. Then

$$\begin{aligned}
\big(d_l^{n-1} \circ s^{n-1}\big)(h \otimes a) &= p^{-1} \sum \big(\mathrm{I}_{H^{m-n}} \otimes m_A^{1,n-1}\big)\big(\Delta_{p-n,1}^H\big(ha_{(1,1)}\big) \otimes a_{(2,n-1)}\big) \\
&= p^{-1} \sum h \otimes a_{(1,1)}a_{(2,n-1)} \\
&\qquad + p^{-1} \sum h_{(1,p-n-1)}a_{(1,1)_{\mathfrak{c}_H}} \otimes h_{(2,1)_{\mathfrak{c}_H}}a_{(2,n-1)} \\
&= np^{-1}h \otimes a \\
&\qquad + p^{-1} \sum h_{(1,p-n-1)}a_{(1,1)_{\mathfrak{c}_H}} \otimes h_{(2,1)_{\mathfrak{c}_H}}a_{(2,n-1)},
\end{aligned}$$

where for the second equality we used (7.2) written for H. A similar computation shows us that

$$\begin{aligned}
\big(s^n \circ d_l^n\big)(h \otimes a) &= p^{-1} \sum \big(m_H^{p-n-1,1} \otimes \mathrm{I}_{A^n}\big)\big(h_{(1,p-n-1)} \otimes \Delta_{1,n}^A\big(h_{(2,1)}a\big)\big) \\
&= p^{-1} \sum h_{(1,p-n-1)}h_{(2,1)} \otimes a \\
&\qquad + p^{-1} \sum h_{(1,p-n-1)}a_{(1,1)_{\mathfrak{c}_A}} \otimes h_{(2,1)_{\mathfrak{c}_A}}a_{(2,n-1)} \\
&= (p-n)p^{-1}h \otimes a \\
&\qquad + p^{-1} \sum h_{(1,p-n-1)}a_{(1,1)_{\mathfrak{c}_A}} \otimes h_{(2,1)_{\mathfrak{c}_A}}a_{(2,n-1)}.
\end{aligned}$$

We conclude the proof by remarking that $\mathfrak{c}_H^{1,1} = -\mathfrak{c}_A^{1,1}$, so (7.4) is true. □

7.3. Trivial extensions. As a first application of Theorem 7.2 we shall investigate some homological properties of trivial extensions. Let V be a non-zero R-bimodule, where R is an algebra over a field of characteristic zero. Obviously, the identity map of $V \otimes V$ is an involutive solution of the braid equation. Hence we can apply the previous theorem for $\mathfrak{c} := \mathrm{I}_{V\otimes V}$. In this case we get that $S_R(V, \mathfrak{c}) = T^a_R(V)$. On the other hand, since the characteristic of $\Bbbk$ is 0, the braided bialgebra $A_R(V) := S_R(V, -\mathfrak{c})$ coincides with the trivial algebra extension of R with kernel V. Thus, $A_R(V) = R \oplus V$, and the product of two elements in V is zero. The comultiplication of $A_R(V)$ is uniquely defined such that any element in $v \in V$ is primitive, that is $\Delta(v) = v \otimes 1 + 1 \otimes v$. Note that the braiding in $A_R(V)$ restricted to $V \otimes V$ is equal to $-\mathrm{I}_{V\otimes V}$.

Corollary 7.4. *If R is a separable algebra over a field $\Bbbk$ of characteristic zero, then $(T^a_R(V), A_R(V))$ and $(A_R(V), T^a_R(V))$ are Koszul. Moreover,* $\operatorname{Hdim} T^a_R(V) = 1$ *and* $\operatorname{Hdim} A_R(V) = \sup\{n \mid V^{\otimes n} \neq 0\}$.

We can now use Corollary 7.4 to compute the Hochschild (co)homology of the trivial extension $A_R(V)$ with coefficients in an $A_R(V)$-bimodule M. For, we apply Theorem 4.3. Since the homogenous component of $T^a_R(V)$ of degree n is $V^{\otimes n}$, we have $\mathrm{K}_n(A_R(V), M) = M \mathbin{\widehat{\otimes}} V^{\widehat{\otimes} n}$. On the other hand, the component $\Delta_{p,n-p}$ of the comultiplication of $T^a_R(V)$ is given by the formula

$$\Delta_{p,n-p}(v_1 \cdots v_n) = \tbinom{n}{p} v_1 \cdots v_p \otimes v_{p+1} \cdots v_n.$$

where $v_1 \cdots v_n$ is a shorthand notation for the tensor monomial $v_1 \otimes \cdots \otimes v_n$, and $\binom{n}{p} = \frac{n!}{p!(n-p)!}$. Hence the differential of $\mathrm{K}_*(A_R(V), M)$ satisfies the relation $\partial_n := n\partial'_n$, where

$$\partial'_n(m \mathbin{\widehat{\otimes}} v_1 \mathbin{\widehat{\otimes}} \cdots \mathbin{\widehat{\otimes}} v_n) = (m \cdot v_1) \mathbin{\widehat{\otimes}} v_2 \mathbin{\widehat{\otimes}} \cdots \mathbin{\widehat{\otimes}} v_n + (-1)^n (v_n \cdot m) \mathbin{\widehat{\otimes}} v_1 \mathbin{\widehat{\otimes}} \cdots \mathbin{\widehat{\otimes}} v_{n-1}.$$

Since the characteristic of $\Bbbk$ is zero, we conclude that the Hochschild homology of $A_R(V)$ with coefficients in M is the homology of the complex $(M \mathbin{\widehat{\otimes}} V^{\widehat{\otimes} *}, \partial'_*)$.

In the more particular case $M := A_R(V)$, we can go further on the computation of Hochschild homology. Indeed, in this setting we can identify $A_R(V) \mathbin{\widehat{\otimes}} V^{\widehat{\otimes} n}$ with $V^{\widehat{\otimes} n} \oplus V^{\widehat{\otimes} n+1}$ via the linear map

$$(r, v_0) \mathbin{\widehat{\otimes}} v_1 \mathbin{\widehat{\otimes}} \cdots \mathbin{\widehat{\otimes}} v_n \longmapsto (r v_1 \mathbin{\widehat{\otimes}} \cdots \mathbin{\widehat{\otimes}} v_n, v_0 \mathbin{\widehat{\otimes}} v_1 \mathbin{\widehat{\otimes}} \cdots \mathbin{\widehat{\otimes}} v_n).$$

Note that the endomorphism λ_n of $V^{\widehat{\otimes} n}$, that maps $v_1 \mathbin{\widehat{\otimes}} \cdots \mathbin{\widehat{\otimes}} v_n$ to

$$(-1)^{n+1} v_n \mathbin{\widehat{\otimes}} v_1 \mathbin{\widehat{\otimes}} \cdots \mathbin{\widehat{\otimes}} v_{n-1},$$

defines an action of the cyclic group $\mathbb{Z}_n$ on $V^{\widehat{\otimes} n}$. With respect to this action and the above identification, $\partial'_n\colon V^{\widehat{\otimes} n} \oplus V^{\widehat{\otimes} n+1} \to V^{\widehat{\otimes} n-1} \oplus V^{\widehat{\otimes} n}$ is given by the formula

$$\partial'_n(x, y) = (0, x - \lambda_n(x)).$$

Hence, for any n, we have

$$\begin{aligned}\mathrm{HH}_n\left(A_R(V), A_R(V)\right) &= \mathrm{Ker}\left(\mathrm{I}_{V^{\widehat{\otimes} n}} - \lambda_n\right) \bigoplus \mathrm{Coker}\left(\mathrm{I}_{V^{\widehat{\otimes} n+1}} - \lambda_{n+1}\right)\\ &= \left(V^{\widehat{\otimes} n}\right)^{\mathbb{Z}_n} \bigoplus \left(V^{\widehat{\otimes} n+1}\right)_{\mathbb{Z}_{n+1}}.\end{aligned}$$

Note that the summands in the above equation are the spaces of invariant and coinvariant elements with respect to the actions induced by λ_n and λ_{n+1}, respectively.

Remark 7.5. The complex from Theorem 4.4 coincides with the one introduced by Cibils in [10] using a different method, based on the work on the rigidity of certain algebras [8,9].

As a more particular case, we now consider the trivial extension associated to a quiver $\mathbf{\Gamma}$ with a finite set $\mathbf{\Gamma}^0$ of vertices, but with an arbitrary set $\mathbf{\Gamma}^1$ of arrows. The source and the target maps of $\mathbf{\Gamma}$ will be denoted by $s, t\colon \mathbf{\Gamma}^1 \to \mathbf{\Gamma}^0$.

By a path of length n (or, equivalently, an n-path) in $\mathbf{\Gamma}$ we mean a sequence of n arrows $\gamma := a_1 \cdots a_n$ such that $t(a_i) = s(a_{i+1})$, for any $i = 1, \ldots, n-1$. The vertices $s(\gamma) := s(a_1)$ and $t(\gamma) := t(a_n)$ will be called the source and the target of γ, respectively. A vertex $v \in \mathbf{\Gamma}^0$ will be regarded as a path of length 0, with the same source and target v. Clearly the paths of length 1 coincide with the arrows of $\mathbf{\Gamma}$. The set of n-paths will be denoted by $\mathbf{\Gamma}^n$. Note that this notation is consistent with the one that we use for the set of vertices and arrows in $\mathbf{\Gamma}$.

Let $\Bbbk$ be a field and let $\Bbbk\mathbf{\Gamma}$-denote the vector space admitting as a basis the set $\bigcup_{n\in\mathbb{N}} \mathbf{\Gamma}^n$ of all paths in $\mathbf{\Gamma}$. Recall that with respect to the multiplication

$$(a_1 \cdots a_n) \cdot (a'_1 \cdots a'_m) = \begin{cases} a_1 \cdots a_n a'_1 \cdots a'_m, & \text{if } t(a_n) = s(a'_1);\\ 0, & \text{otherwise;}\end{cases}$$

$\Bbbk\mathbf{\Gamma}$ is an associative $\Bbbk$-algebra, which is called the *path algebra* of $\mathbf{\Gamma}$. Since $\mathbf{\Gamma}^0$ is finite, the sum of all vertices is a unit of $\Bbbk\mathbf{\Gamma}$. Note that any path can be regarded as the product of its arrows.

There is a standard grading on $\Bbbk\mathbf{\Gamma}$, given by the decomposition $\Bbbk\mathbf{\Gamma} = \oplus_{n\in\mathbb{N}} \Bbbk\mathbf{\Gamma}^n$, where $\Bbbk\mathbf{\Gamma}^n$ is the linear subspace spanned by $\mathbf{\Gamma}^n$. In particular, $\Bbbk\mathbf{\Gamma}$ is a connected $\Bbbk\mathbf{\Gamma}^0$-ring. Note that $\Bbbk\mathbf{\Gamma}^0$ is isomorphic as a $\Bbbk$-algebra with the direct product of $\#\mathbf{\Gamma}^0$ copies of $\Bbbk$, since the vertices of $\mathbf{\Gamma}$ are orthogonal central idempotents in $\Bbbk\mathbf{\Gamma}$. Obviously, this $\Bbbk$-algebra is separable.

Furthermore, $\Bbbk\mathbf{\Gamma}^1$ is a $\Bbbk\mathbf{\Gamma}^0$-bimodule with respect to the structure induced by the multiplication. It is well known that the map $a_1 \cdots a_n \mapsto a_1 \otimes \cdots \otimes a_n$

is an isomorphism of connected $\Bbbk\mathbf{\Gamma}^0$-rings between $\Bbbk\mathbf{\Gamma}$ and $T^a_{\Bbbk\mathbf{\Gamma}^0}(\Bbbk\mathbf{\Gamma}^1)$. Since $(\Bbbk\mathbf{\Gamma}^1)^{\otimes n} \cong \Bbbk\mathbf{\Gamma}^n$, by applying Corollary 7.4 one proves the following.

Corollary 7.6. *Let $\mathbf{\Gamma}$ denote a quiver with a finite number of vertices. The Hochschild dimension of the trivial extension $A_{\Bbbk\mathbf{\Gamma}^0}(\Bbbk\mathbf{\Gamma}^1)$ is given by*

$$\operatorname{Hdim} A_{\Bbbk\mathbf{\Gamma}^0}(\Bbbk\mathbf{\Gamma}^1) = \sup\{n \mid \mathbf{\Gamma}^n \neq \emptyset\}.$$

7.7. Multiparametric quantum spaces. We are going to apply the results that we have obtained in order to give an alternative proof of some known homological properties (cf. for instance [14] and [24]) of the multiparametric quantum spaces $O_q(\mathbb{A}^n)$. Recall from [1] that the affine quantum spaces can be defined as the symmetric $\Bbbk$-algebra $S_\Bbbk(V, \mathfrak{c})$, where V is an n-dimensional vector space and $\mathfrak{c}\colon V \otimes_\Bbbk V \to V \otimes_\Bbbk V$ is the solution of the braid equation given by $\mathfrak{c}(x_j \otimes x_i) = q_{ij} x_i \otimes x_j$. Here, the set $\{x_1, \dots, x_n\}$ denotes a basis on V, and the family of quantum parameters $q = \{q_{ij}\}_{i,j}$ defining $O_q(\mathbb{A}^n)$ is assumed to satisfy the conditions $q_{ij}q_{ji} = 1$, for all $i, j = 1, \dots, n$.

One can think of $O_q(\mathbb{A}^n)$ as the algebra with generators $\{x_1, \dots, x_n\}$ and relations $x_j x_i = q_{ij} x_i x_j$ for all $1 \leq i, j \leq n$, with the natural grading. Note that $x_i^2 = 0$, provided that $q_{ii} = -1$ and $\operatorname{char}(\Bbbk) \neq 2$. Therefore, if the later relation holds for all $i \in \{1, \dots, n\}$ then $O_q(\mathbb{A}^n)$ is a finite dimensional algebra of dimension 2^n, and the homogeneous component of degree n is one dimensional. Clearly, in this case all other components of higher degree vanish. On the other hand, if $q_{ii} = 1$ for a certain i, then the subalgebra generated by x_i is a polynomial ring, so $O_q(\mathbb{A}^n)$ is infinite dimensional.

In view of Theorem 7.2, the pair $(O_q(\mathbb{A}^n), O_{-q}(\mathbb{A}^n))$ is Koszul. By Theorem 4.5, it follows that the Hochschild dimension of $O_q(\mathbb{A}^n)$ is finite if and only if $O_{-q}(\mathbb{A}^n)$ is finite dimensional, i.e. $q_{ii} = 1$ for all $1 \leq i \leq n$. In this situation, $\operatorname{Hdim}(O_q(\mathbb{A}^n)) = n$ and on the basis

$$\{x_{i_1} x_{i_2} \cdots x_{i_r} \mid 1 \leq i_1 < i_2 < \cdots < i_r \leq n\}$$

the comultiplication for $T(O_q(\mathbb{A}^n)) \cong O_{-q}(\mathbb{A}^n)$ is defined by

$$\Delta_{p,q}(x_{i_1} x_{i_2} \cdots x_{i_{p+q}}) = \sum_{\nu \in \operatorname{Sh}(p,q)} q_\nu^{-1} x_{i_{\nu(1)}} \cdots x_{i_{\nu(p)}} \otimes x_{i_{\nu(p+1)}} \cdots x_{i_{\nu(p+q)}},$$

where ν ranges in the set $\operatorname{Sh}(p, q)$ of all shuffles of type (p, q) and the constant q_ν is the q-sign of the shuffle ν, defined as

$$q_\nu = \prod_{\substack{i<j,\\ \nu(j)<\nu(i)}} \big(-q_{\nu(j)\nu(i)}\big).$$

Indeed, $O_{-q}(\mathbb{A}^n) = S_R(V, -\mathfrak{c})$ is a braided bialgebra. Let us denote its braiding by $\mathfrak{c}_O$. As the comultiplication of the multiparametric quantum space is an algebra

map from $O_{-q}(\mathbb{A}^n)$ to the twisted tensor product $O_{-q}(\mathbb{A}^n) \otimes_{\mathfrak{c}_O} O_{-q}(\mathbb{A}^n)$, the desired formula for $\Delta_{p,q}$ can be easily proved by induction using the fact that each x_i is a primitive element.

7.8. The incidence algebra of the power set of a finite set. Let $\Bbbk$ be a field of characteristic 0 and let $X := \{1, \dots, d\}$. The power set $\mathcal{P}(X)$ of X is a poset with respect to the order relation given by inclusion. We are now going to show that the incidence algebra $A(X)$ of this poset is a Koszul R-ring, where $R := \Bbbk^{2^d}$. By definition, $A(X)$ has as a basis the set $\{e_{[I,J]} \mid I \subseteq J\}$, where the interval $[I, J]$ contains all subsets K such that $I \subseteq K \subseteq J$. The multiplication of $A(X)$ is defined by the relation

$$e_{[I,J]} \cdot e_{[I',J']} = \delta_{J,I'} e_{[I,J']}.$$

The unit of $A(X)$ is $\sum_{I \subseteq X} e_{[I,I]}$. The incidence algebra of X is a graded R-ring. Its n-degree component $A^n(X)$ is spanned by all $e_{[I,J]}$ with $|J \setminus I| = n$. In particular, $A(X)$ is connected as an R-ring, as $A^0(X)$ is generated by $\{e_{[I,I]} \mid I \subseteq X\}$, which is a complete set of orthogonal idempotents.

The Koszulity of $A(X)$ will follow as a consequence of the fact that this R-ring is an example of braided symmetric R-bialgebra. Let $V := A^1(X)$. Obviously, V is an R-bimodule as it is a homogeneous component of a connected R-ring. The set of all tensor monomials $e_{I_0 \dots I_n} := e_{[I_0,I_1]} \otimes \cdots \otimes e_{[I_{n-1},I_n]}$ with $I_k \subseteq I_{k+1}$ and $|I_{k+1} \setminus I_k| = 1$ is a basis of $V^{\otimes n}$ regarded as a vector space (recall that by $\otimes$ we mean $\otimes_R$).

Our goal now is to construct an involutive solution $\mathfrak{c}\colon V \otimes V \to V \otimes V$ of the braid equation. As a $\Bbbk$-linear map, $\mathfrak{c}$ is uniquely defined by the elements $\mathfrak{c}(e_{I_0 I_1 I_2})$, where each I_k is a predecessor of I_{k+1} in the poset $\mathcal{P}(X)$, that is I_k is a subset of I_{k+1} and $|I_k| = |I_{k+1}| - 1$. Hence $I_1 = I_0 \bigcup \{i_1\}$ and $I_2 = I_0 \bigcup \{i_1, i_2\}$, where i_1 and i_2 are distinct elements which do not belong to I_0. Let $I_1' := I_0 \bigcup \{i_2\}$. We can now define $\mathfrak{c}$ by

$$\mathfrak{c}(e_{I_0 I_1 I_2}) := e_{I_0 I_1' I_2}. \tag{7.5}$$

Let us note that the Hasse diagram of $\mathcal{P}(X)$ can be identified with the unit cube in $\mathbb{R}^d$, regarded as a quiver $\mathbf{\Upsilon}_d$. The set of vertices of this quiver is the set $\{0, 1\}^d \subseteq \mathbb{R}^d$, and an arrow of this quiver has the source $(i_1, \dots, i_d)$ and the target $(j_1, \dots, j_d)$ if and only if the former vertex is the successor of the latter with respect to the lexicographic ordered on $\{0, 1\}^d$. Thus, from a geometric point of view, the $\Bbbk$-linear map $\mathfrak{c}$ interchanges any pair of oriented 2-paths having the same source and the same target.

In view of the above geometric interpretation of the Hasse diagram, we shall say that I_0 (respectively I_n) is the source (respectively the target) of $e_{I_0 \dots I_n} \in V^{\otimes n}$. Since in the equation (7.5) both elements of the basis on $V^{\otimes 2}$ have the same source

and the same target it follows that $\mathfrak{c}$ is a morphism of R-bimodules. On the other hand, if $e_{I_0I_1I_2I_3} \in V^{\otimes 3}$ and $I_k = I_0 \bigcup \{i_1 \ldots, i_k\}$, then

$$(\mathfrak{c}_1 \circ \mathfrak{c}_2 \circ \mathfrak{c}_1)(e_{I_0I_1I_2I_3}) = e_{I_0I_1'I_2'I_3} = (\mathfrak{c}_2 \circ \mathfrak{c}_1 \circ \mathfrak{c}_2)(e_{I_0I_1I_2I_3}),$$

where $I_1' = I_0 \bigcup \{i_3\}$ and $I_2' = I_0 \bigcup \{i_2, i_3\}$. In conclusion, $(V, \mathfrak{c})$ is a braided R-bimodule. Clearly, $\mathfrak{c}$ is involutive, so it makes sense to consider the braided R-bialgebras $S_R(V, \mathfrak{c})$ and $S_R(V, -\mathfrak{c})$.

Theorem 7.9. *There is an isomorphism of graded R-rings $A(X) \cong S_R(V, \mathfrak{c})$ In particular, $A(X)$ is a 3^d-dimensional Koszul ring of Hochschild dimension d.*

Proof. Let $T := T_R^a(V)$ be the braided R-bialgebra with braiding $\mathfrak{c}_T$, see §7.1. For $n > d$, the n-degree homogeneous component T^n of T vanishes, as any increasing sequence $I_0 \subsetneq \cdots \subsetneq I_k$ has length $k \leq d$. As $S := S(V, \mathfrak{c})$ is a quotient braided R-bialgebra of T we deduce that $S^n = 0$ for any $n > d$.

We claim that $\dim S^n = 2^{d-n}\binom{d}{n}$ for any $n \leq d$. We start the proof of this relation by recalling that the *involutive* braiding $\mathfrak{c}$ induces an action of the symmetric group Σ_n on T^n such that the transposition $\sigma_i := (i, i+1)$ acts on $v_1 \otimes \cdots \otimes v_n$ by

$$\sigma_i \cdot (v_1 \otimes \cdots \otimes v_n) = v_1 \otimes \cdots \otimes \mathfrak{c}(v_i \otimes v_{i+1}) \otimes \cdots \otimes v_n,$$

for any $v_1, \ldots, v_n \in V$. If $e_{I_0 \ldots I_n}$ is an element of the basis on T^n, with $I_k = I_0 \bigcup \{i_1, \ldots i_k\}$, then

$$\sigma \cdot e_{I_0 \ldots I_n} = e_{J_0 \ldots J_n}, \tag{7.6}$$

where $J_0 = I_0$ and $J_k = J_0 \bigcup \{i_{\sigma(1)}, \ldots, i_{\sigma(k)}\}$, for any $k = 1, \ldots, n$.

Regarding each permutation $\sigma \in \Sigma_n$ as an R-bilinear automorphism of $V^{\otimes n}$ and taking into account the definition of the braided symmetric R-bialgebra S, we get

$$S^n = \frac{V^{\otimes n}}{\sum_{i=1}^n \operatorname{Im}(1 - \sigma_i)} = \frac{V^{\otimes n}}{\sum_{\sigma \in \Sigma_n} \operatorname{Im}(1 - \sigma)}.$$

Note that the second equation is a consequence of the relation

$$\mathrm{I}_{V^{\otimes n}} - \sigma_{i_1} \cdots \sigma_{i_n} = (\mathrm{I}_{V^{\otimes n}} - \sigma_{i_n}) + (\sigma_{i_n} - \sigma_{i_{n-1}}\sigma_{i_n}) + \cdots \\ \cdots + (\sigma_{i_2} \cdots \sigma_{i_n} - \sigma_{i_1}\sigma_{i_2} \cdots \sigma_{i_n})$$

and of the fact that $\sigma_1, \ldots, \sigma_{n-1}$ generate Σ_n. Hence, S^n coincides with the coinvariant quotient space $(V^{\otimes n})_{\Sigma_n}$. Since, by assumption $\Bbbk$ is a field of characteristic zero, the canonical linear map from the invariant subspace $(V^{\otimes n})^{\Sigma_n}$ to S^n is a $\Bbbk$-linear isomorphism.

In conclusion, we have to show that $\dim(V^{\otimes n})^{\Sigma_n} = 2^{d-n}\binom{d}{n}$. For, we split the representation $T^n = V^{\otimes n}$ as a direct sum of sub-representations

$$T^n = \oplus_{I \subseteq J} T^n(I, J),$$

where $T^n(I,J)$ denotes the vector space spanned by the elements $e_{I_0,\dots,I_n}$ with source I and target J. The summands are indexed by all pairs (I,J) such that $I \subseteq J \subseteq \{1,\dots,d\}$ and $J \setminus I$ is a set with n elements. Thus the above decomposition has

$$\sum_{k=n}^{d} \binom{d}{k}\binom{k}{n} = 2^{d-n}\binom{d}{n} \tag{7.7}$$

terms. On the other hand $\dim T^n(I,J) = n!$ and, by relation (7.6), it follows that the action of Σ_n on $T^n(I,J)$ is transitive for any $I \subseteq J$ with $|J \setminus I| = n$. We deduce that $T^n(I,J)^{\Sigma_n}$ is a vector space of dimension 1. Hence our claim has been proved, as $\dim S^n = \dim \oplus_{I\subseteq J} T^n(I,J)^{\Sigma_n} = 2^{d-n}\binom{d}{n}$.

We can now prove that S and $A(X)$ are isomorphic. Let $\varphi\colon T \to A(X)$ be the canonical morphism of graded R-rings that extends the identity map $\varphi^0\colon T^0 \to A^0(X)$ and the R-bimodule morphism $\varphi^1\colon T^1 \to A^1(X)$ mapping e_{IJ} to $e_{[I,J]}$, for any $I \subseteq J$ with J a successor of I. Since $\varphi^n(e_{I_0\dots I_n}) = e_{[I_0,I_n]}$, it is easy to see that φ vanishes on the ideal generated by the image of $\mathrm{I}_{T^2} - \mathfrak{c}$. Hence φ induces a surjective graded ring morphism $\overline{\varphi}\colon S \to A(X)$.

To prove that $\overline{\varphi}$ is an isomorphism we notice that $\{e_{[I,J]} \mid I \subseteq J \text{ and } |J\setminus I| = n\}$ is a basis of $A^n(X)$. Hence $\dim A^n(X) = 2^{d-n}\binom{d}{n}$ by the proof of equation (7.7). Therefore, every $\overline{\varphi}^n$ is bijective and

$$\dim A(X) = \sum_{n=0}^{d} \binom{d}{n} 2^{d-n} = (1+2)^d = 3^d.$$

It remains to show that $\operatorname{Hdim} A(X) = d$. Proceeding as above we can show that $S^n(V,-\mathfrak{c}) = 0$ for $n > d$. On the other hand, $\dim S^n(V,-\mathfrak{c}) = 2^{d-n}\binom{d}{n}$, for $n \le d$. We conclude applying Theorem 4.5. □

7.10. A generalization of the Fröberg theorem. Let $\Bbbk$ be a field. By a result of Fröberg [12], the quotient of the free algebra $\Bbbk\langle X_1,\dots,X_n\rangle$ by the ideal generated by a set of non-commuting monomials of degree 2 is a Koszul $\Bbbk$-algebra. As an application of our results on Koszul pairs, we shall prove a similar result for the quotient of a path algebra by an ideal which is generated by 2-paths.

We fix a quiver $\mathbf{\Gamma}$ with a finite number of vertices and a set Φ of 2-paths. The complement of Φ in $\mathbf{\Gamma}^2$ will be denoted by Φ'. We shall also use the following notation: $R := \Bbbk\mathbf{\Gamma}^0$ and $V := \Bbbk\mathbf{\Gamma}^1$. We define the connected R-ring $A(\mathbf{\Gamma},\Phi)$ to be the quotient of the path algebra $\Bbbk\mathbf{\Gamma}$ by the ideal generated by Φ.

For $n = 0$ and $n = 1$ we set $\mathbf{\Gamma}^n_\Phi := \mathbf{\Gamma}^n$. On the other hand, if $n \ge 2$ let $\mathbf{\Gamma}^n_\Phi$ denote the set of n-paths $\gamma = a_1\cdots a_n$ such that $a_i a_{i+1} \in \Phi$, for all i. The sets $\mathbf{\Gamma}^n_{\Phi'}$ are defined in a similar way.

The linear transformation that maps a path in $\mathbf{\Gamma}^n_{\Phi'}$ to its equivalence class in $A(\mathbf{\Gamma},\Phi)^n$ is an isomorphism, for all n. This property allows us to identify $A(\mathbf{\Gamma},\Phi)$

with the connected R-ring $\oplus_{n\geq 0}\Bbbk\mathbf{\Gamma}^n_{\Phi'}$, whose multiplication is given for $a_1\cdots a_n\in\mathbf{\Gamma}^n_{\Phi'}$ and $a'_1\cdots a'_m\in\mathbf{\Gamma}^m_{\Phi'}$ by

$$(a_1\cdots a_n)\cdot(a'_1\cdots a'_m)=\begin{cases}a_1\cdots a_n a'_1\cdots a'_m, & \text{if } t(a_n)=s(a'_1) \text{ and } a_n a'_1\in\Phi';\\ 0, & \text{otherwise.}\end{cases}$$

From now on we shall regard the R-ring $A(\mathbf{\Gamma},\Phi)$ as a subspace of $\Bbbk\mathbf{\Gamma}$ with respect to this multiplication.

The R-bimodule isomorphisms $\Delta_{p,q}\colon \Bbbk\mathbf{\Gamma}^{p+q}\to\Bbbk\mathbf{\Gamma}^p\otimes\Bbbk\mathbf{\Gamma}^q$

$$\Delta_{p,q}(a_1\cdots a_{p+q})=a_1\cdots a_p\otimes a_{p+1}\cdots a_{p+q} \tag{7.8}$$

define an R-coring structure on $\Bbbk\mathbf{\Gamma}$ (recall that $\otimes=\otimes_R$). Let $C(\mathbf{\Gamma},\Phi)_n$ be the linear subspace spanned by $\mathbf{\Gamma}^n_\Phi$. By definition, $C(\mathbf{\Gamma},\Phi):=\oplus_{n\geq 0}C(\mathbf{\Gamma},\Phi)_n$ is a graded R-subcoring of $\Bbbk\mathbf{\Gamma}$.

Theorem 7.11. *The pair* $(A(\mathbf{\Gamma},\Phi),C(\mathbf{\Gamma},\Phi))$ *is Koszul and* $\operatorname{Hdim}A(\mathbf{\Gamma},\Phi)=\sup\{n\mid\mathbf{\Gamma}^n_\Phi\neq\emptyset\}$.

Proof. Let $A:=A(\mathbf{\Gamma},\Phi)$ and $C:=C(\mathbf{\Gamma},\Phi)$. Clearly, by construction, A is a connected R-ring and C is a connected R-coring. Let $\theta_{C,A}:=\mathrm{I}_V$. If $\gamma=aa'$ is a 2-path in Φ then $m^{1,1}\circ\Delta_{1,1}(\gamma)=0$. Indeed, $\Delta_{1,1}(\gamma)=a\otimes a'$ and the product in A of a and a' is 0, since $aa'\notin\Phi'$. Thus (A,C) is almost-Koszul.

By the definition of Koszul pairs, it is enough to prove that $\mathrm{K}^l_*(A,C,m)$ is exact for every $m>0$. Recall that $\mathrm{K}^l_n(A,C,m)=A^{m-n}\otimes C_n$, for any $n\leq m$, and there are no nontrivial n-chains in higher degrees, see §2.21. Note that the set of tensor monomials $\gamma'\otimes\gamma''$, with $\gamma'\in\mathbf{\Gamma}^p_{\Phi'}$ and $\gamma''\in\mathbf{\Gamma}^q_\Phi$ satisfying the condition $t(\gamma')=s(\gamma'')$, is a basis $\mathbf{\Gamma}^{p,q}_{\Phi',\Phi}$ on $A^p\otimes C_q$. Thus $\mathbf{\Gamma}^{m-n,n}_{\Phi',\Phi}$ is a basis on $\mathrm{K}^l_n(A,C,m)$.

The complex $\mathrm{K}^l_*(A,C,m)$ is exact in degree 0, as A is strongly graded and d^l_1 is induced by the multiplication. We now assume that $0<n<m$. Let ω be an n-cycle. Thus

$$\omega=\sum_{i=1}^r\alpha_i\lambda_i\otimes\lambda'_i,$$

for some $\alpha_i\in\Bbbk$ and $\lambda_i\otimes\lambda'_i\in\mathbf{\Gamma}^{m-n,n}_{\Phi',\Phi}$. We may assume without loss of generality that $\lambda_i\otimes\lambda'_i$ and $\lambda_j\otimes\lambda'_j$ are distinct for $i\neq j$. Let $\lambda_i=\gamma_i a_i$, with $\gamma_i\in\mathbf{\Gamma}^{m-n-1}$ and $a_i\in\mathbf{\Gamma}^1$. Similarly, $\lambda'_i=a'_i\gamma'_i$, with $a'_i\in\mathbf{\Gamma}^1$ and $\gamma'_i\in\mathbf{\Gamma}^{n-1}$. By definition of d^l_n and relation (7.8) we get

$$d^l_n(\omega)=\sum_{i=1}^r\alpha_i\gamma_i a_i\cdot a'_i\otimes\gamma'_i.$$

Let I denote the set of all i such that $a_i a'_i\in\Phi$. Then, for any $i\in I$, we have $\gamma_i a_i\cdot a'_i\otimes\gamma'_i=0$. On the other hand, if $i\notin I$ then $\gamma_i a_i\cdot a'_i\otimes\gamma'_i=\gamma_i a_i a'_i\otimes\gamma'_i$ is

an element of $\mathbf{\Gamma}_{\Phi',\Phi}^{m-n+1,n-1}$. Moreover, if i and j are distinct elements which do not belong to I, then $\gamma_i a_i a_i' \otimes \gamma_i'$ and $\gamma_j a_j a_j' \otimes \gamma_j'$ are distinct as well. Since $d_n^l(\omega) = 0$ it follows that $\alpha_i = 0$, for any $i \notin I$. Let us remark that $\gamma_i \otimes a_i a_i' \gamma_i'$ is an element of $\mathbf{\Gamma}_{\Phi',\Phi}^{m-n-1,n+1}$, for any $i \in I$, so it is a chain of degree $n+1$. Since

$$\omega = \sum_{i\in I} \alpha_i \gamma_i a_i \otimes a_i' \gamma_i' = d_{n+1}^l\Big(\sum_{i\in I} \alpha_i \gamma_i \otimes a_i a_i' \gamma_i'\Big)$$

we deduce that any n-cycle is a boundary. As d_m^l maps $1 \otimes a_1 \cdots a_m \in \mathbf{\Gamma}_{\Phi',\Phi}^{0,m}$ to $a_1 \otimes a_2 \cdots a_m \in \mathbf{\Gamma}_{\Phi',\Phi}^{1,m-1}$, this function is injective. In conclusion $\mathrm{K}_*^l(A, C, m)$ is exact.

The computation of the Hochschild dimension of A follows by Theorem 4.5. □

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